Chapter 5 Upper Bounds and Suprema

April 4, 2017

0.1 THEOREM R:

If F is an **ordered field** having the **Least Upper Bound property**, then F has the **Archimedean property** and the following results also hold in F. (The **Least Upper Bound property** is discussed in **Chapter 5**; the **Archimedean property** is discussed in **Chapter 6**.)

- (a) Every **nest** of **closed, bounded intervals** in F has a **nonempty intersection**. (This is called the **Nested Intervals property — Chapter 6**.)
- (b) Every **bounded, infinite subset** of F has a **cluster point**. (This is called the **Bolzano-Weierstrass theorem — Chapter 7**.)
- (c) A **sequence** in F **converges** to an element of F if and only if it is a **Cauchy sequence**. (This is called the **Cauchy criterion — Chapter 10**.)
- (d) A subset of F is **compact** if and only if it is **closed and bounded**. (This is called the **Heine-Borel theorem — Chapter 11**.) +(e) F is **connected**. (**Chapter 12**.)

The **Least Upper Bound property** and parts (a) through (e) of **Theorem R** are not just loosely related statements about the real numbers; they are **equivalent**, that is, they describe the same property of the real numbers. This property is called **completeness** (or **Dedekind completeness**).

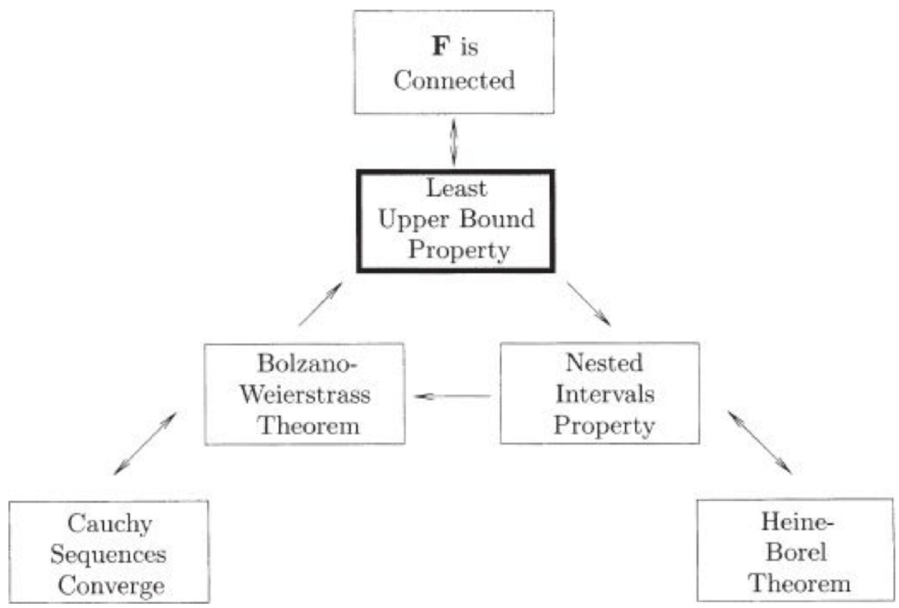
We may define the **real numbers** to be a **complete, Archimedean ordered field**. The **rational numbers** also have the **Archimedean property**; it is the **completeness** that makes the real numbers special.

0.2 THE BIG THEOREM:

If F is an **ordered field**, the following are **equivalent**:

- (a) F has the **Least Upper Bound property**.
- (b) F has the **Archimedean property**, and the **Nested Intervals property**.
- (c) F has the **Archimedean property**, and the **Bolzano-Weierstrass theorem** holds in F.
- (d) The **Heine-Borel theorem** holds in F.
- (e) F has the **Archimedean property**, and the **Cauchy criterion** holds in F.
- (f) F is **connected**.

0.3 THE BIG PICTURE



The big picture