Actuarial Mathematics
SURVIVAL MODELS
Examples

Sheu, Ru-Shuo Ph.D

Department of Applied Mathematics
Chinese Culture University

2021

Example 1-1

$$F_0(t) = \begin{cases} 1 - \left(1 - \frac{t}{120}\right)^{\frac{1}{6}} & \text{for } 0 \leq t \leq 120 \\ 1 & \text{for } t > 120. \end{cases}$$

(A) A newborn life survives beyond 30.

(B) A life aged 30 dies before 50.

(C) A life aged 40 survives beyond 65.

Solution 1-1

(A)

$$S_0(30) = 1 - F_0(50) = 0.9532.$$

(B)

$$F_{30}(20) = \frac{F_0(50) - F_0(30)}{1 - F_0(30)} = 0.0410.$$

(C)

$$S_{40}(25) = \frac{S_0(65)}{S_0(40)} = 0.9395.$$

Example 1-2

As **Example 1-1**, derive μ_x.

Solution 1-2

$$\because S_0(x) = \left(1 - \frac{t}{120}\right)^{\frac{1}{6}},$$

it follows that

$$\frac{d}{dx}S_0(x) = \frac{1}{6}\left(1 - \frac{t}{120}\right)^{-\frac{5}{6}}\left(-\frac{1}{120}\right),$$

$$\therefore \mu_x = \frac{-1}{S_0(x)}\frac{d}{dx}S_0(x) = \frac{1}{720 - 6x}.$$

As an alternative,

$$\mu_x = -\frac{d}{dx}\ln S_0(x) = \frac{1}{720 - 6x}.$$

Example 1-3

Let

$$\mu_x = Bc^x \ ; \quad x > 0,\ 0 < B < 1 \text{ and } c > 1: \ \text{constants.}$$

This model is called **Gompertz's law of mortality**. Derive $S_x(t)$.

Solution 1-3

$$\because S_x(t) = \exp\left[-\int_x^{x+t} Bc^r\,dr\right],$$

thus

$$\begin{aligned}
\int_x^{x+t} Bc^r\,dr &= B\int_x^{x+t} \exp[r\ln c]\,dr \\
&= \frac{B}{\ln c}\exp[r\ln c]\Big|_x^{x+t} \\
&= \frac{B}{\ln c}(c^{x+t} - c^x)
\end{aligned}$$

$$\therefore S_x(t) = \exp\left[-\frac{B}{\ln c}c^x(c^t - 1)\right].$$

Example 1-4 Let $B = 0.0003$ and $c = 1.06$, for $x = 20$, $x = 50$, $x = 80$.

Example 1-5

As **Example 1-1** and **1-2**, let

$$F_0(x) = 1 - \left(1 - \frac{x}{120}\right)^{\frac{1}{6}}; \; 0 \leq x \leq 120.$$

Calculate q_x and $\mu_{x+\frac{1}{2}}$ for $x = 20$ and $x = 110$.

Solution 1-5

We know

$$p_x = \frac{S_0(x+1)}{S_0(x)} = \left(1 - \frac{1}{120-x}\right)^{\frac{1}{6}},$$

giving

$$q_{20} = 0.00167 \text{ and } q_{110} = 0.01741$$

therefore,

$$\mu_{20\frac{1}{2}} = 0.00168 \text{ and } \mu_{110\frac{1}{2}} = 0.01754.$$

Example 1-6

As **Example 1-1**, let

$$F_0(x) = 1 - \left(1 - \frac{x}{120}\right)^{\frac{1}{6}};\ 0 \le x \le 120.$$

Calculate $\mathring{e}_x$ and $V(T_x)$ for (A) $x = 30$ and (B) $x = 80$.

Solution 1-6

(A)

$$\because\ {}_tp_x = \frac{S_0(x+t)}{S_0(x)} = \left(1 - \frac{t}{120-x}\right)^{\frac{1}{6}}$$

therefore

$${}_tp_x = \begin{cases} \left(1 - \frac{t}{120-x}\right)^{\frac{1}{6}} & \text{for } x+t \le 120 \\ 0 & \text{for } x+t > 120. \end{cases}$$

$$\therefore\ \mathring{e}_x = \int_0^{120-x} \left(1 - \frac{t}{120-x}\right)^{\frac{1}{6}} dt = \frac{6}{7}(120-x)$$

$$\Rightarrow\ \mathring{e}_{30} = 77.143 \text{ and } \mathring{e}_{80} = 34.286.$$

Example 1-6 (conti'n)

Solution 1-6

(B)

$$
\begin{aligned}
\because\ E(T_x^2) &= 2\int_0^{120-x} t\,{}_tp_x\,dt \\
&= 2\int_0^{120-x} t\left(1-\frac{t}{120-x}\right)^{\frac{1}{6}} dt \\
&= 2(120-x)^2\left(\frac{6}{7}-\frac{6}{13}\right)
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ V(T_x) &= E(T_x^2)-(\mathring{e}_x)^2 = (120-x)^2\left[2\left(\frac{6}{7}-\frac{6}{13}\right)-\left(\frac{6}{7}\right)^2\right] \\
&= (120-x)^2(0.056515)
\end{aligned}
$$

$$\Rightarrow V(T_{30}) = 457.789 \text{ and } V(T_{80}) = 90.421.$$