Optimal Stopping Problems (Note)

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Contents

1 General Facts

This part is the note based on Chapter 1 of [3] will study the general facts of optimal stopping problems in discrete and continuous cases. The martingale and Markovian approaches will be introduced in both cases.

1.1 Discrete Time

1.1.1 Martingale Approach

In this section, we will study the basic outcomes of optimal stopping problems in discrete time via the Martingale Approach. At the beginning, we will introduce some notations and definitions.

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathsf{P})$, we have a sequence of random variables $G = (G_n)_{n\geq 0}$, and G_n refers to the gain obtained at time n.

Definition 1.1.1 (Markov Time and Stopping Time)**.**

A random variable $\tau : \Omega \to \{0, 1, \cdots, \infty\}$ is called the Markov time if $\{\tau \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$. If $\tau \leq \infty$ $\mathsf{P}_{a.s}$, it is called stopping time.

The family of Markov time will be denoted as $\overline{\mathfrak{M}}$, and the stopping time set will be denoted as $\mathfrak{M}$. The subsets of $\mathfrak{M}$ will be denoted as:

$$\mathfrak{M}_n^N = \{\tau \in \mathfrak{M} : n \leq \tau \leq N\}$$

where $0 \leq n \leq N$. For simplicity, $\mathfrak{M}_0^N = \mathfrak{M}^N$ and $\mathfrak{M}_n^\infty = \mathfrak{M}_n$.

Studying optimal stopping problem is to solve:

$$V_* = \sup_\tau \mathsf{E} G_\tau,$$

which contains two tasks:

1. Compute the value function V_* as explicitly as possible;
2. Exhibit the optimal stopping time τ_*

To ensure the existence of $\mathsf{E}G_\tau$, we also need to make assumptions:

1. $G_N \equiv 0$ when $N = \infty$
2. $\mathsf{E}\left(\sup_{n\leq k\leq N} |G_\tau|\right) < \infty$

With the notation $\mathfrak{M}_n^N$, the value function becomes

$$V_n^N = \sup_{\tau \in \mathfrak{M}_n^N} \mathsf{E} G_\tau \tag{1.1.1}$$

Similarly, we can set $V^N = V_0^N$, $V_n = V_n^\infty$, and $V = V_0^\infty$.

The first method we used to solve the optimal stopping problem (1.1.1) with $N < \infty$ is called the Backward Induction Method, in which we can construct a random sequence $(S_n^N)_{0 \le n \le N}$ solving the problem in a stochastic sense and take expectations the problem will be solved in the original mean-valued sense.

To solve this optimal stopping problem, we can make it more explicit:

$$V_n^N = \sup_{n \le \tau \le N} \mathsf{E} G_\tau. \tag{1.1.2}$$

When $n = N$, the optimal choice is to stop immediately and $S_N^N = G_N$. When $n = N - 1$, if we decide to stop immediately we have $S_{N-1}^N = G_{N-1}$, and if we decide to continue we have $S_{N-1}^N = \mathsf{E}\left(S_N^N | \mathcal{F}_{N-1}\right)$. Therefore, $S_{N-1}^N = \max\left(G_{N-1}, \mathsf{E}\left(S_N^N | \mathcal{F}_{N-1}\right)\right)$. By the method of induction recursively, we can get

$$S_N^N = G_N \tag{1.1.3}$$

$$S_n^N = \max\left(G_n, \mathsf{E}\left(S_{n+1}^N | \mathcal{F}_n\right)\right) \tag{1.1.4}$$

in which $n = 0, 1, 2 \cdots, N - 1$. It suggests an optimal stopping time:

$$\tau_n^N = \inf\{n \le k \le N : S_k^N = G_K\}$$

S_n^N and τ_n^N satisfy the following properties:

Theorem 1.1.1. *For the optimal stopping problem* (1.1.2)*, assuming the existence conditions hold, for all* $0 \le n \le N$ *we have:*

1. $S_n^N \ge \mathsf{E}(G_\tau | \mathcal{F}_n)$ *for each* $\tau \in \mathfrak{M}_n^N$
2. $S_n^N = \mathsf{E}(G_{\tau_n^N} | \mathcal{F}_n)$
3. *The stopping time* τ_n^N *is optimal in optimal stopping problem* (1.1.2)
4. *If* τ_* *is an optimal stopping time in* (1.1.2)*, then* $\tau_n^N \le \tau_*$ $\mathsf{P}_{-a.s.}$
5. $\left(S_k^N\right)_{n \le k \le N}$ *is the smallest supermartingale which dominates* $(G_k)_{n \le k \le N}$
6. *The stopped sequence* $\left(S_{k \cap \tau_n^N}^N\right)_{n \le k \le N}$ *is a martingale.*

The first and second statements in Theorem 1.1.1 can be proved by the method of Backward Induction. For the third statement, we can take expectations in the first and second statements in Theorem 1.1.1, and we can have $V_n^N = \mathsf{E}S_n^N = \mathsf{E}G_{\tau_n^N}$, which implied τ_n^N is optimal. The fourth argument, we can show that for τ_*, if τ_* is optimal, we should have $S_{\tau_*}^N = G_{\tau_*}$ $\mathsf{P}_{a.s}$, and according to the definition of τ_n^N, argument 4 holds. For the fifth statement, it can be concluded from the formulas (1.1.3) and (1.1.4), that S_n^N is a supermartingale that dominates G. To prove S_n^N is the smallest, we can set another supermartingale $\tilde{S}$ also dominates G, and show $\tilde{S}_n^N \geq S_n^N$ by induction. The martingale property can prove the last argument.

Theorem 1.1.2 (Bellman's Principle)**.**

If the stopping rule τ_0^N is optimal for V_0^N, and it was not optimal stop within the time set $\{0, 1, \cdots, n-1\}$, then starting the observation at time n and being based on the information $\mathcal{F}_n$, the same stopping rule is still optimal for the problem V_n^N.

However, the backward induction methods require the time Horizon to be finite, in this part, we will introduce the method of essential supremum. However, it will admit a different characterization of S_n^N,

$$S_n^N = \sup_{\tau \in \mathcal{M}_n^N} \mathsf{E}\left(G_\tau | \mathcal{F}_n\right)$$

which can be directly extended to infinite horizon N. However, if the supremum is taken on the uncountable number of τ, the new characterization of S_n^N will fail. To handle this, we will introduce the idea of essential supremum.

Definition 1.1.2 (Essential Supremum)**.**

We have a family of random variables $\{Z_\alpha : \alpha \in I\}$, in which the index set I could be arbitrary. Then there exists a countable subset J of I such that the random variable

$$Z^* = \sup_{\alpha \in J} Z_\alpha$$

satisfies the following two properties:

1. $\mathsf{P}(Z_\alpha \leq Z^*) = 1$ *for each* $\alpha \in I$.
2. $\tilde{Z}$ *is another random variable with* $\mathsf{P}(Z_\alpha \leq \tilde{Z}) = 1$, *then* $\mathsf{P}(Z^* \leq \tilde{Z}) = 1$.

Z^* is called the essential supremum of Z_α, and is denoted as $Z^* = ess \sup_{\alpha \in I} Z_\alpha$.

Besides, if the set of Z_α is upwards directed that means for any α and β in I there exists γ in I such that $Z_\alpha \vee Z_\beta \leq Z_\gamma$ $\mathsf{P}_{a.s.}$. Then a countable set $J = \{\alpha_n : n \leq 1\}$ can be chosen so that

$$Z^* = \lim_{n\to\infty} Z_{\alpha_n} \quad \mathsf{P}_{a.s.}$$

where $Z_{\alpha_1} \leq Z_{\alpha_2} \leq ... \ \mathsf{P}_{a.s.}$

When N goes to ∞, the optimal stopping problem becomes:

$$V_n = \sup_{\tau \geq n} \mathsf{E} G_\tau \tag{1.1.5}$$

With the concepts of essential supremum, we have S_n as:

$$S_n = ess \sup_{\tau\geq n} \mathsf{E}(G_\tau | \mathcal{F}_n)$$

and the stopping time:

$$\tau_n = \inf\{k \geq n : S_k = G_k\}.$$

The sequence S_n is called the Snell envelope of G. Also, the S_n and τ_n have the following properties:

Theorem 1.1.3.

For the stopping problem (1.1.5)*, we still have the recurrent relationship*

$$S_n = \max\left(G_n, \mathsf{E}(S_{n+1}|\mathcal{F}_n)\right)$$

for all $n \geq 0$*. Assuming*

$$\mathsf{P}(\tau_n < \infty) = 1,$$

then we have

$$S_n \geq \mathsf{E}(G_\tau | \mathcal{F}_n)$$
$$S_n = \mathsf{E}(G_{\tau_n} | \mathcal{F}_n)$$

Moreover, if $n \geq 0$ *is given and fixed, we have the same statements with Theorem 1.1.3.*

Since we have extended the stopping problem to the infinite time horizon, we will explore the connection between the two methods when the time horizon becomes infinite. It can be noticed that S_n^N, τ_n^N and V_n^N are increasing sequence, therefore by Monotone Convergence Theorem, we have

$$S_n^\infty = \lim_{N\to\infty} S_n^N$$
$$\tau_n^\infty = \lim_{N\to\infty} \tau_n^N$$
$$V_n^\infty = \lim_{N\to\infty} V_n^N$$

with $\mathsf{P}_{a.s.}$ for each $n \geq 0$. Also, the following statements hold

$$S_n^\infty \leq S_n$$

$$\tau_n^\infty \leq \tau_n$$

$$V_n^\infty \leq V_n$$

With the loss of condition $\mathsf{E}\left(\sup_{n \leq k \leq N} |G_\tau|\right) < \infty$, the inequalities will become strict.

1.1.2 Markovian Approach

In this part, we will study the basic results of the optimal stopping problem with the discrete time framework and a Markovian process. We will start by introducing some concepts.

$X = (X_n)_{n \geq 0}$ is a time-homogeneous Markov Chain and a shift operator $\theta_n : \Omega \to \Omega$ is well defined by $\theta_n(\omega)(k) = \omega(n+k)$ for $\omega = (\omega(k))_{k \geq 0} \in \Omega$. Given a measurable function $G : E \to \mathbb{R}$ satisfying the following condition:

$$\mathsf{E}\left(\sup_{0 \leq n \leq N} |G(X_n)|\right) < \infty$$

for all $x \in E$, we consider the optimal stopping problem

$$V^N(x) = \sup_{0 \leq \tau \leq N} \mathsf{E}_x G(X_\tau)$$

To solve this problem, we can also utilize the backward induction method to construct a sequence

$$S_n^N = V^{N-n}(X_n)$$

the continuing and stopping set can be defined as

$$C_n = \{x \in E : V^{N-n}(x) > G(x)\}$$

$$D_n = \{x \in E : V^{N-n}(x) = G(x)\}$$

then we can define the stopping time

$$\tau_D = \inf\{0 \leq n \leq N : X_n \in D_n\}$$

and the transition operator T of X is defined as

$$TF(x) = \mathsf{E}_x F(X_1)$$

Theorem 1.1.4.

If X is time-homogeneous, then the value function V and stopping time τ_D have the following properties

1. $V^n(x) = \max(G(x), TV^{n-1}(x))$ *for* $1 \le n \le N$ *where* $V^0 = G$*, which is the Wald-Bellman Equation*
2. *The stopping time* τ_D *is optimal for the stopping problem*
3. *If* τ_* *is another optimal stopping problem, then* $\tau_D \le \tau_*$ *almost surely.*
4. *The sequence* $V^{N-n}(X_n)$ *is the smallest supermartingale which dominates* $G(X_n)$
5. *The stopped sequence is* $(V^{N-n\wedge\tau_D}(X_{n\wedge\tau_D}))$ *is a martingale.*

By introducing the operator Q:

$$QF(x) = \max(G(x), TF(x))$$

we can write the Wald-Bellman equation in a more compact form:

$$V^n(x) = Q^n G(x)$$

Now, we are going to study the case where X is the time-inhomogeneous Markov chain. Setting $Z_n = (n, X_n)$, Z is a time-homogeneous Markov chain. Given a measurable function $G : \{0, 1, \cdots, N\} \times E \to \mathbb{R}$ satisfying the condition:

$$\mathsf{E}_{n,x}\left(\sup_{0\le k\le N-n} |G(n+k, X_{n+k})|\right) < \infty$$

then the stopping problem can be written as:

$$V^N(n,x) = \sup_{0\le\tau\le N-n} \mathsf{E}_{n,x} G(n+\tau, X_{n+\tau}), \tag{1.1.6}$$

and we have

$$S^N_{n+k} = V^N(n+k, X_{n+k})$$

Applying the Markov property, we can rewire the equation (1.1.2) in the form of Wald-Bellman equation

$$V^N(n,x) = \max(G(n,x), TV^n(n,x)).$$

Note, only when X is time-homogeneous, we have $V^N(n,x) = V^{N-n}(x)$. Therefore Theorem 1.1.4 needs to be reformulated. Before the reformulation, we need to introduce the concept of superharmonic function, which is important.

Definition 1.1.3.

A function F is said to be superharmonic if

$$TF(n,x) \leq F(n,x). \tag{1.1.7}$$

It can be noticed that F is a superharmonic if and only if F is a supermartingale. We also have the new notations for the continuation set, stopping set, and the first entry time:

$$C = \{(n,x) \in \{0,1,\cdots,N\} \times E : V(n,x) > G(n,x)\}$$
$$D = \{(n,x) \in \{0,1,\cdots,N\} \times E : V(n,x) = G(n,x)\}$$
$$\tau_D = \inf\{n \leq k \leq N : (n+k, X_{n+k}) \in D\}$$

Therefore the value function satisfies the Wald-Bellman equation and has the following properties:

Theorem 1.1.5.

1. $V^N(n,x) = \max(G(n,x), TV^N(n,x))$ *(Wald-Bellman equations)*
2. *The stopping time τ_D is optimal for the problem 1.1.6*
3. *If τ_* is optimal stopping time for the problem 1.1.6, then $\tau_D \leq \tau_*$ almost surely*
4. *The value function V^N is the smallest superharmonic function which dominates the gain function G*
5. *The stopped sequence $(V^N((n+k) \wedge \tau_D), X_{(n+k)\wedge\tau_D})$ is a martingale.*

We will study the case of the infinite horizon ($N = \infty$), in which the time-inhomogeneous case is not separate from the time-homogeneous case. The optimal stopping problem becomes

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau) \tag{1.1.8}$$

and there are notions:

$$C = \{x : E : V(x) > G(x)\}$$
$$D = \{x : E : V(x) = G(x)\}$$
$$\tau_D = \inf\{t \geq 0 : X_t \in D\}$$

Theorem 1.1.6.

For the optimal stopping problem with an infinite time horizon and $\mathsf{P}_x(\tau_D < \infty) = 1$, we have

1. $V(x) = \max(G(x), TV(x))$

2. τ_D *is optimal in problem 1.1.8*

3. If τ_* *is an optimal stopping time, then* $\tau_D \leq \tau_*$ $\mathsf{P}_{a.s.}$

4. V *is the smallest superharmonic function that dominates the function* G

5. The stopped sequence $V(X_{n\wedge\tau_D})$ *is the martingale.*

Besides if $\mathsf{P}_x(\tau_D = \infty) > 0$*, there is no optimal stopping time with probability* 1*.*

Theorem 1.1.7 (Uniqueness in the Wald-Bellman Equation)**.**

Under the hypothesis of Theorem 1.1.6, suppose F *is a function solving the Wald-Bellman equation*

$$F(x) = \max(G(x), TF(X)),$$

and F *satisfies*

$$\mathsf{E}(\sup_{n\geq 0} F(X_n)) < \infty.$$

Then F *equals* V *iff the following 'boundary conditions at infinity' holds:*

$$\limsup_{n\to\infty} F(X_n) = \limsup_{n\to\infty} G(X_n)$$

with $\mathsf{P}_x - a.s.$

1.2 Continuous Time

In this part, we will study the optimal stopping problem within the continuous time framework. Similar to the previous sections, we will start with the Martingale approach, and then the Markovian approach.

1.2.1 Martingale Approach

Given the optimal stopping problem

$$V_t^T = \sup_{t\leq\tau\leq T} E\, G_\tau, \tag{1.2.1}$$

there is no difference between the finite and infinite time horizons, we will simply set

$$V_t = V_t^T.$$

In the continuous case, we will omit the backward induction method and tackle the problem via the Essential Supreme Method. To solve this problem, we will consider the right-continuous process

$$S_t = \underset{\tau \geq t}{ess\sup}\, \mathsf{E}(G_\tau | \mathcal{F}_t),$$

which is known as the Snell envelope of G. The stopping time is

$$\tau_t = \inf\{s \geq t : S_s = G_s\}$$

Since the time is continuous, then we only have

$$S_t \geq \max(G_t, \mathsf{E}(S_s | \mathcal{F}_t)).$$

However, we have a refinement of the Wald-Bellman equation holds:

$$S_t = \max(G_t, \mathsf{E}(S_{\sigma \wedge \tau_t} | \mathcal{F}_t))$$

where $\sigma \geq t$. For the optimal stopping problem 1.2.1 and given

$$\mathsf{P}(\tau_t < \infty) = 1,$$

Then we have the following statements

Theorem 1.2.1.

1. *$S_t \geq \mathsf{E}(G_\tau | \mathcal{F}_t)$*
2. *$S_t = \mathsf{E}(G_{\tau_t} | \mathcal{F}_t)$*
3. *τ_t is optimal in optimal stopping problem 1.2.1*
4. *If τ_* is optimal, then $\tau_t \leq \tau_*$ $\mathsf{P}_{a.s.}$*
5. *The process $(S_s)_{s \geq t}$ is the smallest right-continuous supermartingale which dominates $(G_s)_{s \geq t}$*
6. *The stopped process $(S_{s \wedge \tau_t})_{s \geq t}$ is a right-continuous supermartingale*
7. *if $\mathsf{P}(\tau_t = \infty > 0)$, then there is no optimal stopping time (with probability 1).*

1.2.2 Markovian Approach

In this part, we will study the basic results of the optimal stopping problem when the process is continuous and Markovian. Considering a Strong Markov process $X = (X_t)_{t \geq 0}$ on a filtered probability space and taking values in the measurable space $(E, \mathcal{B})$. It is assumed that X starts from x under P_x, the sample paths of X are right-continuous and left-continuous over stopping times (if $\tau_n \uparrow \tau$ are stopping times, then $X_{\tau_n} \to X_\tau$ $\mathsf{P}_x - a.s.$ as $n \to \infty$). Given the measurable function G satisfying the following condition

$$E_x(\sup_{0 \le t \le T} |G(X_t)|) < \infty,$$

then we have an optimal stopping problem:

$$V(x) = \sup_{0 \le \tau \le T} \mathsf{E}_x G(X_\tau).$$

When $T = \infty$, we have the optimal stopping problem

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau)$$

where τ is a stopping time. The new notations are

$$C = \{x \in E : V(x) > G(x)\}$$
$$D = \{x \in E : V(x) = G(x)\}$$
$$\tau_D = \inf\{t \ge 0 : X_t \in D\}$$

Then the value function V and the optimal stopping time τ_* have the following properties:

1. $V(x) = \mathsf{E}_x G(X_{\tau_*})$
2. V is the smallest superharmonic function which dominates the gain function G
3. $\tau_D \le \tau_*$ $\mathsf{P}_x - a.s.$
4. The stopped process $(V(X_{t \wedge \tau_D}))_{t \ge 0}$ is right-continuous martingale under P_x for every $x \in E$

2 Solution to Optimal Stopping Theorem

Since we have briefly learned the basic outcomes of the optimal stopping problem, in this part, we will learn how to solve the optimal stopping problem based on Chapter 4 of [3]. We will start with the method of reduction to the free-boundary problem, then the method of pasting condition introduced in the note of [4].

2.1 Method of Reduction to Free-boundary Problem

We will illustrate the idea of the method of reduction to the free-boundary problem, and provide an American option pricing example.

2.1.1 General Idea

Consider a strong Markov process $X = (X_t)_{t\geq 0}$, we have an optimal stopping problem:

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau) \tag{2.1.1}$$

Solving the problem (2.1.1) is equivalent to finding the smallest superharmonic function $\hat{V}$, which dominates G. In this case, the optimal stopping time τ_D can be represented as the first entry time X into the stopping set $D = \{\hat{V} = G\}$, and the continuation set can be $D = \{\hat{V} > G\}$. $\hat{V}$ and C should be the solution to the free-boundary problem:

$$\begin{aligned} \mathbb{L}_X \hat{V} &\leq 0 \ (\hat{V} \text{ minimal}) \\ \hat{V} &\geq G \end{aligned} \tag{2.1.2}$$

where $\mathbb{L}_X$ is the infinitesimal generator of X, $\hat{V} > G$ on C, and $\hat{V} = G$ on D.

Assuming G is smooth in the neighborhood of ∂C, if X starting at ∂C enters immediately into int (D), then above general condition can be transformed into:

$$\begin{aligned} \mathbb{L}_x \hat{V} &= 0 \text{ in } C \\ \hat{V}|_D &= G|_D \\ \frac{\partial \hat{V}}{\partial X}|_{\partial C} &= \frac{\partial G}{\partial X}|_{\partial C} \text{ (Smooth Fit)} \end{aligned}$$

On the other hand, if X starting at ∂C does not enter immediately into int (D), then above general condition can be transformed into:

$$\begin{aligned}
&\mathbb{L}_x\hat{V} = 0 \text{ in } C \\
&\hat{V}|_D = G|_D \\
&\hat{V}|_{\partial C} = G|_{\partial C} \text{ (Continuous Fit)}
\end{aligned}$$

By solving the above system, we can work out the solution to the optimal stopping problem with an infinite time horizon. The finite time horizon problems are more difficult since the equation (2.1.2) contains the $\partial/\partial t$ and in most of them we can not get a closed-form solution. However, we can still formulate $\hat{V}$ and C through this method. To illustrate this method, we will start with this optimal stopping problem:

$$V(t,x) = \sup_{0\leq\tau\leq T-t} E_{t,x}G(t+\tau, X_{t+\tau})$$

In this case when X is a diffusion, and ∂C is sufficiently regular, we can have:

$$\begin{aligned}
\hat{V}_t + \mathbb{L}_X\hat{V} &= 0 \text{ in C} \\
\hat{V}|_D &= G|_D \\
\frac{\partial\hat{V}}{\partial X}|_{\partial C} &= \frac{\partial G}{\partial X}|_{\partial C} \text{ (Smooth Fit)}
\end{aligned}$$

If X has jumps and no diffusion component, and ∂C may be still sufficiently nice, the condition of smooth fit will be replaced by continuous fit

$$\hat{V}|_{\partial C} = G|_{\partial C}$$

The above idea mainly focuses on the Dirichlet problem, but based on different kinds of optimal stopping problems, we have different choices for the differential equation problems, which have been listed in Chapter 3 of [3].

2.1.2 American Option Pricing

Since we have studied the basic idea of reducing the optimal stopping problem to the free boundary problem, we will apply it to the American option pricing problem.

We will start with the optimal stopping problem for the American option pricing with infinite time horizon:

$$V(x) = \sup_{\tau} \mathsf{E}_x(e^{-r\tau}(K - X_\tau)^+)$$

where X is a geometric Brownian motion solving

$$dX_t = rX_t dt + \sigma X_t dB_t$$

The problem above can be solved by making a guess for the solution, and then verify the guessed solution is correct. Firstly, we can ansatz a stopping time

$$\tau_b = \inf\{t \geq 0 : X_t \leq b\} \tag{2.1.3}$$

where $b \in (0, k)$. Based on the strong Markov property of Geometric Brownian motion, we can formulate the free-boundary problem:

$$\begin{aligned}
&\mathbb{L}_X V = rV \text{ for } x > b \\
&V(x) = (K - x)^+ \text{ for } x = b \\
&V'(x) = -1 \text{ for } x = b \text{ (smooth fit)} \\
&V(x) > (K - x)^+ \text{ for } x > b \\
&V(x) = (K - x)^+ \text{ for } 0 < x < b \\
&V(x) \leq K \text{ for any } x
\end{aligned}$$

By solving this system, we have

$$V(x) = \begin{cases} \frac{D}{r}(\frac{K}{1+D/r})^{1+r/D} & \text{if } x \in [b, \infty) \\ K - x & \text{if } x \in (0, b] \end{cases}$$

in which

$$b = \frac{K}{1 + D/r}$$

and

$$D = \sigma^2/2.$$

Then we need to verify the value function V is given explicitly as above and that the solution for τ_b is optimal. By applying the local time-space formula introduced in Chapter 3 of [3], [2], and [1], we can verify the solution is optimal.

2.2 Method of Guess and Verification

In this part, we will study the methods introduced in the note [4]. In the note [4], two methods are introduced, we will start with the simple one and then the complex one.

2.2.1 First Approach

The general idea of the first approach contains two main steps:

1. Guess and prove the optimal stopping region has a particular shape.
2. Find an expression for the associated expected payoff function for each region. The value function should be the largest of these expected payoff functions.

We can illustrate this approach via an example of American option pricing. Given an American put option and the stochastic process is driven by the standard Brownian motion X, the option payoff function is in the form of $g(x) = (K - e^x)^+$. According to the general theory of optimal stopping, the optimal stopping time τ^* should be the first entry time to the stopping region $\mathbf{S}$

$$\mathbf{S} := \{x \in \mathbb{R} | V(x) = g(x)\}$$

Due to the continuity of g and V, $\mathbf{S}$ should be a closed set. Further there exists a $b \in (-\infty, \log(K))$ such that the assumed stopping region can be written as

$$\mathbf{S}_b = (-\infty, b]$$

where $b \in (-\infty, \log(K))$, the associated set of expected payoff functions should be

$$f_b(x) := \mathbb{E}_x(L_{\tau_b})$$

and the associated stopping time should be

$$\tau_b := \inf\{t \geq 0 | X_t \geq b\}$$

The assumption above can be proved by the following process:

Proof.

If we take a $b \in (-\infty, \log(K))$, then we have

$$V(x) = \sup_{\tau} \mathbf{E}_x(L_\tau) \geq \mathbb{E}_x(L_{\tau_b}) > 0. \tag{2.2.1}$$

To prove the shape of $\mathbf{S}$, it is equivalent to proving the statement

$$\mathbf{S} \cap [\log(K), \infty) = \varnothing$$

We can prove this statement by contradiction if $x \in \mathbf{S} \cap [\log(K), \infty)$, then we should stop immediately. According to the option payoff function, we have

$$V(x) = \sup_{\tau} \mathbf{E}_x(L_\tau) = \mathbb{E}_x(L_0) = g(x) = 0,$$

which contradicts the relationship (2.2.1). Also $\mathsf{S} \neq \varnothing$. If $\mathsf{S} = \varnothing$, then $\tau = \infty$, then

$$V(X) = \mathbb{E}_x(L_\infty) = \mathbb{E}_x(0) = 0,$$

which contradicts relationship (2.2.1). Also, by proving $V - g$ is non-decreasing, we can conclude that S is a closed set and we have

$$\mathsf{S} = (-\infty, b].$$

Here is the detailed process, based on the risk-neutral valuation, we have

$$V(x) = \sup_\tau \mathbb{E}_x(e^{-r\tau} g(X_\tau)),$$

which is equivalent to

$$V(x) = \sup_\tau \mathbb{E}(e^{-r\tau} g(x + X_\tau)) \tag{2.2.2}$$

since the law of X under $\mathbb{P}_x$ is equivalent to the law of $(x + X_t)_{t \geq 0}$ under $\mathbb{P} = \mathbb{P}_0$. To prove $V - g$ is non-decreasing, we can simplify g into the form $\hat{g}$

$$\hat{g}(x) = K - e^x$$

Therefore, we have

$$V(x) \geq \sup_\tau \mathbb{E}_x(e^{-r\tau} \hat{g}(X_\tau)).$$

On the other hand

$$V(x) = \mathbb{E}_x(L_{\tau_b}) = \mathbb{E}_x(e^{-r\tau_b} \hat{g}(X_{\tau_b}))$$

Based on the two statements, we have

$$V(x) = \sup_\tau \mathbb{E}_x(e^{-r\tau} \hat{g}(X_\tau)).$$

Further, the stop region becomes

$$\mathbf{S} = \{x \in \mathbb{R} | V(x) = \hat{g}(x)\}$$

By using $V(x)$ (2.2.2), we can conclude $V(x) - \hat{g}(x)$ is not decreasing. Therefore the stopping region is in the form:

$$\mathbf{S} = (-\infty, b].$$

$\square$

Then we can work out the value function V:

$$V(x) = \begin{cases} K - e^x & \text{if } x \leq b \\ (K - e^b)e^{\sqrt{2r}(b-x)} & \text{if } x > b \end{cases}$$

where $b = \log(\sqrt{2r}K/1 + /\sqrt{2r})$.

Proof.

By computing

$$\mathbb{E}_x(L_{\tau_b}) = \mathbb{E}_x(e^{-r\tau_b}(K - e^{X_{\tau_b}})) = (K - e^b)e^{\sqrt{2r}(b-x)}$$

with $b \in (-\infty, \log(K))$. At this stage, we have

$$f_b(x) = \begin{cases} K - e^x & \text{if } x \leq b \\ (K - e^b)e^{\sqrt{2r}(b-x)} & \text{if } x > b \end{cases}$$

Since $V \geq f_b$ all the time, then we can maximize f_b, when $b = \log(\sqrt{2r}K/1 + /\sqrt{2r})$. □

2.2.2 Second Approach

Before studying the second method, we will go through some fundamental theories for this method. We will start with the idea of the first entry time.

Theorem 2.2.1 (Debut Theorem)**.**

If X is an adapted right-continuous process on a filtered probability space satisfying the usual conditions, and $\mathscr{B}$ is any Borel set, then the first entry time of $\mathscr{B}$, defined as:

$$\tau_{\mathscr{B}} = \inf\{t \geq 0 | X_t \in \mathscr{B}\}$$

Based on this theorem, we can have the following lemma.

Lemma 2.2.1.

Given an adapted and right continuous process X and $X_0 = 0$ a.s, $\{b_n\}$ be a non-decreasing sequence of strictly negative numbers so that $b_n \uparrow 0$ with

$$\tau_{b_n} = \inf\{t \geq 0 | X_t \leq b_n\}$$

and

$$\sigma := \inf\{t \geq 0 | X_t < 0\}$$

we have that $\tau_{b_n} \downarrow \sigma$ a.s.

It is important that from Lemma 2.2.1 that the limit of the τ_{b_n} is not $\inf\{t \geq 0 | X_t \leq 0\}$. Similar to the method of reduction to the free-boundary problem introduced in the previous section and Chapter 4 of [3], solving the stopping problem requires us to apply the law of pasting.

Definition 2.2.1 ((Ir)Regularity Upwards/Downwards)**.**

Given a process X, if the optimal stopping region is $(\infty, b]$, it needs to move downwards to enter it. If it happens immediately the stopping time $\tau = 0$, this case is called regular downwards. If $\tau > 0$, this case is called irregular downwards. And vice versa.

Here is an example of how to justify the types of movements. Recall the American put option pricing problem driven by the Spectrally Negative Levy Process, we have the stopping time

$$\tau_b = \inf\{t \geq 0 | X_t \leq b\},$$

and the associated value function is

$$f_b(x) = \mathbb{E}_x(e^{-r\tau_b} g(X_{\tau_b})).$$

Now, we will find how close x to b. For any $h > 0$, we have

$$f_b(b+h) = \mathbb{E}_{b+h}(e^{-r\tau_b} g(X_{\tau_b})) = \mathbb{E}(e^{-r\tau_{-h}} g(b+h+X_{\tau_{-h}}))$$

According to Lemma 2.2.1,

$$\lim_{h \downarrow 0} \tau_{-h} = \sigma > 0 \text{ a.s.}$$

in which $\sigma = \inf\{t \geq 0 | X_t < 0\}$, and it leads

$$e^{-r\tau_{-h}} g(b+h+X_{\tau_{-h}}) \to e^{-r\sigma} g(b+X_{\tau_\sigma}) \text{ } \mathbb{P}\text{-a.s}$$

Then by applying DCT, we have

$$\lim_{h \downarrow 0} f_b(b+h) = \lim_{h \downarrow 0} \mathbb{E}_{b+h}(e^{-r\tau_b} g(X_{\tau_b})) = \mathbb{E}_{b+h}(e^{-r\sigma} g(X_\sigma))$$

Based on Lemma 2.2.1 and $\lim_{h \downarrow 0} f_b(b+h) \neq g(b)$, it can be concluded that the movement is an irregular downwards.

However, if this problem is driven by the Spectrally negative Levy Process with the unbounded variation, the movement will become regular one.

Theorem 2.2.2 (Law of Pasting).

If the movement is regular, we need to use the smooth pasting condition to connect the functions f_b and g to find the optimal boundary b. If the movement is irregular, we need to use the continuous pasting condition to connect the functions f_b and g to find the optimal boundary b.

Based on the knowledge above, we can generate the general idea for this method:

1. Guess and prove the optimal stopping region has a particular shape.
2. Prove the optimal value V should satisfy the expected pasting properties (Ir/Regular up/downwards).
3. Find an expression for the associated expected payoff functions for each assumed region from Step 1.
4. Find the expected payoff functions and stopping region satisfies the pasting condition.

Similar to the previous section, we can apply this method to the American option pricing problem. Holding everything unchanged with the example provided in Section 2.2.1, based on the Lemma 2.2.1, it can be found that X is a regular downwards since X is a standard Brownian motion. Then applying the smooth pasting condition, we can find the optimal region.

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