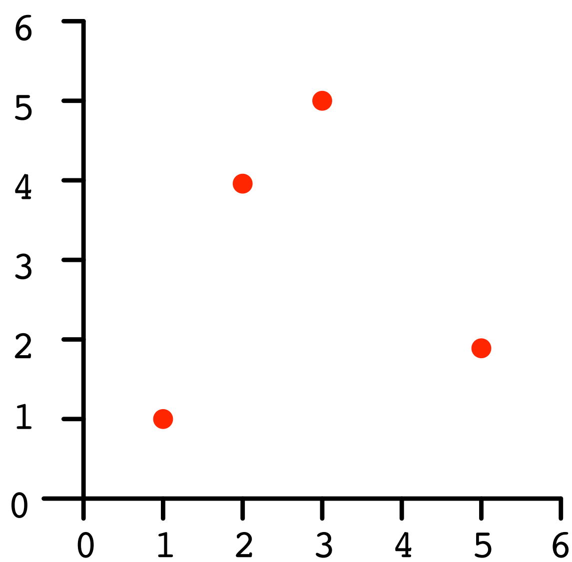
Linear algebra primer

Data as vectors and matrices



Matrix-vector notation

Vector $x \in \mathbb{R}^d$:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_d \end{pmatrix}$$

Matrix $M \in \mathbb{R}^{r \times d}$:

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1d} \\ M_{21} & M_{22} & \cdots & M_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rd} \end{pmatrix}$$

M_{ij} = entry at row i, column j

Transpose of vectors and matrices

$$x = \begin{pmatrix} 1 \\ 6 \\ 3 \\ 0 \end{pmatrix} \text{ has } \textbf{transpose} \quad x^T =$$

$$M = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 9 & 1 & 6 \\ 8 & 7 & 0 & 2 \end{pmatrix} \text{ has } \textbf{transpose} \quad M^T = \qquad .$$

- $(A^T)_{ij} = A_{ji}$
- $(A^T)^T = A$

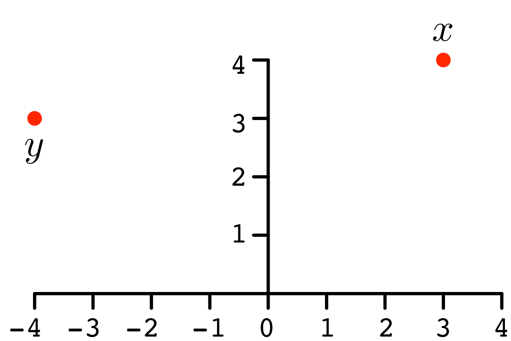
Adding and subtracting vectors and matrices

Dot product of two vectors

Dot product of vectors $x, y \in \mathbb{R}^d$:

$$x \cdot y \; = \; x_1 y_1 + x_2 y_2 + \cdots + x_d y_d.$$

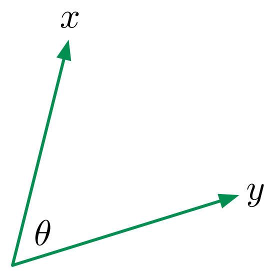
What is the dot product between these two vectors?



Dot products and angles

Dot product of vectors $x, y \in \mathbb{R}^d$: $x \cdot y \ = \ x_1 y_1 + x_2 y_2 + \cdots + x_d y_d$.

Tells us the angle between x and y:



$$\cos\theta \ = \ \frac{x \cdot y}{\|x\| \, \|y\|}.$$

x is **orthogonal** (at right angles) to y if and only if $x \cdot y = 0$ When x, y are **unit vectors** (length 1): $\cos\theta = x \cdot y$ What is $x \cdot x$?

Linear and quadratic functions

1. Linear functions
2. Matrix-vector products
3. Matrix-matrix products

Linear and quadratic functions

In one dimension:

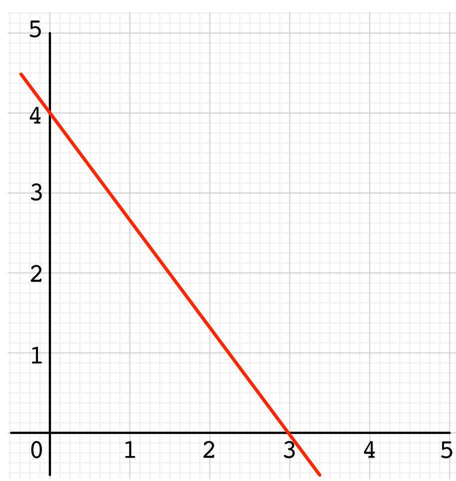
- Linear: $f(x) = 3x + 2$
- Quadratic: $f(x) = 4x^2 - 2x + 6$

In higher dimension, e.g. $x = (x_1, x_2, x_3)$:

- Linear: $3x_1 - 2x_2 + x_3 + 4$
- Quadratic: $x_1^2 - 2x_1x_3 + 6x_2^2 + 7x_1 + 9$

Linear functions and dot products

Linear separator $4x_1 + 3x_2 = 12$:



For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, linear separators are of the form:

$$w_1x_1 + w_2x_2 + \cdots + w_dx_d = c.$$

Can write as $w \cdot x = c$, for $w = (w_1, \ldots, w_d)$.

More general linear functions

A linear function from $\mathbb{R}^4$ to $\mathbb{R}$: $f(x_1, x_2, x_3, x_4) = 3x_1 - 2x_3$

A linear function from $\mathbb{R}^4$ to $\mathbb{R}^3$:
$f(x_1, x_2, x_3, x_4) = (4x_1 - x_2, x_3, -x_1 + 6x_4)$

Matrix-vector product

Product of matrix $M \in \mathbb{R}^{r \times d}$ and vector $x \in \mathbb{R}^d$:

The identity matrix

The $d \times d$ **identity matrix** I_d sends each $x \in \mathbb{R}^d$ to itself.

$$I_d = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Matrix-matrix product

Product of matrix $A \in \mathbb{R}^{r \times k}$ and matrix $B \in \mathbb{R}^{k \times p}$:

Matrix products

If $A \in \mathbb{R}^{r \times k}$ and $B \in \mathbb{R}^{k \times p}$, then AB is an $r \times p$ matrix with (i,j) entry

$$(AB)_{ij} \;=\; (\text{dot product of } i\text{th row of } A \text{ and } j\text{th column of } B) \;=\; \sum_{\ell=1}^{k} A_{i\ell} B_{\ell j}$$

- $I_k B = B$ and $A\, I_k = A$
- Can check: $(AB)^T = B^T A^T$
- For two vectors $u, v \in \mathbb{R}^d$, what is $u^T v$?

Some special cases

For vector $x \in \mathbb{R}^d$, what are $x^T x$ and xx^T?

Associative but not commutative

- Multiplying matrices is **not commutative**: in general, $AB \neq BA$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} =$$

- But it is **associative**: $ABCD = (AB)(CD) = (A(BC))D$, etc.

  Example: if $x \in \mathbb{R}^d$ has length 2, what is $x^T x x^T x x^T x x^T x$?

Square matrices and quadratic functions

1. Square matrices as quadratic functions
2. Special cases of square matrices: symmetric and diagonal
3. Determinant
4. Inverse

A special case

Recall: For vector $x \in \mathbb{R}^d$, we have $x^T x = \|x\|^2$.

What about $x^T M x$, for arbitrary $d \times d$ matrix M?

What is $x^T M x$ for $M = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$?

Quadratic functions

Let M be any $d \times d$ (**square**) matrix.
For $x \in \mathbb{R}^d$, the mapping $x \mapsto x^T M x$ is a **quadratic function** from $\mathbb{R}^d$ to $\mathbb{R}$:

$$x^T M x \ = \ \sum_{i,j=1}^{d} M_{ij} x_i x_j.$$

What is the quadratic function associated with $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix}$?

Write the quadratic function $f(x_1, x_2) = x_1^2 + 2x_1 x_2 + 3x_2^2$ using matrices and vectors.

Special cases of square matrices

- **Symmetric**: $M = M^T$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 3 & 4 & 6 \end{pmatrix}$$

- **Diagonal**: $M = \text{diag}(m_1, m_2, \ldots, m_d)$

$$\text{diag}(1, 4, 7) \; = \; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Determinant of a square matrix

Determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $|A| = ad - bc$.

Example: $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$

Inverse of a square matrix

The **inverse** of a $d \times d$ matrix A is a $d \times d$ matrix B for which $AB = BA = I_d$.
Notation: A^{-1}.

Example: if $A = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$ then $A^{-1} = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 1/4 \end{pmatrix}$. Check!

Inverse of a square matrix, cont'd

The **inverse** of a $d \times d$ matrix A is a $d \times d$ matrix B for which $AB = BA = I_d$. Notation: A^{-1}.

- Not all square matrices have an inverse
- Square matrix A is invertible if and only if $|A| \neq 0$
- What is the inverse of $A = \text{diag}(a_1, \ldots, a_d)$?