ACTUARIAL MATHEMATICS
ANNUITIES

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SUMMARY

For **life contingent annuities**, we consider

- benefit valuation for different payment frequencies;
- relate the valuation of annuity benefits to the valuation of the insurance benefits.

For calculating the **annuity valuation functions**,

- if full survival model information is available, then the calculation can be exact for benefits payable at discrete time point or payable continuously;
- if uses an integer age life table, with benefits payable more frequently than annual (ex. monthly or weekly), then the *UDD assumption* is required to get the approximation.

INTROCTION

A **life annuity** is a series payments to (or from) an individual as long as the individual is **alive** on the payment date.

- The payments are normally made at regular interval.
- The payments are usually of the same amount.

The valuation of annuities appear in the calculation of

- premiums;
- policy values and
- pension benefits.

★ The present value of a life annuity is a random variable, as it depends on the future lifetime.

REVIEW OF ANNUITY-CERTAIN

Recall that, for $n \in Z$ and $i > 0$

- Annuity-due

$$\ddot{a}_{\overline{n}|} = 1 + \nu + \nu^2 + \cdots + \nu^{n-1} = \frac{1-\nu^n}{d}. \quad (1)$$

- Annuity-immediate

$$a_{\overline{n}|} = \nu + \nu^2 + \nu^3 + \cdots + \nu^n = \frac{1-\nu^n}{i} = \ddot{a}_{\overline{n}|} - 1 + \nu^n. \quad (why?)$$

- Annuity-certain payable continuously, $\forall n > 0$,

$$\bar{a}_{\overline{n}|} = \int_0^n \nu^t dt = \frac{1-\nu^n}{\delta}. \quad (2)$$

Recall that, for $n \in Z$ and $i > 0$ (conti'n)

- Annuity-due, payment made every $1/m$ year,

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m}\left(1 + \nu^{\frac{1}{m}} + \nu^{\frac{2}{m}} + \cdots + \nu^{n-\frac{1}{m}}\right) = \frac{1-\nu^n}{d^{(m)}}.$$

- Annuity-immediate, , payment made every $1/m$ year,

$$\begin{aligned} a_{\overline{n}|}^{(m)} &= \frac{1}{m}\left(\nu^{\frac{1}{m}} + \nu^{\frac{2}{m}} + \cdots + \nu^{n}\right) \\ &= \frac{1-\nu^n}{i^{(m)}} \\ &= \ddot{a}_{\overline{n}|}^{(m)} - \frac{1}{m}(1-\nu^n). \quad (why?) \end{aligned} \tag{3}$$

In these equations, we assume that n is an integer multiple of $1/m$.

ANNUAL LIFE ANNUITIES

The annual life annuity is paid once a year, conditional on the survival of a life, the **annuitant**, to the payment date.

Whole life annuity
If the annuity is to be paid throughout the annuitant's life.

Term or temporary annuity
If the annuity has a specified maximum term.

★ Annual annuity is quite rare. We would commonly see annuities payable monthly or weekly.

As with the insurance functions, we are interested in the EPV of a cashflow and the present value random variables in terms of the future lifetime random variables.

Whole life annuity-due

Consider an annuity of 1 per year payable annually **in advance** throughout the lifetime of (x).

That is, for $k \in Z^{+}$, if (x) were to die between ages $x+k$ and $x+k+1$, then the annuity payments would be made at times $0, 1, 2, \cdots, k$, for a total of $k+1$ payments.

Define K_x such that

- (x) dies between ages $x+K_x$ and $x+K_x+1$, so the number of payments is K_x+1;
- if $K_x = k$, the present value random variable, Y, is

$$Y = \ddot{a}_{\overline{K_x+1}|} = \frac{1-\nu^{K_x+1}}{d}.$$

There are three useful ways to derive the expected value of the present value random variable.

First, let $\ddot{a}_x = E(Y)$, that is,

$$\ddot{a}_x = E\left[\frac{1-\nu^{K_x+1}}{d}\right] = \frac{1-E(\nu^{K_x+1})}{d}.$$

By the previous chapter, we know

$$\ddot{a}_x = \frac{1-A_x}{d}. \quad (why?) \tag{4}$$

The variance of Y is

$$\begin{aligned} Var(Y) &= Var\left[\frac{1-\nu^{K_x+1}}{d}\right] = \frac{1}{d^2} Var[\nu^{K_x+1}] \\ &= \frac{{}^2A_x - A_x^2}{d^2}. \quad (why?) \end{aligned} \tag{5}$$

Secondly, we may use the indicator random variable approach to get the same results.

The condition for the payment at time k is that (x) is alive at $x+k$, that is, $T_x > k$. The present value r.v. can be expressed as

$$Y = I(T_x > 0) + \nu I(T_x > 1) + \nu^2 I(T_x > 2) + \nu^3 I(T_x > 3) + \cdots . \quad (6)$$

$$\because \; E[I(T_x > t)] = P(T_x > t) = {}_tp_x,$$

$$\therefore \; \ddot{a}_x = 1 + \nu p_x + \nu^2 {}_2p_x + \nu^3 {}_3p_x + \cdots .$$

That is,

$$\ddot{a}_x = \sum_{k=0}^{\infty} \nu^k {}_kp_x \quad (7)$$

★ It is a very useful equation for $\ddot{a}_x$. However, it does not lead to the expressions for the variance and higher moments of Y.

Finally, we work from the probability function for K_x. That is,

$$P(K_x = k) = {}_{k|}q_x,$$

so that

$$\ddot{a}_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_x. \quad (why?) \tag{8}$$

- ★ (8) is less useful than (4) and (7).
- ★ The difference between (7) and (8) is
  - (7) the summation taken over the possible payment dates;
  - (8) the summation taken over the possible year of death.

Examples

Term annuity-due

The annuity is payable 1 annually to (x) for maximum of n years.

- Payments are made at times $k = 0, 1, 2, \cdots, n-1$, provided that (x) has survived to age $x+k$.

The present value Y is

$$Y = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x = 0, 1, \cdots, n-1; \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n. \end{cases}$$

$$\text{i.e. } \; Y = \ddot{a}_{\overline{\min(K_x+1,n)}|} = \frac{1 - \nu^{\min(K_x+1,n)}}{d}.$$

★ The EPV of Y is denoted $\ddot{a}_{x:\overline{n}|}$.

Thus,

$$\ddot{a}_{x:\overline{n}|} = E(Y) = \frac{1 - E\left[\nu^{\min(K_x+1,n)}\right]}{d}.$$

That is,

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d}. \quad (why?) \tag{9}$$

By summing the EPV's of the individual payments, we have

$$\ddot{a}_{x:\overline{n}|} = 1 + \nu p_x + \nu^2{}_2p_x + \nu^3{}_3p_x + \cdots + \nu^{n-1}{}_{n-1}p_x$$

$$\text{i.e. } \ddot{a}_{x:\overline{n}|} = \sum_{t=0}^{n-1} \nu^t{}_tp_x. \tag{10}$$

Also, by (8), we can write the EPV as

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_x + {}_np_x\,\ddot{a}_{\overline{n}|}. \quad (why?)$$

Whole life immediate annuity

Consider an annuity of 1 per year payable **in arrear**, conditional on the survival of (x) to the payment date.

- The EPV is denoted as a_x.
- Let Y^* be the present value r.v. of the annuity. Then

$$Y^* = \nu\, I(T_x > 1) + \nu^2\, I(T_x > 2) + \nu^3\, I(T_x > 3) + \cdots .$$

- Let $Y^* = Y - 1$, where Y is the present value r.v. of the whole life annuity payable **in advance**,

$$\begin{aligned} \Rightarrow \quad & E(Y^*) = E(Y) - 1 \\ & a_x = \ddot{a}_x - 1. \quad (why?) \end{aligned} \tag{11}$$

- Similarly,

$$Var(Y^*) = Var(Y) = \frac{{}^2A_x - A_x^2}{d^2}.$$

Term immediate annuity

As in the previous section, we have the annuity payments of 1 are made at times $k = 1, 2, \cdots, n-1$, conditional on the survival of the annuitant;

- The EPV is denoted as $a_{x:\overline{n}|}$.
- Let Y be the present value r.v. of the annuity. Then

$$Y = a_{\overline{\min(K_x,n)}|}.$$

- Summing the EPV's of the individual payments, we have

$$a_{x:\overline{n}|} = \nu p_x + \nu^2 {}_2p_x + \nu^3 {}_3p_x + \cdots + \nu^n {}_np_x = \sum_{t=1}^{n} \nu^t {}_tp_x. \quad (12)$$

- By (10) and (12), we have $\ddot{a}_{x:\overline{n}|} - \bar{a}_{x:\overline{n}|} = 1 - \nu^n {}_np_x$

$$\Rightarrow \quad a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + \nu^n {}_np_x. \quad (13)$$

ANNUITIES PAYABLE CONTINUOUSLY

If the discrete time intervals of payments are close together, it is convenient to treat payments as being made continuously.

Whole life continuous annuity

Consider an annuity is payable continuously at rate of 1 per year as long as (x) survives.

- If the annuity is payable weekly, then each week, the annuity payment is $1/52$.
  If the payments were daily, the daily payment would be $1/365$.
- For an infinitesimal interval $(t, t + dt)$, the payment under the annuity is dt provided (x) is alive through the interval.
- The EPV is denoted $\bar{a}_x$.

Let the present value r.v. be Y. That is,

$$Y = \bar{a}_{\overline{T_x}|}.$$

As before, we derive the EPV of the annuity in three different ways.

I. By the annuity-certain formula:

$$\bar{a}_{\overline{n}|} = \frac{1-\nu^n}{\delta} \quad \Rightarrow \quad Y = \frac{1-\nu^{T_x}}{\delta} \text{ and } \bar{a}_x = E(Y) = \frac{1-E\left[\nu^{T_x}\right]}{\delta}.$$

That is,

$$\bar{a}_x = \frac{1-\bar{A}_x}{\delta}. \tag{14}$$

$$Var(Y) = Var\left[\frac{1-\nu^{T_x}}{\delta}\right] = \frac{{}^2\bar{A}_x - \bar{A}_x^2}{\delta^2}.$$

II. By the integral of the product of the paid amount, discount function and probability function: That is,

$$\bar{a}_x = \int_0^\infty e^{-\delta t}\, {}_tp_x dt \tag{15}$$

and the EPV can also be derived using indicator r.v. as

$$Y = \int_0^\infty e^{-\delta t}\ I(T_x > t)dt \ \ (why?)$$

where

- the amount paid in each infinitesimal interval $(t, t + dt)$ is dt;
- the discount factor is $e^{-\delta t}$; and
- the probability of payment is ${}_tp_x$.

III. By the distribution of T_x: That is,

$$\bar{a}_x = \int_0^\infty \bar{a}_{\overline{t}|}\ {}_tp_x \mu_{x+t} dt.$$

$$\because \frac{d}{dt}\bar{a}_{\overline{t}|} = \nu^t = e^{-\delta t}. \quad (why?)$$

Then, by integration by parts,

$$\begin{aligned} \bar{a}_x &= \int_0^\infty \bar{a}_{\overline{t}|} \frac{d}{dt}(-{}_tp_x) dt. \\ &= -\left(\bar{a}_{\overline{t}|}\ {}_tp_x \Big|_0^\infty - \int_0^\infty {}_tp_x e^{-\delta t} dt \right) \\ &= \int_0^\infty e^{-\delta t} {}_tp_x dt. \end{aligned}$$

★ When $\delta = 0$, we see $\bar{a}_x = \mathring{e}_x$, the complete expectation of life.

Term continuous annuity

For term continuous annuity, we have

- the present value r.v. is

$$\bar{a}_{\overline{\min(T_x,n)}|} = \frac{1-\nu^{\min(T_x,n)}}{\delta};$$

- the EPV is denoted by $\bar{a}_{x:\overline{n}|}$.

We have three expressions for this EPV;

1. by the results for endowment insurance functions

$$\bar{a}_{x:\overline{n}|} = \frac{1-\bar{A}_{x:\overline{n}|}}{\delta}. \tag{16}$$

2. by the indicator r.v.
   - ★ integrate over the possible payment dates.

$$\bar{a}_{x:\overline{n}|} = \int_0^n e^{-\delta t}\, {}_tp_x\, dt. \tag{17}$$

3. by the expected value of the present value r.v.
   - ★ integrate over the possible dates of death.

$$\bar{a}_{x:\overline{n}|} = \int_0^n \bar{a}_{\overline{t}|}\, {}_tp_x \mu_{x+t} dt + \bar{a}_{\overline{n}|}\, {}_np_x.$$

ANNUITIES PAYABLE 1/m**thly**

introduction

Recall

- $K_x^{(m)}$:
  the complete future lifetime rounded down to the lower $1/m$th year.
- $\ddot{a}_{\overline{n}|}^{(m)}$:
  the present value of an annuity of 1 per year, payable each year, in m installments of $1/m$ for n years.
  - ★ The first payment is paid at time $t = 0$ and the final payment at $t = n - \frac{1}{m}$.
  - ★ $\ddot{a}_{\overline{n}|}^{(m)}$ is an **annual** factor; that is, be aware of multiplying **annual rate of annuity payment** when necessary.

example

Suppose we have an annuity of \$12,000 per year, payable monthly in advance to a life aged 60. Each monthly payment is \$1,000. The relevant future lifetime r.v. is $K_{60}^{(12)}$.

- If $K_{60}^{(12)} = 0$, then (60) dies in the first month. A single payment \$1,000 was made at $t = 0$ and the present value is

$$12,000 \times \frac{1}{12} = 12,000 \ddot{a}^{(12)}_{\overline{1/12}|} \Rightarrow P.V. = \underset{\text{(annual rate)}}{12,000} \ddot{a}^{(12)}_{\overline{1/12}|}.$$

- In general, the present value is

$$\underset{\text{(annual rate)}}{12,000} \ddot{a}^{(12)}_{\overline{K_{60}^{(12)} + \frac{1}{12}}|}.$$

WHOLE LIFE ANNUITIES PAYABLE 1/mthly

Consider an annuity of total amount of 1 per year, payable in advance m times per year **throughout** the lifetime of (x).

★ Each payment is $\frac{1}{m}$.

- The present value r.v. for the annuity is

$$\ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|} = \frac{1-\nu^{K_x^{(m)}+\frac{1}{m}}}{d^{(m)}}.$$

- The EPV of the annuity is denoted by $\ddot{a}_x^{(m)}$ and is given by

$$\ddot{a}_x^{(m)} = \frac{1-E\left[\nu^{K_x^{(m)}+\frac{1}{m}}\right]}{d^{(m)}}.$$

★ Hence, we have

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}. \tag{18}$$

- Using the indicator r.v. approach, we have

$$\ddot{a}_x^{(m)} = \sum_{r=0}^{\infty} \frac{1}{m} \nu^{r/m} \,_{\frac{r}{m}}p_x. \quad (why?) \tag{19}$$

- Similarly, for annuities payable $1/m$thly in **arrear**, the EPV of the $1/m$thly immediate annuity is

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m}. \quad (why?) \tag{20}$$

TERM ANNUITIES PAYABLE 1/mthly

Consider an annuity of total amount of 1 per year, payable in advance m times per year throughout the lifetime of (x) for a **maximum of n years** .

★ Each payment is $\frac{1}{m}$.

- The present value r.v. for the annuity is

$$\ddot{a}^{(m)}_{\overline{\min\left(K_x^{(m)}+\frac{1}{m},n\right)}|} = \frac{1-\nu^{\min\left(K_x^{(m)}+\frac{1}{m},n\right)}}{d^{(m)}}.$$

- The EPV of the annuity is denoted by $\ddot{a}^{(m)}_{x:\overline{n}|}$ and is given by

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1-E\left[\nu^{\min\left(K_x^{(m)}+\frac{1}{m},n\right)}\right]}{d^{(m)}}.$$

★ Hence, we have

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}}. \tag{21}$$

- Using the indicator r.v. approach, we have

$$\ddot{a}^{(m)}_{x} = \sum_{r=0}^{mn-1} \frac{1}{m} \nu^{r/m} \ {}_{\frac{r}{m}}p_x. \tag{22}$$

- Similarly, for the $1/m$thly term immediate annuity, the EPV is

$$a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{m}(1 - \nu^n \ {}_np_x). \quad (why?) \tag{23}$$

- Furthermore,

$$\begin{aligned} &\text{in (19) and (22), let } m = 1 && \Rightarrow && \text{(7) and (10);} \\ &\text{in (19) and (22), let } m = \infty && \Rightarrow && \text{(15) and (17).} \end{aligned}$$

COMPARISON OF ANNUITIES BY PAYMENT FREQUENCY

example

TABLE: Values of a_x, $a_x^{(4)}$, $\bar{a}_x$, $\ddot{a}_x^{(4)}$, $\ddot{a}_x$

x	a_x	$a_x^{(4)}$	$\bar{a}_x$	$\ddot{a}_x^{(4)}$	$\ddot{a}_x$
20	18.996	19.338	19.462	19.588	19.966
40	17.458	17.829	17.945	18.079	18.458
60	13.904	14.275	14.400	14.525	14.904
80	7.548	7.917	8.042	8.167	8.548

The table values are calculated using the Standard Ultimate Survival Model. We observe that

- each set of values decreasing with age;
- for each age, the ordering

$$a_x < a_x^{(4)} < \bar{a}_x < \ddot{a}_x^{(4)} < \ddot{a}_x.$$

example (conti'n)

There are two reasons for this ordering.

- On average, the payments under the annuity-due are paid earlier. It means that the value of an annuity with earlier payments will be higher than an annuity with later payments (for interest rates greater than zero).
  - ★ The annuity values are in increasing order from the latest average payment date, a_x, to the earliest, $\ddot{a}_x$.
- The different annuities pay different amounts in the year that (x) dies.
  - Under the annual annuity-due, the full year's payment of \$1 is paid.
  - Under the annual annuity-immediate, no payment is made. (Why?)
  - For the $1/m$thly and continuous annuities, less than the full year's annuity may be paid in the year of death.

★ This explains why we can **NOT** make a simple interest adjustment to relate the annuity-due and continuous annuity.

example (conti'n)

- The situation here is different from the *annuity benefits*, A_x and $A_x^{(4)}$, for example.
  - Both value a payment of \$1 in the year of death.
  - A_x pays at the end of the year, and $A_x^{(4)}$ pays at the end of the quarter year of death.
  - ★ There is **no different** in the amount of the payment, only in the timing.
- For annuities, the difference between $\ddot{a}_x$ and $\ddot{a}_x^{(4)}$ arises from differences in both **cashflow timing** and **benefit amount** in the year of death.

Example

DIFERRED ANNUITIES

A deferred annuity is an annuity under which the first payment occurs at some specified future time. Consider

- an individual now aged x;
- an annuity with annual payments of 1 will commence at age $x+u$, $u \in Z$;
- the annuity will continue until the death of (x).

This is an annuity-due deferred u years.

- The EPV of this annuity is denoted by ${}_{u|}\ddot{a}_x$.

- Combining with a u-year term annuity, we have

$$\ddot{a}_x = {}_{u|}\ddot{a}_x + \ddot{a}_{x:\overline{u}|}. \tag{24}$$

or, equivalently,

$$ {}_{u|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{u}|}. \tag{25}$$

- Similarly, the EPV of an annuity payable continuously at rate 1 per year to a life x, commercing at age $x+u$, is denoted by ${}_{u|}\ddot{a}_x$ and given by

$$ {}_{u|}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{u}|}.$$

- Summing the EPV's of the individual payments for the deferred whole life annuity-due gives

$$\begin{aligned} \ddot{a}_x &= \nu^u \, {}_u p_x + \nu^{u+1} \, {}_{u+1} p_x + \nu^{u+2} \, {}_{u+2} p_x + \cdots \\ &= \nu^u \, {}_u p_x \left(1 + \nu \; p_{x+u} + \nu^2 \, {}_2 p_{x+u} + \cdots \right) \end{aligned}$$

so that

$$ {}_{u|}\ddot{a}_x = \nu^u \, {}_u p_x \, \ddot{a}_{x+u} = {}_u E_x \; \ddot{a}_{x+u}. \tag{26}$$

  - *Recall*
    Pure endowment function: ${}_u E_x = \nu^u \, {}_u p_x$.

★ The pure endowment function acts like a discount function.

- In fact, we can use ${}_uE_x$ to find the EPV of any deferred annuity.

  For a deferred term immediate annuity,

$$ {}_{u|}a_{x:\overline{n}|} = {}_uE_x \; a_{x+u:\overline{n}|}. $$

  For an annuity-due payable $1/m$thly,

$$ {}_{u|}\ddot{a}_x^{(m)} = {}_uE_x \; \ddot{a}_{x+u}^{(m)}. \tag{27} $$

- Suppose we have a table with $\ddot{a}_x$ and l_x, and we need $\ddot{a}_{x:\overline{n}|}$. By (24) and (26), we have

$$ \ddot{a}_{x:\overline{n}|} = \ddot{a}_x - \nu^u \; {}_np_x \ddot{a}_{x+n}. \tag{28} $$

  For $1/m$thly payments, we have

$$ \ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - \nu^u \; {}_np_x \ddot{a}_{x+n}^{(m)}. \tag{29} $$

Example

GUARANTEED ANNUITIES

A common feature of pension benefits is that the pension annuity is guarantees to be paid for some period even if the life dies before the end of the period. Suppose

- an annuity-due of 1 per year is paid annually to (x);
- the annuity is guaranteed for a period of n years.

Then

- the payment due at k years is paid whether or not (x) is then alive if $k = 0, 1, 2, \cdots, n-1$;
- but it is paid only if (x) is alive at age $x+k$ for $k = n, n+1, \cdots$.

The present value r.v. for the benefit is

$$Y = \begin{cases} \ddot{a}_{\overline{n}|} & \text{if } K_x \leq n-1; \\ \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \geq n. \end{cases} = \begin{cases} \ddot{a}_{\overline{n}|} & \text{if } K_x \leq n-1; \\ \ddot{a}_{\overline{n}|} + \ddot{a}_{\overline{K_x+1}|} - \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n. \end{cases}$$

$$= \ddot{a}_{\overline{n}|} + \begin{cases} 0 & \text{if } K_x \leq n-1; \\ \ddot{a}_{\overline{K_x+1}|} - \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n. \end{cases} = \ddot{a}_{\overline{n}|} + Y_1,$$

where Y_1 denotes the present value of an n-year deferred annuity-due of 1 per year. Hence

$$E[Y_1] = {}_{n|}\ddot{a}_x = {}_nE_x\ \ddot{a}_{x+n}. \ \ (Why?)$$

The EPV of the annuity is denoted by $\ddot{a}_{x:\overline{n}|}$, therefore

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_{\overline{n}|} + {}_nE_x\ \ddot{a}_{x+n}. \tag{30}$$

Similarly,

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{\overline{n}|} + {}_nE_x\ \ddot{a}^{(m)}_{x+n}.$$

INCREASING ANNUITIES

For **non-level** annuities, we are generally interested in determining the EPV, and are rarely concerned with higher moments.

- To calculate higher moment it is generally necessary to use computers.

★ The best approach for calculating EPV is to use

- the indicator random variable, or
- time-line approach.

ARITHMETICALLY INCREASING ANNUITIES

I. **Whole Life Arithmetically Increasing Annuity**; $(I\ddot{a})x$
- The amount of the annuity is $t+1$ at $t = 0, 1, 2, \cdots$ provided that (x) is alive at time t.
- The EPV of the annuity is denoted by $(I\ddot{a})x$.

We have

$$(I\ddot{a})x = \sum_{t=0}^{\infty} \nu^t (t+1)\ {}_tp_x. \tag{31}$$

II. **Term Arithmetically Increasing Annuity**; $(I\ddot{a})x : \overline{n}|$
- The annuity is payable for a maximum of n payments rather than the whole life of (x).
- The EPV of the annuity is denoted by $(I\ddot{a})x : \overline{n}|$.

We have

$$(I\ddot{a})x : \overline{n}| = \sum_{t=0}^{n-1} \nu^t (t+1)\ {}_tp_x. \tag{32}$$

III. **Continuous Temporary Arithmetically Increasing Annuity**; $(I\bar{a})x : \overline{n}|$

- The annuity is payable continuously with the payments increasing by 1 at each year end.

★ The rate of payment in the tth year is **constant and equal** to t, for $t = 0, 1, 2, \cdots, n$.

- The EPV of the annuity is denoted by $(I\bar{a})x : \overline{n}|$.

★ We may consider the annuity as a sum of **one-year deferred annuities**. Hence, we have

$$(I\bar{a})x : \overline{n}| = \sum_{m=0}^{n-1} (m+1)_{m|}\bar{a}_{x:\overline{1}|}. \ \ (Why?)$$

IV. Arithmetically Increasing Annuity with Rate Changing Continuously

★ The rate of payment at $t > 0$ is t.

- The EPV of the whole life annuity is denoted by $(\bar{I}\bar{a})_x$.
- The EPV of the term annuity is denoted by $(\bar{I}\bar{a})_{x:\overline{n}|}$.

For every infinitesimal interval, $(t, t + dt)$,

- the amount of annuity paid, if (x) is still alive, is tdt;
- the probability of payment is ${}_tp_x$;
- the discount function is $e^{-\delta t} = \nu^t$;

so that

$$(\bar{I}\bar{a})_x = \int_0^n te^{-\delta t}\ {}_tp_x dt. \tag{33}$$

GEOMETRICALLY INCREASING ANNUITIES

Consider an annuity-due with annual payments where

- the amount of the annuity is $(1+j)^t$ at times $t = 0, 1, 2, \cdots, n-1$

provided (x) is alive at t.

★ The EPV of a term annuity-due evaluated at interest rate i^* is denoted by $\ddot{a}_{x:\overline{n}|i^*}$, where

$$i^* = \frac{1+i}{1+j} - 1 = \frac{i-j}{1+j};$$

we have

$$\ddot{a}_{x:\overline{n}|i^*} = \sum_{t=0}^{n-1} (1+j)^t \nu^t \, {}_tp_x. \quad (Why?)$$

EVALUATING ANNUITY FUNCTIONS

If we have full information about the survival function for a life, then we can use summation or numerical integration to compute the EPV of any annuity. We, however, often have only *integer age* information.

- ★ We consider how to evaluate the EPV of $1/m$thly and continuous annuities given only the EPV of annuities at integer ages.

RECURSIONS

We may calculate $\ddot{a}_x$ using a **backward recursion**. We assume

- ${}_tp_x$: available;
- $\exists\, \omega \in Z$, called **limiting age**, so that

$$q_{\omega-1} = 1.$$

Because

$$\begin{aligned}
\ddot{a}_x &= 1 + \nu p_x + \nu^2 {}_2p_x + \nu^3 {}_3p_x + \cdots \\
&= 1 + \nu p_x \left(1 + \nu p_{x+1} + \nu^2 {}_2p_{x+1} + \nu^3 {}_3p_{x+1} + \cdots\right) \\
&= 1 + \nu p_x \ddot{a}_{x+1}.
\end{aligned}$$

Set $\ddot{a}_{\omega-1} = 1$, we have the **backward recursion**

$$\ddot{a}_x = 1 + \nu p_x \ddot{a}_{x+1};\ \text{for } x = \omega - 2, \omega - 3, \cdots. \tag{34}$$

Similarly, for the $1/m$thly annuity-due,

$$\ddot{a}^{(m)}_{\omega-\frac{1}{m}} = \frac{1}{m}, \quad (Why?)$$

the backward recursion for $x = \omega - \frac{2}{m}, \omega - \frac{3}{m}, \cdots$ is

$$\ddot{a}^{(m)}_x = \frac{1}{m} + \nu^{\frac{1}{m}}\, {}_{\frac{1}{m}}p_x\, \ddot{a}^{(m)}_{x+\frac{1}{m}}. \tag{35}$$

- We can calculate EPVs for term annuities and deferred annuities from the whole life annuity EPVs, using (24) and (26).
- We can use numerical integration to find the EPV of an annuity payable continuously.

APPLYING THE UDD ASSUMPTION

We consider the evaluation of $\ddot{a}^{(m)}_{x:\overline{n}|}$ under the assumption of UDD.

★ UDD offers a reasonable approximation at young ages, but may not be sufficiently accurate at older ages.

Under UDD assumption,

$$A^{(m)}_x = \frac{i}{i^{(m)}} A_x \text{ and } \bar{A}^{(m)}_x = \frac{i}{\delta} A_x. \quad (Why?)$$

From (4), (14), and (18), we also have

$$\ddot{a}_x = \frac{1 - A_x}{d}, \ \ddot{a}^{(m)}_x = \frac{1 - A^{(m)}_x}{d^{(m)}}, \text{ and } \bar{a}_x = \frac{1 - \bar{A}_x}{\delta}.$$

Thus, we have

$$\begin{aligned}
\ddot{a}_x^{(m)} &= \frac{1 - A_x^{(m)}}{d^{(m)}} \\
&= \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}} \quad \text{(by UDD)} \\
&= \frac{i^{(m)} - i A_x}{i^{(m)} d^{(m)}} \\
&= \frac{i^{(m)} - i(1 - d\ddot{a}_x)}{i^{(m)} d^{(m)}} \quad \text{(by (4))} \\
&= \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_x - \frac{i - i^{(m)}}{i^{(m)} d^{(m)}} \\
&= \alpha(m) \ddot{a}_x - \beta(m);
\end{aligned}$$

where $\alpha(m) = \dfrac{id}{i^{(m)} d^{(m)}}$ and $\beta(m) = \dfrac{i - i^{(m)}}{i^{(m)} d^{(m)}}$.

For continuous annuities, let $m \to \infty$, so that

$$\bar{a}_x = \frac{id}{\delta^2}\ddot{a}_x - \frac{i-\delta}{\delta^2}. \quad (Why?)$$

By (29), we have

$$\begin{aligned}
\ddot{a}^{(m)}_{x:\overline{n}|} &= \ddot{a}^{(m)}_x - \nu^n{}_np_x\ddot{a}^{(m)}_{x+n} \\
&= \alpha(m)\ddot{a}_x - \beta(m) - \nu^n{}_np_x\left[\alpha(m)\ddot{a}_{x+n} - \beta(m)\right] \quad \text{(by UDD)} \\
&= \alpha(m)(\ddot{a}_x - \nu^n{}_np_x\ \ddot{a}_{x+n}) - \beta(m)(1 - \nu^n{}_np_x) \\
&= \alpha(m)\ddot{a}_{x:\overline{n}|} - \beta(m)(1 - \nu^n{}_np_x).
\end{aligned}$$

★ $\alpha(m)$ and $\beta(m)$ depend only on the frequency of the payments, not on the underlying survival model.

★ It can be showed that $\alpha(m) \approx 1$ and $\beta(m) \approx (m-1)/2$, leading to

$$\ddot{a}^{(m)}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - \nu^n{}_np_x). \quad (36)$$