Chapter 6 Nested Intervals

April 6, 2017

1 The Integer Part of a Number: The Archimedean Property

1.1 THEOREM 6.1 (Archimedean Property):

Let (F, P) be an **ordered field** having the **Least Upper Bound property**, and let x be any element of F. Then there is a natural number n_x with $n_x > x$.

This means **Least Upper Bound property implies Archimedean Property**. The completeness of real numbers is actually about two different things. One is that there should be "**no missing point**", or "**no gap**" according to Dedekind's terminology, in the real line. All six theorems in the BIG THEOREM can achieve this goal. Another point is that there should **NOT be "too many" points**. In such cases of large non-Archimedean fields, real numbers could be isomorphic to a subfield or that ordered field, but not the whole. This property is guaranteed by the **Archimedean property**. Not all theorems listed in the BIG THEOREM implies the Archimedean property. We group the theorems into two groups. The **Least Upper Bound** (also called the **Dedekind complete property**), the **Heine-Borel theorem**, the **connectedness property**, and the **Dedekind Cut property** listed in Chapter 22, implies the Archimedean property. On the other hand, the **Nested Intervals property**, the **Bolzano-Weierstrass theorem** and the **Cauchy Complete property**, do not imply the Archimedean property. This is actually a statement that the **Cauchy completeness is weaker than the Dedekind completeness**.

THEOREM 6.1 states that an ordered field with Least Upper Bound property is Archimedean. However, for ordered fields that Least Upper Bound property is not held, such as rational numbers, it is still possible to be Archimedean.

A **non-Archimedean ordered field** contains numbers that are greater than any natural number, i.e. **infinity**, and therefore **infinitesimals**. The existence of infinitesimals implies existence of infinity. When an ordered field is not Archimedean, one importance property does not hold, i.e. $\lim_{n\to\infty} \frac{1}{n} \neq 0$. This means that $\inf \frac{1}{n}$ is the infinitesimal rather than zero because zero is a lower bound but not the largest lower bound. An example of non-Archimedean field is the **formal rational functions field**.

Exercise 7.6.5 shows that **Bolzano-Weierstrass Theorem** fails in any non-Archimedean ordered field. However, an extended version of "κ-Bolzano-Weierstrass" may hold in certain very large non-Archimedean fields. See *Real analysis in reverse* to find such an example.

1.2 COROLLARY 6.2 (Properties of Archimedean Fields):

If (F, P) is an **ordered field** having the **Least Upper Bound property**, then

- (a) If $x \in P$, there is a natural number n_x with $n_x - 1 < x \leq n_x$.

- (b) If $x \in P$, there is a natural number n_x with $\frac{1}{n_x} < x$.
- (c) For any $x, y \in P$, there is a natural number n so that $nx > y$.
- (d) (**Density Theorem**) Every **nonempty open interval** in F contains both a **rational element** of F and an **irrational element** of F.

A subset of the real numbers is called **dense in the real line** (or just **dense**) if its intersection with any nonempty open interval is nonempty. **Corollary 6.2(d)** says that the rational numbers and the irrational numbers are both dense in the real line. It is often called the **Density theorem** for this reason.

1.3 Exercise 6.1.1 (Formal Rational Function Field is non-Archimedean):

- (a) The **formal rational functions** are an ordered field with the positive set defined by saying $\frac{p(x)}{q(x)}$ is positive if the coefficients of the highest-order terms of $p(x)$ and $q(x)$ have the same sign. It is not hard to show that
  - 1) this set is closed to addition and multiplication,
  - 2) any element either belongs to it, or its negation belongs to it unless it's the additive identity - zero.

By **formal**, it means it is defined by its form, aka what it looks like. It is similar to rational numbers and can be shown that it satisfies the field axioms in a similar way.

- (b) We have $(x/1) - (n/1) \in P,\ \forall n$, which means there exists an element x that is **greater than all natural numbers** in this ordered field.

1.4 Exercise 6.1.2 (Extension of Density Theorem):

- (c) **COROLLARY 6.2.d** can be extended as the following. If (F, P) is an ordered field having the **Least Upper Bound property**, then for $a > b$, both $(a, b) \cap \mathbb{Q}$ and $(a, b) \cap \mathbb{R} \setminus \mathbb{Q}$ are infinite. (**COROLLARY 6.2.d** only states that they are **non-empty**).
- (d) If D is any **dense subset of the real numbers** and $a < b$, then $(a, b) \cap D$ is infinite.
- (e) No **finite set** can be dense in real line.

1.5 Exercise 6.1.3 (Jensen's inequality):

- (a) if $0 < t < 1$ and $a < b$, then $a < at + (1 - t)b < b$

1.6 Exercise 6.1.9 (Coterminal Sets):

- (a) If T is a **linearly ordered set** and , we say that S and T are **coterminal** if for each $t \in T$ there is an $s \in S$ with $s > t$, and vice versa. If T and S are coterminal ordered fields, then T is Archimedean if and only if S is Archimedean.

1.7 Exercise 6.1.10 (Set of Upper Bounds):

Suppose S and U are sets with the following two properties:

- (i) Each element of U is an **upper bound** for S;
- (ii) For any $n \in N$, there are $s \in S$ and $u \in U$ with $u - s < \frac{1}{n}$;

Then,

- (a) Each element of S is a **lower bound** for U.
- (b) $\sup S = \inf U$
- (c) If u is an upper bound a set X with the property that for any $n \in N$ there is an $x \in X$ with $u - x < \frac{1}{n}$, then $u = \sup X$.
- (d) u is an upper bound for a set X if and only if $u + \frac{1}{n}$ is an upper bound for X for all $n \in N$.
- (e) $u = \sup X$ if and only if for all $n \in N$, $u + \frac{1}{n}$ is an upper bound for X but $u - \frac{1}{n}$ is not.

1.8 Exercise 6.1.12 (Properties of Dense Sets):

- (a) If D is **dense** in the real line and $D \subseteq S$, then S is **dense** in the real line as well.
- (b) If S is dense in the real line and a **finite number of points** are removed from S, the resulting set is also dense in the real line.
- (d) Every dense set has a **proper subset** that is also dense.

1.9 Exercise 6.1.13 (Dyadic Rationals):

The **dyadic rationals** are rational numbers of the form $\frac{n}{2^m}$ for some integer n and some natural number m.

- (a) Not every rational number is a dyadic rational.
- (b) The set of dyadic rationals is **countable**.
- (c) The set of dyadic rationals is **dense** in the real line.
- (d) All of this is true if the denominators of the fractions are powers of any natural number greater than 1.

1.10 Exercise 6.1.14 (Natural Numbers in a non-Archimedean Ordered Field):

- (a) No matter Archimedean or not, the set of **natural numbers** does not have a **supremum** in any ordered field.
- (b) Let F be a **non-Archimedean ordered field**, and let

$$U = \{x : \text{ x is an upper bound for N}\}$$

, then we have

– (i) $U \neq \emptyset$
– (ii) U is bounded below but has no infimum.

The nonexistence of the supremum (i.e. (a)) does not mean the nonexistence of upper bounds (i.e. (b)). It is just the case that the infimum of the upper bounds, thus the supremum does not exists. This exactly means that a **non-Archimedean ordered field must fail the Least Upper Bound axiom** because the **natural numbers in a non-Archimedean ordered field must have upper bounds but no the least upper bound**.

2 Nest

2.1 DEFINITION 6.4 (Nest):

A collection of sets, $S_1, S_2, S_3, ...$, is a **nest** if $S_n \subseteq S_{n+1},\ \forall n \in N$.
The **consecutive endpoints** could be the same.

2.2 THEOREM 6.5 (The Nested Intervals Property):

If F is an **ordered field** having the **Least Upper Bound property**, then

- (a) Any nest $\{I_n\}$ of **nonempty, closed, bounded intervals** has a nonempty intersection, i.e. $\bigcap_n I_n \neq \emptyset$.
- (b) If the **infimum of the lengths of the intervals** $\{I_n\}$ is 0, there is an element of F, say x, such that $\bigcap_n I_n = \{x\}$.

The **nested intervals property** implies that ordered field "**has no hole**", i.e. **Dedekind-complete**. The ordered field could still has "**layers**", like those **non-Archimedean** fields.

2.3 Exercise 6.2.1:

Using the notation of **THEOREM 6.5**, we also have

$$\bigcap_n [a_n, b_n] = [\sup a_n, \inf b_n]$$

3 The Fractional Part of a Number: Decimal Expansions

3.1 THEOREM 6.6 (Fractional Part of a Number):

In an **Archimedean ordered field** in which the **Nested Intervals property** holds, there is a **one-to-one correspondence** between the interval $I = (0, 1]$ and the **nonterminating decimal expansions** of the form $0.d_1d_2d_3....$

3.2 COROLLARY 6.7 (Uncountability of Real Numbers):

There are **uncountably many** real numbers. There are uncountably many irrational numbers.

3.3 Exercise 6.3.2:

Every element of a **complete ordered field** has an **expansion in any number base**.

3.4 Exercise 6.3.7 (Repeating Decimals):

- (a) Any **repeating decimal** represents a **rational number** and that every rational number is represented by a repeating decimal.
- (b) There are **countably many** repeating decimals.
- (c) There are **uncountably many non repeating decimals**.