Chapter 5 Upper Bounds and Suprema

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0.1 THEOREM R:

If F is an **ordered field** having the **Least Upper Bound property**, then F has the **Archimedean property** and the following results also hold in F. (The **Least Upper Bound property** is discussed in **Chapter 5**; the **Archimedean property** is discussed in **Chapter 6**.)

- (a) Every **nest** of **closed, bounded intervals** in F has a **nonempty intersection**. (This is called the **Nested Intervals property — Chapter 6**.)
- (b) Every **bounded, infinite subset** of F has a **cluster point**. (This is called the **Bolzano-Weierstrass theorem — Chapter 7**.)
- (c) A **sequence** in F **converges** to an element of F if and only if it is a **Cauchy sequence**. (This is called the **Cauchy criterion — Chapter 10**.)
- (d) A subset of F is **compact** if and only if it is **closed and bounded**. (This is called the **Heine-Borel theorem — Chapter 11**.) +(e) F is **connected**. (**Chapter 12**.)

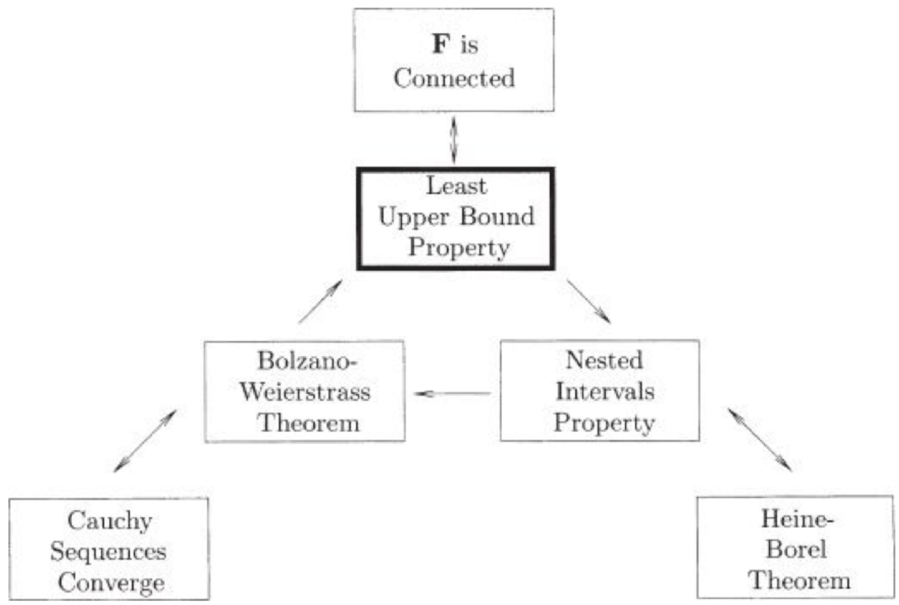
The **Least Upper Bound property** and parts (a) through (e) of **Theorem R** are not just loosely related statements about the real numbers; they are **equivalent**, that is, they describe the same property of the real numbers. This property is called **completeness** (or **Dedekind completeness**).

We may define the **real numbers** to be a **complete, Archimedean ordered field**. The **rational numbers** also have the **Archimedean property**; it is the **completeness** that makes the real numbers special.

0.2 THE BIG THEOREM:

If F is an **ordered field**, the following are **equivalent**:

- (a) F has the **Least Upper Bound property**.
- (b) F has the **Archimedean property**, and the **Nested Intervals property**.
- (c) F has the **Archimedean property**, and the **Bolzano-Weierstrass theorem** holds in F.
- (d) The **Heine-Borel theorem** holds in F.
- (e) F has the **Archimedean property**, and the **Cauchy criterion** holds in F.
- (f) F is **connected**.



The BIG PICTURE

1 Upper and Lower Bounds

1.1 DEFINITION 5.1 (Upper Bound and Lower Bound):

- (a) The number u is an **upper bound** for the set S if $s \leq u$ for each $s \in S$. If S has an upper bound, we say that it is **bounded above**.
- (b) The number w is a **lower bound** for the set S if $s \leq w$ for each $s \in S$. If S has a lower bound, we say that it is **bounded below**.

If a set has **any** upper bound, it has **infinitely many** of them.

1.2 DEFINITION 5.2 (Least Upper Bound and Greatest Lower Bound):

- (a) The number u is the **supremum** (or **least upper bound**) of the set S if
  - (i) u is an **upper bound** for S and
  - (ii) there is no upper bound for S less than u.
- (b) The number w is the **infimum** (or **greatest lower bound**) of S if
  - (i) w is a **lower bound** for S and
  - (ii) there is no lower bound for S greater than w.

The supremum of a set need not be in the set. If we happen to know that the supremum of a set is an element of the set, we will call it the **maximum** and write $\max S$ instead of $\sup S$. Similarly, if we are certain that the infimum of a set is an element of the set, we call it the **minimum** and write $\min S$ instead of $\inf S$.

Formally,

$$u = \sup S \iff (\forall s \in S \ni s \leq u) \text{ and } (v < u \Rightarrow \exists s \in S \ni (v < s))$$

and

$$l = \inf S \iff (\forall s \in S \ni s \geq l) \text{ and } (m > l \Rightarrow \exists s \in S \ni (m > s))$$

1.3 THEOREM 5.3 (Supremum and ε-intervals):

u is the supremum of S if and only if, for any $\epsilon > 0$, it is both the case that there is no element of S greater than $u + \epsilon$ and that there is an element of S greater than $u - \epsilon$.

1.4 Exercise 5.1.4:

- (a) If $S \neq \emptyset$, then $\inf S \leq \sup S$
- (b) $\inf S = \sup S$ if and only if S has only one element.
- (c) If $S = \emptyset$, then $\inf S = +\infty$ and $\sup S = -\infty$

2 The Least Upper Bound Axiom

2.1 THE LEAST UPPER BOUND AXIOM:

Every **nonempty subset** of the real numbers that is bounded above has a **least upper bound** that is a real number.

2.2 THEOREM 5.4 (Square Root of 2):

- (a) There is no **rational number** whose square is 2.
- (b) Any ordered field having the Least Upper Bound property has a positive element whose square is 2.

This theorem proved that **rational numbers does not have the Least Upper Bound property**. The least upper bound may not exist even if the subset if bounded. The upper bounds seem to be approaching to some definite least upper bound that does not exist in rational numbers. It is like convergent to a “hole” in the rational number system even though the rational numbers are **dense**.

2.3 Exercise 5.2.3 (Union and Intersection of Bounded Sets):

- (a) if S and T are **bounded sets**, $S \cup T$ and $S \cap T$ are also bounded.
- (b) if S and T are bounded sets, $\sup(S \cup T) = \max\{\sup S, \sup T\}$
- (c) However, it is **NOT true** that the supremum of the **intersection** is the min of two suprema. The intersection could be empty or neither of the two “supremum portions” is in the intersection.

- (e) let $\{S_\alpha : \alpha \in \mathcal{A}\}$ be collection of bounded sets where $\mathcal{A}$ is a finite set. Then $\bigcup_{\alpha\in\mathcal{A}} S_\alpha$ is bounded.

- (f) the conclusion in (e) cannot be extended infinite $\mathcal{A}$, even if it is countable. One can write **natural numbers** in this way and N is not bounded by **THEOREM 2.4**.

2.4 Exercise 5.2.9 (Bounded Sets and Closed Intervals):

- (a) If S is a nonempty bounded set, then $S \subseteq [\inf S, \sup S]$

- (b) If I is a **closed interval** and $S \subseteq I$, then $[\inf S, \sup S] \subseteq I$

- (c) Take all closed intervals $\{I\}$ as described in (b), then, $[\inf S, \sup S] = \bigcap I$

2.5 Exercise 5.2.10 (Bounded Sets and the Set of Bounds):

- (a) Let S be a nonempty set that is **bounded above**, and let $T = \{x : \text{x is an upper bound for S}\}$. Then, T is **nonempty and bounded below** and we have $\sup S = \inf T$.

- (b) Let S be a nonempty set that is **bounded below**, and let $T = \{x : \text{x is a lower bound for S}\}$. Then, T is nonempty and bounded above and we have $\inf S = \sup T$.

2.6 Exercise 5.2.13 (Irrationality of e):

- (a) Define the **integer part** of a real number x as $[x]$. The definition of the **integer part of a positive number** is as the follow. If $x \in P$ and n is a natural number with $x \in (n-1, n]$, we call $n-1$ the integer part of x. Then x is an **integer** if and only if $[x] = x$.

- (b) A real number x is **rational** if and only if there exists a natural number n such that $[nx] = nx$.

- (c) Since $e = \sum_{k=0}^{\infty} \frac{1}{k!}$, then $[n!e] = n! \sum_{k=0}^{n} \frac{1}{k!}$ because the **remaining part** is strictly smaller than a **geometric series** which is smaller than 1.

- (d) From (c), it is clear that $n! \sum_{k=0}^{n} \frac{1}{k!} < n!e$ because the remaining part is a sum of positive geometric series which is always between 0 and 1. This shows that $n!e$ cannot be an integer. So, e **cannot be rational** because if e is rational, there exist an n such that ne is an integer. Therefore, $n!e$ must be an integer too. This proof was due to Fourier.

2.7 Exercise 5.2.14 (The Seventh Hilbert's Problem):

It is clear that

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^{2} = 2$$

. So, **it is possible to raise an irrational number to an irrational power to get a rational outcome**. Now the question is whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational (or **transcendental**). This is related to **the seventh Hilbert's problem** which is stated like the following:

"Is a^b **transcendental**, for **algebraic** $a \neq 0, 1$ and **irrational algebraic** b?"

The transcendence of $\sqrt{2}^{\sqrt{2}}$ was first proven by Rodion Kuzmin in 1930. Then the full question was answered in the affirmative by Aleksandr Gelfond in 1934, and refined by Theodor Schneider in 1935. This result is known as **Gelfond's theorem** or the **Gelfond–Schneider theorem**. This immediately proves the transcendence of many numbers such e^{π} (**Gelfond's constant**), and i^{i}.