Chapter 3 Algebra of the Real Numbers

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1 The Rules of Arithmetic

1.1 Observations:

- (0) If we add or multiply two real numbers, we get a real number (**closed to addition and multiplication**).
- (1) Addition is **associative**: If x, y, and z are any three real numbers, then $(x + y) + z = x + (y + z)$.
- (2) Addition is **commutative**: If x and y are any two real numbers, then $x + y = y + x$.
- (3) There is a special real number, 0 (the **additive identity**), having the property that $0+x = x$ for any real number x.
- (4) For each real number, x, there is a real number, denoted $-x$, with $x + (-x) = 0$ ($-x$ is the **additive inverse** of x).
- (5) Multiplication is **associative**.
- (6) Multiplication is **commutative**.
- (7) There is a special real number, 1 (the **multiplicative identity**), having the property that $1 \times x = x$ for every real number x.
- (8) For each real number **except 0**, there is another real number, denoted $1/x$ or x^{-1}, with $x \times x^{-1} = 1$ (x^{-1} is the **multiplicative inverse** of x).
- (9) **Multiplication distributes over addition**: If x, y, and z are any numbers, then $x \times (y + z) = (x \times y) + (x \times z)$.

2 Fields

2.1 DEFINITION 3.1 (Field):

A set F, with operations + and $\times$, obeying rules (0) through (9) above (with "real number" replaced by "element of F" everywhere it occurs) is called a **field**.

Both $\mathbb{R}$ and $\mathbb{Q}$ are fields. $\mathbb{Z}$ is not. The integer numbers only form a **ring** (a **commutative ring with unity** to be exactly) but not a field because the absence of **multiplicative inverse**.

$\mathbb{Z}_2$, however, is a field (a **finite field** to be exactly).

In general, $\mathbb{Z}_p$ where p is a **prime number** can form a field. There are **non-prime finite field** as well but they are constructed in a different way.

2.2 EXAMPLE 3.2.1 (The Field of Formal Rational Functions):

A **rational function** is any function which can be defined by a **rational fraction**, i.e. an **algebraic fraction** such that both the numerator and the denominator are **polynomials**. The coefficients of the polynomials need **not** be rational numbers, they may be taken in any field K. In this case, one speaks of a rational function and a rational fraction over K. The values of the variables may be taken in any field L containing K. Then the domain of the function is the set of the values of the variables for which the denominator is not zero and the codomain is L.

This is a very important example because it is an **ordered field** that is **not Archimedean**.

2.3 EXAMPLE 3.2.2 (The Field of Complex Numbers):

All **complex numbers** form a field.

This is an example of a field that **cannot be well ordered** (**linearly ordered**).

2.4 Exercise 3.2.7 (Uniqueness of 0 and 1):

Each element of a field has **only one additive inverse** and that each nonzero element has **only one multiplicative inverse**.

2.5 Exercise 3.2.9 (Trivial Field):

$0 \neq 1$ in **any field with more than one element**. (A field with only one element, aka a **trivial field**, is not very interesting.)