Actuarial Mathematics
LIFE TABLES and SELECTION
Examples

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Example 2-1

x	l_x	d_x
30	10000.00	34.78
31	9964.22	38.10
32	9927.12	41.76
33	9885.35	45.81
34	9839.55	50.26
35	9789.29	55.17
36	9734.12	60.56
37	9673.56	66.49
38	9607.07	72.99
39	9534.08	80.11

(A)l_{40} (B)${}_{10}p_{30}$ (C) q_{35} (D) ${}_5q_{30}$

(E) The probability that a life aged exactly 30 dies between 35 and 36.

Solution 2-1

(A)

$$l_{40} = l_{39} - d_{39} = 9453.97.$$

(B)

$${}_{10}p_{30} = \frac{l_{40}}{l_{30}} = \frac{9453.97}{10000.00} = 0,94540.$$

Solution 2-1 (conti'n)

(C)

$$q_{35} = \frac{d_{35}}{l_{35}} = \frac{55.17}{9789.29} = 0.00564.$$

(D)

$$_5q_{30} = 1 - {}_5p_{30} = 1 - \frac{l_{35}}{l_{30}} = \frac{l_{30} - l_{35}}{l_{30}} = 0.02107.$$

(E)

$$_{5|}q_{30} = {}_5p_{30}\,q_{35} = {}_5p_{30}(1 - p_{35}) = \frac{l_{35}}{l_{30}}\left(1 - \frac{l_{36}}{l_{35}}\right) = \frac{l_{35} - l_{36}}{l_{30}} = \frac{d_{35}}{l_{30}} = 0.00552$$

Example 2-2

Given that $p_{40} = 0.999473$, calculate ${}_{0.4}q_{40.2}$ under UDD assumption.

Solution 2-2

$$
\begin{aligned}
{}_{0.4}q_{40.2} &= 1 - {}_{0.4}p_{40.2} &&= 1 - \frac{l_{40.6}}{l_{40.2}} \\
&= 1 - \frac{{}_{0.6}p_{40}}{{}_{0.2}p_{40}} &&= 1 - \frac{1 - {}_{0.6}q_{40}}{1 - {}_{0.2}q_{40}} \\
&= 1 - \frac{1 - 0.6q_{40}}{1 - 0.2q_{40}} &&= 2.108 \times 10^{-4}.
\end{aligned}
$$

Example 2-3

Use life table in **Example 2-1**, with UDD assumption, to calculate

$$(A)\ {}_{1.7}q_{33} \quad \text{and} \quad (B)\ {}_{1.7}q_{33.5}.$$

Solution 2-3

(A)

$${}_{1.7}q_{33} = 1 - {}_{1.7}p_{33} = 1 - (p_{33})({}_{0.7}p_{34}) = 1 - (p_{33})(1 - {}_{0.7}q_{34}).$$

By given life table and UDD assumption

$${}_{1.7}q_{33} = 1 - \frac{l_{34}}{l_{33}}(1 - 0.7\,q_{34}) = 0.008192.$$

(B)

$$\begin{aligned}
{}_{1.7}q_{33.5} &= 1 - {}_{1.7}p_{33.5} &&= 1 - \frac{l_{35.2}}{l_{33.5}} \\
&= 1 - \frac{l_{35}\,{}_{0.2}p_{35}}{l_{33}\,{}_{0.5}p_{33}} &&= 1 - \frac{l_{35}(1 - {}_{0.2}q_{35})}{l_{33}(1 - {}_{0.5}q_{33})} \\
&= 1 - \frac{l_{35} - 0.2d_{35}}{l_{33} - 0.5d_{33}} &&= 0.008537
\end{aligned}$$

Example 2-4

Use UDD assumption, calculate

$$\lim_{t\to 1^-} \mu_{40+t} \text{ using } p_{40} = 0.999473,$$

and

$$\lim_{t\to 0^+} \mu_{41+t} \text{ using } p_{41} = 0.999429.$$

Solution 2-4

1.

$$\because \mu_{x+t} = -\frac{\frac{d}{dt}\,{}_tp_x}{{}_tp_x} = -\frac{\frac{d}{dt}(1 - {}_tq_x)}{{}_tp_x} = \frac{\frac{d}{dt}\,{}_tq_x}{{}_tp_x}$$

By UDD assumption

$$\therefore \mu_{x+t} = \frac{\frac{d}{dt}\,t\,q_x}{{}_tp_x} = \frac{q_x}{{}_tp_x};\ \forall 0 \le t < 1.$$

Therefore

$$\lim_{t\to 1^-} \mu_{40+t} = \frac{q_{40}}{p_{40}} = 5.273 \times 10^{-4}.$$

2.

$$\lim_{t\to 0^+} \mu_{41+t} = q_{41} = 5.71 \times 10^{-4}.$$

Example 2-5

Given that $q_{70} = 0.010413$ and $q_{71} = 0.011670$, calculate ${}_{0.7}q_{70.6}$ under UDD assumption.

Solution 2-5 Since we are given only q_{70} and q_{71}, we first write

$$ {}_{0.7}q_{70.6} = {}_{0.4}q_{70.6} + {}_{0.4}p_{70.6}\,{}_{0.3}q_{71} = {}_{0.4}q_{70.6} + (1 - {}_{0.4}q_{70.6})\,{}_{0.3}q_{71} $$

and by UDD,

$$ {}_{0.4}q_{70.6} = 1 - {}_{0.4}p_{70.6} = 1 - \frac{l_{71}}{l_{70.6}} = 1 - \frac{p_{70}}{{}_{0.6}p_{70}} = 1 - \frac{1 - q_{70}}{1 - 0.6q_{70}} = 4.191 \times 10^{-3}. $$

With

$$ {}_{0.3}q_{71} = 0.3q_{71} = 3.501 \times 10^{-3} \quad \Rightarrow \quad {}_{0.7}q_{70.6} = 0.3q_{71} = 7.678 \times 10^{-3}. $$

Example 2-6

Given that $p_{40} = 0.999473$, calculate ${}_{0.4}q_{40.2}$ under constant force of mortality.

Solution 2-6

$$\because \quad {}_sp_x = (p_x)^s, \quad \therefore \quad {}_{0.4}q_{40.2} = 1 - {}_{0.4}p_{40.2} = 1 - (p_{40.2})^{0.4} = 2.108 \times 10^{-4}.$$

Example 2-7

Given that $q_{70} = 0.010413$ and $q_{71} = 0.011670$, calculate ${}_{0.7}q_{70.6}$ under constant force of mortality.

Solution 2-7 We know

$${}_{0.7}q_{70.6} = {}_{0.4}q_{70.6} + {}_{0.4}p_{70.6}{}_{0.3}q_{71} = {}_{0.4}q_{70.6} + (1 - {}_{0.4}q_{70.6}){}_{0.3}q_{71}$$

and

$${}_{0.4}q_{70.6} = 1 - (p_{70.6})^{0.4} = 4.178 \times 10^{-3}; \quad {}_{0.3}q_{71} = 1 - (p_{71})^{0.3} = 3.515 \times 10^{-3},$$

thus

$${}_{0.7}q_{70.6} = 7.679 \times 10^{-3}.$$

Example 2-8

Consider men who need undergo surgery because they are suffering a particular disease. The surgery is complicate and there is a probability of only 50% that they will survive for a year following surgery. If they do survive for a year, then they are fully cured and their future mortality follows the Australian Life Table 2000-02, Male, from which you are given the following values:

$$l_{60} = 89777, \quad l_{61} = 89015, \quad l_{70} = 77946.$$

Calculate

(A) the probability that a man aged 60 who is just about to have surgery will be alive at age 70,

(B) the probability that a man aged 60 who had surgery at 59 will be alive at age 70, and

(C) the probability that a man aged 60 who had surgery at 58 will be alive at age 70.

Solution 2-8

In the example, the select age is having surgery at age x.
The select period of the survival model for the group is one year.

(A)

$$\because \ Prob(\text{surviving to } 61) = 0.5.$$

Given that a man survives to 61, the probability of surviving to 70 is

$$\frac{l_{70}}{l_{61}} = \frac{77946}{89015} = 0.8757.$$

$$\therefore \ {}_{70}p_{60} = {}_{61}p_{60}\,{}_{70}p_{61} = (0.5)(0.8757) = 0.4378.$$

(B) Since this man has already survived for one year following surgery, his mortality follows the Australian Life Table 2000-02, Male. Hence

$$ {}_{70}p_{60} = \frac{l_{70}}{l_{60}} = \frac{77946}{89777} = 0.8682.$$

(C) As the probability in (B),

$$ {}_{70}p_{60} = \frac{l_{70}}{l_{60}} = \frac{77946}{89777} = 0.8682.$$

Example 2-9

For

$$y \geq x + d > x + s > x + t \geq x \geq x_0,$$

show that

$${}_{y-x-t}p_{[x]+t} = \frac{l_y}{l_{[x]+t}}$$

and

$${}_{s-t}p_{[x]+t} = \frac{l_{[x]+s}}{l_{[x]+t}}.$$

Solution 2-9

First,

$$\because \; x + t \;\rightarrow\; x + d \;\rightarrow\; y,$$

$$\begin{aligned}
\therefore \quad {}_{y-x-t}p_{[x]+t} &= {}_{d-t}p_{[x]+t} \;\; {}_{y-x-d}p_{[x]+d} = {}_{d-t}p_{[x]+t} \;\; {}_{y-x-d}p_{x+d} \\
&= \frac{l_{x+d}}{l_{[x]+t}} \frac{l_y}{l_{x+d}} \\
&= \frac{l_y}{l_{[x]+t}}.
\end{aligned}$$

EXAMPLES

Solution 2-9 (conti'n)

Second,

$$\because\ [x]+t \rightarrow [x]+s \rightarrow [x]+d,$$

and

$$_{d-t}p_{[x]+t} = {}_{s-t}p_{[x]+t}\ {}_{d-s}p_{[x]+s}.$$

$$\begin{aligned}\therefore \quad {}_{s-t}p_{[x]+t} &= \frac{_{d-t}p_{[x]+t}}{_{d-s}p_{[x]+s}} = \frac{l_{x+d}}{l_{[x]+t}}\frac{l_{[x]+s}}{l_{x+d}} \\ &= \frac{l_{[x]+s}}{l_{[x]+t}}.\end{aligned}$$

Example 2-10

Write an expression for ${}_{2|6}q_{[30]+2}$ in terms of $l_{[x]+t}$ and l_y for appropriate x, t, and y, assuming a select period of five years.

Solution 2-10

$$\because\ [30]+2 \ \rightarrow\ [30]+4 \ \rightarrow\ \text{die between 34 and 40,}$$

$$\begin{aligned}
\therefore\quad {}_{2|6}q_{[30]+2} &= {}_{2}p_{[30]+2}\,{}_{6}q_{[30]+4} \\
&= \frac{l_{[30]+4}}{l_{[30]+2}}\left(1-\frac{l_{[30]+10}}{l_{[30]+4}}\right) \\
&= \frac{l_{[30]+4}-l_{40}}{l_{[30]+2}}.
\end{aligned}$$

Example 2-11

x	l_x
70	80556
71	79026
72	77410
73	75666
74	73802
75	71800

A select survival model has a select period of 3 years. Its ultimate mortality is equivalent to the US Life Table, 2002, Females.
Yo are given that for all ages $x \geq 65$,

$$p_{[x]} = 0.999, \quad p_{[x-1]+1} = 0.998, \quad p_{[x-2]+2} = 0.997.$$

Calculate the probability that a woman currently aged 70 will survive to age 75 given that

(A) she was select at age 67,

(B) she was select at age 68,

(C) she was select at age 69, and

(D) she was select at age 70.

Solution 2-11

(A)

$$ {}_5p_{70} = \frac{l_{75}}{l_{70}} = \frac{71800}{80556} = 0.8913. $$

(B)

$$ \because {}_5p_{[68]+2} = \frac{l_{[68]+2+5}}{l_{[68]+2}} = \frac{l_{75}}{l_{[68]+2}} = \frac{71800}{l_{[68]+2}}. $$

and

$$ p_{[68]+2} = \frac{l_{71}}{l_{[68]+2}} \Rightarrow l_{[68]+2}p_{[68]+2} = l_{71} \Rightarrow l_{[68]+2}(0.997) = 79026 \Rightarrow l_{[68]+2} = 79264. $$

$$ \therefore\ px[5][68] + 2 = 0.9058. $$

EXAMPLES

Solution 2-11 (conti'n)

(C)

$$\because {}_5p_{[69]+1} = \frac{l_{[68]+2+5}}{l_{[69]+1}} = \frac{l_{75}}{l_{[69]+1}} = \frac{71800}{l_{[69]+1}}.$$

and

$$[69]+1 \rightarrow [69]+2 \rightarrow 75.$$

Thus,

$$l_{72} = l_{[69]+1}\ p_{[69]+1}\ p_{[69]+2} \Rightarrow l_{[69]+1} = 77799.$$

$$\therefore {}_5p_{[69]+1} = 0.9229.$$

(D)

$$\because {}_5p_{[70]} = \frac{l_{[70]+5}}{l_{[70]}} = \frac{l_{75}}{l_{[70]+1}} = \frac{71800}{l_{[70]}}.$$

and

$$[70] \rightarrow [70]+2 \rightarrow 75.$$

Thus,

$$l_{73} = l_{[70]}\ p_{[70]}\ p_{[70]+1}\ p_{[70]+2} \Rightarrow l_{[70]} = 76112 \quad \therefore {}_5p_{[70]} = 0.9432.$$

Example 2-12

x	Duration 0 $q_{[x]}$	Duration 1 $q_{[x-1]+1}$	Duration 2+ q_x
60	0.003469	0.004539	0.004760
61	0.003856	0.005059	0.005351
62	0.004291	0.005644	0.006021
63	0.004779	0.006304	0.006781
⋮	⋮	⋮	⋮
70	0.010519	0.014068	0.015786
71	0.011858	0.015868	0.017832
72	0.013401	0.017931	0.020145
73	0.015184	0.020302	0.022759
74	0.017253	0.023034	0.025712
75	0.019664	0.026196	0.029048

CMI table is based on UK data from 1999 to 2002 for male non-smokers who are whole life or endowment insurance policyholders. It has select period of two years.
Calculate the following probabilities:

(A) ${}_4p_{[70]}$,

(B) ${}_3q_{[60]+1}$,

(C) ${}_{2|}q_{73}$.

Solution 2-12

(A)

$$\begin{aligned} {}_4p_{[70]} &= p_{[70]}\ p_{[70]+1}\ p_{[70]+2}\,{}_4p_{[70]+3} \\ &= p_{[70]}\ p_{[70]+1}\ p_{72}\,{}_4p_{73} \\ &= (1-q_{[70]})\,(1-q_{[70]+1})\,(1-q_{72})(1-{}_4q_{73}) \\ &= 0.932447. \end{aligned}$$

(B)

$$\begin{aligned} {}_3q_{[60]+1} &= q_{[60]+1} + p_{[60]+1}\ q_{62} + p_{[60]+1}\ p_{62}\ q_{63} \\ &= 0.017756. \end{aligned}$$

(C)

$$\begin{aligned} {}_{2|}q_{73} &= {}_2p_{73}\ q_{75} \\ &= (1-q_{73})(1-q_{74})\ q_{75} \\ &= 0.027657. \end{aligned}$$