Actuarial Mathematics PREMIUM CALCULATION

Sheu, Ru-Shuo Ph.D

Department of Applied Mathematics
Chinese Culture University

2021

SUMMARY

We discuss principles of premium calculation for insurance policies and annuities.

- **Premium**
    - net premium;
    - gross premium.
- **Present value of future loss random variable**
    - equivalence premium principle;
    - moving contracts from loss to profit or vice versa.
- **Different premium principle**
    - Portfolio percentile premium principle;
- **Insured life subjected to some level of risk**

PRELIMINARIES

- **Net premium**

  The premium **NOT** explicitly allow for the insurance company's expenses. Also, is referred to a **risk premium** or **benefit premium**.
- **Gross premium**

  The premium **DOES** explicitly allow for expenses. Also, is referred to a **office premium** or **expense-loaded premium**.

For any **life insurance policy**, premiums are payable **in advance**, with the first premium payable when the policy is purchased.

- Regular premiums for a policy on a single life cease to be payable on the death of the policyholder.
- The premium paying terms for a policy is the maximum length of time for which premiums are payable.

Premiums are payable to secure **annuity benefits** as well as life insurance benefits.

- **Deferred annuities** may be purchased using a single premium at the start of the deferred period, or by regular premiums payable throughout the deferred period.
- **Immediate annuities** are always purchased by a single premium.

For traditional policies, both gross premiums or net premiums are usually calculated using the **equivalence principal**.
A more contemporary approach, which is commonly used for non-traditional policies, is to consider the **cash flows** from the contract, and to set the premiums to satisfy a specific profit criterion.

THE PRESENT VALUE OF FUTURE LOSS RANDOM VARIABLE

The cash flow for a traditional life insurance contract consist of **the insurance or annuity benefit outgo** (and associated expenses) and **the premium income**.

Definition

The r.v. **present value of the future loss** is represented by the

$$[\text{the future outgo}] - [\text{the future income}].$$

Net future loss L_0^n: expenses are excluded.
Gross future loss L_0^g: expenses are included.
That is

$$\begin{aligned} L_0^n &= PV \text{ of benefit outgo} - PV \text{ of net premium income} \\ L_0^g &= PV \text{ of benefit outgo} + PV \text{ of expenses} \\ &\quad - PV \text{ of gross premium income} \end{aligned}$$

example

An insurer issues a whole life insurance to [60], with sum insured S payable immediately on death. Premiums are payable annually in advance, ceasing at age 80 or early death. The net annual premium is P.

We know
the present value r.v. for the benefit is $S\nu^{T_{[60]}}$ and
the present value r.v. for the premium income is $P\ddot{a}_{\overline{min(K_{[60]}+1,20)|}}$; so

$$L_0^n = S\nu^{T_{[60]}} - P\ddot{a}_{\overline{min(K_{[60]}+1,20)|}}.$$

Since both terms of the r.v. depend on the future lifetime of the same life, [60], they are clearly dependent.

- ★ The last possible premium is payable on the policyholder's 79th birthday. No premium is payable on reaching age 80.

The insurer needs to use **premium principal** to find the suitable premium.

THE EQUIVALENCE PRINCIPAL

NET PREMIUMS

Under the *equivalence principle*, the premium is set such that the expected value of the future loss is zero at the **start of the contract**.

$$E[L_0^n] = 0$$

$$\Rightarrow \quad E[PV \text{ of benefit outgo} - PV \text{ of net premium income}] = 0.$$

That is, under the equivalence principle,

$$\boldsymbol{EPV \text{ of benefit outgo} = EPV \text{ of net premium income}} \tag{1}$$

★ *Equivalence principle* is the most common premium principal in traditional life insurance, and will be our default principal.

example

Consider an endowment insurance with term n years and sum insured S payable at the earlier of the end of the year of death or at maturity, issued to a select life aged x. Premiums of amount P are payable throughout the term of the insurance.
Derive expressions in terms of S, P and standard actuarial functions for

(A) the net future loss, L_0^n,

(B) the mean of L_0^n,

(C) the variance of L_0^n,

(D) the annual net premium for the contract.

Solution

(A)

$$L_0^n = S\nu^{min(K_{[x]}+1,n)} - P\ddot{a}_{\overline{min(K_{[x]}+1,n)}|}.$$

(B)

$$E[L_0^n] = SE\left(\nu^{min(K_{[x]}+1,n)}\right) - PE\left(\ddot{a}_{\overline{min(K_{[x]}+1,n)}|}\right) = SA_{[x]:\overline{n}|} - P\ddot{a}_{[x]:\overline{n}|}.$$

(C)

$$\begin{aligned} L_0^n &= S\nu^{min(K_{[x]}+1,n)} - P\frac{1-\nu^{min(K_{[x]}+1,n)}}{d} \\ &= \left(S+\frac{P}{d}\right)\nu^{min(K_{[x]}+1,n)} - \frac{P}{d}. \end{aligned}$$

So the variance is

$$\begin{aligned} V[L_0^n] &= \left(S+\frac{P}{d}\right)^2 V\left(\nu^{min(K_{[x]}+1,n)}\right) \\ &= \left(S+\frac{P}{d}\right)^2 \left({}^2A_{[x]:\overline{n}|} - [A_{[x]:\overline{n}|}]^2\right). \end{aligned}$$

Solution(cont'n)

(D) By equivalence principal, the net premium is

$$P = S\frac{A_{[x]:\overline{n}|}}{\ddot{a}_{[x]:\overline{n}|}}. \tag{2}$$

$$\because \quad \ddot{a}_{[x]:\overline{n}|} = \frac{1 - A_{[x]:\overline{n}|}}{d} \quad \therefore \quad P = S\left(\frac{1}{\ddot{a}_{[x]:\overline{n}|}} - d\right).$$

★ The only actuarial function needed to calculate P for a given S is $\ddot{a}_{[x]:\overline{n}|}$

Examples

GROSS PREMIUMS

There are three main types of expenses associated with policies:

- **initial expenses**
  incurred by the insurer when a policy is issued. There are two major types of initial expenses
  - commission to agents for selling a policy and
  - underwriting expenses.
- **renewal expenses**
  incurred by the insurer each time when a premium is payable and, in a case of annuity, when an annuity payment is made.
- **termination or claim expenses**
  occur when a policy expires, typically on the death of a policyholder (or annuitant) or on the maturity date of a term insurance or endowment insurance.

PREMIUM CALCULATION

- **Initial** and **renewal expenses** may be proportional to premiums or benefits, meaning that the amount is fixed for all policies and is not related to the size of the contract.
- Often, per policy renewal costs are assumed to be increasing at a compound rate over the terms of the policy, to approximate the effect of inflation.
- Generally, **termination** or **claim expenses** are small and are largely associated with the paperwork required to finalize and pay a claim.

The equivalence principal is applied to the gross premium. That is

$$E[L_0^g] = 0$$

$$\Rightarrow \quad EPV \text{ of benefit outgo} + EPV \text{ of expenses} \\ - EPV \text{ of gross premium income} = 0.$$

In other words, under equivalence principal,

$$\begin{aligned} & \boldsymbol{EPV}\ \textbf{of benefit outgo} + \boldsymbol{EPV}\ \textbf{of expenses} \\ = \ & \boldsymbol{EPV}\ \textbf{of gross premium income} \end{aligned} \tag{3}$$

example

An insurer issues a 25-year annual premium endowment insurance with sum insured \$100 000 to a select life aged 30. The insurer incurs initial expenses of \$2000 plus 50% of the first premium, and renewal expenses of 2.5% of each subsequent premium. The death benefit is payable immediately on death.

(A) Write down the gross future loss random variable.

(B) Calculate the gross premium with 5% per year interest.

Solution(cont'n)

(A) Let $S = 100000$, $x = 30$, $n = 25$ and P denote the annual gross premium. Then

$$\begin{aligned} L_0^n &= S\nu^{min(T_{[x]},n)} + 2000 + 0.475P + 0.025\ddot{a}_{\overline{min(K_{[x]}+1,n)}|} - P\ddot{a}_{\overline{min(K_{[x]}+1,n)}|} \\ &= S\nu^{min(T_{[x]},n)} + 2000 + 0.475P - 0.975\ddot{a}_{\overline{min(K_{[x]}+1,n)}|}. \end{aligned}$$

(B) The *EPV* of premium income is

$$P\ddot{a}_{[30]:\overline{25}|} = 14.73113P.$$

The *EPV* of all expenses is

$$2000 + 0.475P + 0.025 + P\ddot{a}_{[30]:\overline{25}|} = 2000 + 0.843278P.$$

The *EPV* of death benefit is

$$\bar{A}_{[30]:\overline{25}|} = 100000(0.298732) = 298732.2.$$

Thus, the equivalence principal gives

$$P = \frac{29873.2 + 2000}{14.73113 - 0.843278} = 2295.04.$$

example

Calculate the monthly gross premium for a 10-year term insurance with sum insured \$50 000 payable immediately on death, issued to a select life aged 55, using the following basis:

- **Interest**
  5% per year
- **Initial expenses**
  \$500 + 10% of each monthly premium in the first year
- **Renewal expenses**
  1% of each monthly premium in the second and the subsequent policy years

Solution(cont'n)

Let P denote the monthly premium. Then the *EPV* of premium income is $12P\ddot{a}^{(12)}_{[55]:\overline{n}|}$ and the *EPV* of all expenses is

$$500 + 0.09(12P\ddot{a}^{(12)}_{[55]:\overline{1}|}) + 0.01(12P\ddot{a}^{(12)}_{[55]:\overline{10}|}).$$

So the equivalence principal gives

$$12P\left(0.99\ddot{a}^{(12)}_{[55]:\overline{10}|} - 0.09\ddot{a}^{(12)}_{[55]:\overline{1}|}\right) = 500 + 50000\bar{A}^{\;1}_{[55]:\overline{10}|}.$$

We find tha $\ddot{a}^{(12)}_{[55]:\overline{10}|} = 7.8341$, $\ddot{a}^{(12)}_{[55]:\overline{1}|} = 0.9773$, and $\bar{A}^{\;1}_{[55]:\overline{10}|} = 0.024954$, giving that $P = \$18.99$ per month.

Examples

PREMIUM CALCULATION

- In the first examples, the annual premium is \$2295.04 and the expense at time 0 are

$$2000 + 0.5(2295.04) = 3146.75,$$

which exceeds the first premium.

- Similarly, in the second example, the total premium in the first year is \$227.88 and the total expenses in the first year are

$$500 + 0.1(\text{premium in the first year}).$$

The premium income in the first year is insufficient to cover expenses in the first year.

This situation is common in practice and is referred to as **new business strain**.

★ These early expenses are paid off by the expense loadings in the future premiums.

PROFIT

- ★ The equivalence principal does **NOT** allow explicitly for a loading for profit.
- ★ In traditional insurance, we often load for profit implicitly, by margins in the valuation assumptions.

Each individual policy sold will generate a profit or loss.

- ★ For each individual policy, the experienced mortality rate in any year can take only the values **0** or **1**.
- ★ For the actual profit from a group of policies to be reliably close to the expected profit, we need to sell **a large number of contracts** to individuals. whose future lifetime can be regarded as **statistically independent**, so that the losses and profits from individual policies are combined.

example

Consider a life who purchases a one-year term insurance with

- sum insured \$1000 payable at the end of the year of death,
- mortality of rate of 0.01 over the year,
- insurer earning interest rate at 5% per year,
- no expenses.

By *equivalence Principal*, the premium is

$$P = 1000(0.01)/1.05 = 9.52.$$

The future loss random variable is

$$L_0^n = \begin{cases} 1000\nu - P = 942.86 & \text{if } T_x \leq 1, \text{ with probability } 0.01, \\ -P & \text{if } T_x > 1, \text{ with probability } 0.99. \end{cases}$$

The expected loss is

$$0.01(942.86) + 0.99(-9.52) = 0, \text{ as required.}$$

example(cont'n)

Only if the insurer issues a large number of policies, so that the overall proportion of policies becoming claims will be close to the assumed proportion of 0.01.

- Suppose the insurer were to issue 100 such policies to independent lives.
- The insurer would expect to make a (small) profit on 99 of them.
- If all lives survive for the year, then the insurer makes a profit.
- If one life dies, then there is no profit or loss.

Let D be the number of death in the portfolio, so that $D \sim B(100, 0.01)$.

* The $P(\text{profit } \geq 0)$ on the **whole portfolio** is $P(D \leq 1) = 0.73576$.
* The profit on the **individual contract** is 0.99.
* As the number of policies issued increases, the probability of profit will tend, monotonically, to 0.5. (why?)
* While the probability of loss is increasing with the portfolio size, the probability of vary large aggregate loses is much smaller for a large portfolio. ($\because$ **diversification**)

PREMIUM CALCULATION

Let us consider a whole insurance policy with

- sum insured S payable at the end of the year of death,
- initial expenses of I, and
- the renewal expenses of e associated with each premium payment (including the first)

issued to a select life x by annual premium P. Then we have

$$L_0^g = S\nu^{K_{[x]}+1} + I + e\ddot{a}_{\overline{K_{[x]}+1}|} - P\ddot{a}_{\overline{K_{[x]}+1}|},$$

where $K_{[x]}$ denote the curtate future lifetime of $[x]$.

★ We can use L_0^g to find the minimum future lifetime for the policyholder in order the insurer makes a profit on this policy. That is

$$P(L_0^g < 0).$$

$$\begin{aligned}
P(L_0^g < 0) &= P(S\nu^{K_{[x]}+1} + I + e\ddot{a}_{\overline{K_{[x]}+1}|} - P\ddot{a}_{\overline{K_{[x]}+1}|} < 0) \\
&= P\left(\nu^{K_{[x]}+1} < \frac{\frac{P-e}{d} - I}{S + \frac{P-e}{d}}\right) \\
&= P\left[K_{[x]} + 1 > \frac{1}{\delta}\log\left(\frac{P-e+Sd}{P-e-Id}\right)\right].
\end{aligned} \tag{4}$$

Let the RHS term of (4) by τ, so that the contract generates a profit for the insurer if $K_{[x]} + 1 > \tau$.

- In general, τ is not an integer.
- Thus let $\lfloor \tau \rfloor$ be the integer part of τ.

The insurer makes a profit if the life survives at least $\lfloor \tau \rfloor$ years, the probability of which is

$$_{\lfloor \tau \rfloor}p_{[x]}.$$

example

Let us continue the previous illustration by assuming that

$x = 30$, $S = 100000$, $I = 1000$, $e = 50$.

Then we find $P = \$498.45$, and by (4) we can find

$$K_{[3]} + 1 > 52.57 \quad \Rightarrow \quad {}_{\lfloor 52.57 \rfloor}p_{[30]} = {}_{52}p_{[30]} = 0.70704.$$

There is a profit if the life survives for 52 years; the probability of which is 0.70704.

- ★ In general, a large losses occurs in the early years of the policy, and even larger profits occur if the policyholder dies in an advanced age.
- ★ The probability of realizing either a large loss or profit is small.

example

A life insurer is about to issue a 25-year endowment insurance with a basic sum insured of \$250000 to a select life aged exactly 30.
Premiums are payable throughout the term of the policy. Initial expenses are \$1200 plus 40% of the first premium and renewal expenses are 1% of the second and subsequent premiums.
The insurer allows for a compound reversionary bonus of 2.5% of the basic sum insured, vesting on each policy anniversary (including the last).
The death benefit is payable at the end of the year of death. Assume the interest at 5% per year.

(A) Derive an expression for the future loss r.v., L_0^g, for this policy.

(B) Calculate the annual premium for this policy.

(C) Calculate $L_0(k)$ (define later).

(D) Calculate the probability that the insurer makes a profit on this polivy.

(E) Calculate $V(L_0^g)$.

Solution

(A) Let the policyholder's curtate future lifetime be

$$K_{[30]} = k; k = 0, 1, 2, \cdots, 24,$$

then

the number of bonus additions is k,
the death benefit is payable $k + 1$ years from issue

and hence the present value of the death benefit is

$$250000(1.025)^{K_{[30]}} \nu^{K_{[30]+1}}.$$

However, if the policyholder survives for 25 years, then 25 bonuses are applied Thus, if P denotes the annual premium,

$$\begin{aligned} L_0^g &= 250000(1.025)^{min(K_{[30]},25)} \nu^{min(K_{[30]+1},25)} + 1200 + 0.39P \\ &\quad -0.99P\ddot{a}_{\overline{min(K_{[x]}+1,25)}|}. \end{aligned}$$

Solution(cont'n)

(B) The *EPV* of the premium, less premium expenses, is

$$0.99P\ddot{a}_{[30]:\overline{25}|} = 14.5838P.$$

As the death benefit is $250000(1.025)^t$ if the policyholder dies in the tth policy year, the *EPV* of the death benefit is

$$250000\sum_{t=0}^{24} \nu^{t+1}{}_{t|}q_{[30]}(1.025)^t = 250000\left(\frac{1}{1.025}A^{1}_{[30]:\overline{25}|j}\right) = 3099.37$$

where $1+j = \frac{1+i}{1.025}$, so that $j = 0.02439$.
The *EPV* of the survival benefit is

$$250000\nu^{25}{}_{25}p_{[30]}(1.025)^{25} = 134295.43,$$

and the *EPV* of the remaining expenses is

$$1200 + 0.39P.$$

Hence, by $14.5838P = 3099.37 + 134295.43 + 1200 + 0.39P$, we find that

$$P = \$9764.44$$

.

Solution(cont'n)

Let

- $L_0(k)$ be the present value of the loss on the policy given that $K_{[30]} = k$ for $k \leq 24$ and
- $L_0(25)$ be the present value on the policy given that the policyholder survives to age 55.

(c)

$$L_0(k) = 250000(1.025)^k \nu^{k+1} + 1200 + 0.39P - 0.99P\ddot{a}_{\overline{k+1}|}.$$

If the policyholder survives to age 55, there is one extra bonus payment, and

$$L_0(25) = 250000(1.025)^{25}\nu^{25} + 1200 + 0.39P - 0.99P\ddot{a}_{\overline{25}|}.$$

Solution(cont'n)

(D) By the results in (C), the present value of the loss, show that there is a profit if and only if the policyholder survives 24 years and pays the premium at the start of the 25th year

$$(\because\ L_0(24) = -4517\ \&\ L_0(25) = -1179).$$

Hence, the probability of a profit is

$$_{24}p_{[30]} = 0.98297.$$

(E) By the results in (C), the full set of values for $L_0(k)$, we have

$$E[(L_0^g)^2] = \sum_{k=0}^{24} [L_0(k)]^2 {}_{t|}q_{[30]} + [L_0(25)]^2 {}_{25}p_{[30]} = (12115.55)^2.$$

Example

THE PORTFOLIO PERCENTILE PREMIUM PRINCIPAL

The **portfolio percentile premium principal** is an alternative to the *equivalence principal*. We assume

- a large portfolio of **identical** and **independent** policies.
- **Identical** means that the policies are identical.
- **Independent** means that the policyholders are independent.

We wish to find an appropriate premium. Let

- N: the number of policies in the portfolio;
- $L_{0,i}$: the future loss r.v. for the ith policy in the portfolio, $i = 1, 2, \cdots, N$;
- L: the total future loss in the portfolio.

We have

$$
\begin{aligned}
L &= \sum_{i=1}^{N} L_{0,i}; \\
E[L] &= \sum_{i=1}^{N} E[L_{0,i}] = NE[L_{0,1}]; \\
V[L] &= \sum_{i=1}^{N} V[L_{0,i}] = NV[L_{0,1}]; (why?)
\end{aligned}
$$

★ The portfolio percentile premium principal sets a premium so that there is a specified probability, say α, that the total future loss is **negative**. That is,

$$P(L < 0) = \alpha.$$

If $N \to \infty$ (say, greater than 30), then $L \sim N(NE[L_{0,1}], NV[L_{0,1}])$. In this case, portfolio percentile premium can be calculated from

$$P(L < 0) = \left(\frac{L - E[L]}{\sqrt{V[L]}} < \frac{-E[L]}{\sqrt{V[L]}} \right) = \Phi \left(\frac{-E[L]}{\sqrt{V[L]}} \right) = \alpha.$$

example

An insurer issues whole life insurance policies to select lives aged exactly 30. The sum insured of $10000 is paid at the end of the month of death and level monthly premiums are payable throughout the term of the policy.
Initial expenses, incurred at the issue of the policy, are 15% of the total of the first year's premium. Renewal expenses are 4% of every premiums.
Assume the interest at 5% per year.

(A) Calculate the monthly by the equivalence principal.

(B) Calculate the monthly by the portfolio percentile premium principal, with $N = 10000$ and $\alpha = 95\%$.

Solution(cont'n)

(A) Let P be the monthly premium. Then the *EPV* of the premium is

$$12P\ddot{a}^{(12)}_{[30]} = 227.065P.$$

The *EPV* of benefits is

$$100000A^{(12)}_{[30]} = 7866.18$$

and the *EPV* of the expenses is

$$0.15(12P) + 0.04(12P\ddot{a}^{(12)}_{[30]}) = 10.8826P.$$

Hence, by $227.065P = 7866.18 + 10.8826P$, we find that

$$P = \$36.39 \text{ per month.}$$

Solution(cont'n)

(B) The future loss r.v. for the ith policy is

$$L_{0,i} = 100000\nu^{K_{[30]}^{(12)}+\frac{1}{12}} + 0.15(12P) - 0.96\left(12P\ddot{a}^{(12)}_{\overline{K_{[30]}^{(12)}+\frac{1}{12}}|}\right),$$

and its expected value can be calculated by using the results of (A) as

$$E[L_{0,i}] = 7866.18 - (227.065 - 10.8826)P = 7866.18 - 216.18P.$$

$$\because \; L_{0,i} = \left(100000 + \frac{0.96(12P)}{d^{(12)}}\right)\nu^{K_{[30]}^{(12)}+\frac{1}{12}} + 0.15(12P) - \frac{0.96(12P)}{d^{(12)}}$$

so that

$$\begin{aligned} V[L_{0,i}] &= \left(100000 + \frac{0.96(12P)}{d^{(12)}}\right)^2 \left[{}^2A_{[30]}^{(12)} - (A_{[30]}^{(12)})^2\right] \\ &= (100000 + 236.59P)^2(0.0053515) \end{aligned}$$

giving

$$\sqrt{V[L_{0,i}]} = (100000 + 236.59P)(0.073154).$$

Solution(cont'n)

(B) (cont'n)

$$\therefore L = \sum_{i=1}^{100000} L_{0,i}$$
$$\Rightarrow E[L] = 100000(7866.18 - 216.18P)$$

and

$$V[L] = 100000(100000 + 236.59P)^2(0.0053515).$$

Thus,

$$P(L<0) = \Phi\left(\frac{-E[L]}{\sqrt{V[L]}}\right) = \Phi\left[\frac{100000(216.18P - 7866.18)}{100(100000 + 236.59P)(0.073154)}\right] = 0.95,$$

which gives $P = \$36.99$.

Note

- ★ The portfolio percentile premium depends on the
  - number of policies in the portfolio (100000) and
  - the level of probability we set for the future loss being negative (0.95).
- ★ With α fixed, we note that P decreases as N increases. In fact, as

$$N \to \infty \;\Rightarrow\; P \to \text{ equivalence principal premium.}$$

EXTRA RISKS

There are different ways in which we can model the extra mortality risk in a premium calculation if underwriting determines that an individual should not be offered insurance at standard rates.

AGE RATING

Impaired life
An individual suffers from a medical condition and might not be offered insurance at standard rates.
For example, an impaired life aged 40 might be asked to pay the same premium paid by a non-impaired life aged 45.

This approach to modeling extra risk involves no new idea in premium calculation and we simply change the policyholder's age. This is referred to as **age rating**.

CONSTANT ADDITION TO μ_x

Individuals can be deemed to be ineligible to the standard rates if they regularly participate in hazard pursuits. The extra risk is largely independent of age and so we could model this extra risk by adding a **constant** to the force of mortality. That is

$$\mu'_{[x]+s} = \mu_{[x]+s} + \phi.$$

- The superscript $'$ relates to the impaired life.
- ϕ is the **constant addition** to the force of mortality.

Then

$$ {}_tp'_{[x]} = \exp\left\{-\int_0^t \mu'_{[x]+s} ds\right\} = \exp\left\{-\int_0^t (\mu_{[x]+s} + \phi) ds\right\} = e^{-\phi t}\, {}_tp_{[x]}.$$

$$\because e^{-\delta t}\, {}_tp'_{[x]} = e^{-(\delta+\phi)t}\, {}_tp_{[x]},$$

so , for example, the *EPV* of a survival benefit is

$$\ddot{a}'_{[x]:\overline{n}|} = \sum_{0}^{n-1} e^{-\delta t}\, {}_tp'_{[x]} = \sum_{0}^{n-1} e^{-(\delta+\phi)t}\, {}_tp_{[x]} = \ddot{a}_{[x]:\overline{n}|\ j} \tag{5}$$

where $j = e^{\delta+\phi} - 1$, the rate of interest.
Let $K'_{[x]}$ be the curtate impaired future lifetime. We know that

$$\ddot{a}'_{[x]:\overline{n}|} = E\left[\ddot{a}_{\overline{min(K'_{[x]}+1,n)}|}\right] = \frac{1 - E\left[\nu^{min(K'_{[x]}+1,n)}\right]}{d} = \frac{1 - A'_{[x]:\overline{n}|}}{d}.$$

So

$$A'_{[x]:\overline{n}|} = 1 - d\,\ddot{a}'_{[x]:\overline{n}|} = 1 - d\,\ddot{a}_{[x]:\overline{n}|\ j}. \tag{6}$$

It is important to note here that for insurance benefit we can **NOT** just change the interest rate.

- ★ In (6), the annuity is evaluated at rate j, but the function d uses the original rate of interest, that is $d = \frac{i}{1+i}$.
- ★ It is simplest to calculate the annuity function first, using a simple adjustment of interest, then use (6) for any insurance factor.

example

Calculate the annual premium for a 20-year endowment insurance with sum insured of \$200000 issued a life ages 30 whose force of mortality at age $30 + s$ is given by $\mu_{[30]+s} + 0.01$. Allow for initial expenses of \$2000 plus 40% of the first premium, and renewal expenses of 2% of the second and subsequence premiums. Assume the interest at 5% per year.

Solution(cont'n)

Let P be the annual premium. By (5), the *EPV* of premium income is

$$P\sum_{0}^{19} \nu^{t}\,{}_{t}p'_{[30]} = P\ddot{a}_{[30]:\overline{20|}\ j};\quad j = 1.05e^{0.01} - 1 = 0.06055.$$

Similarly, the *EPV* of expenses is

$$2000 + 0.38P + 0.02P\ddot{a}_{[30]:\overline{20|}\ j}.$$

($*$ The above interest rate are using j)

The *EPV* of the benefit is $200000A'_{[30]:\overline{20|}}$ and $i = 0.05$. By (6),

$$A'_{[30]:\overline{20|}} = 1 - d\ddot{a}_{[30]:\overline{20|}\ j}.$$

As $\ddot{a}_{[30]:\overline{20|}\ j} = 12.072$ and $d = \frac{0.05}{1.05}$, we find $A'_{[30]:\overline{20|}} = 0.425158$ and hence $P = \$7600.84$. $\square$

CONSTANT MULTIPLE OF MORTALITY RATE

A third method is to assume that lives are subject to mortality rates that are higher than the standard lives' mortality rates. For example,

$$q'_{[x]+t} = 1.1q_{[x]+t}.$$

- ★ A computational disadvantage of this approach is that we have to apply approximations in calculating *EPVs* if payments are at other than annual intervals.

example

Calculate the monthly premium for a 10-year term insurance with sum insured of \$100000 payable immediately on death, issued to a life aged 50.

- Assume that each year throughout the 10-year term the live is subject to mortality rates that are 10% higher than for a standard life of the same age.
- Allow for initial expenses of \$1000 plus 50% of the first monthly premium.
- Renewal expenses are of 3% of the second and subsequence monthly premiums.
- Use the UDD assumption where appropriate.
- Assume the interest at 5% per year.

Solution

Let P be the total premium **per year**.
The *EPV* of premium income is $P\ddot{a}^{(12)'}_{50:\overline{10}|}$ and, by UDD,

$$\ddot{a}^{(12)'}_{50:\overline{10}|} = \alpha(12)\ddot{a}'_{50:\overline{10}|} - \beta(12)(1 - \nu^{10}{}_{10}p'_{50}),$$

where

$$\alpha(12) = \frac{id}{i^{(12)}d^{(12)}} = 1.0002$$

and

$$\beta(12) = \frac{i - i^{(12)}}{i^{(12)}d^{(12)}} = 0.4665.$$

As the initial expenses of \$1000 plus 50% of the first premium, which is $\frac{1}{12}P$, the *EPV* of expenses is

$$1000 + \frac{0.47P}{12} + 0.03P\ddot{a}^{(12)'}_{50:\overline{10}|}.$$

Solution(cont'n)

The *EPV* of the death benefit is $100000(\bar{A}^{1}_{50:\overline{10|}})'$ and, by UDD,

$$\begin{aligned}(\bar{A}^{1}_{50:\overline{10|}})' &= \frac{i}{\delta}(A^{1}_{50:\overline{10|}})' \\ &= \frac{i}{\delta}\left[(A_{50:\overline{10|}})' - \nu^{10}\,{}_{10}p'_{50}\right] \\ &= \frac{i}{\delta}\left[1 - d\,\ddot{a}'_{50:\overline{10|}} - \nu^{10}\,{}_{10}p'_{50}\right]\end{aligned}$$

$$\because\ \ddot{a}'_{50:\overline{10|}} = \sum_{t=0}^{9} \nu^{t}\,{}_{t}p'_{50}$$

where

$${}_{t}p'_{50} = \prod_{r=0}^{t-1}(1 - 1.1\,q_{[50]+r}). \tag{7}$$

Hence $\ddot{a}'_{50:\overline{10|}} = 8.0516$, $\ddot{a}^{(12)'}_{50:\overline{10|}} = 7.8669$ and $\bar{A}^{1}_{50:\overline{10|}})' = 0.01621$ which gives $P = \$345.18$ and so the monthly premium is \$28.76.