3.1

(a) Since $f: \mathbb{R} \to \mathbb{R}$ is convex and $a, b \in \operatorname{dom} f$ with $a < b$, we have

$$f(\theta a + (1-\theta)y) \le \theta f(a) + (1-\theta) f(b) \quad \text{for all } x, y \in \operatorname{dom} f, \quad 0 \le \theta \le 1.$$

Then we can set $\theta = \dfrac{b-x}{b-a}$, which implies $(1-\theta) = \dfrac{x-a}{b-a}$. Since $x \in [a, b]$, we have $0 \le \theta \le 1$. Then by the definition of convex function, we have

$$f(\theta a + (1-\theta)y) = f\left(\frac{b-x}{b-a} a + \frac{x-a}{b-a} b\right) = f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b).$$

(b)

From (a), we substract $f(a)$ on both sides, then we have:

$$f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) \iff f(x) - f(a) \le \frac{b-x}{b-a} f(a) - f(a) + \frac{x-a}{b-a} f(b)$$

$$\iff f(x) - f(a) \le \frac{x-a}{b-a} (f(b) - f(a))$$

$$\iff \frac{f(x) - f(a)}{x-a} \le \frac{f(b) - f(a)}{b-a} \quad (\text{since } x \in (a, b),\ x-a > 0) \quad \text{——} ①$$

We can also substract $f(b)$ on both sides, then we have:

$$f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) \iff f(x) - f(b) \le \frac{b-x}{b-a} f(a) + \frac{x-b}{b-a} f(b)$$

$$\iff f(x) - f(b) \le \frac{x-b}{b-a} (f(b) - f(a))$$

$$\iff \frac{f(x) - f(b)}{x-b} \ge \frac{f(b) - f(a)}{b-a} \quad (\text{since } x \in (a, b),\ x-b < 0)$$

$$\iff \frac{f(b) - f(x)}{b-x} \ge \frac{f(b) - f(a)}{b-a} \quad \text{——} ②$$

Then combine ① and ②, we get the inequality

$$\frac{f(x) - f(a)}{x-a} \le \frac{f(b) - f(a)}{b-a} \le \frac{f(b) - f(x)}{b-x}$$

(c)

First, we take limit $x \to a$ on the both sides of first inequality from (b), we have:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \le \lim_{x \to a} \frac{f(b) - f(a)}{b - a} \Leftrightarrow f'(a) \le \frac{f(b) - f(a)}{b - a} \quad \text{--- ③}$$

Then, we take limit $x \to b$ on the both sides of the second inequality from (b), we have

$$\lim_{x \to b} \frac{f(b) - f(a)}{b - a} \le \lim_{x \to b} \frac{f(b) - f(x)}{b - x} \Leftrightarrow \frac{f(b) - f(a)}{b - a} \le f'(b) \quad \text{--- ④}$$

Therefore, we can combine ③ and ④ to get

$$f'(a) \le \frac{f(b) - f(a)}{b - a} \le f'(b)$$

(d)

From (c), we know that $f'(b) \ge f'(a)$, so we have

$$f'(b) - f'(a) \ge 0$$

$$\Leftrightarrow \frac{f'(b) - f'(a)}{b - a} \ge 0$$

$$\Leftrightarrow \lim_{b \to a} \frac{f'(b) - f'(a)}{b - a} \ge \lim_{b \to a} 0$$

$$\Leftrightarrow f''(a) \ge 0.$$

also

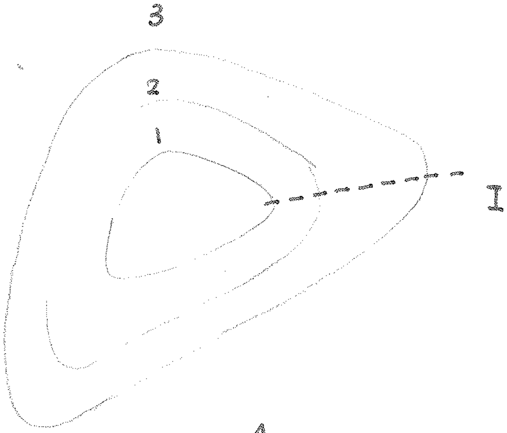
$$\frac{f'(b) - f'(a)}{b - a} \ge 0$$

$$\Leftrightarrow \lim_{a \to b} \frac{f'(a) - f'(b)}{a - b} \ge \lim_{a \to b} 0$$

$$\Leftrightarrow f''(b) \ge 0.$$

3.2

The first function can not be convex. We plot the function values along the dashed line I as the Fig 1. The function is not convex along the dashed line I clearly.



And the first function can't be concave or quasiconcave since the super level sets aren't convex.

The first function can be quasi convex since it sub level sets appear to be convex.

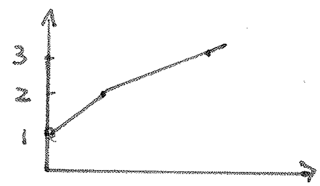


Fig 1.

Then we do the same analysis on the second function.

The second function can not be convex since its sub level sets are not convex. The second function can not be quasiconvex neither.

The second function can be concave clearly by observing its level sets, and therefore it can be quasi concave.

3.22

(a) Let $g(x) = \log\left(\sum_{i=1}^{m} e^{a_i^T x + b_i}\right)$ and $h(y) = -\log y$. Since $g(x)$ is a composition of log-sum-exp and an affine function, $g(x)$ is convex which implies that $-g(x)$ is concave. The function $h(y) = -\log y$ is convex and $\tilde{h}(y)$ is not increasing. Therefore, $f(x) = h(-g(x)) = -\log\left(-\log\left(\sum_{i=1}^{m} e^{a_i^T x + b_i}\right)\right)$ is convex.

(b) We can rewrite $f(x,u,v)$ as $f(x,u,v) = -\sqrt{u(v - x^T x/u)}$. Let function $h(x_1, x_2) = -\sqrt{x_1 x_2}$ and it is convex on $\mathbb{R}^2_{++}$, and $\tilde{h}$ is nonincreasing. Let $g_1(x,u,v) = u$ and $g_2(x,u,v) = v - x^T x/u$. We know that g_1 and g_2 are concave, so $g(x,u,v) = g_1(x,u,v) \times g_2(x,u,v) = uv - x^T x$ is concave. Therefore, $f(x,u,v) = h(g(x,u,v))$ is convex.

(c) We can denote $f(x,u,v)$ as $f(x,u,v) = -\log(uv - x^Tx) = -\log\left(u\left(v - \frac{x^Tx}{u}\right)\right)$

Let $g_1(x,u,v) = u$, $g_2(x,u,v) = v - \frac{x^Tx}{u}$ then $g(x,u,v) = g_1(x,u,v) \times g_2(x,u,v)$

And let $h(y) = -\log(y)$. We know that $g_2(x,u,v)$ is concave since v is linear and $\frac{x^Tx}{u}$ is convex, $g_2(x,u,v) = v - \frac{x^Tx}{u}$ is concave. And since $g_1(x,u,v) = u$ is linear, we have $g(x,u,v) = g_1(x,u,v) \times g_2(x,u,v)$ is concave and $g(x,u,v) = u \times \left(v - \frac{x^Tx}{u}\right) = uv - x^Tx$. We also know that $h(y) = -\log y$ is convex and $\tilde{h}(y)$ is nonincreasing. Therefore, $f(x,u,v) = h(g(x,u,v)) = -\log(uv - x^Tx)$ is convex.

(d) We can write $f(x,t)$ as : $f(x,t) = -(t^p - \|x\|_p^p)^{\frac{1}{p}} = -\left(t - \frac{\|x\|_p^p}{t^{p-1}}\right)^{\frac{1}{p}} \cdot t^{1-\frac{1}{p}}$

It is the composition of $h(y_1, y_2) = -y_1^{\frac{1}{p}} \cdot y_2^{1-\frac{1}{p}}$ and $g_1(x,t) = \left(t - \frac{\|x\|_p^p}{t^{p-1}}\right)$ and $g_2(x,t) = t$. We know that t is linear and $\frac{\|x\|_p^p}{t^{p-1}}$ is convex, so $g_1(x,t) = t - \frac{\|x\|_p^p}{t^{p-1}}$ is concave. And $h(y_1, y_2)$ is convex and nonincreasing in each argument. Therefore, $f(x,t) = -(t^p - \|x\|_p^p)^{1/p}$ is convex since $h(y_1,y_2)$ is convex and nonincreasing with two concave $g_1(x,t)$ and $g_2(x,t)$.

(e) We can write $f(x,t)$ as :

$$f(x,t) = -\log\left(t^p - \|x\|_p^p\right) = -\log t^{p-1} - \log\left(t - \frac{\|x\|_p^p}{t^{p-1}}\right)$$

$$= -(p-1)\log t - \log\left(t - \frac{\|x\|_p^p}{t^{p-1}}\right)$$

The first term is $-\log$ function, so it is convex.
And we know that $\frac{\|x\|_p^p}{t^{p-1}}$ is convex, so $t - \frac{\|x\|_p^p}{t^{p-1}}$ is concave. And $-\log$ is a convex and nonincreasing function, so the second term $-\log\left(t - \frac{\|x\|_p^p}{t^{p-1}}\right)$ is convex. Therefore, $f(x,t)$ is convex.

3.28

(a) The point $(x, f(x))$ is in the boundary of $\mathbf{epi}\, f$. If the point were in $\mathbf{int\, epi}\, f$, then for small, positive ϵ we would have $(x, f(x)-\epsilon) \in \mathbf{epi}\, f$, which is impossible. And we know that there is a supporting hyperplane to $\mathbf{epi}\, f$ at $(x, f(x))$ such that

$$a^T z + bt \geq a^T x + bf(x) \text{ for all } (z,t) \in \mathbf{epi}\, f, \quad a \in \mathbb{R}^n \text{ and } b \in \mathbb{R}.$$

Since t can be arbitrary large if $(z,t) \in \mathbf{epi}\, f$, we conclude $b \geq 0$.

If $b=0$, then

$$a^T z \geq a^T x \text{ for all } z \in \mathbf{dom}\, f,$$

which contradicts $x \in \mathbf{int\, dom}\, f$. Therefore $b>0$. Then we can express the inequality as:

$$a^T z + bt \geq a^T x + bf(x) \iff t \geq f(x) + \frac{a^T}{b}(x-z) \text{ for all } (z,t) \in \mathbf{epi}\, f$$

Therefore, the affine function $g(z) = f(x) + \frac{a^T}{b}(x-z)$. is an affine global underestimator of f. By definition of $\tilde{f}$,

$$f(x) \geq \tilde{f}(x) \geq g(x).$$

However, $g(x) = f(x)$, so we have $f(x) = \tilde{f}(x)$.

(b) A closed convex set is the intersection of all halfspaces that contains it. Then we apply this result to $\mathbf{epi}\, f$. Define $H = \{(a,b,c) \in \mathbb{R}^{n+2} \mid (a,b) \neq 0, \inf_{(x,t)\in \mathbf{epi}\, f} (a^T x + bt) \geq c\}$. It is clear that all elements of H satisfy $b \geq 0$. If $b>0$, the affine function $h(x) = -\frac{a^T}{b}x + \frac{c}{b}$ minorize f, since $t \geq f(x) \geq -\frac{a^T}{b}x + \frac{c}{b} = h(x)$. for all $(x,t) \in \mathbf{epi}\, f$. We need to prove that $\mathbf{epi}\, f = \bigcap_{(a,b,c)\in H,\, b>0} \{(x,t) \mid a^T x + bt \geq c\}$. Since H may contain element with $b=0$, this does not immediately follow from $\mathbf{epi}\, f = \bigcap_{(a,b,c)\in H} \{(x,t) \mid a^T x + bt \geq c\}$. We want to show that $\bigcap_{(a,b,c)\in H,\, b>0} \{(x,t) \mid a^T x + bt \geq c\} = \bigcap_{(a,b,c)\in H} \{(x,t) \mid a^T x + bt \geq c\}$. Assume $(\bar{x}, \bar{t})$ in the left set, that $a^T \bar{x} + b\bar{t} \geq c$ for all $a^T x + bt \geq 0$ that are nonvertical $(b>0)$. Assume that $(\bar{x}, \bar{t})$ is not in the right set, i.e., there exists a $(\tilde{a}, \tilde{b}, \tilde{c}) \in H$, $\tilde{b}=0$ such that $\tilde{a}^T \bar{x} < \tilde{c}$. H contains at least one element (a_0, b_0, c_0) with $b_0>0$, otherwise $\mathbf{epi}\, f$ would be an intersection of vertical half spaces. Consider the halfspace defined by $(\tilde{a}, 0, \tilde{c}) + \epsilon(a_0, b_0, c_0)$ for a small ϵ. This halfspace is nonvertical and it contains $\mathbf{epi}\, f$: $(\tilde{a} + \epsilon a_0)^T x + \epsilon b_0 t \geq \tilde{a}^T x + \epsilon(a_0^T x + b_0 t) \geq \tilde{c} + \epsilon c_0$, for all $(x,t) \in \mathbf{epi}\, f$, because the halfspaces $\tilde{a}^T x \geq \tilde{c}$ and $a_0^T x + b_0 t \geq c_0$ both contain $\mathbf{epi}\, f$.

However, $(\tilde{a} + \epsilon a_0)^T \bar{x} + \epsilon b_0 \bar{t} = \tilde{a}^T \bar{x} + \epsilon (a_0^T \bar{x} + b_0 \bar{t}) < \tilde{c} + \epsilon c_0$ for small ϵ, so the halfspace does not contain $(\bar{x}, \bar{t})$. This contradicts the assumption that $(\bar{x}, \bar{t})$ is in the intersection of all nonvertical halfspaces containing epi f. We conclude that $\bigcap_{\substack{(a,b,c) \in H \\ b>0}} \{(x,t) \mid a^T x + bt \geq c\} = \bigcap_{(a,b,c) \in H} \{(x,t) \mid a^T x + bt \geq c\}$ holds.

3.39

(a)

$$g^*(y) = \sup \left(y^T x - g(x) \right) = \sup \left(y^T x - f(x) - c^T x - d \right)$$

$$= \sup \left((y-c)^T x - f(x) \right) - d$$

$$= f^*(y-c) - d.$$

(b) The perspective of a function $f: \mathbb{R}^n \to \mathbb{R}$ is the function $g: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$

$$g(x,t) = t f\left(\frac{x}{t}\right), \quad \operatorname{dom} g = \left\{ (x,t) \mid \frac{x}{t} \in \operatorname{dom} f,\ t > 0 \right\}$$

$$g^*(y,s) = \sup_{\substack{\frac{x}{t} \in \operatorname{dom} f, \\ t>0}} \left(y^T x + st - t f\left(\frac{x}{t}\right) \right)$$

$$= \sup_{\substack{\frac{x}{t} \in \operatorname{dom} f \\ t>0}} \left(t \left(y^T \frac{x}{t} + s - f\left(\frac{x}{t}\right) \right) \right)$$

$$= \sup_{t>0} t \left(s + \sup_{\frac{x}{t} \in \operatorname{dom} f} \left(y^T \frac{x}{t} - f\left(\frac{x}{t}\right) \right) \right)$$

$$= \sup_{t>0} t \left(s + f^*(y) \right) = \begin{cases} 0, & \text{if } s + f^*(y) \leq 0 \\ \infty, & \text{otherwise.} \end{cases}$$

(c)

$$g^*(y) = \sup_x \left(y^T x - \inf_z f(x,z) \right) = \sup_{x,z} \left(y^T x - f(x,z) \right) = f^*(y, 0)$$

Then we apply the result to $f(x,z) = \begin{cases} h(z), & Az + b = x \\ \infty, & \text{otherwise} \end{cases}$, we have

$$f^*(y, v) = \sup \left(y^T x + v^T z - f(x,z) \right)$$

$$= \sup_{Az+b=x} \left(y^T x + v^T z - h(z) \right)$$

$$= \sup_z \left(y^T (Az + b) + v^T z - h(z) \right)$$

$$= y^T b + \sup_z \left(y^T A z + v^T z - h(z) \right) = y^T b + h^* (A^T y + v)$$

Therefore, $g^*(y) = f^*(y, 0) = y^T b + h^*(A^T y)$

(d)

$f^*(y) = \sup_x \left(y^T x - f(x) \right)$. If $y^* \in \text{dom } f^*$, then the affine function $h(x) = y^T x - f^*(y)$ minorizes f. Conversely, if $h(x) = a^T x + b$ minorizes f, then $a \in \text{dom} f^*$ and $f^*(a) \leq -b$.

The set of all affine functions that minorize f is therefore exactly equal to the set of all functions $h(x) = y^T x + c$ where $y \in \text{dom } f^*$, $c \leq - f^*(y)$.

Therefore, by the result of 3.28, we have

$$f(x) = \sup_{y \in \text{dom} f^*} \left(y^T x - f^*(y) \right) = f^{**}(y).$$

A2.23

(a)

$f(x) = -e^{-g(x)}$ where $g: \mathbb{R}^n \to \mathbb{R}$.

$$\frac{\partial f(x)}{\partial x_i} = e^{-g(x)} \cdot \frac{\partial g(x)}{\partial x_i} \Leftrightarrow \nabla f(x) = e^{-g(x)} \nabla g(x)$$

$$\nabla^2 f(x) = -e^{-g(x)} \nabla g(x) \cdot \nabla g(x)^T + e^{-g(x)} \nabla^2 g(x)$$

$$= e^{-g(x)} \left(\nabla^2 g(x) - \nabla g(x) \nabla g(x)^T \right)$$

Since $\begin{bmatrix} \nabla^2 g(x) & \nabla g(x) \\ \nabla g(x) & 1 \end{bmatrix} \succeq 0$, we have $\nabla^2 f(x) = e^{-g(x)} \left(\nabla^2 g(x) - \nabla g(x) \nabla g(x)^T \right) \succeq 0$.

Therefore, $f(x)$ is convex.

(b)

$f(x)$ is a composition of a norm function (convex) and an affine function (convex), parameterized by $\mathbb{P}$.

A2.42

(a) Since the vehicle moves at a constant speed, we can denote s as: $s = \frac{d}{t}$.

And $g(d,t) = \int_0^t f(s)\,d\tau = \int_0^t f(\frac{d}{t})\,d\tau = t\, f(\frac{d}{t})$

$$\nabla g = \begin{bmatrix} f'(\frac{d}{t}) \\ f(\frac{d}{t}) - \frac{d}{t} f'(\frac{d}{t}) \end{bmatrix}, \quad \nabla^2 g = \begin{bmatrix} \frac{f''(\frac{d}{t})}{t} & \frac{-d \cdot f''(\frac{d}{t})}{t^2} \\ \frac{-d \cdot f''(\frac{d}{t})}{t^2} & \frac{d^2 \cdot f''(\frac{d}{t})}{t^3} \end{bmatrix}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we compute $x^T \nabla^2 g\, x = \frac{f''(\frac{d}{t})}{t^3} \left(t^2 x_1^2 - 2t x_1 x_2 d + d^2 x_2^2 \right)$

$$= \frac{f''(\frac{d}{t})}{t^3} (t x_1 - d x_2)^2$$

Since f is convex and positive increasing, $f''(\frac{d}{t}) \geq 0$, we have $x^T \nabla^2 g\, x \geq 0$ implies $\nabla^2 g \succeq 0$. Therefore, g is convex.

(b) Since the vehicle moves at a constant speed s, the total time $t = \frac{d}{s}$. Hence,

$h(d) = f(s) \cdot \frac{d}{s}$. $\nabla h = \frac{f(s)}{s}$, $\nabla^2 h = 0 \geq 0$. Therefore, h is convex.

A2.46

(a) First, let $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $y = \begin{bmatrix} 10 \\ 2 \\ 2 \end{bmatrix}$, $\theta = 0.5$. Then we have median$(x) = 2$,

median$(y) = 2$, median$(\theta x + (1-\theta)y)$ = median$\left(\begin{bmatrix} 5.5 \\ 2 \\ 2.5 \end{bmatrix}\right) = 2.5$,

$\theta \cdot$ median(x) + $(1-\theta)$ median(y) = 2. We see that median$(\theta x + (1-\theta)y) \not\leq \theta \cdot$ median(x) + $(1-\theta)$ median(y)

Hence, median(x) is not convex.

Secondly, let $x = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$, $y = \begin{bmatrix} -10 \\ -2 \\ -2 \end{bmatrix}$, $\theta = 0.5$. Then we have median$(x) = -2$

median$(y) = -2$, median$(\theta x + (1-\theta)y)$ = median$\left(\begin{bmatrix} -5.5 \\ -2 \\ -2.5 \end{bmatrix}\right) = -2.5$,

$\theta \cdot$ median(x) + $(1-\theta)$ median$(y) = -2$. We see that $-$median$(\theta x + (1-\theta)y) \not\leq \theta \cdot (-$median$(x))$ + $(1-\theta)(-$median$(y))$

Hence, median(x) is no concave.

In conclusion, median(x) is neither convex nor concave.

(b)

Range$(x) = x_{[1]} - x_{[n]} = x_{[1]} + (-x)_{[1]}$

Since the sum of r largest components of x = $f(x) = x_{[1]} + \dots + x_{[r]}$ is convex,

we can set $r = 1$, so $f(x) = x_{[1]}$ is convex, so as $(-x)_{[1]}$ is convex.

And the sum of two convex function is convex, so Range$(x) = x_{[1]} - x_{[n]} = x_{[1]} + (-x)_{[1]}$

is convex.

(c) First, let $x = \begin{bmatrix} -5 \\ 0 \\ 5 \end{bmatrix}$, $y = \begin{bmatrix} 5 \\ 4 \\ -5 \end{bmatrix}$, $\theta = 0.5$. Denote the midpoint of the range as $MR(\cdot)$.

$MR(x) = \frac{5+(-5)}{2} = 0$, $MR(y) = \frac{5+(-5)}{2}$. $MR(\theta x + (1-\theta)y) = MR\left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) = \frac{2+0}{2} = 1$

$\theta MR(x) + (1-\theta)MR(y) = 0$. We see that $MR(\theta x + (1-\theta)y) \not\leq \theta MR(x) + (1-\theta)MR(y)$. It is not convex.

Secondly, let $x = \begin{bmatrix} -5 \\ 0 \\ 5 \end{bmatrix}$, $y = \begin{bmatrix} 5 \\ -4 \\ -5 \end{bmatrix}$, $\theta = 0.5$. Then $MR(x) = 0$, $MR(y) = 0$, $MR(\theta x + (1-\theta)y) = -1$

$\theta MR(x) + (1-\theta)MR(y) = 0$. We see that $-MR(\theta x + (1-\theta)y) = 1 \not\leq \theta \cdot (-MR(x)) + (1-\theta)(-MR(y)) = 0$

It is not concave.

Therefore, the midpoint of range is neither convex nor concave.

(d) We denote interquartile range as $IR(\cdot)$ in the following text.

First, let $x = [0\ 2\ 2\ 2]^T$, $y = [2\ 2\ 2\ 0]^T$, $\theta = 0.5$.

Then $IR(x) = 0$, $IR(y) = 0$, $IR(\theta x + (1-\theta)y) = IR([1\ 2\ 2\ 1]^T) = 1$

$\theta IR(x) + (1-\theta) IR(y) = 0$. We see that $IR(\theta x + (1-\theta)y) \nleq \theta IR(x) + (1-\theta) IR(y)$. It is not convex.

Secondly, let $x = [1\ 1\ 3\ 3]^T$, $y = [3\ 3\ 1\ 1]^T$, $\theta = 0.5$. Then $IR(x) = 2$, $IR(y) = 2$,

$IR(\theta x + (1-\theta)y) = IR([2\ 2\ 2\ 2]^T) = 0$. $\theta IR(x) + (1-\theta) IR(y) = 2$.

We see that $-IR(\theta x + (1-\theta)y) = 0 \nleq \theta(-IR(x)) + (1-\theta)(-IR(y)) = -2$. It is not concave.

Hence, interquartile range is neither convex nor concave.

(e) We denote Symmetric trimmed mean as $STM(\cdot)$ in the following text.

First, let $x = [100\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ -100]^T \in \mathbb{R}^{20}$

$y = [-98\ 2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ -2\ 102]^T \in \mathbb{R}^{20}$

$\theta = 0.5$

$STM(x) = \frac{-2}{17}$ $STM(y) = \frac{-2}{17}$, $STM(\theta x + (1-\theta)y) = STM([1\ 2\ 0\ 0 \cdots 0 \cdots 0\ 0\ -2\ 1]^T) = \frac{1}{17}$

$\theta STM(x) + (1-\theta) STM(y) = \frac{-2}{17}$ We see that $STM(\theta x + (1-\theta)y) \nleq \theta STM(x) + (1-\theta) STM(y)$

It is not convex.

Secondly, let $x = [100\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ 2\ -100]^T \in \mathbb{R}^{20}$

$y = [-102\ 2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ -2\ 2\ 98]^T \in \mathbb{R}^{20}$

$\theta = 0.5$.

Then we have $STM(x) = \frac{2}{17}$, $STM(y) = \frac{2}{17}$ $STM(\theta x + (1-\theta)y) = STM([-1\ 2\ 0\ 0 \cdots 0\ 0\ 2\ -1]^T) = \frac{-1}{17}$

$\theta STM(x) + (1-\theta) STM(y) = \frac{2}{17}$ We see that $-STM(\theta x + (1-\theta)y) \nleq \theta \cdot (-STM(x)) + (1-\theta)(-STM(y))$

It is not concave.

Therefore, symmetric trimmed mean is neither convex nor concave.

(f) Let $f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[9n/10]}$. We know that it is sum of $\frac{9n}{10}$ largest components of $x \in \mathbb{R}^n$, so it is convex. And $0.9n+1$ is a constant.

Therefore, lower trimmed mean $\dfrac{x_{[1]} + \cdots + x_{[\frac{9n}{10}]}}{0.9n+1}$ is convex.