Chapter 3: Probability Concepts and Distributions

Statistics, Data Analysis, and Decision Modeling, Fifth Edition

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Probability

- Probability – the likelihood that an outcome occurs
- Probabilities are values between 0 and 1. The closer the probability is to 1, the more likely it is that the outcome will occur.
- Some convert probabilities to percentages
  - The statement "there is a 10% chance that oil prices will rise next quarter" is another way of stating that "the probability of a rise in oil prices is 0.1."

Experiments and Outcomes

- Experiment – a process that results in some outcome
  - Roll dice
  - Observe and record weather conditions
  - Conduct market survey
  - Watch the stock market
- Outcome – an observed result of an experiment
  - Sum of the dice
  - Description of the weather
  - Proportion of respondents who favor a product
  - Change in the Dow Jones Industrial Average

Sample Space

- Sample space - all possible outcomes of an experiment
  - Dice rolls: 2, 3, …, 12
  - Weather outcomes: clear, partly cloudy, cloudy
  - Customer reaction: proportion who favor a product (a number between 0 and 1)
  - Change in DJIA: positive or negative real number

Three Views of Probability

- Classical definition: based on theory
- Relative frequency: based on empirical data
- Subjective: based on judgment

Classical Definition

- Probability = number of favorable outcomes divided by the total number of possible outcomes
- Example: There are six ways of rolling a 7 with a pair of dice, and 36 possible rolls. Therefore, the probability of rolling a 7 is 6/36 = 0.167.

Relative Frequency Definition

- Probability = number of times an event has occurred in the past divided by the total number of observations
- Example: Of the last 10 days when certain weather conditions have been observed, it has rained the next day 8 times. The probability of rain the next day is 0.80

Subjective Definition

- What is the probability that the New York Yankees will win the World Series this year?
- What is the probability your school will win its conference championship this year?
- What is the probability the NASDAQ will go up 2% next week?

Basic Probability Rules and Formulas

1. Probability associated with any outcome must be between 0 and 1
   - $0 \leq P(O_i) \leq 1$ for each outcome O_i
2. Sum of probabilities over all possible outcomes must be 1.0
   - $P(O_1) + P(O_2) + \ldots + P(O_n) = 1$

Example: Flip a coin three times

Outcomes: HHH, HHT, HTH, THH, HTT, THT, TTH, TTT

Each has probability of $(1/2)^3 = 1/8$

Events

- An event is a collection of one or more outcomes from
  - Obtaining a 7 or 11 on a roll of dice
  - Having a clear or partly cloudy day
  - The proportion of respondents that favor a product is at least 0.60
  - Having a positive weekly change in the Dow
- If *A* is any event, the complement of *A*, denoted A^c, consists of all outcomes in the sample space not in *A*.

Rule 1

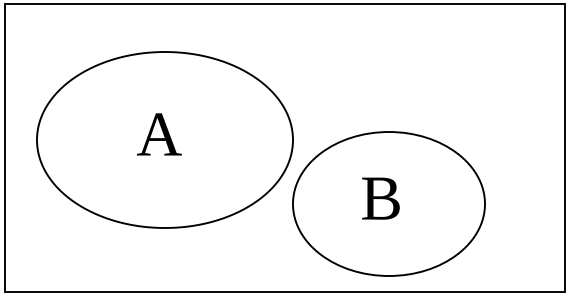
- The probability of any event is the sum of the probabilities of the outcomes that compose that event.

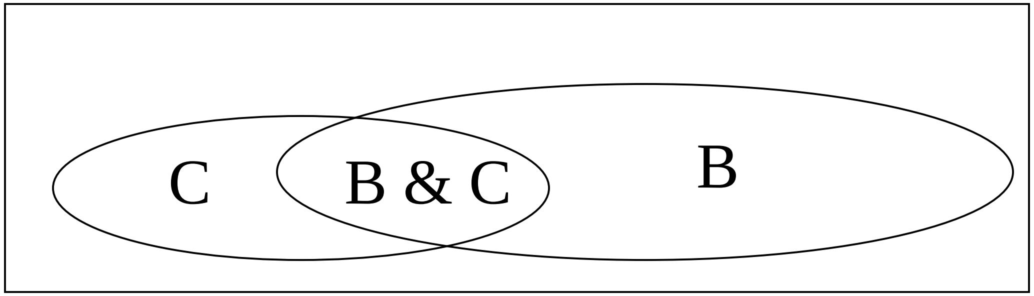
Rule 2

- The probability of the complement of any event A is $P(A^c) = 1 - P(A)$.

Mutually Exclusive Events

- Two events are mutually exclusive if they have no outcomes in common.





A and B are mutually exclusive

B and C are not

Rule 3

- If events A and B are mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B).$$

Rule 4

- If two events *A* and *B* are not mutually exclusive, then

  $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$

Example

- What is the probability of obtaining exactly two heads or exactly two tails in 3 flips of a coin?
  - These events are mutually exclusive. Probability = 3/8 + 3/8 = 6/8
- What is the probability of obtaining at least two tails or at least one head?
  - A = {TTT, TTH, THT, HTT}, B = {TTH, THT, THH, HTT, HTH, HHT, HHH} The events are *not* mutually exclusive.
  - P(A) = 4/8; P(B) = 7/8; P(A&B) = 3/8. Therefore, P(A or B) = 4/8 + 7/8 – 3/8 = 1

Conditional Probability

- Conditional probability – the probability of the occurrence of one event, *A*, given that another event *B* is known to be true or have already occurred.

$$P(A/B) = P(A \text{ and } B)/P(B)$$

Example

Cross-Tabulation	Brand 1	Brand 2	Brand 3	Total
Male	25	17	21	63
Female	9	6	22	37
Total	34	23	43	100

M = respondent is male
F = respondent is female
$B1$ = respondent prefers brand 1
$B2$ = respondent prefers brand 2
$B3$ = respondent prefers brand 3

$$P(B1 \mid M) = P(B1 \text{ and } M)/P(M) = (25/100)/(63/100) = 25/63 = 0.397.$$

$$P(B1 \mid F) = P(B1 \text{ and } F)/P(F) = (9/100)/(37/100) = 9/37 = 0.243.$$

P(Brand | Gender)	Brand 1	Brand 2	Brand 3
Male	0.397	0.270	0.333
Female	0.243	0.162	0.595

Multiplication Law of Probability

$$P(A \text{ and } B) = P(A \mid B)\, P(B) = P(B \mid A)\, P(A)$$

- Two events A and B are independent if

$$P(A \mid B) = P(A)$$

- If A and B are independent, then

$$P(A \text{ and } B) = P(B)P(A) = P(A)P(B)$$

Random Variables

- Random variable – a numerical description of the outcome of an experiment. Random variables are denoted by capital letters, X, Y, …; specific values by lower case letters, x, y, …
  - Discrete random variable – the number of possible outcomes can be counted
  - Continuous random variable – outcomes over one or more continuous intervals of real numbers

Examples of Random Variables

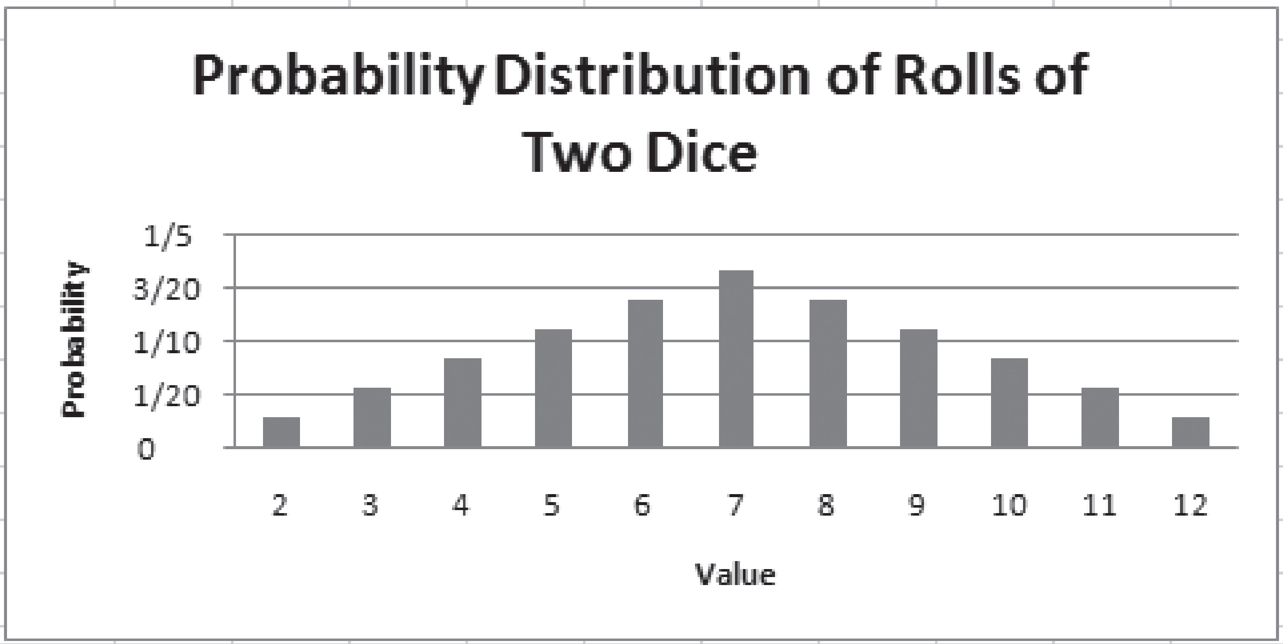
- Experiment: flip a coin 3 times.
  - Outcomes: TTT, TTH, THT, THH, HTT, HTH, HHT, HHH
  - Random variable: X = number of heads.
  - X can be either 0, 1, 2, or 3.
- Experiment: observe end-of-week closing stock price.
  - Random variable: Y = closing stock price.
  - X can be any nonnegative real number.

Probability Distributions

- Probability distribution – a characterization of the possible values a random variable may assume along with the probability of assuming these values.
  - Probability distributions may be defined for both discrete and continuous random variables.

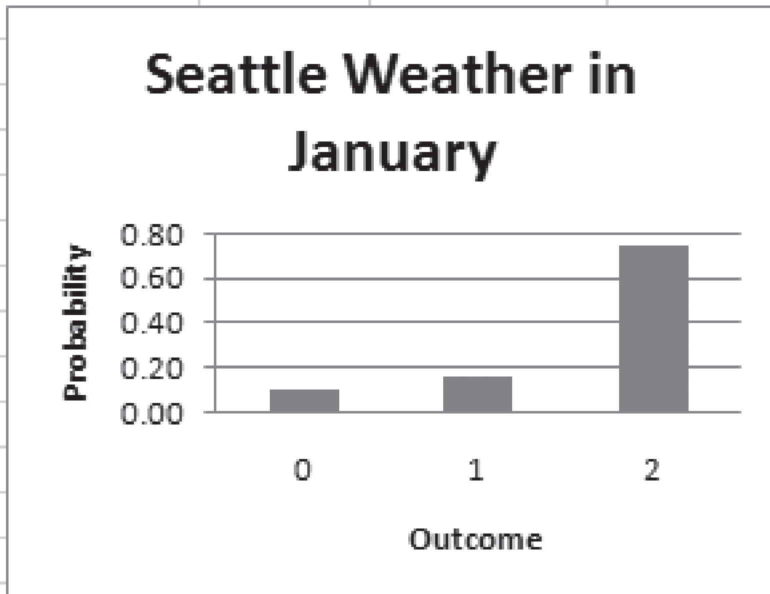
Example: Theoretical Probability Distribution

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	Outcome	2	3	4	5	6	7	8	9	10	11	12
3	Probability	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36



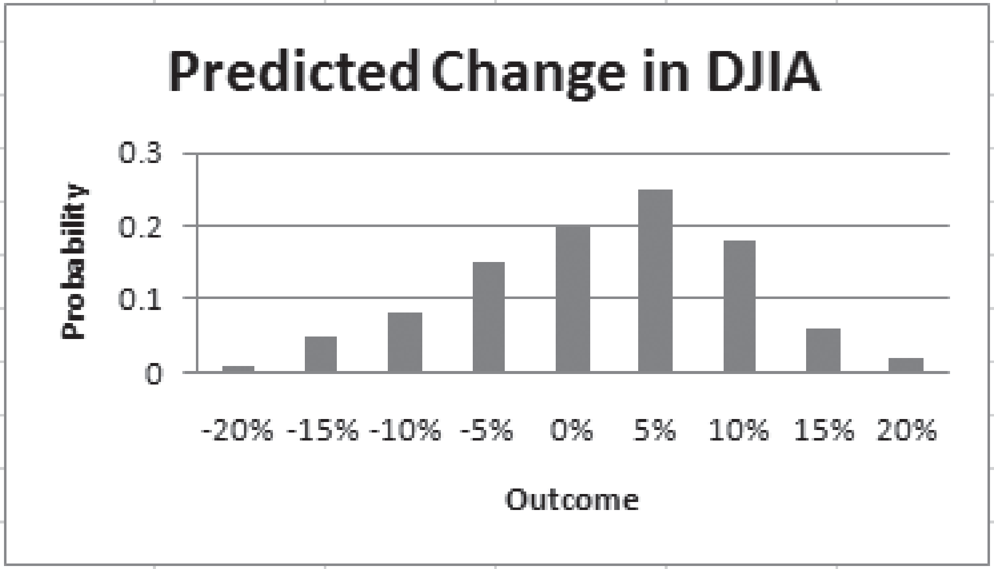
Example: Empirical Probability Distribution

	C	D	E	F
19	**Outcome**	Clear	Partly Cloudy	Cloudy
20	**Probability**	3/31	5/31	23/31



Example: Subjective Probability Distribution

	A	B
1	Change in DJIA	Subjective Probability
2	-20%	0.01
3	-15%	0.05
4	-10%	0.08
5	-5%	0.15
6	0%	0.2
7	5%	0.25
8	10%	0.18
9	15%	0.06
10	20%	0.02
11		



Discrete Random Variables

- Probability mass function f(x): specifies the probability of each discrete outcome
- Two properties:

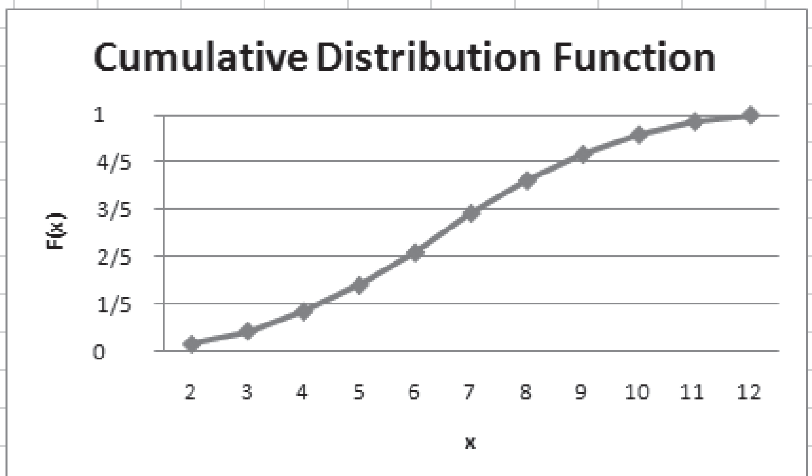
$$0 \le f(x_i) \le 1 \qquad \text{for all } i$$

$$\sum_i f(x_i) = 1$$

Cumulative Distribution Function, F(x)

- Specifies the probability that the random variable X will be *less than or equal to* x, denoted as $P(X \leq x)$.

	A	B	C	D	E	F	G	H	I	J	K	L
18	Outcome	2	3	4	5	6	7	8	9	10	11	12
19	Probability	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36
20	Cumulative Probability	1/36	1/12	1/6	5/18	5/12	7/12	13/18	5/6	11/12	35/36	1



Expected Value and Variance of Discrete Random Variables

- Expected value of a random variable X is the theoretical analogy of the mean, or weighted average of possible values:

$$E[X] = \sum_{i=1}^{\infty} x_i f(x_i)$$

- Variance and standard deviation of a random variable X:

$$Var[X] = \sum_{j=1}^{\infty} (x_j - E[X])^2 f(x_j)$$

$$\sigma_X = \sqrt{\sum_{j=1}^{\infty} (x_j - E[X])^2 f(x_j)}$$

Example

- You play a lottery in which you buy a ticket for $50 and are told you have a 1 in 1000 chance of winning $25,000. The random variable X is your net winnings, and its probability distribution is

x	f(x)
-$50	0.999
$24,950	0.001

Calculations

- $E[X] = -\$50(0.999) + \$24{,}950(0.001) = -\$25.00$
- $Var[X] = (-50 - [-25.00])^2(0.999) + (24{,}950 - [-25.00])^2(0.001) = 624{,}375$

Discrete Probability Distributions

- Bernoulli
- Binomial
- Poisson

Bernoulli Distribution

- A random variable with two possible outcomes: "success" ($x = 0$) and "failure" ($x = 1$)
- p = probability of "success"

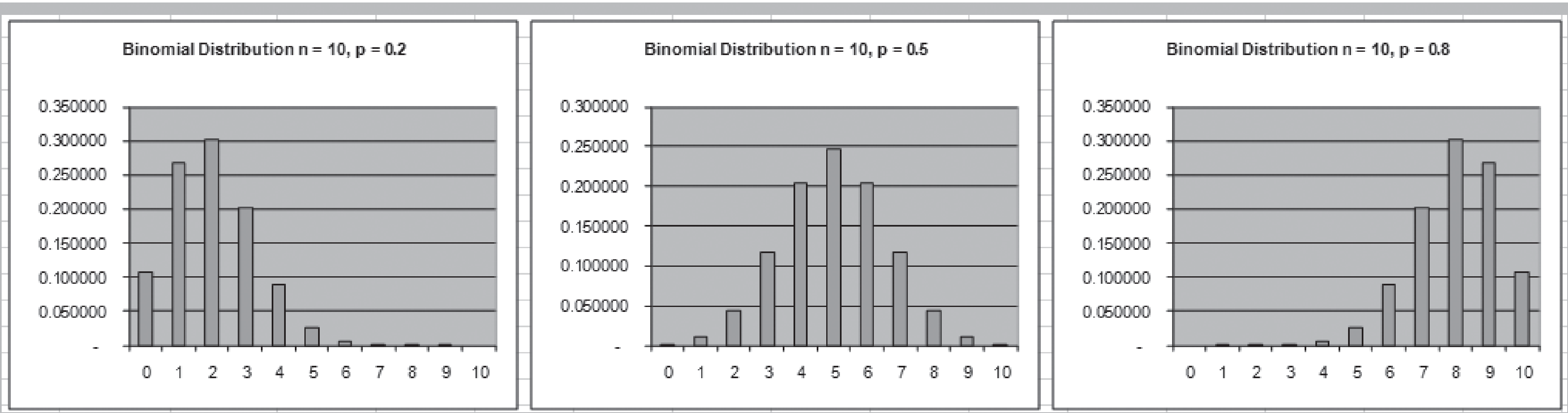
$$f(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

- Expected value = p; variance = $p(1 - p)$

Binomial Distribution

- n independent replications of a Bernoulli experiment, each with constant probability of success p
- X represents the number of successes in n experiments.
- Expected value $= np$
- Variance $= np(1-p)$

Examples of the Binomial Distribution



Excel Function

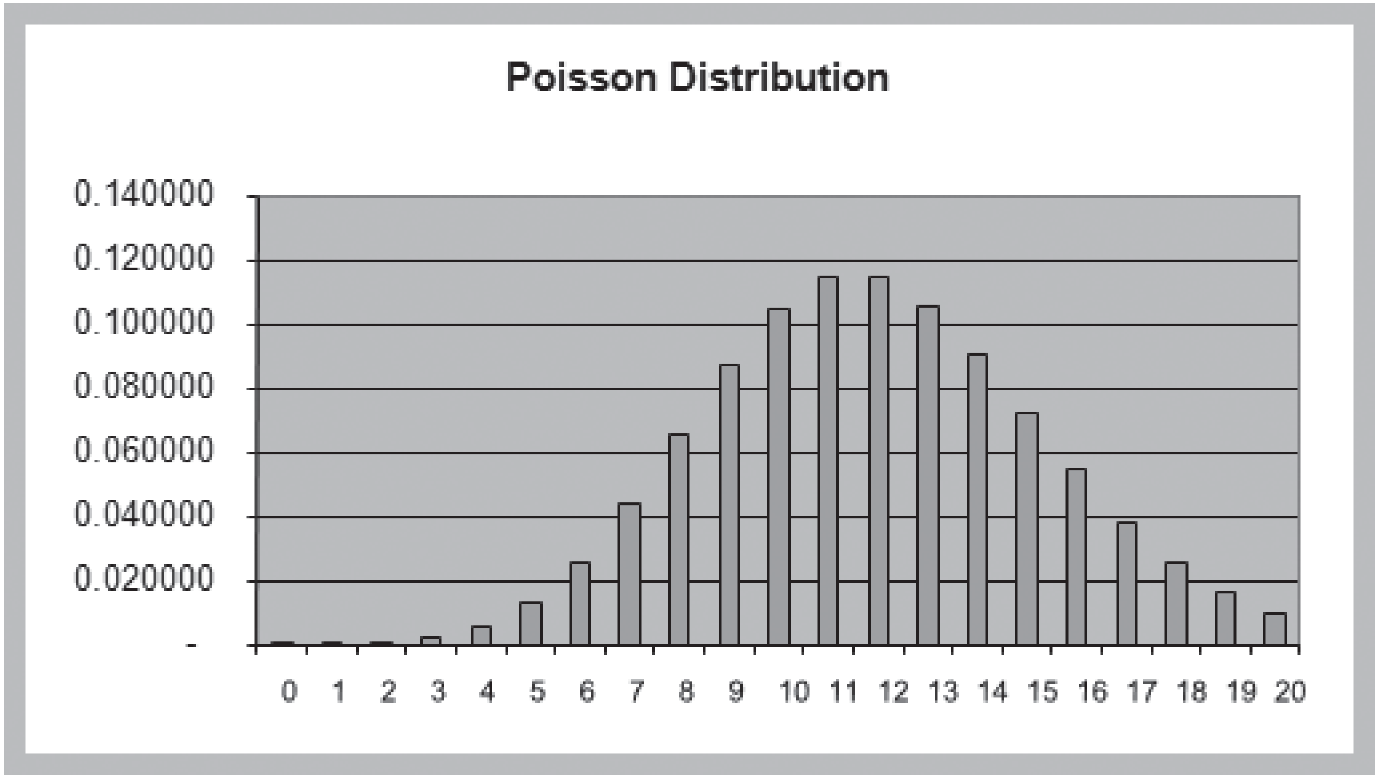
- BINOM.DIST(*number_s, trials, probability_s, cumulative*)

	A	B	C	D	E	F
1	Binomial Probabilities					
2			=BINOMDIST(A7,B3,B4,FALSE)			
3	n	10				
4	p	0.2	=BINOMDIST(A7,B3,B4,TRUE)			
5						
6	x	f(x)	F(x)			
7	0	0.107374	0.107374			
8	1	0.268435	0.375810			
9	2	0.301990	0.677800			
10	3	0.201327	0.879126			
11	4	0.088080	0.967207			
12	5	0.026424	0.993631			
13	6	0.005505	0.999136			
14	7	0.000786	0.999922			
15	8	0.000074	0.999996			
16	9	0.000004	1.000000			
17	10	0.000000	1.000000			

Poisson Distribution

- Models the number of occurrences in some unit of measure, e.g., events per unit time, number of items per order
- X = number of events that occur; $x = 0, 1, 2,$
- Expected value = λ; variance = λ
- Poisson approximates binomial when n is large and p small

Example of the Poisson Distribution ($\lambda = 12$)



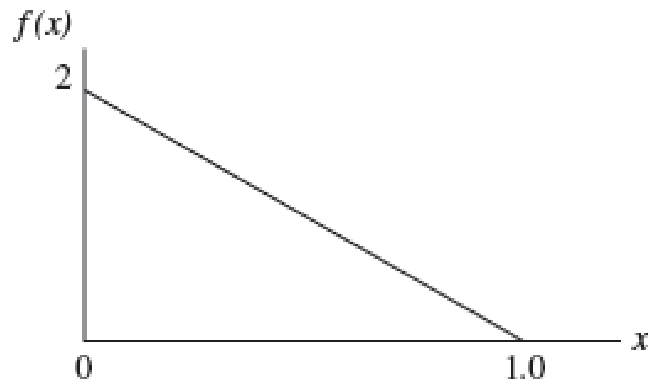
Excel Function

- POISSON.DIST(x, mean, cumulative)

	A	B	C	D	E	F
1	Poisson Probabilities					
2			=POISSON(A7,B3,FALSE)			
3	Mean	12				
4			=POISSON(A7,B3,TRUE)			
5						
6	**x**	**f(x)**	**F(x)**			
7	0	0.000006	0.000006			
8	1	0.000074	0.000080			
9	2	0.000442	0.000522			
10	3	0.001770	0.002292			
11	4	0.005309	0.007600			
12	5	0.012741	0.020341			
13	6	0.025481	0.045822			
14	7	0.043682	0.089504			
15	8	0.065523	0.155028			
16	9	0.087364	0.242392			
17	10	0.104837	0.347229			
18	11	0.114368	0.461597			
19	12	0.114368	0.575965			
20	13	0.105570	0.681536			
21	14	0.090489	0.772025			
22	15	0.072391	0.844416			
23	16	0.054293	0.898709			
24	17	0.038325	0.937034			
25	18	0.025550	0.962584			
26	19	0.016137	0.978720			
27	20	0.009682	0.988402			

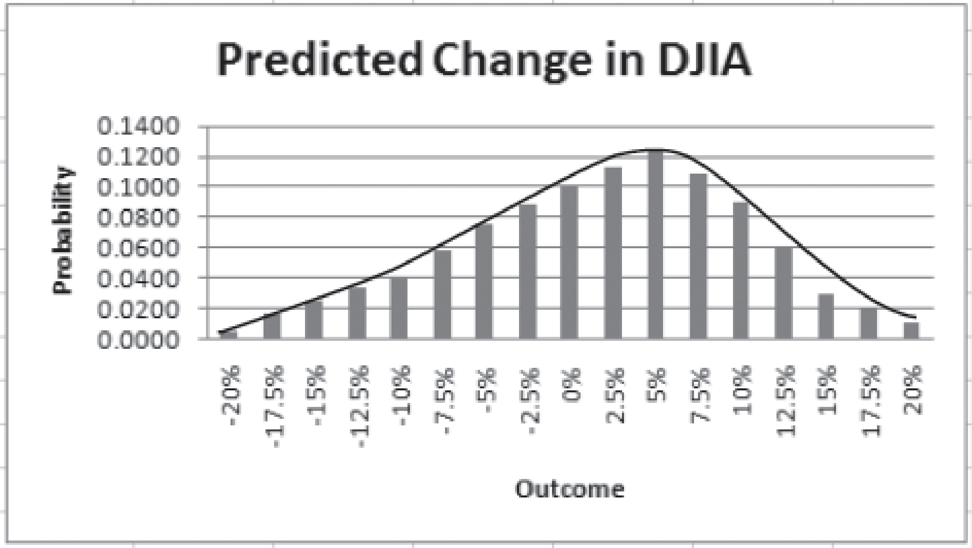
Continuous Probability Distributions

- Probability density function, *f(x)*, a continuous function that describes the probability of outcomes for a continuous random variable *X*. A histogram of sample data approximates the shape of the underlying density function.



Refining Subjective Probabilities Toward a Continuous Distribution

Change in DJIA	Subjective Probability
-20%	0.0050
-17.5%	0.0159
-15%	0.0250
-12.5%	0.0334
-10%	0.0400
-7.5%	0.0584
-5%	0.0750
-2.5%	0.0884
0%	0.1000
2.5%	0.1134
5%	0.1250
7.5%	0.1084
10%	0.0900
12.5%	0.0609
15%	0.0300
17.5%	0.0209
20%	0.0100



Properties of Probability Density Functions

- $f(x) \geq 0$ for all x
- Total area under $f(x) = 1$
- There are always infinitely many values for X
- $P(X = x) = 0$
- We can only define probabilities over intervals:

  $P(a \leq X \leq b)$, $P(X < c)$, or $P(X > d)$

Cumulative Distribution Function

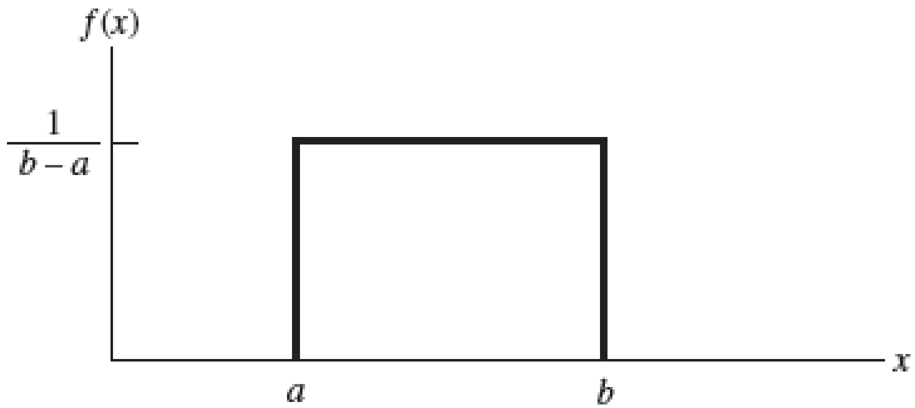
- $F(x)$ specifies the probability that the random variable X will be less than or equal to x; that is, $P(X \leq x)$.
- $F(x)$ is equal to the area under $f(x)$ to the left of x
- The probability that X is between a and b is the area under $f(x)$ from a to b:

$$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$$

Properties of Continuous Distributions

- Continuous distributions have one or more parameters that characterize the density function:
  - Shape parameter – controls the shape of the distribution
  - Scale parameter – controls the unit of measurement
  - Location parameter – specifies the location relative to zero on the horizontal axis

Uniform Distribution



$EV[X] = (a + b)/2$

$V[X] = (b – a)^2/12$

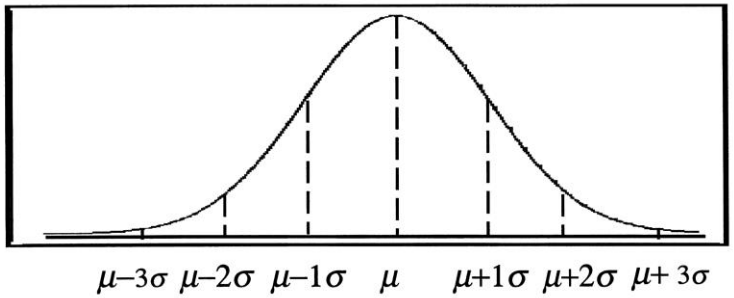
a = location

b = scale for fixed *a*

Normal Distribution

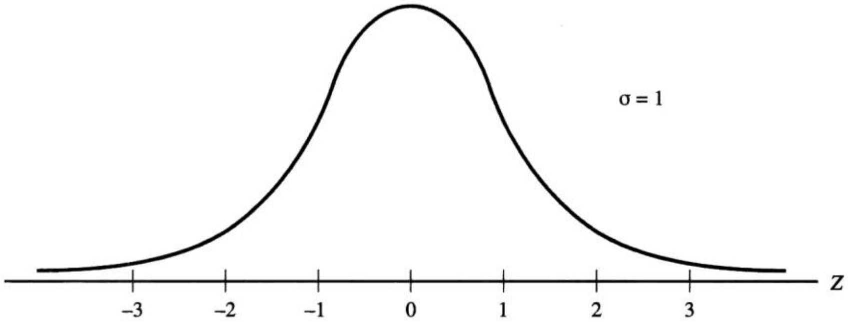
- Familiar bell-shaped curve.
- Symmetric, median = mean = mode; half the area is on either side of the mean
- Range is unbounded: the curve never touches the x-axis Parameters
  - Mean, μ (location)
  - Variance $\sigma^2 > 0$ (scale)
- Density function:

$$f(x) = \frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$



Standard Normal Distribution

- Standard normal: mean = 0, variance = 1, denoted as N(0,1)



- See Appendix Table A.1

Excel Functions

- NORM.DIST(*x, mean, standard_deviation, cumulative*)
  - NORM.DIST(*x, mean, standard_deviation, TRUE*) calculates the cumulative probability $F(x) = P(X \leq x)$ for a specified mean and standard deviation.
- NORM.S.DIST(*z*)
  - NORM.S.DIST(*z*) generates the cumulative probability for a standard normal distribution.

Example

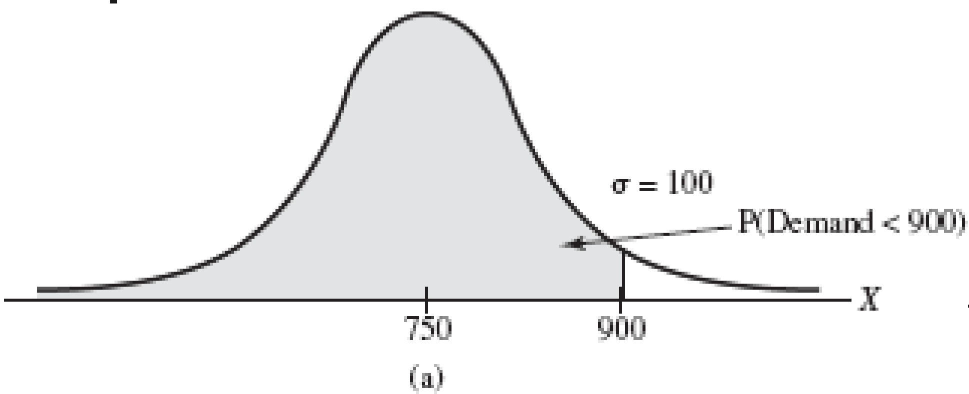
- Customer demand (X) is normal with a mean of 750 units/month and a standard deviation of 100 units/month.

1. What is the probability that demand will be at most 900 units?
2. What is the probability that demand will exceed 700 units?
3. What is the probability that demand will be between 700 and 900 units?
4. What level of demand would be exceeded at most 10% of the time?

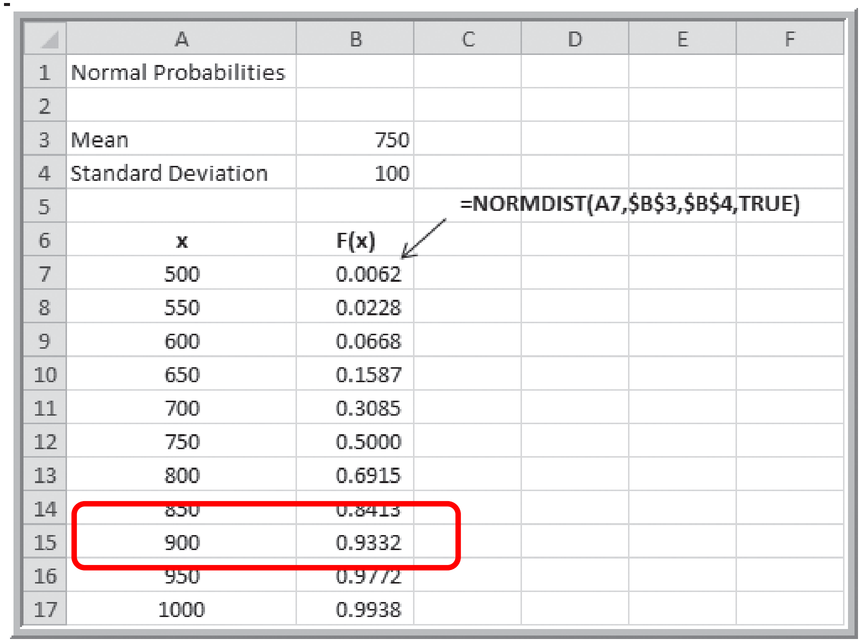
Excel Probability Tabulation

	A	B	C	D	E	F
1	Normal Probabilities					
2						
3	Mean	750				
4	Standard Deviation	100				
5			=NORMDIST(A7,B3,B4,TRUE)			
6	**x**	**F(x)**				
7	500	0.0062				
8	550	0.0228				
9	600	0.0668				
10	650	0.1587				
11	700	0.3085				
12	750	0.5000				
13	800	0.6915				
14	850	0.8413				
15	900	0.9332				
16	950	0.9772				
17	1000	0.9938				

P(X>900)

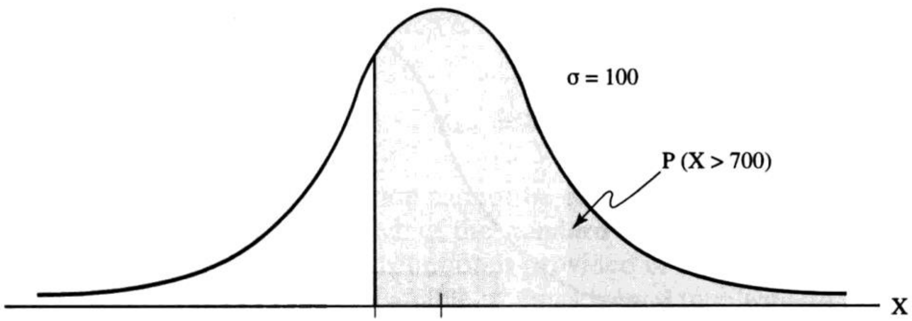


NORM.DIST(900,750,100,TRUE) = 0.9332.



	A	B	C	D	E	F
1	Normal Probabilities					
2						
3	Mean	750				
4	Standard Deviation	100				
5				=NORMDIST(A7,B3,B4,TRUE)		
6	x	F(x)				
7	500	0.0062				
8	550	0.0228				
9	600	0.0668				
10	650	0.1587				
11	700	0.3085				
12	750	0.5000				
13	800	0.6915				
14	850	0.8413				
15	900	0.9332				
16	950	0.9772				
17	1000	0.9938				

P(X>700)

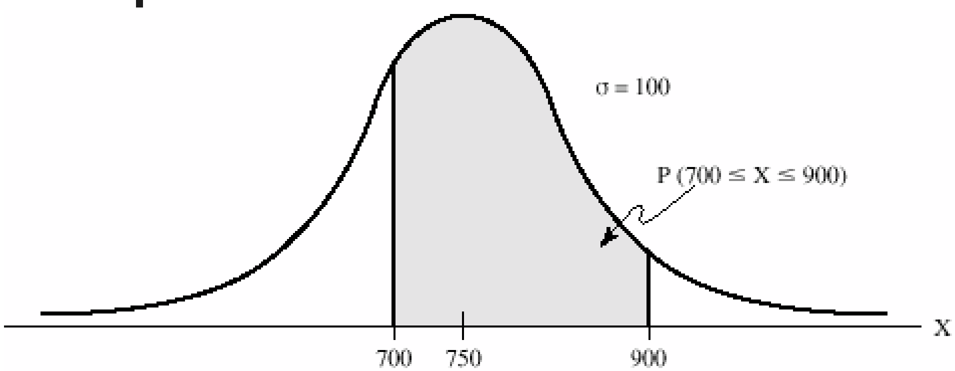


(a)

$P(X > 700) = 1 - P(X < 700)$
$= 1 - F(700)$
$= 1 - 0.3085$
$= 0.6915$
= 1 - NORM.DIST(700,750,100,TRUE)

	A	B	C	D	E	F
1	Normal Probabilities					
2						
3	Mean	750				
4	Standard Deviation	100				
5			=NORMDIST(A7,B3,B4,TRUE)			
6	x	F(x)				
7	500	0.0062				
8	550	0.0228				
9	600	0.0668				
10	650	0.1587				
11	700	0.3085				
12	750	0.5000				
13	800	0.6915				
14	850	0.8413				
15	900	0.9332				
16	950	0.9772				
17	1000	0.9938				

P(700<X<900)

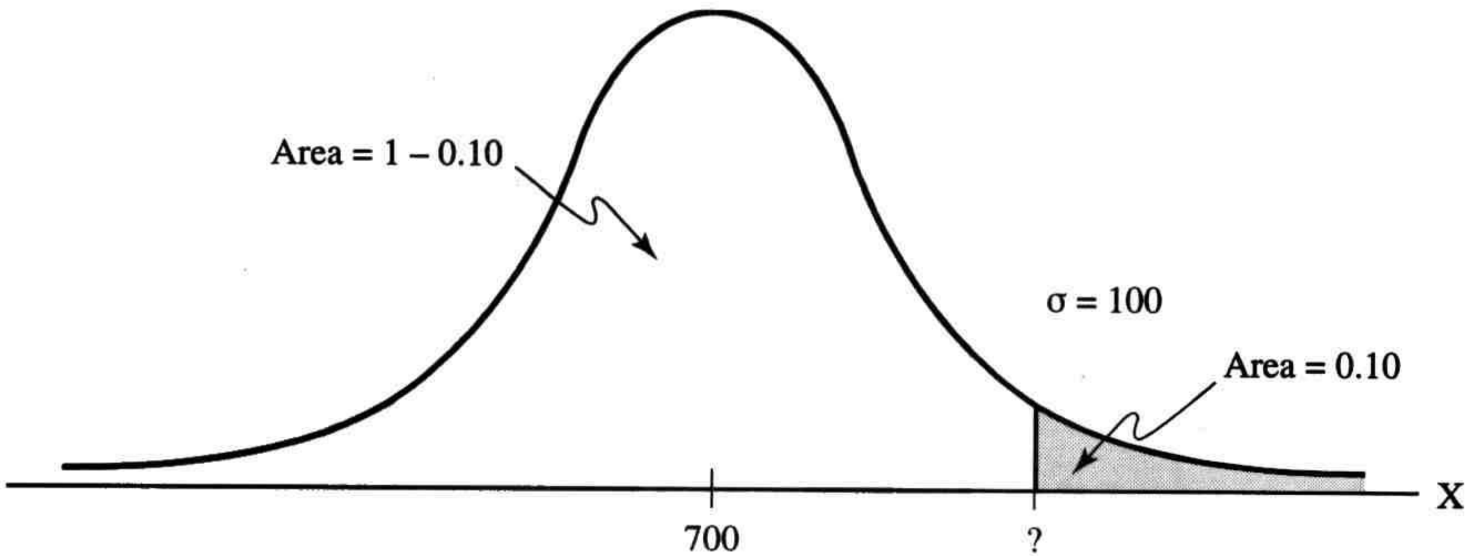


$P(700 < X < 900)$
$= F(900) – F(700)$
$= 0.9332 - 0.3085$
$= 0.6247$
= NORM.DIST(900,750,100,TRUE) – NORM.DIST(700,750,100,TRUE)

	A	B	C	D	E	F
1	Normal Probabilities					
2						
3	Mean	750				
4	Standard Deviation	100				
5			=NORMDIST(A7,B3,B4,TRUE)			
6	**x**	**F(x)**				
7	500	0.0062				
8	550	0.0228				
9	600	0.0668				
10	650	0.1587				
11	700	0.3085				
12	750	0.5000				
13	800	0.6915				
14	850	0.8413				
15	900	0.9332				
16	950	0.9772				
17	1000	0.9938				

$P(X>x) > 0.10$

NORM.INV(0.90,750,100) = 878.155



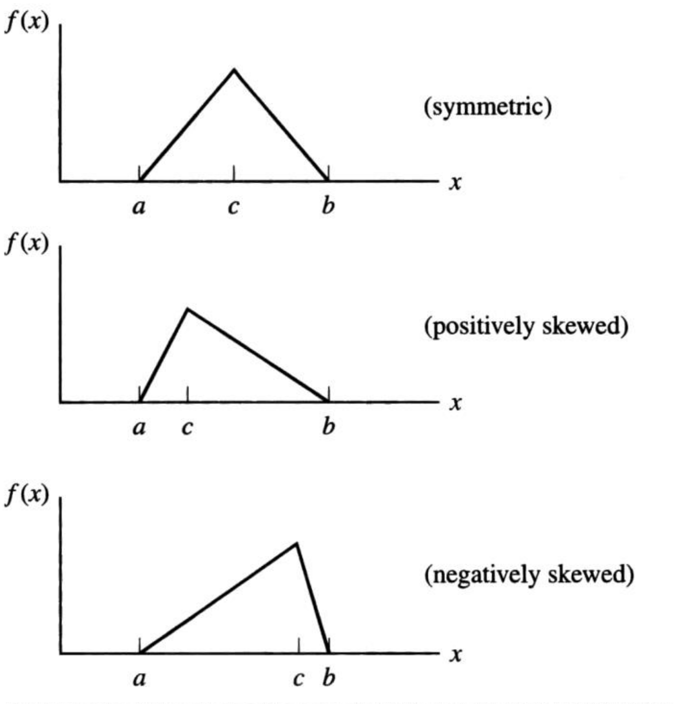
x = 878

Standard Normal Probabilities

	A	B	C	D	E
1	Standard Normal Probabilities				
2					
3	z	F(z)		=NORMSDIST(A4)	
4	-3	0.0013			
5	-2	0.0228			
6	-1	0.1587			
7	0	0.5000			
8	1	0.8413			
9	2	0.9772			
10	3	0.9987			

Triangular Distribution

- Three parameters:
  - Minimum, a
  - Maximum, b
  - Most likely, c
- a is the location parameter;
- b is the scale parameter for fixed a;
- c the shape parameter.

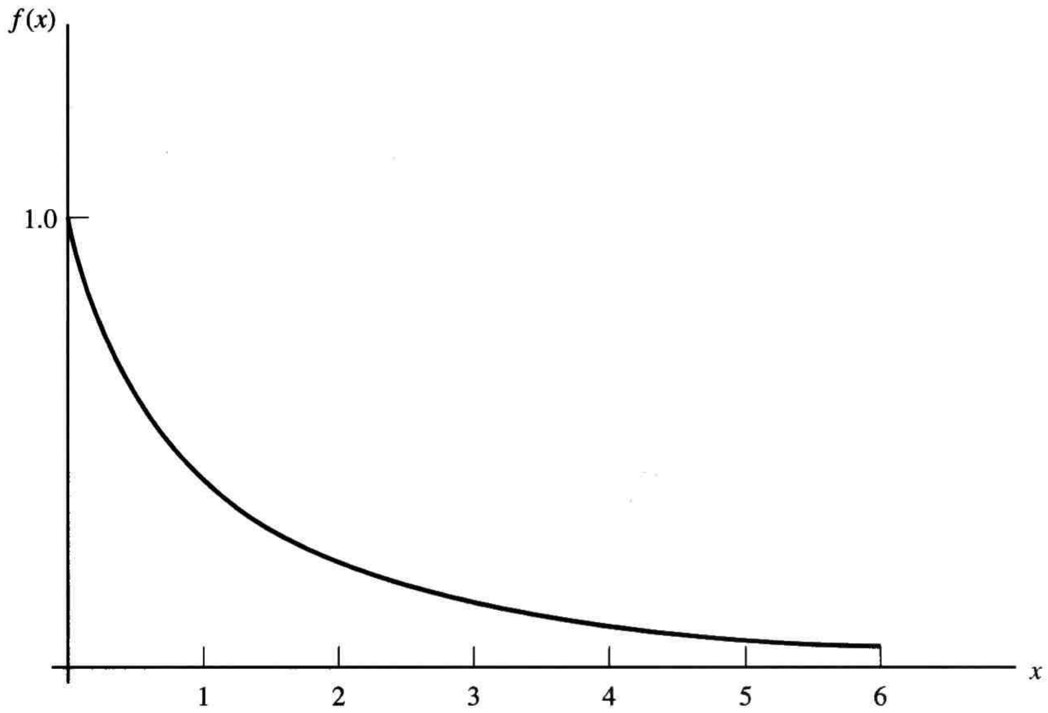


$EV[X] = (a + b + c)/3$

$V[X] = (a^2 + b^2 + c^2 - ab - ac - bc)/18$

Exponential Distribution

- Models events that occur randomly over time
  - Customer arrivals, machine failures



Properties of Exponential

- Excel function EXPONDIST(*x, lambda, cumulative*).
- $EV[X] = 1/\lambda$
- $V[X] = 1/\lambda^2$

	A	B	C	D	E	F
1	Exponential Probabilities					
2						
3	Mean	8000				
4			=EXPONDIST(A6,1/B3,TRUE)			
5	x	F(x)				
6	1000	0.117503				
7	2000	0.221199				
8	3000	0.312711				
9	4000	0.393469				
10	5000	0.464739				
11	6000	0.527633				
12	7000	0.583138				
13	8000	0.632121				
14	9000	0.675348				
15	10000	0.713495				
16	11000	0.74716				
17	12000	0.77687				
18	13000	0.803088				
19	14000	0.826226				
20	15000	0.846645				

Other Useful Distributions

- Lognormal
- Gamma
- Weibull/Erlang
- Beta
- Geometric
- Negative Binomial
- Hypergeometric
- Logistic
- Pareto
- Extreme value

Joint and Marginal Probability Distributions

- Joint probability distribution – A probability distribution that specifies the probabilities of outcomes of two different random variables, X and Y, that occur at the same time, or jointly
- Marginal probability – probability associated with the outcomes of each random variable regardless of the value of the other

Example

		DJIA Up	DJIA Unchanged	DJIA Down	Marginal Probability
	Joint Probabilities	$X = 1$	$X = 0$	$X = -1$	
Nasdaq Up	$Y = 1$	0.26	0.03	0.05	0.29
Nasdaq unchanged	$Y = 0$	0.04	0.02	0.03	0.06
Nasdaq down	$Y = -1$	0.10	0.05	0.42	0.15
	Marginal Probability	0.40	0.10	0.50	1

Binomial Distribution Theory

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & \text{for } x = 0, 1, 2, \ldots, n \\ 0, & \text{otherwise} \end{cases} \tag{3A.3}$$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} \tag{3A.4}$$

Poisson Distribution Theory

$$f(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!}, & \text{for } x = 0, 1, 2, \ \ldots \\ 0, & \text{otherwise} \end{cases} \qquad \textbf{(3A.5)}$$

Uniform Distribution Theory

$$f(x) = \frac{1}{b - a} \quad \text{if } a \leq x \leq b \qquad \textbf{(3A.7)}$$

$$F(x) = \begin{cases} 0, & \text{if } x < a \\ \dfrac{x - a}{b - a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } b < x \end{cases} \qquad \textbf{(3A.8)}$$

Normal Distribution Theory

- Standardized normal values (z-values):

$$z = \frac{x - \mu}{\sigma} \qquad (3A.10)$$

Exponential Distribution Theory

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \qquad \textbf{(3A.11)}$$

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0 \qquad \textbf{(3A.12)}$$