Chapter 8 Topology of the Real Numbers

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1 Open Sets

1.1 DEFINITION 8.1 (Open Set):

A set is **open** if it is a **neighborhood** of each of its points.

1.2 Example 8.1 (Open Intervals and Empty Set):

- (1) **Open intervals** are open. The **whole real line** is open.
- (2) The **empty set** is open.

1.3 THEOREM 8.2 (Union of two open sets):

The **union** of two open sets is open.

1.4 THEOREM 8.3 (Union of open sets):

The **union of any collection** of open sets is open.

1.5 THEOREM 8.4 (Intersection of finite collection of open sets):

The **intersection of any finite collection** of open sets is open.

Note the difference between **THEOREM 8.3** and **THEOREM 8.4**. The union allows infinitely many open sets but the intersection has to be a finite collection. **Exercise 8.1.4** is the counter example if we do **infinitely many intersection**.

Compare **THEOREM 8.3** and **THEOREM 8.4** to **THEOREM 8.8** and **THEOREM 8.9**, there are an interesting mirror image between the properties of **open sets** and **closed sets**.

1.6 Exercise 8.1.4:

$$\bigcap_{n=1}^{\infty}(-\frac{1}{n}, 1+\frac{1}{n}) = [0,1]$$

2 General Topologies

2.1 DEFINITION 8.5 (Topology and Topological Space):

Let X be a set and $\mathcal{T}$ a collection of subsets of X such that:

- (i) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
- (ii) The **union of any collection** of sets from $\mathcal{T}$ is in $\mathcal{T}$. and
- (iii) The **intersection of any finite collection** of sets from $\mathcal{T}$ is in $\mathcal{T}$.

Then is called a **topology** on X. The pair $(X, \mathcal{T})$ is called a **topological space**, and the elements of $\mathcal{T}$ are called **open subsets** of X.

Note that union is *ANY* union while intersection is *FINITE* intersection.

Note also that, it doesn't require **complement** of a set to be in the topology. In fact, the complement of an open set is a **closed set** which should not be in the topology.

This definition draws some analogue to the definition of **sigma-algebra**. In sigma algebra, it requires closed under countable unions and complements.

An open set in one topology might not be open in another topology over the same universal set X. If we talk about open set/interval, we should be specific which topogoly we are talking about. **Real anlysis** concerns about the **Euclidean topology**. But that is not the only topology over the real numbers.

A proof or definition that is **topological**, that is, one that uses only facts about open sets, is usually *preferable* to one that uses **other structures** (such as **order**, **algebra**, or **distance**).

2.2 Exercise 8.2.1 (Finite Complement Topology):

- (e)/(g) If X is any set, $\mathcal{T} = \{S \subseteq X : X \setminus S \text{ is finite}\} \cup \{\emptyset\}$ is topology on X, which is called the **finite complement topology**.

2.3 Exercise 8.2.2 (Dense Sets):

- (a) A set is **dense in the real line** if its **intersection** with **any open set** is **nonempty**.
- (c) Let $\mathcal{T}$ consist of $\emptyset$, $\mathbb{R}$ and all sets of the form (a, ∞) for any $a \in \mathbb{R}$, then $\mathcal{T}$ is a topology on $\mathbb{R}$.

2.4 Exercise 8.2.3 (Discrete Topology and Indiscrete Topology):

- (a) If $\mathcal{T} = \{X, \emptyset\}$, this topology is called the **indiscrete topology**, which is the **smallest** or **coarsest** topology on X. If $\mathcal{T} = \mathcal{P}(X)$, i.e. the **power sets** of X, this topology is called the **discrete topology**, which is the **biggest** or **finest** topology on X.
- (c) Any **nonempty set** is **dense** in any space having the **indiscrete topology**.
- (e) If X is **finite**, the **finite complement topology** on X is the same as the **discrete topology**.
- (f) If X is **infinite**, the **finite complement topology** on X is **NOT** the same as the **discrete topology**.

3 Closed Sets

3.1 THEOREM 8.6 (Closed Sets):

A set is **closed** if it contains all its **cluster points**, that is $S' \subseteq S$.

3.2 Example 8.3:

- (1) **Closed intervals** and the **whole real line** are closed sets.
- (2) The **empty set** is closed since it has no cluster points.
- (4) "**Not closed**" is not the same as "**open**".

3.3 THEOREM 8.7:

A set having **no cluster points** is **closed**.

3.4 THEOREM 8.8 (Union of closed sets):

The **union** of **finitely many** closed sets is closed.

3.5 THEOREM 8.9 (Intersection of closed sets):

The **intersection** of **any collection** of closed sets is closed.

3.6 THEOREM 8.10 (Traditional Definition of Closed Sets):

A set is **open** if and only if its **complement** is **closed**.

Again, **closed** does not mean "**not open**". There are sets that are **neither open nor closed**.

3.7 Exercise 8.3.1 (Bounded Closed Set Contains Its Supremum and Infimum):

A **nonempty**, **closed** and **bounded set** contains its **supremum** and **infimum**.

The converse of this is not true. **THEOREM 5.3** implies that an open interval cannot contains its supremum or infimum. But it could be the case that the set contains an open interval which makes the set neither open nor closed.

3.8 Exercise 8.3.2 (Closure, Interior and Boundary):

The **closure** of a set S, denoted S^-, is the **intersection** of **all closed sets containing** S. The **interior** of S, denoted S^o, is the **union** of **all open sets contained in** S. The **boundary** of S, denoted ∂S, is $S^- \setminus S^o$.

- (a) S^- is the **smallest closed set containing** S, and S^o is the **largest open set contained in** S.
- (b) $x \in \partial S$ if and only if every **neighborhood** of x contains points of S and of $C(S)$. Elements of ∂S are called **boundary points**.
- (f) S is **closed** if and only if $S = S^-$. S is **open** if and only if $S = S^o$.
- (g) ∂S is **closed**.

3.9 Exercise 8.3.3 (Closure, Interior and Boundary of Rational Numbers):

- (a) Under **standard topology**,

$$\mathbb{Q}^o = \emptyset \text{ and } \mathbb{Q}^- = \mathbb{R}, \text{ therefore } \partial\mathbb{Q} = \mathbb{R}$$

  .

- (b) For the topology in **Exercise 8.2.2 (c)**, which is

$$\mathcal{T} = \{(a, \infty), \forall a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

  , we also have

$$\mathbb{Q}^o = \emptyset \text{ and } \mathbb{Q}^- = \mathbb{R}, \text{ therefore } \partial\mathbb{Q} = \mathbb{R}$$

  .

3.10 Exercise 8.3.4 (Boundary Points):

- (a) Let S be a **bounded set**. Then $\inf S \in \partial S$ and $\sup S \in \partial S$.
- (b) If S is nonempty and S has **no boundary points**, then S is **unbounded**.
- (c) An **interior point** of a set cannot be a **boundary point**.

3.11 Exercise 8.3.5 (Kuratowski's Problem):

- (b) Construct a set, take the **closure** of the set, then the **complement** of its closure, and the closure of the complement of its closure. Repeat this over and over: closure -> complement -> closure -> ... The question is how many **different** sets can we obtain in this way. This is called **Kuratowski's problem**, and the answer is 14. A subset realizing the maximum of 14 is called a **14-set**. The space of real numbers under the usual topology contains 14-sets.

4 The Structure of Open Sets

4.1 THEOREM 8.11 (Structure of Open Sets):

A nonempty set S is **open** if and only if there is a **countable collection** of **mutually disjoint** open intervals such that $\{U_1, U_2, ...\}$ such that $S = \bigcup_n U_n$.

4.2 Exercise 8.4.3 (Open Sets a countable union of closed sets):

- (a) Any **open interval** is a **countable union** of **closed intervals** (not necessarily disjoint).
- (b) Any **open set** is a **countable union** of **closed sets**.

4.3 Exercise 8.4.5 (Countability of Open Sets):

If a topological space $(X, \mathcal{T})$ has a **countable** and **dense** subset, then any collection of **mutually disjoint open subsets** of X is **countable**.

4.4 Exercise 8.4.6 (First Countable and Second Countable):

A **topology** is called **first countable** if it has the following property:

For each point x in the **topological space**, there is a **countable collection** of **open sets** $\{U_n^x\}$, each containing x and such that if U is any open set containing x, there is an n so that $U_n^x \subseteq U$. The collection is called a **local neighborhood base** at x. Notice that the collection $\{U_n^x\}$ probably changes from point to point.

A **topology** is called **second countable** if:

There is a **single countable collection** $\{U_n\}$ such that, if x is in the space and U is an open set containing x, then there is an n so that $x \in U_n \subseteq U$. Such a collection $\{U_n\}$ is called a **base** (or **basis**) for the topology. Note that **a basis need not be countable**.

- (a) The **standard topology** on the real line is **first countable**.
- (b) The standard topology on the real line is **second countable**.
- (c) Any second countable topology is also first countable.
- (f) **Cofinite topology** is an example of a topology that is **not first countable**. Any **uncountable discrete space** is **first countable** but **not second countable**.
- (g) If $\{U_\alpha\}$ is a basis for a topology $\mathcal{T}$, then every element of $\mathcal{T}$ can be written as a union of sets in $\{U_\alpha\}$.
- (h) If $(X, \mathcal{T})$ is a topological space and $\{V_\alpha\}$ is a collection of subsets of X with the property that every element in $\mathcal{T}$ can be written as a union of sets, each of which is the **intersection of finitely many** of the sets in $\{V_\alpha\}$. $\{V_\alpha\}$ is called a **subbasis** for $\mathcal{T}$. The collection of **open rays** is a subbasis for the standard topology on the real line.

4.5 Exercise 8.4.7 (Cantor Sets):

The **Cantor set** $\mathcal{C}$ is constructed as the following:

- Let $C_0 = [0, 1]$, and $S_1 = (1/3, 2/3)$ (we refer to S_1 as the "**open middle third**" of C_0).
- Let $C_1 = C_0 \setminus S_1$. Then C_1 consists of two closed intervals. Now each of these intervals also has an open middle third.
- Let $S_2 = (1/9, 2/9) \cup (7/9, 8/9)$ — the two open middle thirds of C_1 and let $C_2 = C_1 \setminus S_2$. Then C_2 consists of four closed intervals.
- Remove the four open middle thirds of C_2 to obtain C_3, and so on.
- Note that $C_n \supseteq C_{n+1}$ for all n.
- Let
$$\mathcal{C} = \bigcap_n C_n$$
.

We have the following observations of the cantor sets:

- (a) $\mathcal{C}$ is **closed**.
- (b) If $x, y \in \mathcal{C}$ and $x < y$, there is a number $z \notin \mathcal{C} \ni x < z < y$.
- (c) The sum of removed intervals $\{S_n\}$ is 1.

- (d) The **length** of $\mathcal{C}$ is therefore 0.
- (e) $\mathcal{C}$ consists of all elements of C_0 whose **ternary expansion** contains no 1s.
- (f) $\mathcal{C}$ is **uncountable**. Therefore $\mathcal{C}$ is an example whose **cardinality** is as high as $\aleph_1$ but whose **length** is 0. The connection between cardinality and length is mystery.
- (g) The **end points** of $\mathcal{C}$ is **countable**. So, there are **non-end-point elements** in $\mathcal{C}$. An example would be 1/4.
- (i) A **fat Cantor set** (or **Smith-Volterra-Cantor set**) can be constructed by removing 1/4 instead of 1/3. This fat Cantor set has a **non-trivial length** of 1/2. More interestingly, the set is still **uncountable** and **not dense in any place**. This is a very intriguing example. By construction, the fat Cantor set contains no intervals and therefore has **empty interior**. It is also the intersection of a sequence of closed sets, which means that it is **closed**. However, the part removed only has a **positive Lebesgue measure** of 1/2. This makes the fat Cantor set an example of a **closed set** whose **boundary** has **positive Lebesgue measure** of 1-1/2=1/2.
- (j) There is a **one-to-one correspondence** between $\mathcal{C}$ and C_0 by converting the ternary expansion to binary expansion, which is why $\mathcal{C}$ is **uncountable**.
- (k) Since $\mathcal{C}$ is **closed**, it contains all its **cluster points**, that is $\mathcal{C}' \subseteq \mathcal{C}$. $\mathcal{C}$ is also **perfect**, which means $\mathcal{C}' = \mathcal{C}$.
- (l) Every **nonempty perfect set** is **uncountable**.

5 Functions - Direct and Inverse Images

5.1 DEFINITION 8.12 (Direct Image, range, onto and inverse function):

Let $f : A \to B$, and $S \subseteq A$. The **direct image of** S **under** f is given by

$$f(S) = \{y \in B : \exists x \in S \ni (y = f(x))\}$$

The direct image of the **entire domain**, i.e. $f(A)$, is sometimes called the **range** of f. The set B, in this case sometimes called the range, but not necessarily all are output of f, is called **codomain of** f to avoid confusion.

So, f is an **onto** if and only if $B = f(A)$.

If $f : A \to B$ is a **one-to-one correspondence**, it has an **inverse function** denoted f^{-1}, whose domain is B and whose range is A and which is defined by:

$$x = f^{-1}(y) \iff y = f(x)$$

.

The **direct image** of **a subset of the range** of f under f^{-1}, i.e. $S \subseteq f(A) \subseteq B$, is given by

$$f^{-1}(S) = \{x \in A : f(x) \in S\}$$

. A little bit of relaxation on the requirement of S leads to the following definition of **inverse image**.

5.2 DEFINITION 8.13 (Inverse Image):

Let $f : A \to B$, and $S \subseteq B$. The **inverse image of S under f** is given by

$$f^{-1}(S) = \{x \in A : f(x) \in S\}$$

.

Inverse images can *always* be found, and so they are much *more useful* to us than **direct images under inverse functions**. The direct image of the inverse function may not exist if the inverse function does not exist (in other words, the function itself is not an onto).

5.3 THEOREM 8.14 (Intersection of Direct and Inverse Image):

Suppose $f : A \to B$. Then,

- (a) If $C, D \subseteq A$, then $f(C \cap D) \subseteq f(C) \cap f(D)$
- (b) If $E, F \subseteq B$, then $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$

Always keep in mind, the relationship between inverse and direct images and algebraic operations on functions is very subtle. The inverse image has a much better property than the direct image.

5.4 THEOREM 8.15 (Composition of functions):

If $f : A \to B$, $g : B \to C$ and $S \subseteq C$, then $(g \circ f)^{-1} = f^{-1}(g^{-1}(S))$.

5.5 Exercise 8.5.4 (Set Algebra on Direct and Inverse Images):

- (a) $f(C \cup D) = f(C) \cup f(D)$
- (b) $f(C) \setminus f(D) \subseteq f(C \setminus D)$
- (c) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
- (d) $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$. Comparing (d) to (b), it is obvious that the inverse image has better behavior than the direct image.
- (g) $f(f^{-1}(C)) \subseteq C$
- (h) $C \subseteq f^{-1}(f(C))$
- (i) if f is a **one-to-one**, then $f(C) \setminus f(D) = f(C \setminus D)$

5.6 Exercise 8.5.7 (One-to-one):

- (a) Suppose $f : A \to B$ and $S \subseteq f(A)$. The **cardinality** of $f^{-1}(S)$ is no less than the cardinality of S.
- (b) A function is **one-to-one** if and only if it has the property that the **inverse image** of any set with **one element** has **at most one element**.

6 Continuous Functions

6.1 DEFINITION (Continuous function: the ϵ-δ definition)

Function f is **continuous** at a if for every $\epsilon > 0$, there is a $\delta > 0$ so that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. We can call this that f is **continuous(ϵ)**.

The ϵ-δ definition depends on the concept of **distance** which in turn implicitly refers to the concept of **ordering**. It would be nice to have a **topological definition** of this concept of continuous function that does not depend on these extra structures. **Neighborhood** will replace **intervals** in the topological view of this familiar concept.

6.2 DEFINITION 8.16 (Continuous function: the topological definition):

A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** if $f^{-1}(S)$ is **open** whenever S is **open**.

Formally, a function f is **continuous** if $\forall S \subseteq \mathbb{R}, (S \text{ is open} \Rightarrow f^{-1}(S) \text{ is open})$, and we call that function f is **continuous($\mathcal{T}$)**.

A function f that is **not continuous** if $\exists S \subseteq \mathbb{R} \ni (S \text{ is open and} f^{-1}(S) \text{ is not open})$. Alternatively, this definition says "**inverse images of open sets are open**".

For example, a single jump point could map an open interval whose one end is in the gap to a closed interval.

6.3 THEOREM 8.17 (Composition of Continuous Functions):

Compositions of continuous functions are continuous.

6.4 THEOREM 8.18 (Equivalence of two definitions):

A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous(ϵ)** if and only if it is **continuous($\mathcal{T}$)**.

6.5 Exercise 8.6.3 (Inverse image of closed sets):

If f is continuous, the **inverse images of closed sets** are **closed**.

The proof of this needs the result in **Exercise 8.5.4 (d)**.

6.6 Exercise 8.6.6 (Continuous function and discrete and indiscrete topology):

- (a) If X has the **discrete topology**, every function whose **domain** is X is continuous. This is true regardless of the topology of the **codomain** because every set in the domain is an open set.

- (b) If Y has the **indiscrete topology**, every function whose **codomain** is Y is continuous. This is true regardless of the topology of the **domain** because the only open sets here are the whole and the empty set whose inverse image is also the whole and the empty set.

- (c) **Constant functions** are **always continuous** because the inverse images are either empty or the whole.

- (d) If X has the **indiscrete topology** and Y does not, the only functions $f : X \to Y$ that are continuous are **constants**.

- (e) The **identify function** $f(x) = x$ from a **topological space** to itself is always continuous.

- (f) Suppose X_1 and X_2 are **equal as sets** but have **different topologies** (and so they are **not equal as topological spaces**). The identify function $f : X_1 \to X_2$ need not be continuous.

- (g) The condition we need for (f) to guarantee the identity function to be continuous is that X_1 has a **finer topology** than X_2, i.e. every **open set** in X_2 is also an **open set** in X_1.

7 Relative Topologies

7.1 DEFINITION 8.19 (*open and relative topology):

If $T \subseteq S \subseteq \mathbb{R}$, we say T is ***open** in S if there is an open subset U of $\mathbb{R}$ so that $T = S \cap U$.

The collection of ***open subsets** of a set S is called the **relative topology on** S **inherits from** $\mathbb{R}$.

7.2 DEFINITION 8.20 (Continuous under relative topology):

If $S, T \subseteq \mathbb{R}$, a function $f : S \to T$ is **continuous** if the inverse image of any *open set in T is a *open set in S.

This definition can be applied to any function between **topological spaces**.

7.3 EXAMPLE 8.7:

1. (***open under N**): Each **natural number** is a ***open subset of N**. Every subset of N is *open. Therefore, **every function whose domain is N is *open**. Also see **Exercise 8.7.4 (c)** for another view of the problem.
2. Let $S = [0, 1]$. Then $(1/2, 1]$ is *open in S.
3. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, it is also continuous when its domain is **restricted** to $[0, 1]$. In fact, every continuous function on $[0, 1]$ arises in this way. We can always **extend** a function continuous on $[0, 1]$ to continuous on $\mathbb{R}$ (see **Exercise 8.7.5 (a)**). This is also related to **Exercise 8.7.3 (a)**.
4. Let $S = (0, 1)$. A subset of S is *open if and only if it is open when considered as a subset of $\mathbb{R}$. A function could be continuous on S but could not obtained through a restriction to S of a continuous function on $\mathbb{R}$. An example of that would be $f(x) = 1/x$.

7.4 Exercise 8.7.1 (Relative topology is a topology):

If $S \subseteq \mathbb{R}$, the ***open subsets** of S are a **topology** on S.

7.5 Exercise 8.7.2 (open and *open):

- (a) If S is **open** and $T \subseteq S$, T is ***open** if and only if T is **open**.

- (c) if $T \subseteq S$ and T is **open**, T is ***open** no matter what S is.

7.6 Exercise 8.7.3 (Continuous on relative topology):

- (a) Suppose $Y \subseteq X$ and f is **continuous** on X. f is **continuous** on Y.
- (b) f could be continuous on Y but not continuous on X. **EXAMPLE 8.7.4** is an example.
- (c) If f is continuous on $A \cup B$, it is continuous on A and B.
- (d) If f is continuous on $\bigcup_{\alpha \in S} A_\alpha$ for any set S, it is continuous on A_α.
- (e) The analogue of (c) and (d) for intersection is **NOT** always true. A counterexample is $A = [0, 2]$ and $B = [1, 3]$ and a function that is continuous on $[1, 2]$ but not continuous either on $[0, 1)$ or $(2, 3]$.
- (f) If f is continuous on A and on B, then it is continuous on $A \cap B$.
- (g) the analogue of (f) for union is **NOT** always true. A counterexample is a function that is continuous on $(-\infty, 1)$ and $[1, \infty)$ and jump happened on 1.

7.7 Exercise 8.7.4 (Isolated points and continuous function):

- (a) Suppose $x \in S$ is an **isolated point** of S. Then $\{x\}$ is ***open** in S.
- (b) With S and X as in (a), every function defined on S is **continuous(ϵ)** at x.
- (c) If S is a set with **no cluster points**, every function defined on S is **continuous**.

7.8 Exercise 8.7.5 (Extension of a continuous function):

- (a) Suppose $f : [a, b] \to \mathbb{R}$ is **continuous**. The following function

  $$g(x) = \begin{cases} f(a) & \text{if } x < a \\ f(x) & \text{if } a \leq x \leq b \\ f(b) & \text{if } x > b \end{cases}$$

  is continuous on the **whole real line**. This is called an **extension** of f. There are **many ways** to extend a function.
- (b) If the **domain** of f in (a) is an **open interval**, it might not be possible to extend it to a continuous function on the whole real line. A counterexample is presented in **EXAMPLE 8.7.4**.
- (c) Suppose $f : \mathbb{R} \to \mathbb{R}$ is **continuous** and that $B = f(\mathbb{R})$. Then f considered as a function $f : \mathbb{R} \to B$ is continuous.

7.9 Exercise 8.7.6 (Isolated points and discrete topology):

- (a) The **topology** that N inherits from $\mathbb{R}$ is the same as the **discrete topology**.
- (b) If $S \subseteq \mathbb{R}$ and every point of S is an **isolated point**, the **topology** that S inherits from $\mathbb{R}$ is the **discrete topology**.
- (c) If $S \subseteq X$ and X has the **discrete topology**, then S has the discrete topology; and if X has the **indiscrete topology**, then S inherits the indiscrete topology.