CSE 151: Machine learning **Winter 2019**

Homework 3

Submission instructions:

- Please type up your solutions.
- If a problem asks for a numerical answer, you need only provide this answer. There is no need to show your work, unless you would like to.
- Upload the PDF file for your homework to `gradescope` by 6pm on Tuesday January 29.

Part A: Probability and generative modeling

1. The TryMe smartphone company has three factories making its phones. They are all fairly unreliable: 10% of the phones from factory 1 are defective, 20% of the phones from factory 2 are defective, and 24% of the phones from factory 3 are defective. The factories do not produce the same numbers of phones: factory 1 produces 1/2 of TryMe's phones, while factories 2 and 3 each produce 1/4.
   (a) What is the probability that a TryMe phone chosen at random is defective?
   (b) Given that a TryMe phone is defective, what is the probability that it came from factory 1? Factory 2? Factory 3?
2. Here are some statistics collected by a doctor about patients who walk into her office.
   - 25% of the patients have the flu.
   - Among patients with the flu, 75% have a fever.
   - Among patients who don't have the flu, 50% have a fever.

   A new person walks into the doctor's office and turns out to have a fever. What is the probability that he has the flu?
3. A fair die is rolled twice. Let X_1 and X_2 denote the outcomes, and define random variable X to be the minimum of X_1 and X_2.
   (a) Determine the distribution of X.
   (b) What is $\mathbb{E}(X)$?
   (c) What are the variance and standard deviation of X?
4. In each of the following cases, say whether X and Y are independent.
   (a) Randomly pick a card from a pack of 52 cards. Define X to be 1 if the card is a Jack, and 0 otherwise. Define Y to be 1 if the card is a spade, and 0 otherwise.
   (b) Randomly pick two cards from a pack of 52 cards. X is 1 if the first card is a spade, and 0 otherwise. Y is 1 if the second card is a spade, and 0 otherwise.

5. Would you expect the following pairs of random variables to be uncorrelated, positively correlated, or negatively correlated?

   (a) The amount of rainfall on a given day and the amount of rainfall the following day.

   (b) The number of people at the beach on a given day and the number of people skiing that day.

   (c) A person's age and social security number.

6. Each of the following scenarios describes a joint distribution (x, y). In each case, give the parameters of the (unique) bivariate Gaussian that satisfies these properties.

   (a) x has mean 2 and standard deviation 1, y has mean 2 and standard deviation 0.5, and the correlation between x and y is -0.5.

   (b) x has mean 1 and standard deviation 1, and y is equal to x.

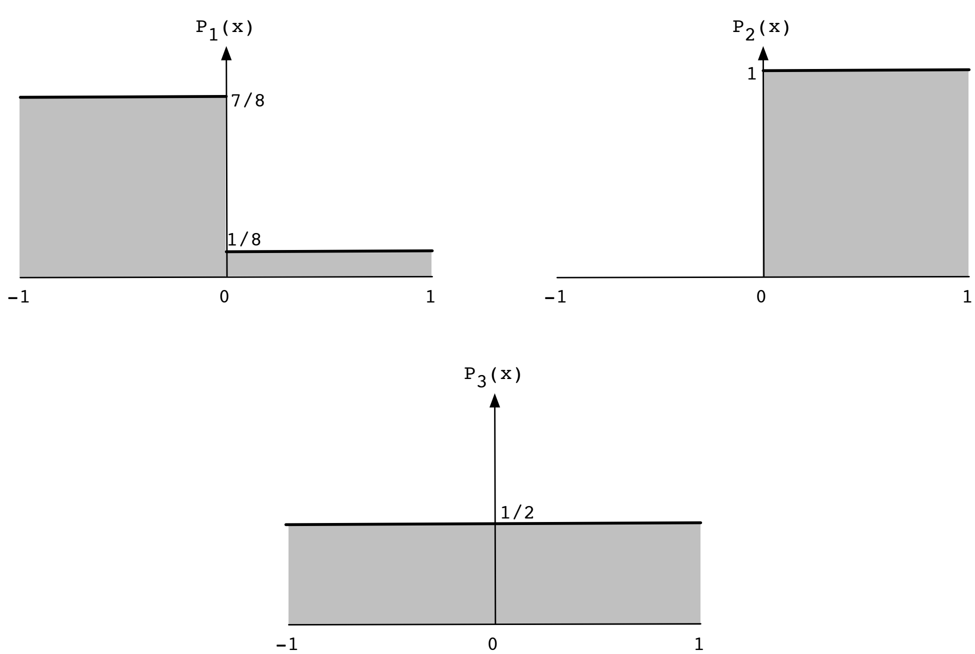
7. Roughly sketch the shapes of the following Gaussians $N(\mu, \Sigma)$. You only need to show a representative contour line which is qualitatively accurate (has approximately the right orientation, for instance).

   (a) $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$

   (b) $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 1 & -0.75 \\ -0.75 & 1 \end{pmatrix}$

8. For each of the two Gaussians in the previous problem, check your answer using Python: draw 100 random samples from that Gaussian and plot them.

9. Suppose $\mathcal{X} = [-1, 1]$ and $\mathcal{Y} = \{1, 2, 3\}$, and that the individual classes have weights $\pi_1 = \frac{1}{3}, \pi_2 = \frac{1}{6}, \pi_3 = \frac{1}{2}$ and densities P_1, P_2, P_3 as shown below.



   What is the optimal classifier h^*? Specify it as a function from $\mathcal{X}$ to $\mathcal{Y}$.

Part B: Linear algebra

1. Find the unit vector in the same direction as $x = (1, 2, 3)$.
2. Find all unit vectors in $\mathbb{R}^2$ that are orthogonal to $(1, 1)$.
3. How would you describe the set of all points $x \in \mathbb{R}^d$ with $x \cdot x = 25$?
4. The function $f(x) = 2x_1 - x_2 + 6x_3$ can be written as $w \cdot x$ for $x \in \mathbb{R}^3$. What is w?
5. For a certain pair of matrices A, B, the product AB has dimension 10×20. If A has 30 columns, what are the dimensions of A and B?
6. We have n data points $x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}^d$ and we store them in a matrix X, one point per row.
   (a) What is the dimension of X?
   (b) What is the dimension of XX^T?
   (c) What is the (i, j) entry of XX^T, simply?
7. Vector x has length 10. What is $x^T x x^T x x^T x$?
8. For $x = (1, 3, 5)$ compute $x^T x$ and xx^T.
9. Vectors $x, y \in \mathbb{R}^d$ both have length 2. If $x^T y = 2$, what is the angle between x and y?
10. The quadratic function $f : \mathbb{R}^3 \to \mathbb{R}$ given by
$$f(x) = 3x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_3^2$$
can be written in the form $x^T M x$ for some **symmetric** matrix M. What is M?
11. Which of the following matrices is necessarily symmetric?
    (a) AA^T for arbitrary matrix A.
    (b) $A^T A$ for arbitrary matrix A.
    (c) $A + A^T$ for arbitrary square matrix A.
    (d) $A - A^T$ for arbitrary square matrix A.
12. Let $A = \text{diag}(1, 2, 3, 4, 5, 6, 7, 8)$.
    (a) What is $|A|$?
    (b) What is A^{-1}?
13. Vectors $u_1, \ldots, u_d \in \mathbb{R}^d$ all have unit length and are orthogonal to each other. Let U be the $d \times d$ matrix whose rows are the u_i.
    (a) What is UU^T?
    (b) What is U^{-1}?
14. Matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & z \end{pmatrix}$ is singular. What is z?