Chapter 9 Sequences

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1 An Approximation Problem

1.1 DEFINITION 9.1 (Real Sequence):

A **real sequence** is a function $x : N \to \mathbb{R}$. **Real** here refers to the **range** of the function. We should distinguish the **sequence** (x_n) in which the **order** is important from the **range** $\{x_n\}$ which is a **set** and doesn't have order by itself.

2 Convergence

2.1 DEFINITION 9.2 (Convergence and Limit):

- (a) A sequence (x_n) is said to **converge** to L if, for every $\epsilon > 0$, there is a number N_ϵ so that $|x_n - L| < \epsilon$ whenever $n > N_\epsilon$. If this is the case, we say L is the **limit** of (x_n) and write $\lim x_n = L$. (This is usually called the **"ϵ-N version" of the definition**.)
- (b) A sequence (x_n) is said to **converge** if there is a number L so that (x_n) converges to L. A sequence that does not converge is said to **diverge**.

Note that L must be a number. A sequence cannot converge to ∞. Also, it is reasonable to assume N_ϵ depends on ϵ. If we make our "*error tolerance*" smaller, we expect to be required go further along in the process to achieve it.

2.2 ALTERNATE DEFINITION 9.3 (Convergence and Limit):

(x_n) **converges** to L if, for each neighborhood V of L, there is a number N_V so that $x_n \in V$ whenever $n > N_V$.

2.3 DEFINITION 9.4 (Eventually and Frequently):

- (a) A sequence (x_n) is **eventually** in the set S if there is a number N so that $x_n \in S$ whenever $n > N$.
- (b) A sequence (x_n) is **frequently** in a set S if for any natural number N, there is an $n > N$ for which $x_n \in S$.

2.4 FINAL DEFINITION 9.5 (Convergence):

The sequence (x_n) **converges** to L if it is **eventually** in **any neighborhood** of L.

The deep part of the **theory of convergent sequences** can be stated as "*if the terms in a sequence get close to a limit, they must get close to each other*". This is what behind **Cauchy convergence**.

2.5 Exercise 9.2.3 (Eventually vs Frequently):

- (a) **Eventually** $\Rightarrow$ **Frequently**
- (b) The inverse of (a) is not true.
- (c) A sequence is **eventually** in a set if there are only **finitely many** values of n for which it is **NOT** in the set.
- (d) A sequence is **frequently** in a set if it is in the set for **infinitely many values** of n.

2.6 Exercise 9.2.7:

A sequence cannot be **eventually** in both of two **disjoint sets**.

2.7 Exercise 9.2.8:

If there is an $\epsilon > 0$ so that (x_n) is **not eventually** in any interval of length ϵ, then (x_n) **divergences**.

2.8 Exercise 9.2.9 (Limit of a sequence and the limit of its rolling average):

Given a sequence of (x_n), define a sequence (a_n) by

$$a_n = \frac{x_1 + x_2 + ... + x_n}{n}$$

. If $\lim x_n = L$, then $\lim a_n = L$.

The inverse of above is not true. It is possible that (a_n) converges to L but (x_n) diverges. An example would be (x_n) is the sum of a convergent sequence (y_n) and an error term of alternating $+1$ and -1, i.e. $x_n = y_n + (-1)^n$. The arithmetic average of the error term converges to zero. So (a_n) converges to the limit of (y_n). But (x_n) does not converge.

3 Convergent Sequences

3.1 THEOREM 9.6 (Uniqueness of the limit of a sequence and bounded sequences):

A sequence can have **at most one limit**. We say a seqeunce (x_n) is **bounded** if its **range** $\{x_n\}$ is a **bounded set**.

It is worth noting that the proof of this theorem depends on the result of **Exercise 4.10.4**, which states "If $x \neq y$, there are neighborhoods U of x and V of y such that $U \cap V = \emptyset$". This is *not* true in all **topological spaces**. It depends very much on the **separation property** of the topological space. A space with this property is called a **Hausdorff space**. See also **Exercise 9.3.6**.

3.2 THEOREM 9.7:

A **convergent** sequence is **bounded**.

The converse of above is not true. A bounded sequence could well be divergent, such as the counterexample in **Exercise 9.2.9**.

3.3 THEOREM 9.8 (Convergence of a sequence and convergence its distance to its limit):

Let (x_n) be sequence, $L \in \mathbb{R}$, and $d_n = |x_n - L|$. Then $\lim x_n = L \iff \lim d_n = 0$.

3.4 Exercise 9.3.2 (Alternative condition for convergence):

$\lim x_n = L$ if and only if the following holds: given $\epsilon > 0$ and any positive real number b, there is an $n \in N$ so that $|x_n - L| < b\epsilon$ whenever $n > N$.

3.5 Exercise 9.3.4:

- (a) If $\lim x_n = 0$ and (y_n) is bounded, then $\lim x_n y_n = 0$ no matter (y_n) converges or not.

3.6 Exercise 9.3.5 (Convergence of the absolute value of a sequence):

- (a) $\lim x_n = L \Rightarrow \lim |x_n| = L$
- (b) the converse of (a) is not true
- (c) $\lim |x_n| = 0 \Rightarrow \lim x_n = 0$

3.7 Exercise 9.3.6 (Hausdorff Space):

A **topological space** is called a **Hausdorff space** if it has the following property:

If $x \neq y$, there are neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

Suppose X is a topological space that does not have above property, there must a sequence in X that converges to *two different limits*.

4 Sequences and Order

4.1 THEOREM 9.9 (Squeeze theorem):

- (a) If (x_n) converges and $x_n \geq 0, \forall n$, then $\lim x_n \geq 0$.
- (b) If (x_n) and (y_n) converge and $x_n \leq y_n, \forall n$, then $\lim x_n \leq \lim y_n$.
- (c) If (x_n) and (y_n) both converge to L and $x_n \leq z_n \leq y_n, \forall n$, then (z_n) converges and $\lim z_n = L$.

(c) is often called the **Squeeze Theorem**. An example is the following.

4.2 EXAMPLE 9.4:

Since

$$-\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n};$$

then,

$$\lim \frac{\cos(n)}{n} = 0.$$

4.3 COROLLARY 9.10:

If $|x_n - L| \leq b_n$, and $\lim b_n = 0$, then $\lim x_n = L$.

4.4 Exercise 9.4.4:

- (a) If $x_n \geq a, \forall n$, and (x_n) converges, then $\lim x_n \geq a$.

4.5 Exercise 9.4.6 (Positive set of R):

The **positive set of** $\mathbb{Q}$ is defined as; for $p/q \in \mathbb{Q}$, it is in the positive set if p and q are either both natural number of both additive inverse of natural numbers. Then the **positive set of** $\mathbb{R}$ is defined as:

$$\{x \in \mathbb{R} : x \neq 0 \text{ and } \exists (r_n) \ni (r_n \in \mathbb{Q}, r_n > 0, \text{ and } \lim r_n = x)\}$$

5 Sequences and Algebra

5.1 THEOREM 9.11 (Algebra of the limits of sequences):

Suppose $\lim x_n = L$ and $\lim y_n = M$. Then

- (a) $\lim(x_n + y_n) = L + M$
- (b) $\lim(cx_n) = cL, \forall c \in \mathbb{R}$
- (c) $\lim(x_n y_n) = LM$
- (d) $\lim(x_n/y_n) = L/M,$ provided $y_n \neq 0, \forall n$ and $M \neq 0$

6 Sequences and Topology

6.1 DEFINITION (Cluster point of a seqeunce):

A point is a **cluster point of a sequence** if it is a cluster point of its **range**.

6.2 LEMMA 9.12 (Convergent sequences and cluster points):

A **convergent sequence** can have **at most one cluster point**, its **limit**.

However, a cluster point of a sequence is not necessary its limit (the sequence could be not convergent). The limit of a convergent sequence is not necessary a cluster point.

6.3 THEOREM 9.13 (Limits and cluster points):

If $\lim x_n = L$, one of the following is true:

- (i) (x_n) is eventually equal to L, or
- (ii) L is the only cluster point of (x_n).

6.4 LEMMA 9.14 (Cluster point of a set):

The point c is a **cluster point** of a set S if and only if there is a **sequence** of elements of S, all different from c and converging to c.

The proof of the *"only if"* part depends on the Archimedean property because it uses the $\frac{1}{n}$ to define a next of smaller and smaller neighborhood. If the field is not Archimedean, $\frac{1}{n}$ does not converge to 0. On a deeper level, the result relies on the fact that the topology of the real number line is **first countable**, which makes sure that the convergence can be done by a sequence, which is indexed and therefore countable.

6.5 THEOREM 9.15 (Define closed sets by convergent sequences):

A subset S of $\mathbb{R}$ is **closed** if and only if $\lim x_n \in S$ whenever (x_n) is a **convergent sequence** whose terms are all in S.

6.6 THEOREM 9.16 (Define continuous function by convergent sequences):

A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** if and only if, for each **convergent sequence** (x_n),

$$\lim f(x_n) = f(\lim(x_n))$$

.

6.7 Exercise 9.6.4:

Suppose S is a **nonempty open set** that isn't the whole real line. Then, there is a sequence of elements of S that converges to an element of $C(S)$.

6.8 Exercise 9.6.6:

- (a) A sequence that is **eventually constant** converges.
- (b) If the topological space X has the **discrete topology**, the only sequences that converge are those that are **eventually constant**.
- (c) If the topological space X has the **indiscrete topology**, every sequence converges to every element in X.

7 Subsequences

7.1 DEFINITION 9.17 (Subsequence):

- (a) A function $n : N \to N$ is said to be **strictly increasing** if
$$n(k+1) > n(k)\ \forall n \in N.$$
- (b) If $x : N \to \mathbb{R}$ is a sequence and $n : N \to N$ is **strictly increasing**, then $x \circ n : N \to \mathbb{R}$ is called a subsequence of x.

The whole point of having the **strictly increasing function** n is that the subsequence, though consists of infinitely many terms of (x_n), are kept in the **same order** of (x_n).

An interesting result is that a **divergent sequence** could have **convergent subsequences**.

7.2 THEOREM 9.18 (Convergence of subsequences):

If the sequence (x_n) **converges**, so does **every subsequence** of it and they all converge to the **same limit**.

7.3 COROLLARY 9.19 (Divergent sequences):

If a sequence has **subsequences** that converge to **two different limits**, the sequence **diverges**.

7.4 Exercise 9.7.1 (Subsequence of subsequences):

If Y is a subsequence of X and Z is a subsequence of Y, then Z is a subsequence of X.

7.5 Exercise 9.7.2 (Strictly increasing function):

- (a) If $g : N \to N$ is **strictly increasing**, then $g(k) \geq k$.
- (b) A **real function** $g : \mathbb{R} \to \mathbb{R}$ is strictly increasing if $x > y \Rightarrow g(x) > g(y)$. Even if g is **strictly increasing**, it is *not* necessarily the case that $g(x) \geq x, \forall x \in \mathbb{R}$.

7.6 LEMMA 9.7.3 (Irrational number's fractional part):

Given $e > 0$ and an **irrational number** a, there exists a **natural number** n such that $(na) < e$, where (na) means the **fractional part** of na.

7.7 Exercise 9.7.3 (Subsequences of $\cos(n)$ and $\sin(n)$):

The subsequences of $\cos(n)$ can converge to **any real number** in $[-1, 1]$.

7.8 Exercise 9.7.4 ($\sin(\frac{1}{x})$ at 0):

$f(x) = \sin(\frac{1}{n})$ **cannot be defined** at 0 in such a way to make it **continuous** because there are subsequences that can converge any value between -1 and 1. **COROLLARY 9.19** states that it will **diverge** at 0.