Change Measure and Risk Neutral

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1 Introduction

In this note, we will study measure change and risk-neutral pricing, which is a key part of financial mathematics.

2 Background Knowledge

Before going to the topics we mainly focused on, we need to review some previous knowledge.

Definition 1 (Function Composition)**.** The composition operator $\circ$ combines the two functions f and g, and return a new function h:

$$h(x) = g(f(x)) = (g \circ f)(x)$$

Definition 2 (Brownian Motion)**.** A random process B defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ such that:

1. $B_0 = 0$;
2. $B = B_t$ is continuous;
3. $B = B_t$ is a square-integrable martingale wrt $\mathcal{F}_t$ such that

$$\mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] = t - s, \quad s \leq t \tag{1}$$

is a Brownian motion.

Theorem 3 (Itô's Isometry)**.** If $\phi(t, \omega)$ is bounded and elementary, then

$$\mathbb{E}\Big[\Big(\int_s^T \phi(t,\omega) dB_t(\omega)\Big)\Big] = \mathbb{E}\Big[\Big(\int_s^T \phi(t,\omega)^2 dt\Big)\Big].$$

3 Basic Transformations

In this part, we will study the transformations of the process and related equations, which are important in solving optimal stopping problems with an analytical solution. This part is mainly based on Section 5 Chapter 2 of [9]. In this note, we will study the following transformations:

1. Change of time,
2. Change of space,
3. Change of measure.

3.1 Change of Time

In this section, we will study the transformation of the time, which is close to the speed measure we have studied. At the beginning, we will briefly introduce what is the change of time. Suppose there is a complicated process X_t:

$$dX_t = \sigma(t, X_t)dW_t.$$

This process can be represented in the form:

$$X = \hat{X} \circ T,$$

which can also be written as:

$$X_t = \hat{X}_{T(t)}$$

where $\hat{X} = (\hat{X}_\theta)_{\theta \geq 0}$. $\hat{X}_\theta$ is a simple process where θ is a new time, and $\theta = T(t)$ is the change of time leading the transformation of the old time t to the new time θ. For instance, $\hat{X} = (\hat{X}_\theta)_{\theta \geq 0}$ could be Brownian motion. In summary, in the time change, we have this equality:

$$X_t = \hat{X}_{T(t)} = \hat{X}_\theta, \tag{2}$$

where $\hat{X}_\theta$ is a Brownian motion.

We have studied the basics of the time change, and we will illustrate it mathematically. Firstly, we can define:

$$T(t) = \int_0^t \sigma^2(u, X_u)du$$

and

$$\hat{T}(\theta) = \inf\{t : T(t) > \theta\}.$$

Assuming $\sigma^2(u, X_u) > 0$, then for any $0 < t < \infty$, we have $\int_0^t \sigma^2(u, X_u)du < \infty$ and $\int_0^\infty \sigma^2(u, X_u)du = \infty$ $\mathbb{P}$-a.s. Then $T(t)$ is a continuously increasing process, so $\hat{T} = \inf\{t : T(t) = \theta\}$ and

$$\int_0^{\hat{T}(\theta)} \sigma^2(u, X_u)du = T(\hat{T}(\theta)) = \theta, \tag{3}$$

which can also be expressed

$$\frac{d\hat{T}(\theta)}{d\theta} = \frac{1}{\sigma^2(\hat{T}(\theta), X_{\hat{T}.(\theta)})}$$

When $\sigma = \sigma(x)$, this is the time-homogeneous case, we can have a measure $m = m(dx)$:

$$m(dx) = \frac{1}{\sigma^2(x)}dx$$

which is the speed measure. By this new time θ, we can define the new process:

$$\hat{X}_\theta = X_{\hat{T}(\theta)} = \int_0^{\hat{T}(\theta)} \sigma(u, X_u)dW_u. \tag{4}$$

Since this local martingale is square integrable, it is a martingale. Therefore we have:

$$\mathbb{E}(\hat{X}_\theta) = 0,$$

and

$$\mathbb{E}^2(\hat{X}_\theta) = \int_0^{\hat{T}(\theta)} \sigma^2(u, X_u)du = \theta,$$

by Itô's isometry and integral (3). Therefore, we can conclude that $\hat{X}_\theta$ is a Brownian motion. According to the integral (4), we have:

$$\hat{X} = X \circ \hat{T},$$

and

$$X = \hat{X} \circ T.$$

Therefore, we can have the conclusion (2) in detail:

$$X_t = \hat{X}_{T(t)} = \hat{X}_{\int_0^t \sigma^2(s, X_s)ds}.$$

Therefore, we can conclude that a local martingale $X_t = \int_0^t \sigma(s, X_s)dW_s)$ can be resented by a time change $\theta = T(t)$ of a new Brownian motion $\hat{X}$. In general, X is aligned with the time-change Brownian motion we studied in [10] and included in the scale function and speed measure note.

In addition, we can also study the change of time from the perspective of transition probabilities of a (Markov) process X. We can define $f = f(s, x; t, y)$ being its transition density:

$$f(s, x; t, y) = \frac{\partial \mathbb{P}(X_t \leq y \mid X_s = x)}{\partial y}.$$

Then this density function is the solution to the Kolmogorov forward and backward equations:

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(t, y)f) \text{ (if } t > s)\\ \frac{\partial f}{\partial t} &= -\frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(s, x)f) \text{ (if } t < s)\end{aligned}$$

By introducing the new time variable $\theta = T(t) = \int_0^t \sigma^2(u, X_u)du$, we can define $\theta' = T(s)$ for $s < t$. Then we can define a new transition probability:

$$f^{'}(\theta', x; \theta, y) = f(s, x; t, y).$$

By chain rule, we have

$$\frac{\partial f'}{\partial \theta} = \frac{1}{2}\frac{\partial^2 f'}{\partial y^2}$$

and

$$\frac{\partial f'}{\partial \theta'} = -\frac{1}{2}\frac{\partial^2 f'}{\partial y^2}.$$

We know that $\hat{X}$ is a Brownian motion, therefore

$$f^{'}(\theta', x; \theta, y) = \frac{1}{\sqrt{2\pi(\theta - \theta')}}e^{-\frac{(y-x)^2}{2(\theta-\theta')}}.$$

We have some general principles included in page 109 of [9]. but we will omit it, provide some results and introduce lemmas about the results on the change of time in stochastic analysis. For the variables $T(t)$, $\hat{T}(t)$, and θ, we have

1. $\hat{T}(T(t)) = t$
2. $T(\hat{T}(\theta)) = \theta$
3. $\hat{T}(\theta) = T^{-1}(\theta)$
4. $T(t) = \hat{T}^{-1}(t)$

For non-negative functions $f = f(t)$, we have

$$\begin{aligned}\int_0^{\hat{T}(b)} f(t)d(T(t)) &= \int_0^b f(\hat{T}(\theta))d\theta\\ \int_0^{T(a)} f(\theta)d(\hat{T}(\theta)) &= \int_0^a f(T(t))dt \qquad (5)\end{aligned}$$

Lemma 4 (Dambis-Dubins-Schwarz)**.** Let $X = (X_t)_{t\geq 0}$ be a continuous local martingale defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ starts from 0 and $\langle X \rangle_\infty = \infty$, and let

$$T(t) = \langle X \rangle_t$$
$$\hat{T}(\theta) = \inf\{t \geq 0 : T(t) > \theta\}$$

Then

1. The process $\hat{X} = (\hat{X}_\theta)_{\theta \geq 0}$ given by

$$\hat{X}_\theta = X_{\hat{T}(\theta)}$$

is a Brownian motion;

2. The process X can also be reconstructed from the Brownian motion $\hat{X}$ by formulation:

$$X_\theta = \hat{X}_{T(t)}.$$

3.2 Change of Space

In the previous section, we studied that by changing the time from t to θ a square-integrable local martingale M_t can be represented by a Brownian motion $\hat{X}_\theta$. In this section, we will study the change of space, which allows us to transform the diffusion into a Brownian motion. A. N. Kolmogorov initially applied this transformation [12]. They utilise the transformation of time and space at the same time $(s, x; t, y) \to (\theta, \xi; \vartheta, \eta)$, which leads to a transition function in transition functions $f(s, x; t, y) \to g(\theta, \xi; \vartheta, \eta)$. We will illustrate this idea by an Ornstein-Uhlenbeck process $X = (X_t)_{t\geq 0}$ with $X_0 = x_0$, which follows:

$$dX_t = (\alpha(t) - \beta(t)X_t)dt + \sigma(t)dW_t$$

Therefore, we have

$$X_t = \gamma(t)\left[x_0 + \int_0^t \frac{\alpha(s)}{\gamma(s)}ds + \int_0^t \frac{\sigma(s)}{\gamma(s)}dW_s\right]$$

where $\gamma(s) = \exp(-\int_0^t \beta(u)du)$. We assume $\int_0^t \frac{\alpha(s)}{\gamma(s)}ds \in \mathcal{L}^1$ and $\int_0^t \frac{\sigma(s)}{\gamma(s)}dW_s \in \mathcal{L}_2$ when $t < \infty$. As $t \to \infty$, we have $\int_0^t |\frac{\sigma(s)}{\gamma(s)}|^2 ds \to \infty$. According to what we have studied in the change of time, we can define a new time $\theta = T(t)$:

$$T(t) = \int_0^t \left(\frac{\sigma(s)}{\gamma(s)}\right)^2 ds.$$

Based on lemma 4, we have a Brownian motion:

$$\hat{B}_\theta = \int_0^{\hat{T}(\theta)} \frac{\sigma(s)}{\gamma(s)} dW_s.$$

Therefore,

$$X_t = \varphi(t) + \gamma(t)\hat{B}_{T(t)},$$

where

$$\varphi(t) = \gamma(t)\left[x_0 + \int_0^t \frac{\alpha(s)}{\gamma(s)} ds\right].$$

Then we have a function to transform the space:

$$\Psi(t, y) = \frac{y}{\gamma(t)} - \int_0^t \frac{\alpha(u)}{\gamma(u)} du$$

and the related density function becomes:

$$g(\theta, \xi; \vartheta, \eta) = \frac{1}{\frac{\partial \Psi(t,y)}{\partial y}} f(s, x; t, y)$$

where

$$\begin{aligned}
\eta &= \Psi(t, y) \\
\xi &= \Psi(s, x) \\
\vartheta &= \int_0^s \left(\frac{\sigma(u)}{\gamma(u)}\right)^2 du \\
\theta &= \int_0^t \left(\frac{\sigma(u)}{\gamma(u)}\right)^2 du
\end{aligned}$$

and $x_0 = 0$. According to this formula, we have transformed the time and space and $\Psi(t, X_t) = \int_0^t \frac{\sigma(s)}{\gamma(s)} dW_s$. Therefore.

$$\Psi(\hat{T}(\theta), X_{\hat{T}(\theta)}) = \int_0^{\hat{T}(\theta)} \frac{\sigma(s)}{\gamma(s)} dW_s = \hat{B}_{T(\hat{T}(\theta))} = \hat{B}_\theta. \tag{6}$$

In summary, by applying the transformation in time ($t \to \theta$) and space change by utilizing the formula $\Psi(t, y)$, we could construct a new Brownian motion $\hat{B}$. Also, this transformation is reversible.

In general, the change of space holds for the general one-dimensional diffusion. When the diffusion becomes the time-homogeneous Markov diffusion process:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \tag{7}$$

the space transformation and time transformation are known as the scale function introduced in [7] and [11] (Scale function Note).

3.3 Change of Measure

Unlike the previous change in time and space that transforms the trajectories of the process, the change of measure only transforms the probability measure $\mathbb{P}$ to $\mathbb{Q}$, such that:

$$\mathrm{Law}(X \mid \mathbb{Q}) = \mathrm{Law}(\tilde{X} \mid \mathbb{P}). \tag{8}$$

It means that under the transformed measure $\mathbb{Q}$ the complicated stochastic process X behaves like the simple process $\tilde{X}$ under the original probability measure $\mathbb{P}$. In mathematical finance, we transform the objective probability to the equivalent martingale measure (risk-neutral measure), which we will illustrate in the next section.

Assuming we have a Itô's process:

$$dX_t = b_t dt + dB_t,$$

we can construct a new probability measure $\tilde{\mathbb{P}}$ from the objective probability measure $\mathbb{P}$ such that

$$d\tilde{\mathbb{P}}_t = Z_t d\mathbb{P}$$

where

$$Z_t = \exp\left\{ \int_0^t (\tilde{b}_s(\omega) - b_s(\omega)) dB_s - \frac{1}{2}\int_0^t \left(\tilde{b}_s(\omega) - b_s(\omega)\right)^2 ds \right\} \tag{9}$$

where $\tilde{b}$ is another adapted integrable process. With this Z_t, we can have equation (8). According to Girsanov's result, under $\tilde{\mathbb{P}}$,

$$\tilde{B}_t = B_t - \int_0^t (\tilde{b}_s - b_s) ds$$

is a Brownian motion, and

$$dX_t = \tilde{b}_t(\omega)dt + d\tilde{B}_t.$$

Then under $\mathbb{P}$,

$$d\tilde{X}_t = b_t(\omega)dt + dB_t$$

Therefore, we have the equation (8). In particular, if $\tilde{b} \equiv 0$, we have that X_t is a Brownian motion under $\tilde{P}$. Therefore, we can summarize the Girsanov theorem of measure transformation, which is included in Page 48 of [4].

Theorem 5 (Girsanov Theorem)**.** Assume we have a process $X_t = B_t - \int_0^t \mu(s)ds$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, which is driven by:

$$dX_t = -\mu(t)dt + dB_t$$

where B_t is a standard Brownian motion. If $int_0^t \mu(s)dB_s$ is a martingale under $\mathbb{P}$, then

$$Z_t = \exp\left(\int_0^t \mu(s)dB_s - \frac{1}{2}\int_0^t \mu^2(s)ds\right)$$

is also a martingale under $\mathbb{P}$. This is also the case when the Novikov condition:

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T \mu^2(s)ds\right)\right) \leq +\infty$$

holds, Z_t is also a martingale. Then we can define a new probability measure $\mathbb{Q}$, such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_t.$$

4 Risk Neutral

We have studied the fundamental theorem in stochastic analysis, in this section, we will study the idea of risk-neutral mainly based on Chapter 7 of [1]. Firstly, we will introduce the concept of no arbitrage opportunity, Black-Scholes model, and study the concept of risk-neutral through this model.

4.1 Arbitrage

The price of financial derivatives is determined by its market condition, which include the coefficients like attitudes to risk on the market and expectations about the future stock prices. When Black and Scholes derive the Black Scholes formula [3], they assume the market is efficient, which means free of arbitrage probabilities.

We will start with the definition of self-financing portfolio [6]:

Definition 6 (Self-financing Portfolio)**.** A portfolio V_t is said to be self-financing if we have two adapted process H_t^0 and H_t denoting the quantities of riskless and risky assets held at time t, satisfying

1. $$\int_0^T |H_t^0|dt + \int_0^T (H_t)^2 dt < \infty$$

   almost surely;

2. $$H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u dS_u$$

   almost surely.

In this definition S_t^0 refers to the riskless assets and $0 \leq t \leq T$.

According to this definition, we can define the arbitrage:

Definition 7 (Arbitrage Opportunity)**.** A financial market is said to have an arbitrage opportunity if a self-financed portfolio V such that

$$\begin{aligned} V_0 &= 0 \\ \mathbb{P}(V_T \geq 0) &= 1 \\ \mathbb{P}(V_T > 0) &> 0 \end{aligned}$$

A market is arbitrage free if it is without arbitrage opportunity.

In [1], they illustrate that if there is an arbitrage opportunity, we can make money without taking risk, which means there is free lunch on the financial market and will lead us to misprice the asset. Therefore, Black and Scholes [3] assume the market is efficient. Therefore, under the arbitrage-free market, we always have

Proposition 8. Consider a self-financing portfolio V, the related dynamics in the form:

$$dV_t = rV_t dt.$$

This proposition must hold. If it does not hold, it will lead an opportunity of arbitrage.

4.2 Black Scholes Formula

We can model a market by the following equations:

$$\begin{aligned} dB_t &= rB_t dt \\ dS_t &= S_t\mu(t, S_t)dt + S_t\sigma(t, S_t)dW_t \end{aligned} \tag{10}$$

and a contingent claim of the form $\mathcal{X} = \Phi(S_T)$. In this market, the claim can be traded and its price process $\Pi_t[\Phi] = F(t, S_t)$, in which F are some kind of smooth function. We aim to find the expression of F in the market $[S_t, B_t, \Pi_t[\Phi]]$ without arbitrage possibilities.

We will start with deriving the pricing equation of $F(t, S_t)$, which is a simplified version of the derivation introduced in [5]. Applying Itô's formula on $F(t, S_t)$, we can get:

$$\begin{aligned} dF(t, S_t) &= F_t(t, S_t)dt + F_{S_t}(t, S_t)S_t\mu(t, S_t)dt + F_{S_t}(t, S_t)S_t\sigma(t, S_t)dW_t + \frac{1}{2}F_{S_tS_t}(t, S_t)S_t^2\sigma^2(t, S_t)dt \\ &= \big(F_t(t, S_t) + F_{S_t}(t, S_t)S_t\mu(t, S_t) + \frac{1}{2}F_{S_tS_t}(t, S_t)S_t^2\sigma^2(t, S_t)\big)dt + F_{S_t}(t, S_t)S_t\sigma(t, S_t)dW_t \end{aligned}$$

Consider that we have a portfolio Π by short an option and long $F_{S_t}(t, S_t)$ shares:

$$\Pi_t = -F(t, S_t) + F_{S_t}(t, S_t)S_t$$

then its value changes following:

$$\begin{aligned} d\Pi_t &= -dF(t, S_t) + F_{S_t}(t, S_t)dS_t \\ &= -\big(F_t(t, S_t) + \frac{1}{2}F_{S_tS_t}(t, S_t)S_t^2\sigma^2(t, S_t)\big)dt \text{ (Substitute)} \end{aligned}$$

It means that the portfolio is riskless since its change is only related to the time change. Since this market is without arbitrage opportunity and based on proposition 8, we have

$$d\Pi_t = r\Pi_t dt,$$

which is also equivalent to

$$-\big(F_t(t, S_t) + \frac{1}{2}F_{S_tS_t}(t, S_t)S_t^2\sigma^2(t, S_t)\big)dt = r(-F(t, S_t) + F_{S_t}(t, S_t)S_t)dt.$$

Then rearranging and simplifying the equation, we can get

$$(F_t(t, S_t) + \frac{1}{2}F_{S_tS_t}(t, S_t)S_t^2\sigma^2(t, S_t) + rF_{S_t}(t, S_t)S_t) = rF(t, S_t). \tag{11}$$

Equation (11) is the Black Scholes equation for option pricing, and we have a boundary condition when $F(T, S_T) = \Phi(T, S_T)$. This system can be directly solved by the Feynman-Kac formula:

Theorem 9 (Feynman-Kac Formula)**.** Given a PDE:

$$\frac{\partial}{\partial t}u(x, t) + \mu(x, t)\frac{\partial}{\partial x}u(x, t) + \frac{1}{2}\sigma^2(x, t)\frac{\partial^2}{\partial^2 x}u(x, t) - V(x, t)u(x, t) + f(x, t) = 0, \tag{12}$$

when $t \in [0, T]$ and a terminal condition:

$$\mu(x, T) = \psi(x),$$

then the solution u can be written as:

$$u(x, t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T V(X_\tau, \tau)d\tau}\psi(X_T) + \int_t^T e^{\int_t^\tau V(X_s, s)ds}f(X_\tau, \tau)d\tau \mid X_t = x\right]$$

where X follows:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t^{\mathbb{Q}}. \tag{13}$$

W_t is a Brownian motion under the probability measure $\mathbb{Q}$.

It can be noticed that from equations (12) and (11) that $u = F\ \sigma = \sigma$, and $V = r$, $f = 0$, and $\psi = \Phi$. However, the μ in (13) disagrees with rS in (11). Therefore, we need to perform some transformations on the process S. We have the discounted stock price process $\tilde{S}_t = B_t^{-1} S_t$, then

$$\begin{aligned} d\tilde{S}_t &= d(B_t^{-1} S_t) \\ &= S_t dB_t^{-1} + B_t^{-1} dS_t \\ &= \sigma(t, S_t)\tilde{S}_t \Big(dW_t + \frac{\mu(t, S_t) - r}{\sigma(t, S_t)} dt \Big). \end{aligned} \tag{14}$$

Therefore, according to the measure change we have studied, by utilizing the Girsanov theorem 5, we can construct a probability $\mathbb{Q}$, under which the stochastic process $dW_t + \frac{\mu - r}{\sigma} dt$ is a Brownian motion $W_t^{\mathbb{Q}}$. Under $\mathbb{Q}$, the discounted stock price $\tilde{S}_t$ is governed by

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t^{\mathbb{Q}}.$$

Then

$$\begin{aligned} dS_t &= \mu'(t, S_t) S_t dt + \sigma(t, S_t) S_t dW_t \\ &= \mu'(t, S_t) S_t dt + \sigma(t, S_t) S_t dW_t^{\mathbb{Q}} - (\mu'(t, S_t) - r) S_t dt \\ &= r S_t dt + \sigma(t, S_t) S_t dW_t^{\mathbb{Q}} \end{aligned} \tag{15}$$

where μ' is to avoid duplication. By this transformation, we have all parameters in equation(11) and dS_t (15) consistent with the theorem 9 under probability measure $\mathbb{Q}$, and then we have

$$F(x, t) = \mathbb{E}^{\mathbb{Q}} \Bigg[e^{-r(T-t)} \Phi(S_T) \Bigg].$$

According to the SDE (15), we have:

$$S_T = S_t \exp \Bigg\{ \Big(r - \frac{1}{2}\sigma^2 \Big)(T - t) + \sigma \Big(W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}} \Big) \Bigg\}$$

and under $\mathbb{Q}$, we have

$$\begin{aligned} F(x, t) &= \mathbb{E}^{\mathbb{Q}} \Bigg[e^{-r(T-t)} \Phi(S_T) \Bigg] \\ &= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \Bigg[\Phi(S_T) \Bigg] \\ &= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \Bigg[\Phi(S_t e^z) \Bigg] \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} \Phi(S_t e^z) f(z) dz \end{aligned} \tag{16}$$

where $f(z)$ is the PDF of normal distribution with mean $((r - \frac{1}{2}\sigma^2)(T - t))$ and standard deviation $\sigma\sqrt{T - t}$. Expanding the equation (16), we can work out the Black-Scholes formula for European option pricing.

4.3 Comments and Notes on Risk Neutral

In this section, we will make some comments on risk-neutral pricing and provide some results from some books and papers to help us better understand the idea of this tool in asset pricing.

In the previous section 4.2, we can notice that under the objective probability measure $\mathbb{P}$, the market is driven by the SDE (10). We can derive the BS equation for option pricing by constructing a portfolio containing the options and stocks in the same form as the Feynman-Kac formula. However, the SDE of the stock price in (10) does not agree with X_t (13) in theorem 9. To solve the option pricing equation by the Feynman-Kac formula, we use the measure change technique to define a new probability measure $\mathbb{Q}$ by the theorem 5 (Girsanov Theorem), under which we have:

$$W_t^{\mathbb{Q}} = W_t + \frac{\mu - r}{\sigma} t$$

and

$$\text{Law}(W_t^{\mathbb{Q}} \mid \mathbb{Q}) = \text{Law}(W_t \mid \mathbb{P}).$$

Also, under $\mathbb{Q}$, we have

$$dS_t = rS_t dt + \sigma(t, S_t)S_t dW_t^{\mathbb{Q}}.$$

It can be concluded that from the section 3.3 and books [9] and [1], the change of measure does not change the trajectories of S under $\mathbb{P}$, but we can use some 'good properties' of S under $\mathbb{Q}$ to work out the solution. It is a trick, and we do not make any 'physical' changes on S_t! In addition to the name risk-neutral measure, $\mathbb{Q}$ also refers to the Martingale measure, under which the discounted stock price process S_t/B_t is a martingale and under the Black-Scholes model, the discounted price process Π_t/B_t is also a $\mathbb{Q}$ martingale.

The above is the underlying mathematical reasons for risk-neutral measure $\mathbb{Q}$. In this paragraph, we will provide some economic explanations of what is risk-neutral. Under $\mathbb{P}$, the SDE of the stock price (10) contains the local mean rate of return $\mu(t, s)$ of the underlying assets, but it is useless in the option pricing that can be concluded from the BS equation (11), which only the σ from (10) being reserved. As a result, the different agents may have different opinion on their estimation of μ, but they will agree on the valuation of option pricing (only σ and no μ). The resulting option pricing formula (16) will only work under $\mathbb{Q}$. In this world ($\mathbb{Q}$), the agents will have no preference for the risk, since they will not be rewarded for taking risk (we can formulate the portfolio Π, which the risk of option can be hedged). The risk neutral agents will price the asset based on the probability without preference. For instance, we have a lottery, under which we can 1% to win \$1000 and 99% worthless. For a risk-neutral agent, it will be worth $(\$1000 \times 0.01 + \$0 \times 0.99)e^{-rt} = \$10e^{-rt}$. However, a risk-reverse agent will value this lottery cheaper than $10e^{-rt}$, and a risk-seeking agent will pay more than $10e^{-rt}$ to this lottery. Due to this (under this measure, the agent is risk-neutral), we call $\mathbb{Q}$ risk-neutral measure, which in mathematics is called equivalent measure. For details, we can look through the finance book like [8] and mathematical finance book [1], and also visit https://quant.stackexchange.com/questions/74319/conceptual-problem-with-risk-neutrality-what-is-a-risk-neutral-world-exactly.

A Fundamental Theorems of Asset Pricing

In mathematical finance, the fundamental theorems of asset pricing plays an important role. This section is cited from the Chapter 11 of [1]

Theorem 10 (First Fundamental Theorem of Asset Pricing (FTAP I))**.** A market is arbitrage free iff there exists a equivalent martingale measure $\mathbb{Q}$ such that

$$\mathbb{Q} \sim \mathbb{P}$$

and the discounted stock process is a martingale.

Theorem 11 (Second Fundamental Theorem of Asset Pricing (FTAP II))**.** The market is said to be complete iff $\mathbb{Q}$ is unique.

The proof of this two theorems is complicated, we can review the proof in the Chapter 11 of [2] (Page 154). It will show us why we require $\mathbb{Q} \sim \mathbb{P}$.

Corollary 12 (Consequence of Complete Market)**.** In a complete market, we can have the following consequence:

1. We can hedge everything
2. There is an unique risk neutral measure
3. No arbitrage opportunity.

References

[1] Tomas Björk. *Arbitrage theory in continuous time.* Oxford university press, 2009.

[2] Tomas Bjork and Agatha Murgoci. A general theory of markovian time inconsistent stochastic control problems. *Available at SSRN 1694759*, 2010.

[3] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–654, 1973.

[4] A. N. Borodin and Paavo. Salminen. *Handbook of Brownian motion : facts and formulae.* Probability and its applications. Birkhauser, Basel, 2nd ed. edition, 2002.

[5] Raymond Chan. Math4210 financial mathematics.

[6] Alison Etheridge. Stochastic calculus for finance.

[7] Alison Etheridge. Stochastic analysis and pdes. 2016.

[8] John C Hull and Sankarshan Basu. *Options, futures, and other derivatives.* Pearson Education India, 2016.

[9] Goran. Peskir and Albert. Shiryaev. *Optimal Stopping and Free-Boundary Problems.* Lectures in Mathematics. ETH Zurich. Birkhauser Basel, Basel, 2006.

[10] Philip E. Protter. *Stochastic Differential Equations*, pages 249–361. Springer Berlin Heidelberg, Berlin, Heidelberg, 2005.

[11] Leonard CG Rogers and David Williams. *Diffusions, markov processes, and martingales: Volume 1, foundations.* Cambridge university press, 2000.

[12] A. N. Shiryayev. *On Analytical Methods In Probability Theory*, pages 62–108. Springer Netherlands, Dordrecht, 1992.