Chapter 7 Cluster Points

April 18, 2017

1 Points and Sets

1.1 Exercise 7.1.1 (Neighborhood and Supremum):

A set cannot be a **neighborhood** of its **supremum**.

2 One Point of View

2.1 THEOREM 7.1 (Intervals, all points vs excluding finite points):

Take a set S and a point s and consider the following statements:

- (1) There is an $\epsilon > 0$ so that S contains all points of $(s - \epsilon, s + \epsilon)$.
- (2) There is an $\epsilon > 0$ so that S contains all but finitely many points of $(s - \epsilon, s + \epsilon)$.
- (3) There is an $\epsilon > 0$ so that S contains infinitely many points of $(s - \epsilon, s + \epsilon)$.
- (4) There is an $\epsilon > 0$ so that S contains a point other than s of $(s - \epsilon, s + \epsilon)$.
- (5) For every $\epsilon > 0$, S contains infinitely many points of $(s - \epsilon, s + \epsilon)$.
- (6) For every $\epsilon > 0$, S contains a point other than s of $(s - \epsilon, s + \epsilon)$.

Following observations can be made:

- (a) Statement (1) implies statement (2);
- (b) If $s \in S$, then statement (1) and statement (2) are equivalent.

If s is one of the '*missing points*" in statement (2), S is called a **deleted neighborhood** of s.

2.2 THEOREM 7.2 (Intervals, infinite vs one excluded the center):

Statement (5) and (6) in **THEOREM 7.1** are equivalent. Statement (3) and (4) are preparation for the following two statements.

3 Another Point of View

N/A

4 Cluster Points

4.1 DEFINITION 7.3 (Cluster Point):

The point s is a **cluster point** of the set S if, for every $\epsilon > 0$, the set $(s-\epsilon, s+\epsilon) \cap S$ is **infinite**. Some use the phrase "**accumulation point**" or "**limit point**," while others use the same words with slight differences in meaning.

THEOREM 7.2 says that s is a **cluster point** of a set S if and only if, $\forall \epsilon$, $(s-\epsilon, s+\epsilon) \cap S \setminus \{s\} \neq \emptyset$.

Knowing that a set is infinite would seem to be more useful than knowing just that it is not empty. Curiously, it is often easier to show that a set is infinite than to show that it is not empty. It is rare that the more useful bit of information is easier to come by.

Here, the definition of cluster point depends on intervals, which in turn depends on the definition of **distance**, or a **metric space**. In a more general view of **topological spaces**, this will be replaced by **open sets**.

4.2 EXAMPLE 7,4 (Finite Set has no Cluster Point, Cluster Point Not Necessarily be In the Set):

- (1) A **finite set** has no cluster points.
- (2) If S is **neighborhood** of s, then s is a cluster point of S. However, if s is a cluster point of S, S is *NOT* necessary a neighborhood of s (for example the end point of a closed interval). More generally, **it is not necessary for a point to be en element of a set for the point to be a cluster point of the set**.

4.3 Exercise 7.4.1:

s is a **cluster point** of a set S if and only if $S \cap U$ is **infinite** whenever U is a **neighborhood** of s.

4.4 Exercise 7.4.2 (Supremum and Cluster Point):

If S is a set that is **bounded above**, the **supremum** of S is either an **element** of S or is a **cluster point** of S.

5 Derived Sets

5.1 DEFINITION (Derived Sets):

The set of cluster points of a set S is called the **derived set** of S and is denoted S'.

5.2 EXAMPLE 7.5:

- (1) Most **closed intervals** are equal to their derived sets. The only exceptions are those that consist of a single point.
- (2) The derived set of an infinite set can be finite or empty.
- (3) $\mathbb{Q}' = \mathbb{R}$. That means the derived set can be much larger than the set itself. Here the derived set of a **countable set** is **uncountable**. Along this line, $(\mathbb{R} \setminus \mathbb{Q})' = \mathbb{R}$. Two different sets may have the same derived sets. There are **no antiderivatives**.

5.3 THEOREM 7.4 (Derived Set of Derived Set):

For any set S, we have $S'' \subseteq S'$.

In other words, derived sets might be more **well-behaved** than are typical sets.

5.4 Exercise 7.5.2:

$$\text{if } S \subseteq T, \text{ then } S' \subseteq T'$$

5.5 Exercise 7.5.3 (Cluster points of sin(n)):

- (b) The cluster points of $\sin(n)$ is **all real numbers** in $[-1, 1]$.

5.6 Exercise 7.5.5 (Adding or removing finitely many points has no effect on its derived set):

Adding **finitely many points** to a set or deleting finitely many points from a set does not change its derived set.

5.7 Exercise 7.5.6 (Set operation's effect on derived sets):

- (a) $(A \cup B)' = A' \cup B'$
- (b) The relationship in (a) hold for a union of **finitely many** sets.
- (c) $\bigcup_{\alpha \in \mathcal{A}} (A_\alpha)' \subseteq (\bigcup_{\alpha \in \mathcal{A}} A_\alpha)'$, and the containment might be **strict** if $\mathcal{A}$ is **infinite**.
- (d) $(\bigcap_{\alpha \in \mathcal{A}} A_\alpha)' \subseteq \bigcap_{\alpha \in \mathcal{A}} (A_\alpha)'$, and the containment might be **strict** even if $\mathcal{A}$ is **finite**.

5.8 Exercise 7.5.8 (Derived Set of a Dense Set):

S is **dense** in $\mathbb{R}$ if and only if $S' = \mathbb{R}$.

5.9 Exercise 7.5.9 (Isolation Point):

- (b) There is no such a set whose derived set is $\mathbb{Q}$.
- (c) If $x \in S$ and x is not a cluster point of S, then x is called an **isolation point** of S.
- (d) Suppose that $\{x_\alpha\}$ is the collection of **isolated point** of a set S. There exist positive numbers ϵ_α so that the ϵ**-neighborhood** of $\{x_\alpha\}$ are **mutually disjoint**.
- (e) A set can have **at most countably many** isolated points. This is because any discrete subset S of Euclidean space must be countable, since the isolation of each of its points together with the fact the rationals are dense in the reals means that the points of S may be mapped into a set of points with rational coordinates, (i.e. open intervals with rational endpoints) of which there are only countably many. However, **not every countable set** is discrete, of which the rational numbers under the usual Euclidean metric are the canonical example (none of the rational numbers is an isolated point).
- (f) If S is a set that **contains all of its cluster points**. S is the **derived set** of some (possibly different) set. This is an interesting comparison to **THEOREM 7.4**.

6 The Bolzano-Weierstrass Theorem

6.1 THEOREM 7.5 (The Bolzano-Weierstrass Theorem):

If F is an **Archimedean ordered field** in which the **Nested Intervals property** holds, then any **bounded, infinite subset** of F has a **cluster point**.

It can also be phrased as: **Every bounded, infinite set has a cluster point**. It is important to note that the **Archimedean property** is necessary to make it happen. In a **non-Archimedean ordered field**, such as the **formal rational function fields**, the **natural numbers** are **bounded and infinite**, but it has **no cluster point**.

6.2 Exercise 7.6.3 (Sets without cluster points):

If a set S has **no cluster points**, then S must be **either finite or unbounded**.

6.3 Exercise 7.6.4 (Uncountable subset of R has a cluster point):

- (d) Every **uncountable subset** of the real line has a cluster point. This is similar to **Exercise 7.5.9(e)**

6.4 Exercise 7.6.5 (Bolzano-Weierstrass Theorem fails in any non-Archimedean ordered field):

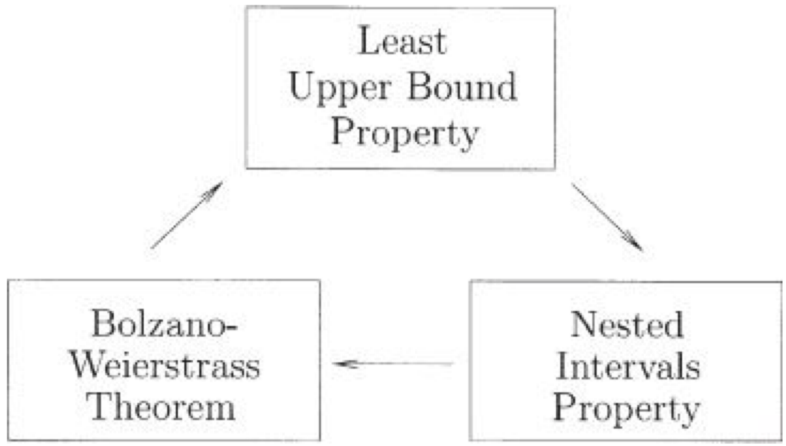
- (b) The **Bolzano-Weierstrass Theorem** fails in any ordered field that is **not Archimedean**. This is because **natural numbers**, which according to **THEOREM 4.13** must be contained in any ordered field, is **bounded** in non-Archimedean ordered field, which does not has a cluster point.

6.5 Exercise 7.6.6 (Limit Superior and Limit Inferior):

- (a) Let S be a **bounded, infinite set**. S' is then also **bounded**.
- (b) We define the **limit superior** of S by $\limsup S = \sup S'$. Then, $a = \limsup S$ if and only if $(a - \epsilon, \infty) \cap S$ is **infinite** for all $\epsilon > 0$ and $(a + \epsilon, \infty) \cap S$ is **finite** for all $\epsilon > 0$.
- (c) If S is **bounded above**, then $a = \limsup S$ is a cluster point of S.
- (d) Similarly, if S is **bounded below**, we define the **limit inferior** as $\liminf S = \inf S'$. Corresponding statements of (b) and (c) also hold for limit inferior.
- (g) If $S' = \emptyset$, then $\liminf S = \infty$ and $\limsup S = -\infty$

6.6 Exercise 7.6.7:

- (a) If S is any **bounded infinite set**, then $\liminf S \leq \limsup S$.
- (b) If in addition S has more than one cluster point, then $\liminf S < \limsup S$.



A local closed loop in the Big Theorem

6.7 Exercise 7.6.9 (Cluster Points and Countability):

- (a) A bounded set having **exactly one cluster point is denumerable**.
- (b) **Unbounded set** having **exactly one cluster point** is also **denumerable**.
- (c) A set having **finitely many cluster points** is **denumerable**.
- (d) If a set has **denumerably many cluster points**, the set is **denumerable** (because the **isolation points** are **countable**).
- (e) A set who has **uncountably many cluster points** *could* be countable (for example, $\mathbb{Q}$).

7 Closing the Loop

7.1 THEOREM 7.6 (Bolzona-Weierstrass in an Archimedean Ordered Field Implies Least Upper Bound property):

If F is an **Archimedean ordered field** in which the **Bolzano-Weierstrass theorem** holds, then the **Least Upper Bound property** also holds in F.

7.2 Exercise 7.7.2:

Suppose U and S are sets with the properties:

- (i) each element of U is an **upper bound** for S;
- (ii) for any $\epsilon > 0$, there are elements u of U and s of S with $|u - s| < \epsilon$.

(a) Then $\inf U = \sup S$.