ACTUARIAL MATHEMATICS
INSURANCE BENEFITS

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SUMMARY

We develop formulae for the valuation of the following traditional insurance benefits:

- whole life insurance;
- term insurance;
- endowment insurance.

The valuation functions for benefits, are based on

- T_x: continuous future lifetime;
- K_x: curtate future lifetime;
- $K_x^{(m)}$: the number of complete periods of length $1/m$ years

lived by a life (x).

INTRODUCTION

Because of the dependence on death or survival of a policyholder or a pension plan member, the **present value** of the insurance benefit is modeled as a *random variable*.
In this chapter

$$\begin{aligned} & \text{Survival model } + \text{ Time value of money function} \\ \Rightarrow\ & \text{The distribution of the present value of an uncertain,} \\ & \text{life contingent future benefit.} \end{aligned}$$

Assuming that the interest is **constant** and **fixed**, so that

- in case of a **death benefit**, death payment is paid at the exact time of death;
- in case of an **annuity**, a continuous benefit of, say, \$1 per year would pay $\$dt$ in every $(t, t + dt)$.

ASSUMPTIONS

A set of assumptions used in life insurance or pension calculations is denoted as **basis**.

- The interest rates are constant.
- Reminds of interest theory, with $\nu = \frac{1}{1+i}$ and δ being the *force of interest*,

$$\delta = \ln(1+i), \quad 1+i = e^{\delta}, \quad \text{and } \nu = e^{-\delta}.$$

$$i^{(p)} = p\left[(1+i)^p - 1\right] \;\Leftrightarrow\; 1+i = \left(1 + \frac{i^{(p)}}{p}\right)^p.$$

$$d = 1-\nu = i\nu = 1 - e^{-\delta};\; d: \text{ rate of discount.}$$

$$d^{(p)} = p\left(1 - \nu^{1/p}\right) \;\Leftrightarrow\; \nu = \left(1 - \frac{d^{(p)}}{p}\right)^p.$$

VALUATION OF INSURANCE BENEFITS

Whole life insurance: the continuous case, $\bar{A}_x$

For **whole life insurance policy**, the benefit will be paid when the policyholder actually *dies* and the policy becomes a claim.

- The present value of the benefit is a random variable.
- Considering the value of a benefit of $1 payable following the death of a life currently aged x.
- Assume that the benefit is payable *immediately* on the death of (x). That is known as the **continuous case**.

Therefore, for (x), The **present value** of a benefit of $1 payable immediately on the death is a random variable denoted as Z where

$$Z = \nu^{T_x} = e^{-\delta T_x}.$$

Whole life insurance: the continuous case, $\bar{A}_x$

We are generally most interested in the *expected value of the present value* random variable for some future time and refer to this the **Expected Present Value** or **EPV** (or *Actuarial Value* or *Actuarial Present Value*).

Definition

The EPV of the whole life insurance benefit payment with sum insured \$1 is $E(e^{-\delta T_x})$ and is denoted as $\bar{A}_x$ in actuarial notation. That is,

$$\bar{A}_x = E(e^{-\delta T_x}) = \int_0^\infty e^{-\delta t} {}_tp_x \mu_{x+t} dt \tag{1}$$

with $T_x \sim \ f_x(t) = {}_tp_x \nu_{x+t}$.

Whole life insurance: the continuous case, $\bar{A}_x$

The second moment (about zero) of the present value of the death benefit is

$$\begin{aligned} E(Z^2) &= E[(e^{-\delta T_x})^2] = E[e^{-2\delta T_x}] \\ &= \int_0^\infty e^{-2\delta t} {}_tp_x \mu_{x+t} dt \\ &= {}^2\bar{A}_x. \end{aligned} \tag{2}$$

- ★ The superscript ^{2} indicates that the calculation is at force of interest 2δ; that is, at rate of interest j, where $j = e^{2\delta} = (1+i)^2$.
- ★ Hence, the variance of the present value of a unit benefit payable immediately on death is

$$Var(Z) = Var(e^{-\delta T_x}) = E(Z^2) - [E(Z)]^2 = {}^2\bar{A}_x - (\bar{A}_x)^2. \tag{3}$$

Whole life insurance: the continuous case, $\bar{A}_x$

In fact, we can calculate any probability associated with Z from the probability associated with T_x.

example

$$\begin{aligned} P(Z \leq 0.5) &= P[e^{-\delta T_x} \leq 0.5] = P[-\delta T_x \leq \ln(0.5)] \\ &= P[\delta T_x > \ln(2)] \\ &= P[T_x > \ln(2)/\delta] \\ &= {}_u p_x \quad ; \qquad u = \ln(2)/\delta. \end{aligned}$$

★ Low values of Z are associated with high values of Tx. This makes sense because the benefit is more expensive to the insurer if it is paid early, as there has been little opportunity to earn interest.

Whole life insurance: the annual case, A_x

Suppose now that the benefit of \$1 is payable at the end of the year of death of (x), rather than immediately on death.

★ The time to the end of the year of death of (x) is $K_x + 1$. (Why?)

$$i.e.\ Z = \nu^{K_x+1}.$$

The EPV of the benefit, $E(Z)$, is denoted by A_x.

$$A_x = E(\nu^{K_x+1}) = \sum_{k=0}^{\infty} \nu^{k+1}\,{}_{k|}q_x = \nu q_x + \nu^2\,{}_{1|}q_x + \nu^3\,{}_{2|}q_x + \cdots \tag{4}$$

Whole life insurance: the annual case, A_x

Similarly, the second moment of the present value is

$$\sum_{k=0}^{\infty} \nu^{2(k+1)}{}_{k|}q_x = \sum_{k=0}^{\infty} (\nu^2)^{(k+1)}{}_{k|}q_x = (\nu)^2 q_x + (\nu^2)^2{}_{1|}q_x + (\nu^2)^3{}_{2|}q_x + \cdots .$$

Define

$$^2A_x = \sum_{k=0}^{\infty} \nu^{2(k+1)}{}_{k|}q_x, \tag{5}$$

so the variance of the present value of a benefit of S payable at the end of the year of death is

$$S^2\left[{}^2A_x - (A_x)^2\right]. \tag{6}$$

Whole life insurance: the $1/m$th ly case, $A_x^{(m)}$

$K_x^{(m)}$:

the $1/m$thly curtate future lifetime random variable, where $m > 1$ is an integer. That is, it is the future lifetime of (x) in years rounded to the lower $\frac{1}{m}$th of a year.

- ★ The most common m are 2,4, and 12.

Let $\lfloor\rfloor$ be the integer part function, then

$$K_x^{(m)} = \frac{1}{m} \lfloor mT_x \rfloor . \tag{7}$$

example

Suppose (x) lives exactly 23.675 years. Then

$$K_x = 23, \quad K_x^{(2)} = 23.5, \quad K_x^{(4)} = 23.5, \quad K_x^{(12)} = 23.6667.$$

Whole life insurance: the $1/mth$ly case, $A_x^{(m)}$

- ★ $K_x^{(m)}$ is a discrete random variable.
- ★ $K_x^{(m)} = k$ indicates that (x) dies in the interval $[k, k+\frac{1}{m})$ for $k = 0, \frac{1}{m}, \frac{2}{m}, \cdots$.

For $k = 0, \frac{1}{m}, \frac{2}{m}, \cdots$,

$$P\left[K_x^{(m)} = k\right] = P\left(k \le T_x < k + \frac{1}{m}\right) = {}_{k|\frac{1}{m}}q_x = {}_k p_x - {}_{k+\frac{1}{m}}p_x.$$

example

Let $m = 12$, then $Z = \nu^{K_x^{(12)}+\frac{1}{12}}$. Hence,

$$E(Z) = A_x^{(12)} = \nu^{\frac{1}{12}}\, {}_{\frac{1}{12}}q_x + \nu^{\frac{2}{12}}\, {}_{\frac{1}{12}|\frac{1}{12}}q_x + \nu^{\frac{3}{12}}\, {}_{\frac{2}{12}|\frac{1}{12}}q_x + \nu^{\frac{4}{12}}\, {}_{\frac{3}{12}|\frac{1}{12}}q_x + \cdots.$$

Whole life insurance: the $1/mth$ly case, $A_x^{(m)}$

In general, $\forall$ m,

$$A_x^{(m)} = \nu^{\frac{1}{m}} {}_{\frac{1}{m}}q_x + \nu^{\frac{2}{m}} {}_{\frac{1}{m}|\frac{1}{m}}q_x + \nu^{\frac{3}{m}} {}_{\frac{2}{m}|\frac{1}{m}}q_x + \nu^{\frac{4}{m}} {}_{\frac{3}{m}|\frac{1}{m}}q_x + \cdots \quad (8)$$

$$= \sum_{k=0}^{\infty} \nu^{\frac{k+1}{m}} {}_{\frac{k}{m}|\frac{1}{m}}q_x. \quad (9)$$

Similarly,

$$E(Z^2) = E\left[\nu^{2(K_x^{(m)}+\frac{1}{m})}\right] = E\left[(\nu^2)^{(K_x^{(m)}+\frac{1}{m})}\right] = {}^2A_x^{(m)},$$

so the variance is

$${}^2A_x^{(m)} - (A_x^{(m)})^2.$$

RECURSIONS

We can calculate the values of A_x using **backwards recursion**.

- Let all lives expire by age ω, that is, $q_{\omega-1} = 1$.

Therefore, we know that $K_{\omega-1} = 0$ and

$$A_{\omega-1} = E\left(\nu^{K_{\omega-1}+1}\right) = \nu. \quad (Why?)$$

Hence,

$$\begin{aligned} A_x &= \sum_{k=0}^{\omega-x-1} \nu^{k+1} {}_kp_x q_{x+k} \\ &= \nu q_x + \nu^2 p_x q_{x+1} + \nu^3 {}_2p_x q_{x+2} + \cdots \\ &= \nu q_x + \nu p_x \left(\nu q_{x+1} + \nu^2 p_{x+1} q_{x+2} + \nu^3 {}_2p_{x+1} q_{x+3} + \cdots\right). \end{aligned}$$

We have

$$A_x = \nu q_x + \nu p_x A_{x+1}. \tag{10}$$

RECURSIONS

★ The intuition of (10) is that

$$\begin{aligned}
& \text{the EPV of the whole life insurance} \\
= \quad & \text{(the value of benefit due in the first year)} \\
+ \quad & \text{(discount value from age} x+1 \text{ to } x) \\
& \text{(the probability of surviving to age } x+1) \\
& \text{(the value at age } x+1 \text{ of all subsequent benefits)}
\end{aligned}$$

Similarly, we can have

$$A_x^{(m)} = \nu^{\frac{1}{m}} {}_{\frac{1}{m}}q_x + \nu^{\frac{1}{m}} {}_{\frac{1}{m}}p_x \left(\nu^{\frac{1}{m}} {}_{\frac{1}{m}}q_{x+\frac{1}{m}} + \nu^{\frac{2}{m}} {}_{\frac{1}{m}}p_{x+\frac{1}{m}} {}_{\frac{1}{m}}q_{x+\frac{2}{m}} + \cdots \right).$$

Hence

$$A_x^{(m)} = \nu^{1/m} {}_{\frac{1}{m}}q_x + \nu^{\frac{1}{m}} {}_{\frac{1}{m}}p_x A_{x+\frac{1}{m}}^{(m)}.$$

Examples

Term insurance: the continuous case, $\bar{A}^{1}_{x:\overline{n}|}$

Under **term insurance policy**,

- ★ the death benefit is payable only if the policyholder dies within a fixed term of , say, n year;
- ★ in the continuous case, the benefit is payable immediately on death;
- ★ the present value of \$1 is

$$Z = \begin{cases} \nu^{T_x} & \text{if } T_x \leq n, \\ 0 & \text{if } T_x > n. \end{cases}$$

Hence

- ★ The EPV of Z, $E(Z)$, is denoted $\bar{A}^{1}_{x:\overline{n}|}$.
- ★ The 1 above x indicates that (x) must die before the term of n years expires in order for the benefit is payable.
- ★ $$\bar{A}^{1}_{x:\overline{n}|} = \int_0^n e^{-\delta t}\,{}_tp_x\mu_{x+t}dt. \tag{11}$$

  Similarly,

  $$^2\bar{A}^{1}_{x:\overline{n}|} = \int_0^n e^{-2\delta t}\,{}_tp_x\mu_{x+t}dt.$$

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Term insurance: the annual case, $A^{1}_{x:\overline{n}|}$

In annual case,

- the death benefit of \$1 is payable at the end on the year of death, provided it occurs within n years;
- the present value of the benefit is

$$Z = \begin{cases} \nu^{K_x+1} & \text{if } K_x \le n-1, \\ 0 & \text{if } K_x \ge n. \end{cases}$$

- The EPV of Z, $E(Z)$, is denoted $A^{1}_{x:\overline{n}|}$ and

$$A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \nu^{k+1} {}_{k|}q_x. \tag{12}$$

Term insurance: the $1/m$th case, $A^{(m)}{}^{1}_{x:\overline{n|}}$

Now consider a death benefit of \$1 is payable at the end of the $1/m$th year of death, provided it occurs within n years.

- ★ the present value of the benefit is

$$Z = \begin{cases} \nu^{K_x^{(m)} + \frac{1}{m}} & \text{if } K_x^{(m)} \le n - \dfrac{1}{m}, \\ 0 & \text{if } K_x^{(m)} \ge n. \end{cases}$$

- ★ The EPV of Z, $E(Z)$, is denoted $A^{(m)}{}^{1}_{x:\overline{n|}}$ and

$$A^{(m)}{}^{1}_{x:\overline{n|}} = \sum_{k=0}^{mn-1} \nu^{\frac{k+1}{m}} \, {}_{\frac{k}{m}|\frac{1}{m}}q_x. \tag{13}$$

Examples

Pure endowment

Pure endowment benefits are conditional on the *survival* of the policyholder at a policy maturity date.

- ★ Pure endowment benefits are **NOT** sold as stand-alone policy, but may be sold in conjunction with term insurance benefits to create the **endowment insurance** benefit.
- ★ The pure endowment benefit of \$1, issued to a life aged x, with a term of n years has present value

$$Z = \begin{cases} 0 & \text{if } T_x < n, \\ \nu^{T_x} & \text{if } T_x \geq n. \end{cases}$$

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Hence

★ The EPV of Z, $E(Z)$, is denoted $\bar{A}_{x:\overset{1}{\overline{n|}}}$ or ${}_nE_x$.

★ The 1 above n indicates that the term must expire before the life does for the benefit to be paid.

★ The present value Z can be rewrite as

$$Z = \begin{cases} 0 & \text{with probability } 1 - {}_np_x, \\ \nu^{T_x} & \text{with probability } {}_np_x. \end{cases} \tag{14}$$

★

$$\bar{A}_{x:\overset{1}{\overline{n|}}} = {}_nE_x = \nu^n {}_np_x. \tag{15}$$

★ Pure endowment is no need to specify continuous and discrete time versions.

Endowment insurance

Endowment insurance provides a combination of a *term insurance* and a *pure endowment*.

(I) **Death benefit payable immediately on death**

- ★ The present value

$$
\begin{aligned}
Z &= \begin{cases} \nu^{T_x} = e^{-\delta T_x} & \text{if } T_x < n, \\ \nu^n & \text{if } T_x \geq n. \end{cases} \\
&= \nu^{\min(T_x, n)} \\
&= e^{-\delta \min(T_x, n)}.
\end{aligned}
$$

(I) **Death benefit payable immediately on death** (cont'n)

- The EPV of Z is

$$\begin{aligned} E(Z) &= \int_0^n e^{-\delta t}\,{}_tp_x\mu_{x+t}dt + \int_n^\infty e^{-\delta n}\,{}_tp_x\mu_{x+t}dt \\ &= \int_0^n e^{-\delta t}\,{}_tp_x\mu_{x+t}dt + e^{-\delta n}\,{}_np_x \\ &= \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}. \end{aligned}$$

- That is

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}. \tag{16}$$

- Similarly,

$${}^2\bar{A}_{x:\overline{n}|} = E(Z^2) = \int_0^n e^{-2\delta t}\,{}_tp_x\mu_{x+t}dt + e^{-2\delta n}\,{}_np_x.$$

(II) Death benefit payable at the end of the year of death

★ The present value

$$Z = \begin{cases} \nu^{K_x+1} & \text{if } K_x \leq n-1, \\ \nu^n & \text{if } K_x \geq n. \end{cases} = \nu^{\min(K_x+1,n)}.$$

★ The EPV of Z is

$$E(Z) = \sum_{k=0}^{n-1} \nu^{k+1} {}_{k|}q_x + \nu^n P(K_x \geq n) = A^1_{x:\overline{n}|} + \nu^n {}_np_x. \tag{17}$$

★ That is,

$$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}. \tag{18}$$

Similarly,

$${}^2A_{x:\overline{n}|} = E(Z^2) = \sum_{k=0}^{n-1} \nu^{2(k+1)} {}_{k|}q_x + \nu^{2n} {}_np_x.$$

(III) Death benefit payable at the end of the $1/m$th year of death

★ The present value

$$Z = \begin{cases} \nu^{K_x^{(m)}+\frac{1}{m}} & \text{if } K_x^{(m)} \leq n - \dfrac{1}{m}, \\ \nu^n & \text{if } K_x^{(m)} \geq n. \end{cases} = \nu^{\min(K_x^{(m)}+\frac{1}{m}, n)}.$$

★ The EPV of Z is

$$E(Z) = \sum_{k=0}^{mn-1} \nu^{\frac{k+1}{m}} {}_{\frac{k}{m}|\frac{1}{m}}q_x + \nu^n P(K_x^{(m)} \geq n) = A^{(m)}{}^{\,1}_{x:\overline{n}|} + \nu^n {}_np_x.$$

★ That is,

$$A^{(m)}_{x:\overline{n}|} = A^{(m)}{}^{\,1}_{x:\overline{n}|} + A_{x:\overset{\,1}{\overline{n}|}}. \tag{19}$$

Examples

Deferred insurance benefits

Deferred insurance refers to insurance which does not begin to offer death benefit cover until the end of a *deferred period*, say u.
Suppose a benefit of \$1 is payable **immediately** on the death of (x) provided that (x) dies between ages $x+u$ and $x+u+n$.

★ The present value

$$Z = \begin{cases} 0 & \text{if } T_x < u \text{ or } T_x > u+n, \\ e^{-\delta T_x} & \text{if } u \le T_x \le u+n. \end{cases}$$

★ The EPV of Z is

$$ {}_{u|}\bar{A}^{1}_{x:\overline{n}|} = E(Z) = \int_{u}^{u+n} e^{-\delta t} {}_{t}p_{x} \mu_{x+t} dt. \tag{20} $$

Let $s = t - u$,

$$ \begin{aligned} {}_{u|}\bar{A}^{1}_{x:\overline{n}|} &= \int_{0}^{n} e^{-\delta(s+u)} {}_{s+u}p_{x} \mu_{x+s+u} ds \\ &= e^{-\delta u} {}_{u}p_{x} \int_{0}^{n} e^{-\delta s} {}_{s}p_{x+u} \mu_{x+s+u} ds \\ &= e^{-\delta u} {}_{u}p_{x} \bar{A}^{\;1}_{x+u:\overline{n}|} \\ &= \nu^{n} {}_{u}p_{x} \bar{A}^{\;1}_{x+u:\overline{n}|} \\ &= {}_{u}E_{x} \bar{A}^{\;1}_{x+u:\overline{n}|}. \end{aligned} \tag{21} $$

★ Furthermore, by (20), because

$$\begin{aligned}&\int_u^{u+n} e^{-\delta t}\,{}_tp_x\mu_{x+t}dt\\ =\ &\int_0^{u+n} e^{-\delta t}\,{}_tp_x\mu_{x+t}dt - \int_0^u e^{-\delta t}\,{}_tp_x\mu_{x+t}dt,\end{aligned}$$

therefore

$$_{u|}\bar{A}^{1}_{x:\overline{n|}} = \bar{A}^{1}_{x:\overline{u+n|}} - \bar{A}^{1}_{x:\overline{u|}}. \tag{22}$$

That is, the EPV of a *deferred term insurance* benefit can be found by differencing the EPVs of *term insurance* benefits for terms $u+n$ and u.

★ In (21), ${}_uE_x = \nu^n{}_up_x$ acts similarly to a discount function.

★ An n-year term insurance can be decomposed as the sum of n deferred term insurance policies, each with a term of one year.

$$\begin{aligned}\bar{A}^{1}_{x:\overline{n}|} &= \int_0^n e^{-\delta t}\,{}_tp_x\mu_{x+t}dt \\ &= \sum_{r=0}^{n-1}\int_r^{r+1} e^{-\delta t}\,{}_tp_x\mu_{x+t}dt \\ &= \sum_{r=0}^{n-1} {}_{r|}\bar{A}^{1}_{x:\overline{1}|}. \qquad (23)\end{aligned}$$

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- Similarly, for a *whole life insurance policy*, we have

$$\bar{A}_x = \sum_{r=0}^{\infty} {}_{r|}\bar{A}^{1}_{x:\overline{1}|}.$$

- $$\because A_x = A^{1}_{x:\overline{n}|} + {}_{n|}A_x \quad (why?)$$

$$\begin{aligned} \therefore A^{1}_{x:\overline{n}|} &= A_x - {}_{n|}A_x \\ &= A_x - \nu^n {}_np_x A_{x+n} \end{aligned}$$

which can be used to calculate $A^{1}_{x:\overline{n}|}$ for $x,\ n \in Z$ given a table of values of A_x and l_x.

RELATING $\bar{A}_x$, A_x, AND $A_x^{(m)}$

- Even though the death benefits payable at the end of the year of death are very unusual, A_x are still useful.
- If the only information is a life table with *integer age* functions only, we can approximate $\bar{A}_x$ or $A_x^{(m)}$ from A_x.
- We approximate these relationship between the continuous function and discrete function using the following two assumptions:
  - **uniform distribution of death**, **UDD**, assumption;
  - **claims acceleration approach**.

Using the uniform distribution of deaths assumption

UDD is the most convenient fractional age assumption for the approximation. Under UDD, we have

$$q_y = {}_sp_y \mu_{y+s}; \quad 0 \leq s \leq 1 \text{ and } y \in Z. \quad (why?)$$

Therefore,

$$\begin{aligned}
\bar{A}_x &= \int_0^\infty e^{-\delta t} {}_tp_x \mu_{x+t} dt = \sum_{k=0}^{\infty} \int_k^{k+1} e^{-\delta t} {}_tp_x \mu_{x+t} dt \\
&= \sum_{k=0}^{\infty} {}_kp_x \nu^{k+1} \int_0^1 e^{(1-s)\delta} {}_sp_{x+k} \mu_{x+k+s} ds \quad (why?) \\
&= \sum_{k=0}^{\infty} {}_kp_x q_{x+k} \nu^{k+1} \int_0^1 e^{(1-s)\delta} ds \quad (why?) \\
&= A_x \frac{e^\delta - 1}{\delta}.
\end{aligned}$$

Because

$$e^{\delta} = 1 + i \ \ (why?)\ ,$$

under the UDD assumption, we have

$$\bar{A}_x = \frac{i}{\delta} A_x. \tag{24}$$

That is, the exact result under the UDD assumption gives rise to the approximation

$$\bar{A}_x \approx \frac{i}{\delta} A_x. \tag{25}$$

Similarly, under the UDD assumption, we have

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x; \quad (assignment) \tag{26}$$

for endowment insurance, under the UDD assumption,

$$\bar{A}_{x:\overline{n}|} \approx \frac{i}{\delta} A^{1}_{x:\overline{n}|} + \nu^{n}\,{}_{n}p_x. \quad (assignment) \tag{27}$$

Using the claims acceleration approach

The only difference between A_x, $A_x^{(m)}$, and $\bar{A}_x$ is the **timing of the payment**.

example

Consider A_x and $A_x^{(4)}$. If (x) dies in the year of age $x + K_x$ to $x + K_x + 1$.

Valued by A_x:	the sum insured is paid at $K_x + 1$.
Valued $A_x^{(4)}$:	the sum insured is paid at $K_x + \frac{1}{4}$, $K_x + \frac{2}{4}$, $K_x + \frac{3}{4}$, or $K_x + 1$.

If the death occur evenly over the year, then, on average, the benefit is paid at time $K_x + \frac{5}{8}$ (*why?*), which is $\frac{3}{8}$ years earlier than the end of year of death benefit.

In general, for an $1/m$thly death benefit, assuming deaths are uniformly distributed over the year of age,

$$\because \quad \frac{1}{m}+\frac{2}{m}+\cdots+\frac{m}{m}=\frac{m(m+1)}{m} \Rightarrow \frac{\left[\frac{m(m+1)}{m}\right]}{m}=\frac{(m+1)}{2m},$$

$\therefore$ the average time of payment of the death benefit is $\frac{(m+1)}{2m}$ in the year of death.

So we have

$$\begin{aligned} A_x^{(m)} &\approx q_x \nu^{\frac{m+1}{2m}} + {}_{1|}q_x \nu^{1+\frac{m+1}{2m}} + {}_{2|}q_x \nu^{2+\frac{m+1}{2m}} + \cdots \quad (why?) \\ &= \sum_{k=0}^{\infty} {}_{k|}q_x \nu^{k+\frac{m+1}{2m}} = (1+i)^{\frac{m-1}{2m}} \sum_{k=0}^{\infty} {}_{k|}q_x \nu^{k+1}. \end{aligned}$$

That is

$$A_x^{(m)} \approx (1+i)^{\frac{m-1}{2m}} A_x. \tag{28}$$

For continuous EPV, $\bar{A}_x$, we let $m \to \infty$ in (28), to give the approximation

$$\bar{A}_x^{(m)} \approx (1+i)^{\frac{1}{2}} A_x. \quad (why?) \tag{29}$$

★ This explains the fact that, if the benefit is paid immediately on death, and lives die uniformly through the year, then, on average, the benefit is paid half-way through the year of death, which is half s year earlier than the benefit valued by A_x.

Similarly, for an endowment insurance using the claims acceleration approach, we have

$$\bar{A}_{x:\overline{n}|} \approx (1+i)^{\frac{1}{2}} A^{1}_{x:\overline{n}|} + \nu^{n}\,{}_{n}p_{x}. \tag{30}$$

★ Note that both UDD and claims acceleration approach give values for $A_x^{(m)}$ or $\bar{A}_x$ such that $\frac{A_x^{(m)}}{A_x}$ and $\frac{\bar{A}_x}{A_x}$ are **independent** of x.

VARIABLE INSURANCE BENEFITS

For all insurance benefits studied so far, we know

$$
\begin{aligned}
& \text{EPV of the benefit} \\
= \; & \text{(the amount of benefit paid)} \\
& \text{(the appropriate discount factor for the payment date)} \\
& \text{(the probability that the benefit will be paid at that date).}
\end{aligned}
$$

- ★ This approach works for the EPV of any *traditional benefit*- that is, where the **lifetime is the sole source of uncertainty**.
- ★ It will **NOT** generate higher moments or probability distribution.

The approach can be justified using **indicator random variables**. Let $\boldsymbol{E}$ be the event that (x) dies in $(k, k+1]$. The *indicator random variables* is

$$I(\boldsymbol{E}) = \begin{cases} 1 & \text{if } \boldsymbol{E} \text{ is true,} \\ 0 & \text{if } \boldsymbol{E} \text{ is false.} \end{cases}$$

Hence, $P(\boldsymbol{E} \text{ is true}) = {}_{k|}q_x$ and

$$E[I(\boldsymbol{E})] = 1({}_{k|}q_x) + 0(1 - {}_{k|}q_x) = {}_{k|}q_x.$$

example

An insurance pays \$1,000 after 10 years if (x) has dead by that time, and \$2,000 after 20 years if (x) dies in the second 10-year period, with no benefit otherwise.

$$r.v.(\text{present value}) = 1000\, I(T_x \leq 10)\nu^{10} + 2000\, I(10 < T_x \leq 20)\nu^{20}.$$

$$EPV = 1000\,{}_{10}q_x \nu^{10} + 2000\,{}_{10|}q_{x10}\nu^{20}.$$

Indicator random variables can be used for continuous benefits. Consider

$$I(t < T_x \leq t + dt), \text{ for infinitesimal } dt,$$

$$\begin{aligned} E[I(t < T_x \leq t + dt)] &= P(t < T_x \leq t + dt) \\ &= P(T_x > t)P(T_x \leq t + dt | T_x > t) \\ &\approx {}_tp_x \mu_{x+t} dt. \end{aligned}$$

example

An increasing insurance policy with a death benefit of T_x payable at the moment of death. That is, the death benefit is exactly equal to the years lived by an insured life from age x to his/her death.

- ★ It is a continuous whole life insurance.
- ★ The death benefit is a linearly increasing function.

example (cont'n)

We note that

- the payment may be made $(t, t+dt)$, $0 \leq t < \infty$;
- the amount paid in $(t, t+dt)$ is t;
- the discount factor of the payment in $(t, t+dt)$ is $e^{-\delta t}$;
- the probability that the benefit is paid in $(t, t+dt)$ is approximate to ${}_tp_x \mu_{x+t} dt$.

Hence, the EPV of the benefit, denoted by $(\overline{I}\overline{A})_x$, is

$$(\overline{I}\overline{A})_x = \int_0^\infty t e^{-\delta t} {}_tp_x \mu_{x+t} dt. \tag{31}$$

★ The I in the actuarial notation here stands for "*increasing*".

example (cont'n)

An alternative approach for (31)is to identify the present value *random variable* for the benefit.
Let the the present value random variable for the benefit be

$$Z = T_x e^{-\delta T_x}.$$

Then any moment of Z can be found from

$$E(Z^k) = \int_0^\infty (te^{-\delta t})^k {}_tp_x \mu_{x+t} dt.$$

★ The EPV of the death benefit of a policy term which ceases after a fixed term of n years is

$$(\bar{I}\bar{A})^1_{x:\overline{n|}} = \int_0^n te^{-\delta t} {}_tp_x \mu_{x+t} dt.$$

Examples