Chapter 11 Compact Sets

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1 The Extreme Value Theorem

1.1 THEOREM (Extreme Value Theorem)

If f is a **continuous function** whose **domain** is a **closed, bounded** interval, then f assumes a **maximum** on its domain. This is equivalent to say for a **continuous function** f and its **closed, bounded** domain S, $\sup f(S) \in f(S)$.

By a **bounded function**, we mean that its **range** is bounded. The purpose of the investigation in this chapter is to study what is special about a closed and bounded interval.

1.2 DEFINITION 11.1 (Compact Set):

The set $K \subseteq \mathbb{R}$ is **compact** if every continuous function $f : K \to \mathbb{R}$ assumes a **maximum**.

1.3 EXAMPLE 11.1:

- **Closed, bounded intervals** are compact.
- **Finite sets** are compact.
- The **whole real line** is *not* compact.
- $\mathbb{N}$ is *not* compact.
- $H = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}...\}$ is *not* compact.
- $S = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}...\}$ is compact.

1.4 THEOREM 11.2 (Union of Compact Sets):

If A and B are **compact**, the $A \cup B$ is compact.

1.5 THEOREM 11.3 (Union of Finite Number of Compact Sets):

The union of any **finite collection** of compact sets is compact.

1.6 THEOREM 11.4 (Compact Set is Closed and Bounded):

A **compact set** is closed and bounded.

1.7 Exercise 11.1.6:

- (a) The union of an infinite collection of compact sets need not be compact. For example, any open set is an (uncoutably) infinite collection of inidividual points, which are compact.
- (b) However, an infinite collection of compact sets could still be compact such as the S in **EXAMPLE 11.1**.

1.8 Exercise 11.1.8:

The **difference** of two compact sets need not be compact. Such as the S in **EXAMPLE 11.1**, the difference $S \setminus \{0\} = H$. Both S and $\{0\}$ are compact but H is not.

1.9 Exercise 11.1.10 (Compactness on Discrete and Indiscrete Topology):

- (a) If X is an infinite set with the **discrete topology**, X then is **not compact**.
- (b) If X is any set with **indiscrete** topology, then X is **compact**.

2 The Covering Property

DEFINITION 11.1 is only **partially topological**. The issue of **continuous function** is **topological**. But we must also consider the **nontopological** question of **maximum** which needs the **ordering** of the real line to make sense. The purpose of introducing the concept of **covering property** is to make a completely topological definition of compactness.

2.1 DEFINITION 11.5 (Open Cover):

A collection of **open sets** $\{U_\alpha : \alpha \in \mathcal{A}\}$ is an **open cover** of the set S if $S \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. If $\{U_\alpha\}$ is an open cover of S, we say that we have **covered** S with $\{U_\alpha\}$.

The reason that we made α as a subscript because we made no commitment as to the **cardinality** of the **index set** $\mathcal{A}$.

2.2 DEFINITION 11.6 (Covering Property):

A set has the **covering property** if any open cover of it has a **finite subcover**.

Remember that producing a finite open cover is not a challenge. It becomes one only when we must cover our set with finitely many sets **chosen from** a previously specified collection, i.e. a **finite subcover**.

2.3 Exercise 11.2.1 (Union and Covering Property):

The union of two sets having the covering property has the covering property.

2.4 Exercise 11.2.3 (Closed and Bounded vs Covering Property):

- (a) A **closed, bounded** set having **exactly one cluster point** has the covering property.
- (c) A **closed, bounded** set having **finitely many cluster points** has the covering property.
- (d) If S is **closed and bounded** and S' is **finite**, then S has the covering property. And if S is **closed and bounded** and S" is **finite**, then S has the covering property.

2.5 Exercise 11.2.4 (Non-compact Sets don't have Covering Property):

If the set is not compact, there is a continuous function f defined on it that attains no maximum. Hence, if $f(x)$ is any value of f, there is a number y in the set with $f(y) > f(x)$. So, we can always construct an open cover of this set with no finite subcover.

2.6 Exercise 11.2.5:

- (a) A set of the form $[a, \infty)$ is not compact.
- (b) A set of the form $[a, \infty)$ does not have the covering property.

2.7 Exercise 11.2.7:

Suppose S is an infinite set having the covering property and that $\{U_\alpha\}$ is an open cover of S. Then, there is an α^* so that $U_{\alpha^*} \cap S$ is infinite.

2.8 Exercise 11.2.8 (Covering Property and Discrete and Indiscrete Topology):

- (a) An infinite set with the **discrete topology** fails to have the covering property.
- (b) Any set with the **indiscrete topology** has the covering property.

2.9 Exercise 11.2.9 (One-point Compactification of $\mathbb{R}$)

Consider the set consisting of the real numbers and **another symbol** ∞. This set is denoted $\mathbb{R} \cup \{\infty\}$. We say that a subset of $\mathbb{R} \cup \{\infty\}$ is a **neighborhood of** ∞ if it contains the **complement of a compact set**. Neighborhood of the other elements of the set are defined in the usual way.

- (a) A subset of $\mathbb{R} \cup \{\infty\}$ that does not contain ∞ is open if and only if it is an open subset of $\mathbb{R}$ in the usual sense.
- (b) A subset of $\mathbb{R} \cup \{\infty\}$ that contains ∞ is open if they are closed to any union and finite number of intersection.
- (c) The open subsets of $\mathbb{R} \cup \{\infty\}$ form a topology.
- (d) A function $f : \mathbb{R} \cup \{\infty\} \to \mathbb{R}$ is continuous if
  - it is countinuous on $\mathbb{R}$ in the usual sense, and
  - $f(\infty) = \lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x)$.
- (f) $\mathbb{R} \cup \{\infty\}$ is **compact**.

- (g) $\mathbb{R} \cup \{\infty\}$ has the covering property.
- (h) $\mathbb{R} \cup \{\infty\}$ is called the **one-point compactification of** $\mathbb{R}$. And this procedure works for ***any*** **noncompact topological space**.

2.10 Exercise 11.2.10 (Two-point Compactification of $\mathbb{R}$)

Consider the set consisting of the real numbers and **two new symbols** ∞ and $-\infty$, and the sign does not indicate arithmetic operation. This set is denoted $\mathbb{R} \cup \{-\infty, \infty\}$. We say that a subset of $\mathbb{R} \cup \{-\infty, \infty\}$ is a **neighborhood of** ∞ if it contains the **complement of a closed set that is bounded above**, and a subset of $\mathbb{R} \cup \{-\infty, \infty\}$ is a **neighborhood of** $-\infty$ if it contains the **complement of a closed set that is bounded below**.

- (a) A subset of $\mathbb{R} \cup \{-\infty, \infty\}$ that does not contain $-\infty$ or ∞ is open if and only if it is an open subset of $\mathbb{R}$ in the usual sense.
- (c) The open subsets of $\mathbb{R} \cup \{-\infty, \infty\}$ form a **topology**.
- (d) A function $f : \mathbb{R} \cup \{-\infty, \infty\} \to \mathbb{R}$ is continuous if
  - it is countinuous on $\mathbb{R}$ in the usual sense, and
  - $f(-\infty) = \lim_{x \to -\infty} f(x)$, and
  - $f(\infty) = \lim_{x \to \infty} f(x)$.
- (g) $\mathbb{R} \cup \{-\infty, \infty\}$ is compact.
- (h) $\mathbb{R} \cup \{-\infty, \infty\}$ has the covering property.
- (i) $\mathbb{R} \cup \{-\infty, \infty\}$ is called the **two-point compactification of** $\mathbb{R}$. This procedure does not make sense in the general topological setting. The definition of the neighborhood around the two infinities requires the concept of "bounded above" and "bounded below", which means we need at least a **partial order**.

3 The Haine-Borel Theorem

3.1 THEOREM 11.7 (Heine-Borel Theorem):

If F is an **Archimedean ordered field** having the **Nested Intervals property**, then the following is also true: A subset of F is **closed and bounded** if and only if it has the **covering property**.

3.2 THEOREM 11.8 (Continuous Image of Covering Property):

If the set S has the **covering property** and $f : S \to \mathbb{R}$ is **continuous**, then $f(S)$ has the **covering property**.

3.3 THEOREM 11.9 (Covering Property and Compactness):

If S has the **covering property**, it is **compact**.

3.4 THEOREM 11.10 (Intersection of Compact Sets):

The **intersection** of **any collection** of **compact sets** is compact.

3.5 THEOREM 11.11 (Continuous Image of Compact Sets is Compact):

If the set S is **compact** and $f : S \to \mathbb{R}$ is **continuous**, then $f(S)$ is compact.

3.6 Exercise 11.3.5.d (Countable Subcover):

Any open cover of $\mathbb{R}$ can be reduced to a **countable subcover**. A **topological space** with the property that any open cover can be reduced to a countable subcover is called a **Lindelöf space**.

3.7 Exercise 11.3.6 (Range of Convergent Sequences):

- The set made up of the **range of a convergent sequence together with its limit** is **closed and bounded**.
- The set made up of the **range of a convergent sequence together with its limit** has the covering property.

3.8 Exercise 11.3.7 (Lebesgue Number Lemma):

Suppose K is **compact**, $\{U_\alpha\}$ is an **open cover** of K with finite subcover $\{U_{\alpha_n}\}$, and $x \in K$. Let

$$\epsilon_x = \sup\{\epsilon : (x - \epsilon, x + \epsilon) \subseteq U_{\alpha_n} \text{ for some n}\}.$$

- (a) $\epsilon_x > 0$ for all $x \in K$.
- (b) The function $f : K \to \mathbb{R}$ defined by $f(x) = \epsilon_x$ is **continuous**.
- (c) $f(x)$ assumes a positive minimum on K.
- (d) For any open cover of a compact set K, there is a positive number δ so that any interval of length less than δ and containing a point of K is contained in a **single element of the cover**. This is called the **Lebesgue Number Lemma**, and δ is called the **Lebesgue Number** of the cover.

3.9 Exercise 11.3.8 (Cauchy Convergence and Compactness):

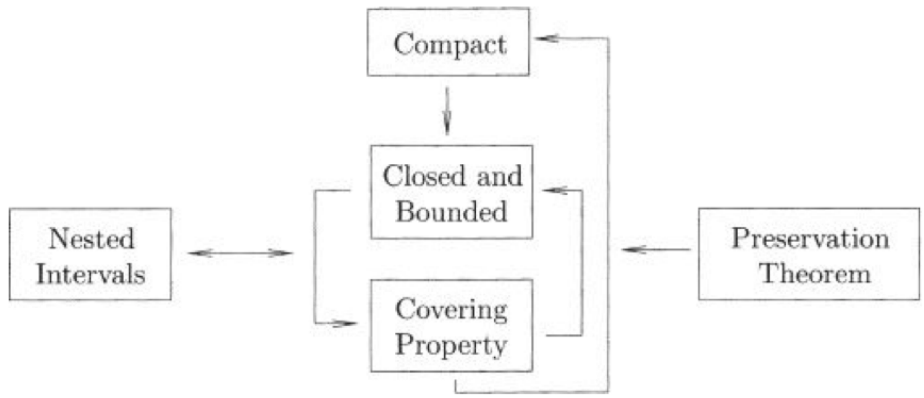
A set is **compact** if and only if any sequence contained in it has a subsequence that converges to an element of the set.

3.10 Exercise 11.3.9 (Nested Intervals Property and the Compact Sets):

If $K_1 \supseteq K_2 \supseteq ...$ is a **nest of non-empty compact sets**, then $\bigcap_n K_n \neq \emptyset$.

3.11 Exercise 11.3.10 (Bolzano-Weierstrass and Compact Sets):

An **infinite compact set** must have a cluster point that is in the set.



A local closed loop in the Big Theorem

3.12 Exercise 11.3.11 (Supremum and Infinum of Compact Sets):

A **nonempty compact set** has and contains a supremum and infimum.

4 Closing the Loop

4.1 THEOREM 11.12 (Heine-Borel Indicates Nested Interval):

An **ordered field** in which **Heine-Borel theorem** holds also has the **Archimedean property** and the **Nested Interval property**.