CSCE 670 - Information Storage and Retrieval

Week 3: Link Analysis

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Course Website: https://yuzhang-teaching.github.io/CSCE670-S26.html

Adapted from the slides by Prof. Jure Leskovec (Stanford)

Recap: BM25

$$\mathrm{BM25}(q,d) = \sum_{t \in q} \mathrm{IDF}(t) \cdot \frac{\mathrm{TF}(t,d) \cdot (k_1 + 1)}{\mathrm{TF}(t,d) + k_1 \left(1 - b + b \cdot \frac{|d|}{\mathrm{avgdl}}\right)}$$

- k_1 controls term frequency scaling
  - $k_1 = 0$: binary model
  - k_1 very large: raw term frequency
- b controls document length normalization
  - $b = 0$: no document length normalization
  - $b = 1$: relative frequency (full document length normalization)
- Typically, k_1 is set between 1.2 and 2; b is set around 0.75

- $|d|$ is the length of d (in words); avgdl = average document length (in words)

Our Plan: Ranking

- ✅ Why is ranking important?
- ✅ What factors impact ranking?
- Two foundational text-based approaches
  - ✅ TF-IDF
  - ✅ BM25
- Two foundational link-based approaches
  - PageRank
  - HITS
- Machine-learned ranking (“learning to rank”)

Recap: What factors impact ranking?

- Query: "*TAMU 2025 Fall Break*"
- Document 1: https://registrar.tamu.edu/academic-calendar/fall-2025

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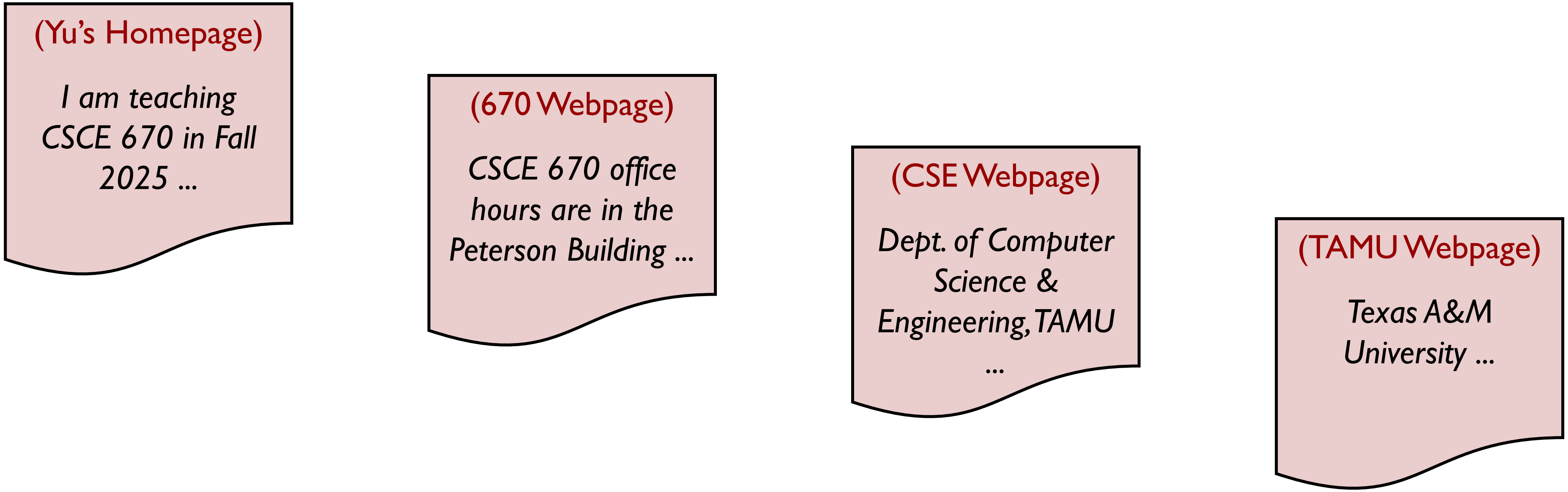
- Document 2: A social media post written by an account with 10 followers mentioning the time of TAMU 2025 Fall Break
- Document 1 should be ranked higher than Document 2 because it has a higher "reputation".
  - But how can we know the "reputation" of a website?

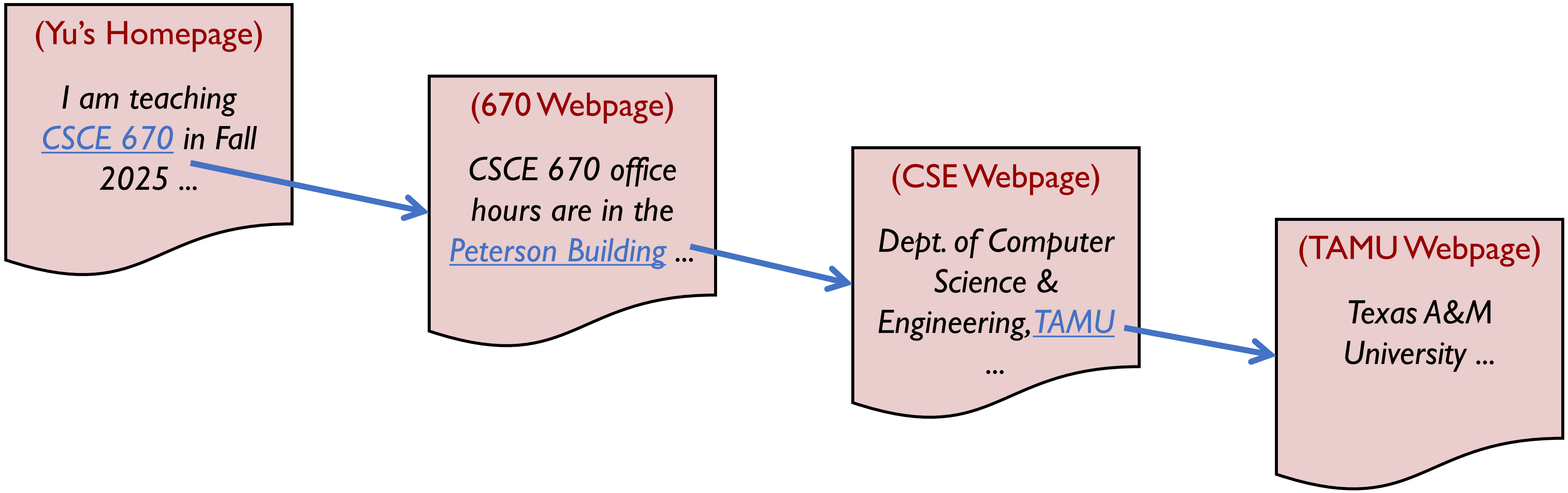
Web as a Directed Graph

- Nodes: Webpages

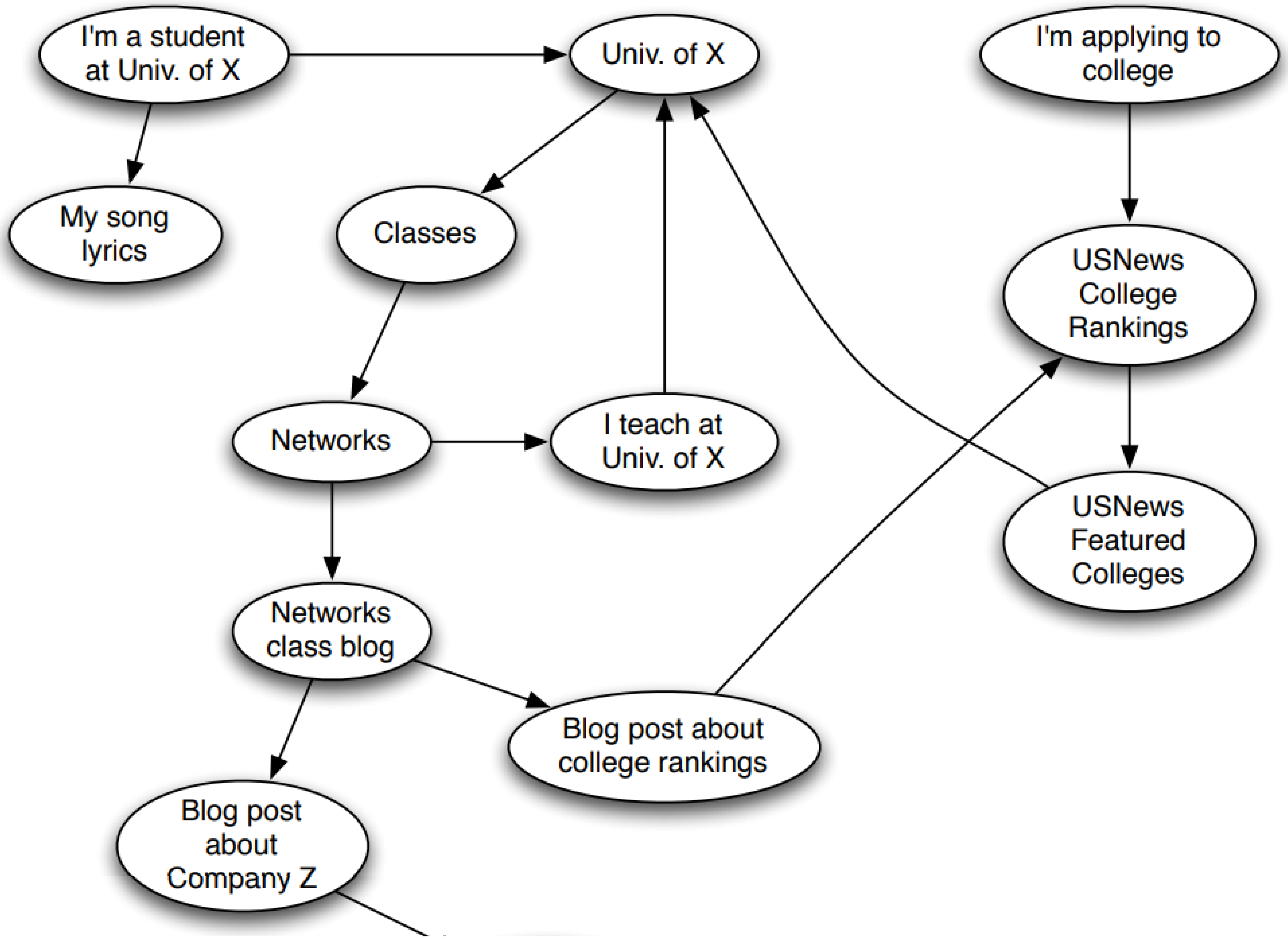


Web as a Directed Graph

- Nodes: Webpages
- Edges: Hyperlinks



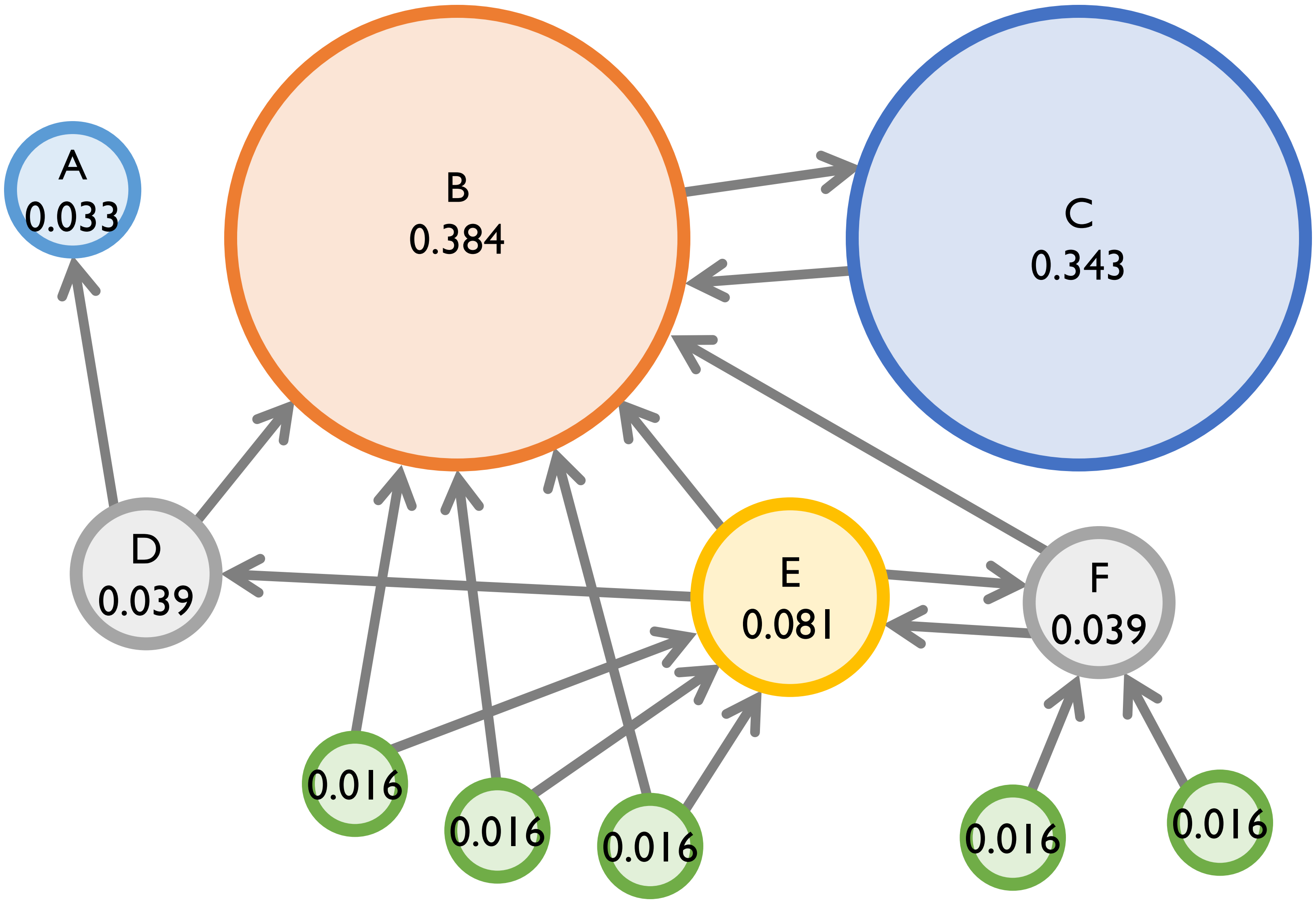
Web as a Directed Graph



Links as Votes

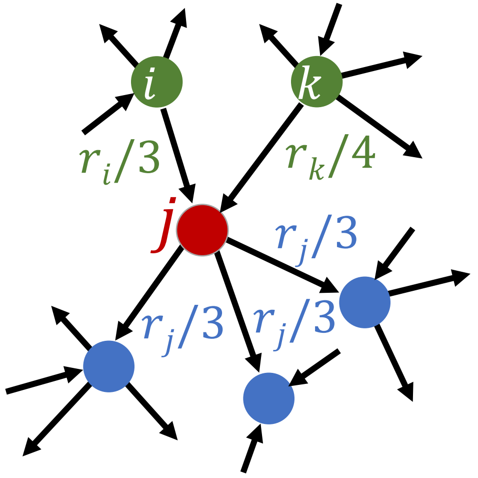
- Rough Idea: A webpage is more important if it has more links
  - In-coming links? Out-going links?
  - Out-going links can be easily manipulated by the webpage creator.
- Think of in-links as votes:
  - www.stanford.edu has 23,400 in-links
  - www.joe-schmoe.com has 1 in-link
- Are all in-links equal?
  - Links from important webpages count more.
  - Recursive question!

Example: PageRank Scores



Simple Recursive Formulation

- Each link's vote is proportional to the importance of its source page.
- If page j with importance r_j has n out-links, each link gets r_j/n votes
  - A vote from an important page is worth more.
- Page j's own importance is the sum of the votes on its in-links.
  - A page is important if it is pointed to by other important pages



$$r_j = \frac{r_i}{3} + \frac{r_k}{4}$$

In general, $r_j = \sum_{i \to j} \frac{r_i}{d_i}$

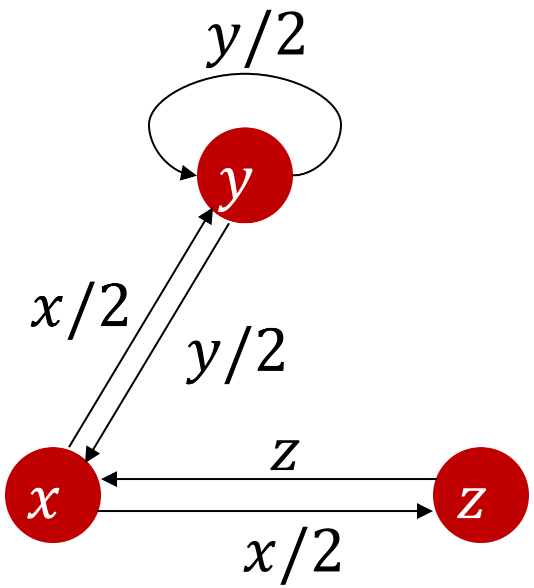
where d_i is the out-degree of i

Example

- $x = \frac{y}{2} + z$ (1)
- $y = \frac{y}{2} + \frac{x}{2}$ (2)
- $z = \frac{x}{2}$ (3)

- 3 equations, 3 unknowns. Looks like we can solve it!
- BUT if you add (1) and (2) together,
  - You will get (3).
  - Essentially, we have only 2 equations, so there exist infinitely many sets of solutions.

- Additional constraint forces uniqueness:
  - $x + y + z = 1$

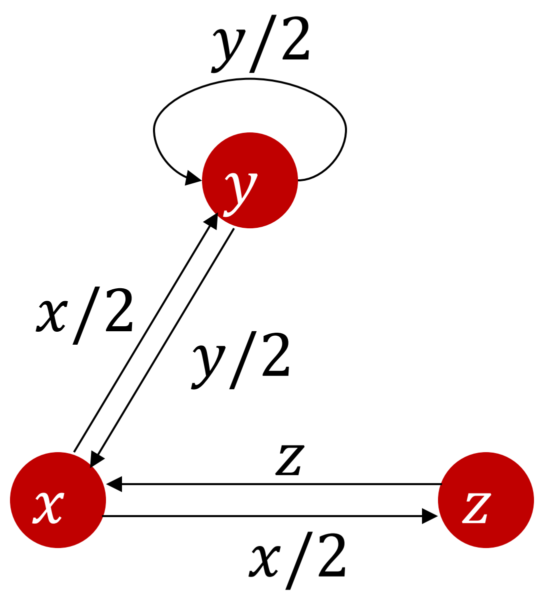


Example

- $x = \frac{y}{2} + z$ (1)
- $y = \frac{y}{2} + \frac{x}{2}$ (2)
- $x + y + z = 1$ (3)

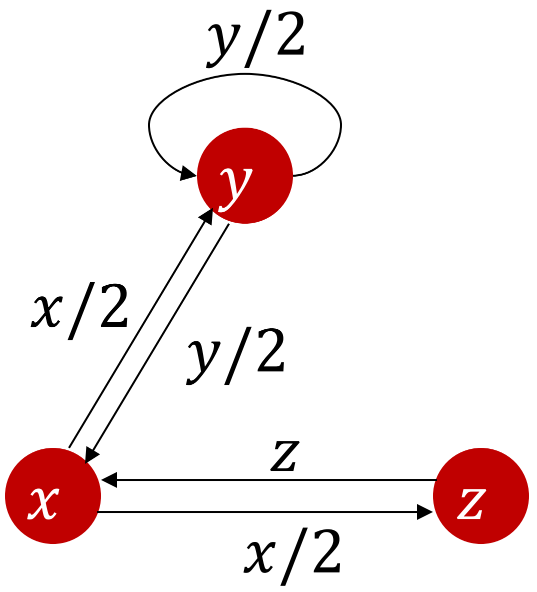
- Solution:
  - $x = \frac{2}{5}, y = \frac{2}{5}, z = \frac{1}{5}$.

- Gaussian elimination method works for small examples, but we need a better method for large web-size graphs.
  - We need a new formulation!



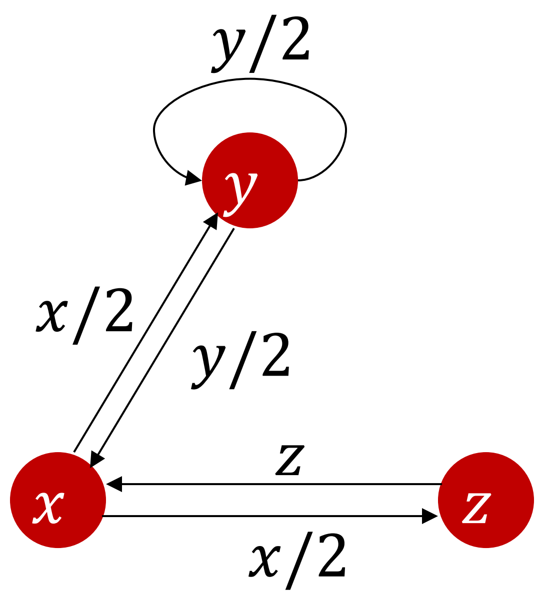
PageRank: Matrix Formulation

- Stochastic adjacency matrix $\boldsymbol{M}$
  - Assume page i has d_i out-links
  - If $i \rightarrow j$, then $M_{ji} = \frac{1}{d_i}$, else $M_{ji} = 0$.
  - Entries in each column of $\boldsymbol{M}$ sum to 1
  - Example: $\boldsymbol{M} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$



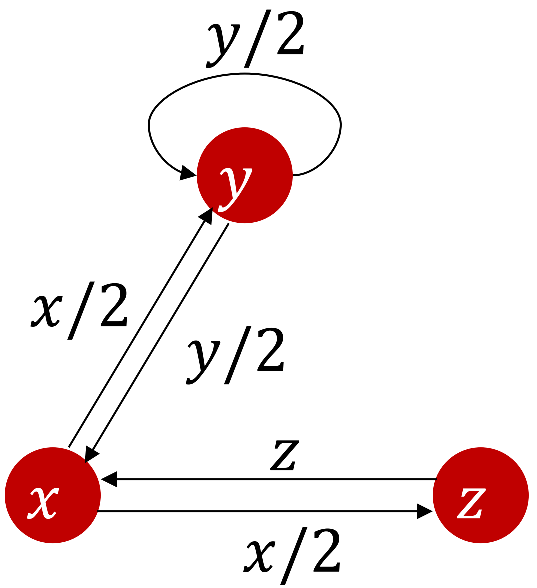
PageRank: Matrix Formulation

- Rank vector $\boldsymbol{r}$
  - r_i is the importance score of page i
  - Entries in $\boldsymbol{r}$ sum to 1
  - Example: $\boldsymbol{r} = \begin{bmatrix} 2/5 \\ 2/5 \\ 1/5 \end{bmatrix}$



PageRank: Matrix Formulation

- Equations:
  - $r_j = \sum_{i \to j} \frac{r_i}{d_i}$
  - Matrix form: $\boldsymbol{Mr} = \boldsymbol{r}$
  - Example: $\begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2/5 \\ 2/5 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 2/5 \\ 1/5 \end{bmatrix}$



- PageRank task:
  - Given the stochastic adjacency matrix $\boldsymbol{M}$, we need to find a rank vector $\boldsymbol{r}$ (whose entries sum to 1), so that

$$\boldsymbol{Mr} = \boldsymbol{r}$$

Solving $\boldsymbol{Mr} = \boldsymbol{r}$: Power Iteration Method

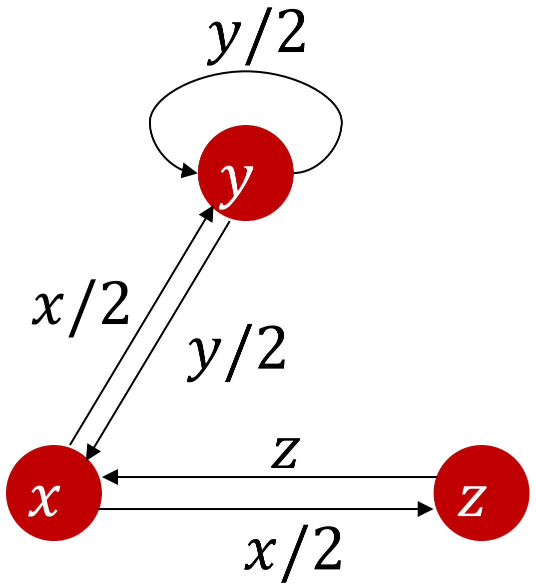
- *(Let's first assume this algorithm is correct. We will show why it works later.)*
- Power Iteration: a simple iterative scheme
  - Suppose there are N web pages in total
  - Initialize: $\boldsymbol{r}^{(0)} = [1/N, \dots, 1/N]^T$
  - Iterate: $\boldsymbol{r}^{(t+1)} = \boldsymbol{M}\boldsymbol{r}^{(t)}$
  - Stop when $\left\|\boldsymbol{r}^{(t+1)} - \boldsymbol{r}^{(t)}\right\| < \epsilon$ (a very small number, e.g., 0.001)

- If the algorithm stops, we have a good solution $\boldsymbol{r}^{(t)}$
  - $\boldsymbol{M}\boldsymbol{r}^{(t)}$ is very close to $\boldsymbol{r}^{(t)}$

Example

- Power Iteration:
  - Initialize: $\boldsymbol{r}^{(0)} = [1/N, \ldots., 1/N]^T$
  - Iterate: $\boldsymbol{r}^{(t+1)} = \boldsymbol{M}\boldsymbol{r}^{(t)}$
  - Stop when $\left\|\boldsymbol{r}^{(t+1)} - \boldsymbol{r}^{(t)}\right\| < \epsilon$

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$

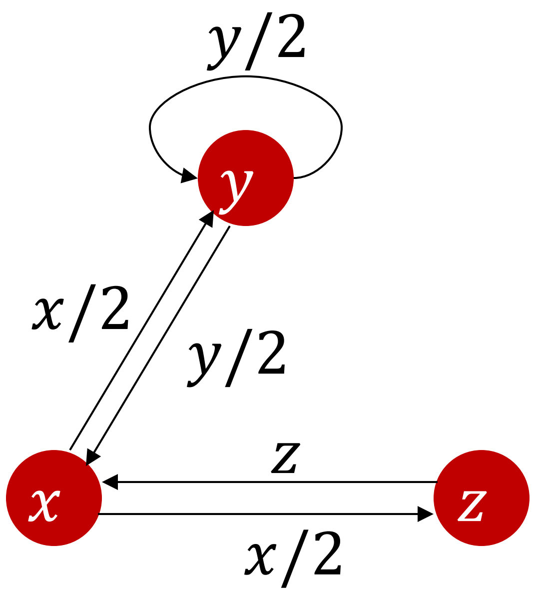


	$\boldsymbol{r}^{(0)}$
x	1/3 (0.33)
y	1/3 (0.33)
z	1/3 (0.33)

Example

- Power Iteration:
  - Initialize: $\boldsymbol{r}^{(0)} = [1/N, \ldots., 1/N]^T$
  - Iterate: $\boldsymbol{r}^{(t+1)} = \boldsymbol{M}\boldsymbol{r}^{(t)}$
  - Stop when $\left\|\boldsymbol{r}^{(t+1)} - \boldsymbol{r}^{(t)}\right\| < \epsilon$

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$

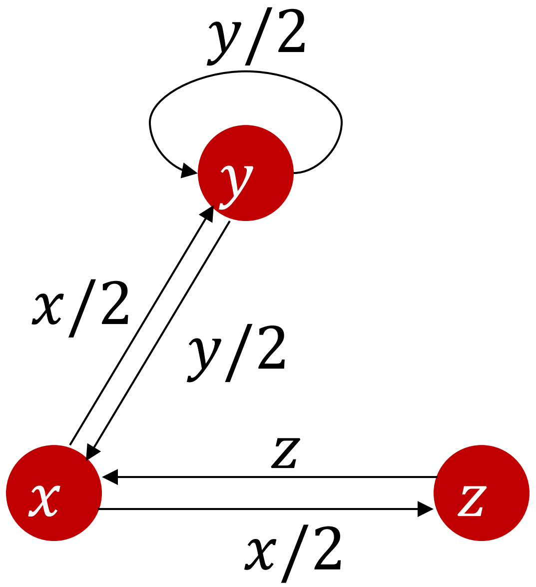


	$\boldsymbol{r}^{(0)}$	$\boldsymbol{r}^{(1)}$
x	1/3 (0.33)	1/2 (0.50)
y	1/3 (0.33)	1/3 (0.33)
z	1/3 (0.33)	1/6 (0.17)

Example

- Power Iteration:
  - Initialize: $\boldsymbol{r}^{(0)} = [1/N, \ldots., 1/N]^T$
  - Iterate: $\boldsymbol{r}^{(t+1)} = \boldsymbol{M}\boldsymbol{r}^{(t)}$
  - Stop when $\left\|\boldsymbol{r}^{(t+1)} - \boldsymbol{r}^{(t)}\right\| < \epsilon$

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$



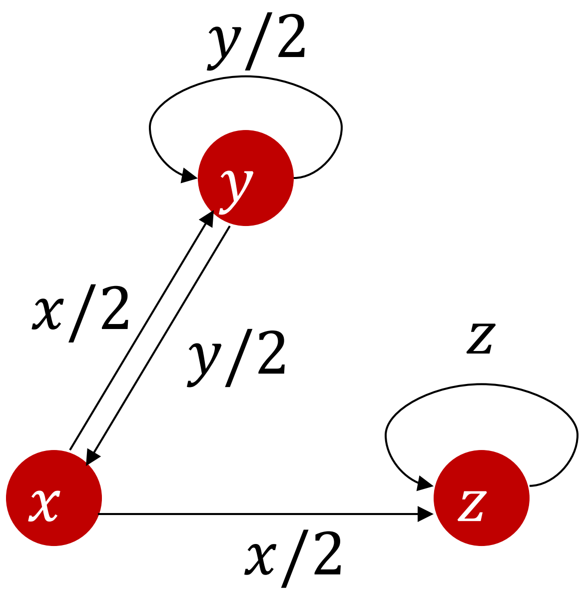
	$\boldsymbol{r}^{(0)}$	$\boldsymbol{r}^{(1)}$	$\boldsymbol{r}^{(2)}$	$\boldsymbol{r}^{(3)}$	…	Finally
x	1/3 (0.33)	1/2 (0.50)	1/3 (0.33)	11/24 (0.46)	…	0.40
y	1/3 (0.33)	1/3 (0.33)	5/12 (0.42)	3/8 (0.38)	…	0.40
z	1/3 (0.33)	1/6 (0.17)	1/4 (0.25)	1/6 (0.17)	…	0.20

Questions?

Another Example

- Power Iteration:
  - Initialize: $\boldsymbol{r}^{(0)} = [1/N, \dots, 1/N]^T$
  - Iterate: $\boldsymbol{r}^{(t+1)} = \boldsymbol{M}\boldsymbol{r}^{(t)}$
  - Stop when $\left\|\boldsymbol{r}^{(t+1)} - \boldsymbol{r}^{(t)}\right\| < \epsilon$

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

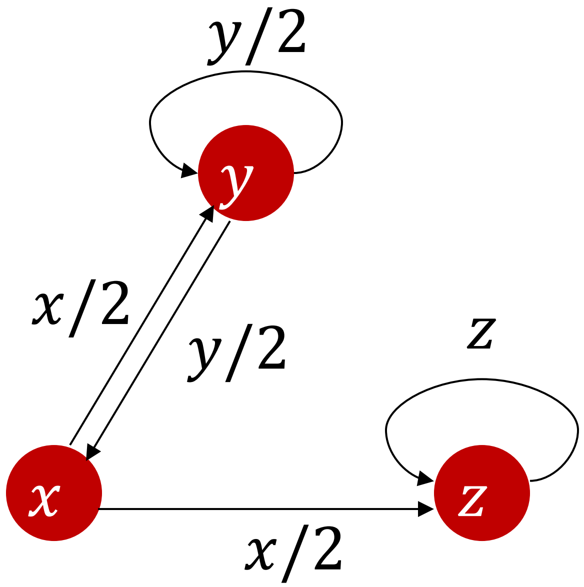


	$\boldsymbol{r}^{(0)}$
x	1/3 (0.33)
y	1/3 (0.33)
z	1/3 (0.33)

Another Example

- Power Iteration:
  - Initialize: $\boldsymbol{r}^{(0)} = [1/N, \ldots, 1/N]^T$
  - Iterate: $\boldsymbol{r}^{(t+1)} = \boldsymbol{M}\boldsymbol{r}^{(t)}$
  - Stop when $\left\|\boldsymbol{r}^{(t+1)} - \boldsymbol{r}^{(t)}\right\| < \epsilon$

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$



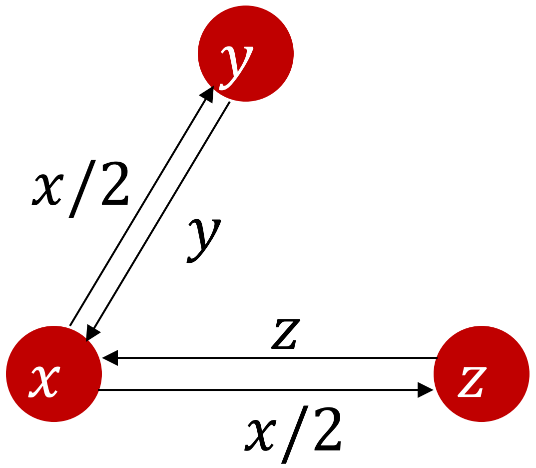
All the PageRank scores get "trapped" in node z.

	$\boldsymbol{r}^{(0)}$	$\boldsymbol{r}^{(1)}$	$\boldsymbol{r}^{(2)}$	$\boldsymbol{r}^{(3)}$	...	Finally
x	1/3 (0.33)	1/6 (0.17)	1/6 (0.17)	1/8 (0.13)	...	0.00
y	1/3 (0.33)	1/3 (0.33)	1/4 (0.25)	5/24 (0.21)	...	0.00
z	1/3 (0.33)	1/2 (0.50)	7/12 (0.58)	2/3 (0.67)	...	1.00

An Even Worse Example

- Power Iteration:
  - Initialize: $\boldsymbol{r}^{(0)} = [1/N, \dots, 1/N]^T$
  - Iterate: $\boldsymbol{r}^{(t+1)} = \boldsymbol{M}\boldsymbol{r}^{(t)}$
  - Stop when $\left\|\boldsymbol{r}^{(t+1)} - \boldsymbol{r}^{(t)}\right\| < \epsilon$

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$



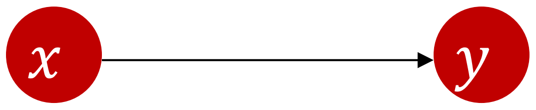
The algorithm falls into an infinite loop and will not terminate!
Root cause: the graph is bipartite.

	$\boldsymbol{r}^{(0)}$	$\boldsymbol{r}^{(1)}$	$\boldsymbol{r}^{(2)}$	$\boldsymbol{r}^{(3)}$	…	Finally
x	1/3	2/3	1/3	2/3	…	?
y	1/3	1/6	1/3	1/6	…	?
z	1/3	1/6	1/3	1/6	…	?

Yet Another Even Worse Example

- Power Iteration:
  - Initialize: $\boldsymbol{r}^{(0)} = [1/N, \ldots., 1/N]^T$
  - Iterate: $\boldsymbol{r}^{(t+1)} = \boldsymbol{M}\boldsymbol{r}^{(t)}$
  - Stop when $\left\|\boldsymbol{r}^{(t+1)} - \boldsymbol{r}^{(t)}\right\| < \epsilon$

$$\boldsymbol{M} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

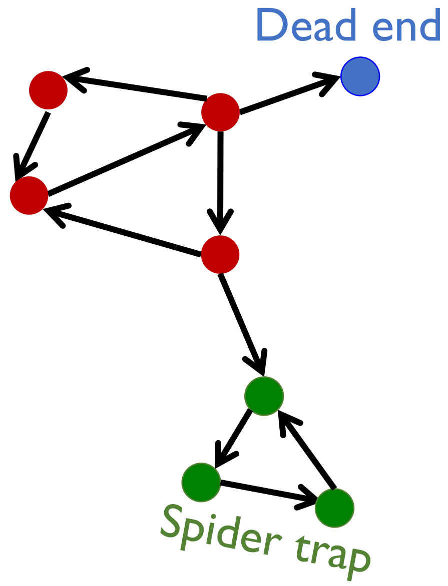


All the PageRank scores get "leaked"!
Root cause: the graph has a dead-end node (i.e., no out-links).

	$\boldsymbol{r}^{(0)}$	$\boldsymbol{r}^{(1)}$	$\boldsymbol{r}^{(2)}$	$\boldsymbol{r}^{(3)}$
x	1/2	0	0	0
y	1/2	1/2	0	0

Summary of the Challenges

- Spider traps
  - All out-links are within the group
  - Can have more than one node
  - Eventually spider traps absorb all importance
- Dead ends
  - The node has no out-links, therefore its importance score has nowhere to go
  - Eventually dead ends cause all importance to "leak out"
- Bipartite graph
  - If the graph is bipartite and the two partitions have different numbers of nodes, the algorithm will not converge.



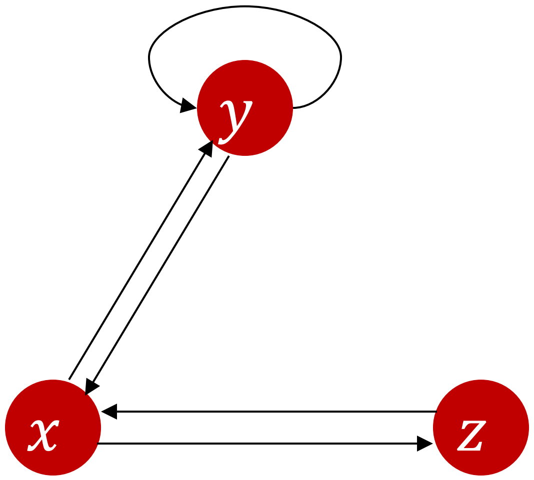
PageRank: Google Formulation

- Google's solution for spider traps: Teleportation!
  - Each node must contribute a portion of its importance score and distribute it evenly to all other nodes.

- Without teleports, $\boldsymbol{M} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$

- With teleports, $\boldsymbol{M} = \beta \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} + (1-\beta) \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

- In practice, $\beta = 0.8, 0.85$, or 0.9

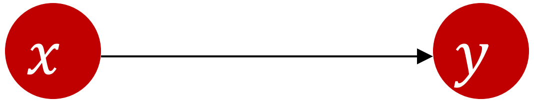


How about dead ends?

- Dead ends must contribute ALL of its importance score and distribute it evenly to all other nodes.

- Without teleports, $\boldsymbol{M} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

- Without teleports, $\boldsymbol{M} = \beta \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} + (1-\beta) \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$



- Why do we call this solution "teleportation"?
  - Part of the importance score still flows according to the graph's defined neighborhoods
  - While the other part can instantly "teleport" to any node in the graph

Why does teleportation solve the problems?

- Spider traps: with traps, PageRank scores are not what we want
  - Solution: Never get stuck in a spider trap by teleporting out of it

- Dead ends: the matrix $\boldsymbol{M}$ is no longer column-stochastic (entries in a column may sum to 0 rather than 1)
  - Solution: Make $\boldsymbol{M}$ column-stochastic by always teleporting when there is nowhere else to go

- Wait, how about the bipartite-graph issue?
  - Teleportation makes the graph fully-connected (with different edge weights) and naturally non-bipartite.

PageRank: Google Formulation [Brin and Page, WWW 1998]

- Node-wise form:

$$r_j = \beta\left(\sum_{i\to j}\frac{r_i}{d_i}\right) + (1-\beta)\frac{1}{N}$$

- Note 1: Each node i in the graph teleports a score of $(1-\beta)\frac{1}{N}r_i$ to node j, so the total score node j receives through teleportation is exactly $(1-\beta)\frac{1}{N}\sum_i r_i = (1-\beta)\frac{1}{N}$.

- Note 2: This formulation assumes the graph has no dead ends. If there is a dead end, we can first link it to all the nodes (include itself).

PageRank: Google Formulation [Brin and Page, WWW 1998]

- Matrix form:

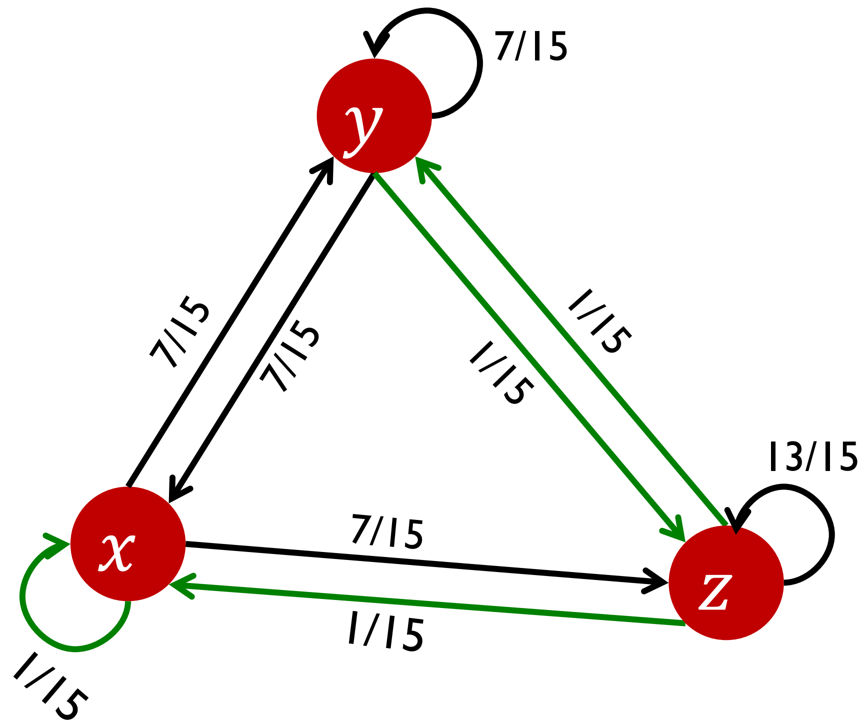
$$\boldsymbol{A} = \beta \boldsymbol{M} + (1-\beta)\frac{\mathbf{1}}{N}$$

- Note: $\mathbf{1}$ is an $N \times N$ matrix where all entries are 1.

- Now we need to solve $\boldsymbol{Ar} = \boldsymbol{r}$
  - We can still use Power Iteration

Example ($\beta = 0.8$)

$$\boldsymbol{A} = 0.8 \times \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} + 0.2 \times \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$
$$= \begin{bmatrix} 1/15 & 7/15 & 1/15 \\ 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 13/15 \end{bmatrix}$$



	$\boldsymbol{r}^{(0)}$	$\boldsymbol{r}^{(1)}$	$\boldsymbol{r}^{(2)}$	$\boldsymbol{r}^{(3)}$	…	Finally
x	1/3	0.20	0.20	0.18	…	0.15
y	1/3	0.33	0.28	0.26	…	0.21
z	1/3	0.47	0.52	0.56	…	0.64

Extended Content
(will not appear in quizzes or the exam)

Why does Power Iteration work?

- $\boldsymbol{Ar} = \boldsymbol{r}$
- In other words, $\boldsymbol{r}$ is an eigenvector of $\boldsymbol{A}$ with the corresponding eigenvalue $\lambda = 1$

- Why does $\boldsymbol{A}$ necessarily have an eigenvalue of 1?
- How about other eigenvalues of $\boldsymbol{A}$?

- Perron–Frobenius Theorem: Let $\boldsymbol{A}$ be a square matrix with all entries strictly positive, and entries in each column sum to 1, then
  - $\boldsymbol{A}$ has an eigenvalue of 1
  - 1 is the unique "largest" eigenvalue of $\boldsymbol{A}$. That is, for all other eigenvalues λ of $\boldsymbol{A}$, we have $|\lambda| < 1$.

Why does Power Iteration work?

- Power Iteration:
  - Initialize: $\boldsymbol{r}^{(0)}$= $[1/N, \ldots., 1/N]^T$
  - Iterate: $\boldsymbol{r}^{(t+1)} = \boldsymbol{A}\boldsymbol{r}^{(t)}$

$$\boldsymbol{r}^{(1)} = \boldsymbol{A}\boldsymbol{r}^{(0)}$$
$$\boldsymbol{r}^{(2)} = \boldsymbol{A}\boldsymbol{r}^{(1)} = \boldsymbol{A}\left(\boldsymbol{A}\boldsymbol{r}^{(1)}\right) = \boldsymbol{A}^2\boldsymbol{r}^{(0)}$$
$$\boldsymbol{r}^{(3)} = \boldsymbol{A}\boldsymbol{r}^{(2)} = \boldsymbol{A}\left(\boldsymbol{A}^2\boldsymbol{r}^{(0)}\right) = \boldsymbol{A}^3\boldsymbol{r}^{(0)}$$
$$\ldots$$

- We have a sequence of vectors $\boldsymbol{A}\boldsymbol{r}^{(0)}, \boldsymbol{A}^2\boldsymbol{r}^{(0)}, \boldsymbol{A}^3\boldsymbol{r}^{(0)}, \ldots$
- We need to prove that this sequence converges to the eigenvector of $\boldsymbol{A}$ with the eigenvalue $\lambda = 1$

Proof

- Let's assume $\boldsymbol{A}$ has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$, where $1 = \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_N|$
- The eigenvectors corresponding to $\lambda_1, \lambda_2, \dots, \lambda_N$ are $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N$
  - Let's also assume that $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N$ are linearly independent
  - If $\lambda_1, \lambda_2, \dots, \lambda_N$ are different from each other, this assumption always holds.

- $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N$ form a basis, so we can write $\boldsymbol{r}^{(\mathbf{0})} = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_N\boldsymbol{x}_N$
- $$\begin{aligned}\boldsymbol{A}\boldsymbol{r}^{(\mathbf{0})} &= \boldsymbol{A}(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_N\boldsymbol{x}_N) \\ &= c_1\boldsymbol{A}\boldsymbol{x}_1 + c_2\boldsymbol{A}\boldsymbol{x}_2 + \cdots + c_N\boldsymbol{A}\boldsymbol{x}_N \\ &= c_1\lambda_1\boldsymbol{x}_1 + c_2\lambda_2\boldsymbol{x}_2 + \cdots + c_N\lambda_N\boldsymbol{x}_N\end{aligned}$$

- Repeated multiplication on both sides
- $\boldsymbol{A}^2\boldsymbol{r}^{(\mathbf{0})} = c_1\lambda_1^2\boldsymbol{x}_1 + c_2\lambda_2^2\boldsymbol{x}_2 + \cdots + c_N\lambda_N^2\boldsymbol{x}_N$
- $\boldsymbol{A}^k\boldsymbol{r}^{(\mathbf{0})} = c_1\lambda_1^k\boldsymbol{x}_1 + c_2\lambda_2^k\boldsymbol{x}_2 + \cdots + c_N\lambda_N^k\boldsymbol{x}_N$

Proof

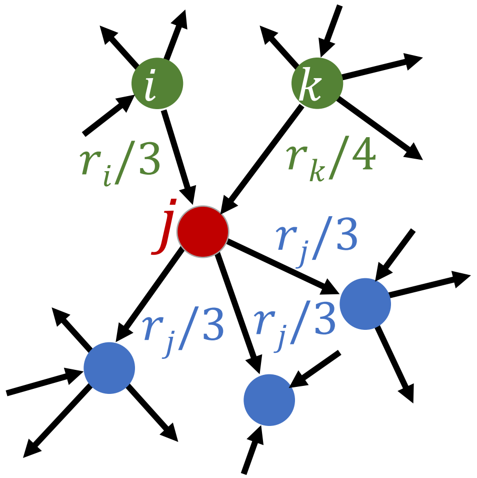
- Let's assume $\boldsymbol{A}$ has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$, where $1 = \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_N|$
- The eigenvectors corresponding to $\lambda_1, \lambda_2, \dots, \lambda_N$ are $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N$

- Repeated multiplication on both sides
- $$\begin{aligned} \boldsymbol{A}^k \boldsymbol{r}^{(\mathbf{0})} &= c_1 \lambda_1^k \boldsymbol{x}_1 + c_2 \lambda_2^k \boldsymbol{x}_2 + \cdots + c_N \lambda_N^k \boldsymbol{x}_N \\ &= \lambda_1^k \left(c_1 \boldsymbol{x}_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k \boldsymbol{x}_2 + \cdots + c_N \left(\frac{\lambda_N}{\lambda_1}\right)^k \boldsymbol{x}_N \right) \end{aligned}$$
- Note that $\left|\left(\frac{\lambda_i}{\lambda_1}\right)^k\right| = \left|\frac{\lambda_i}{\lambda_1}\right|^k \to 0$ when $k \to \infty$ (because $|\lambda_i| < |\lambda_1|$)
- Therefore, $\boldsymbol{A}^k \boldsymbol{r}^{(\mathbf{0})} \to \lambda_1^k (c_1 \boldsymbol{x}_1 + 0 + \cdots + 0) = c_1 \boldsymbol{x}_1$, which is the eigenvector of $\boldsymbol{A}$ with the eigenvalue $\lambda_1 = 1$.

Note: This proof does not apply to the case where $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N$ are NOT linearly independent, which may happen when $\boldsymbol{A}$ does not have N distinct eigenvalues.

PageRank: Random Walk Interpretation

- Imagine there is a random web surfer
  - At time t, the surfer is on a page i
  - At time $t+1$, the surfer has two options
    - With probability β, it follows an out-link from i uniformly at random (i.e., ends up on some page j linked from i)
    - With probability $1-\beta$, it jumps to a random page in the graph (can be i, j, or any other node)



- The process repeats indefinitely
- Let $\boldsymbol{p}(t)$ be the vector whose i-th coordinate is the probability that the surfer is at page i at time t
  - So $\boldsymbol{p}(t)$ is a probability distribution over pages

The Stationary Distribution

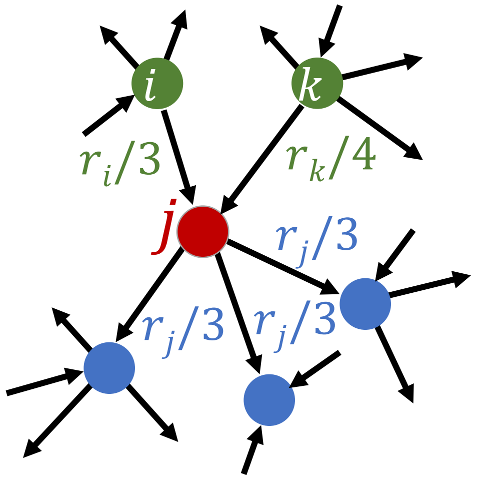
- Where is the surfer at time $t+1$?

$$\boldsymbol{p}(t+1) = \boldsymbol{A} \cdot \boldsymbol{p}(t)$$

- Suppose the random walk reaches a state

$$\boldsymbol{p}(t+1) = \boldsymbol{A} \cdot \boldsymbol{p}(t) = \boldsymbol{p}(t)$$

then $\boldsymbol{p(t)}$ is stationary distribution for the random walk



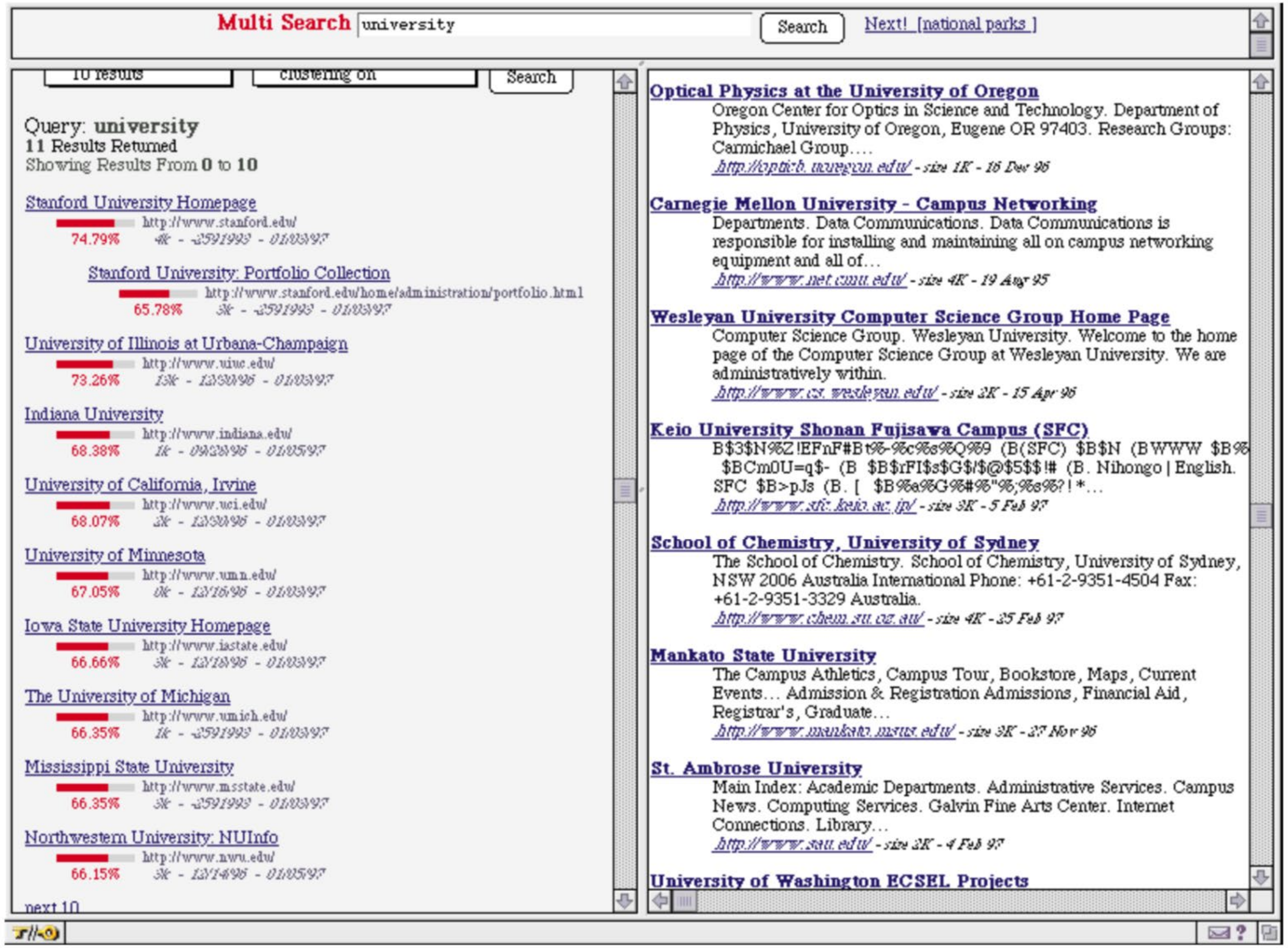
- The PageRank vector $\boldsymbol{r}$ satisfies $\boldsymbol{r} = \boldsymbol{A} \cdot \boldsymbol{r}$
  - So $\boldsymbol{r}$ is a stationary distribution for the random walk

A central result from the theory of random walks (Markov processes):
For graphs that satisfy certain conditions (connected and non-bipartite), the stationary distribution is unique and eventually will be reached no matter what the initial probability distribution is at time $t = 0$

Back to the Broader Story of Ranking

Boolean + PageRank results for the query "*university*" [Page et al., 1999]

- With the rise of the Web, traditional text-based signals (e.g., TF-IDF and BM25) may not be sufficient.
- Many early web search engines relied on classic text-based ranking plus some rudimentary link-based signals.



Back to the Broader Story of Ranking

- In practice, we will build a scoring function that considers many features.
- Typically, we have:
  - Query-dependent features: e.g., TF-IDF, BM25, # of times a query word occurs in a document, …
  - Query-independent features: e.g., PageRank, # of in-links to a webpage, popularity of an album, …
    - Many query-independent features are proxies for "reputation"

- How to jointly consider these features?
  - Week 5

Our Plan: Ranking

- ✅ Why is ranking important?
- ✅ What factors impact ranking?
- Two foundational text-based approaches
  - ✅ TF-IDF
  - ✅ BM25
- Two foundational link-based approaches
  - ✅ PageRank
  - HITS
- Machine-learned ranking ("learning to rank")

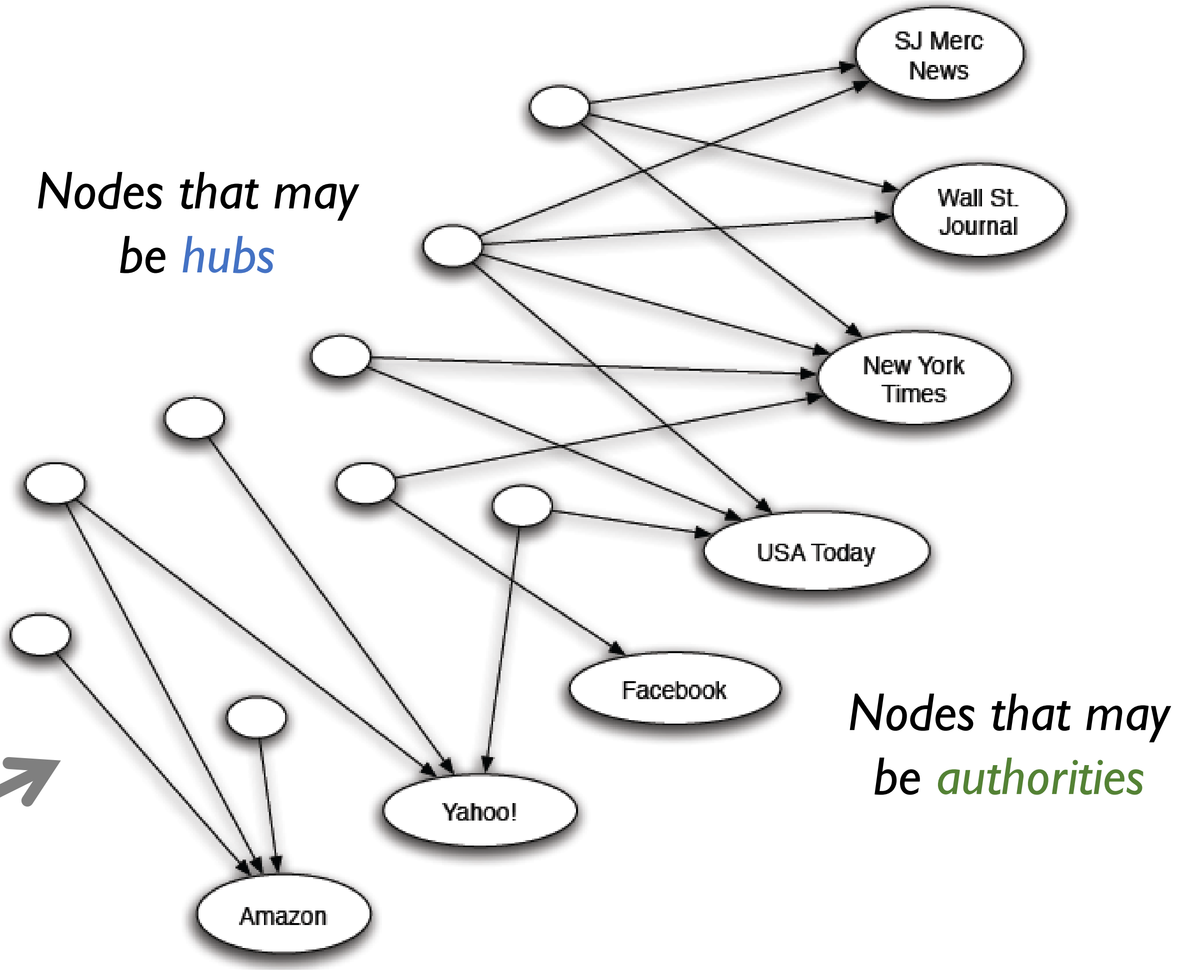
HITS

- HITS (Hypertext-Induced Topic Selection) [Kleinberg, SODA'98]
  - Is a measure of webpage importance, similar to PageRank
  - Proposed at around same time as PageRank
- Goal: Say we want to find good newspapers
  - Don't just find newspapers.
  - Find "experts" – people who link in a coordinated way to good newspapers
- Idea: Links as votes
  - Page is more important if it has more links
  - In-coming links? Out-going links?

Finding Newspapers

- Each page has 2 scores
  - Quality as content (authority)
  - Quality as an expert (hub)
- Interesting pages fall into two classes:
  - Authorities are pages containing useful information
  - Hubs are pages that link to authorities

Note this is idealized example. In practice, the graph is not bipartite, and each page has both hub and authority scores.

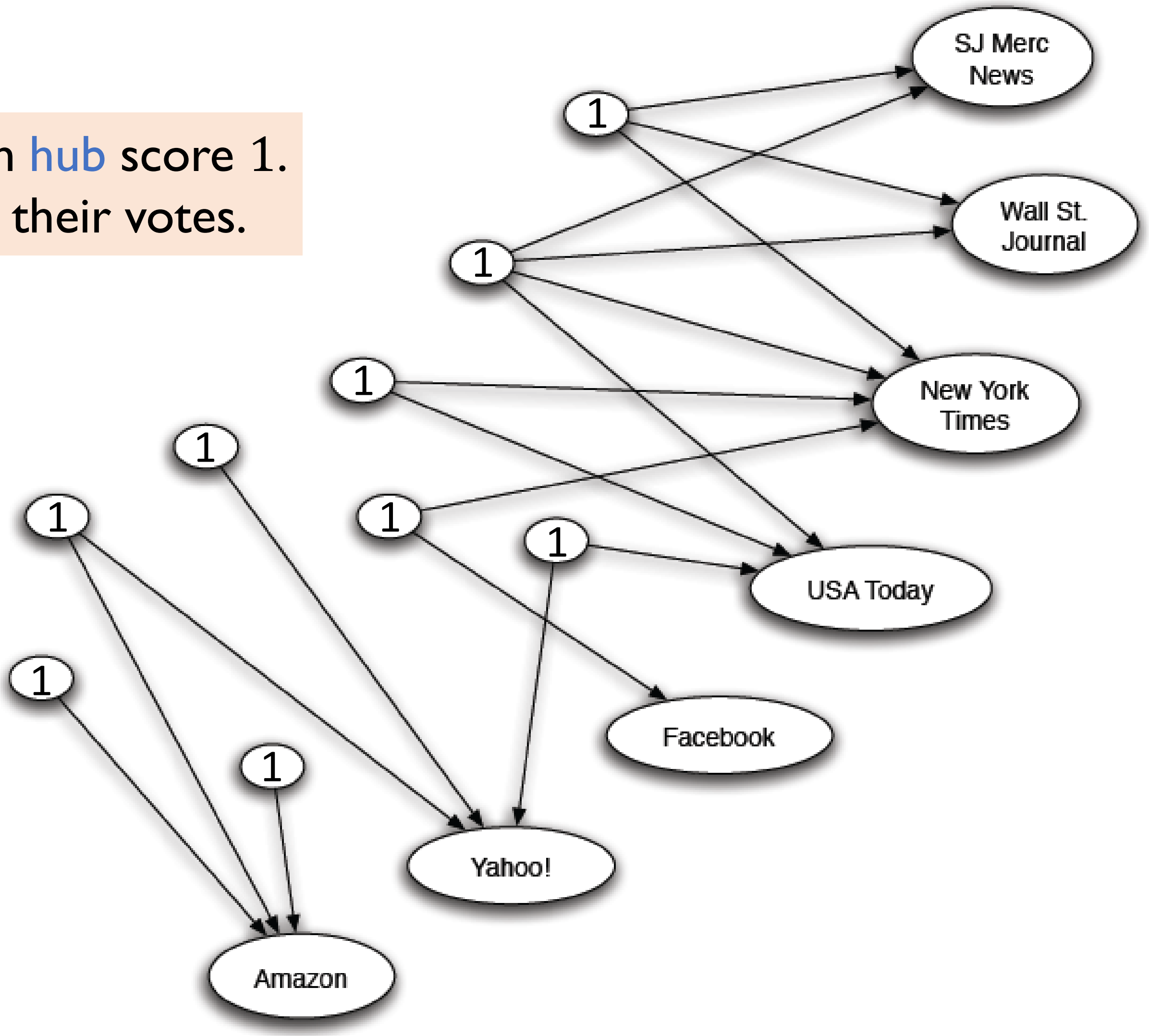


Hubs and Authorities

- Authorities are pages containing useful information
  - Newspaper homepages
  - Course homepages
  - Homepages of auto manufacturers
- Hubs are pages that link to authorities
  - List of newspapers
  - Course bulletin
  - List of US auto manufacturers
- Mutually recursive definition
  - A good hub links to many good authorities
  - A good authority is linked from many good hubs

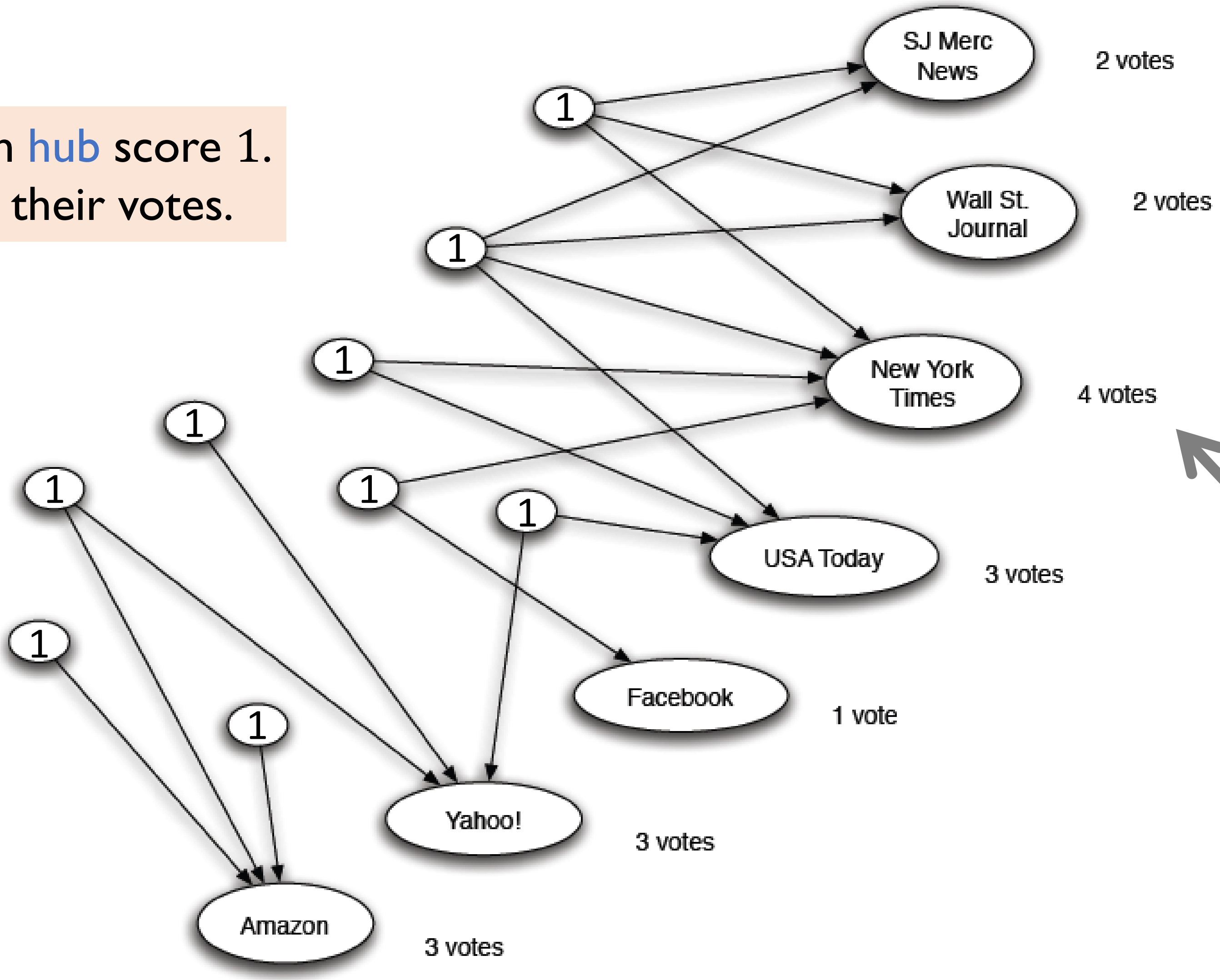
Principle of Repeated Improvement

Each page starts with hub score 1.
Authorities collect their votes.



Principle of Repeated Improvement

Each page starts with hub score 1.
Authorities collect their votes.

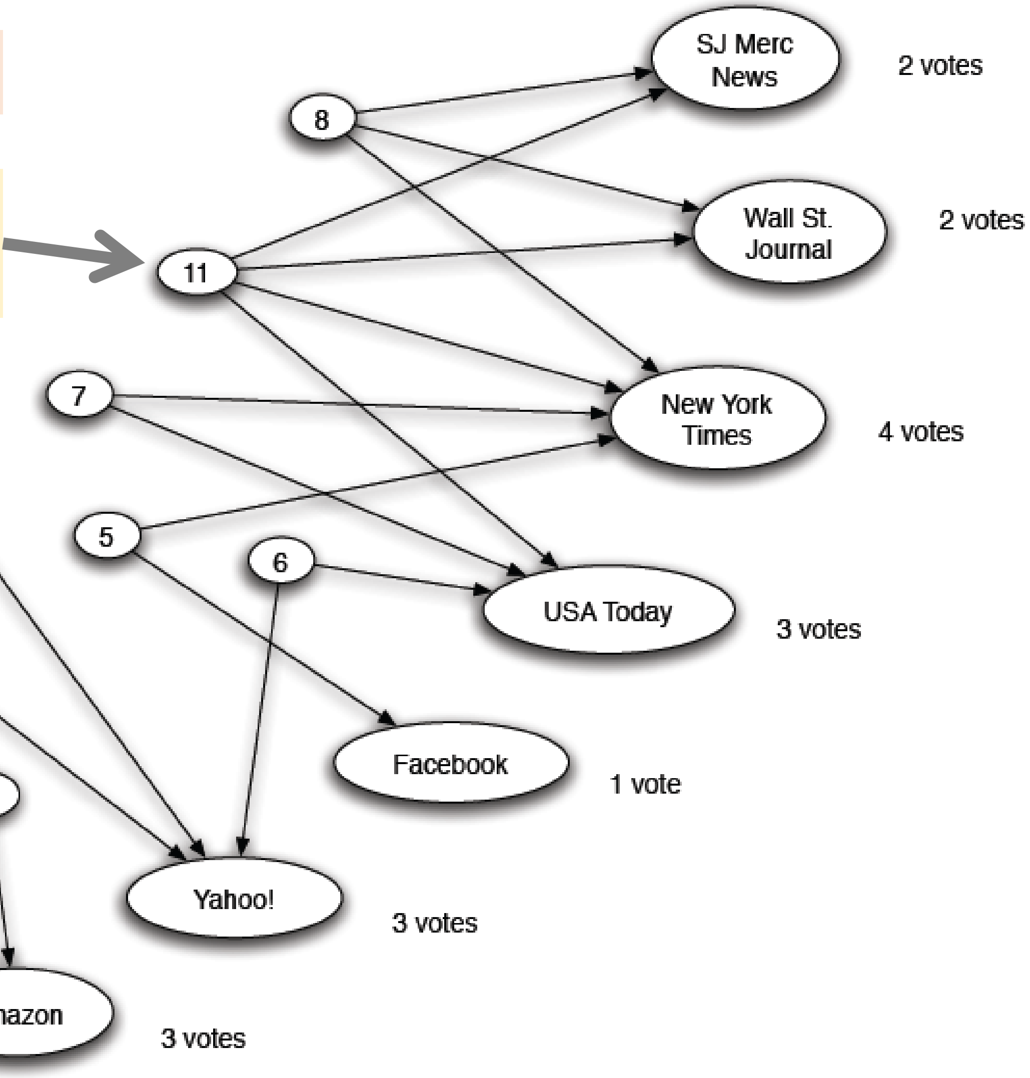


Sum of hub scores of nodes pointing to NYT

Principle of Repeated Improvement

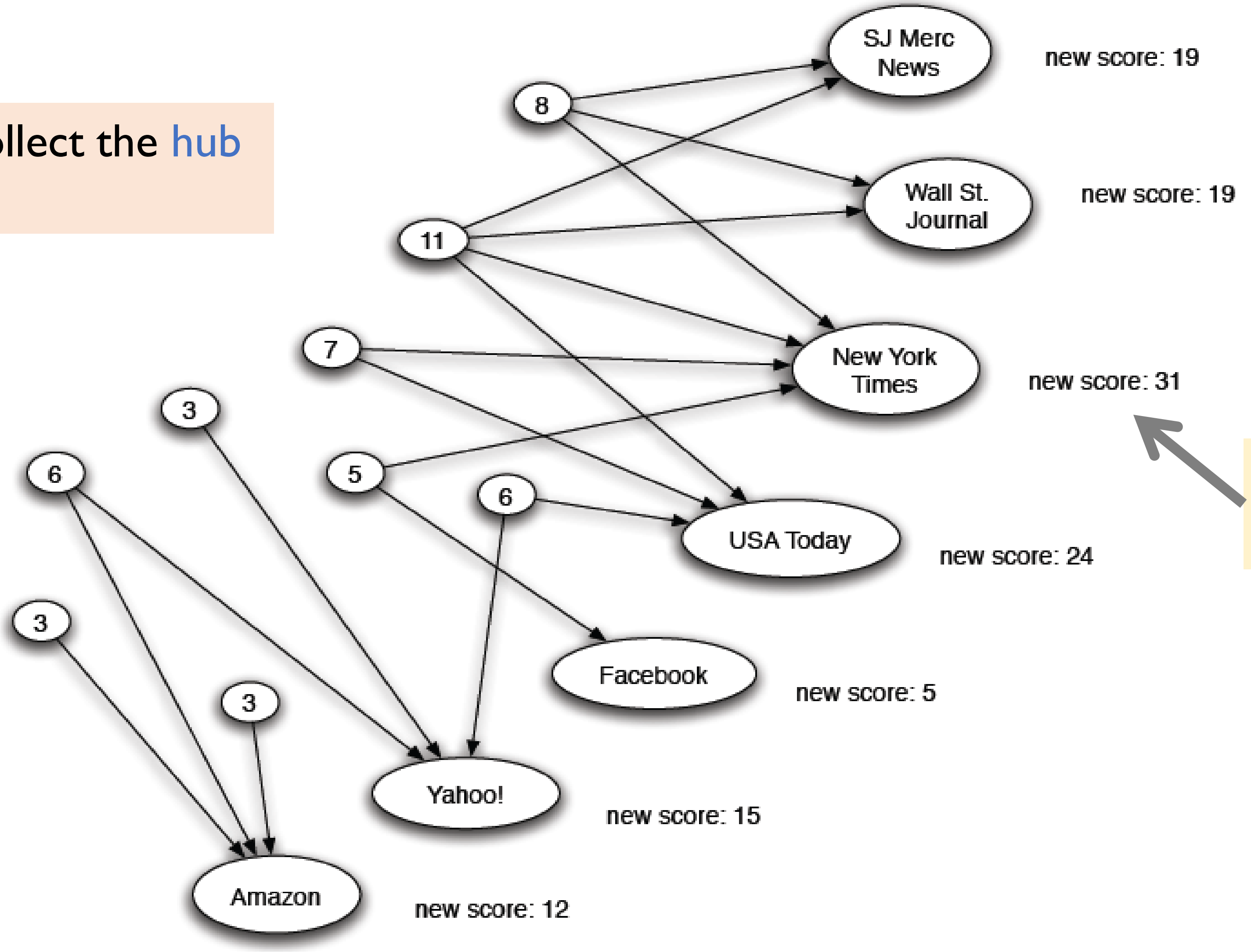
Hubs collect authority scores.

Sum of authority scores of nodes that the node points to.



Principle of Repeated Improvement

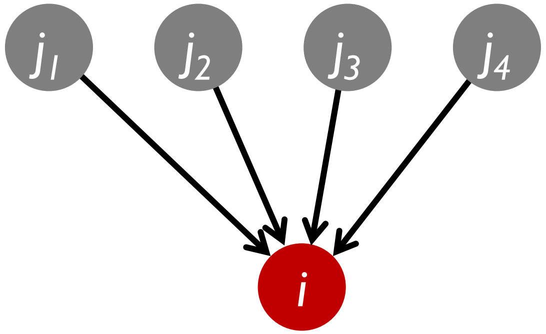
Authorities again collect the hub scores.



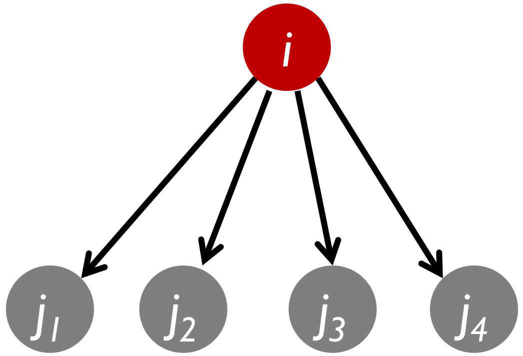
Updated score of NYT

HITS Algorithm: Formal Description

- Each page i has 2 scores:
  - Authority score: a_i
  - Hub score: h_i
- HITS algorithm
  - Initialize: $a_j^{(0)} = 1/\sqrt{N},\ h_j^{(0)} = 1/\sqrt{N}$
  - Then keep iterating until convergence:
    - $\forall i$, update the authority score: $a_i^{(t+1)} = \sum_{j \to i} h_j^{(t)}$
    - $\forall i$, update the hub score: $h_i^{(t+1)} = \sum_{i \to j} a_j^{(t)}$
    - $\forall i$, normalize: $\sum_i \left(a_i^{(t+1)}\right)^2 = 1, \sum_j \left(h_j^{(t+1)}\right)^2 = 1$



$$a_i = \sum_{j \to i} h_j$$



$$h_i = \sum_{i \to j} a_j$$

Matrix Version

- Notation:
  - Vectors $\boldsymbol{a} = \begin{pmatrix} a_1 \\ \cdots \\ a_n \end{pmatrix}$ and $\boldsymbol{h} = \begin{pmatrix} h_1 \\ \cdots \\ h_n \end{pmatrix}$ denote the authority/hub scores of all pages
  - Adjacency matrix $\boldsymbol{A}$, where $A_{ij} = \begin{cases} 1, & \text{if } i \to j \\ 0, & \text{otherwise} \end{cases}$
- Then, $h_i = \sum_{i \to j} a_j$ can be rewritten as $h_i = \sum_j A_{ij} a_j$
  - In other words, $\boldsymbol{h} = \boldsymbol{A}\boldsymbol{a}$
- Similarly, $a_i = \sum_{j \to i} h_j$ can be rewritten as $a_i = \sum_j A_{ji} h_j$
  - In other words, $\boldsymbol{a} = \boldsymbol{A}^T \boldsymbol{h}$

Matrix Version

- $\boldsymbol{h} = \boldsymbol{A}\boldsymbol{a}$
- $\boldsymbol{a} = \boldsymbol{A}^T\boldsymbol{h}$

- If we ignore the normalization step
  - $\boldsymbol{a} = \boldsymbol{A}^T\boldsymbol{h} = \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{a}$
    - Power Iteration with the matrix $\boldsymbol{A}^T\boldsymbol{A}$
  - $\boldsymbol{h} = \boldsymbol{A}\boldsymbol{a} = \boldsymbol{A}\boldsymbol{A}^T\boldsymbol{h}$
    - Power Iteration with the matrix $\boldsymbol{A}\boldsymbol{A}^T$

Recall Power Iteration in PageRank

- Given the adjacency matrix $\boldsymbol{A}$,
  - The authority vector $\boldsymbol{a}$ we are looking for is an eigenvector of $\boldsymbol{A}^T\boldsymbol{A}$
  - The hub vector $\boldsymbol{h}$ we are looking for is an eigenvector of $\boldsymbol{A}\boldsymbol{A}^T$

Extended Content
(will not appear in quizzes or the exam)

Existence and Uniqueness

- Theorem: Under reasonable assumptions about $\boldsymbol{A}$, HITS converges to hub/authority vectors $\boldsymbol{h}^*$ and $\boldsymbol{a}^*$, where
  - $\boldsymbol{h}^*$ is the eigenvector of matrix $\boldsymbol{A}\boldsymbol{A}^T$ corresponding to its largest eigenvalue
  - $\boldsymbol{a}^*$ is the eigenvector of matrix $\boldsymbol{A}^T\boldsymbol{A}$ corresponding to its largest eigenvalue

- Proof (similar to PageRank but easier):
  - Both $\boldsymbol{A}\boldsymbol{A}^T$ and $\boldsymbol{A}^T\boldsymbol{A}$ are real symmetric matrices
    - The eigenvalues of a real symmetric matrix are all real numbers: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$
    - The eigenvectors of a real symmetric matrix are orthogonal to each other and form a basis of the entire vector space: $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$
      - When considering eigenvectors of a real symmetric matrix, we often normalize $\boldsymbol{x}_i$ so that $||\boldsymbol{x}_i||^2 = {\boldsymbol{x}_i}^T \boldsymbol{x}_i = 1$
      - This explains why we use $1/\sqrt{N}$ for initialization and normalize the vectors to unit length after each iteration in HITS

Existence and Uniqueness

- Proof (Cont'd)
- $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N$ form a basis, so we can write $\boldsymbol{h}^{(\mathbf{0})} = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_N\boldsymbol{x}_N$
- $\boldsymbol{AA}^T\boldsymbol{h}^{(\mathbf{0})} = \boldsymbol{AA}^T(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_N\boldsymbol{x}_N)$

$$= c_1\boldsymbol{AA}^T\boldsymbol{x}_1 + c_2\boldsymbol{AA}^T\boldsymbol{x}_2 + \cdots + c_N\boldsymbol{AA}^T\boldsymbol{x}_N$$

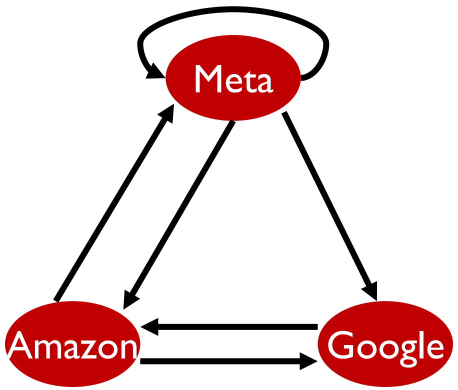
$$= c_1\lambda_1\boldsymbol{x}_1 + c_2\lambda_2\boldsymbol{x}_2 + \cdots + c_N\lambda_N\boldsymbol{x}_N$$

- Repeated multiplication on both sides
- $(\boldsymbol{AA}^T)^k\boldsymbol{h}^{(\mathbf{0})} = c_1\lambda_1^k\boldsymbol{x}_1 + c_2\lambda_2^k\boldsymbol{x}_2 + \cdots + c_N\lambda_N^k\boldsymbol{x}_N$

$$= \lambda_1^k\left(c_1\boldsymbol{x}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\boldsymbol{x}_2 + \cdots + c_N\left(\frac{\lambda_N}{\lambda_1}\right)^k\boldsymbol{x}_N\right)$$

$$\to \lambda_1^k c_1\boldsymbol{x}_1 \qquad \text{(when } k \to \infty\text{, if } \lambda_1 > \lambda_2\text{)}$$

Example



$$\begin{array}{cccc} & \text{Meta} & \text{Amazon} & \text{Google} \\ \boldsymbol{A} = & \multicolumn{3}{c}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \end{array}$$

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Hub	$\boldsymbol{h}^{(0)}$	$\boldsymbol{h}^{(1)}$	$\boldsymbol{h}^{(2)}$	$\boldsymbol{h}^{(3)}$	…	Finally
Meta	0.58	0.80	0.80	0.79	…	0.788
Amazon	0.58	0.53	0.53	0.57	…	0.577
Google	0.58	0.27	0.27	0.23	…	0.211

Authority	$\boldsymbol{a}^{(0)}$	$\boldsymbol{a}^{(1)}$	$\boldsymbol{a}^{(2)}$	$\boldsymbol{a}^{(3)}$	…	Finally
Meta	0.58	0.58	0.62	0.62	…	0.628
Amazon	0.58	0.58	0.49	0.49	…	0.459
Google	0.58	0.58	0.62	0.62	…	0.628

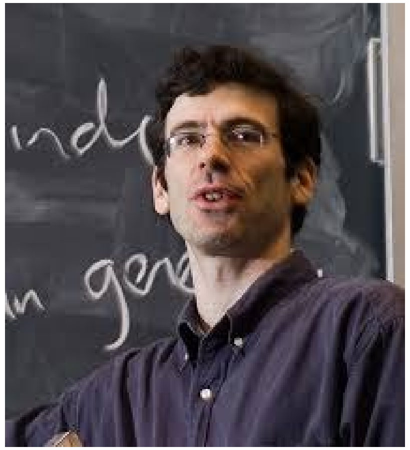
PageRank and HITS

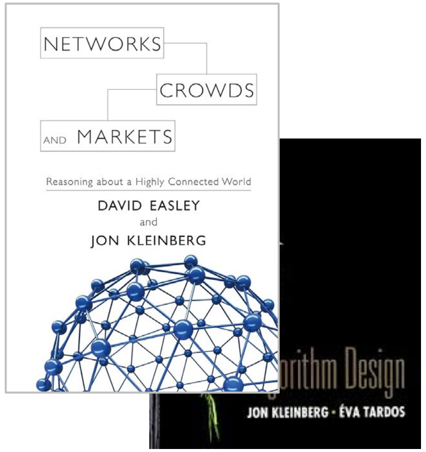
- PageRank and HITS are two solutions to the same problem:
  - How to identify important pages given the hyperlink graph of webpages?
- The destinies of PageRank and HITS after 1998 were very different

Sergey Brin

Larry Page

Co-founders of Google





Jon Kleinberg

Professor at Cornell University
Member of NAS and NAE

Topic-Sensitive PageRank

Topic-Sensitive PageRank (a.k.a., Personalized PageRank)

- PageRank measures generic importance of a page
  - Can we measure page importance within a topic?
- Goal: Evaluate Web pages not just according to their popularity, but by how close they are to a particular topic, e.g., "*sports*" or "*history*"
  - Allow search queries to be answered based on interests of the user
- Idea: Modify the teleportation mechanism
  - Standard PageRank: The random surfer can teleport to any page with equal probability
    - To avoid dead-end and spider-trap problems
  - Topic-Sensitive PageRank: The random surfer can only teleport to a topic-specific set of "relevant" pages

Topic-Sensitive PageRank (a.k.a., Personalized PageRank)

- Topic-Sensitive PageRank: The random surfer can only teleport to a topic-specific set of "relevant" pages (denoted as S)
  - S contains only pages that are relevant to the topic
    - E.g., Open Directory (DMOZ) pages for a given topic/query

Matrix Formulation

- Standard PageRank

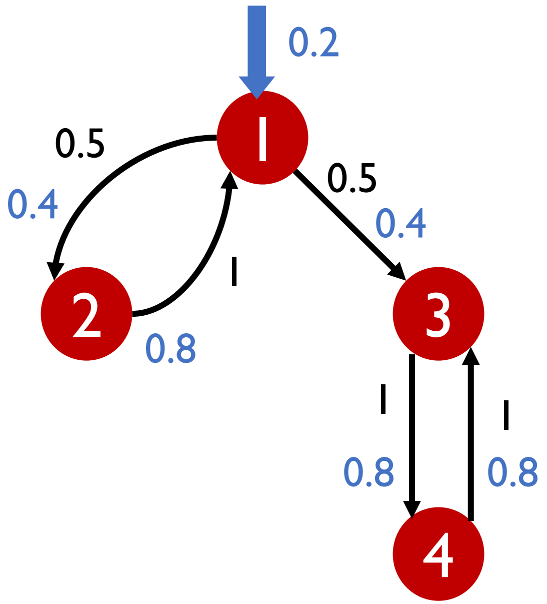
$$A_{ij} = \beta M_{ij} + (1-\beta)\frac{1}{N}, \qquad \forall \text{ pages } i, j$$

- Topic-Sensitive PageRank

$$A_{ij} = \begin{cases} \beta M_{ij} + (1-\beta)\frac{1}{|S|}, & \text{if } i \in S \\ \beta M_{ij}, & \text{otherwise} \end{cases}$$

- We weighted all pages in S equally
  - Could also assign different weights to pages!
- The computation is similar to that of standard PageRank
  - Power Iteration

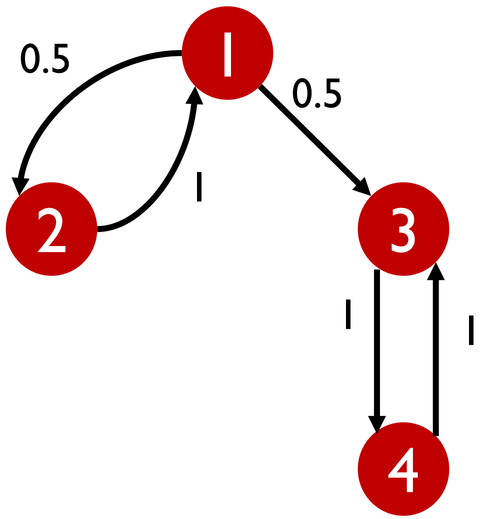
Example



Suppose $S = \{1\}$ and $\beta = 0.8$

	$r^{(0)}$	$r^{(1)}$	$r^{(2)}$	…	Finally
1	0.25	0.40	0.28	…	0.294
2	0.25	0.10	0.16	…	0.118
3	0.25	0.30	0.32	…	0.327
4	0.25	0.20	0.24	…	0.261

Example



$S = \{1\}$
$\beta = 0.9$

Node	Score
1	0.17
2	0.07
3	0.40
4	0.36

$S = \{1\}$
$\beta = 0.8$

Node	Score
1	0.29
2	0.12
3	0.33
4	0.26

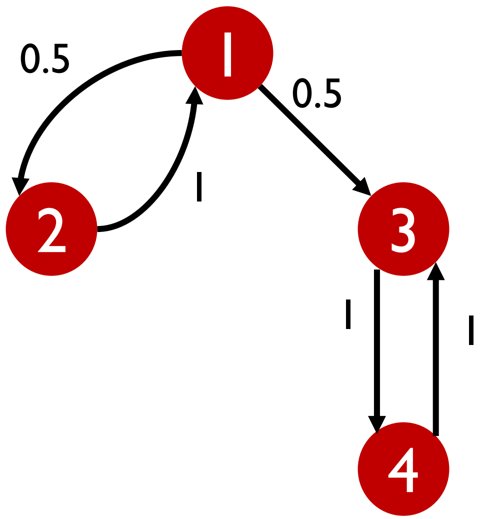
$S = \{1\}$
$\beta = 0.7$

Node	Score
1	0.39
2	0.14
3	0.27
4	0.19

Trend?

- The more you want to emphasize relevance to the topic node set S, the smaller you should set β.
  - A smaller β directs more votes $(1-\beta)$ toward S in each iteration.
  - Drawback: The general importance of each page is also considered less

Example



$S = \{1\}$
$\beta = 0.8$

Node	Score
1	0.29
2	0.12
3	0.33
4	0.26

$S = \{1,2\}$
$\beta = 0.8$

Node	Score
1	0.26
2	0.20
3	0.29
4	0.23

$S = \{1,2,3\}$
$\beta = 0.8$

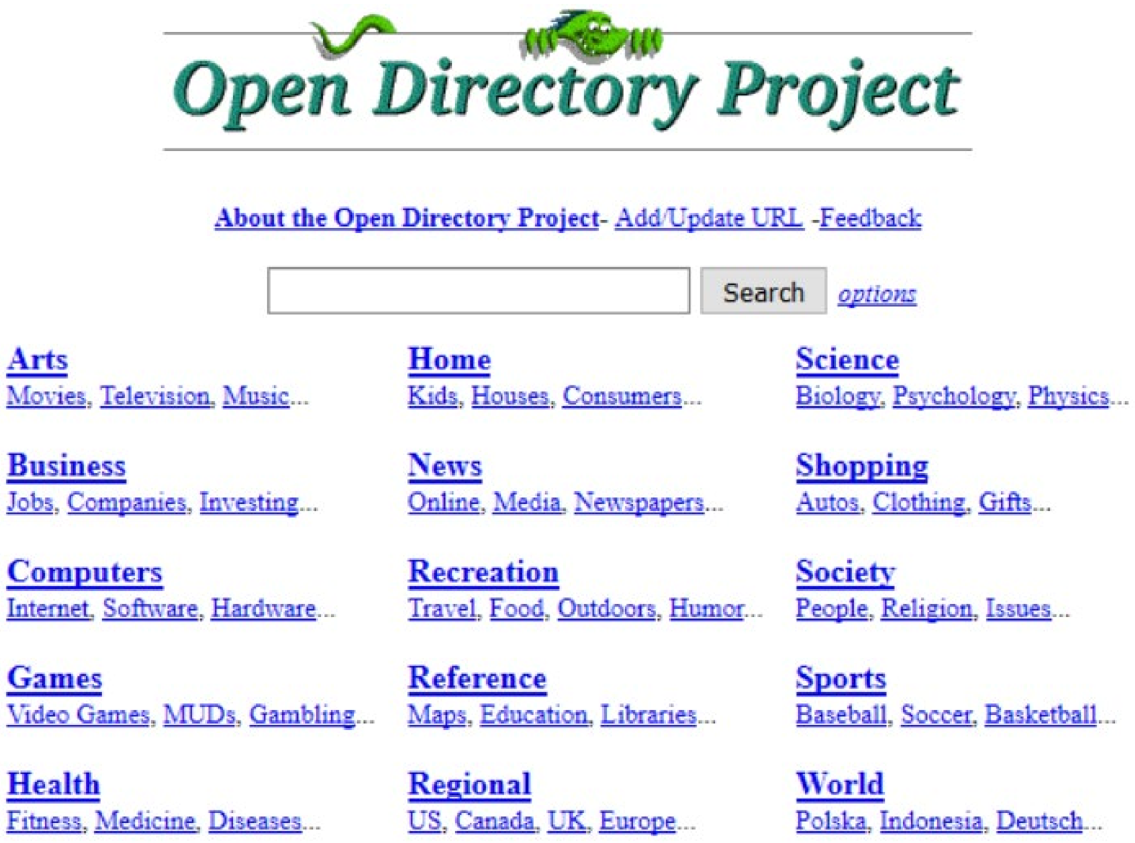
Node	Score
1	0.17
2	0.13
3	0.38
4	0.30

Trend?

- As S covers more nodes, relevance to the topic becomes increasingly less important.
- When S includes all nodes, topic-sensitive PageRank reduces to standard PageRank.

How to get *S*?

- The 15 DMOZ top-level categories:
  - arts, business, sports, …
  - Compute different PageRank scores for different topics
- Which topic ranking to use?
  - Users can pick from a menu
  - Classify the query into a topic
  - Query context, e.g., search history
  - User context, e.g., user's bookmarks



Questions?

Link Spamming

- Once Google became the dominant search engine, spammers began to work out ways to fool Google.
  - Imagine an "evil" user who, after creating his personal homepage, tries to manipulate its PageRank score to make it appear higher in people's search results.

- Spam farms were developed to concentrate PageRank on a single page.

- Link spam: Creating link structures that boost PageRank of a particular page

Link Spamming

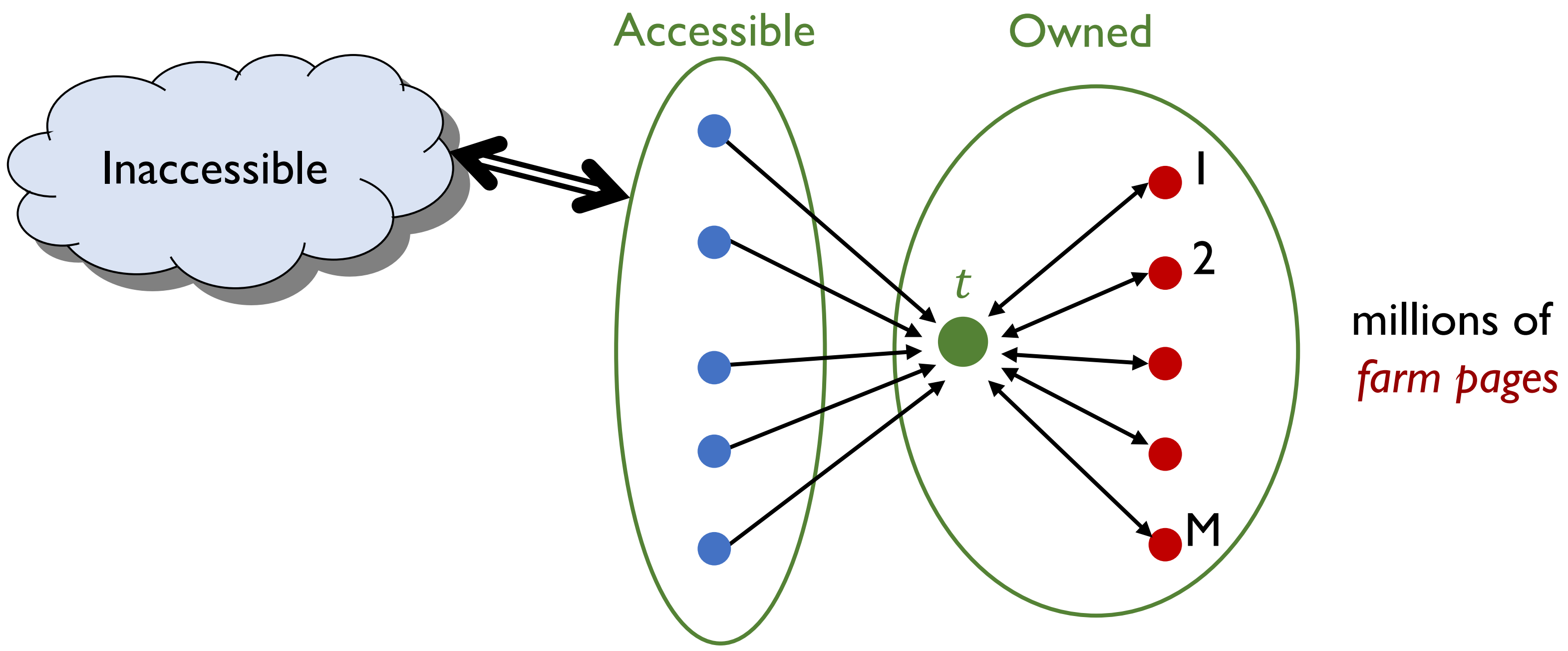
- Three kinds of web pages from a spammer's point of view
  - Inaccessible pages
    - E.g., official homepage of CNN
  - Accessible pages
    - E.g., social media comment pages
    - The spammer can post links to his pages
  - Owned pages
    - Completely controlled by spammer
    - E.g., register several new GitHub accounts, and use each account to create a personal homepage.

Reply: https://XXX.github.io

…

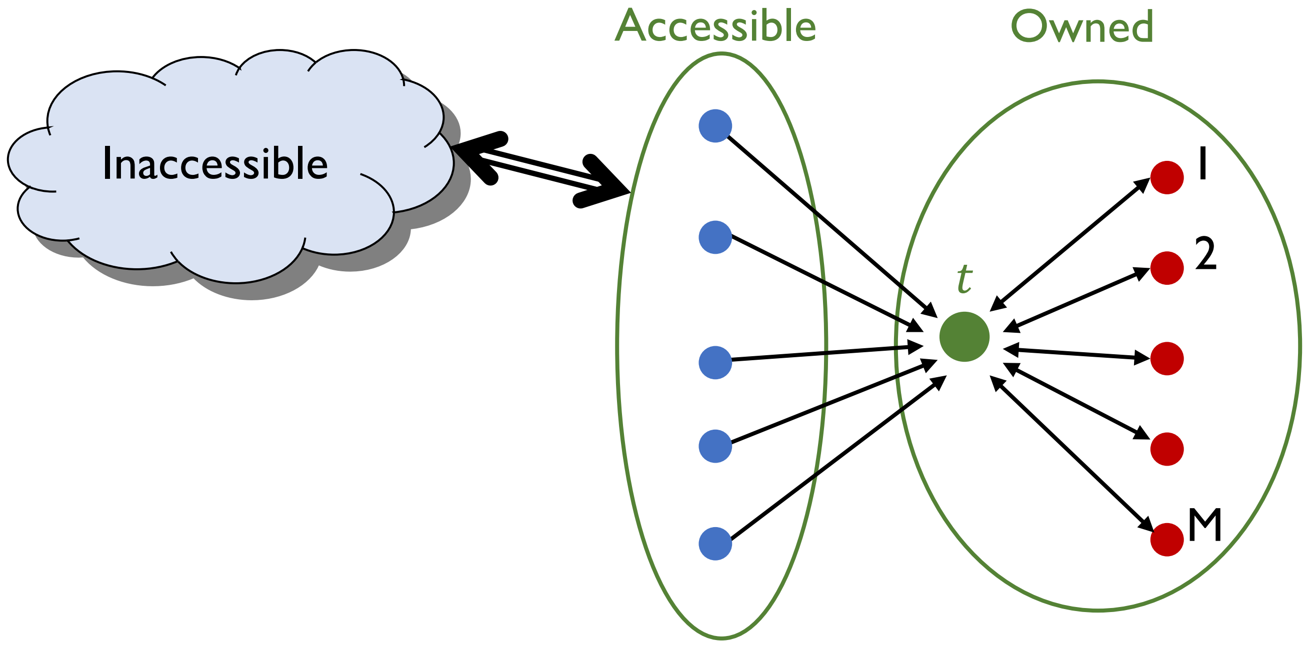
Link Farms

- Spammer's goal: Maximize the PageRank score of a target page t
- Technique:
  - Get as many links from accessible pages as possible to the target page t
  - Construct a "link farm" to get a PageRank multiplier effect



Analysis

- Let x be the PageRank score of the target page t
  - What is the PageRank score of each "farm" page? $\beta\frac{x}{M} + (1-\beta)\frac{1}{N}$
- Let y be the PageRank scores contributed by accessible pages to t
- So $x = y + \beta M\left[\beta\frac{x}{M} + (1-\beta)\frac{1}{N}\right] + (1-\beta)\frac{1}{N}$



millions of *farm pages*

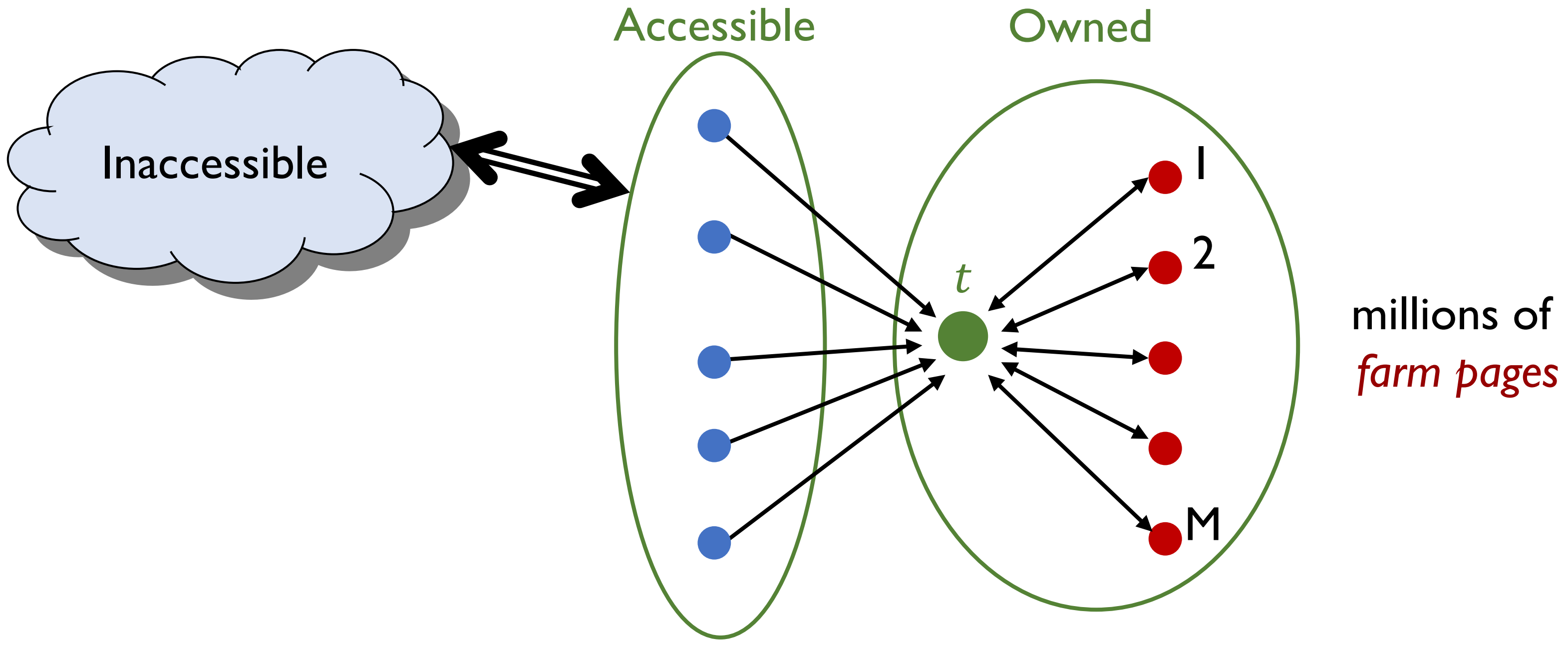
Analysis

- Let x be the PageRank score of the target page t
- $x = y + \beta M \left[\beta \frac{x}{M} + (1-\beta)\frac{1}{N}\right] + (1-\beta)\frac{1}{N}$

$$= y + \beta^2 x + \frac{\beta(1-\beta)M}{N} \boxed{+ (1-\beta)\frac{1}{N}}$$

very small, can be ignored

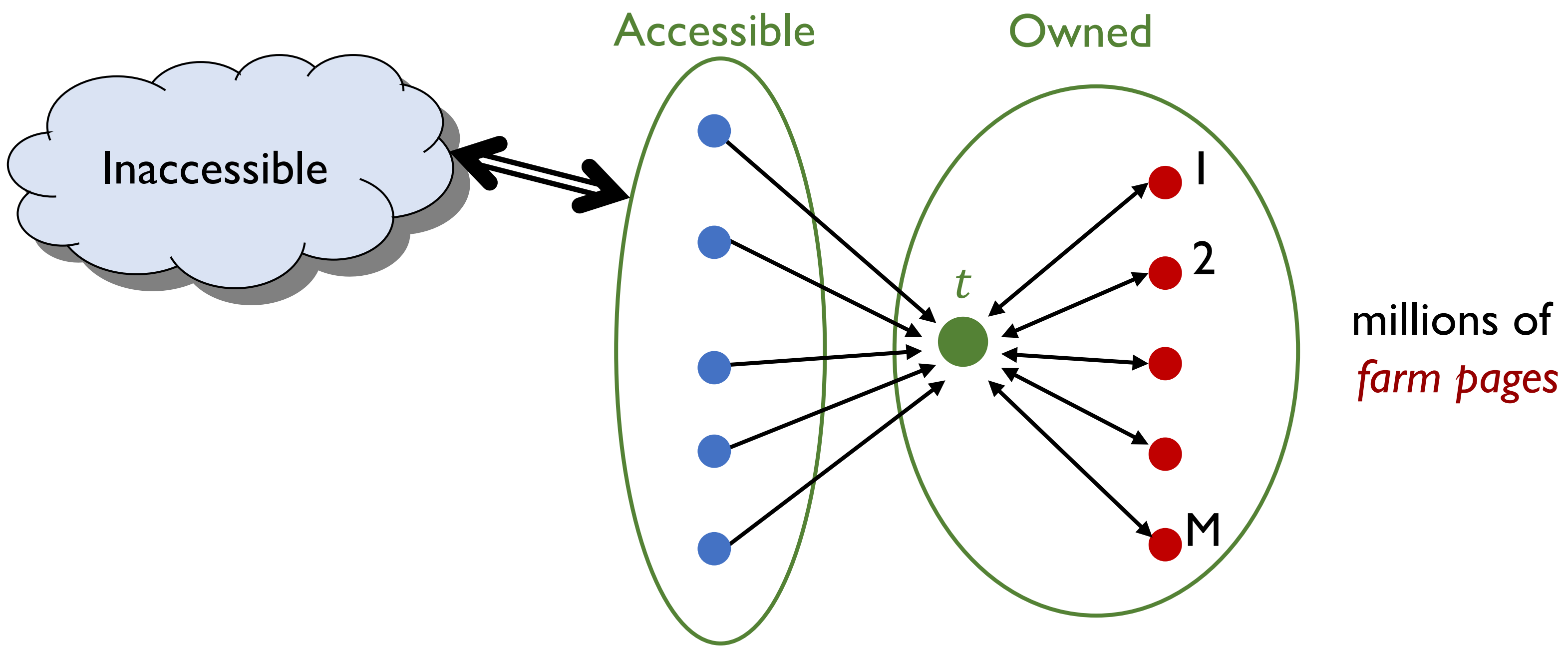
$$\Rightarrow \quad x = \frac{y}{1-\beta^2} + \frac{\beta}{1+\beta}\frac{M}{N}$$



Analysis

$$x = \frac{y}{1-\beta^2} + \frac{\beta}{1+\beta}\frac{M}{N}$$

- If $\beta = 0.8$, then $x = 2.78y + 0.44\frac{M}{N}$
- By making M large, we can make x as large as we want



Extended Content
(will not appear in quizzes or the exam)

How to combat link spamming?

- Naïve Idea: detecting and blacklisting structures that look like spam farms
  - Leads to another war: hiding and detecting spam farms
- More Advanced Idea: Topic-Sensitive PageRank with teleportation to trusted pages
  - Example of trusted pages: *.edu* domains
- Step 1: Sample a set of seed pages from the web
  - Each page can be good (i.e., trusted) or bad (i.e., spam)
- Step 2: Ask humans to identify the good/bad pages in the seed set
  - An expensive task, so we must make seed set as small as possible

How to combat link spamming?

- Step 1: Sample a set of seed pages from the web
- Step 2: Ask humans to identify the good/bad pages in the seed set
- Step 3: Perform Topic-Sensitive PageRank with $S = \{\text{seed pages identified as good}\}$
  - Essentially propagate trust through links
  - Each page gets a trust value between 0 and 1

- Given a webpage, how to judge whether it is spam or not?
- Solution 1: Use a threshold value and mark all pages below the trust threshold as spam
  - Why should this work?
  - Are there cases where this may not work?

Why should Topic-Sensitive PageRank work here?

- Basic principle: Approximate isolation
  - It is rare for a trusted page to point to a spam page
- Trust attenuation: The degree of trust conferred by a trusted page decreases with the distance in the graph
- Trust splitting: The larger the number of out-links from a page, the less scrutiny the page author gives each out-link
  - Trust is split across out-links

How to pick the seed set?

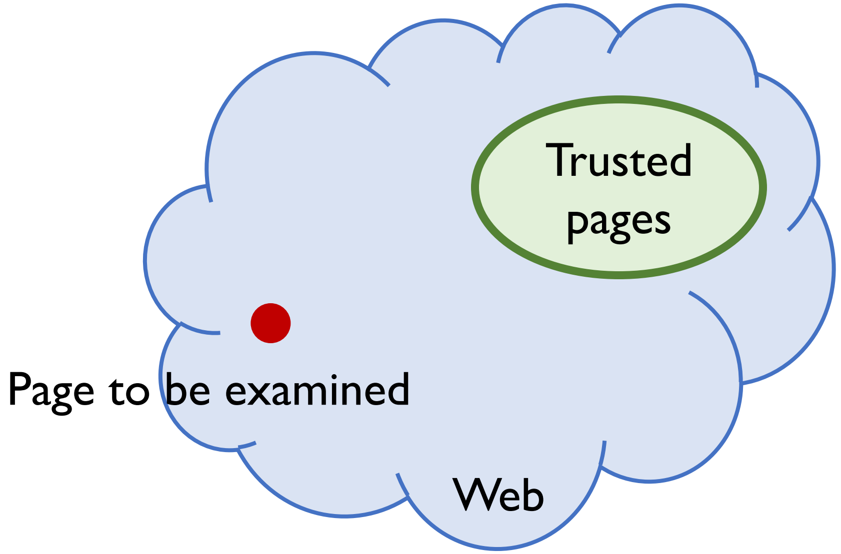
- Two conflicting considerations:
  - Humans have to inspect each seed page, so the seed set must be as small as possible
  - Must ensure every good page gets adequate trust rank, so need make all good pages reachable from seed set by short paths

- How to pick the seed set then?
  - PageRank: Pick the top k pages according to the standard PageRank score. The intuition is that you cannot get a bad page's rank really high
  - Use trusted domains whose membership is controlled, like *.edu*, *.mil*, and *.gov*

Spam Mass

- Solution 1: Use a threshold value and mark all pages below the trust threshold as spam
  - Are there cases where this may not work?
  - When will a node get a low Topic-Sensitive PageRank score?
    - Case 1: It is far away from S (i.e., trusted page)
    - Case 2: It has a low Standard PageRank score
      - This does not imply the node is a spam. Maybe it is just newly created.

- Solution 2: We can calculate what fraction of a page's PageRank comes from spam pages
  - In practice, we do not know all the spam pages, so we need to estimate.

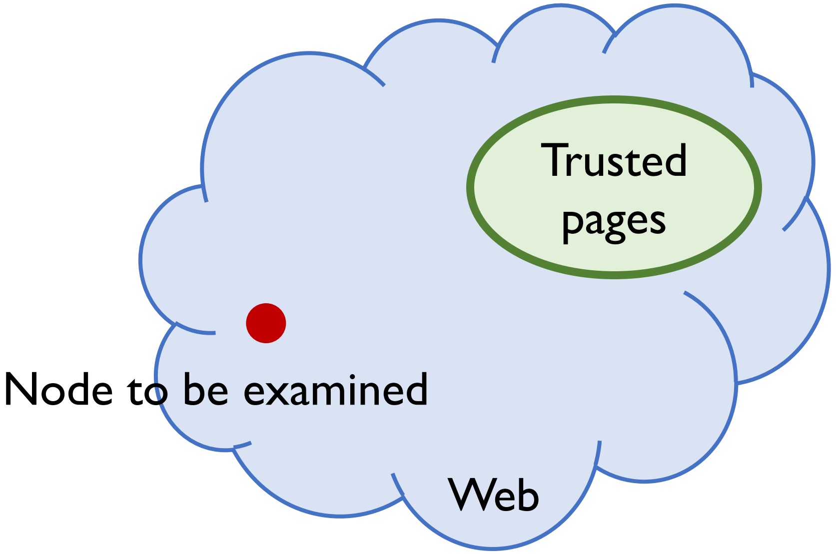


Spam Mass Estimation

- r_p = Standard PageRank score of page p
- r_p^+= Topic-Sensitive PageRank of page p with teleportation into trusted pages only
  - r_p^+ may be small simply because r_p is small. We need to exclude this case.
- What fraction of a page's PageRank comes from spam pages?

$$r_p^- = r_p - r_p^+$$

- Spam mass of p is defined as $\frac{r_p^-}{r_p}$.
- Pages with high spam mass are judged as spam.



Thank You!

Course Website: https://yuzhang-teaching.github.io/CSCE670-S26.html