# 1 Effective Field

The Hamiltonian of $i$th site is given by:

$$
\begin{aligned}
\mathscr{H}_i = &- \sum_{j\in N} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j \\
&- \sum_{j\in N} \mathbf{D}_{ij}\mathbf{S}_i \times \mathbf{S}_j \\
&- \mu_i \mathbf{S}\cdot\mathbf{H} \\
&- \frac{\mu_i}{2}\mathbf{S}\cdot\mathbf{H}_d \\
&- \frac{\mu_0\mu_i^2}{4\pi}\sum_j \frac{3\left(\mathbf{S}_i\cdot\hat{\mathbf{r}}_{ij}\right)\left(\mathbf{S}_j\cdot\hat{\mathbf{r}}_{ij}\right) - \mathbf{S}_i\cdot\mathbf{S}_j}{r_{ij}^3} \\
&- K_1\left(\mathbf{S}_i\cdot\hat{\mathbf{A}}\right)^2 \\
&- K_2\left(S_x^4 + S_y^4 + S_z^4\right)
\end{aligned}
\tag{1}
$$

Where $\mathbf{S}_i = \dfrac{\boldsymbol{\mu}_i}{\mu_i}$, with $\mu_i = g\mu_B S$, where $g$ is the Lande factor, $\mu_B$ is the Bohr magneton and $S$ the average spin magnitude, and:
The effective field is given by:

$$
\mathbf{H}_i^{\text{eff}} = -\frac{1}{\mu_i}\frac{\partial\mathscr{H}}{\partial\mathbf{S}_i} \tag{2}
$$

Which turns out to be:

$$
\begin{aligned}
\frac{\partial\mathscr{H}}{\partial\mathbf{S}_i} = &- \sum_{j\in N} J_{ij}\mathbf{S}_j \\
&- \sum_{j\in N} \mathbf{S}_j \times \mathbf{D}_{ij} \\
&- \mu_i\mathbf{H} \\
&- \frac{\mu_i}{2}\mathbf{H}_d \\
&- \frac{\mu_0\mu_i^2}{4\pi}\sum_j \frac{3\left(\mathbf{S}_j\cdot\hat{\mathbf{r}}_{ij}\right)\hat{\mathbf{r}}_{ij} - \mathbf{S}_j}{r_{ij}^3} \\
&- 2K_1\left(\mathbf{S}_i\cdot\hat{\mathbf{A}}\right)\hat{\mathbf{A}} \\
&- 4K_2\left(S_x^3\hat{\mathbf{i}} + S_y^3\hat{\mathbf{j}} + S_z^3\hat{\mathbf{k}}\right)
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
\implies -\frac{1}{\mu_i}\frac{\partial\mathscr{H}}{\partial\mathbf{S}_i} = &\ \frac{1}{\mu_i}\sum_{j\in N} J_{ij}\mathbf{S}_j \\
&+ \frac{1}{\mu_i}\sum_{j\in N} \mathbf{S}_j \times \mathbf{D}_{ij} \\
&+ \mathbf{H} \\
&+ \frac{1}{2}\mathbf{H}_d \\
&+ \frac{\mu_0\mu_i}{4\pi}\sum_j \frac{3\left(\mathbf{S}_j\cdot\hat{\mathbf{r}}_{ij}\right)\hat{\mathbf{r}}_{ij} - \mathbf{S}_j}{r_{ij}^3} \\
&+ \frac{2K_1}{\mu_i}\left(\mathbf{S}_i\cdot\hat{\mathbf{A}}\right)\hat{\mathbf{A}} \\
&+ \frac{4K_2}{\mu_i}\left(S_x^3\hat{\mathbf{i}} + S_y^3\hat{\mathbf{j}} + S_z^3\hat{\mathbf{k}}\right)
\end{aligned}
\tag{4}
$$

Right now, dipolar interaction (5 term), demagnetizing field (4 term) and cubic anisotropy (last term) are not implemented, so the effective field becomes:

$$
\begin{aligned}
\implies -\frac{1}{\mu_i}\frac{\partial \mathscr{H}}{\partial \mathbf{S}_i} = \;& \frac{1}{\mu_i}\sum_{j\in N} J_{ij}\mathbf{S}_j \\
&+ \frac{1}{\mu_i}\sum_{j\in N}\mathbf{S}_j \times \mathbf{D}_{ij} \\
&+ \mathbf{H} \\
&+ \frac{2K_1}{\mu_i}\left(\mathbf{S}_i \cdot \hat{\mathbf{A}}\right)\hat{\mathbf{A}}
\end{aligned} \tag{5}
$$

All derivations are present on Extras section.

# 2 LLGS equation

The LLGS equation is given by:

$$
\begin{aligned}
\frac{\partial \mathbf{S}_i}{\partial t} =& -\gamma \mathbf{S}_i \times \mathbf{H}^{\text{eff}} + \alpha \mathbf{S}_i \times \frac{\partial \mathbf{S}_i}{\partial t} \\
&+ \frac{pa^3}{2q_e S}\left(\mathbf{j}_c \cdot \nabla\right)\mathbf{S}_i - \frac{pa^3\beta}{2q_e S^2}\mathbf{S}_i \times \left(\mathbf{j}_c \cdot \nabla\right)\mathbf{S}_i \\
&+ \frac{\gamma\hbar pa^3}{2q_e d\mu_i}\mathbf{S}_i \times \left(\mathbf{j}_c \times \mathbf{S}_i\right) + \frac{\beta\gamma\hbar pa^3}{2q_e d\mu_i}\mathbf{j}_c \times \mathbf{S}_i
\end{aligned} \tag{6}
$$

Changing the time units to $t = \frac{\hbar}{|J|}\tau \implies dt = \frac{\hbar}{|J|}d\tau$:

$$
\begin{aligned}
\frac{\partial \mathbf{S}_i}{\frac{\hbar}{|J|}\partial \tau} =& -\gamma \mathbf{S}_i \times \mathbf{H}^{\text{eff}} + \alpha \mathbf{S}_i \times \frac{\partial \mathbf{S}_i}{\frac{\hbar}{|J|}\partial \tau} \\
&+ \frac{pa^3}{2q_e S}\left(\mathbf{j}_c \cdot \nabla\right)\mathbf{S}_i - \frac{pa^3\beta}{2q_e S^2}\mathbf{S}_i \times \left(\mathbf{j}_c \cdot \nabla\right)\mathbf{S}_i \\
&+ \frac{\gamma\hbar pa^3}{2q_e d\mu_i}\mathbf{S}_i \times \left(\mathbf{j}_c \times \mathbf{S}_i\right) + \frac{\beta\gamma\hbar pa^3}{2q_e d\mu_i}\mathbf{j}_c \times \mathbf{S}_i
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\frac{|J|}{\hbar}\frac{\partial \mathbf{S}_i}{\partial \tau} =& -\gamma \mathbf{S}_i \times \mathbf{H}^{\text{eff}} + \frac{|J|}{\hbar}\alpha \mathbf{S}_i \times \frac{\partial \mathbf{S}_i}{\partial \tau} \\
&+ \frac{pa^3}{2q_e S}\left(\mathbf{j}_c \cdot \nabla\right)\mathbf{S}_i - \frac{pa^3\beta}{2q_e S^2}\mathbf{S}_i \times \left(\mathbf{j}_c \cdot \nabla\right)\mathbf{S}_i \\
&+ \frac{\gamma\hbar pa^3}{2q_e d\mu_i}\mathbf{S}_i \times \left(\mathbf{j}_c \times \mathbf{S}_i\right) + \frac{\beta\gamma\hbar pa^3}{2q_e d\mu_i}\mathbf{j}_c \times \mathbf{S}_i
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{S}_i}{\partial \tau} =& -\frac{\gamma\hbar}{|J|}\mathbf{S}_i \times \mathbf{H}^{\text{eff}} + \alpha \mathbf{S}_i \times \frac{\partial \mathbf{S}_i}{\partial \tau} \\
&+ \frac{pa^3\hbar}{2q_e S\,|J|}\left(\mathbf{j}_c \cdot \nabla\right)\mathbf{S}_i - \frac{pa^3\hbar\beta}{2q_e S^2\,|J|}\mathbf{S}_i \times \left(\mathbf{j}_c \cdot \nabla\right)\mathbf{S}_i \\
&+ \frac{\gamma\hbar^2 pa^3}{2q_e d\mu_i\,|J|}\mathbf{S}_i \times \left(\mathbf{j}_c \times \mathbf{S}_i\right) + \frac{\beta\gamma\hbar^2 pa^3}{2q_e d\mu_i\,|J|}\mathbf{j}_c \times \mathbf{S}_i
\end{aligned} \tag{9}
$$

Changing the current units, so that $\mathbf{j}_c = \frac{2q_e S\,|J|}{a^2\hbar}\mathbf{j}'_c$, we obtain:

$$
\begin{aligned}
\frac{\partial \mathbf{S}_i}{\partial \tau} =& -\frac{\gamma\hbar}{|J|}\mathbf{S}_i \times \mathbf{H}^{\text{eff}} + \alpha \mathbf{S}_i \times \frac{\partial \mathbf{S}_i}{\partial \tau} \\
&+ \frac{pa^3\hbar}{2q_e S\,|J|}\left(\left(\frac{2q_e S\,|J|}{a^2\hbar}\mathbf{j}'_c\right) \cdot \nabla\right)\mathbf{S}_i - \frac{pa^3\hbar\beta}{2q_e S^2\,|J|}\mathbf{S}_i \times \left(\left(\frac{2q_e S\,|J|}{a^2\hbar}\mathbf{j}'_c\right) \cdot \nabla\right)\mathbf{S}_i \\
&+ \frac{\gamma\hbar^2 pa^3}{2q_e d\mu_i\,|J|}\mathbf{S}_i \times \left(\left(\frac{2q_e S\,|J|}{a^2\hbar}\mathbf{j}'_c\right) \times \mathbf{S}_i\right) + \frac{\beta\gamma\hbar^2 pa^3}{2q_e d\mu_i\,|J|}\left(\frac{2q_e S\,|J|}{a^2\hbar}\mathbf{j}'_c\right) \times \mathbf{S}_i
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{S}_i}{\partial \tau} = & -\frac{\gamma\hbar}{|J|}\mathbf{S}_i \times \mathbf{H}^{\text{eff}} + \alpha \mathbf{S}_i \times \frac{\partial \mathbf{S}_i}{\partial \tau} \\
& + pa\left(\mathbf{j}_c' \cdot \nabla\right)\mathbf{S}_i - \frac{pa\beta}{S}\mathbf{S}_i \times \left(\mathbf{j}_c' \cdot \nabla\right)\mathbf{S}_i \\
& + \frac{\gamma\hbar paS}{d\mu_i}\mathbf{S}_i \times \left(\mathbf{j}_c' \times \mathbf{S}_i\right) + \frac{\beta\gamma\hbar paS}{d\mu_i}\mathbf{j}_c' \times \mathbf{S}_i
\end{aligned} \tag{11}
$$

To transform back in the LL form, we do the simplification:

$$
\mathbf{V}_i = -\frac{\gamma\hbar}{|J|}\mathbf{S}_i \times \mathbf{H}^{\text{eff}} + pa\left(\mathbf{j}_c' \cdot \nabla\right)\mathbf{S}_i - \frac{pa\beta}{S}\mathbf{S}_i \times \left(\mathbf{j}_c' \cdot \nabla\right)\mathbf{S}_i + \frac{\gamma\hbar paS}{d\mu_i}\mathbf{S}_i \times \left(\mathbf{j}_c' \times \mathbf{S}_i\right) + \frac{\beta\gamma\hbar paS}{d\mu_i}\mathbf{j}_c' \times \mathbf{S}_i \tag{12}
$$

So:

$$
\frac{\partial \mathbf{S}_i}{\partial \tau} = \mathbf{V}_i + \alpha \mathbf{S}_i \times \frac{\partial \mathbf{S}_i}{\partial \tau} \tag{13}
$$

$$
\frac{\partial \mathbf{S}_i}{\partial \tau} = \mathbf{V}_i + \alpha \mathbf{S}_i \times \left(\mathbf{V}_i + \alpha \mathbf{S}_i \times \frac{\partial \mathbf{S}_i}{\partial \tau}\right) \tag{14}
$$

$$
\frac{\partial \mathbf{S}_i}{\partial \tau} = \mathbf{V}_i + \alpha \mathbf{S}_i \times \mathbf{V}_i + \alpha^2 \mathbf{S}_i \times \mathbf{S}_i \times \frac{\partial \mathbf{S}_i}{\partial \tau} \tag{15}
$$

$$
\frac{\partial \mathbf{S}_i}{\partial \tau} = \mathbf{V}_i + \alpha \mathbf{S}_i \times \mathbf{V}_i + \alpha^2 \left[\mathbf{S}_i \left(\mathbf{S}_i \cdot \frac{\partial \mathbf{S}_i}{\partial \tau}\right) - \frac{\partial \mathbf{S}_i}{\partial \tau}\left(\mathbf{S}_i \cdot \mathbf{S}_i\right)\right] \tag{16}
$$

Since $\mathbf{S}_i = \frac{\boldsymbol{\mu}_i}{\mu_i}$, $\mathbf{S}_i \cdot \mathbf{S}_i = 1$, and $\mathbf{S}_i \perp \frac{\partial \mathbf{S}_i}{\partial \tau} \implies \mathbf{S}_i \cdot \frac{\partial \mathbf{S}_i}{\partial \tau} = 0$:

$$
\frac{\partial \mathbf{S}_i}{\partial \tau} = \mathbf{V}_i + \alpha \mathbf{S}_i \times \mathbf{V}_i - \alpha^2 \frac{\partial \mathbf{S}_i}{\partial \tau} \tag{17}
$$

$$
\therefore \frac{\partial \mathbf{S}_i}{\partial \tau} = \frac{1}{1+\alpha^2}\left[\mathbf{V}_i + \alpha \mathbf{S}_i \times \mathbf{V}_i\right] \tag{18}
$$

# Extras