# Scale Function and Speed Measure (Note)

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## 1 Introduction

This note is based on the section 6.1 of [1]. In this note, we will study the scale function and speed measure, which can help us transform a diffusion process $Z_t$ into a standard Brownian motion $B_t$. Both of them are very important in computing some quantities from the diffusion process. The diffusion process is connected to second-order parabolic equations. Suppose we can have an analytical solution to the equations, which means that we have the explicit expression for the transition densities. For instance, the Fokker–Planck equation's solution is the probability density function for a stochastic process. However, in many cases, we can not find an explicit solution to the differential equation for many diffusion processes. Luckily, all one-dimensional diffusion processes can be transformed into the standard Brownian motion, by using the scale function to get a time-changed BM, then applying the speed measure, we can have a SBM eventually.

## 2 Some Background Knowledge

In this section, we will study some knowledge as the background for us to study the scale function and speed measure.

According to [3], the time-changed Brownian motion can be defined as:

**Theorem 2.1** (Time Changed Brownian Motion). *$M = (M_t)_{t\geq 0}$ is a continuous local martingale starts from $0$ and $\lim_{t\to\infty}\langle M, M\rangle_t = \infty$ almost surely. Defining a stopping time:*

$$T_s = \inf\{t > 0 : \langle M, M\rangle_t > s\},$$

*$\mathcal{G}_s = \mathcal{F}_{T_s}$, and $B_s = M_{T_s}$. Then $(B_s, \mathcal{G}_s)_{s\geq 0}$ is a SBM. Also, $(\langle M, M\rangle_t)_{t\geq 0}$ are stopping times for $(\mathcal{G}_s)_{s\geq 0}$ and*

$$M_t = B_{\langle M,M\rangle_t} \text{ a.s. } 0 \geq t \geq \infty.$$

*Therefore $M$ can be represented as a time-changed Brownian motion (TCBM).*

In mathematical finance, we usually concentrate on the evolution of a stochastic process. For the Feller process, we usually use the infinitesimal generator to grasp such information.

**Definition 2.2** (Infinitesimal Generator)**.** *Given a feller process $X = (X_t)_{t\geq 0}$ (a continuous-time Markov process satisfying certain regularity conditions) and a state space $E$, we can define a generator $(A, D(A))$ as:*

1. *$D(A) = \{f \in C_0(E) : \lim_{t\downarrow 0} \frac{T_t f - f}{t}$ exists as uniform limit$\}$*
2. *$Af = \lim_{t\downarrow 0} \frac{T_t f - f}{t}$, for any $f \in D(A)$*

*where $T_t f(x) = \mathbb{E}_x(f(X_t))$.*

This definition is for the general case. Usually, we have a diffusion process

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

where $X_0 = x$, then the infinitesimal generator can be written as

$$Af(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x).$$

For specified cases, we can refer to [2].

# 3 Scale Function and Speed Measure

We have studied the time-changed Brownian motion. In this section, we firstly will study the scale function and how it can transform the diffusion process into a time-changed Brownian motion. Then use the speed measure to get an SBM from a TCBM.

At the first stage, we can define a diffusion process $Z_t$ with generator $A_Z$, infinitesimal drift $\mu_Z(x)$, and infinitesimal variance $\sigma_Z^2(x)$. Defining an operator $\Delta_h$ to represent the increment of a process over the next time step with a sufficiently small length $h$, then we can have

$$\mathbb{E}[\Delta_h Z(0) \mid Z_0 = y] = \mu_Z(y)h + o(h)$$
$$\mathbb{E}[(\Delta_h Z(0))^2 \mid Z_0 = y] = \sigma_Z^2(y)h + o(h).$$

We also define another new process $Y_t = Z_{\tau(t)}$ where

$$\tau(t) = \int_0^t \beta(Y_s)ds.$$

. We can assume $\beta$ is a continuous, bounded and strictly positive function. Therefore, if $Y_0 = Z_0$ the time change from 0 to $dt$ in $Z_t$ becomes from 0 to $d\tau(t)$ in $Y_t$ that is also 0 to $\beta(Y_0)dt$. Then we have

$$\mathbb{E}[\Delta_h Y(0) \mid Y_0 = y] = \beta(Y_0)\mu_Z(y)h + o(h) = \beta(y)\mu_Z(y)h + o(h)$$
$$\mathbb{E}[(\Delta_h Y(0))^2 \mid Z_0 = y] = \beta(Y_0)\sigma_Z^2(y)h + o(h) = \beta(y)\sigma_Z^2(y)h + o(h).$$

Therefore, it can concluded that

$$A_Y f(x) = \beta(x) A_Z f(x) \tag{1}$$

For instance, if $\beta$ is a constant function, it can noticed that we have changed the time units without changing the space. In general, the time units depend on the space, which is not so detailed but we will illustrate it. In general, we can have a diffusion process $X_t$ governed by the infinitesimal generator:

$$Af(x) = \frac{1}{2}\sigma^2(x)\frac{d^2 f}{dx^2} + \mu(x)\frac{df}{dx}$$

where $f$ is twice continuously differentiable ($C^2$) on interval $(a,b)$, and the parameter $\mu(x)$ and $\sigma(x)$ are bounded and locally Lipchitz on $(a,b)$. Assuming we have a strictly increasing function $S(x)$ on $(a,b)$, we can define a new process $Z_t = S(X_t)$. Then we can calculate

$$\begin{aligned}
A_Z f(x) &= \frac{d}{dt}\mathbb{E}[f(Z_t) \mid Z_0 = x]\Big|_{t=0} \\
&= \frac{d}{dt}\mathbb{E}[f(S(X_t)) \mid S(X_0) = x]\Big|_{t=0} \\
&= A_X(f \circ S)(S^{-1}(x)) \\
&= \frac{1}{2}\sigma^2(S^{-1}(x))\frac{d^2}{dx^2}(f \circ S)(S^{-1}(x)) + \mu(S^{-1}(x))\frac{d}{dx}(f \circ S)(S^{-1}(x)) \\
&= \frac{1}{2}\sigma^2(S^{-1}(x))\{(S'(S^{-1}(x)))^2\frac{d^2 f}{dx^2}(x) + S''(S^{-1}(x))\frac{df}{dx}(x)\} + \mu(S^{-1}(x))S'(S^{-1}(x))\frac{df}{dx}(x) \\
&= \frac{1}{2}\sigma^2(S^{-1}(x))(S'(S^{-1}(x)))^2\frac{d^2 f}{dx^2}(x) + AS(S^{-1}(x))\frac{df}{dx}(x) \qquad (2)
\end{aligned}$$

It can be noticed from (2), that if we can find a $S$ such that $AS = 0$, the $A_Z f(x)$ will only contain the diffusion term, and it becomes a time-changed Brownian motion. Based on this fact, we can work out the definition of scale function.

**Definition 3.1** (Scale Function)**.** *Given a diffusion process $X_t$ on $(a, b)$ with drift $\mu$ and variance $\sigma^2$. the scale function is defined by*

$$S(x) = \int_{x_0}^{x} exp(-\int_{\eta}^{y} \frac{2\mu(z)}{\sigma^2(z)} ds)dy, \tag{3}$$

*in which $x_0$ and $\eta$ are arbitrary points in $(a, b)$.*

Now, we can find a function $S$, which can generate a new process $Z_t = S(X_t)$ on the interval $(a, b)$, and this mapping transform $X_t$ on $(a, b)$ into a TCMB $Z_t$ on $(S(a), S(b))$. According to [4], we can use the scale function to compute some probabilities.

**Theorem 3.2.** *$X$ is a diffusion on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_x)$, and we have a scale function $S$ on $X$ such that for the interval $(a, b)$, we have*

$$\mathbb{P}_x(\tau_b < \tau_a) = \frac{s(x) - s(a)}{s(b) - s(a)}.$$

If $S(x) = x$, we can claim $X_t$ is a natural scale, and this definition can be generalized.

**Definition 3.3** (Natural Scale)**.** *If the scale function for a diffusion $X_t$ is linear, then the diffusion is said to be in natural scale.*

By the scale function (3), we can transform $X_t$ into the form of TCBM $S(X_t)$. Now, we will utilise the speed measure to change $S(X_t)$ into the form of SBM.

**Definition 3.4** (Speed Measure)**.** *The function $m(\xi) = \frac{1}{\sigma^2(\xi)S'(\xi)}$ is the density of the speed measure of the process $X_t$, then we write*

$$M(x) = \int_{x0}^{x} m(\xi)d\xi.$$

Note: Function $m$ has the same role as the $\beta$ in the previous paragraphs.

In conclusion, we have

$$Af = \frac{1}{2}\frac{1}{dM/dS}\frac{d^2 f}{dS^2} = \frac{1}{2}\frac{d}{dM}(\frac{df}{dS})$$

# References


[1] Alison Etheridge. Stochastic analysis and pdes. 2016.

[2] Majnu John and Yihren Wu. Calculating infinitesimal generators. *Journal of Stochastic Analysis*, 2(4):4, 2021.

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[4] Leonard CG Rogers and David Williams. *Diffusions, markov processes, and martingales: Volume 1, foundations.* Cambridge university press, 2000.