Multiclass linear prediction

Topics we'll cover

1. Multiclass logistic regression
2. Multiclass Perceptron
3. Multiclass support vector machines

Multiclass classification

Of the classification methods we have studied so far, which seem inherently binary?

- Nearest neighbor?
- Generative models?
- Linear classifiers?

The main idea

Remember Gaussian generative models...

From binary to multiclass logistic regression

Binary logistic regression: for $\mathcal{X} = \mathbb{R}^d$, classifier given by $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$:

$$\Pr(y = 1|x) = \frac{e^{w \cdot x + b}}{1 + e^{w \cdot x + b}}$$

Labels $\mathcal{Y} = \{1, 2, \ldots, k\}$: specify a classifier by $w_1, \ldots, w_k \in \mathbb{R}^d$ and $b_1, \ldots, b_k \in \mathbb{R}$:

$$\Pr(y = j|x) \;\propto\; e^{w_j \cdot x + b_j}$$

- What is the fully normalized form of the probability?
- Given a point x, which label to predict?

Multiclass logistic regression

- **Label space**: $\mathcal{Y} = \{1, 2, \ldots, k\}$
- **Parametrized classifier**: $w_1, \ldots, w_k \in \mathbb{R}^d$, $b_1, \ldots, b_k \in \mathbb{R}$:

$$\Pr(y = j|x) = \frac{e^{w_j \cdot x + b_j}}{e^{w_1 \cdot x + b_1} + \cdots + e^{w_k \cdot x + b_k}}$$

- **Prediction**: given a point x, predict label $\arg\max_j \, (w_j \cdot x + b_j)$.
- **Learning**: Given: $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$.
  Find: $w_1, \ldots, w_k \in \mathbb{R}^d$ and $b_1, \ldots, b_k$ that maximize the likelihood

$$\prod_{i=1}^{n} \Pr(y^{(i)}|x^{(i)})$$

  Taking negative log gives a convex minimization problem.

Multiclass Perceptron

Setting: $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{Y} = \{1, 2, \ldots, k\}$

Model: $w_1, \ldots, w_k \in \mathbb{R}^d$ and $b_1, \ldots, b_k \in \mathbb{R}$

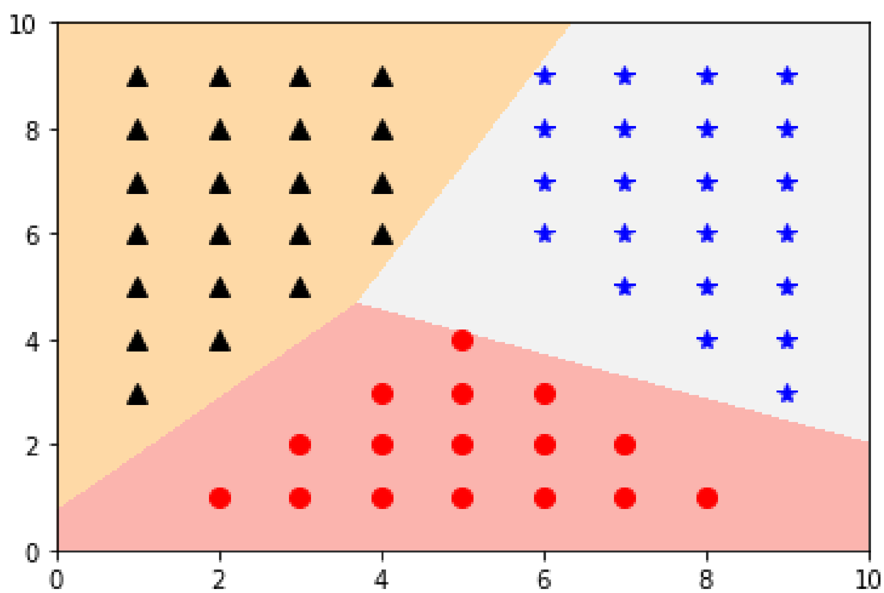
Prediction: On instance x, predict label $\arg\max_j (w_j \cdot x + b_j)$

Learning. Given training set $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$:

- Initialize $w_1 = \cdots = w_k = 0$ and $b_1 = \cdots = b_k = 0$
- Repeat while some training point (x, y) is misclassified:

$$\begin{aligned} \text{for correct label } y: \quad & w_y = w_y + x \\ & b_y = b_y + 1 \\ \text{for predicted label } \widehat{y}: \quad & w_{\widehat{y}} = w_{\widehat{y}} - x \\ & b_{\widehat{y}} = b_{\widehat{y}} - 1 \end{aligned}$$

Multiclass Perceptron: example



Multiclass SVM

Model: $w_1, \ldots, w_k \in \mathbb{R}^d$ and $b_1, \ldots, b_k \in \mathbb{R}$

Prediction: On instance x, predict label $\arg\max_j (w_j \cdot x + b_j)$

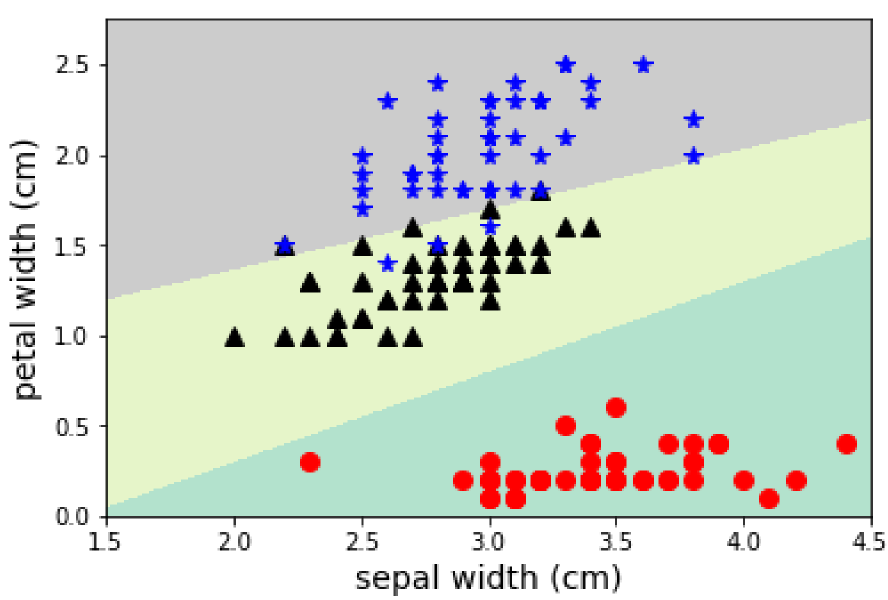
Learning. Given training set $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$:

$$\min_{w_1,\ldots,w_k \in \mathbb{R}^d, b_1,\ldots,b_k \in \mathbb{R}, \xi \in \mathbb{R}^n} \sum_{j=1}^{k} \|w_j\|^2 + C \sum_{i=1}^{n} \xi_i$$

$$w_{y^{(i)}} \cdot x^{(i)} + b_{y^{(i)}} - w_y \cdot x^{(i)} - b_y \geq 1 - \xi_i \quad \text{for all } i, \text{ all } y \neq y^{(i)}$$

$$\xi \geq 0$$

Multiclass SVM example: iris



Multiclass SVM

Given training set $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$:

$$\boxed{\begin{gathered} \min_{w_1,\ldots,w_k \in \mathbb{R}^d, b_1,\ldots,b_k \in \mathbb{R}, \xi \in \mathbb{R}^n} \quad \sum_{j=1}^{k} \|w_j\|^2 + C \sum_{i=1}^{n} \xi_i \\ w_{y^{(i)}} \cdot x^{(i)} + b_{y^{(i)}} - w_y \cdot x^{(i)} - b_y \geq 1 - \xi_i \quad \text{for all } i, \text{ all } y \neq y^{(i)} \\ \xi \geq 0 \end{gathered}}$$

Once again, a convex optimization problem.

Question: how many variables and constraints do we have?