Chapter 12 Connected Sets

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1 The Intermediate Value Theorem

1.1 THEOREM (Intermediate Value Theorem):

If f is a **continuous, real-valued function** defined on an interval, and f takes on a positive value at some point a and a negative value at some point b, then there is a point c between a and b where $f(c) = 0$.

1.2 DEFINITION 12.1 (Connected Sets):

A set C is **connected** if every continuous, real-valued function $f : C \to \mathbb{R}$ has the **Intermediate Value property** on C.

1.3 EXAMPLE 12.1:

- (5) The **Intermediate Value Theorem** indicates that **intervals** are **connected**.
- (6) A set with a **single point** is **connected**.

1.4 THEOREM 12.2 (Intervals and Connectedness):

If S is a **connected subset** of the real line, then S is an **interval**.

However, **THEOREM 12.2** leaves open the possibility that, except for sets with a single point, there are *no* connected subsets of the real line at all (which is *true* for **rational numbers**).

1.5 Exercise 12.1.3 (Property of Connected Ordered Set):

A set S is **connected** if and only if it has the property that whenever $x < z < y$ and x and y are elements of S, then z is an element of S.

1.6 Exercise 12.1.4 (Discrete and Indiscrete Topology and Connectedness):

- (a) If the topological space X has the **indiscrete topology**, every subset of X is **connected**.
- (b) If X has the **discrete topology**, the only connected subsets of X are thos having only one element.

2 Disconnections

2.1 THEOREM 12.3 (Disconnected Set):

The set $S \subseteq \mathbb{R}$ is **disconnected** if and only if there are open sets A and B such that:

- (i) A and B are **disjoint**
- (ii) $A \cap S \neq \emptyset$ and $B \cap S \neq \emptyset$, and
- (iii) $S \subseteq A \cup B$.

We call A and B are a **disconnection** of S.

The beautiful part of above definition is that, while **connectedness** is entangled with the **order structure of the real line**, this definition of **disconnectedness** is purely **topological**.

2.2 Exercise 12.2.1 (Alternative Definition of Disconnected Sets):

A set is **disconnected** if and only if it has a **proper nonempty subset** that is **both *open and *closed**.

2.3 Exercise 12.2.2 (Formal Rational Functions are Disconnected):

The field of **formal rational functions** is disconnected.

3 The Big Theorem Sails into the Sunset

3.1 THEOREM 12.4 (Least Upper Bound Property Indicates Connectedness):

An ordered field with the **Least Upper Bound property** is **connected**.

3.2 COROLLARY 12.5 (The Intermediate Value Theorem):

Intervals in the real line are **connected**.

3.3 Exercise 12.3.2 (Roots of Polynomial):

- (a) A positive number a has all possible nth roots, i.e. the equation $x^n - a = 0$ has a solution.
- (b) Any polynomial of **odd degree** has a real root, i.e. there is a real number c so that $f(c) = 0$.

3.4 Exercise 12.3.3 (Fixed Point):

- (a) Let $f : [0, 1] \to [0, 1]$ be continuous. There must be a number x for which $f(x) = x$. And this point is called a **fixed point** for function f.
- (b) If we constrain ourselves to rational numbers, (a) will be no longer true.
- (c) if the range in (a) is $(0, 1)$ instead of $[0, 1]$, it is no longer true.

3.5 Exercise 12.3.4 (Crossing of Two Functions):

Suppose $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both **continuous** and that $f(a) < g(a)$ and $f(b) > g(b)$. Then there must be a number x such that $f(x) = g(x)$.

3.6 Exercise 12.3.5 (Equal Points on a Closed Loop):

- (a) Suppose $f : [0, 2] \to \mathbb{R}$ is **continuous** and $f(0) = f(2)$. Then there are elements of $[0, 2]$, say a and b, such that $|a - b| = 1$ and $f(a) = f(b)$.
- (b) Suppose $f : C \to \mathbb{R}$ is **continuous**, where C is the circle $x^2 + y^2 = 1$. Then there are points a and b on C that are **diametrically opposed** such that $f(a) = f(b)$.

4 Closing the Loop

4.1 THEOREM 12.6 (Connectedness Indicates Least Upper Bound Property):

Any **connected ordered field** has the **Least Upper Bound** property.

5 Continuous Functions and Intervals

5.1 THEOREM 12.7 (Image of Connected Sets):

If S is **connected** and $f : S \to \mathbb{R}$ is **continuous**, then $f(S)$ is **connected**.

5.2 THEOREM 12.8 (Continuous Image of Closed and Bounded Intervals):

If $f : [a, b] \to \mathbb{R}$ is **continuous**, there exist real numbers c and d so that $f([a, b]) = [c, d]$.

5.3 Exercise 12.5.1:

Suppose $f : [a, b] \to \mathbb{R}$ is a **continuous function** whose range includes only **rational numbers**, then f is **constant**.

6 A Comments on Calculus

6.1 THEOREM 12.9 (Rational Numbers are Totally Disconnected):

The only **connected subsets** of the **rational numbers** are sets with a single element. A set with the property that its only connected subsets are those with one element is called **totally disconnected**.

6.2 THEOREM 12.10 (Sets that are both Open and Closed):

The only subsets of the real line that are **both open and closed** are the **empty set** and the **whole real line**.

6.3 Exercise 12.6.3 (Connectedness Implies Archimedean):

The **connectedness** of an **ordered field** implies it has the **Archimedean property**.

6.4 Exercise 12.6.4 (Archimedean Property):

- **Connectedness**, the **Least Upper Bound** property and the **Heine-Borel Theorem** implies the **Archimedean property**.
- The **Nested Intervals** property, the **Bolzano-Weierstrass Theorem** and the **Cauchy Criterion** do *not* imply the **Archimedean property**.

6.5 Exercise 12.6.5 (Closure of Connected Sets):

- (a) if S is a **connected set**, then the **closure** of S, S^-, is **connected**.
- (b) If S is a **connected set** and T is such that $S \subseteq T \subseteq S^-$, then T is **connected**.

6.6 Exercise 12.6.6 (Open and Closed Subsets of the Real Line):

- (a) Suppose S is a **nonempty open set** that isn't the whole real line. There is a **cluster point** of S that is an element of $C(S)$.
- (b) Suppose S is a **nonempty closed set** that isn't the whole real line. Then S must contain a point of which it is not a **neighborhood**.