Chapter 8 Topology of the Real Numbers

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DEFINITION 8.1 (Open Set): A set is open if it is a neighborhood of each of its points.

Example 8.1 (Open Intervals and Empty Set): (1) Open intervals are open. The whole real line is open. (2) The empty set is open.

THEOREM 8.2 (Union of two open sets): The union of two open sets is open.

THEOREM 8.3 (Union of open sets): The union of any collection of open sets is open.

THEOREM 8.4 (Intersection of finite collection of open sets): The intersection of any finite collection of open sets is open.

Note the difference between THEOREM 8.3 and THEOREM 8.4. The union allows infinitely many open sets but the intersection has to be a finite collection. Exercise 8.1.4 is the counter example if we do infinitely many intersection.

Compare THEOREM 8.3 and THEOREM 8.4 to THEOREM 8.8 and THEOREM 8.9, there are an interesting mirror image between the properties of open sets and closed sets.

Exercise 8.1.4:

DEFINITION 8.5 (Topology and Topological Space): Let X be a set and a collection of subsets of X such that: (i) . (ii) The union of any collection of sets from is in . and (iii) The intersection of any finite collection of sets from is in . Then is called a topology on X. The pair is called a topological space, and the elements of are called open subsets of X.

Note that union is ANY union while intersection is FINITE intersection.

Note also that, it doesn't require complement of a set to be in the topology. In fact, the complement of an open set is a closed set which should not be in the topology.

This definition draws some analogue to the definition of sigma-algebra. In sigma algebra, it requires closed under countable unions and complements.

An open set in one topology might not be open in another topology over the same universal set X. If we talk about open set/interval, we should be specific which topogoly we are talking about. Real anlysis concerns about the Euclidean topology. But that is not the only topology over the real numbers.

A proof or definition that is topological, that is, one that uses only facts about open sets, is usually preferable to one that uses other structures (such as order, algebra, or distance).

Exercise 8.2.1 (Finite Complement Topology): (e)/(g) If X is any set, is topology on X, which is called the finite complement topology.

Exercise 8.2.2 (Dense Sets): (a) A set is dense in the real line if its intersection with any open set is nonempty. (c) Let consist of , and all sets of the form for any , then is a topology on .

Exercise 8.2.3 (Discrete Topology and Indiscrete Topology): (a) If , this topology is called the indiscrete topology, which is the smallest or coarsest topology on X. If , this topology is called the discrete topology, which is the biggest or finest topology on X. (c) Any nonempty set is dense in any space having the indiscrete topology. (e) If X is finite, the finite complement topology on X is the same as the discrete topology. (f) If X is infinite, the finite complement topology on X is not the same as the discrete topology.

THEOREM 8.6 (Closed Sets): A set is closed if it contains all its cluster points, that is .

Example 8.3: (1) Closed intervals and the whole real line are closed sets. (2) The empty set is closed since it has no cluster points. (4) “Not closed” is not the same as “open”.

THEOREM 8.7: A set having no cluster points is closed.

THEOREM 8.8 (Union of closed sets): The union of finitely many closed sets is closed.

THEOREM 8.9 (Intersection of closed sets): The intersection of any collection of closed sets is closed.

THEOREM 8.10 (Traditional Definition of Closed Sets): A set is open if and only if its complement is closed.

Again, closed does not mean “not open”. There are sets that are neither open nor closed.

Exercise 8.3.1 (Bounded Closed Set Contains Its Supremum and Infimum): A nonempty, closed and bounded set contains its supremum and infimum.

The converse of this is not true. THEOREM 5.3 implies that an open interval cannot contains its supremum or infimum. But it could be the case that the set contains an open interval which makes the set neither open nor closed.

Exercise 8.3.2 (Closure, Interior and Boundary): The closure of a set S, denoted , is the intersection of all closed sets containing S. The interior of S, denoted , is the union of all open sets contained in S. The boundary of S, denoted , is . (a) is the smallest closed set containing S, and is the largest open set contained in S. (b) if and only if every neighborhood of x contains points of S and of C(S). Elements of are called boundary points. (f) S is closed if and only if . S is open if and only if . (g) is closed.

Exercise 8.3.3 (Closure, Interior and Boundary of Rational Numbers): (a) Under standard topology, . (b) For the topology in Exercise 8.2.2 (c), which is , we also have .

Exercise 8.3.4 (Boundary Points): (a) Let S be a bounded set. Then (b) If S is nonempty and S has no boundary points, then S is unbounded. (c) An interior point of a set cannot be a boundary point.

Exercise 8.3.5 (Kuratowski’s Problem): (b) Construct a set, take the closure of the set, then the complement of its closure, and the closure of the complement of its closure. Repeat this over and over: closure -> complement -> closure -> ... The question is how many different sets can we obtain in this way. This is called Kuratowski’s problem, and the answer is 14. A subset realizing the maximum of 14 is called a 14-set. The space of real numbers under the usual topology contains 14-sets.

THEOREM 8.11 (Structure of Open Sets): A nonempty set S is open if and only if there is a countable collection of mutually disjoint open intervals such that .

Exercise 8.4.3 (Open Sets a countable union of closed sets): (a) Any open interval is a countable union of closed intervals (not necessarily disjoint). (b) Any open set is a countable union of closed intervals.

Exercise 8.4.5 (Countability of Open Sets): If a topological space has a countable and dense subset, then any collection of mutually disjoint open subsets of X is countable.

Exercise 8.4.6 (First Countable and Second Countable): A topology is called first countable if it has the following property: For each point x in the topological space, there is a countable collection of open sets , each containing x and such that if U is any open set containing x, there is an n so that . The collection is called a local neighborhood base at x. Notice that the collection probably changes from point to point. A topology is called second countable if there is a single countable collection such that, if x is in the space and U is an open set containing x, then there is an n so that . Such a collection is called a base (or basis) for the topology. Note that a basis need not be countable.

(a) The standard topology on the real line is first countable. (b) The standard topology on the real line is second countable. (c) Any second countable topology is also first countable. (f) Cofinite topology is an example of a topology that is not first countable. Any uncountable discrete space is first countable but not second countable. (g) If is a basis for a topology , then every element of can be written as a union of sets in . (h) If is a topological space and is a collection of subsets of X with the property that every element in can be written as a union of sets, each of which is the intersection of finitely many of the sets in . is called a subbasis for . The collection of open rays is a subbasis for the standard topology on the real line.

Exercise 8.4.7 (Cantor Sets): The Cantor set C is constructed as the following: Let , and (we refer to as the “open middle third” of). Let . Then consists of two closed intervals. Now each of these intervals also has an open middle third. Let — the two open middle thirds of — and let . Then consists of four closed intervals. Remove the four open middle thirds of to obtain , and so on. Note that for all n. Let . (a) C is closed. (b) If , there is a number . (c) The sum of removed intervals is 1. (d) The length of C is therefore 0. (e) C consists of all elements of whose ternary expansion contains no 1s. (f) C is uncountable. Therefore C is an example whose cardinality is as high as but whose length is 0. The connection between cardinality and length is mystery. (g) The end points of C is countable. So, there are non-end-point elements in C. An example would be 1/4. (i) A fat Cantor set (or Smith-Volterra-Cantor set) can be constructed by removing 1/4 instead of 1/3. This fat Cantor set has a non-trivial length of 1/2. More interestingly, the set is still uncountable and not dense in any place. This is a very intriguing example. By construction, the fat Cantor set contains no intervals and therefore has empty interior. It is also the intersection of a sequence of closed sets, which means that it is closed. However, the part removed only has a positive Lebesgue measure of 1/2. This makes the fat Cantor set an example of a closed set whose boundary has positive Lebesgue measure of 1-1/2=1/2. (j) There is a one-to-one correspondence between C and by converting the ternary expansion to binary expansion, which is why C is uncountable. (k) Since C is closed, it contains all its cluster points, that is . C is also perfect, which means . (l) Every nonempty perfect set is uncountable.

DEFINITION 8.12 (Direct Image, range, onto and inverse function): Let , and . The direct image of S under f is given by

The direct image of the entire domain, i.e. f(A), is sometimes called the range of f. The set B, in this case sometimes called the range, but not necessarily all are output of f, is called codomain of f to avoid confusion.

So, f is an onto if and only if B=f(A).

If is a one-to-one correspondence, it has an inverse function denoted , whose domain is B and whose range is A and which is defined by: .

The direct image of a subset of the range of f under , i.e. , is given by . A little bit of relaxation on the requirement of S leads to the following definition of inverse image.

DEFINITION 8.13 (Inverse Image): Let and . The inverse image of S under f is given by .

Inverse images can always be found, and so they are much more useful to us than direct images under inverse functions. The direct image of the inverse function may not exist if the inverse function does not exist (in other words, the function itself is not an onto).

THEOREM 8.14 (Intersection of Direct and Inverse Image): Suppose . Then, (a) If , then (b) If , then

Always keep in mind, the relationship between inverse and direct images and algebraic operations on functions is very subtle. The inverse image has a much better property than the direct image.

THEOREM 8.15 (Composition of functions): If , and , then .

Exercise 8.5.4 (Set Algebra on Direct and Inverse Images): (a) (b) (c) (d) . Comparing (d) to (b), it is obvious that the inverse image has better behavior than the direct image. (g) (h) (i) if f is a one-to-one, then

Exercise 8.5.7 (One-to-one): (a) Suppose and . The cardinality of is no less than the cardinality of S. (b) A function is one-to-one if and only if it has the property that the inverse image of any set with one element has at most one element.

DEFINITION (Continuous function: the ε-δ definition) Function f is continuous at a if for every $\varepsilon > 0$, there is a $\delta > 0$ so that $\mid f(x) - f(a) \mid < \varepsilon$ whenever $\mid x - a \mid < \delta$. We can call this that f is .

The ε-δ definition depends on the concept of distance which in turn implicitly refers to the concept of ordering. It would be nice to have a topological definition of this concept of continuous function that does not depend on these extra structures. Neighborhood will replace intervals in the topological view of this familiar concept.

DEFINITION 8.16 (Continuous function: the topological definition): A function is continuous if is open whenever S is open. Formally, a function f is continuous if , and we call that function f is . A function f that is not continuous if Alternatively, this definition says "inverse images of open sets are open".

For example, a single jump point could map an open interval whose one end is in the gap to a closed interval.

THEOREM 8.17 (Composition of Continuous Functions): Compositions of continuous functions are continuous.

THEOREM 8.18 (Equivalence of two definitions): A function is if and only if it is .

Exercise 8.6.3 (Inverse image of closed sets): If f is continuous, the inverse images of closed sets are closed. The proof of this needs the result in Exercise 8.5.4 (d).

Exercise 8.6.6 (Continuous function and discrete and indiscrete topology): (a) If X has the discrete topology, every function whose domain is X is continuous. This is true regardless of the topology of the codomain because every set in the domain is an open set. (b) If Y has the indiscrete topology, every function whose codomain is Y is continuous. This is true regardless of the topology of the domain because the only open sets here are the whole and the empty set whose inverse image is also the whole and the empty set. (c) Constant functions are always continuous because the inverse images are either empty or the whole. (d) If X has the indiscrete topology and Y does not, the only functions that are continuous are constants. (e) The identify function f(x) = x from a topological space to itself is always continuous. (f) Suppose and are equal as sets but have different topologies (and so they are not equal as topological spaces). The identify function need not be continuous. (g) The condition we need for (f) to guarantee the identity function to be continuous is that has a finer topology than , i.e. every open set in is also an open set in .

DEFINITION 8.19 (*open and relative topology): If , we say T is* open in S if there is an open subset U of so that . The collection of *open subsets of a set S is called the relative topology on S inherits from .

DEFINITION 8.20 (Continuous under relative topology): If , a function is continuous if the inverse image of any *open set in T is a* open set in S. This definition can be applied to any function between topological spaces.

EXAMPLE 8.7: 1. (*open under N): Each natural number is a* open subset of N. Every subset of N is *open. Therefore, every function whose domain is N is* open. Also see Exercise 8.7.4 (c) for another view of the problem. 2. Let S=[0,1]. Then (1/2 ,1] is *open in S. 3. If is continuous, it is also continuous when its domain is restricted to [0,1]. In fact, every continuous function on [0,1] arises in this way. We can always extend a function continuous on [0,1] to continuous on (see Exercise 8.7.5 (a)). This is also related to Exercise 8.7.3 (a). 4. Let S=(0,1). A subset of S is* open if and only if it is open when considered as a subset of . A function could be continuous on S but could not obtained through a restriction to S of a continuous function on . An example of that would be .

Exercise 8.7.1 (Relative topology is a topology): If , the *open subsets of S are a topology on S.

Exercise 8.7.2 (open and *open): (a) If S is open and , T is* open if and only if T is open. (c) if and T is open, T is *open no matter what S is.

Exercise 8.7.3 (Continuous on relative topology): (a) Suppose and f is continuous on X. f is continuous on Y. (b) f could be continuous on Y but not continuous on X. EXAMPLE 8.7.4 is an example. (c) If f is continuous on , it is continuous on A and B. (d) If f is continuous on for any set S, it is continuous on . (e) the analogue of (c) and (d) for intersection is not always true. A counterexample is A = [0,2] and B = [1,3] and a function that is continuous on [1,2] but not continuous either on [0,1) or (2,3]. (f) If f is continuous on A and on B, then it is continuous on . (g) the analogue of (f) for union is not always true. A counterexample is a function that is continuous on (-inf, 1) and [1, inf) and jump happened on 1.

Exercise 8.7.4 (Isolated points and continuous function): (a) Suppose is an isolated point of S. Then {x} is *open in S. (b) With S and X as in (a), every function defined on S is at x. (c) If S is a set with no cluster points, every function defined on S is continuous.

Exercise 8.7.5 (Extension of a continuous function): (a) Suppose is continuous. The following function

is continuous on the whole real line. This is called an extension of f. There are many ways to extend a function. (b) If the domain of f in (a) is an open interval, it might not be possible to extend it to a continuous function on the whole real line. A counterexample is presented in EXAMPLE 8.7.4. (c) Suppose is continuous and that . Then f considered as a function is continuous.

Exercise 8.7.6 (Isolated points and discrete topology): (a) The topology that N inherits from is the same as the discrete topology. (b) If and every point of S is an isolated point, the topology that S inherits from is the discrete topology. (c) If and X has the discrete topology, then S has the discrete topology; and if X has the indiscrete topology, then S inherits the indiscrete topology.