Chapter 9 Sequences

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1 An Approximation Problem

1.1 DEFINITION 9.1 (Real Sequence):

A real sequence is a function . Real here refers to the range of the function. We should distinguish the sequence in which the order is important from the range which is a set and doesn't have order by itself.

2 Convergence

2.1 DEFINITION 9.2 (Convergence and Limit):

- (a) A sequence is said to converge to L if, for every $\varepsilon > 0$, there is a number so that . If this is the case, we say L is the limit of and write . (This is usually called the "ε-N version" of the definition.)
- (b) A sequence is said to converge if there is a number L so that converges to L. A sequence that does not converge is said to diverge.

Note that L must be a number. A sequence cannot converge to . Also, it is reasonable to assume depends on ε. If we make our "error tolerance" smaller, we expect to be required go further along in the process to achieve it.

2.2 ALTERNATE DEFINITION 9.3 (Convergence and Limit):

converges to L if, for each neighborhood V of L, there is a number so that .

2.3 DEFINITION 9.4 (Eventually and Frequently):

- (a) A sequence is eventually in the set S if there is a number N so that .
- (b) A sequence is frequently in a set S if for any natural number N, there is an n > N for which .

2.4 FINAL DEFINITION 9.5 (Convergence):

The sequence converges to L if it is eventually in any neighborhood of L. The deep part of the theory of convergent sequences can be stated as "if the terms in a sequence get close to a limit, they must get close to each other". This is what behind Cauchy convergence.

2.5 Exercise 9.2.3 (Eventually vs Frequently):

- (a) Eventually => Frequently
- (b) The inverse of (a) is not true.
- (c) A sequence is eventually in a set if there are only finitely many values of n for which it is not in the set.
- (d) A sequence is frequently in a set if it is in the set for infinitely many values of n.

2.6 Exercise 9.2.7:

A sequence cannot be eventually in both of two disjoint sets.

2.7 Exercise 9.2.8:

If there is an $\varepsilon > 0$ so that is not eventually in any interval of length ε, then divergences.

2.8 Exercise 9.2.9 (Limit of a sequence and the limit of its rolling average):

Given a sequence of , define a sequence by . If , then .

The inverse of above is not true. It is possible that converges to L but diverges. An example would be is the sum of a convergent sequence and an error term of alternating +1 and -1, i.e. . The arithmetic average of the error term converges to zero. So converges to the limit of .

3 Convergent Sequences

3.1 THEOREM 9.6 (Uniqueness of the limit of a sequence and bounded sequences):

A sequence can have at most one limit. We say a seqeunce is bounded if its range is a bounded set.

It is worth noting that the proof of this theorem depends on the result of Exercise 4.10.4, which states "If $x \neq y$, show that there are neighborhoods U of x and V of y such that ". This is not true in every topological space. It depends very much on the separation property of the topological space. A space with this property is called a Hausdorff space. See also Exercise 9.3.6.

3.2 THEOREM 9.7:

A convergent sequence is bounded.

The converse of above is not true. A bounded sequence could well be divergent, such as the counterexample in Exercise 9.2.9.

3.3 THEOREM 9.8 (Convergence of a sequence and convergence its distance to its limit):

Let be sequence, , and . Then .

3.4 Exercise 9.3.2 (Alternative condition for convergence):

if and only if the following holds: given $\varepsilon > 0$ and any positive real number b, there is an so that .

3.5 Exercise 9.3.4:

- (a) If and is bounded, then no matter converges or not.

3.6 Exercise 9.3.5 (Convergence of the absolute value of a sequence):

- (a)
- (b) the converse of (a) is not true
- (c)

3.7 Exercise 9.3.6 (Hausdorff Space):

A topological space is called a Hausdorff space if it has the following property: If $x \neq y$, there are neighborhoods U of x and V of y such that .

Suppose X is a topological space that does not have above property, there must a sequence in X that converges to two different limits.

4 Sequences and Order

4.1 THEOREM 9.9 (Squeeze theorem):

- (a) If converges and , then .
- (b) If and converge and , then .
- (c) If and both converge to L and , then converges and .

(c) is often called the Squeeze Theorem. An example is the following.

4.2 EXAMPLE 9.4:

Since ; then, .

4.3 COROLLARY 9.10:

If , then .

4.4 Exercise 9.4.4:

- (a) If , and converges, then

4.5 Exercise 9.4.6 (Positive set of R):

The positive set of is defined as; for , it is in the positive set if p and q are either both natural number of both additive inverse of natural numbers. Then the positive set of is defined as:

5 Sequences and Algebra

5.1 THEOREM 9.11 (Algebra of the limits of sequences):

Suppose . Then

- (a)
- (b)
- (c)
- (d)

6 Sequences and Topology

6.1 DEFINITION (Cluster point of a seqeunce):

A point is a cluster point of a sequence if it is a cluster point of its range.

6.2 LEMMA 9.12 (Convergent sequences and cluster points):

A convergent sequence can have at most one cluster point, its limit.

However, a cluster point of a sequence is not necessary its limit (the sequence could be not convergent). The limit of a convergent sequence is not necessary a cluster point.

6.3 THEOREM 9.13 (Limits and cluster points):

If , one of the following is true:

- (i) is eventually equal to L,
- or (ii) L is the only cluster point of .

6.4 LEMMA 9.14 (Cluster point of a set):

The point c is a cluster point of a set S if and only if there is a sequence of elements of S, all different from c and converging to c.

The proof of the "only if" part depends on the Archimedean property because it uses the $1/n$ to define a next of smaller and smaller neighborhood. If the field is not Archimedean, $1/n$ does not converge to 0. On a deeper level, the result relies on the fact that the topology of the real number line is first countable, which makes sure that the convergence can be done by a sequence, which is indexed and therefore countable.

6.5 THEOREM 9.15 (Define closed sets by convergent sequences):

A subset S of is closed if and only if whenever is a convergent sequence whose terms are all in S.

6.6 THEOREM 9.16 (Define continuous function by convergent sequences):

A function is continuous if and only if, for each convergent sequence , .

6.7 Exercise 9.6.4:

Suppose S is a nonempty open set that isn't the whole real line. Then, there is a sequence of elements of S that converges to an element of C(S).

6.8 Exercise 9.6.6:

- (a) A sequence that is eventually constant converges.
- (b) If the topological space X has the discrete topology, the only sequences that converge are those that are eventually constant.
- (c) If the topological space X has the indiscrete topology, every sequence converges to every element in X.

7 Subsequences

7.1 DEFINITION 9.17 (Subsequence):

- (a) A function is said to be strictly increasing if
- (b) If is a sequence and is strictly increasing, then is called a subsequence of x.

The whole point of having the strictly increasing function n is that the subsequence, though consists of infinitely many terms of , are kept in the same order of .

An interesting result is that a divergent sequence could have convergent subsequences.

7.2 THEOREM 9.18 (Convergence of subsequences):

If the sequence converges, so does every subsequence of it and they all converge to the same limit.

7.3 COROLLARY 9.19 (Divergent sequences):

If a sequence has subsequences that converge to two different limits, the sequence diverges.

7.4 Exercise 9.7.1 (Subsequence of subsequences):

If Y is a subsequence of X and Z is a subsequence of Y, then Z is a subsequence of X.

7.5 Exercise 9.7.2 (Strictly increasing function):

- (a) If is strictly increasing, then .
- (b) A real function is strictly increasing if . Even if g is strictly increasing, it is not necessarily the case that .

7.6 LEMMA 9.7.3 (Irrational number's fractional part):

Given e>0 and an irrational number a, there exists a natural number n such that (na) < e, where (na) means the fractional part of na.

7.7 Exercise 9.7.3 (Subsequences of cos(n) and sin(n)):

The subsequences of can converge to any real number in [-1,1].

7.8 Exercise 9.7.4 (sin(1/x) at 0):

cannot be defined at 0 in such a way to make it continuous because there are subsequences that can converge any value between -1 and 1. COROLLARY 9.19 states that it will diverge at 0.