Chapter 2 Finite, Infinite and Bigger

April 3, 2017

1 Cardinalities

1.1 DEFINITION 2.1 (One-to-one Correspondence, Finite and Infinite Sets)

- (a) A function c : S $\rightarrow$ T is a **one-to-one correspondence** if it has the following properties:
  - (i) $\forall x \in S, \forall y \in S, (x \neq y \Rightarrow c(x) \neq c(y))$, that is, c is **one-to-one**, and
  - (ii) $\forall z \in T, \exists x \in S \ni (c(x) = z)$, that is, c is **onto**.
- (b) If a one-to-one correspondence exists between two sets, we say that the **sets are (or can be put) in one-to-one correspondence**.
- (c) A set is **finite** (and contains n elements) if it can be put in one-to-one correspondence with an **initial segment of natural numbers {1, 2, ... n}**.
- (d) A set that is not finite is **infinite**.

Here we take **natural numbers** as something that are intuitive and defined like an **axiom**. This is actually a **circular definition** given the light of **Definition 2.3**. A preferred way of defining a **finite set** is in **Exercise 2.3.7 (c)** and **Exercise 2.1.4 (e)** below, then use **Definition 2.3** to define natural numbers.

1.2 DEFINITION 2.2 (Comparing Cardinality):

- (a) Two sets have the **same cardinality** if they can be put in one-to-one correspondence with each other.
- (b) If such sets are finite, we say they have the **same number of elements**.

1.3 DEFINITION 2.3 (Finite Cardinality and Natural Numbers):

- (a) A **cardinality** is **the property common to a collection of sets that can be put in one-to-one correspondence with each other**, but that is not shared by any set that can't be put in one-to-one correspondence with a set in the collection.
- (b) If the sets in such a collection are finite, the cardinality is also said to be finite. A finite cardinality is called a **natural number**.

1.4 Exercise 2.1.4 (Alternative Definition of Finite Sets)

- (e) It is impossible to have a one-to-one correspondence between a finite set and one of its **proper subsets**. Sets that have this property are **finite sets**.

1.5 Exercise 2.1.5 (Equivalence Relation and Equivalence Class)

The relation $\approx$ is called an **equivalence relation** on X if it has the properties:

- (i) (**Reflexivity**) $x \approx x, \forall x$
- (ii) (**Symmetry**) $x \approx y \Rightarrow y \approx x$
- (iii) (**Transitivity**) $x \approx y$ and $y \approx z \Rightarrow x \approx z$.

So, = is an equivalence relationship, but $\leq$ is not because it failed ii). is actually a **partial order relation**, which has the **antisymmetry property**: $x \leq y$ and $y \leq x \Rightarrow x = y$. Similarly, $\subseteq$ is a partial order relation rather than an equivalence relation.

Now, let X be a set and $\approx$ an equivalence relation on X. For any element a of X, let $X_a = \{x \in X : x \approx a\}$. Then:

- (i) $X_a \neq \emptyset \; \forall a$
- (ii) if $X_a \cap X_b \neq \emptyset,$ then $X_a = X_b$, and
- (iii) $X = \bigcup_a X_a$.

The set X_a is called the **equivalence class** of a. Effectively, an equivalence class "groups" all elements that are equivalent to a together to form a set. A collection of subsets $\{X_a\}$ of a set X having properties (i), (ii), and (iii) is called a **partition** of X.

2 Infinite Sets

2.1 THEOREM 2.4 (Infinity of Natural Numbers):

The set of all **natural numbers**, N, is **infinite**.

2.2 Definition (Countable and Uncountable Infinity):

We are generally interested in sets with only three types of cardinalities: Natural numbers (the cardinalities of finite sets), the cardinality of N (which is denoted $\aleph_0$, and those that are bigger. A set with cardinality $\aleph_0$ is said to be **denumerable**. A set that is finite or denumerable is said to be countable. If a set is not **countable**, it is **uncountable**.

2.3 THEOREM 2.5 (Algebra of Countable Cardinality):

The **union** of a countable collection of countable sets is countable.

2.4 Exercise 2.2.1 (Cantor-Bernstein-Dedekind Theorem):

In set theory, the **Schröder–Bernstein theorem** (named after Felix Bernstein and Ernst Schröder, also known as **Cantor–Bernstein theorem**, or **Cantor–Schröder–Bernstein** after Georg Cantor who first published it without proof) states that, if there exist **injective functions** $f : A \to B$ and $g : B \to A$ between the sets A and B, then there exists a **bijective function** $h : A \to B$. In terms of the cardinality of the two sets, this means that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$; that is, A and B are **equivalent**. This is a useful feature in the **ordering of cardinal numbers** (the cardinal numbers has the **anti-symmetry property** and therefore **partially ordered** (See **Exercise 2.1.5**)).

2.5 Exercise 2.2.5 (Algebraic Numbers):

If a number is a solution to a **polynomial equation with coefficients that are integers**, it is called **algebraic**.

- (a) All **rational numbers** are algebraic.
- (c) The set of algebraic numbers is **countable**.

2.6 Exercise 2.2.6 (Algebra of $\aleph_0$):

- (a) $\aleph_0 + \aleph_0 = \aleph_0$
- (b) $\aleph_0 + n = \aleph_0$
- (c) $\aleph_0 \times \aleph_0 = \aleph_0$

3 Uncountable Sets

3.1 THEOREM (Cantor Diagonalization):

In set theory, **Cantor's diagonal argument**, also called **the diagonalisation argument**, **the diagonal slash argument** or **the diagonal method**, was published in 1891 by Georg Cantor as a mathematical proof that there are infinite sets which cannot be put into one-to-one correspondence with the infinite set of natural numbers. Such sets are now known as **uncountable sets**, and the size of infinite sets is now treated by **the theory of cardinal numbers** which Cantor began.

The diagonal argument was not Cantor's first proof of the uncountability of the real numbers; it was actually published much later than his **first proof**, which appeared in 1874. However, it demonstrates a powerful and general technique that has since been used in a wide range of proofs, including the first of **Gödel's incompleteness theorems** and **Turing's answer to the Entscheidungsproblem**. Diagonalization arguments are often also the source of contradictions like **Russell's paradox** and **Richard's paradox**.

3.2 Exercise 2.3.4 (Power Set):

The **power set** of a set S, denoted $P(S)$, is the set of all subsets of S.

- (c) If S is any set at all, the **cardinality** of $P(S)$ is larger than the cardinality of S.

3.3 Exercise 2.3.5 (Infinity of Aleph Numbers):

- (a) There are **infinitely many** different **infinite cardinals** (**Aleph numbers**).

3.4 Exercise 2.3.7 (Alternative Definition of Infinite Sets):

- (b) Every **infinite set** has a **denumerable subset**. (The proof of this theorem is beyond the scope of this book.)
- (c) Every **infinite set** can be put in **one-to-one correspondence** with a **proper subset of itself** (this is an alternative definition of "infinite").
- (d) Forming the union of an infinite set with a finite set does not increase the first set's cardinality.
- (e) Forming the union of an infinite set with a countable set does not increase the first set's cardinality.

3.5 Exercise 2.3.8 (Transcendental Numbers):

Numbers that are not algebraic are **transcendental**.

3.6 Exercise 2.3.9 (Finite subsets):

- (b) The set of **finite subsets of a denumerable set** is **denumerable**. (This draws an analogue to rational numbers whose digit expansion are terminating or repeating - finite subsets vs real numbers whose digit expansion are in general non-repeating - the power set).