Chapter 10 Sequences and the Big Theorem

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1 Convergence without Limits

N/A

2 Monotone Sequences

2.1 DEFINITION 10.1 (Monotonous sequences):

A sequence (x_n) is **increasing** if $x_{n+1} \geq x_n, \forall n \in N$, and it is **decreasing** if $x_{n+1} \leq x_n, \forall n \in N$. A sequence is **monotone** if it is either increasing or decreasing. If the relationship in this definition is strict, it is called **strictly increasing** or **strictly decreasing**. If this relationship only holds after a certain index k, it is called **eventually increasing** or **eventually decreasing**.

2.2 LEMMA 10.2 (Cluster point of monotone sequences):

- (a) Any **cluster point** of an **increasing sequence** is an **upper bound** for the sequence.
- (b) Any **cluster point** of a decreasing sequence is a **lower bound** for the sequence.

2.3 THEOREM 10.3 (Convergence of monotone sequences):

A **bounded, monotone** sequence **converges**.

2.4 COROLLARY 10.4 (Limits and bounds of monotone sequences):

If (x_n) is **increasing** and **bounded**, then $\lim x_n = \sup\{x_n\}$. If (x_n) is **decreasing** and **bounded**, then $\lim x_n = \inf\{x_n\}$.

2.5 Exercise 10.2.3 (Basel Problem and Limits of Upbounds):

Let

$$s_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}.$$

Then it can be shown by induction that

$$s_n < 2 - \frac{1}{n}, \text{ for } n = 2, 3, ...,$$

or $2-\frac{1}{n}$ is an upper bound of s_n. Apparently, $2-\frac{1}{n}$ converges to 2, but this does not mean that $\lim s_n$ is 2 because 2 is only an upper bound rather than a **supremum**. There is no indication that the gap between $2-\frac{1}{n}$ and s_n is shrinking to zero.

This is actually the famous **Basel problem** which was solved by young Euler. It is also a special case of the **Riemann zeta function** at 2. The correct limit is $\frac{\pi^2}{6}$.

2.6 Exercise 10.2.7 (Supremum and infimum of bounded sets):

If S is a **bounded set**, then there is an **increasing sequence** of elements of S converging to $\sup(S)$ and a **decreasing sequence** of elements of S converging to $\inf(S)$.

2.7 Exercise 10.2.8 (Bounded function on half-open domains):

Suppose f is an **increasing, bounded function** whose domain contains ray $[a, \infty)$. Then $\lim_{x\to\infty} f(x)$ exists.

2.8 Exercise 10.2.9.a-c (Harmonic series and geometric series):

Define the following partial sums:

(**Harmonic series**):

$$h_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$$

(**Geometric series**):

$$g_n = 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n}$$

Both the harmonic series and geometric series are increasing. (g_n) converges while (h_n) diverges. The limit of is the following. Since

$$g_n = \frac{1-\frac{1}{2}^n}{1-\frac{1}{2}}$$

then

$$\lim g_n = \frac{1}{1-\frac{1}{2}} = 2$$

Although the harmonic series diverges, the limit of the gap between the harmonic series and the logarithm function exists.

$$\gamma = \lim(h_n - \ln(n))$$

See **Exercise 10.2.9.f** for details.

2.9 Exercise 10.2.9.d-e (Convergence of series):

- (d) Suppose (a_n) is sequence of **positive** numbers and let $S = (s_n)$ be defined by $s_n = \sum_i^n a_i$, which is a **series**. S is **monotone** and it converges if and only if it is **bounded**.

- (e) Suppose (a_n) and (b_n) are sequences of **positive** numbers, with $a_n \leq b_n$ for all n. Let S and T be series based on (a_n) and (b_n). Then if T converges, S converges. If S diverges, T diverges. It is possible that S converges while T diverges.

2.10 Exercise 10.2.9.f (Euler-Mascheroni constant):

The limit $\gamma = \lim(h_n - \ln(n))$ exists because the **harmonic series** is strictly smaller than $1 + \ln(n)$ therefore this difference is **bounded**. The limit is called **Eulear-Mascheroni constant**. It is approximately 0.577. But it is still *unknown whether it is rational or irrational*.

3 A Recursively Defined Sequence

3.1 THEOREM 10.5 (Convergence of eventually monotone sequences):

A **bounded, eventually monotone** sequence **converges**. This is a direct extension of **THEOREM 10.3**.

3.2 Exercise 10.3.5:

Let $x_1 = a > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$. So, (x_n) diverges.

3.3 Exercise 10.3.8 (Newton's Method for finding the square root):

$$x_1 = 1;\ x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}),\ \forall n \geq 1$$

Then,

$$\lim x_n = \sqrt{a}$$

3.4 Exercise 10.3.9 (Newton's Method for finding the root of an equation):

Suppose we have a function $y = f(x)$ and we want to find its root, i.e. $f(x) = 0$. The **recursive sequence**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

has its limit at its root if it converges. The result from **Exercise 10.3.8** is consistent with this formula in the sense:

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{x_n^2 - a}{2x_n} = \frac{1}{2}(x_n - \frac{a}{x_n})$$

.

3.5 Exercise 10.3.11 (Fixed point of a function):

The point a is called a **fixed point** of the function $y = f(x)$ if $f(a) = a$. Then, if a convergent sequence is defined as $x_1 = a_0,\ x_{n+1} = f(x_n)$ and if the function f is continuous at the limit of the sequence, then the limit is a **fixed point** of f.

3.6 Exercise 10.3.13 (Continued fraction):

- (a)

  $$x_1 = 0,\ x_{n+1} = \frac{1}{2 + x_n}$$

  is recursively defined sequence and its limit is $\sqrt{2} - 1$.

- (b) The sequence of (a) can be rewritten in the form of **continued fraction** as

$$\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}}.$$

Irrational numbers will have **infinite continued fraction** (**rational numbers** will always have a **terminated continued fraction**), but the pattern could be quite simple. Above continued fraction can be defined as $[2, 2, 2, ...]$. Another example is the golden ratio, which is

$$[1, 1, 1, ...] = \frac{\sqrt{5}-1}{2}.$$

This is a deep topic and refer to this wiki page for further discussion.

4 The Bolzano-Weierstrass Theorem (Revisited)

4.1 THEOREM 10.6 (Cluster point of a set and convergent subsequences):

If c is a **cluster point** of set $\{x_n\}$, there is a **subsequence** (x_n) **converging** to c. This theorem is related to **LEMMA 9.14** which states that "The point c is a cluster point of a set S if and only if there is a sequence of elements of S, all different from c and converging to c".

4.2 DEFINITION 10.7 (Sequential cluster point):

A point c is a **sequential cluster point** of (x_n) if $\{n : x_n \in (c-\epsilon, c+\epsilon)\}$ is infinite for each $\epsilon > 0$.

This concept is different from the cluster point of a set in the sense that it only requires that the number of terms of the sequence is infinite in each neighborhood. It is **not necessarily** a cluster point of the range of the sequence. An example is a sequence that is **eventually constant**. So, the sequential cluster point is more relevant when we consider the convergence of a sequence.

4.3 THEOREM 10.8 (Sequential cluster point and convergent subsequence):

c is a **sequential cluster point** of (x_n) if and only if (x_n) has a subsequence converging to c.

4.4 THEOREM 10.9 (Bolzano-Weierstrass Theorem for Sequences):

Every **bounded** sequence has a **sequential cluster point**.

4.5 COROLLARY 10.10:

Every **bounded** sequence has a **convergent subsequence**.

4.6 Exercise 10.4.5 (Sequential cluster points and divergent sequences):

A **bounded** sequence **diverges** if and only if it has (**at least**) **two sequential cluster points**.

4.7 Exercise 10.4.6 (Monotone subsequence):

Every sequence has a **monotone subsequence**.

4.8 Exercise 10.4.7 (Monotone sequences and their sequential cluster point):

A **monotone sequence** can have **only one sequential cluster point**.

4.9 Exercise 10.4.8 (Finite range sequences):

If the range of a sequence is **finite**, at least one element of it must be repeated for infinitely many values of n.

4.10 Exercise 10.4.9:

A sequence with an **ordinary cluster point** cannot be **eventually constant**.

4.11 Exercise 10.4.10 (Limit superior and limit inferior):

The **limit superior** of a sequence (a_n), denoted $\limsup a_n$, is the **supremum** of its set of **sequential cluster points**; if (a_n) is not bounded above, $\limsup a_n = \infty$. Formally, if $E \subseteq \mathbb{R}$ is the set of **all sequential cluster points** of (a_n), then

$$\limsup_{n\to\infty} a_n = \sup E$$

and

$$\liminf_{n\to\infty} a_n = \inf E.$$

(Alternative Definition 1)

$$\limsup a_n = \lim_{n\to\infty} \sup_{k\geq n}\{a_k\}$$

and

$$\liminf a_n = \lim_{n\to\infty} \inf_{k\geq n}\{a_k\}.$$

(Alternative Definition 2)

$$\limsup_{n\to\infty} x_n := \inf_{n\geq 0} \sup_{m\geq n} x_m = \inf\{\, \sup\{\, x_m : m \geq n \,\} : n \geq 0 \,\}$$

and

$$\liminf_{n\to\infty} x_n := \sup_{n\geq 0} \inf_{m\geq n} x_m = \sup\{\, \inf\{\, x_m : m \geq n \,\} : n \geq 0 \,\}$$

For the proof of the equivalence of the three definitions, see my separate notes titled "*Limit Superiors and Limit Inferiors*".

- (d)

$$\lim a_n = L \iff \limsup a_n = \liminf a_n = L$$

- (e)

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n,$$

  and

$$\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n.$$

5 The Converse of Theorem 9.18?

5.1 THEOREM 10.11 (Partial converse of THEOREM 9.18):

If (x_n) is a **bounded sequence** with the property that every **convergent subsequence** converges to the same number L, then (x_n) converges to L.

The reason that we do a *partial* reversion instead of the *full* reversion is that the full reversion is a *trivial* statement. The full reversion states, "If every subsequence of (x_n) converges to L, then (x_n) converges to L." Since (x_n) is a subsequence of itself, this statement is trivial.

6 Cauchy Sequences

6.1 DEFINITION 10.12 (Cauchy Sequence):

The sequence (x_n) is a **Cauchy sequence** if, for any $\epsilon > 0$, there is a natural number N so that $|x_m - x_n| < \epsilon$ whenever $m > N$ and $n > N$.

The intuition of this definition is that "*the terms in the sequence get close to each other.*"

One counter example is that $(\ln(n))$ is not a Cauchy sequence. Although $\ln(n) - \ln(n-1)$ does go to 0, $\ln(n) - \ln(m)$ for any $n > m$ is not bounded because $\ln(n) - \ln(m) = \ln(\frac{n}{m})$, and $\frac{n}{m}$ is unbounded and therefore $\ln(\frac{n}{m})$ is unbounded.

6.2 THEOREM 10.13 (Cauchy Sequence and Convergency):

A **convergent sequence** is a **Cauchy sequence**.

6.3 LEMMA 10.14 (Cauchy Sequence and Subsequence):

If any **subsequence of a Cauchy sequence** converges, the sequence does also, and by **THEOREM 9.18**, they all converge to the same limit.

6.4 LEMMA 10.15 (Cauchy Sequence is Bounded)

A **Cauchy sequence** is **bounded**.

6.5 THEOREM 10.16 (Cauchy Completeness):

If F is an **ordered field** in which the **Bolzano-Weierstrass theorem** holds, then a **Cauchy sequence** of elements of F converges to an element of F.

Note that the **Archimedean property** was not mentioned in **THEOREM 10.16** because it was mentioned separately in both part (c), i.e. the **Bolzano-Weierstrass theorem**, and part (e), i.e. **Cauchy completeness**. This is indeed a result that neither the **Bolzano-Weierstrass theorem** nor **Cauchy completeness** is as strong as the **Dedekind completeness**.

6.6 EXAMPLE 10.6 (Decimal Expansion as a Cauchy Sequence):

The **decimal expansion** of a real number $x = 0.d_1d_2d_3....$ Let x_n is the cut up to the nth decimal. (x_n) forms a **Cauchy sequence** and it therefore converges. This is another proof that every decimal expansion corresponds to a real number.

6.7 Exercise 10.6.2 (Algebra of Cauchy Sequences):

- (a) If (a_n) and (b_n) are **Cauchy sequences**, then $(a_n + b_n)$ and $(a_n b_n)$ are Cauchy sequences too.
- (b) Same setting to (a), $(\frac{a_n}{b_n})$ is not necessarily a Cauchy sequence.

6.8 Exercise 10.6.3 (Cauchy Sequence of Integers):

- (a) A **Cauchy sequence of integers** must be **eventually constant**.

6.9 Exercise 10.6.4 (Unique Sequential Cluster Point of Cauchy Sequence):

- (a) A Cauchy sequence cannot have more than one sequential cluster point.

6.10 Exercise 10.6.5 (Contractive Sequence):

Suppose that (x_n) is a sequence wiht the property that there is a number $k < 1$ so that $|x_{n+1} - x_{n+2}| \leq k|x_n - x_{n+1}|$. Such a sequence is called **contractive**.

- (a) A **contractive sequence converges**.
- (b) An example of a **non-contractive convergent sequence** is the sequence $x_n = \frac{1}{n}$. If there is such a k that
  $$|x_{n+1} - x_{n+2}| \leq k|x_n - x_{n+1}|,$$
  then
  $$\frac{1}{(n+1)(n+2)} \leq \frac{k}{n(n+1)},$$
  which means
  $$\frac{n}{n+2} \leq k < 1.$$
  Since the ratio $\frac{n}{n+2}$ has its supremum at 1, no such k exists.

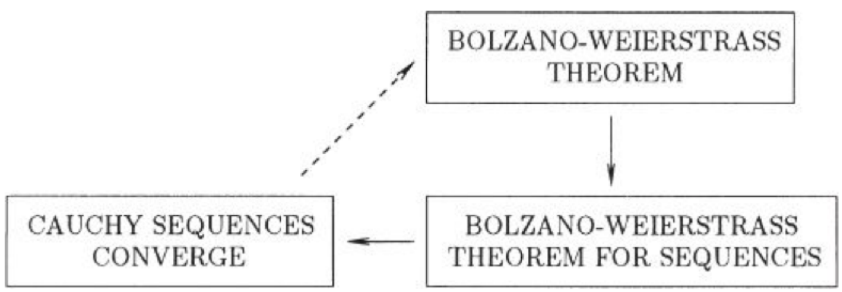
6.11 Exercise 10.6.7 (Bounded, Monotone Sequence is Cauchy)

A **bounded, monotone sequence** is a **Cauchy sequence**.

7 Closing the Loop

7.1 THEOREM 10.17 (Cauchy Completeness Indicates Bolzano-Weierstrass)

If F is an ordered field in which every **Cauchy sequence** converges to an element of F, then every **bounded, infinite** subset of F has a cluster point that is an element of F.



A local closed loop in the Big Theorem