1. Method

We will use a neural network to approximate both the **Runge function** and its **derivative**.

$$f(x) = \frac{1}{1+25x^2} \qquad f'(x) = -\frac{50x}{(1+25x^2)^2} \qquad x \leftarrow [-1,1]$$

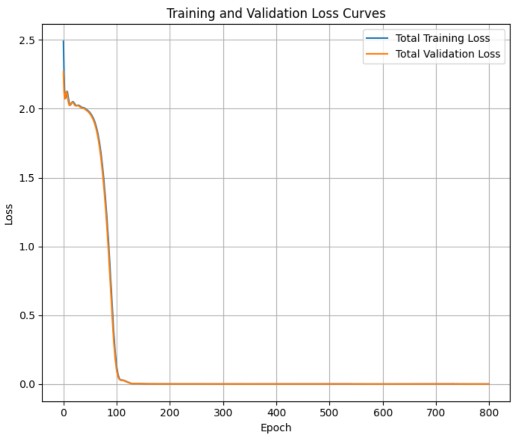
Loss Function:

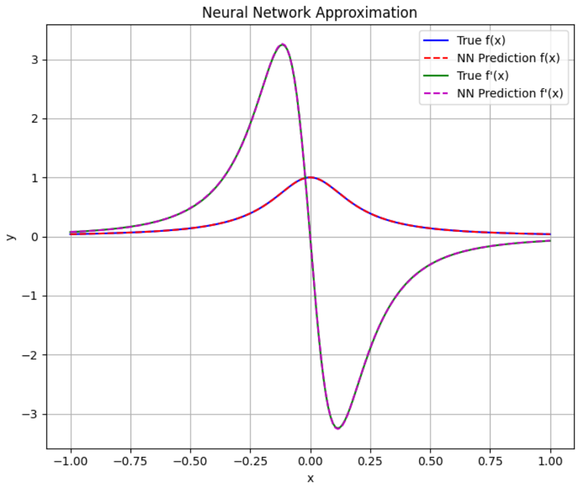
$$L(\theta) = \left(\frac{1}{N}\right) * \sum_{i}^{N} (y_i - \check{y}_i)^2 + \left(y'_i - \check{y}'_i\right)^2$$

And its gradient:

$$\nabla_\theta L_{total} = -\frac{2}{N}\sum(y_i - \hat{y}_i)\nabla_\theta \hat{f}(x_i,\theta) - \frac{2}{N}\sum(y'_i - \hat{y}'_i)\nabla_\theta\left(\frac{\partial \hat{f}(x_i,\theta)}{\partial x_i}\right)$$

2. result





Epoch 100/800 -> Train Loss: 0.15499093, Val Loss: 0.12372499

Epoch 200/800 -> Train Loss: 0.00056474, Val Loss: 0.00055885

Epoch 300/800 -> Train Loss: 0.00031473, Val Loss: 0.00031329

Epoch 400/800 -> Train Loss: 0.00020587, Val Loss: 0.00020491

Epoch 500/800 -> Train Loss: 0.00012884, Val Loss: 0.00012814

Epoch 600/800 -> Train Loss: 0.00008002, Val Loss: 0.00007869

Epoch 700/800 -> Train Loss: 0.00004824, Val Loss: 0.00004796

Epoch 800/800 -> Train Loss: 0.00003201, Val Loss: 0.00003128

--- 最終誤差評估 ---

訓練集 - 函數 MSE: 0.00000028

訓練集 - 導數 MSE: 0.00003172

驗證集 - 函數 MSE: 0.00000012

驗證集 - 導數 MSE: 0.00003116

Coding assisted by Gemini.