Chapter 4 Ordering, Intervals and Neighborhoods

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1 Orderings

1.1 DEFINITION 4.1 (Linear Order):

An **ordering** on a set is linear if, for **any pair of elements** of the set (a, b) one and only one of the following holds:

- (i) $a < b$;
- (ii) $a = b$; or
- (iii) $b < a$.

This is called the **trichotomy**.

1.2 Exercise 4.1.1 (Least Element):

If S is a subset of an **ordered set**, a **least element** of S is an element x, if there is one, such that (i) $x \in S$ and (ii) if $y \in S$ and y is comparable to x, then $x \leq y$.

It is important to notice that a **linearly ordered set might not have a least element**, thus linearly ordered set are **not** necessarily **well ordered**.

1.3 Exercise 4.1.8 (Directed Set):

- (b) A set with an ordering is a **directed set** if $S, T \in X$, then $\exists U \in X$ such that $S < U$ and $Y < U$. Intuitively, this says "S and T may not be *directly* comparable, but there is something comparable to, and bigger than, both of them".
- (c) The **power set** $P(X)$ with ordering $\subseteq$ is a directed set.

2 The Ordering of the Natural Numbers

2.1 DEFINITION 4.2 (The Ordering of Natural Numbers):

If m and n are **natural numbers**, we say $m < n$ if either

- (a) n is among the natural numbers: $m + 1, m + 1 + 1, m + 1 + 1 + 1, \ldots$ or
- (b) no function whose **domain** is a set with m elements and whose **range** is a set with n elements is **onto**.

3 Well-ordering and Induction

3.1 DEFINITION 4.3 (Well-ordering):

A **linearly ordered set** is said to be **well-ordered** if every nonempty subset of it has a **least element**.

3.2 AXIOM (Well ordering property of natural numbers):

The **natural numbers** are **well-ordered**.

This is the foundation for mathematical induction. **Rational numbers**, on contrary, are **not** well ordered by <.

3.3 THEOREM 4.4 (Validity of Induction):

Suppose $S \subseteq N$ is such that

- (i) $1 \in S$ and
- (ii) $k \in S \Rightarrow k+1 \in S$ whenever $k \geq 1$

Then $S = N$.

The proof of this theorem used the **well-ordering property of natural numbers** (through contradiction). Therefore, the **well-ordering property** and the **validity of induction** are **equivalent**.

3.4 THEOREM 4.5 (Mathematical Induction):

Suppose $P(n)$ is an open statement, where n can be any natural number. If

- (i) $P(1)$ is true and
- (ii) $P(k) \Rightarrow P(k+1)$ whenever $k \geq 1$, then $P(n)$ is true for all.

4 Organizaing Proofs by Induction

N/A

5 Strong Induction

5.1 THEOREM 4.6 (Strong Induction):

Induction is equivalent to the following: Let $P(n)$ be an open statement, where n can be any natural number. If

- (i) $P(1)$ is true and
- (ii) $(P(1), ...,$ and $P(k)) \Rightarrow P(k+1)$ whenever $k \geq 1$, then $P(n)$ is true for all.

Strong induction is important not so much as a separate technique of proof (by constructing our propositions carefully, we can avoid using it explicitly), but as a signpost to bigger and better things. If we rephrase strong induction in the language we first used to describe induction itself, it would look like this:

If $S \subseteq N$ is such that $1 \in S$ and, for each $n > 1$, $\{k : k < n\} \subseteq S \Rightarrow n \in S$, then $S = N$.

Notice that this statement makes sense with N replaced by any **well-ordered set** and 1 replaced by **the least element** of the set (and there are well-ordered sets that are bigger and more complicated than we can possibly imagine just now). The resulting statement is a very deep and powerful tool called **transfinite induction**.

Transfinite induction is an extension of mathematical induction to well-ordered sets, for example to sets of ordinal numbers or cardinal numbers. Proofs or constructions using **induction** and **recursion** often use **the axiom of choice** to produce a well-ordered relation that can be treated by transfinite induction. However, if the relation in question is already well-ordered, one can often use transfinite induction without invoking the axiom of choice.

5.2 Exercise 4.5.5 (Goldbach Conjecture):

Q: What is the smallest natural number that can be written as a sum of three primes but can't be written as a sum of two primes?

A: This is related to the **Goldbach conjecture**. The **weak Goldbach conjecture** states that *any odd number greater than 5 can be expressed as the sum of three primes*. This has been proven in 2012. The **strong version** of the conjecture states *every even number greater than 2 can be expressed as the sum of two prime numbers*. This remains unsolved.

But it is clear that a number that can be written as the sum of three prime numbers but can't be written as the sum of two prime numbers must be an odd number. And an odd number is the sum of an odd number and an even number. The only even prime number is 2. So, we need to find the smallest odd number k such that k-2 is not a prime number. It turns out that k=11.

5.3 Exercise 4.5.7 (Induction Implies Well-ordering):

This is the inverse of **THEOREM 4.4**. Together, it is saying that **well-ordering property** and the **validity of induction** is **equivalent**.

5.4 Exercise 4.5.11 (Fibonacci Numbers and the Golden Ratio):

Definition 1:

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$$

Definition 2:

$$F_n = \frac{1}{\sqrt{5}} \left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n \right)$$

The ratio of two consecutive terms converges and the limit is the **golden ratio**:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

5.5 Exercise 4.5.16 (Algebraic-Geometric Mean Inequality):

- (b)

$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}$$

5.6 Exercise 4.5.17 (Pascal's Triangle):

- (a)

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

5.7 Exercise 4.5.18 (Telescoping Series):

- (b)

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + ... + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

This is an example of **telescoping series** which is a series whose partial sums eventually only have a fixed number of terms after cancellation. Notice that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

.

5.8 Exercise 4.5.19 (Bernoulli's Inequality):

$$(1+a)^n \geq 1 + na, \ \forall a \geq -1$$

5.9 Exercise 4.5.21 (Two Series):

- (a)

$$(1 \times 2) + (2 \times 3) + ... + (n \times (n+1)) = \frac{n(n+1)(n+2)}{3}, \ \forall n \in N$$

- (b)

$$1^3 + 2^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2, \ \forall n \in N$$

5.10 Exercise 4.5.26 (Inequality of Three Pythagorean means):

Define the arithmetic mean (AM), the geometric mean (GM), and the harmonic mean(HM) as follows:

$$AM(x_1, ..., x_n) = \frac{1}{n}(x_1 + ... + x_n)$$
$$GM(x_1, ..., x_n) = \sqrt[n]{x_1 \cdot ... \cdot x_n}$$
$$HM(x_1, ..., x_n) = \frac{n}{\frac{1}{x_1} + ... + \frac{1}{x_n}}$$

The inequality is:

$$\min \leq HM \leq GM \leq AM \leq \max$$

6 Ordered Fields

6.1 DEFINITION 4.7 (Positive Set and Ordered Field):

- (a) Let P be a nonempty subset of a field F. Suppose
  - (i) $\forall a \in P$ and $b \in P \Rightarrow a + b \in P$ and $a \times b \in P$, and
  - (ii) For each $x \in F$, **exactly one** of $x \in P,\ x = 0,$ or $-x \in P$ holds. Then P is called a **positive set**.
- (b) A pair (F, P), where F is a field and P is a positive set, is called an **ordered field**.
- (c) In an ordered field, we say $a < b,$ if $b + (-a) \in P$. If $x \in P$, we say x is **positive**, and if $-x \in P$, we say x is **negative**.

This definition is related to the **field structure** in a big way. The only substantial part of the definition of a field we don't see is the **multiplicative inverse**.

There are two *equivalently common definitions* of an ordered field. The **definition of total order** appeared first historically and is a first-order axiomatization of the ordering $\leq$ as a binary predicate. Artin and Schreier gave the **definition in terms of positive cone** in 1926, which axiomatizes the sub-collection of nonnegative elements. Although the latter is higher-order, viewing positive cones as maximal prepositive cones provides a larger context in which field orderings are extremal partial orderings.

The equivalence is proved as following:

Let F be a field. There is a bijection between the field orderings of F and the positive cones of F. Given a field ordering $\leq$ as in the first definition, the set of elements such that $x \geq 0$ forms a positive cone of F. Conversely, given a positive cone P of F as in the second definition, one can associate a total ordering $\leq$ on F by setting $x \leq y$ to mean $y - x \in P$. This total ordering $\leq$ satisfies the properties of the first definition.

6.2 THEOREM 4.8:

If (F, P) is an ordered field, $(a \in F, a \neq 0) \Rightarrow (a^2 \in P)$.

6.3 COROLLARY 4.9:

In an ordered field, $1 \in P$.

The proof of this corollary uses **Exercise 3.2.9** that if a field has more than 1 element, $1 \neq 0$.

6.4 THEOREM 4.10:

The product of a **positive element** of an ordered field and a **negative element** is negative.

6.5 COROLLARY 4.11:

In an ordered field, $x \in P \iff x^{-1} \in P$.

The proof of this corollary uses **THEOREM 4.8** and **THEOREM 4.10**.

6.6 THEOREM 4.12:

Let a, b and c be elements of an ordered field.

- (a) If $a < b$, then $a + c < b + c$.
- (b) If $a < b$ and , then $ac < bc$.
- (c) If $a < b$ and , then $bc < ac$.
- (d) If $a < b$, then $a < (a + b)/2 < b$.
- (e) If $a < b$ and $b < c$, then $a < c$.

6.7 THEOREM 4.13 (Every Ordered Field Contains a "Copy" of Natural Numbers):

In an ordered field, the elements $1, 1 + 1, 1 + 1 + 1, ...$ are **all positive** and **all different**.

This theorem is inspired by the observation that every field contains the elements: $1, 1 + 1, 1 + 1+1, ...$ In some fields, these elements aren't all different (in $\mathbb{Z}_2$, for instance, we have $1+1+1 = 1$). This theorem tells us that this can't happen in an **ordered field**, and consequently that **every ordered field contains a "copy" of** N. Note that a field that contains such a copy of N also contains copies of $\mathbb{Z}$ and $\mathbb{Q}$ (due to the closeness to addition and multiplication), and so it makes sense to refer to "**integers**" and "**rational elements**" in any ordered field.

6.8 COROLLARY 4.14 (Ordered Field is Infinite):

An ordered field is infinite.

6.9 Exercise 4.6.2 (Sum of Squares):

Suppose a and b are elements of an ordered field. Then

$$a^2 + b^2 = 0 \iff a = 0, b = 0$$

This is a very important property of ordered field. If we can find non-trivial squares sum up to zero in a field, it then can't be ordered. So, **complex numbers cannot be ordered**.

6.10 Exercise 4.6.3 (Zp cannot be ordered):

None of the fields $\mathbb{Z}_p$ can be ordered.

This is an interesting results that can be proved by **Exercise 4.6.2**. To prove cannot be ordered, we just find a pair of non-trivial squares that sum up to zero. **Fermat's little theorem** tells that x^{p-1} is congruent to 1, or x^p is congruent to x. It is possible to prove every element in the finite field can be expressed as sum of two squares. The it is possible to prove every finite field of the same order are **isomorphic**.

6.11 Exercise 4.6.8 (Archimedean Property):

Given any rational number $r = \frac{p}{q}$, there is a natural number n with $r < n$. An ordered field in which the natural numbers are distributed in this way is said to have the **Archimedean property**.

The reason by saying "*distributed in this way*" is because **THEOREM 4.13** guarantees that every ordered field contains a copy of natural numbers. So, an ordered field is **Archimedean** or not completely depends on how this copy is distributed. In non-Archimedean ordered field, the natural numbers need to be "*clustered in a region*" so that **infinity** and **infinitesimal** are contained in the ordered field.

6.12 Exercise 4.6.15 (Non-uniqueness of Order):

It is possible to define two positive sets $P_1 \neq P_2$ for the **same field** that both satisfy the definition of a **positive set**.

Here is an interesting example, which is also stated in the **Example 1.2 in Counterexamples in Analysis**.

7 Absolute Value and Distance

7.1 DEFINITION 4.15 (Absolute Value):

The **absolute value** of an element x of an ordered field (F, P) is given by:

$$|x| = \begin{cases} x & \text{if } x \in P \text{ or } x = 0 \\ -x & \text{if } -x \in P \end{cases}$$

7.2 THEOREM 4.16 (The Triangle Inequality):

If x and y are elements of an ordered field then:

$$\big||x| - |y|\big| \leq |x + y| \leq |x| + |y|$$

Similarly (**shortest path between two points is a straight line**):

$$|x - y| \leq |x - z| + |y - z|$$

7.3 Exercise 4.7.3 (Metric):

A **metric** in the sense of linear algebra is a function $d(x, y)$ over a set that satisfies the following properties:

- (i) (**Non-negativity/Separation Axiom and Identity of Indiscernibles**)

$$\forall x, y,\ d(x, y) \geq 0 \text{ and } d(x, y) = 0 \iff x = y$$

- (ii) (**Symmetry**)

$$d(x, y) = d(y, x)$$

- (iii) (**Triangle Inequality**)

$$d(x, z) \leq d(x, y) + d(y, z)$$

The **absolute value** defined in **DEFINITION 4.15** is an example of **metric**.

7.4 Exercise 4.7.4 (L1 norm and L2 norm):

If $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$, let $D_1 : \mathbb{R}^2 \to \mathbb{R}$ and $D_2 : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$D_1(\vec{x}, \vec{y}) = |y_1 - x_1| + |y_2 - x_2|$$

and

$$D_2(\vec{x}, \vec{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$$

Both are examples of **metric** defined in **Exercise 4.7.3**.

8 Intervals

8.1 DEFINITION 4.18 (Intervals):

Let S be a **linearly ordered set**. The set $I \subseteq S$ is an interval if I has one of the following forms:

- (1) $\{x \in S : a < x < b\}$ denoted (a, b)
- (2) $\{x \in S : a \leq x \leq b\}$ denoted $[a, b]$
- (3) $\{x \in S : a \leq x < b\}$ denoted $[a, b)$
- (4) $\{x \in S : a < x \leq b\}$ denoted $(a, b]$
- (5) $\{x \in S : a < x\}$ denoted (a, ∞)
- (6) $\{x \in S : x < b\}$ denoted $(-\infty, b)$
- (7) $\{x \in S : a \leq x\}$ denoted $[a, \infty)$
- (8) $\{x \in S : x \leq b\}$ denoted $(-\infty, b]$
- (9) all of S (sometimes) $(-\infty, \infty)$

The symbol ∞ should not be endowed with any meaning except that given it here. It is particularly important to remember that **this symbols ∞ does not represent an element of S**. (1), (5) and (6) are called **open**, (2), (7) and (8) are called **closed**. (5), (6), (7) and (8) are sometimes called **rays**.

8.2 THEOREM 4.19 (Distance Inside an Interval):

If x and y are elements of (a, b), then $|x - y| < b - a$.

8.3 THEOREM 4.20 (Property of Nested Intervals):

If $x \in (a, b) \cap (c, d)$, and $(d - c) < \min\{b - x, x - a\}$, then $(c, d) \subseteq (a, b)$.

9 When Should We Picture?

9.1 Exercise 4.9.2 (End Points of Intervals):

- (a) if $c < d$ and (a, b) contains neither c nor d, then either $(a, b) \cap (c, d) = \emptyset$ or $(a, b) \subseteq (c, d)$.
- (b) Suppose that A and B are **open intervals** such that neither contains an **endpoint** of the other. Then A and B are either **disjoint** or **identical**.

9.2 Exercise 4.9.3:

Suppose S is a set with the property that $|x - y| \geq 1$ for any two different elements x and y of S. Then, an interval $(a, a+1)$ can contain **at most one element** of S.

9.3 Exercise 4.9.5 (Union and Intersection of Intervals):

- (a) The **intersection** of two intervals is an interval.
- (b) If I and J are intervals and $I \cap J \neq \emptyset$, then $I \cup J$ is an interval.

10 Neighborhoods

10.1 THEOREM 4.21 (Interval in terms of Center and Radius):

If $a < b$. Let $c = (a+b)/2$ and $\varepsilon = (b-a)/2$. Then, $(a, b) = \{x : |x - c| < \epsilon\}$.

10.2 DEFINITION 4.22 (ε-neighborhood):

An interval of the form $\{x : |x - c| < \epsilon\}$ for some real number c and some positive real number ε is called an **ϵ-neighborhood of c** (or an **ϵ-interval around c**).

10.3 DEFINITION 4.23 (Neighborhood):

The set U is a **neighborhood of c** if there exists $\epsilon > 0$ so that U contains the **ϵ-neighborhood of c**.

Note that every ϵ-neighborhood of a point is a neighborhood of it, but not every neighborhood is an ϵ-neighborhood.

If a set is a neighborhood of a point, the point must be an element of the set, but a set can contain points of which it is not a neighborhood.

10.4 THEOREM 4.24 (An Open Interval is Neighborhood of All Its Elements):

An **open interval** is a **neighborhood** of each of its points.

10.5 Exercise 4.10.3 (Uncountability of Neighborhood):

Any set that is a **neighborhood** of some point is **uncountable**.

A **bijection** can be constructed between a **neighborhood** to the **whole real line**.

10.6 Exercise 4.10.4 (Separation of distinct points):

If $x \neq y$, then there are **neighborhoods** U of x and V of y such that $U \cap V = \emptyset$.