Chapter 10 Sequences and the Big Theorem

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1 Convergence without Limits

N/A

2 Monotone Sequences

2.1 DEFINITION 10.1 (Monotonous sequences):

A sequence is increasing if , and it is decreasing if . A sequence is monotone if it is either increasing or decreasing. If the relationship in this definition is strict, it is called strictly increasing or strictly decreasing. If this relationship only holds after a certain index k, it is called eventually increasing or eventually decreasing.

2.2 LEMMA 10.2 (Cluster point of monotone sequences):

- (a) Any cluster point of an increasing sequence is an upper bound for the sequence.
- (b) Any cluster point of a decreasing sequence is a lower bound for the sequence.

2.3 THEOREM 10.3 (Convergence of monotone sequences):

A bounded, monotone sequence converges.

2.4 COROLLARY 10.4 (Limits and bounds of monotone sequences):

If is increasing and bounded, then . If is decreasing and bounded, then .

2.5 Exercise 10.2.3 (Basel Problem and Limits of Upbounds):

Let . Then it can be shown by induction that , or is an upper bound of . Apparently, converges to 2, but this does not mean that is 2 because 2 is only an upper bound rather than a supremum. There is no indication that the gap between and is shrinking to zero.

This is actually the famous Basel problem which was solved by young Euler. It is also a special case of the Riemann zeta function at 2. The correct limit is .

2.6 Exercise 10.2.7 (Supremum and infimum of bounded sets):

If S is a bounded set, then there is an increasing sequence of elements of S converging to sup(S) and a decreasing sequence of elements of S converging to inf(S).

2.7 Exercise 10.2.8 (Bounded function on half-open domains):

Suppose f is an increasing, bounded function whose domain contains ray . Then exists.

2.8 Exercise 10.2.9.a-c (Harmonic series and geometric series):

Define the following partial sums: (Harmonic series):
(Geometric series): Both the harmonic series and geometric series are increasing. converges while diverges. The limit of is the following: Since
then

2.9 Exercise 10.2.9.d-e (Convergence of series):

- (d) Suppose is sequence of positive numbers and let be defined by , which is a series. S is monotone and it converges if and only if it is bounded.
- (e) Suppose and are sequences of positive numbers, with for all n. Let S and T be series based on and . Then if T converges, S converges. If S diverges, T diverges. It is possible that S converges while T diverges.

2.10 Exercise 10.2.9.f (Euler-Mascheroni constant):

The limit exists because the harmonic series is strictly smaller than therefore this difference is bounded. The limit is called Eulear-Mascheroni constant. It is approximately 0.577. But it is still unknown whether it is rational or irrational.

3 A Recursively Defined Sequence

3.1 THEOREM 10.5 (Convergence of eventually monotone sequences):

A bounded, eventually monotone sequence converges. This is a direct extension of THEOREM 10.3.

3.2 Exercise 10.3.5:

Let and . So, diverges.

3.3 Exercise 10.3.8 (Newton's Method for finding the square root):

Then,

3.4 Exercise 10.3.9 (Newton's Method for finding the root of an equation):

Suppose we have a function and we want to find its root, i.e. . The recursive sequence has its limit at its root if it converges. The result from Exercise 10.3.8 is consistent with this formula in the sense: .

3.5 Exercise 10.3.11 (Fixed point of a function):

The point a is called a fixed point of the function if . Then, if a convergent sequence is defined as and if the function f is continuous at the limit of the sequence, then the limit is a fixed point of f.

3.6 Exercise 10.3.13 (Continued fraction):

- (a) is recursively defined sequence and its limit is .
- (b) The sequence of (a) can be rewritten in the form of continued fraction as . Irrational numbers will have infinite continued fraction (rational numbers will always have a terminated continued fraction), but the pattern could be quite simple. Above continued fraction can be defined as . Another example is the golden ratio, which is .

This is a deep topic and refer to this wiki page for further discussion.

4 The Bolzano-Weierstrass Theorem (Revisited)

4.1 THEOREM 10.6 (Cluster point of a set and convergent subsequences):

If c is a cluster point of set , there is a subsequence converging to c. This theorem is related to LEMMA 9.14 which states that The point c is a cluster point of a set S if and only if there is a sequence of elements of S, all different from c and converging to c.

4.2 DEFINITION 10.7 (Sequential cluster point):

A point c is a sequential cluster point of if is infinite for each .

This concept is different from the cluster point of a set in the sense that it only requires that the number of terms of the sequence is infinite in each neighborhood. It is not necessarily a cluster point of the range of the sequence. An example is a sequence that is eventually constant. So, the sequential cluster point is more relevant when we consider the convergence of a sequence.

4.3 THEOREM 10.8 (Sequential cluster point and convergent subsequence):

c is a sequential cluster point of if and only if has a subsequence converging to c.

4.4 THEOREM 10.9 (Bolzano-Weierstrass Theorem for Sequences):

Every bounded sequence has a sequential cluster point.

4.5 COROLLARY 10.10:

Every bounded sequence has a convergent subsequence.

4.6 Exercise 10.4.5 (Sequential cluster points and divergent sequences):

A bounded sequence diverges if and only if it has (at least) two sequential cluster points.

4.7 Exercise 10.4.6 (Monotone subsequence):

Every sequence has a monotone subsequence.

4.8 Exercise 10.4.7 (Monotone sequences and their sequential cluster point):

A monotone sequence can have only one sequential cluster point.

4.9 Exercise 10.4.8 (Finite range sequences):

If the range of a sequence is finite, at least one element of it must be repeated for infinitely many values of n.

4.10 Exercise 10.4.9:

A sequence with an ordinary cluster point cannot be eventually constant.

4.11 Exercise 10.4.10 (Limit superior and limit inferior):

The limit superior of a sequence , denoted , is the supremum of its set of sequential cluster points; if is not bounded above, . Formally, if is the set of all sequential cluster points of , then

and .

(Alternative Definition 1)

and . (Alternative Definition 2)

and

For the proof of the equivalence of the three definitions, see my separate notes titled "Limit Superiors and Limit Inferiors".

- (d)
- (e) , and

5 The Converse of Theorem 9.18?

6 Cauchy Sequences

7 Closing the Loop