ACTUARIAL MATHEMATICS
POLICY VALUES

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SUMMARY

Policy values are a fundamental tool in insurance risk management, as they are used

- to determine the economic or regulatory capital needed to remain solvent,
- and also to determine the profit or loss for the company over any time period.

We start by considering the case where all the cash flows take place at the start or end of a year. We

- define **policy values**
    - Gross Premium Policy Values
    - Net Premium Policy Values;

POLICY VALUES

- calculate policy values recursively from year to year
  - calculating the profit from a policy in each policy year
  - introducing the **Asset Share** for a policy.

We then extend

- the analysis to policies where the cash flows are continuous and
- derive **Thiele's differential equation** for policy values.

We consider how policy values can be used to evaluate *policy alterations*

- e.g. a policyholder chooses to withdraw or stop paying premiums.

We show

- how a **retrospective valuation** has connections with asset shares and the policy values determined at looking at future cash flows;
- how the *net premium policy value* can be used to approximate the *gross premium policy value*;
- why the approximation is useful when **acquisition expenses** are high.

POLICIES WITH ANNUAL CASH FLOWS

In this chapter, we are concerned with the estimation of future losses at *intermediate times* during the term of policy, not just at inception.
(***recall***: L_0 is the future loss r.v. at time 0.)

THE FUTURE LOSS RANDOM VARIABLE

Consider a policy which is still in force t years after it was issued. Define

L_t^n: present value of future **net** loss random variable at time t.

L_t^g: present value of future **gross** loss random variable at time t.

That is

$$L_t^n = [\text{Present value, at } t\text{, of future benefits}] - [\text{Present value, at } t\text{, of future } \textbf{net premiums}]$$

$$L_t^g = [\text{Present value, at } t\text{, of future benefits}] + [\text{Present value, at } t\text{, of future expenses}] - [\text{Present value, at } t\text{, of future } \textbf{gross premiums}]$$

★ L_t is defined only if the contract is still in force t years after issue.

Assume that the premiums are payable annually and the sum insured is payable at the end of the year of death, so that all the cash flows are at the start or end of each year.

example

Consider a 20-year endowment policy purchased by a life aged 50.
Level premiums are payable annually throughout the term of the policy and the sum insured, \$500000, is payable at the end of the year of death or at the end of the term, which is sooner.
The interest rate is 5% per year and no allowance for expenses.

(A) Calculate the annual premium, P, by the equivalence principal.

(B) Calculate the $E(L_t^n)$ for $t = 10$ and $t = 11$, in both cases just before the premium due at t is paid.

Solution

(A) The equation of value for P is

$$P\ddot{a}_{[50]:\overline{20}|} - 500000A_{[50]:\overline{20}|} = 0 \tag{1}$$

and we have

$$\ddot{a}_{[50]:\overline{20}|} = 12.8456 \text{ and } A_{[50]:\overline{20}|} = 0.38830,$$

giving

$$P = \frac{500000A_{[50]:\overline{20}|}}{\ddot{a}_{[50]:\overline{20}|}} = \$15114.33.$$

(B) We have

$$\begin{aligned} L_{10}^{n} &= 500000\nu^{min(K_{60}+1,10)} - P\ddot{a}_{\overline{min(K_{60}+1,10)}|} \\ L_{11}^{n} &= 500000\nu^{min(K_{61}+1,9)} - P\ddot{a}_{\overline{min(K_{61}+1,9)}|} \end{aligned}$$

then, with $\ddot{a}_{60:\overline{10}|} = 7.9555$ and $\ddot{a}_{61:\overline{9}|} = 7.3282$,

$$\begin{aligned} E(L_{10}^{n}) &= 500000A_{60:\overline{10}|} - P\ddot{a}_{60:\overline{10}|} = \$190339, \\ E(L_{11}^{n}) &= 500000A_{61:\overline{9}|} - P\ddot{a}_{61:\overline{9}|} = \$214757. \end{aligned}$$

□

POLICY VALUES

- For a policy which is still in force at t, the future premiums (from t) are not expected to be sufficient to pay for the future benefits and expenses.

> The amount needed to cover this shortfall is called the **policy value** for the policy at time t.

- With regular level premium and increasing level of risk, the premium in each of the early years is more than sufficient to pay the expected benefits in that year, given that the life has survived to the start of the year.

  e.g. in the first year, $P = \$15114.33 > \$492.04 = 500000\nu q_{[50]}$.

  In fact, in the example,

$$P > 500000\nu q_{[50]+t}; \ t = 0, 1, \cdots, 18.$$

$$P < 500000\nu = 476190; \ t = 19 \text{ (the final year)}.$$

POLICY VALUES

- The concept of a **policy value**-
  we need to hold capital during the term of a policy to meet the liabilities in the period when *outgo on the benefits* exceeds *income from premiums*.

Suppose the insurer issues a large number, say N, of policies identically to the previous **example** to independent lives all aged 50. Suppose that the experience of this group of policyholders is precisely as assumed in the example. Consider the situation of the insurer after these policies have been in force for 10 years.

- ${}_{10}p_{[50]}N$ policyholders still be alive.
- $q_{[50]}N$ policyholders have died in the 1st year,
  ${}_{1|}q_{[50]}N$ policyholders have died in the 2nd year, $\cdots$,
  ${}_{9|}q_{[50]}N$ policyholders have died in the 10th year.

POLICY VALUES

- To $t = 10$ at 5% interest rate, *the accumulated premiums received* (not including the premium due at time 10) minus *all sums insured which have been paid* is

$$
\begin{aligned}
& NP(1.05^{10} + p_{[50]}1.05^{9} + \cdots + {}_{9}p_{[50]}1.05) \\
& \quad -500000N(q_{[50]}1.05^{9} + {}_{1|}q_{[50]}1.05^{8} + \cdots + {}_{9|}q_{[50]}) \\
= & \; 1.05^{10}NP(1 + p_{[50]}1.05^{-1} + \cdots + {}_{9}p_{[50]}1.05^{-9}) \\
& \quad -1.05^{10}(500000N)(q_{[50]}1.05^{-1} + {}_{1|}q_{[50]}1.05^{-2} + \cdots + {}_{9|}q_{[50]}1.05^{-10}) \\
= & \; 1.05^{10}N(P\ddot{a}_{[50]:\overline{10|}} - 500000A^{\;1}_{[50]:\overline{10|}}) \\
= & \; 186634N
\end{aligned}
$$

  so the share of this fund for each surviving policyholder is

$$\frac{186634N}{{}_{10}p_{[50]}N} = \$190339 = E(L^{n}_{10}).$$

  This happens because
  - the premium was calculated using the equivalence principal;
  - we assume the experience up to time 10 was exactly as in the calculation of the premium.

Another Approach

By (1), we have

$$P\ddot{a}_{[50]:\overline{20}|} = 500000A_{[50]:\overline{10}|}$$

$$\begin{aligned}
&\Rightarrow \quad P(\ddot{a}_{[50]:\overline{10}|} + \nu^{10}\,{}_{10}p_{[50]}\,\ddot{a}_{[60]:\overline{10}|}) = 500000(A^{\;1}_{[50]:\overline{10}|} + \nu^{10}\,{}_{10}p_{[50]}\,A_{[60]:\overline{10}|}) \\
&\Rightarrow \quad P\ddot{a}_{[50]:\overline{10}|} - 500000A^{\;1}_{[50]:\overline{10}|} = \nu^{10}\,{}_{10}p_{[50]}(500000A_{[60]:\overline{10}|} - P\ddot{a}_{[60]:\overline{10}|}) \\
&\Rightarrow \quad \frac{1.05^{10}}{{}_{10}p_{[50]}}(P\ddot{a}_{[50]:\overline{10}|} - 500000A^{\;1}_{[50]:\overline{10}|}) = 500000A_{[60]:\overline{10}|} - P\ddot{a}_{[60]:\overline{10}|}. \qquad (2)
\end{aligned}$$

- The LHS is the fund built up at time 10 for each surviving policyholder;
- the RHS is the expected value of the future loss r.v. at time 10, $E(L^n_{10})$.

The equation holds because

A. the premium was calculated using the equivalence principal;
B. the expected value of the future loss r.v. was calculated using the premium basis, and
C. we assume the experience followed precisely the assumptions in the premium basis.

In practice,

★ (A), (B) may or may not apply; (C) is extremely unlike to hold.

POLICY VALUES FOR POLICIES WITH ANNUAL CASH FLOWS

Notation

$_tV$: general notation for a policy value t years after a policy was issued.

- ★ The policy value for a policy in force at t years after it was purchased is the **expected value** at that time of the future loss random variable.

We have the equation

$$\boxed{\begin{aligned} &{}_tV + EPV \text{ at } t \text{ of future premiums} \\ &\quad = EPV \text{ at } t \text{ of future benefits } + \text{ expenses.} \end{aligned}}$$

Valuation

An important element in the financial control of a insurance company is the calculation at regular interval, usually at least annually, of *the sum of the policy values for all policies in force* at that time and also *the value of all the company's investments*. The process is called a **valuation** of the company.

- ★ For a company to be financial sound, the investments should have a greater value than the total policy value.

Reserve, **Prospective reserve**, **Prospective policy value**

These are sometimes used in place of *policy value*.

- ★ We use **policy value** to mean the **expected value** of the future loss random variable.
- ★ We restrict **reserve** to mean the **actual capital** held in respect of a policy.

Definition

The **gross premium policy value** for a policy in force at duration t years after it was purchased is the *expected value* at that time of the *gross future loss random variable* on a specified basis.

★ The premiums used in the calculation are the **actual premiums** payable under the contract.

Definition

The **net premium policy value** for a policy in force at duration t years after it was purchased is the *expected value* at that time of the *net future loss random variable* on a specified basis (which makes no allowance for expenses).

★ The premiums used in the calculation are the **net premiums** calculated on the policy value basis using the **equivalence principle**, not the actual premiums.

POLICY VALUES

Notation

(1) The numerical value of a net or gross policy value depends on the assumptions

$$-\textit{survival model},\ \textit{interest},\ \textit{expenses},\ \textit{futurebonuses}-$$

used in its calculation. These assumptions are called **policy value basis**.

- ★ These assumptions may different from those used to calculate premium, that is, the **premium basis**.

(2) A net premium policy value can be regarded as a special case of a gross premium policy value.

- ★ Suppose the gross premium policy value is calculated assuming no expenses. Then at issue, the gross and net *premiums* are the same.
- ★ t years later, the *gross premium policy value* and the *net premium policy value* are the same only if the *policy value basis* is the same as the *original premium basis*, which is unlikely the case.

Notation (conti'n)

- ★ The net premium policy value will use a *net premium* based on the *policy value basis*, not the *original premium basis*.
- ★ The gross premium policy value uses a the *actual contract premium*, with no recalculation.

(3) So, when the policy value basis changes, the *net premium policy value* requires the recalculation of the premium.

(4) it is the usual practice to regard a *premium* and any *premium-related expenses* due at that time as **future payments** and any *insurance benefits* and *related expenses* as **past payments**.

- ★ Under annuity contracts, the *annuity payments* and *related expenses* may be treated either as **future payments** or **past payments**, so we have to particularly careful to specify in such cases.

(5) If an insurance policy has a finite term, n years, then ${}_nV = 0$. (Why?)

- ★ ${}_0V = E(L_0) = 0$ if the premium is calculated using the equivalence principle *and* the policy value basis is the same as the premium basis.

Notation (conti'n)

(6) For an endowment insurance which is still in force at the maturity date and S is the sum insured. Then

$$_{n^-}V = S \text{ and } {}_nV = 0. \ (Why?)$$

(7) In practice, the amount of reserve required for a policy is usually set by calculating a policy value on a specified basis.

- ★ The funds for the reserve will come from accumulating *past premiums*, *net of past benefits* and *expense costs*.
- ★ if the past experience of a portfolio is better than the basis assumptions (from the insurer's perspective), eg. higher interest rates and lower mortality (for death benefits), then the accumulated cash flows will be more than needed to fund the reserve, and some profit can be taken.

example

An insurer issues a whole life insurance policy to a life aged 50. The insured of \$100000 is payable at the end of the year of death. Level premium s of \$1300 are payable annually in advance through the term of the contract.

(A) Calculate the *gross premium policy value* five years after the inception of the contract, assuming the policy is still in force, using the following basis:
- Interest: 5% per year effective
- Expenses: 12.5% of each premium

(B) Calculate the *net premium policy value* five years after the issue of the contract, assuming the policy is still in force, using the following basis:
- Interest: 4% per year effective

Solution

(A) The gross future loss random variable at time 5 is

$$L_5^g = 100000\nu^{K_{55}+1} - (0.875)\mathbf{1300}\,\ddot{a}_{\overline{K_{55}+1}|}$$

so

$$_5V^g = E(L_5^g) = 100000A_{55} - (0.875)\mathbf{1300}\ddot{a}_{55} = \$5256.35.$$

Solution (conti'n)

(B) For net premium policy values we always calculate the (*hypothetical*) net premium for the contract on the policy value basis.
Let P be the net premium for the policy value. At 4% per year,

$$P = 100000 \frac{A_{[55]}}{\ddot{a}_{[55]}} = \$1321.31.$$

So, at 4% per year,

$$L_5^n = 100000 \nu^{K_{55}+1} - \mathbf{1321.31}\, \ddot{a}_{\overline{K_{55}+1}|}$$

and hence

$$_5V^n = 100000 A_{55} - \mathbf{1321.31}\, \ddot{a}_{55} = \$6704.75.$$

★ The hypothetical net premium is greater than the gross premium of the contract. This would generally not be permitted under the regulations on valuation, as we would not allow it happens. □

example

A woman aged 60 purchases a 20-year endowment insurance with a sum insured of \$100000 payable at the end of the year of death or on survival to age 80, whichever occurs first. An annual premium of \$5200 is payable for at most 10 years. The insurer uses the following basis:

- Interest: 5% per year effective
- Expenses: 10% of the 1st premium, 5% of the subsequent premiums, plus \$200 on payment of the sum insured

Calculate ${}_0V$, ${}_5V$, ${}_6V$ and ${}_{10}V$, that is, the gross premium policy values at $t = 0, 5, 6, 10$.

Solution

We know

$$\begin{array}{lll} \ddot{a}_{[60]:\overline{10}|} = 7.9601 & \ddot{a}_{65:\overline{5}|} = 4.4889 & \ddot{a}_{66:\overline{4}|} = 3.6851 \\ A_{[60]:\overline{20}|} = 0.41004 & A_{65:\overline{15}|} = 0.51140 & A_{66:\overline{14}|} = 0.53422 \\ & A_{70:\overline{10}|} = 0.63576. & \end{array}$$

Solution (conti'n)

Therefore

$$L_0 = 100200\nu^{min(K_{[60]}+1,20)} + 0.05P - 0.95P\ddot{a}_{\overline{min(K_{[60]}+1,20)}|}$$

where $P = \$5200$ (*actual premium*). Hence

$${}_0V = E(L_0) = 100200A_{[60]:\overline{20}|} - (0.95\ddot{a}_{[60]:\overline{10}|} - 0.05)P = \$2023.$$

Similarly,

$$L_5 = 100200\nu^{min(K_{65}+1,15)} - 0.95P\ddot{a}_{\overline{min(K_{65}+1,5)}|}$$

$$\Rightarrow \quad {}_5V = E(L_5) = 100200A_{65:\overline{15}|} - 0.95P\ddot{a}_{65:\overline{5}|} = \$29068.$$

$$L_6 = 100200\nu^{min(K_{66}+1,14)} - 0.95P\ddot{a}_{\overline{min(K_{66}+1,4)}|}$$

$$\Rightarrow \quad {}_6V = E(L_6) = 100200A_{66:\overline{14}|} - 0.95P\ddot{a}_{66:\overline{4}|} = \$35324.$$

Finally, as no premiums are payable after $t = 9$,

$$L_{10} = 100200\nu^{min(K_{70}+1,10)}$$

$$\Rightarrow \quad {}_{10}V = E(L_{10}) = 100200A_{70:\overline{10}|} = \$63703. \qquad \square$$

POLICY VALUES FOR POLICIES WITH ANNUAL CASH FLOWS

Notation

- ${}_0V > 0$, it means that from the outset the insurer expects to make a loss on this policy. The explanation is that the policy value basis may be more conservative than the premium basis.
  In the example, the insurer may assume an interest rate of 6% in the premium calculation, but, for policy value calculations, assumes investments will earn only 5%.

example

A man aged 50 purchases a deferred annuity policy. The annuity will be paid annually for life, with the first payment on his 60th birthday. Each annual payment will be \$10000. Level payments of \$11900 are payable annually for at most 10 years. On death before 60, all premiums paid will be returned, without interest, at the end of the year of death. The insurer uses the following basis:

- Interest: 5% per year effective
- Expenses: 10% of the 1st premium, 5% of the subsequent premiums, \$25 each time an annuity payment is paid, and \$100 when a death claim is paid.

Calculate the gross premium policy value

- at the start of the policy;
- at the end of the 5th year of the policy;
- at the end of the 15th year of the policy just *before* and just *after* the annuity payment and expenses due at that time.

Solution

We know

$$\ddot{a}_{[50]:\overline{10}|} = 8.0566 \qquad \ddot{a}_{55:\overline{5}|} = 4.5268 \qquad \ddot{a}_{60} = 14.9041 \qquad \ddot{a}_{65} = 13.5498$$

$$\nu^{5}\,{}_{5}p_{55} = 0.77382 \qquad \nu^{10}\,{}_{10}p_{[50]} = 0.60196$$

$$A^{\;1}_{[50]:\overline{10}|} = 0.01439 \quad (IA)^{\;1}_{[50]:\overline{10}|} = 0.08639 \quad A^{\;1}_{[55]:\overline{5}|} = 0.01062 \quad (IA)^{\;1}_{[55]:\overline{5}|} = 0.03302$$

and

- let ${}_{15^-}V$ and ${}_{15^+}V$ be the policy values at duration 15 years just before and after the annuity payment and expenses due at that time, respectively;
- $P = \$11900$, given;

then the policy value at t is

$$\textit{EPV at } t \text{ of future benefits + expenses} \;-\; \textit{EPV at } t \text{ of future premiums.}$$

Solution (conti'n)

At the inception of $t = 0$, we have

- $P(IA)^{\,1}_{[50]:\overline{10}|}$: the *EPV* of the death benefit;
- $100A^{\,1}_{[50]:\overline{10}|}$: the *EPV* of the death claim expenses;
- the *EPV* of the annuity benefit and associated expenses is

$$10025\nu^{10}\,{}_{10}p_{[50]}\,\ddot{a}_{60};$$

- the *EPV* of future premiums less associated expenses is

$$0.95P\ddot{a}_{[50]:\overline{10}|} - 0.05P;$$

so that

$$\begin{aligned} {}_0V &= P(IA)^{\,1}_{[50]:\overline{10}|} + 100A^{\,1}_{[50]:\overline{10}|} + 10025\nu^{10}\,{}_{10}p_{[50]}\,\ddot{a}_{60} - (0.95\ddot{a}_{[50]:\overline{10}|} - 0.05)P \\ &= \$485. \end{aligned}$$

Solution (conti'n)

At the inception of $t = 5$, assuming it is still in force,

- the future death benefit is $6p, 7P, \cdots, 10P$ depending on whether the life dies in the 6th, 7th, $\cdots$, 10th years, respectively.
- We write the benefit as a level benefit of $5P$ plus an increasing benefit of $p, 2P, \cdots, 5P$.

So,

- the *EPV* of the death benefit is
$$P\left[(IA)^{\ \ 1}_{[55]:\overline{5}|} + 5A^{\ \ 1}_{[55]:\overline{5}|}\right],$$
- the *EPV* of the death claim expenses is $100A^{\ \ 1}_{[55]:\overline{5}|}$;
- the *EPV* of the annuity benefit and associated expenses is $10025\nu^5\,{}_5p_{55}\,\ddot{a}_{60}$;
- the *EPV* of future premiums less associated expenses is $0.95P\ddot{a}_{55:\overline{10}|}$;

so that

$$\begin{aligned} {}_5V &= P\left[(IA)^{\ \ 1}_{[55]:\overline{5}|} + 5A^{\ \ 1}_{[55]:\overline{5}|}\right] + 100A^{\ \ 1}_{[55]:\overline{5}|} + 10025\nu^5\,{}_5p_{55}\,\ddot{a}_{60} - 0.95P\ddot{a}_{55:\overline{10}|} \\ &= \$65470. \end{aligned}$$

Solution (conti'n)

Once the premium payment period of 10 years is completed there are no future premiums to value, so

$$_{15^-}V = 10025\ddot{a}_{65} = \$135873, \quad (why?)$$

and

$$_{15^+}V = 10025 a_{65} = {}_{15^-}V - 10025 = \$125812. \quad (why?) \qquad \square$$

Notation

- ★ For an endowment insurance, the policy values build up over time to provide the sum insured on maturity.
- ★ For a term insurance, the policy values increase then decrease.
- ★ The reason why small policy values occur for the term insurance policy is that there is a small probability of death benefit being paid.

RECURSIVE FORMULAE FOR POLICY VALUES

In this section we see how to derive recursive formulae for policy values for policies with discrete cash flows.

- ★ These formulae also provide an understanding of how **the policy value builds up** and **how profit emerges** while the policy is in force.

example

For the first **example** and for $t = 0, 1, 2, \cdots, 19$, show that

$$({}_tV + P)(1+i) = 500000\, q_{[50]+t} + p_{[50]+t}\ {}_{t+1}V \tag{3}$$

where $P = \$15114.33$, $i = 5\%$ and the policy value is calculated on the basis specified in the first **example**.

Solution

From the first **example**, we have

$$ {}_tV = 500000A_{[50]+t:\overline{20-t}|} - P\ddot{a}_{[50]+t:\overline{20-t}|};\ t = 0, 1, 2, \cdots, 19. $$

Then

$$
\begin{aligned}
{}_tV &= 500000\left(\nu q_{[50]+t} + \nu p_{[50]+t}A_{[50]+t+1:\overline{19-t}|}\right) - P\left(1 + \nu p_{[50]+t}\ddot{a}_{[50]+t+1:\overline{19-t}|}\right) \\
&= \nu\left[500000q_{[50]+t} + p_{[50]+t}\left(500000A_{[50]+t+1:\overline{19-t}|} - P\ddot{a}_{[50]+t+1:\overline{19-t}|}\right)\right] - P \\
&= \nu\left[500000q_{[50]+t} + p_{[50]+t}\left({}_{t+1}V\right)\right] - P
\end{aligned}
$$

Therefore

$$ {}_tV + P = \nu\left[500000q_{[50]+t} + p_{[50]+t}\left({}_{t+1}V\right)\right] $$

$$ \Rightarrow\quad ({}_tV + P)(1+i) = (1+i)\nu\left[500000q_{[50]+t} + p_{[50]+t}\left({}_{t+1}V\right)\right] $$

$$ \Rightarrow\quad ({}_tV + P)(1+i) = 500000q_{[50]+t} + p_{[50]+t}\ {}_{t+1}V. $$

example

For the 4th **example** and for $t = 0, 1, 2, \cdots, 9$, show that

$$({}_tV + 0.95P)(1+i) = [(t+1)P + 100]\, q_{[50]+t} + p_{[50]+t}\ {}_{t+1}V \qquad (4)$$

where $P = \$11900$, $i = 5\%$ and the policy value is calculated on the basis specified in the fourth **example**.

Solution

From the 4th **example**, we have

$$\begin{aligned} {}_tV &= P(IA)^{\ 1}_{[50]+t:\overline{10-t}|} + (tP + 100)(IA)^{\ 1}_{[50]+t:\overline{10-t}|} \\ &\quad + 10025\nu^{10-t}\ {}_{10-t}p_{[50]+t}\,\ddot{a}_{60} - 0.95\,\ddot{a}_{[50]+t:\overline{10-t}|};\ t = 0, 1, 2, \cdots, 9. \end{aligned}$$

Recall

$$\begin{aligned} \ddot{a}_{[x]+t:\overline{n-t}|} &= 1 + \nu p_{[x]+t}\,\ddot{a}_{[x]+t+1:\overline{n-t-1}|} \\ A^{\ 1}_{[x]+t:\overline{n-t}|} &= \nu q_{[x]+t} + \nu p_{[x]+t} A^{\ 1}_{[x]+t+1:\overline{n-t-1}|} \\ (IA)^{\ 1}_{[x]+t:\overline{n-t}|} &= \nu q_{[x]+t} + \nu p_{[x]+t}\left[(IA)^{\ 1}_{[x]+t+1:\overline{n-t-1}|} + A^{\ 1}_{[x]+t+1:\overline{n-t-1}|}\right] \end{aligned}$$

Solution (conti'n)

Then, for $t = 0, 1, 2, \cdots, 9$,

$$\begin{aligned} {}_tV &= P\left\{\nu q_{[50]+t} + \nu p_{[50]+t}\left[(IA)^{\ 1}_{[50]+t+1:\overline{10-t-1|}} + A^{\ 1}_{[50]+t+1:\overline{10-t-1|}}\right]\right\} \\ &\quad +(tP+100)\left(\nu q_{[50]+t} + \nu p_{[50]+t} A^{\ 1}_{[50]+t+1:\overline{10-t-1|}}\right) \\ &\quad +10025\nu p_{[50]+t}\left(\nu^{10-t-1}\,{}_{10-t-1}p_{[50]+t+1}\,\ddot{a}_{60}\right) \\ &\quad -0.95\left(1+\nu p_{[50]+t}\ddot{a}_{[50]+t+1:\overline{10-t-1|}}\right) \end{aligned}$$

implies

$$\begin{aligned} {}_tV &= \nu q_{[50]+t}\left[(t+1)P+100\right] - 0.95P \\ &\quad +\nu p_{[50]+t}\left\{P(IA)^{\ 1}_{[50]+t+1:\overline{10-t-1|}} + [(t+1)P+100]A^{\ 1}_{[50]+t+1:\overline{10-t-1|}}\right. \\ &\quad \left. +10025\,{}_{10-t-1}p_{[50]+t+1}\nu^{10-t-1}\ddot{a}_{60} - 0.95P\ddot{a}_{[50]+t+1:\overline{10-t-1|}}\right\} \\ &= \nu q_{[50]+t}\left[(t+1)P+100\right] - 0.95P + \nu p_{[50]+t}\ {}_{t+1}V. \end{aligned}$$

Therefore (4) holds as required. □

Remarks

- The methods we use to derive (3) and (4) can be used for other policies:

  Step 1. write down formula for ${}_tV$;
  Step 2. break up

$$\begin{aligned} EPVs \quad = & \quad [EPVs \text{ of the payments in } t \text{ to } t+1] \\ & + [EPVs \text{ of the payments from } t+1 \text{ onwards}]. \end{aligned}$$

In a more general setting, consider

- a policy issued to a life (x)
- where cash flows -premiums, expenses and claims- can occur only at the starts or ends of a year.
- suppose the police has been in force for t years.

Consider the $t+1$ year, and let

POLICY VALUES

P_t	:	the premium payable at t.
e_t	:	the premium-related expenses payable at t.
S_{t+1}	:	the sum insured at $t+1$ if the policyholder dies in the year.
E_{t+1}	:	the expense of paying sum insured at $t+1$.
${}_tV$	:	the *gross premium policy value* for a policy in force at t.
${}_{t+1}V$	:	the *gross premium policy value* for a policy in force at $t+1$.

The quantities e_t, E_t, $q_{[x]+t}$ and i_t are all assumed in the policy basis.

Then, by considering what can happen in the year, we have

$$L_t = \begin{cases} (1+i_t)^{-1}(S_{t+1}+E_{t+1}) - P_t + e_t & \text{if } P(K_{[x]+t}=0) = q_{[x]+t}; \\ \\ (1+i_t)^{-1}L_{t+1} - P_t + e_t & \text{if } P(K_{[x]+t}\geq 1) = p_{[x]+t}. \end{cases}$$

Taking expected value, we have

$$
\begin{aligned}
{}_tV = E(L_t) \quad &= \quad q_{[x]+t}(1+i_t)^{-1}S_{t+1} + E_{t+1}) - (q_{[x]+t} + p_{[x]+t})(P_t - e_t) \\
&\quad + p_{[x]+t}(1+i_t)^{-1}E(L_{t+1}),
\end{aligned}
$$

and because

$$ {}_{t+1}V = E(L_{t+1}), $$

which gives the **important equation**:

$$ \boxed{({}_tV + P_t - e_t)(1+i_t) = q_{[x]+t}(S_{t+1} + E_{t+1}) + p_{[x]+t}\ {}_{t+1}V} \qquad (5) $$

★ For policies with cash flows **only at the start/end of each year**, the recursive formulae always have the same general form. This form can be explained by examining (5).

Remarks

- We can interpret ${}_tV$ as the value of assets the insurer should have at t (in respect of a policy still in force) in order to expect to break even the future course of the policy.
- In (5), ${}_tV + P_t - e_t$,
  - $P_t - e_t$:
    the net cash flow received by insurer at t as assumed in the policy value basis.
  - ${}_tV + P_t - e_t$:
    the amount of the insurer's assets at t just after the cash flow.
  - There are no further cash flows until the end of the year.
- The LHS of (5), $({}_tV + P_t - e_t)(1 + i_t)$,
  - The assets are rolled up to the end of the year with interest rate assumed in the policy basis, i_t. This also gives the LHS's of (3) and (4).

Remarks (conti'n)

- The RHS of (5), $q_{[x]+t}(S_{t+1}+E_{t+1})+p_{[x]+t}\ {}_{t+1}V$,
  - If the policy holder is alive at $t+1$ the insurer needs to have assets of amount ${}_{t+1}V$ at that time, $p_{[x]+t}\ {}_{t+1}V$.
  - if the policyholder has died during the year, the insurer must pay the death benefit and related expenses,$q_{[x]+t}(S_{t+1}+E_{t+1})$.

We can rewrite (5), (3) and (4) as follows:

$$\begin{aligned}
({}_tV+P_t-e_t)(1+i_t) &= {}_{t+1}V+q_{[x]+t}(S_{t+1}+E_{t+1}-{}_{t+1}V) && (6)\\
({}_tV+P)(1+i) &= {}_{t+1}V+q_{[x]+t}(500000-{}_{t+1}V) && (how?)\\
({}_tV+0.95P)(1+i) &= {}_{t+1}V+q_{[x]+t}\left[(t+1)P\ q_{50+t}-{}_{t+1}V\right].
\end{aligned}$$

The RHS's of these formulae can be interpreted slightly differently.

POLICY VALUES

- For policy in force at t the insurer needs to provide the policy value, ${}_{t+1}V$, at $t+1$, whether the life died during the year or not.

★ If the policyholder has died in the year, the insurer must also provide the **extra amount to increase the policy value** to the death benefit payable plus related expenses,
$$S_{t+1} + E_{t+1} - {}_{t+1}V.$$

The extra amount required to increase the policy value to *death benefit* is called the **Death Strain at Risk** (DSAR) or the **Sum at Risk** or the **Net Amount at Risk**, at $t+1$. In general ,

$$[\text{DSAR at } t] = S_t - {}_tV.$$

★ DSAR is an important measure of the insurer's risk if mortality exceeds the basis assumption.

★ DSAR is useful in determining risk management strategy, including reinsurance.

POLICY VALUES

- ★ Until now, we are able to calculate the policy value directly. In more complicate examples, however, in particular where **the benefits are defined in terms of policy value**, it may **NOT** be possible.
- ★ In these cases, the recursive formulae for policy values, (5), can be very useful.

example

Consider a 20-year endowment insurance policy purchased by a life aged 50. Level premiums of $23500 per year are payable through the term of the contract. a sum insured $700000 is payable at the end of the term if the life survives to age 70. On death before age 70 a sum insured is payable at the end of the year of death equal to the *policy value at the start of the year* in which the policyholder dies.

- Interest: 3.5% per year effective
- Expenses: nil

Calculate ${}_{15}V$, the policy value for a policy in force at the start of the 16th year.

Solution

By (5), with $P = \$23500$,

$$({}_tV + P)1.035 = q_{[50]+1}\ S_{t+1} + p_{[50]+1}\ {}_{t+1}V; \quad t = 0, 1, 2, \cdots, 19.$$

Putting $t = 19$, the final year of the policy, the survival benefit is \$700000 and it gives

$$({}_{19}V + P)1.035 = q_{69}\ {}_{19}V + p_{69}\ 700000. \quad (* \ S_{20} = {}_{19}V; \ why?)$$

We can work backwards as:

$$\begin{aligned}
{}_{19}V &= (p_{69}\ 700000 - 1.035P)/(1.035 - q_{69}) = 652401. \\
{}_{18}V &= (p_{68}\ {}_{19}V - 1.035P)/(1.035 - q_{68}) = 562145. \\
\vdots &= \vdots \\
{}_{15}V &= (p_{65}\ {}_{16}V - 1.035P)/(1.035 - q_{65}) = 478036.
\end{aligned}$$

Hence, the answer is \$478036. $\square$

ANNUAL PROFIT BY SOURCE

If the policy value basis are not met, it happens very likely in general, then

- the insurer's assets at $t+1$ may be more than sufficient to pay any benefits due at that time and to provide a policy value of ${}_{t+1}V$ for those policies still in force. The insurer will have made a **profit** in the year;
- the insurer's assets at $t+1$ may be not sufficient to pay any benefits due at that time and to provide a policy value of ${}_{t+1}V$ for those policies still in force. The insurer will have made a **loss** in the year.

example

An insurer issues a large number of policies identical to the policy in the 3rd **example** to women aged 60. Five years after they were issued, a total of 100 of those policies were still in force. In the following year,

- expenses of 6% of each premium paid were incurred;
- interest was earned at 6.5% on all assets;
- one policyholder died;
- expenses of $250 were incurred on the payment of the sum insured for the policyholder who died.

(A) Calculate the profit or loss on this group of policies for this year.

(B) Determine how much of this profit/loss is attributable to profit/loss from mortality, from interest and from expenses.

Solution

(A) Consider the cash flows in the 6th year.
For each of the 100 policies still in force at $t = 5$ and $P = \$5200$, the total assets at $t = 5$ after receiving premiums and paying premium-related expenses were

$$100{}_5V + 100(0.94P) = \$3395551; \quad * \ 100{}_5V: \text{ insurer's assets at } t = 5\,.$$

At $t = 6$, before paying any death claims and expenses, the setting up policy value is

$$[100{}_5V + 100(0.94P)](1.065) = \$3616262.$$

Because, at the end of $t = 5$ (the beginning of $t = 6$),

- the death claim plus related expenses was \$100250;
- there are 99 policies still in force, so the insurer is required $99\,{}_6V$ (${}_6V$ is from the 3rd **example**) for the total policy value at that time.

Solution (conti'n)

(A) (conti'n)
Therefore, the total amount the insurer requires at the end of $t = 5$ is

$$100250 + 99\,{}_6V = \$3597342.$$

Hence the insurer has made a **profit** in the sixth year of

$$[100{}_5V + 100(0.94P)](1.065) - (100250 + 99\,{}_6V) = \$18919.$$

(B) The sources of profit and loss in the 6th year are as follows.

I. Interest: **profit**;

$$\underset{\text{(the actual interest)}}{6.5\%} > \underset{\text{(assumed policy value interest)}}{5\%}.$$

II. Expenses: **loss**;

$$\begin{aligned} \underset{\text{(premium related)}}{6\%} &> \underset{\text{(assumed premium related)}}{5\%}. \\ \underset{\text{(claim related)}}{\$250} &> \underset{\text{(assumed claim related)}}{\$200}. \end{aligned}$$

Solution (conti'n)

(B) (conti'n)

III. Mortality: **loss**;
assumed $q_{65} = 0.0059$, then

$$100q_{65} = 0.59_{\text{(expected death)}} < 1_{\text{(actual death)}}$$

and

$$\begin{aligned} & \text{the amount required for a death } \$100250 \\ > \quad & \text{the amount required on survival } {}_6V = \$35324. \end{aligned}$$

★ Since the overall profit is positive, (I) has had a greater effect than (II) and (III) combined.

We can attribute the total profit to the three sources as follows.

Solution (conti'n)

(B) (conti'n)

Interset

Assumed in the policy value basis,

- the expenses at the start of the year are $0.05P$ per policy still in force;
- the interest is earned at 5% ;

the total interest received in the year is

$$\mathbf{0.05}[100{}_5V + 100(0.95P)] = \$170038.$$

The actual interest earned, before allowing for actual expenses, is

$$\mathbf{0.065}[100{}_5V + 100(0.95P)] = \$221049.$$

Hence, there is a profit of $51011 attributable to interest.

Solution (conti'n)

(B) (conti'n)

Expenses
we allow for the actual interest rate earned during the year but use the expected mortality. That is

- the interest is earned at 6.5%;
- the number of death is $100q_{65}$.

The expected expenses on this basis, valued at the year end, are

$$100(\mathbf{0.05}\boldsymbol{P})(1.065) + 100q_{65}(\mathbf{200}) = \$27808.$$

The actual expenses, if death were as expected, are

$$100(\mathbf{0.06}\boldsymbol{P})(1.065) + 100q_{65}(\mathbf{250}) = \$33376.$$

The loss from expenses is $5568.

Solution (conti'n)

(B) (conti'n)

Mortality
We use

- the actual interest earned, 6.5%;
- the actual expenses.

for each death, the cost to the insurer is the death strain at risk, is

$$100000 + 250 - {}_6V,$$

so the mortality profit is

$$(\mathbf{100q_{65} - 1})(100000 + 250 - {}_6V) = -\$26524.$$

This gives a total profit of

$$51011 - 5568 - 26524 = \$18919$$

which is the mount calculated earlier. □

POLICY VALUES

We have calculated the profit from the three sources in the order: interest, expenses, mortality.
At each step

- ★ we assume that factors **not yet considered** are as specified in the **policy value basis**
- ★ whereas factors **already considered** are as **actually occurred**.
- ★ This avoid *double counting* and gives the correct total.

We can follow the same principle, building from expected to actual, one basis element at a time, but change the order of the calculation as follows.

Expenses

$$100[(0.06-0.05)P](\mathbf{1.05})+100q_{65}(250-200)=\$5490.$$

Interest

$$(0.065-0.05)[100\,{}_5V+\mathbf{100}(\mathbf{0.94}P)]=\$50933.$$

Mortality

$$(100q_{65}-1)(100000+250-{}_6V)=-\$26524.$$

This also gives a total profit of

$$50933-5490-26524=\$18919$$

★ The exercise of breaking down the profit or loss into its component parts is called **analysis of surplus**.

ASSET SHARES

We have learned that it is virtually impossible for the experience of a policy or a portfolio of policies to follow exactly the assumptions in the premium basis.

- The invested premium nay have earned a greater or smaller rate of return that used in the premium basis.
- The expenses and mortality experiences will differ from the premium basis.

Each policy contributes to the total assets of the **insurer** through the actual investment, expenses and mortality experience.

Definition

The share of the insurer's assets attributable to *each policy* in force at *any given time* is called as the **asset share** of the policy at that time.

★ **Asset share** is calculated by assuming

- the policy is one of a large group of **identical policies** issued **simultaneously**.
- For this group of policies

  $$\text{Premium} \; - \; \text{claims, expenses}$$

  are accumulated using values based on the insurer's experience for similar policies over the period.
- At any given time,

  $$\frac{\text{the accumulated fund}}{\text{the number of survivors}}$$

  gives **asset share** at that time for each surviving policyholder.

Notation

- ★ If the insurer's experience is close to the assumptions in the policy value basis, then we would expect the **asset share** to be close to the **policy value**.
- ★ The **policy value** at duration t represents the amount the insurer *need to have* at that time in respect of each survival policyholder.
- ★ The **asset share** represents an estimate of the amount the insurer *actually does have* at that time in respect of each survival policyholder.

example

Consider a policy identical to the policy studied 4th **example** and suppose that this policy has now been in force for 5 year. Suppose that over the past 5 years the insurer's experience in respect of similar policies has been as follows.

- Annual interest earned on investments has been as shown in the following table.

Year	1	2	3	4	5
Interest %	4.8	5.6	5.2	4.9	4.7

- Expenses at the start of the year in which a policy was issued were 15% of the premium.
- Expenses at the start of the year after a policy was issued were 6% of the premium.
- The expense of paying a death claim was, on average, $120.
- The mortality rate, $q_{[50]+t}$ for $t = 0, 1, \cdots, 4$, has been approximately 0.0015.

Calculate the asset share for the policy at the start of each of the first six years.

Solution

Let

- N: number of identical policies issued simultaneously.
- AS_t: the asset share per policy surviving at $t = 0, 1, \cdots, 5$.

We calculate, by t and by the insurer's actual experience over this period,

$$AS_t = \frac{[(\text{total premiums received}) - (\text{the claims and expenses paid})]}{\text{the number of surviving policies}}$$

★ AS_t does **NOT** include the premium and related expense due at t. Therefore,

$$AS_0 = 0. \text{ (recall: } {}_0V = \$490)$$

Solution (conti'n)

We have $P = \$11900$,

- at $t = 0$, the [(premiums) − (expenses received)] are

$$0.85(11900N) = 10115N.$$

- At the end of the year (i.e. $t = 1$) with $i = 4.8\%$, this amount accumulates to

$$10115N(1 + 0.048) = 10601N.$$

At the end of the year (i.e. $t = 1$), because there are $0.0015N$ policyholders die in the first year so that death claims plus expenses are

$$0.0015(11900 + 120)N = 18N$$

which leaves $10601N - 18N = 10582N$ at the end of the year. Therefore

$$AS_1 = \frac{10582N}{0.9985N} = 10598.$$

Solution (conti'n)

you are asked to check out the following results.

Year t	Fund at start of t	Cash flow at start of t	Fund at end of t before death claims	Death claims and expenses	Fund at end of t	Survivors	AS_t
1	0	$10115N$	$10601N$	$18N$	$10582N$	$0.9985N$	10589
2	$10582N$	$11169N$	$22970N$	$36N$	$22934N$	0.9985^2N	23003
3	$22934N$	$11152N$	$35859N$	$54N$	$35805N$	0.9985^3N	35967
4	$35805N$	$11136N$	$49241N$	$71N$	$49170N$	0.9985^4N	49466
5	$49170N$	$11119N$	$63123N$	$89N$	$63034N$	0.9985^5N	63509

□

Notation

★ N does not affect AS_t.

★ The experience of the insurer over the five years has been close to the assumptions in the policy value basis in the 4th **example**. Therefore, $AS_5 = \$63509$ is reasonable close to ${}_5V = \$65470$ in this example.

POLICY VALUES WITH CASH FLOWS AT $1/m$thly INTERVALS

In practice, for example, premiums are often payable monthly and death benefits are usually payable immediately following, or soon after death.

example

A life aged 50 purchases a 10-year term insurance with sum insured \$500000 payable at the end of the month of death. Level quarterly premiums, each of amount $P = \$460$ are payable for at most five years.
Calculate the gross premium policy values at durations 2.75, 3 and 6.5 years using the following basis.

- Interest: 5% per year
- Expenses: 10% of each gross premium

Solution

$$\because \; A^{(12)}_{\substack{1\\52.75:\overline{7.25}|}} = 0.01327 \quad \text{and} \quad \ddot{a}^{(4)}_{52.75:\overline{2.25}|} = 2.14025$$

Hence

$$\begin{aligned} {}_{27.5}V &= 500000A^{(12)}_{\substack{1\\52.75:\overline{7.25}|}} - 0.9(4)P\ddot{a}^{(4)}_{52.75:\overline{2.25}|} \\ &= \$3091.02. \end{aligned}$$

Similarly,

$$\begin{aligned} {}_{3}V &= 500000A^{(12)}_{\substack{1\\53:\overline{7}|}} - 0.9(4)P\ddot{a}^{(4)}_{53:\overline{2}|} \\ &= \$3357.94 \end{aligned}$$

and

$$\begin{aligned} {}_{6.5}V &= 500000A^{(12)}_{\substack{1\\56.5:\overline{3.5}|}} = 500000(0.008532) \\ &= \$4265.63. \end{aligned}$$ $\square$

RECURSIONS

We can derive recursive formulae for policy values for policies with cash flows at discrete times other than annually.

- ★ We need to be careful because the premiums and benefits may be paid with different frequency, as we did in the previous example.

So, for example, we can use a recurrence relationship to generate the policy value at each month end and allowing for premiums only every third month.

$$\begin{aligned} & [{}_{2.75}V + 460 - 0.1(460)](1.05)^{(\frac{1}{12})=0.083} \\ = \ & 500000 {}_{0.083}q_{52.75} + {}_{0.083}p_{52.75} \ {}_{(2.75+0.083=)2.833}V. \end{aligned}$$

Similarly,

$$\begin{aligned} {}_{2.833}V\,(1.05)^{0.083} &= 500000 {}_{0.083}q_{52.833} + {}_{0.083}p_{52.833} \ {}_{2.917}V \\ {}_{2.917}V\,(1.05)^{0.083} &= 500000 {}_{0.083}q_{52.917} + {}_{0.083}p_{52.917} \ {}_{3}V. \end{aligned}$$

VALUATION BETWEEN PREMIUM DATES

We often need to calculate policy values between premium dates.

★ Because we will value **all policies on the same calendar day** each year as part of the insurer's liability valuation process.

example

For the contract described in 9th **example**, calculate the policy value after

(A) 2 yeas and 10 months and

(B) 2 yeas and 9.5 months

assuming the policy is still in force at that time in each case.

Solution

(A) The *EPV* of future benefits is

$$SA^{(12)}_{\overset{1}{52\frac{10}{12}:\overline{7\frac{2}{12}}|}} = S(0.0132012) = 6600.58; \quad S = 500000.$$

The *EPV* of future premiums less premium expenses is

$$0.9(4)P[\nu^{\frac{1}{12}}\ {}_{\frac{2}{12}}p_{52\frac{10}{12}}]\ddot{a}^{(4)}_{53:\overline{2}|} = 0.9(4)P(1.898466) = 3143.86.$$

So the policy value is ${}_{2\frac{10}{12}}V = \$3456.72$.

- ★ Note that $A^{(m)}_{\overset{1}{x:\overline{n}|}}$ and $\ddot{a}^{(m)}_{x:\overline{n}|}$ are defined only if n is an integer multiple of $\frac{1}{m}$, so that $A^{(12)}_{\overset{1}{52\frac{10}{12}:\overline{7\frac{2}{12}}|}}$ is well defined, but $\ddot{a}^{(4)}_{\frac{10}{12}:\overline{\frac{2}{12}}|}$ is not.
- ★ The comparison time is at $t = 2\frac{10}{12}$.

Solution (conti'n)

(B) Now, the valuation date is at neither a benefit nor a premium date.

$$\because\ EPV(\text{benefits - premiums at } t = 2\tfrac{10}{12} \text{ years}) = {}_{2\frac{10}{12}}V,$$

$$\therefore\ EPV(\text{benefits - premiums at } t = 2\tfrac{19}{24} \text{ years})$$

$$= \begin{cases} {}_{2\frac{10}{12}}V\,\nu^{\frac{1}{24}} & \text{if the life survives at } t = 2\frac{19}{24}; \\ S\nu^{\frac{1}{24}} & \text{if the life dies at } t = 2\frac{19}{24}. \end{cases}$$

Therefore, the policy value at 2 yeas and 9.5 months is

$${}_{2\frac{19}{24}}V = {}_{\frac{1}{24}}q_{52\frac{19}{24}}\,S\nu^{\frac{1}{24}} + {}_{\frac{1}{24}}p_{52\frac{19}{24}}\,({}_{2\frac{10}{12}}V)\,\nu^{\frac{1}{24}} = \$3480.99. \qquad \square$$

Notation

- It would **NOT** be appropriate to apply simple interpolation to the two policy values corresponding to the premium dates before and after the valuation date.
  For our example,

  $${}_{2\frac{9}{12}}V = \$3091.02, \ {}_{2\frac{19}{24}}V = \$3480.99 \ \text{ and } \ {}_{3}V = \$3357.94;$$

  is not linear.
- Function ${}_tV$ is not smooth if premiums are paid at discrete intervals, since the policy value will jump immediately after each premium payment by the amount of that payment.
- A reasonable approximation to the policy value between premium dates can be achieved by interpolation between the policy value **just after the previous premium** and the policy value **just before the next premium**.
  That is, suppose the premium dates are k years apart, for $s < k$, we have

  $${}_{t+k+s}V \ \approx \ ({}_{t+k}V + P_{t+k} - E_{t+k})\left(1 - \frac{s}{k}\right) + {}_{t+2k}V\left(\frac{s}{k}\right).$$

POLICY VALUES WITH CONTINUOUS CASH FLOWS

THIELE'S DIFFERENTIAL EQUATION

Policy values with successive cash flow time points usually extend to contracts where regular payments and sums insured are payable immediately on death.

- ★ *Thiele's differential equation* is a continuous time version of the recursive equation.
  Recall that for discrete life insurance (6)

$$({}_tV + P_t - e_t)(1 + i_t) = {}_{t+1}V + q_{[x]+t}(S_{t+1} + E_{t+1} - {}_{t+1}V).$$

POLICY VALUES

Consider a policy

- issued to a select life aged x;
- premiums and premium-related expenses are payable continuously;
- the sum insured, together with any related expenses, is payable immediately on death;
- has been in force for t years, $t \geq 0$.

Let

P_t	the annual rate of premium payable at t;
e_t	the annual rate of payment-related expense payable at t;
S_t	the sum insured at time t if the policyholder dies at exact t;
E_t	the expense of paying the sum insured at t;
$\mu_{[x]+t}$	the force of mortality at age $[x]+t$;
δ_t	force of interest per year assumed earned at t;
${}_tV$	the policy value for a policy in force at t.

★ P_t, e_t, S_t, $\mu_{[x]+t}$, δ_t: continuous functions of t;

★ e_t, E_t, $\mu_{[x]+t}$, δ_t: assumed in the policy value basis.

Thus, we have

- $\nu(t)$ = the present value of payment 1 at t,

$$\nu(t) = \exp\left\{-\int_0^t \delta_s ds\right\}; \tag{7}$$

- ${}_tV$= EPV (benefits + benefit related expenses)-EPV(premium-premium related expenses),

$${}_tV = \int_0^\infty \frac{\nu(t+s)}{\nu(t)}(S_{t+s}+E_{t+s})_s p_{[x]+t}\mu_{[x]+t+s}ds - \int_0^\infty \frac{\nu(t+s)}{\nu(t)}(P_{t+s}-e_{t+s})_s p_{[x]+t}ds$$

★ The sum insured is payable at $t+s$ and we are discounting back to t.

Let $r = t+s$, gives

$${}_tV = \int_t^\infty \frac{\nu(r)}{\nu(t)}(S_r + E_r)_{r-t}p_{[x]+t}\mu_{[x]+r}dr - \int_t^\infty \frac{\nu(r)}{\nu(t)}(P_r - e_r)_{r-t}p_{[x]+t}dr \tag{8}$$

We can calculate ${}_tV$ by numerical integration. However, we are going to turn this identity into a differential equation.

$$\because \ {}_{r-t}p_{[x]+t} = \frac{{}_rp_{[x]}}{{}_tp_{[x]}} \quad (why?)$$

so that

$${}_tV = \frac{1}{\nu(t)\,{}_tp_{[x]}}\left(\int_t^\infty \nu(r)(S_r + E_r)\,{}_rp_{[x]}\mu_{[x]+r}dr - \int_t^\infty \nu(r)(P_r - e_r)\,{}_rp_{[x]}dr\right)$$

which we can write as

$$\nu(t)\,{}_tp_{[x]}\,{}_tV = \int_t^\infty \nu(r)(S_r + E_r)\,{}_rp_{[x]}\mu_{[x]+r}dr - \int_t^\infty \nu(r)(P_r - e_r)\,{}_rp_{[x]}dr \qquad (9)$$

Differentiation of (9) w.r.t. t leads toThiele's differential equation.

First, the differentiation of the RHS of (9) yields

$$-\nu(t)(S_t+E_t)_t p_{[x]}\mu_{[x]+t}+\nu(t)(P_t-e_t)_t p_{[x]} = \nu(t)_t p_{[x]}\left[P_t - e_t - (S_t + E_t)\mu_{[x]+t}\right]. \quad (10)$$

Differentiation of the LHS is done by product rule. First

$$\frac{d}{dt}(\nu(t)_t p_{[x]t}V) = \nu(t)_t p_{[x]}\frac{d}{dt}{}_tV + {}_tV\frac{d}{dt}(\nu(t)_t p_{[x]}).$$

Next

$$\frac{d}{dt}(\nu(t)_t p_{[x]}) = \nu(t)\frac{d}{dt}{}_t p_{[x]} + {}_t p_{[x]}\frac{d}{dt}(\nu(t).$$

We know that

$$\frac{d}{dt}{}_t p_{[x]} = -{}_t p_{[x]}\mu_{[x]+t} \quad (why?)$$

and by (7)

$$\frac{d}{dt}\nu(t) = -\delta_t \exp\left[-\int_0^t \delta_s ds\right] = -\delta_t \nu(t).$$

Thus, the derivative of the LHS of (9) is

$$\begin{aligned}\frac{d}{dt}(\nu(t)\,{}_tp_{[x]}\,{}_tV) &= \nu(t)\,{}_tp_{[x]}\frac{d}{dt}{}_tV - {}_tV\left[\nu(t)\,{}_tp_{[x]}\mu_{[x]+t} + {}_tp_{[x]}\delta_t\nu(t)\right] \\ &= \nu(t)\,{}_tp_{[x]}\left(\frac{d}{dt}{}_tV - {}_tV(\mu_{[x]+t}+\delta_t)\right).\end{aligned}$$

Equating this to (10) yields **Thiele's differential equation**, namely

$$\boxed{\frac{d}{dt}{}_tV = \delta_t\,{}_tV + P_t - e_t - (S_t + E_t - {}_tV)\mu_{[x]+t}.} \qquad (11)$$

The LHS of (11), $\frac{d}{dt}{}_tV$, is the rate of increase in the policy value at t. We can derive a formula for this rate of increase by considering the individual factors affecting ${}_tV$ as following.

By (11), we have

$$h\frac{d}{dt}{}_tV = h\delta_t {}_tV + h(P_t - e_t) - h(S_t + E_t - {}_tV)\mu_{[x]+t}$$

★ From time t to $t+h$, the interest earned is $h\delta_t {}_tV$, so that the rate of increase at t is $\delta_t {}_tV$.

Then,

$$h\frac{d}{dt}{}_tV + h\mu_{[x]+t}(S_t + E_t - {}_tV) = h\delta_t {}_tV + h(P_t - e_t)$$

because

$$\frac{d}{dt}{}_tV \approx \frac{1}{h}({}_{t+h}V - {}_tV) \tag{12}$$

$$\Rightarrow \quad ({}_{t+h}V - {}_tV) + h\mu_{[x]+t}(S_t + E_t - {}_tV) \approx h\delta_t {}_tV + h(P_t - e_t)$$

$$\Rightarrow \quad (1 + h\delta_t){}_tV + h(P_t - e_t) \approx {}_{t+h}V + h\mu_{[x]+t}(S_t + E_t - {}_tV).$$

POLICY VALUES

- Because $h \to 0$ the LHS is, from time t to $t+h$,

$$
\begin{aligned}
&(\text{The accumulation of the policy value at } t)\\
&+(\text{the accumulation of the premium income})\\
&-(\text{premium-related expenses in } (t,\ t+h)).
\end{aligned}
$$

- This total accumulation must provide

$$
\begin{aligned}
&(\text{the policy value at } t+h,)\\
&\qquad\qquad \text{if death occurs in } (t,\ t+h),\\
&+(\text{ the excess } S_t + E_t - {}_tV \text{ over the policy value}).
\end{aligned}
$$

The probability of death in $(t,\ t+h)$ is approximately $h\mu_{[x]+t}$.

NUMERICAL SOLUTION OF THIELE'S DIFFERENTIAL EQUATION

To evaluate policy values by Thiele's differential equation, the key is to apply (12) as an identity rather than an approximation, assuming $h \to 0$.
This leads to

$$ {}_{t+h}V - {}_{t}V = h[\delta_{t\,t}V + P_t - e_t - \mu_{[x]+t}(S_t + E_t - {}_{t}V)]. \ (why?) \tag{13} $$

- ★ The smaller the value of h, the better this approximation is likely to be.
- ★ If δ_t, P_t, e_t, $\mu_{[x]+t}$, S_t, E_t are known then (13) allows us
  - calculate ${}_{t}V$ provided we know ${}_{t+h}V$ or
  - calculate ${}_{t+h}V$ provided we know ${}_{t}V$.

★ But we always know ${}_tV$ as t approaches the end of the policy term since, in the limit, it is the amount that should be held in respect of a policyholder who is still alive.

- For an **endowment policy** with term n years and sum insured S.

$$\lim_{t \to n^-} {}_tV = S. \quad (why?)$$

- For an **term insurance** with term n years and sum insured S.

$$\lim_{t \to n^-} {}_tV = 0. \quad (why?)$$

- For an **whole life insurance** with sum insured S.

$$\lim_{t \to \omega^-} {}_tV = S. \quad (why?)$$

Using endowment policy as an example, (13) with $t = n - h$ gives us

$$S - {}_{n-h}V = h[\delta_{n-h}\,{}_{n-h}V + P_{n-h} - e_{n-h} - \mu_{[x]+n-h}(S_{n-h} + E_{n-h} - {}_{n-h}V)],$$

from which we can have ${}_{n-h}V$. Another application of (13) with $t = n - 2h$ gives ${}_{n-2h}V$, and so on.
This method for the numerical solution of a differential equation is known as **Euler's method**.

example

Consider a 20-year endowment insurance issued to a life aged 30. The sum insured, \$100000, is payable immediately on death, or on survival to the end of the term, whichever occurs sooner. Premiums are payable continuously at a constant rate of \$2500 per year throughout the term of the policy. The policy value basis uses a constant force of interest, δ, and makes no allowance for expenses.

(A) Evaluate ${}_{10}V$.

(B) Use Euler's method with $h = 0.05$ years to calculate ${}_{10}V$.

$\delta = 0.04$ per year.

Solution

(A) We have

$$ {}_{10}V = 100000\bar{A}_{40:\overline{10}|} - 2500\,\bar{a}_{40:\overline{10}|} $$

and as

$$ \bar{A}_{40:\overline{10}|} = 1 - \delta\,\bar{a}_{40:\overline{10}|}, $$

we have

$$ {}_{10}V = 100000 - (100000\delta + 2500)\,\bar{a}_{40:\overline{10}|} \quad \text{and} \quad \bar{a}_{40:\overline{10}|} = 8.2167, $$

hence ${}_{10}V = 46591$.

(B)

$$ \because\ \delta_t = 0.004,\ e_t = 0 = E_t,\ P_t = 2500,\ \mu_{45.95} = 0.0011471, $$

$$ \begin{aligned} \Rightarrow \quad & 100000 - {}_{19.95}V \\ = \quad & (0.05)[(0.04)\,{}_{19.95}V + 2500 - (0.003204)(100000 - {}_{19.95}V)] \\ \Rightarrow \quad & {}_{19.95}V = 99676. \end{aligned} $$

$\therefore$ calculating recursively ${}_{19.90}V, {}_{19.85}V,$ we have $\cdots, {}_{10}V = 46635.$ $\square$

POLICY ALTERATIONS

So far we have assumed that the terms of the contract are never broken or altered in any way. In practice, it is not uncommon for the policyholder to request a change in the terms of the policy.

(1) The policyholder wishes to cancel the policy with immediate effect.

- The insurance company may pay a lump sum immediately to the policyholder; If the policy has a significant investment component (e.g. endowment insurance or whole life insurance).
- This kind of policy is said to **lapse** or to be **surrendered**.
- The immediate payment by the insurance is called a **surrender value** or a **cash value**.

Usually, we use

lapse	:	a voluntary cessation with no surrender value;
surrendered	:	with a return of assets of some amount to the policyholder.

(2) The policyholder wishes to pay no more premiums but does not want to cancel the policy.
- Any policy for which no further payments are payable is said to be **paid-up**.
- The reduced sum insured for a paid-up policy is called a **paid-up sum insured**.

(3) A whole life policy may be converted to a paid-up term insurance policy for the original sum insured.

(4) Other types of alternations may be requested:
reducing or increasing premiums; changing the amount of the benefits; converting a whole life insurance to an endowment insurance; converting a non-participating policy to a with-profit policy; etc.

★ These changes are common requested by the policyholder and were not part of the original terms of the policy.

★ If a contract is issued with a substantive investment objective, rather than solely offering protection against ultimately death, then at least part of the funds should be the policyholder's under the stewardship of the insurer. (it is the reason why insurance companies agree to the alterations)

In particular, fixed or minimum cash surrender values, as a percentage of the sum insured, are specified in advance in the contract terms.

Let CV_t be the **cash surrender value** at duration t.

- If surrender values are not set in advance, actuary would determine an appropriate value for CV_t at the time of alteration.

★ Calculating CV_t by ${}_tV$ if CV_t is calculated in advance;

★ Calculating CV_t by AS_t if CV_t is not pre-specified.

Recall

- AS_t: the cash the insurer actually has at t;
- V_t: the amount the insurer should have at t;

If the policy value basis is close to the actual experience, then V_t will be numerically close to AS_t.

Setting CV_t equal to either AS_t or V_t could be regarded as over-generous:

- The insurer should insure that surrendering policyholders do not benefit at the expense of the continuing policyholders. That is, the policyholder may be acting on knowledge that is not available to the insurer. For example, a policyholder may alter a whole life policy to a term insurance if he or she becomes aware that their health is failing.
  This is called **anti-selection** or **selection against the insurer**.
- The insurance company will incur some expenses in making the alterations to the policy.
- The alteration may, at least in principle, cause the insurance company to realize assets it would otherwise have held, especially if the alteration is a surrender.
  This **liquidity risk** may lead to reduced investment returns for the company.

For these reasons, CV_t is usually less than 100% of either AS_t or V_t and may include an explicit allowance for the expense of making the alteration.

For alterations other than cash surrenders, we can apply CV_t as if

- it were a single premium,
- or an extra preliminary premium,

for the future benefits. That is

$$\begin{aligned} & CV_t + EPV \text{ at } t \text{ of future premiums, altered contract} \\ = \; & EPV \text{ at } t \text{ of future benefits plus expenses, altered contract.} \end{aligned} \tag{14}$$

★ The rationale behind (14) is :
together with the cash currently available, CV_t, the future premiums are expected to provide the future benefits and pay for the future expenses.

example

Consider the policy discussed in the 4th and 9th **example**'s. You are given hat the insurer's experience in the five years following the issue of this policy is as in the 9th **example**. At the start of the sixth year, before paying the premium then due, the policyholder requests that the policy be altered on one of the following three ways.

(A) The policy is surrendered immediately.
(B) No more premiums are paid and a reduced annuity is payable from 60. In this case, all premiums paid are refunded at the end of the year of death if the policyholder dies before age 60.
(C) Premiums continue to pay, but benefit is altered from an annuity to a lump sum (pure endowment) payable on reaching 60. Expenses and benefits on death before age 60 follow the original policy terms. There is an expense of \$100 associated with paying the sum insured at the new maturity date.

Calculate

(A) the surrender value,
(B) the reduced annuity and
(C) the sum insured .

assuming the insurer uses

(I) 90% of the asset share less a charge of \$200, or
(II) 90% of the policy value less a charge of \$200

together with the assumptions the policy value basis when calculating revised benefits and premiums.

Solution

We have

$$ {}_5V = 65470 \text{ and } AS_5 = 63509. $$

(A) By (14), the amount CV_5 is

$$ 0.9(AS_5) - 200 = 56958, \quad 0.9({}_5V) - 200 = 58723. $$

The surrender values are CV_5, so we have

$$ \text{(i)\$56958}, \quad \text{(ii)\$58723}. $$

(B) Let X be the revised annuity amount. By (14),

$$ CV_5 \text{ (no premium) } = 5(11900 A^{1}_{55:\overline{5}|}) + 100 A^{1}_{55:\overline{5}|} + (X+25)\nu^5 {}_5p_{55}\, \ddot{a}_{60}. $$

So we have

$$ \text{(i)\$4859}, \quad \text{(ii)\$5012}. $$

(C) Let S be the new sum insured. By (14),

$$ CV_5 + 0.95(11900 \ddot{a}_{55:\overline{5}|}) = 11900[(IA)^{1}_{55:\overline{5}|}) + 5A^{1}_{55:\overline{5}|}] + 100A^{1}_{55:\overline{5}|} + \nu^5 {}_5p_{55}(S+100). $$

So we have

$$ \text{(i)\$138314}, \quad \text{(ii)\$140594}. \qquad \square $$

example

Ten years ago a man aged 40 purchased a with-profit whole life insurance. The basic sum insured, payable at the end of the year of death, was \$200000. Premiums of \$1500 were payable annually for life.
The policyholder now requests that the policy be changed to a with-profit endowment insurance with a remaining term of 20 years, with same premiums payable annually, but now for maximum of 20 further years.
The insurer uses the following basis for calculation of policy value and policy alterations,

Interest	:	5% per year
Expenses	:	None
Bonus	:	Compound reversionary bonus at rate 1.2% per year at the start of each policy year, including the first.

The insurer uses the **full policy value less an expense of** \$100 when calculating revised benefits. You are given the actual bonus rate declared in each of the past 19 years has been 1.6%.

(A) Calculate the revised sum insured, to which future bonuses will be added, assuming the premium now due has not been paid and the bonus now due has not been declared.

(B) Calculate the revised sum insured, to which future bonuses will be added, assuming the premium now due has not been paid and the bonus now due has been declared to be 1.6% .

Solution

(A) Before the declaration of the bonus now due, the sum insured for the original policy is

$$200000(1.016)^{10} = 234405.$$

Hence, the policy value for the original policy, ${}_{10}V$, is

$${}_{10}V = 234404 A_{40\ j} - P\ddot{a}_{40}$$

where $P = 1500$ and

$$\because (1 + 0.012)k = 1.05 \ \Rightarrow \ k = 1.0375494$$

the subscript j indicates the rate of interest to be used is 3.75494%. Let S be the revised sum insured. Then, by (14),

$${}_{10}V - 1000 = S\ A_{40:\overline{20}|j} - P\ddot{a}_{40:\overline{20}|}. \tag{15}$$

Because

$$\begin{aligned} A_{40\ j} &= 0.19569, & \ddot{a}_{40} &= 18.4578, \\ A_{40:\overline{20}|j} &= 0.48233, & \ddot{a}_{40:\overline{20}|} &= 12.9935. \end{aligned}$$

Hence, $S = \$76039$.

Solution (conti'n)

(B) Let ${}_{10^+}V$ be the policy value just after the premium has been paid and the bonus has been declared at $t = 10$.
The sums insured are

$t = 11$, 234405(1.016) ($\because$ Given the bonus declared at $t = 10$ is 1.6%)
$*$ the value is known

$t = 12$, 234405(1.016)(1.012) $*$it is an assumed value since it assumes the bonus declared at the start of the 12th year will be 1.2%.

Hence,

$$\begin{aligned} {}_{10^+}V &= \left(\tfrac{1.016}{1.012}\right) 234405 A_{40\ j} - P a_{40} \\ &= \left(\tfrac{1.016}{1.012}\right) 234405 A_{40\ j} - P \ddot{a}_{40} + P. \end{aligned}$$

Let S' be the revised sum insured for the endowment policy. Then, by (14),

$$\begin{aligned} {}_{10^+}V - 1000 &= \left(\frac{S'}{1.012}\right) A_{40:\overline{20}|j} - P a_{40:\overline{19}|} \\ &= \left(\frac{S'}{1.012}\right) A_{40:\overline{20}|j} - P \ddot{a}_{40:\overline{20}|} + P. \end{aligned}$$

Hence,

$$S' = \$77331. \qquad \square$$

RETROSPECTIVE POLICY VALUES

PROSPECTIVE AND RETROSPECTIVE VALUATION

A *policy value* is also called a **prospective policy value** for looking to the future.

Retrospective Policy Value

We define the **retrospective policy value** at duration t as

$$\frac{\text{the accumulated value of the } \textbf{past premiums} \text{ received} \; - \; \text{the value of the } \textbf{past insurance}}{\text{expected number of survivors}}$$

assuming the experience follows precisely the assumptions in the policy value basis.

Recall

- ★ The purpose of the **policy value** is to assess future needs, it is natural to take the *prospective* approach.
- ★ Policy value is what actuaries call the **reserve** at t for the policy.
- ★ **Asset share** tracks the accumulated contribution of each surviving policy to the insurer's funds.

The **retrospective policy value** measures the value at t of all the cash flows from time 0 to t, expressed per surviving policyholder. It is connected with **asset share**.
The differences between **retrospective policy value** and **asset share** are

- ★ The **asset share** at t uses the *actual experience* up to t. It can **NOT** be calculated until t.
- ★ The **retrospective policy value** can use any basis, and can be calculated at any time.
  - If the retrospective policy value basis exactly matches the experience, then it will be equal to the asset share.

- ★ The **prospective policy value** measures the funds *needed* at t.

  $$* \text{ Policy value} = E(\text{ future loss random variable}).$$

- ★ The **retrospective policy value** measures the funds *expected to be acquired* at t.
- ★ The **reserve** must be prospective to meet natural requirements that *assets should be sufficient to meet future liabilities*.
- ★ We have an exact measure of **asset share** at t.

Under very specific conditions, the prospective policy value and retrospective policy value are equal.

Let, for an n-year insurance policy,

$$
\begin{array}{lcl}
{}_tV^R & : & \text{the retrospective policy value at } t. \\
{}_tV^P & : & \text{the prospective policy value at } t.
\end{array}
$$

The following two conditions for ${}_tV^P$ to be equal to ${}_tV^R$:

- the premiums for the contract are determined using the equivalence principle, and
- the same basis is used for ${}_tV^R$ and ${}_tV^P$ and the equivalence principle premium.

$*$ In most cases, these conditions are very unlikely to be satisfied.

DEFINING THE RETROSPECTIVE NET PREMIUM POLICY VALUE

Consider an insurance sold to (x) at $t=0$ with term n (n may be ∞ for a whole life contract). For a policy in force at t, let

- $$L_t = \text{PV}[(\text{all the future benefits}) - (\text{net premiums})] \text{ at } t.$$

  Therefore, the prospective policy value at $t<n$ is

  $$_tV^P = E(L_t).$$

  If (x) does not survive to t then L_t is undefined.

- We have the random variable

$$\begin{aligned} & I(T_x > t)\nu^t L_t \\ = & \quad \text{The value at issue of} \\ & [(\text{all the future benefits}) - (\text{premiums payable from } t \text{ onwards})], \end{aligned}$$

  where I is the indicator function.

- For $t \leq n$, define

$$\begin{aligned} & L_{0,t} \\ = \quad & PV(\text{At issue, of future benefits payable up to } t) \\ & - PV(\text{At issue, of future net premiums payable up to } t). \end{aligned}$$

  * If premiums and benefits are paid at discrete intervals, and t is a premium or benefit payment date, then

    $L_{0,t}$ would include benefits payable at t, but not premiums.

- At issue (time 0),

$$L_0 = L_{0,t} + I(T_x > t)\nu^t L_t.$$

We now define

$$ {}_tV^R = \frac{-E[L_{0,t}](1+i)^t}{{}_tp_x} = \frac{-E[L_{0,t}]}{{}_tE_x}. $$

* Recall the conditions for equality of the retrospective and prospective values
  (1) the premium is calculated using the equivalence principal,
  (2) the same basis is used for prospective policy values, retrospective policy value and the equivalence principal premium.

By the equivalence principal

$$
\begin{aligned}
& E[L_0] = E[L_{0,t} + I(T_x > t)\nu^t L_t] = 0 \\
\Rightarrow\quad & -E[L_{0,t}] = E[I(T_x > t)\nu^t L_t] \\
\Rightarrow\quad & -E[L_{0,t}] = {}_tp_x \nu^t {}_tV^P \ (why?) \\
\Rightarrow\quad & {}_tV^R = {}_tV^P. \qquad (16)
\end{aligned}
$$

example

An insurer issues a whole life insurance policy to a life aged 40. The death benefit for the first five years of the contract is $5000. In subsequent years, the death benefit is $100000. The death benefit is payable at the end of the year of death. Premiums are paid annually for a maximum of 20 years. Premiums are level for the first five years, then increase by 50%.

(A) Write the equation of value for calculating the net premium, using standard actuarial function.

(B) Write equations of value for the premium policy value at $t = 4$ using
  (I) the retrospective policy value approach;
  (II) the prospective policy value approach.

(C) Write equations of value for the premium policy value at $t = 20$ using
  (I) the retrospective policy value approach;
  (II) the prospective policy value approach.

Solution

(A) The equivalence principal premium is P for the first 5 years and $1.5P$ thereafter, where

$$P = \frac{5A^{\;1}_{40:\overline{5}|} + 100\,{}_5E_{40}\,A_{45}}{\ddot{a}_{40:\overline{5}|} + 1.5\,{}_5E_{40}\,\ddot{a}_{45:\overline{15}|}}. \tag{17}$$

(B) At $t = 4$,

The retrospective policy value equation is

$${}_4V^R = \frac{P\ddot{a}_{40:\overline{4}|} - 5A^{\;1}_{40:\overline{4}|}}{{}_4E_{40}}. \tag{18}$$

The prospective policy value equation is

$${}_{20}V^P = 5A^{\;1}_{44:\overline{1}|} + 100\,{}_1E_{44}\,A_{45} - P\left(\ddot{a}_{44:\overline{1}|} + 1.5\,{}_1E_{44}\,\ddot{a}_{45:\overline{15}|}\right). \tag{19}$$

(C) At $t = 20$, The retrospective policy value and prospective policy value equations are

$${}_{20}V^R = \frac{\left(P\ddot{a}_{40:\overline{5}|} + 1.5\,{}_5E_{40}\,\ddot{a}_{45:\overline{15}|}\right) - 5A^{\;1}_{40:\overline{5}|} - 100\,{}_5E_{40}\,A_{45:\overline{15}|}}{{}_{20}E_{40}} \quad \text{and} \quad {}_{20}V^P = 100A_{60}.$$

example

For **example** above, show that the retrospective policy value and prospective policy value at $t = 4$, given in (18) and (19), are equal under the standard assumptions (premium and policy values are all use the same basis, and the equivalence principal premium).

Solution

assuming all calculations use the same basis,

$$\begin{aligned}
A^{1}_{40:\overline{5}|} &= A^{1}_{40:\overline{4}|} + {}_4E_{40}\, A^{1}_{44:\overline{1}|} \\
\ddot{a}_{40:\overline{5}|} &= \ddot{a}_{40:\overline{4}|} + {}_4E_{40}\, \ddot{a}_{44:\overline{1}|} \\
{}_5E_{40} &= {}_4E_{40}\, {}_1E_{44}.
\end{aligned}$$

Now we use these to rewrite (17),

$$\begin{aligned}
& P\left(\ddot{a}_{40:\overline{5}|} + 1.5\, {}_5E_{40}\, \ddot{a}_{45:\overline{15}|}\right) = 5A^{1}_{40:\overline{5}|} + 100\, {}_5E_{40}\, A_{45} \\
\Rightarrow\quad & P\left(\ddot{a}_{40:\overline{4}|} + {}_4E_{40}\, \ddot{a}_{44:\overline{1}|} + 1.5\, {}_4E_{40}\, {}_1E_{44}\, \ddot{a}_{45:\overline{15}|}\right) = 5\left(A^{1}_{40:\overline{4}|} + {}_4E_{40}\, A^{1}_{44:\overline{1}|}\right) + 100\, {}_4E_{40}\, {}_1E_{44}\, A_{45}.
\end{aligned}$$

Rearranging gives

$$P\ddot{a}_{40:\overline{4}|} - 5A^{1}_{40:\overline{4}|} = {}_4E_{40}\left[5A^{1}_{44:\overline{1}|} + 100\, {}_1E_{44}\, A_{45} - P\left(\ddot{a}_{44:\overline{1}|} + 1.5\, {}_1E_{44}\, \ddot{a}_{45:\overline{15}|}\right)\right]$$

Dividing both sides by ${}_4E_{40}$ gives ${}_4V^R = {}_4V^P$ as required. $\square$

NEGATIVE POLICY VALUES

Negative gross policy values are not unusual **in the first few months** of a contract.

- It would be unusual for policy values to be negative after the early period of the contract.

Recall that

$$ {}_tV = EPV_{\text{future benefits at } t} + \text{Expenses} - EPV_{\text{future premiums at } t}, $$

the only way for a negative value to arise is if the **future benefits are worth less than the future premiums**.

- In practice, negative policy values would generally be set to zero when carrying out a valuation of the insurance company.

POLICY VALUES

Negative policy values arise when a contract is poorly design, so that the value of benefits in early years exceeds the value of premiums.

- ★ If the policyholder lapses then the policyholder will have benefitted from the higher benefits in the early years without waiting around to pay for the benefit in the later years.
- ★ The policyholder may be able to achieve the same benefit at a cheaper price by lapsing and buying a new policy- called the **lapse and re-entry option**.

DEFERRED ACQUISITION EXPENSES AND MODIFIED PREMIUM RESERVES

While most jurisdictions use a **gross premium policy value** approach to the principles of reserve calculation, the **net premium policy value** is still used in the USA.

- The use of the net premium policy value approach can offer some advantages in computation and in smoothing results.However, it can be quite severe standard when there are large initial expenses, called **acquisition expenses** (commission, underwriting and administrative), incurred by the insurer.

★ To reduce the impact, the reserve is not calculated directly as the net premium policy value, but can be modified, to approximate a gross premium policy value approach, whilst maintaining the advantages of the net premium policy value approach.

Let

$$
\begin{aligned}
{}_tV^n &: \quad \text{net premium policy for a contract in force } t \text{ years after issue;} \\
{}_tV^g &: \quad \text{gross premium policy for the same contract.}
\end{aligned}
$$

Thus

$$
\begin{aligned}
{}_tV^n &= EPV_{\text{future benefits}} - EPV_{\text{future net premiums}} \\
{}_tV^g &= EPV_{\text{future benefits}} + EPV_{\text{future expenses}} \\
&\quad - EPV_{\text{future gross premiums}} \\
{}_0V^n &= {}_0V^g = 0.
\end{aligned}
$$

So we have

$$
\begin{aligned}
{}_tV^g = \quad & EPV_{\text{future benefits}} + EPV_{\text{future expenses}} \\
& - \left(EPV_{\text{future net premiums}} + EPV_{\text{future expense loadings}} \right).
\end{aligned}
$$

That is,

$$
{}_tV^g = {}_tV^n + {}_tV^e,
$$

where ${}_tV^e$ is the **expense** policy value,

$$
{}_tV^e = EPV_{\text{future expenses}} - EPV_{\text{future expense loadings}}.
$$

POLICY VALUES

★ In general, ${}_tV^e$ is negative. That is

$$ {}_tV^n > {}_tV^g; \quad \forall t > 0. $$

- This is because the gross premium approach offsets the *higher future outgo* with the *higher future premiums*.

- If expenses incurred and premiums are *level* through out the policy terms, then

$$ \text{the net premium} = \text{the gross premium}. $$

- This is because the extra expenses valued in the gross premium case would be exactly offset by the extra premiums.

★ In general, expenses are **NOT** incurred at a flat rate. This results in ${}_tV^e < 0$.

Let

P^g : Gross premium for a level premium contract.
P^n : Net premium for the same level premium contract.
P^e : The **expense loading** (or expense premium) for the contract

$$P^e = P^g - P^n.$$

- P^e is the level annual amount paid by the policyholder to cover the policy expenses.
- If expenses are incurred as a level sum, P^e would equal those incurred expenses.
- If expenses are weighted to the start of the contract, as in general cases, then P^e will be greater than the renewal expenses as it must fund both the renewal and initial expenses.

example

An insurer issues a whole life insurance policy to a life aged 50. The sum of insured of $100000 is payable at the end of the year of death. Level premiums are payable annually in advance throughout the term of the contract. All premiums and policy are calculated using the Standard Select Survival Model, and an interest rate of 4% per year effective. Initial expenses are 50% of the gross premium plus $250. Renewal expenses are 3% of the gross premium plus $25 at each premium date after the first. Calculate

(A) the expense loading, P^e;

(B) ${}_{10}V^e$, ${}_{10}V^n$ and ${}_{10}V^g$.

Solution

(A) The expense loading, P^e, depends on the gross premium, P^g, which we calculate first first as

$$P^g = \frac{100000 A_{[50]} + 25 \ddot{a}_{[50]} + 225}{0.97 \ddot{a}_{[50]} - 0.47} = \$1435.89.$$

P^e can be calculated by the following two ways:

1. Find the *EPV* of future expenses, and calculate the premium to fund those expenses. That is

$$P^e \ddot{a}_{[50]} = 25 \ddot{a}_{[50]} + 225 + 0.03 P^g \ddot{a}_{[50]} + 0.47 P^g.$$

2. Calculate

$$P^n = 100000 \frac{A_{[50]}}{\ddot{a}_{[50]}} = 1321.31$$

and $P^e = P^g - P^n = \$114.58$.

Solution (Cont'n)

(A) (Cont'n)
Compare the expense premium with the incurred expense.

- The annual renew expenses are

$$1435.89(0.03) + 25 = \$68.08.$$

- The rest of the expense loading, $114.58 - 68.08 = \$46.50$ at each premium date, reimburses the *acquisition expenses*,

$$1435.89(0.5) + 250 = \$967.94$$

at inception.

Thus, at any premium date after the first, the value of the *future expenses* will smaller than the *future expense loadings*. That is ${}_tV^e < 0$.

Solution (Cont'n)

(B) The expense reserve at $t = 10$ for an in-force contract is

$$_{10}V^e = (25\ddot{a}_{60} + 0.03P^g\ddot{a}_{60}) - P^e\ddot{a}_{60} = -46.50P^g\ddot{a}_{60} = -770.14,$$

the net premium policy value is

$$_{10}V^n = 100000A_{60} - P^n\ddot{a}_{60}) = 14416.12,$$

and the gross premium policy value is

$$_{10}V^g = 100000A_{60} + 25\ddot{a}_{60} - 0.97P^g\ddot{a}_{60} = 13645.98.$$

As we expected, the expense reserve is negative, and that

$$_{10}V^g = {}_{10}V^n + {}_{10}V^e.$$

- The negative expense reserve is referred to as **deferred acquisition cost**, or **DAC**.
  - The use of the *net premium reserve* can be viewed as being overly conservative, as it does not allow for the DAC reimbursement.
  - The idea is that an insurer should not be required to hold the full net premium policy value as a reserve, when the true future liability value is smaller because of the DAC.
- One solution would to use a *gross premium reserve*.
- An alternative method, which maintains most of the numerical simplicity of the net premium approach, is to modify the net premium method to allow DAC as approximate correction.
  - The most common method of adjusting the net premium policy value is the **Full Preliminary Term** (**FPT**) approach.

POLICY VALUES

Consider a whole life insurance with level annual premiums payable throughout the term. Let

$P_{[x]+s}$:	the net premium for a contract issued to a life $[x]+s$, who was select at x.
${}_1P_{[x]}$:	the single premium to fund the benefits payable during the first year of the contract, called the **first year Cost of Insurance**.

The FPT reserve for a contract issued to a select life x is the *net premium policy value* assuming that the net premium in the first year is ${}_1P_{[x]}$ and in all subsequent year is $P_{[x]+1}$.

- ★ This is equivalently considering the policy as two policies, a one-year term insurance, and a separate contract issued to the same life one year later, if the life survives.

example

(A) Calculate the premiums ${}_1P_{[50]}$ and $P_{[50]+1}$ for the previous **example**.

(B) Compare the net premium policy value, the gross policy value and the FPT reserve in the previous **example** at durations 0,1 and 10.

Solution

(A) At $t = 0$,

$$ {}_1P_{[50]} = 100000 A^{\;1}_{50:\overline{1}|} = 100000 \nu q_{[50]} = 99.36. $$

The modified net premium assumed paid at all subsequent premium dates is

$$ P_{[50]+1} = \frac{100000 A_{[50]+1}}{\ddot{a}_{[50]+1}} = 1387.89. $$

POLICY VALUES

Solution(Cont'n)

(B) Let P^g be the gross premium for the policy. It is calculated in the previous **example**, payable annually throughout the term.
At $t = 0$,

$$
\begin{aligned}
{}_0V^n &= 100000A_{[50]} - P_{[50]}\ddot{a}_{[50]} = 0, \\
{}_0V^g &= 100000A_{[50]} + 225 + 25\ddot{a}_{[50]} + 0.47P^g - 0.97P^g\ddot{a}_{[50]} = 0 \\
{}_0V^{FPT} &= 100000A_{[50]} - {}_1P_{[50]} - P_{[50]+1}\nu p_{[50]}\ddot{a}_{[50]+1} \\
&= 100000\left(A^{\,1}_{50:\overline{1}|} + \nu p_{[50]}A_{[50]+1}\right) \\
&\quad -100000A^{\,1}_{50:\overline{1}|} - \left(\frac{100000A_{[50]+1}}{\ddot{a}_{[50]+1}}\right)\nu p_{[50]}A_{[50]+1} \\
&= 0.
\end{aligned}
$$

Solution(Cont'n)

(B) At $t = 1$,

$$
\begin{aligned}
{}_1V^n &= 100000A_{[50]+1} - P_{[50]}\ddot{a}_{[50]+1} = 1272.15,\\
{}_1V^g &= 100000A_{[50]+1} + 25\ddot{a}_{[50]+1} - 0.97P^g\ddot{a}_{[50]+1} = 383.73,\\
{}_1V^{\text{FPT}} &= 100000A_{[50]+1} - P_{[50]+1}\ddot{a}_{[50]+1} = 0.
\end{aligned}
$$

At $t = 2$,

$$
\begin{aligned}
{}_2V^n &= 100000A_{52} - P_{[50]}\ddot{a}_{52} = 2574.01,\\
{}_2V^g &= 100000A_{52} + 25\ddot{a}_{52} - 0.97P^g\ddot{a}_{52} = 1697.30,\\
{}_2V^{\text{FPT}} &= 100000A_{52} - P_{[50]+1}\ddot{a}_{52} = 1318.63.
\end{aligned}
$$

At $t = 10$, by the previous **example**,

$$\because\ {}_{10}V^n = 14416.12 \text{ and } {}_{10}V^g = 13645.98,$$

$$\therefore\ {}_{10}V^{\text{FPT}} = 100000A_{60} - P_{[50]+1}\ddot{a}_{60} = 13313.34.$$

<u>**Solution**</u>(Cont'n)

- The FPT reserve is intended to approximate the gross premium policy value, particular in the first few years of the contract.
- In the **example**, the insurer would benefit significantly in the first year from using FPT approach rather than the net premium policy value.
- The FPT method implicitly assumes that the whole first year premium is spent on the cost of insurance and acquisition expenses. In this case, the assumption overstates the acquisition expenses slightly, with the result that the FPT reserve is actually a little *lower* than the gross premium policy value. □