Introduction to Linear Dynamical Systems

Engineering Examples and State-Space Conversion

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Engineering Examples of Linear Dynamical Systems

Linear dynamical systems arise in many fields of engineering and science:

- Electrical circuits (RLC networks, filters)
- Mechanical systems (mass-spring-damper, suspension)
- Aerospace (aircraft stability, flight control)
- Robotics (joint and manipulator dynamics)
- Communications (filters, channel models)
- Power systems (generator swing dynamics, grid stability)
- Economics (linearized market and supply-demand models)

General State-Space Model

Continuous-Time State-Space Equations:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

Definitions:

- $x(t) \in \mathbb{R}^n$: State vector (internal variables that capture system dynamics)
- $u(t) \in \mathbb{R}^m$: Input vector (external signals that drive the system)
- $y(t) \in \mathbb{R}^p$: Output vector (measured or controlled variables)

The Matrices in State-Space Form

System Matrix $A \in \mathbb{R}^{n \times n}$

- Governs how the state evolves on its own (without input)
- Encodes natural dynamics, eigenvalues, and stability
- Example: In a mass–spring–damper system, A determines natural frequency and damping
- *Intuition:* A tells you "if I leave the system alone, how do states evolve?"

Input Matrix $B \in \mathbb{R}^{n \times m}$

- Maps the external input $u(t)$ into the state equations
- Determines how each input affects each state
- *Intuition:* B tells you "when I push on the system, which states move and how strongly?"

The Matrices in State-Space Form (cont.)

Output Matrix $C \in \mathbb{R}^{p \times n}$

- Maps the internal states to the measurable outputs
- Determines what part of the state vector is visible at the output
- *Intuition:* C tells you "which states am I actually measuring or observing?"

Feedthrough/Direct Matrix $D \in \mathbb{R}^{p \times m}$

- Maps the inputs directly to the outputs (without going through the state)
- $D = 0$ in many physical systems (no instantaneous effect), but not always
- Example: In resistor circuits, input voltage can instantly affect output current
- *Intuition:* D tells you "does the input show up instantly in the output, bypassing states?"

Example: Electrical Circuit (RLC)

System: Series RLC circuit driven by voltage $u(t)$.

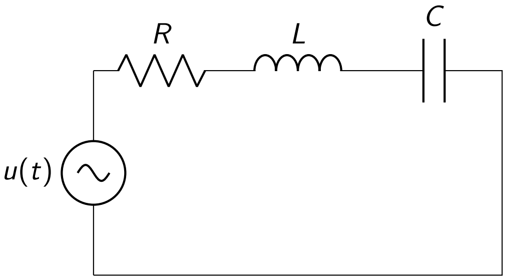
Governing equations:

$$Ri(t) + L\frac{di(t)}{dt} + u_c(t) = u(t)$$

state-space form:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\tfrac{R}{L}x_2 - \tfrac{1}{LC}x_1 + \tfrac{1}{L}u(t),\end{aligned}$$

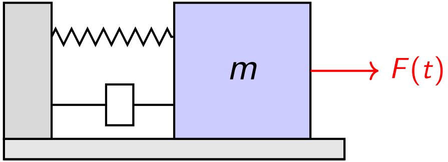
where $x_1 = Cu_c(t)$, $x_2 = i(t)$.



Linear ODE system from KVL/KCL.

Mechanical System: Mass–Spring–Damper

- Block of mass m attached to wall with spring (stiffness k) and damper (damping c).
- The spring (like a rubber band) wants to restore position. The damper (shock absorber) resists motion and kills vibrations, which **improves stability and comfort**
- External force $F(t)$ applied to the block. Displacement from equilibrium: $x(t)$.



Models vibrations and suspension dynamics.

The damper provides a force proportional to velocity that opposes motion:

- **Dissipates energy** $\rightarrow$ prevents endless oscillations.
- **Controls response type:**
  - Underdamped ($c < c_{\text{crit}}$): oscillatory decay.
  - Critically damped ($c = c_{\text{crit}}$): fastest return to equilibrium without oscillation.
  - Overdamped ($c > c_{\text{crit}}$): no oscillation, but sluggish return.

Governing Equation

Newton's Second Law:

$$m\ddot{x}(t) = F_{\text{spring}} + F_{\text{damper}} + F_{\text{external}}.$$

- Spring force: $F_{\text{spring}} = -kx(t)$ (Hooke's Law).
- Damper force: $F_{\text{damper}} = -c\dot{x}(t)$.
- External force: $F_{\text{external}} = F(t)$.

Final governing ODE:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t).$$

Why a Linear Dynamical System?

- The equation is **linear** in $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$.
- It is **dynamical**: the future state depends on displacement, velocity, and the input force.
- Compact representation possible using state-space form.

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

State-Space Representation

Define states:

$$x_1 = x(t), \quad x_2 = \dot{x}(t).$$

Then:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}F(t).$$

state-space matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = [0].$$

Physical Interpretation

- **Matrix** A**:** Intrinsic system dynamics (interaction of displacement and velocity).
- **Matrix** B**:** How external input force $F(t)$ affects the system.
- **Matrix** C**:** Chooses displacement $x(t)$ as output.
- **Matrix** D**:** Zero (input does not appear directly in output).

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

Engineering Applications

The mass–spring–damper model underlies many real systems:

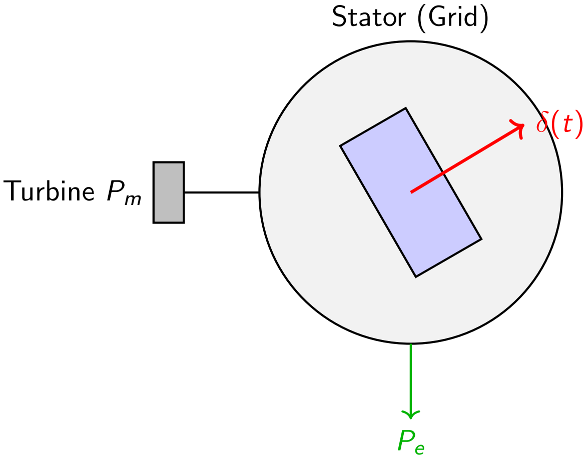
- **Automotive suspensions:** ride comfort and vibration isolation.
- **Civil engineering:** building dynamics under wind or earthquake loads.
- **Robotics/aerospace:** vibration damping in structures and arms.
- **Control theory:** canonical test system for PID and state-feedback design.

Power Grid Example: Swing Equation (Generator Rotor Dynamics)

- Rotor angle $\delta(t)$ is measured relative to the synchronous reference.
- First derivative: $\dot{\delta}(t) = \Delta\omega(t)$ is speed deviation.
- Second derivative follows from the **Swing Equation**:

$$M\,\ddot{\delta}(t) + D\,\dot{\delta}(t) \;=\; P_m(t) - P_e(\delta(t)),$$

  where:
  - M inertia, resists acceleration.
  - D damping, resists oscillations.
  - P_m mechanical input power, P_e electrical output power.



Operating Point and Small Deviations

- Choose an operating point $(\delta_0,\ \dot{\delta} = 0)$ with power balance

$$P_{m0} \;=\; P_e(\delta_0).$$

- Define small deviations:

$$\Delta\delta = \delta - \delta_0, \quad \Delta\omega = \dot{\delta}, \quad \Delta P_m = P_m - P_{m0}.$$

- Linearize electrical power around δ_0:

$$P_e(\delta) \;\approx\; P_e(\delta_0) \;+\; K_s\,\Delta\delta, \qquad K_s \;=\; \left.\frac{dP_e}{d\delta}\right|_{\delta_0}.$$

Linearization around the Operating Point

Start from

$$M\,\ddot{\delta} + D\,\dot{\delta} \;=\; P_m - P_e(\delta).$$

Subtract the equilibrium and use the linearization:

$$M\,\Delta\ddot{\delta} + D\,\Delta\dot{\delta} \;=\; \Delta P_m \;-\; K_s\,\Delta\delta.$$

State-space form with $x = \begin{bmatrix} \Delta\delta \\ \Delta\omega \end{bmatrix}$, $\Delta\omega = \Delta\dot{\delta}$, $u = \Delta P_m$:

$$\begin{aligned} \Delta\dot{\delta} &= \Delta\omega, \\ M\,\Delta\dot{\omega} &= \Delta P_m - K_s\,\Delta\delta - D\,\Delta\omega, \end{aligned} \qquad \Longrightarrow \qquad \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K_s}{M} & -\frac{D}{M} \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{B} u.$$

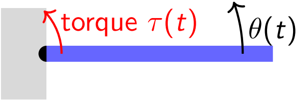
Robotics Example: Single-Joint Position Control

Joint Dynamics:

$$J\ddot{\theta}(t) + b\dot{\theta}(t) = \tau(t).$$

State-space:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{J} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} \tau(t), \qquad x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$



Interpretation:

- Models one rotary robotic joint.
- J: inertia $\rightarrow$ resists acceleration.
- b: damping $\rightarrow$ frictional losses.
- τ: motor torque (control input).

From ODE to State-Space (1/2)

General idea: Any higher-order ODE can be rewritten as a first-order system.

General nth-order ODE:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = b_0 u(t).$$

Step 1. Define the states

$$x_1(t) = y(t), \quad x_2(t) = \dot{y}(t), \quad \ldots, \quad x_n(t) = y^{(n-1)}(t).$$

Step 2. Write the state equations

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = x_3(t), \quad \ldots, \quad \dot{x}_{n-1}(t) = x_n(t),$$

$$\dot{x}_n(t) = -\frac{a_0}{a_n} x_1(t) - \frac{a_1}{a_n} x_2(t) - \cdots - \frac{a_{n-1}}{a_n} x_n(t) + \frac{b_0}{a_n} u(t).$$

Step 3. Express in matrix form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t),$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{b_0}{a_n} \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad D = [0].$$

This representation is the *companion form* (or canonical form), which is useful for analysis (stability, modes) and controller design. This works for any linear constant-coefficient ODE.

Example: 2nd-Order ODE to state-space

Mass–spring–damper ODE:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t).$$

Define states:

$$x_1 = x(t), \quad x_2 = \dot{x}(t).$$

State equations:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}F(t).$$

Matrix form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Example: 3rd-Order ODE to state-space

Consider:

$$y^{(3)} + a_2 y^{(2)} + a_1 y^{(1)} + a_0 y = b_0 u(t).$$

Define states:

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}.$$

State equations:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = -a_0 x_1 - a_1 x_2 - a_2 x_3 + b_0 u.$$

Matrix form:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x.$$

Linearity in ODEs

Definition

A system is called *linear* if its equations are linear in the unknown variables and their derivatives. This means the **superposition principle** holds.

High-order Linear ODEs

An nth-order ODE of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y(t) = b_0 u(t) + b_1 \frac{du}{dt} + \cdots + b_m \frac{d^m u}{dt^m}$$

is linear because

- $y(t)$ and its derivatives appear only to the first power.
- No products like y^2, $y\dot{y}$, or $\sin(y)$.
- Coefficients $\{a_i, b_i\}$ are constants (or functions of t, but not of y).

Linearity in State-Space Models

State-Space Models

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

This is linear because both equations are linear functions of $x(t)$ and $u(t)$. There are no products (e.g., $x_1 x_2$) or nonlinear functions (e.g., $\sin(x)$).

Superposition Principle

If $(x_1(t), y_1(t))$ is the solution for input $u_1(t)$, and $(x_2(t), y_2(t))$ is the solution for input $u_2(t)$, then for any scalars α, β:

$$u(t) = \alpha u_1(t) + \beta u_2(t)$$

produces

$$x(t) = \alpha x_1(t) + \beta x_2(t), \qquad y(t) = \alpha y_1(t) + \beta y_2(t).$$