ACTUARIAL MATHEMATICS
SURVIVAL MODELS

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2021

BACKGROUND

- **Insurance Policy**
  a contract between the insurer and the policyholder, which determines the claims which the insurer is legally required to pay.
- **Insurer**
  a financial institution that sells insurance.
- **Insured**
  the person, group of people, or organization that is insured in a particular agreement.
- **Beneficiary**
  a person or group who receives money, advantages, etc. as a result of something else.

BACKGROUND (cont'n)

- **Policyholder**
  The policyholder is a person or entity who owns or controls an insurance policy and has the privilege to exercise the rights outlined in the contract.
  - ★ This party is often, but not always, the insured and may or may not be one of the policy's beneficiaries.
- **Insurance Premium**
  the amount of money an individual or business pays for an insurance policy.
  - ★ Policy holder is usually the party who pays the premium.
- **Insurance Interest**
  Insurable interest is a type of investment that protects anything subject to a financial loss.

BACKGROUND (cont'n)

- **Underwriting**
  Underwriting (UW) services are provided by some large financial institutions, such as banks, insurance companies and investment houses, whereby they guarantee payment in case of damage or financial loss and accept the financial risk for liability arising from such guarantee.
  - **Preferred lives**: low mortality risk based on standard information.
  - **Normal lives**: may have some higher rated risk than *Preferred lives*.
  - **Rated lives**: have one or more risk factors at raised levels and are **NOT** acceptable at standard premium rates. However, they can be insured for a higher premium.
  - **Uninsurable lives**: have significant risk that the insurer will not enter an insurance policy at any price.

BACKGROUND (cont'n)

- **Pension Benefits**
  The *pension plan* is usually sponsored by an employer and offers employees (or **pension plan holders**) either lump sums or annuity benefits or both on retirement, or deferred benefits on earlier withdrawal.
  - **Defined Benefit (DB)**
    DB is based on service and salary with an employee, using a defined formula to determine the pension. The BD funded by contributions paid by the **employer** and (usually) the **employee** over the working lifetime of the employee.
  - **Defined Contribution (DC)**
    DC works more like a bank account. The employer and employee pay a predetermined contribution (usually a fixed percentage of salary) into a fund, and the fund earns interest. When the employee retires or leaves, the proceeds are available to provide income throughout retirement.

SUMMARY

In this chapter we

- represent the future lifetime of an individual as a random variable;
- show how probabilities of death or survival can be calculated under this framework.

Furthermore, we

- define *force of mortality*;
- introduce some actuarial notations;
- discuss properties of the distribution of future lifetime;
- introduce *curtate future lifetime random variable* and explain why it is useful.

THE FUTURE LIFETIME RANDOM VARIABLE

Let

- (x): a life aged x; $x \geq 0$.
- T_x: a continuous r.v. represents the future lifetime of (x);
  - $x + T_x$: age-at-death r.v. for (x).
- F_x: the distribution function of T_x. That is,

$$F_x(t) = P(T_x \leq t)$$

  and is called **lifetime distribution** from age x.
- S_x: the probability of survival. Defined the probability that (x) survives for at least t years as

$$S_x(t) = 1 - F_x(t) = P(T_x > t)$$

  and is called **survival function**.

* If $S_0(t) = 0$ then t is called a **limiting age** which is denoted as the Greek letter ω.

To interpret the collection of $\{T_x\}_{x\geq 0}$, $\forall x$, we need a connection between any pair of T_x and x.
Let T_0 be the future lifetime at birth for (x) then

$$T_x \leq t \equiv T_0 \leq x + t.$$

Hence,

$$\bigstar \quad P(T_x \leq t) = P(T_0 \leq x + t | T_0 > x). \tag{1}$$

That is,

$$P(T_x \leq t) = \frac{P(x < T_0 \leq x + t)}{P(T_0 > x)} \Rightarrow F_x(t) = \frac{F_0(x+t) - F_0(x)}{S_0(x)}$$

We have

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} \quad (Why?) \tag{2}$$

$$\bigstar \quad S_0(x+t) = S_0(x)S_x(t). \tag{3}$$

SURVIVAL MODELS

Similarly,

$$P(T_x > t) = P(T_0 > x + t | T_0 > x).$$

Furthermore, for any survival probability for (x), we have

$$S_x(t+u) = \frac{S_0(x+t+u)}{S_0(x)}$$

$$\Rightarrow \quad S_x(t+u) = \frac{S_0(x+t)}{S_0(x)} \frac{S_0(x+t+u)}{S_0(x+t)}$$

$$\Rightarrow \quad S_x(t+u) = S_x(t) S_{x+t}(u). \tag{4}$$

- ★ Equation (3): if we know survival probability from birth then we know survival probability from any future age x.
- ★ Equation (4): if we know survival probability from any age $x \geq 0$ then we know survival probability from any future age $x + t$ $(\geq x)$.

Conditions of survival function for a lifetime distribution

1. $S_x(0) = 1$.
2. $\lim\limits_{t\to\infty} S_x(t) = 0$.
3. $S_x(t)$ is a non-increasing function of t.
   - ★ These conditions are both necessary and sufficient, any function satisfies these three conditions defines a survival function for a lifetime distribution.

Additional assumptions

1. $S_x(t)$ is differentiable for all $t > 0$, so

$$\frac{d}{dt} S_x(t) \le 0, \ \forall t > 0.$$

2. $\lim\limits_{t\to\infty} tS_x(t) = 0$ and $\lim\limits_{t\to\infty} t^2 S_x(t) = 0$.
   - ★ The last assumption ensures that the mean and variance of the distribution function of T_x exist.

Example

THE FORCE OF MORTALITY

Definition

The **Force of Mortality** at age x, μ_x, is defined as

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} P(T_0 \leq x + dx | T_0 > x). \tag{5}$$

From (1), we see the the following equivalent way of defining μ_x;

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} P(T_x \leq dx) = \lim_{dx \to 0^+} \frac{F_x(dx)}{dx}.$$

or

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} [1 - S_x(dx)]. \tag{6}$$

★ μ_x depends, numerically, on the unit of time; usually measured per *year*.

Equation (5) gives the approximation

$$\mu_x dx \approx P(T_0 \le x + dx | T_0 > x). \tag{7}$$

★ For $dx \to 0$, $\mu_x dx$ can be interpreted as

$$Prob(\text{a life has attained age} x \text{ dies before attaining age } x + dx).$$

Force of mortality, μ_x, & survival function from birth, S_0

$$\because S_x(dx) = \frac{S_0(x+dx)}{S_0(x)} \text{ and by (6)}$$

$$\begin{aligned} \Rightarrow \quad \mu_x &= \frac{1}{S_0(x)} \lim_{dx \to 0^+} \frac{S_0(x) - S_0(x+dx)}{dx} = \frac{1}{S_0(x)} \left[-\frac{d}{dx} S_0(x) \right]. \\ \therefore \mu_x &= \frac{-1}{S_0(x)} \frac{d}{dx} S_0(x). \end{aligned} \tag{8}$$

Let

$$f_x(t) = \frac{d}{dt}F_x(t) = -\frac{d}{dt}S_x(t);$$

that is, $f_x(t)$ is the density function for T_x. Then, by (8), we have

$$\mu_x = \frac{f_0(x)}{S_0(x)}.$$

Force of mortality at any age $x+t,\ t>0$

$$\begin{aligned} \text{For } x \text{ fixed,} \quad \mu_{x+t} &= -\frac{1}{S_0(x+t)}\frac{d}{dt}S_0(x+t) \\ &= -\frac{1}{S_x(t)}\frac{d}{dt}S_x(t) \quad (Why?) \end{aligned}$$

$$\text{hence} \qquad \mu_{x+t} = \frac{f_x(t)}{S_x(t)}. \tag{9}$$

$S_x(t)$ in terms of the force of mortality

$$\because \quad \mu_x = -\frac{d}{dx}\ln S_0(x) \quad (Why?)$$

$$\Rightarrow \quad \int_0^y \mu_x dx = -[\ln S_0(y) - \ln S_0(0)]$$

$$\Rightarrow \quad S_0(y) = \exp\left\{-\int_0^y \mu_x dx\right\}$$

$$\begin{aligned} \therefore S_x(t) &= \frac{S_0(x+t)}{S_0(x)} = \exp\left\{-\int_x^{x+t} \mu_r dr\right\} \\ &= \exp\left\{-\int_0^t \mu_{x+s} ds\right\}. \end{aligned} \tag{10}$$

- If $\mu_x, \ \forall x \geq 0$, is known, then $S_x(t), \ \forall x, t$, can be calculated.
- μ_x fully describes the lifetime distribution, just as S_0 does.
- It is often more convenient to describe the lifetime distribution using the force of mortality μ_x than the survival function S_x.

Examples

ACTUARIAL NOTATIONS

International actuarial notations

$S_x(t)$, $F_x(t)$ and $f_x(t)$ are standard in statistics.

$$\mu_x \quad : \quad \text{force of mortality.}$$

$$_tp_x = p(T_x > t) = S_x(t). \tag{11}$$

$$_tq_x = p(T_x \leq t) = 1 - S_x(t) = F_x(t). \tag{12}$$

$$_{u|t}q_x = p(u < T_x \leq u + t) = S_x(u) - S_x(u + t). \tag{13}$$

- ★ $_1p_x = p_x$
- ★ $_1q_x = q_x$: called the **mortality rate** at age x.
- ★ $_{u|t}q_x$: called a **deferred mortality probability**.

We have

$$
\begin{aligned}
1 &= {}_tp_x + {}_tq_x. \\
{}_{u|t}q_x &= {}_up_x - {}_{u+t}p_x. \\
{}_{t+u}p_x &= {}_tp_x \; {}_up_{x+t}. \quad (Why?) && (14) \\
\mu_x &= -\frac{1}{{}_xp_0}\frac{d}{dx}{}_xp_0. \quad (Why?) && (15)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\mu_{x+t} &= -\frac{1}{{}_tp_x}\frac{d}{dx}{}_tp_x \Rightarrow \frac{d}{dx}{}_tp_x = -{}_tp_x\mu_{x+t}. && (16) \\
\mu_{x+t} &= \frac{f_x(t)}{S_x(t)} \Rightarrow f_x(t) = {}_tp_x\mu_{x+t}. \quad (Why?) && (17) \\
{}_tp_x &= \exp\left\{-\int_0^t \mu_{x+s}ds\right\}. \quad (Why?) && (18)
\end{aligned}
$$

As for F_x and f_x,

$$F_x(t) = \int_0^t f_x(s)ds \;\Rightarrow\; {}_tq_x = \int_0^t {}_sp_x\mu_{x+s}ds. \quad (Why?) \tag{19}$$

★ ${}_sp_x\mu_{x+s}ds$ can be interpreted as the probability that (x) dies between ages $x+s$ and $x+s+dx$. (Why?)

When $t = 1$, (19) becomes

$$q_x = \int_0^1 {}_sp_x\mu_{x+s}ds.$$

When $q_x \to 0$, it follows $p_x \to 1$, and hence ${}_sp_x \to 1$, $\forall 0 \le s < 1$. Thus

$$q_x \approx \int_0^1 \mu_{x+s}ds \approx \mu_{x+\frac{1}{2}}.$$

Example

MEAN AND STANDARD DEVIATION OF T_x

Definition

$$\mathring{e}_x = E(T_x)$$

is the expected future lifetime of (x) and is called **complete expectation of life**.

To evaluate $\mathring{e}_x$, we have

$$f_x(t) = {}_tp_x\mu_{x+t} = -\frac{d}{dx}{}_tp_x. \quad (Why?) \tag{20}$$

Then

$$\begin{aligned} \mathring{e}_x &= \int_0^\infty t f_x(t)dt = \int_0^\infty t\,{}_tp_x\mu_{x+t}dt \\ &= -\int_0^\infty t\left[\frac{d}{dt}{}_tp_x\right]dt. \end{aligned}$$

By integration by parts, we have

$$\mathring{e}_x = \int_0^\infty {}_tp_x dt. \quad (Why?) \tag{21}$$

Similarly, for $E(T_x^2)$,

$$\begin{aligned} E(T_x^2) &= \int_0^\infty t^2 {}_tp_x \mu_{x+t} dt \\ &= -\int_0^\infty t^2 \left[\frac{d}{dt} {}_tp_x\right] dt. \\ &= 2\int_0^\infty t {}_tp_x dt. \quad (Why?) \end{aligned} \tag{22}$$

Thus, we are able to evaluate

$$Var(T_x) = E(T_x^2) - (\mathring{e}_x)^2.$$

Example

Term expectation of life

We are sometimes interested in the future lifetime r.v. subjected to a cap of n years, $min(T_x, n)$, and define

$$\mathring{e}_{x:\overline{n}|} = min(T_x, n).$$

e.g.

Suppose that (x) is entitled to a benefit payable continuously for a maximum of n years, conditional on survival. Then $min(T_x, n)$ would represent the payment period for the benefit.

$$\begin{aligned}
\mathring{e}_{x:\overline{n}|} &= \int_0^n t\,{}_tp_x\mu_{x+t}dt + \int_n^\infty n\,{}_tp_x\mu_{x+t}dt \\
&= -\int_0^n t\left[\frac{d}{dt}{}_tp_x\right]dt + n\,{}_np_x. \quad (Why?) \\
\Rightarrow \quad \mathring{e}_{x:\overline{n}|} &= \int_0^n {}_tp_x dt. \quad (Why?)
\end{aligned}$$

CURTATE FUTURE LIFETIME

Definition

The **curtate futuure lifetime** random variable is defined as the integer part of future lifetime and is denoted by K_x for (x). That is, we have

$$K_x = \lfloor x \rfloor; \quad \lfloor x \rfloor: \text{ floor function of } x.$$

Thus, for $k = 0, 1, 2, \cdots$,

$$\begin{aligned} P(K_x = k) &= P(k \le T_x < k+1) \\ &= {}_{k|}q_x \\ &= {}_k p_x - {}_{k+1}p_x \\ &= {}_k p_x - {}_k p_x p_{x+k} \\ &= {}_k p_x q_{x+k} \end{aligned}$$

Definition

$$e_x = E(K_x)$$

is the expected value of K_x and is called **curtate expectation of life**.

So,

$$\begin{aligned} e_x &= E(K_x) = \sum_{k=0}^{\infty} kP(K_x = k) \\ &= \sum_{k=0}^{\infty} k({}_kp_x - {}_{k+1}p_x) \\ &= ({}_1p_x - {}_2p_x) + 2({}_2p_x - {}_3p_x) + 3({}_3p_x - {}_4p_x) + \cdots \\ &= \sum_{k=1}^{\infty} {}_kp_x. \end{aligned} \tag{23}$$

★ The lower limit of summation is 1.

Similarly,

$$
\begin{aligned}
E(K_x^2) &= \sum_{k=0}^{\infty} k^2({}_kp_x - {}_{k+1}p_x) \\
&= ({}_1p_x - {}_2p_x) + 4({}_2p_x - {}_3p_x) + 9({}_3p_x - {}_4p_x) + 16({}_4p_x - {}_5p_x) + \\
&= 2\sum_{k=1}^{\infty} k\,{}_kp_x - \sum_{k=1}^{\infty} k\,{}_kp_x \\
&= 2\sum_{k=1}^{\infty} k\,{}_kp_x - e_x.
\end{aligned}
$$

The expected value of $min(K_x, n)$ is denoted $e_{x:\overline{n}|}$ and

$$e_{x:\overline{n}|} = \sum_{k=1}^{\infty} {}_kp_x; \quad n \in Z. \quad (Why?)$$

$\mathring{e}_{x:\overline{n}|}$ and $e_{x:\overline{n}|}$

We can obtain an approximate relationship between $\mathring{e}_{x:\overline{n}|}$ and $e_{x:\overline{n}|}$;

$$\mathring{e}_{x:\overline{n}|} = \int_0^{\infty} {}_tp_x dt = \sum_{j=0}^{\infty} \int_j^{j+1} {}_tp_x dt.$$

By trapezium rule, we have

$$\int_j^{j+1} {}_tp_x dt \approx \frac{1}{2}({}_jp_x - {}_{j+1}p_x),$$

and hence

$$\mathring{e}_{x:\overline{n}|} \approx \sum_{j=0}^{\infty} \frac{1}{2}({}_jp_x - {}_{j+1}p_x) = \frac{1}{2} + \sum_{j=1}^{\infty} {}_jp_x.$$

that is

$$\mathring{e}_{x:\overline{n}|} \approx e_x + \frac{1}{2} \tag{24}$$