Local Time (Note)

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1 Introduction

This note is based on section 7 of chapter IV [3] and the local time part of [1], we will study local time and the related extension of Itô's formula (Itô-Meyer formula), which will help us to perform Itô's lemma on functions that are not twice differentiable.

2 Background Knowledge

In this section, we will study some back ground knowledge about this note. According to Chapter 2 of [2], we have a definition of semimartinagle

Definition 2.1 (Semimartingale)**.** *A process X_t is a semimartingale if it admits the following representations:*

$$X = X_0 + M + A$$

where M is a local martingale and A is a càdlàg (illustrated below) process of bounded variation.

In [1], the process A is also called as the FV process: The Itô's formula allows us to write a function f of continuous semimartingale X in the form of the stochastic integral:

$$f(X) = f(X_0) + \int f^{''}(X)dX + \frac{1}{2}\int f^{''}(X)d\langle x\rangle. \tag{1}$$

However, we can only apply formula (1) on continuous semimartingale X and twice differentiable function f. In [3], we have a generalised Itô's formula, in which the semimartingale does not need to be continuous and it is a càdlàg process in general. càdlàg process is a right continuous process with left limits, which can be shown in Figure 1.

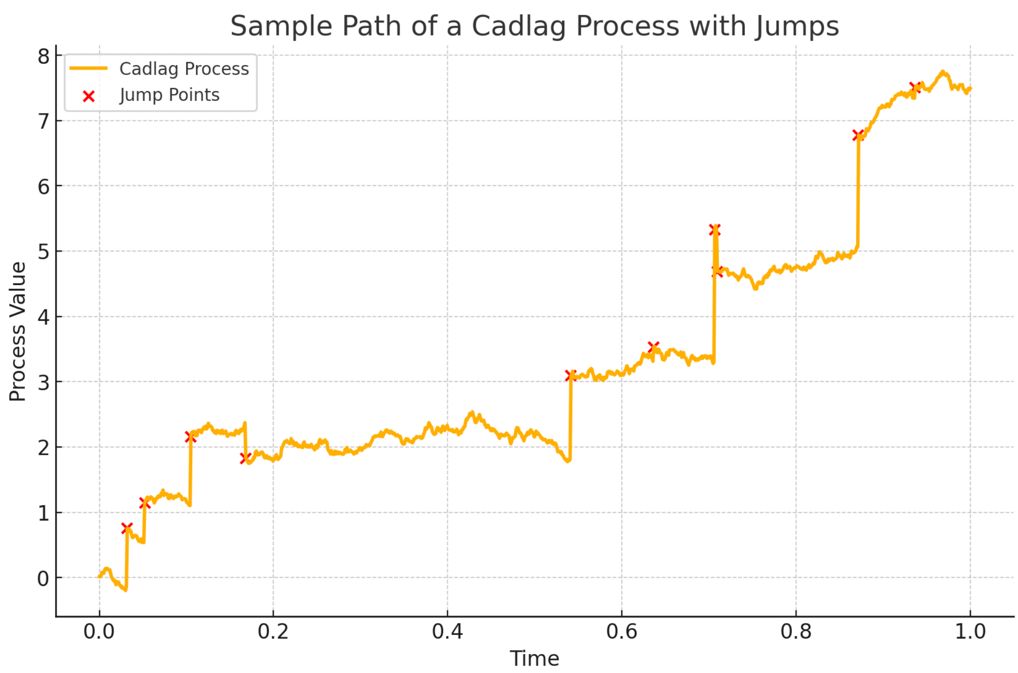

Figure 1: This group draws the càdlàg process X_t, which is a right continuous process with left limits. It can be observed that there are some jumps between at $f(X_t)$, which does not equal the left limits of the process and has been marked by the crosses.

Theorem 2.2 (Generalized Itô's Formula). *Given a semimartingale X and a $\mathcal{C}^2$ functions, then $f(X)$ is also a semimartingale:*

$$f(X_t) = f(X_0) + \int_{0+}^{t} f^{'}(X_{s-})dX_s + \frac{1}{2}\int_{0+}^{t} f^{''}(X_{s-})d\langle X_s\rangle^c + \sum_{0\leq s\leq t}\{f(X_s) - f(X_{s-}) - f^{'}(X_{s-})\Delta X_s\} \tag{2}$$

$$= f(X_0) + \int_{0+}^{t} f^{'}(X_{s-})dX_s + \frac{1}{2}\int_{0+}^{t} f^{''}(X_{s-})d\langle X_s\rangle + \sum_{0\leq s\leq t}\{\Delta f(X_s) - f^{'}(X_{s-})\Delta X_s - \frac{1}{2}f^{''}(X_{s-})\Delta X_s^2\} \tag{3}$$

where X_{s-} is the left limits of càdlàg process, $\Delta X_s = X_s - X_{s-}$ is the jump of X at time s, and $\Delta f(X_s) = f(X_s) - f(X_{s-})$ is the jump of $f(X)$ at time s.

Note: $\sum_{0\leq s\leq t} f^{''}(X_{s-})\Delta X_s^2$ is the jump part of $\int_{0+}^{t} f^{''}(X_{s-})d\langle X_s\rangle$. The difference between formulas (1) and (3) is that the latter one deletes the sum of jumps of $f(X_t) - f(X_0)$, $\int_{0+}^{t} f^{'}(X_{s-})dX_s$, and $\frac{1}{2}\int_{0+}^{t} f^{''}(X_{s-})d\langle X_s\rangle$.

3 Local Time

Since we have studied the generalized Itô's formula for semimartingale (3), it shows that if X is a semimartingale, then $f(x)$ is also a semimartingale where f is a $\mathcal{C}^2$ function from $\mathbb{R} \to \mathbb{R}$. In this section, we will study the concept of local time, which can replace the quadratic variation term in Itô's formula and further relax the requirement on differentiability.

Theorem 3.1. *Given a convex function $f : \mathbb{R} \to \mathbb{R}$ and a semimartingale X, $f(X)$ is a semimartingale, and we have*

$$f(X_t) - f(X_0) = \int_{0+}^{t} f'(X_{s-})dX_s + A_t \tag{4}$$

where f' is the left derivative of f, and A is an adapted, right continuous increasing process with $\Delta A_t = f(X_t) - f(X_{t-}) - f'(X_{t-})\Delta X_t$.

Proof. The proof of this theorem is based on [1] and can be divided into two parts: $f \in \mathcal{C}^2$ or $f \notin \mathcal{C}^2$.

1. When $f \in \mathcal{C}^2$, we can use the Itô's formula (2) to expand $f(X_t)$:

$$\begin{aligned} f(X_t) - f(X_0) &= \int_{0+}^{t} f'(X_{s-})dX_s + \frac{1}{2}\int_{0+}^{t} f''(X_{s-})d\langle X_s\rangle^c + \sum_{0\le s\le t}\{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\} \\ &= \int_{0+}^{t} f'(X_{s-})dX_s + A_t \end{aligned}$$

   where $A_t = \frac{1}{2}\int_{0+}^{t} f''(X_{s-})d\langle X_s\rangle^c + \sum_{0\le s\le t}\{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\}$. Due to the convexity, f'' and $\Delta f(X) - f'(X_)\Delta X$ are positive, and therefore it is increasing.

2. When $f \notin \mathcal{C}^2$, we can convolve f with any $\mathcal{C}^2$ nonnegative function $\theta : (0,\infty) \to \mathbb{R}$ that $\int \theta(x)dx = 1$, then we can have a sequence of convex and positive f_n:

$$f_n(x) = \int_0^\infty f(x - \frac{y}{n})\theta(y)dy.$$

   Since $f_n(x) \in \mathcal{C}^2$, we can directly apply theorem 3.1 and get

$$f_n(X_t) - f_n(X_0) = \int_{0+}^{t} f_n'(X_{s-})dX_s + A_t^n$$

   Then by convergence in probability, we can finish the proof.

□

Based on this theorem, we have the following statements:

Corollary 3.2. *1. If X is a semimartingale, then $|X|$, X^+, and X^- are semimartingales.*

2. If X and Y are semimartingales, then $X \vee Y$ and $X \wedge Y$ are semimartingales.

3. According to [1], if X is a semimartingale and f is a convex function, then $f(x)$ is also a semimartingale.

For instance, we have a sign function

$$sign(x) = \begin{cases} 1 & \text{if } X > 0 \\ -1 & \text{if } X \leq 0 \end{cases}$$

which is the left derivative of the convex function $h_0(x) = |x|$. Further for $h_a(x) = |x - a|$, the left derivative is $sign(x - a)$. Due to the convexity of $h_a(x)$, for the semimartingale X, we have

$$h_a(X_t) = |X_t - a| = |X_0 - a| + \int_{0+}^{t} sign(X_{s-} - a) dX_s + A_t^a$$

which is according to theorem 3.1. Based on this example, we can define a semimartingale named local time.

Definition 3.3 (Local Time)**.** *For a semimartingale X, and h_a and A^a we defined in the example. The local time of semimartingale X at level a can be denoted as $L_t^a = L^a(X)_t$, and it can formulated as*

$$L_t^a = A_t^a - \sum_{0<s\leq t} \{h_a(X_s) - h_a(X_{s-}) - h_a^{'}(X_{s-})\Delta X_s\}. \tag{5}$$

Note: From the formula (5), L^a is the continuous part of the increasing process A^a.

Since we have known the definition of local time, now we can incorporate it with theorem 3.1.

Theorem 3.4. *Given a semimartingale X and its stopping time at a can be defined as L^a, then*

$$(X_t - a)^+ - (X_0 - a)^+ = \int_{0+}^{t} \mathbf{1}_{X_{s-}>a} dX_s + \sum_{0<s\leq t} \mathbf{1}_{X_{s-}>a}(X_s - a)^- + \sum_{0<s\leq t} \mathbf{1}_{X_{s-}\leq a}(X_s - a)^+ + \frac{1}{2} L_t^a$$

and

$$(X_t - a)^- - (X_0 - a)^- = -\int_{0+}^{t} \mathbf{1}_{X_{s-}\leq a} dX_s + \sum_{0<s\leq t} \mathbf{1}_{X_{s-}>a}(X_s - a)^- + \sum_{0<s\leq t} \mathbf{1}_{X_{s-}\leq a}(X_s - a)^+ + \frac{1}{2} L_t^a$$

Proof. This theorem can be proved by theorem 3.1. Define $f(x) = (x-a)^+$ and $g(x) = (x-a)^-$, applying theorem 3.1, we can get:

$$
\begin{aligned}
f(X_t) &= f(X_0) + \int_{0+}^{t} f^{'}(X_{s-})dX_s + C_t^+ \\
g(X_t) &= g(X_0) + \int_{0+}^{t} g^{'}(X_{s-})dX_s + C_t^-
\end{aligned}
$$

Then we have

$$
\begin{aligned}
D_t^+ &= C_t^+ - \sum_{0<s\leq t} \{f(X_s) - f(X_{s-}) - f^{'}(X_{s-})\Delta X_s\} \\
D_t^- &= C_t^- - \sum_{0<s\leq t} \{g(X_s) - g(X_{s-}) - g^{'}(X_{s-})\Delta X_s\}
\end{aligned}
$$

By subtracting the formulas, we can get $C_t^+ = C_t^-$, then $D_t^+ = D^-(t)$. Also, $D_t^+ + D^-(t) = L_t^a$, so $D^-(t) = D^+(t) = \frac{1}{2}L_t^a$ □

Since we have studied how the semimartingale is preserved with the convex transformation in theorem 3.1, the definition of Local time 3.3, and how they are combined in theorem 3.4. We will introduce a new form of Itô's formula, which can use the local time to represent the quadratic variation term in Itô's formula (2). According to [1], this is a more generalized form of Itô's formula since it has relaxed the condition in differentiability of f, which is developed by Tanaka and Meyer.

Theorem 3.5 (The Itô-Tanaka-Meyer Formula)**.** *Given a difference of two convex functions f, f' refers to its left derivative, and μ is the signed measure which is the second derivative of f in the generalized function sense. Then we have the following equation:*

$$
\begin{aligned}
f(X_t) &= f(X_0) + \int_0^t f^{'}(X_-)dX + \frac{1}{2}\int_{-\infty}^{\infty} L_t^x f''(dx) + \sum_{s\leq t}(\Delta f(X_s) - f'(X_{s-})\Delta X_s) \ \textit{(from [1])} \\
&= f(X_0) + \int_0^t f^{'}(X_-)dX + \frac{1}{2}\int_{-\infty}^{\infty} \mu(da)L_t^a + \sum_{s\leq t}(\Delta f(X_s) - f'(X_{s-})\Delta X_s) \ \textit{(from [3])}
\end{aligned}
$$

References

[1] George Lowther. The ito-tanaka-meyer formula, Oct 2020.

[2] Goran. Peskir and Albert. Shiryaev. *Optimal Stopping and Free-Boundary Problems*. Lectures in Mathematics. ETH Zurich. Birkhauser Basel, Basel, 2006.

[3] Philip E. Protter. *Stochastic Differential Equations*, pages 249–361. Springer Berlin Heidelberg, Berlin, Heidelberg, 2005.