Actuarial Mathematics
LIFE TABLES AND SELECTION

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2021

SUMMARY

In this chapter we

- define a life table (integer ages only);
- show, using fractional age assumptions, how to calculate survival probabilities for all ages and durations;
- consider life tables appropriate to individuals who have purchased particular type of life insurance policy;
- discuss why the survival probabilities differ from those in the corresponding national life table.

Furthermore, we

- consider the effect of *selection* of lives for insurance policies (e.g. through underwriting);
- define a select survival model and derive some formulae for the model.

LIFE TABLES

Let

- x_0: initial age (arbitrary);
- ω: maximum age;

given a survival model with survival probability ${}_tp_x$, we can construct the **life table** by a function $\{l_x\}$ for the model from age x_0 to age ω.

For $\{l_x\};\ x_0 \leq x \leq \omega$, let

- l_{x_0}: an arbitrary positive number, called **radix** for the table;
- $0 \leq t \leq \omega - x_0$.

Define

$$l_{x_0+t} = l_{x_0}\ {}_tp_{x_0}.$$

For $x_0 \leq x \leq x+t \leq \omega$,

$$\begin{aligned} l_{x+t} &= l_{x_0}\ {}_{x+t-x_0}p_{x_0} \\ &= l_{x_0}\ {}_{x-x_0}p_{x_0}\, {}_tp_x \\ &= l_x\ {}_tp_x, \end{aligned}$$

so that

$$ {}_tp_x = \frac{l_{x+t}}{l_x}. \tag{1}$$

- ★ l_{x+t} can be interpreted as the expected number of survivors at age $x+t$ from l_x *independent individuals* aged x, $\forall x \geq x_0$.
- ★ Let L_t be the number of survivors to age $x+t$, then

$$L_t \sim b(l_x, {}_tp_x).$$

Hence

$$E(L_t) = l_x\ {}_tp_x = l_{x+t}.$$

- We always use the table in the form $\frac{l_y}{l_x}$, that is why the radix of the table is arbitrary.
- From (1), We are able to calculate both survival and mortality probabilities; e.g.

$$q_{30} = 1 - \frac{l_{31}}{l_{30}} = \frac{l_{30} - l_{31}}{l_{30}}$$

and

$$_{15|30}q_{40} = {}_{15}p_{40}\ {}_{30}q_{55} = \frac{l_{55}}{l_{40}}(1 - \frac{l_{85}}{l_{55}}) = \frac{l_{55} - l_{85}}{l_{40}}.$$

- A life table is defined for all x from the initial age, x_0, to the limiting age, ω, at **integer ages** only.

In this, it allow us to calculate surviving or dying over integer numbers of years starting from an integer age. Let

$$d_x = l_x - l_{x+1} \tag{2}$$

We have

$$d_x = l_x \left(1 - \frac{l_{x+1}}{l_x}\right) = l_x(1 - p_x) = l_x q_x.$$

★ d_x can be interpreted as the **expected number of deaths** in the year of age x to $x+1$ from a group of l_x lives aged exactly x, so that, using the binomial distribution again

$$d_x = l_x q_x. \tag{3}$$

<u>**Example**</u>

FRACTIONAL AGE ASSUMPTIONS

Given values of l_x at integer ages only, we need to make some assumption about the probability distribution for the future r.v. between integer ages to calculate probabilities for **non-integer** ages or distributions.

★ These assumptions are described using the term **fractional age assumptions** and can be specified in terms of *force of mortality function* or *survival or mortality probabilities*.

In this section, we

- assume a life table with integer ages only;
- describe two most useful fractional age assumptions:

  uniform distribution of deaths
  and
  constant force of mortality.

Uniform Distribution of Deaths

The **uniform distribution of deaths** (**UDD**) assumption is the most common fractional age assumption. It can be formulated in two different, but equivalent, ways.

UDD1

$$ {}_sq_x = s\,q_x; \quad \forall x \in Z;\ 0 \leq s < 1. \tag{4}$$

UDD2

$$T_x = K_x + R_x \tag{5}$$

with

K_x: the integer part of T_x

and

R_x: a random variable with

$$R_x \sim U(0,1) \quad \wedge \quad R_x \perp\!\!\!\perp T_x.$$

UDD1 $\Rightarrow$ UDD2

For $x \in Z$ and $0 \leq s < 1$,

- $(R_x \sim U(0,1))$

$$
\begin{aligned}
P(R_x \leq s) &= \sum_{k=0}^{\infty} P(R_x \leq s \ \wedge \ K_x = k) = \sum_{k=0}^{\infty} P(k \leq T_x \leq k+s) \\
&= \sum_{k=0}^{\infty} {}_kp_x \ {}_sq_{x+k} = \sum_{k=0}^{\infty} {}_kp_x \ s q_{x+k} \quad \text{(by UDD1)} \\
&= s \sum_{k=0}^{\infty} {}_kp_x \ q_{x+k} \\
&= s \sum_{k=0}^{\infty} P(K_x = k) \\
&= s.
\end{aligned}
$$

UDD1 $\Rightarrow$ UDD2 (conti'n)

- $(R_x \perp\!\!\!\perp K_x)$

$$\begin{aligned} P(R_x \le s \ \wedge \ K_x = k) &= P(k \le T_x \le k+s) \\ &= {}_k p_x \, {}_s q_{x+k} \\ &= s \, {}_k p_x \, q_{x+k} \\ &= P(R_x \le s) P(K_x = k). \end{aligned}$$

UDD2 $\Rightarrow$ UDD1

$$\begin{aligned} {}_s q_x &= P(T_x \le s) = P(K_x = 0 \ \wedge \ R_x \le s) \\ &= P(R_x \le s) P(K_x = 0) \quad (Why?) \\ &= s \, q_x. \quad (Why?) \end{aligned}$$

★ In practice, UDD1 is the more useful of the two (UDD2 explains why the name is).

An immediate consequence is that

$$l_{x+s} = l_x - s\, d_x; \quad \forall\, 0 \leq s < 1. \tag{6}$$

Proof

$$\because {}_s q_x = 1 - {}_s p_x = 1 - \frac{l_{x+s}}{l_x}$$

then, by UDD,

$$s\; q_x = 1 - \frac{l_{x+s}}{l_x},$$

that is,

$$s \frac{d_x}{l_x} = 1 - \frac{l_{x+s}}{l_x}$$

$$\therefore l_{x+s} = l_x - s\, d_x.$$

★ l_{x+s} is a linear decreasing function of s.

By (5),

$$\frac{d}{ds}{}_sq_x = q_x, \quad \forall\, 0 \le s < 1.$$

Hence'

$$q_x = {}_sp_x\,\mu_{x+s} \quad \forall\, 0 \le s < 1. \quad (Why?) \tag{7}$$

- ★ The left-hand side of (7) does not depend on s, which means that $f_x(s)$ is a constant for $0 \le s < 1$, which also follows from the uniform assumption for R_x.
- ★ From (7),

$$\because\ q_x:\ \text{constant} \ \wedge\ {}_sp_x \in \searrow \text{ in } s \quad \therefore\ \mu_{x+s} \in \nearrow \text{ in } s.$$

It is appropriate for ages of interest to insurers.

Examples

Constant Force of Mortality

A second fractional age assumption is that the force of mortality is constant between integer ages.Let

$$\mu_{x+s} = \mu_x^*; \quad \forall\, x \in Z,\ 0 \leq s < 1; \quad i.e.\ \mu_{x+s} \perp\!\!\!\perp s$$

$$\begin{aligned} \because\ p_x &= \exp\left[-\int_0^1 \mu_{x+s} ds\right] \ \wedge\ \mu_{x+s} = \mu_x^*. \\ &\Rightarrow \quad p_x = e^{-\mu_x^*} \ \text{ or } \ \mu_x^* = -\ln p_x. \\ \therefore\ {}_sp_x &= \exp\left[-\int_0^s \mu_x^* du\right] = (p_x)^s. \end{aligned}$$

Similarly, for $t,\ s > 0$ and $t + s < 1$,

$${}_sp_{x+t} = \exp\left[-\int_0^s \mu_x^* du\right] = (p_x)^s. \quad (Why?)$$

Under the constant force assumption,

- ★ the probability of surviving for a period of $s < 1$ year from age $x + t$ is independent of t provided $t + s < 1$;
  - the assumption between integer ages leads to a step function for the force of mortality over successive years of age, whereas we would expect the force of mortality to increase smoothly.
- ★ In general, the assumption s of a uniform distribution of death and a constant force mortality produce very similar solutions to problems.

$$\because \; q_x = 1 - p_x = 1 - e^{-\mu^*} \approx \; \mu^* \; \text{ if } \; \mu^* \to 0.$$

$$\therefore \; {}_tq_x = 1 - e^{-\mu^* t} \approx \; \mu^* t, \; \text{ for } \; 0 < t < 1.$$

It is what we obtain under the uniform distribution of death assumption.

Examples

NATIONAL LIFE TABLES

We note the following points in a national life table.

- The value of q_0 is relative high which is called **perinatal mortality**. In England, The value of q_x does not reach this value until about age 55.
- The rate of mortality is much lower after the first year and declines until around age 10.
- The pattern of male and female mortality in the late teenage years diverges significantly, with a steeper incline in male mortality. it is sometimes called **accident hump**.
- The accident hump creates a relatively large increase for the male mortality rate between ages 10 and 20, a modest increase from ages 20 to 40.
- The rate of mortality for a female almost always less than that for a male of the same age.

SURVIVAL MODELS FOR LIFE INSURANCE POLICYHOLDERS

In the countries with well-developed life insurance markets, the mortality experience of people who purchase life insurance policies tend to be different from the population as a whole.

- ★ The mortality of different types of insurance policyholders is investigated separately, and life tables appropriate for these groups are published.
- ★ In general, the mortality rates for the insurance policyholders are much lower than those from whole population.

There are reasons for this difference.

LIFE TABLES AND SELECTION

- Within any large group, there are likely to be variation in mortality rates in subgroups. For example, social class, defined in terms of occupation, has a significant effect on mortality.
- Given that people who purchase term insurance policies are likely to be among the better paid people in population, and hence have a lower mortality rate.
- The most significant reason arises from the **selection process** which policyholders must complete before the insurer will issue the insurance policy.The **selection** or **underwriting** process ensures that people who purchase life insurance cover are healthy when the insurance was purchased.
- A national life tables, on the other hand, are based on data from both healthy and unhealthy lives.

SELECT AND ULTIMATE SURVIVAL MODELS

- **Aggregate survival models**
  The probabilities of future survival depend only on the individual's current age. That is, given survival model and t

  $$ {}_tp_x \text{ depends only on } x. $$

- **Select and ultimate survival model**
  The model is usually shortened to be called as **Select survival model**.
  The probabilities of future survival depend **NOT** only on the individual's current age but also on how long ago the individual entered the group of policyholders, i.e. when the policy was purchased.

Definition ⟨ Select survival model ⟩

(A) Future survival probabilities for an individual in the group depend on the individual's *current age* and on the *age at which the individual joined the group*.

(B) There is d, usually $d \in Z$, such that if an individual joined the group more than d years ago, future survival probabilities depend only on current age. The initial selection effect is assumed to be worn off after d years.

★ For a term insurance policyholder, being selected at x, means that the mortality rate for the individual was lower than that of a term insurance policyholder of the same age who had been selected some years earlier.

Example

NOTATION AND FORMULAE FOR SELECT SURVIVAL MODELS

Recall

For select survival model, probabilities of survival dependent on current age and (within the select period d) age at selection, i.e. age at join the group.

- ★ The survival model for those individuals **all selected at the same age**, say x, depends only on their current age.
- ★ Hence, provided we fix and specify the age at selection, we can adapt the notation and formulae developed in the previous Chapter to a select survival model.

Define

$$\begin{aligned} & {}_t p_{[x]+s} \\ = \; & P(\text{ a life currently aged } x+s \text{ who was selected at age } x \\ & \quad \text{survives to age } x+s+t.) \end{aligned}$$

$$\begin{aligned} & {}_t q_{[x]+s} \\ = \; & P(\text{ a life currently aged } x+s \text{ who was selected at age } x \\ & \quad \text{dies before age } x+s+t.) \end{aligned}$$

$\mu_{[x]+s}$: the force of mortality at age $x+s$ for an individual who was selected at age x,

$$\mu_{[x]+s} = \lim_{h \to 0} \left(\frac{1 - {}_h p_{[x]+s}}{h} \right). \quad (Why?)$$

$${}_t p_{[x]+s} = \exp\left[-\int_0^t \mu_{[x]+s+u} du \right]. \quad (Why?)$$

For a select survival model with a select period d,

- for duration at or beyond the select period, $t \geq d$, called the **ultimate part** of the survival model, the following values do not depend on t, depend only on the current age x. That is,

$$\mu_{[x-t]+t} \equiv \mu_x, \quad {}_sp_{[x-t]+t} \equiv {}_sp_x, \quad {}_{u|s}q_{[x-t]+t} \equiv {}_{u|s}q_x;$$

- for $t < d$, called the **select part** of the survival model.

SELECT LIFE TABLE

Recall

For **ultimate survival model**, we use $\{l_x\}$ to construct a life table.

We can construct a **select life table** in a similar way but we need the table to reflect

$\bigstar$ *duration* as well as *age*, $\bigstar$

during the **select period**.

- The construction in this table is for select life table specified at **all** ages and not at just integer ages.
- Select life tables are usually presented at integer ages only, as is the case for ultimate life tables.

Let

- x_0: initial select age, $\forall x_0 \geq 0$;
- d: select period, $d \in Z$ and $\forall\ l_{x_0+d} \in Z^+$.

I. For $y \geq x_0 + d$ (i.e. the **ultimate part**)

★ That is, the minimum age of these people is $x_0 + d$. For these people, we can construct an **ultimate life table** since their future survival probabilities depend only on their current ages.

Define

$$
{}_{y-(x_0+d)}p_{x_0+d} = \frac{l_y}{l_{x_0+d}}
$$

$$
\Rightarrow \quad l_y = {}_{y-(x_0+d)}p_{x_0+d}\ l_{x_0+d}. \tag{8}
$$

For $y > x \geq x_0 + d$;

$$\begin{aligned}
\because l_y &= \left[{}_{y-(x_0+d)}p_{x_0+d}\right]\ l_{x_0+d} \\
&= \left[{}_{x-(x_0+d)}p_{x_0+d}\right]\ \left[{}_{y-x}p_{[x_0]+x-x_0}\right]\ l_{x_0+d} \\
&\quad \text{(because } x_0 + d \to x \to y\text{)} \\
&= \left[{}_{x-(x_0+d)}p_{x_0+d}\right]\ \left[{}_{y-x}p_x\right]\ l_{x_0+d} \\
&= {}_{y-x}p_x\ l_x\ \ (Why?)
\end{aligned}$$

$$\therefore l_y = {}_{y-x}p_x\ l_x. \tag{9}$$

★ In the ultimate part of the model, l_y can be interpreted as the expected number of survivors to age y out of l_x lives currently aged x ($< y$), who were selected at least d years ago.

★ (8) defines the life table within the **ultimate part** of the table.

II. For $x \geq x_0$ and $0 \leq t \leq d$ (i.e. the **select part**)

- ★ We define the life table within the **select period** for a life select at age x.
- ★ We calculate it by *working backwards* from l_{x+d}.

Since $x \to x+t \to x+d$, we have

$$_{d-t}p_{[x]+t} = \frac{l_{([x]+t)+(d-t)}}{l_{[x]+t}} = \frac{l_{[x]+d}}{l_{[x]+t}}.$$

Therefore,

$$l_{[x]+t} = \frac{l_{x+d}}{_{d-t}p_{[x]+t}}. \tag{10}$$

- ★ If we have $l_{[x]+t}$ lives aged $x+t$, selected t years ago, then the expected number of survivors to age $x+d$ is l_{x+d}.
- ★ (10) defines the life table within the **select part** of the table.

example

For

$$y \geq x+d > x+s > x+t \geq x \geq x_0,$$

show that

$$ {}_{y-x-t}p_{[x]+t} = \frac{l_y}{l_{[x]+t}} \tag{11}$$

and

$$ {}_{s-t}p_{[x]+t} = \frac{l_{[x]+s}}{l_{[x]+t}}. \tag{12}$$

★ (11) and (12), together with (9), show that the construction preserves the interpretations of the l's as the expected number of survivors within both the ultimate and select parts of the model.

proof.

1. Since $x+t \to x+d \to y$, we have

$$\begin{aligned} {}_{y-x-t}p_{[x]+t} &= \left[{}_{t-d}p_{[x]+t}\right]\left[{}_{y-(x+d)}p_{[x]+d}\right] \\ &= \left[{}_{t-d}p_{[x]+t}\right]\left[{}_{y-(x+d)}p_{x+d}\right] \\ &= \frac{l_{x+d}}{l_{[x]+t}}\frac{l_y}{l_{x+d}} = \frac{l_y}{l_{[x]+t}}. \end{aligned}$$

2. Since $x+t \to x+s \to x+d$, we have

$$\begin{aligned} {}_{s-t}p_{[x]+t} &= \frac{{}_{d-t}p_{[x]+t}}{{}_{d-s}p_{[x]+s}} = \frac{l_{x+d}}{l_{[x]+t}}\frac{l_{[x]+s}}{l_{x+d}} \\ &= \frac{l_{[x]+s}}{l_{[x]+t}}. \end{aligned}$$

□

Examples

SOME COMMENTS ON HETEROGENEITY IN MORTALITY

Studies of mortality have shown that the following principles apply quite generally.

- Wealthier lives experience lighter mortality overall than less wealthier lives.
- Those buying **term insurance** might be expected to have slightly heavier mortality than those buying whole life insurance, and those buying **annuities** might be expected to have lighter mortality. That is, there will be some impact on mortality experience from **self-selection**.
- The more rigorous the underwriting, the lighter the resulting mortality experience. For *group insurance*, there will be minimal underwriting.

MORTALITY TRENDS

Survival probabilities are **NOT** constant over time. Commonly, each generation, on average, lives longer than the previous generation.

The changes of mortality over time are sometimes separated into three components:

- **trend**
  the gradual reduction in mortality rates over time;
- **shock**
  a short term jump in mortality rates from war or pandemic disease;
- **idiosyncratic**
  year to year random variation that does not come from trend or shock (often difficult to distinguish these changes).

★ Shock and idiosyncratic are unpredictable; trends can be identified by examining mortality pattern over a number of years.

Reduction factor

The **constant factor**, r_x, that mortality rates at each age are assumed to decrease annually; which depends on the *age* and *sex* of the individual aged x.

Let

$$q(x, Y): \text{ the mortality rate for } (x) \text{ in year } Y.$$

That is

$$q(x, 0): \text{ the mortality rate for } (x) \text{ for a baseline year, } Y = 0.$$

Then, the estimated one-year mortality probability for (x) at $Y = s$ is

$$q(x, s) = q(x, 0)\, r_x^s\,; \;\; 0 < r_x^s \leq 1.$$

Remarks

- ★ Typical values of r_x are in the range 0.95 to 1, where higher values tend to apply at order ages.
- ★ Using $r_x = 1$ for the oldest age reflects that although people living longer than previous generation,
  - there is little or no increase in the maximum age attained;
  - the change is that a greater proportion of lives survive to older ages.
- ★ r_x shows the greatest reduction in mortality rates often occur at the youngest ages, and as age increases from around 60, r_x's are increasing.

Remarks (conti'n)

★ Let $q(x,0) = q_x$ and a set of age-based r_x, we can calculate survival probabilities from the baseline year, ${}_tp_{(x,0)}$, as

$$\begin{aligned} & {}_tp_{(x,0)} \\ = \; & p(x,0)\, p(x+1,1) \cdots p(x+t-1,t-1) \qquad (13) \\ = \; & (1-q_x)(1-q_{x+1}r_{x+1})(1-q_{x+2}r_{x+2}^2)\cdots(1-q_{x+t-1}r_{x+t-1}^{t-1}). \end{aligned}$$

★ **Longevity risk**

This risk, is great recent interest, as mortality rates have declined in many countries at much faster rate than anticipated.