Chapter 1 Building Proofs

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0.1 Exercise 1.4.3 (Logics):

- **de Morgan's laws**
  1. not(A or B) $\Leftrightarrow$ ((not A) and (not B))
  2. not(A and B) $\Leftrightarrow$ ((not A) or (not B))
- **Contrapositive**
  1. (A $\Rightarrow$ B) $\Leftrightarrow$ (not B $\Rightarrow$ not A)
  2. ((A or B) $\Rightarrow$ C) $\Leftrightarrow$ ((A $\Rightarrow$ C) and (B $\Rightarrow$ C))
  3. (A $\Leftrightarrow$ (B and C)) $\Leftrightarrow$ ((A $\Rightarrow$ B) and (A $\Rightarrow$ C))
  4. not (not A) $\Leftrightarrow$ A

0.2 Exercise 1.4.6

1. "A $\Rightarrow$ (B $\Rightarrow$ C)" $\Leftrightarrow$ "(A and B) $\Rightarrow$ C."
2. (A $\Leftrightarrow$ B) $\Leftrightarrow$ (not A $\Leftrightarrow$ not B).
3. "A $\Rightarrow$ (B $\Rightarrow$ C)" $\Leftrightarrow$ "(A and B) $\Rightarrow$ C."

0.3 Exercise 1.7.1

(A $\Rightarrow$ (B or C)) $\Leftrightarrow$ ((A and not C) $\Rightarrow$ B)

0.4 Exercise 1.8.1

((A $\Rightarrow$ C) and (B $\Rightarrow$ D)) $\Rightarrow$ ((A or B) $\Rightarrow$ (C or D)).

0.5 Exercise 1.15.1 (Set operations)

1. $A \subseteq B \Leftrightarrow C(B) \subseteq C(A)$
2. $C(A \cup B) = C(A) \cap C(B)$
3. $C(A \cap B) = C(A) \cup C(B)$
4. $A \subseteq (B \cap C) \Leftrightarrow (A \subseteq B)$ and $(A \subseteq C)$
5. $(A \cup B) \subseteq C \Leftrightarrow (A \subseteq C)$ and $(B \subseteq C)$
6. $A \subseteq (B \cup C) \Leftrightarrow (A \setminus C) \subseteq B$
7. if $A \subseteq B$, then $B \setminus (B \setminus A) = A$

0.6 Exercise 1.15.5 (Set difference)

If S and T are sets, $S \setminus T = \emptyset \iff S \subseteq T$

0.7 Exercise 1.15.6 (Symmetric Difference)

Define **Symmetric Difference** as

$$S \triangle T = (S \setminus T) \cup (T \setminus S)$$

, then

$$S \triangle T = (S \cup T) \setminus (T \cap S)$$

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0.8 Exercise 1.15.7 (Countable union and intersection)

Definition (**Countable union and intersection**):

$$\bigcup_{\alpha \in \mathcal{A}} S_\alpha = \{x : \exists \alpha \in \mathcal{A} \ni (x \in S_\alpha)\}$$

$$\bigcap_{\alpha \in \mathcal{A}} S_\alpha = \{x : \forall \alpha \in \mathcal{A} \ni (x \in S_\alpha)\}$$

1. **Distributive law** also holds for countable union and intersection
2. The union of a collection of sets is the smallest set that contains all the sets in the collection.
3. The intersection of a collection of sets is the largest set that is contained in all the sets in the collection.

0.9 Exercise 1.15.9 (Limit Superior and Limit Inferior of Sets)

Definition (**Limit Superior and Limit Inferior of Sequence of Sets**)

$$\limsup\{S_n\} = \bigcap_{k=1}^{\infty}(\bigcup_{n=k}^{\infty} S_n)$$

$$\liminf\{S_n\} = \bigcup_{k=1}^{\infty}(\bigcap_{n=k}^{\infty} S_n)$$

This interesting definition should be looked across with the definition in **Exercise 10.4.10 (Limit Superior and Limit Inferior of Sequence of Real Numbers)**. The union here acts as the sup in subsequences and the intersection is the inf of subsequences. The connection can be drawn by defining the set in **Exercise 1.15.9** as the **range** of the subsequences in **Exercise 10.4.10**. See the separate note I wrote for **Exercise 10.4.10** titled *Limit Superiors and Limit Inferiors*.

A limit superior is the smallest (intersect) of the largest (union) of the collection of infinite number of sets. A limit inferior is the largest (union) of the smallest (intersection) of the collection of infinite number of sets.

1. $\limsup\{S_n\} = \{x : x \in S_n$ for infinitely many n$\}$
2. $\liminf\{S_n\} = \{x : x \in S_n$ for all but finitely many n$\}$
3. $\liminf\{S_n\} \subseteq \limsup\{S_n\}$, and they might not be equal
4. if $S_1 \subseteq S_2 \subseteq S_3 \subseteq ...,$ then $\limsup\{S_n\} = \bigcup_{n=1}^{\infty} S_n$
5. if $S_1 \supseteq S_2 \supseteq S_3 \supseteq ...,$ then $\limsup\{S_n\} = \bigcap_{n=1}^{\infty} S_n$