

ECE240 Introduction to Linear Dynamical Systems

Lecture 11: Observability, Controllability, and Reachability

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- 2 Controllability and Reachability
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Linear Dynamical System (LDS) Model

Consider a discrete-time LDS:

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ state; $\mathbf{u}(t) \in \mathbb{R}^m$ input; and $\mathbf{y}(t) \in \mathbb{R}^p$ output.

system matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, input matrix $\mathbf{B} \in \mathbb{R}^{n \times m}$:

output matrix $\mathbf{C} \in \mathbb{R}^{p \times n}$, feedthrough matrix $\mathbf{D} \in \mathbb{R}^{p \times m}$.

High-Level Overview of Key Concepts

We will explore:

- **Observability:** whether the internal state can be uniquely inferred from output measurements.
- **Detectability:** all unobservable modes are stable, so the system's unstable behavior is still detectable. In other words, even if some states cannot be estimated, do all unobservable states naturally decay so the system still behaves well?
- **Controllability:** whether any state can be reached from any initial condition using appropriate inputs.
- **Reachability:** the set of states that can be reached from the origin with finite inputs.
- **Stabilizability:** all uncontrollable modes are stable, so the system can still be stabilized with feedback.
- **Duality:** observability of (\mathbf{A}, \mathbf{C}) mirrors controllability of $(\mathbf{A}^T, \mathbf{B}^T)$.

State Estimation Problem

Understanding **state estimation** helps motivate all the aforementioned key concepts.

Goal: estimate the state $\hat{\mathbf{x}}(s \mid t - 1)$ using past data $\mathbf{u}_{0:t-1}$ and $\mathbf{y}_{0:t-1}$.

- $s = 0$: estimate the *initial state* (possible only if the system is observable).
- $s = t - 1$: estimate the *current state* (central task of filtering and observers).
- $s = t$: predict the *next state* (needed for control, forecasting, and model-based decisions).

Observability Matrix Derivation

Stacked outputs:

$$\begin{bmatrix} \mathbf{y}(0) \\ \vdots \\ \mathbf{y}(t-1) \end{bmatrix} = \mathbf{O}_t \mathbf{x}(0) + \mathbf{T}_t \begin{bmatrix} \mathbf{u}(0) \\ \vdots \\ \mathbf{u}(t-1) \end{bmatrix}$$

where

$$\mathbf{O}_t = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{t-1} \end{bmatrix}, \quad \mathbf{T}_t = \begin{bmatrix} \mathbf{D} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{CB} & \mathbf{D} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{CAB} & \mathbf{CB} & \mathbf{D} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{CA}^{t-2}\mathbf{B} & \mathbf{CA}^{t-3}\mathbf{B} & \dots & \mathbf{CB} & \mathbf{D} \end{bmatrix}.$$

- \mathbf{O}_t captures how the initial state affects the output sequence.
- \mathbf{T}_t captures how the known input sequence affects the output sequence.

Original Definition of Observability

Consider the LTI system:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k, \quad \mathbf{y}_k = \mathbf{C}\mathbf{x}_k.$$

Definition

The system is **observable** if the initial state \mathbf{x}_0 can be *uniquely determined* from the outputs $\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{n-1}$.

Equivalently:

$$\mathbf{y}_k = \mathbf{0}, \forall k = 0, \dots, n-1 \implies \mathbf{x}_0 = \mathbf{0}.$$

Intuition: What Does Observability Mean?

Each initial state \mathbf{x}_0 generates a unique output sequence:

$$\mathbf{y}_0 = \mathbf{C}\mathbf{x}_0, \quad \mathbf{y}_1 = \mathbf{C}\mathbf{A}\mathbf{x}_0, \quad \mathbf{y}_2 = \mathbf{C}\mathbf{A}^2\mathbf{x}_0, \dots$$

Intuition

A system is observable if every nonzero state leaves a unique “fingerprint” in the outputs. Unobservable states lie in a hidden subspace that never affects \mathbf{y}_k .

Thus, observability asks:

$$\mathbf{C}\mathbf{A}^k\mathbf{x}_0 = \mathbf{0}, \quad k = 0, \dots, n-1 \quad \implies \quad \mathbf{x}_0 = \mathbf{0}.$$

From Outputs to the Observability Matrix

Stack the first n outputs:

$$\begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \mathbf{x}_0 = \mathbf{O} \mathbf{x}_0.$$

Thus reconstructing \mathbf{x}_0 from output data is equivalent to solving

$$\mathbf{O} \mathbf{x}_0 = \mathbf{y}_{0:n-1}.$$

Key Idea

The map $\mathbf{x}_0 \mapsto \mathbf{y}_{0:n-1}$ is one-to-one iff $\text{null}(\mathbf{O}) = \{\mathbf{0}\}$.

Unobservable Subspace

If the nullspace of the observability matrix is nontrivial; i.e., $\mathbf{x}(0) \in \text{null}(\mathbf{O}) \neq \{\mathbf{0}\}$, then

$$\mathbf{y}(0) = \mathbf{y}(1) = \cdots = \mathbf{y}(n-1) = \mathbf{0}.$$

Interpretation:

- State components in $\text{null}(\mathbf{O})$ never affect output.
- They cannot be estimated by any measurements.

Observability and rank(\mathbf{O})

Observability Condition

The system is observable $\iff \text{null}(\mathbf{O}) = \{\mathbf{0}\} \iff \text{rank}(\mathbf{O}) = n.$

- If $\text{rank}(\mathbf{O}) = n$: no hidden directions; every state affects the output.
- If $\text{rank}(\mathbf{O}) < n$: a nontrivial nullspace exists; some state components never appear in \mathbf{y}_k .

Interpretation

Full column rank means every state dimension produces a detectable signature in the output over n steps.

Observability for Large t

By Cayley–Hamilton: $\text{null}(\mathbf{O}_t) = \text{null}(\mathbf{O})$, $\forall t \geq n$,

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \in \mathbb{R}^{(np) \times n}.$$

Observability Condition

System observable iff $\text{rank}(\mathbf{O}) = n$. That is, observability matrix \mathbf{O} is full column rank.

Numeric Example: System is observable

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = [1 \ 0] \implies \mathbf{O} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{rank}(\mathbf{O}) = 2.$$

Physical Example: Inverted Pendulum Observability (1/2)

Consider the linearized inverted pendulum on a cart (about upright position).

State: $\mathbf{x} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, output measured: θ .

Linearized dynamics (small angle approximation):

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Observability matrix:

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{rank}(\mathbf{O}) = 2.$$

Physical Example: Inverted Pendulum Observability (2/2)

Interpretation

Even though we measure only θ , the angular velocity $\dot{\theta}$ becomes observable because:

- The dynamics couple θ and $\dot{\theta}$ ($\dot{\theta} = \mathbf{A}\mathbf{x}$ contains $\dot{\theta}$);
- Future measurements of $\theta(t)$ implicitly contain information about $\dot{\theta}(t)$.

Hence the full state $(\theta, \dot{\theta})$ is observable.

Why $\text{null}(\mathbf{O}_t)$ Stops Changing for $t \geq n$ (1/2)

Cayley–Hamilton Theorem: $\mathbf{A}^n = c_{n-1}\mathbf{A}^{n-1} + \cdots + c_1\mathbf{A} + c_0\mathbf{I}$. Multiply by \mathbf{C} :

$$\mathbf{CA}^n = c_{n-1}\mathbf{CA}^{n-1} + \cdots + c_1\mathbf{CA} + c_0\mathbf{C}.$$

Key Consequence

Every block row \mathbf{CA}^k for $k \geq n$ is a linear combination of $\{\mathbf{C}, \mathbf{CA}, \dots, \mathbf{CA}^{n-1}\}$.

Thus, adding more rows to

$$\mathbf{O}_t = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{t-1} \end{bmatrix} \quad \text{for } t \geq n$$

does *not* introduce new independent directions.

Why $\text{null}(\mathbf{O}_t)$ Stops Changing for $t \geq n$ (2/2)

Nullspace Stabilization

All additional rows for $t \geq n$ are dependent, so

$$\text{null}(\mathbf{O}_t) = \text{null}(\mathbf{O}), \quad \mathbf{O} := \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}.$$

Hence the system is observable iff $\text{rank}(\mathbf{O}) = n$.

Detectability Definition

A system is detectable if all unobservable modes are asymptotically stable.

Detectability

A system is detectable if every eigenvalue of \mathbf{A} with $\text{Re}(\lambda) \geq 0$ (continuous-time) or $|\lambda| \geq 1$ (discrete-time) is observable.

Meaning:

- Unobservable states may exist but must converge to zero.
- Needed for observer design.

Detectable but not Observable: Two Examples (1/2)

Example 1

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

- Second state is unobservable ($\text{rank}(\mathbf{O}) = 1 < 2$).
- Its eigenvalue 0.9 is stable.

Thus, the system is **detectable but not observable**.

Detectable but not Observable: Two Examples (2/2)

Example 2

$$\mathbf{A} = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

- $\text{rank}(\mathbf{O}) = 1 < 2$, so the system is not observable.
- The unobservable mode corresponds to eigenvalue 0.5, which is stable.
- The unstable eigenvalue 1.1 belongs to an observable state.

Thus, the system is **detectable but not observable**.

Controllability Matrix \mathbf{C}_t Derivation

Consider a discrete-time LTI system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, starting from $\mathbf{x}(0) = \mathbf{0}$.

The state at time t is

$$\mathbf{x}(t) = \mathbf{A}^{t-1}\mathbf{B}\mathbf{u}(0) + \mathbf{A}^{t-2}\mathbf{B}\mathbf{u}(1) + \cdots + \mathbf{A}\mathbf{B}\mathbf{u}(t-2) + \mathbf{B}\mathbf{u}(t-1).$$

Stack the inputs into one vector:

$$\mathbf{u}_{0:t-1} = \begin{bmatrix} \mathbf{u}(t-1) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}, \quad \mathbf{x}(t) = \mathbf{C}_t \mathbf{u}_{0:t-1},$$

where the finite-time controllability matrix is defined as

$$\mathbf{C}_t = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{t-1}\mathbf{B} \end{bmatrix} \in \mathbb{R}^{n \times mt}.$$

Why Talk About the Reachable Set?

Goal: Understand which states can be reached by choosing inputs.

Controllability vs. Observability

- **Observability:** Which states can we infer from outputs?
- **Controllability:** Which states can we reach using inputs?

To study controllability, we must describe the set of states that are reachable from $\mathbf{x}(0) = \mathbf{0}$:

$$\mathcal{R}_t = \{\mathbf{x}(t) \mid \mathbf{x}(0) = \mathbf{0}, \mathbf{u}(0), \dots, \mathbf{u}(t-1) \text{ arbitrary}\}.$$

Reachable Set at Time t

We showed that $\mathbf{x}(t) = \mathbf{C}_t \mathbf{u}_{0:t-1}$.

Reachable Set at Time t

All states that can be reached at time t from $\mathbf{x}(0) = \mathbf{0}$ are $\mathcal{R}_t = \text{range}(\mathbf{C}_t)$.

Interpretation: \mathcal{R}_t is the subspace spanned by the columns of $\mathbf{B}, \mathbf{AB}, \dots, \mathbf{A}^{t-1}\mathbf{B}$. Each column direction represents how an input at some time affects the state.

By Cayley–Hamilton, for $k \geq n$ we have

$$\mathbf{A}^k \mathbf{B} = \text{linear combination of } \mathbf{B}, \mathbf{AB}, \dots, \mathbf{A}^{n-1} \mathbf{B}.$$

Thus, for $t \geq n$:

$$\text{range}(\mathbf{C}_t) = \text{range}(\mathbf{C}_{\text{ctr}}), \quad \mathbf{C}_{\text{ctr}} := \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1} \mathbf{B} \end{bmatrix}.$$

Controllability Condition

Reachable Subspace and Controllability

$$\mathcal{R}_t = \mathcal{R} := \text{range}(\mathbf{C}_{\text{ctr}}), \quad t \geq n.$$

The system is **controllable** iff

$$\mathcal{R} = \mathbb{R}^n \iff \text{rank}(\mathbf{C}_{\text{ctr}}) = n.$$

Full rank means every state direction is reachable by some input sequence.

Interpretation:

- Inputs excite all modes.
- Entire state space reachable.

Numeric Example: Not Controllable

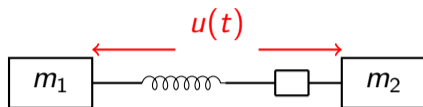
Given

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies \mathbf{C}_{\text{ctr}} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{rank}(\mathbf{C}_{\text{ctr}}) = 1 < 2.$$

Interpretation:

- The system is not controllable.
- $\mathbf{AB} = \mathbf{B}$, so both columns of \mathbf{C}_{ctr} are identical.
- Reachable states lie in $\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, the line $x_1 = x_2$.
- Input affects both states equally, so the difference $x_1 - x_2$ cannot be changed.

Physical Example: Two Masses (Internal Actuation)



Two masses are connected by a spring–damper pair. The input $\mathbf{u}(t)$ is an *internal force*: it pulls m_1 and m_2 in opposite directions. For two masses m_1 and m_2 at positions x_1, x_2 , the center of mass is

$$x_{\text{COM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad \ddot{x}_{\text{COM}} = \frac{m_1 \ddot{x}_1 + m_2 \ddot{x}_2}{m_1 + m_2} = 0.$$

Internal forces **cannot move the center of mass**. Thus, the reachable subspace is

$$\mathcal{R} = \{\text{differential motions} = x_1 - x_2\},$$

and the center-of-mass coordinate is **uncontrollable**.

Minimum Energy Input

- With the finite-time controllability matrix

$$\mathbf{C}_t = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{t-1}\mathbf{B} \end{bmatrix} \in \mathbb{R}^{n \times mt},$$

to reach $\mathbf{x}(t) = \mathbf{x}_{\text{des}}$ from $\mathbf{x}(0) = \mathbf{0}$, we have

$$\mathbf{x}_{\text{des}} = \mathbf{C}_t \mathbf{u}_{0:t-1}.$$

- Least energy control (least-norm solution):

$$\mathbf{u}_{\text{ln}} = \mathbf{C}_t^T \left(\mathbf{C}_t \mathbf{C}_t^T \right)^{-1} \mathbf{x}_{\text{des}}.$$

- Minimum input energy required:

$$E_{\text{min}} = \|\mathbf{u}_{\text{ln}}\|_2^2 = \mathbf{x}_{\text{des}}^T \left(\mathbf{C}_t \mathbf{C}_t^T \right)^{-1} \mathbf{x}_{\text{des}}.$$

Intuition of Stabilizability

Stabilizability

A system is stabilizable if all uncontrollable modes are stable.

Intuition:

- Some modes are controllable: input can move and stabilize them.
- Some modes may be uncontrollable: input has no effect.
- If an uncontrollable mode is *stable*, it decays on its own — no problem.
- If an uncontrollable mode is *unstable*, nothing can fix it — system cannot be stabilized.

Thus, stabilizability asks:

Are all uncontrollable directions already harmless (stable)?

Stabilizability Example

Consider

$$\mathbf{A} = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Interpretation:

- The mode at 0.5 is controllable and stable.
- The mode at 1.2 is uncontrollable (first state not actuated).
- Since 1.2 is unstable, the system is **not stabilizable**.

If we replace 1.2 with 0.8:

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix},$$

the uncontrollable mode becomes stable \Rightarrow system is now stabilizable.

PBH (Popov–Belevitch–Hautus) Tests for Observability and Controllability

PBH Test for Observability

The pair (\mathbf{A}, \mathbf{C}) is observable if and only if

$$\text{rank} \begin{bmatrix} \lambda \mathbf{I} - \mathbf{A} \\ \mathbf{C} \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C}.$$

Equivalently: no eigenvector of \mathbf{A} lies in the null space of \mathbf{C} .

PBH Test for Controllability

The pair (\mathbf{A}, \mathbf{B}) is controllable if and only if

$$\text{rank} \begin{bmatrix} \lambda \mathbf{I} - \mathbf{A} & \mathbf{B} \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C}.$$

Equivalently: each eigenvector of \mathbf{A} can be excited by the input \mathbf{u} .

PBH Tests: It Suffices to Check Eigenvalues

Key Fact

If λ is *not* an eigenvalue of \mathbf{A} , then $\lambda \mathbf{I} - \mathbf{A}$ is invertible \Rightarrow rank is already n . Thus, only eigenvalues of \mathbf{A} can cause a rank drop.

Practical PBH Tests

$$(\mathbf{A}, \mathbf{C}) \text{ observable} \iff \text{rank} \begin{bmatrix} \lambda_i \mathbf{I} - \mathbf{A} \\ \mathbf{C} \end{bmatrix} = n \quad \text{for each eigenvalue } \lambda_i(\mathbf{A}).$$

$$(\mathbf{A}, \mathbf{B}) \text{ controllable} \iff \text{rank} \begin{bmatrix} \lambda_i \mathbf{I} - \mathbf{A} & \mathbf{B} \end{bmatrix} = n \quad \text{for each eigenvalue } \lambda_i(\mathbf{A}).$$

Why PBH is Preferred in Practice

Both PBH tests and the classical rank tests are *mathematically equivalent*, but PBH is often preferred in practice.

1. Computationally Simpler

- Avoids computing $\mathbf{A}^2, \mathbf{A}^3, \dots, \mathbf{A}^{n-1}$.
- Checks at most n matrices of size $n \times (n + m)$ or $(n + p) \times n$.
- Much more numerically stable for large n .

2. Eigenstructure Insight

- PBH tests whether any eigenvector \mathbf{v} satisfies $\mathbf{C}\mathbf{v} = 0$ (unobservable) or $\mathbf{B}^\top \mathbf{v} = 0$ (uncontrollable).
- Reveals *which mode* is unobservable or uncontrollable.
- Classical rank tests do not identify specific problematic modes.

Additional Advantages of PBH Tests

3. Scales Better to Large Systems

- Observability and controllability matrices may be huge (size $np \times n$).
- PBH uses only $n \times (n + m)$ or $(n + p) \times n$ matrices.

4. Handles Repeated Eigenvalues Naturally

- PBH works directly on each eigenvalue λ_i , including repeated ones.
- Classical tests often become numerically ill-conditioned with Jordan blocks.

5. Numerically Robust

- Powers of \mathbf{A} (in classical tests) amplify numerical errors.
- PBH avoids \mathbf{A}^k entirely and is more stable in computation.

Duality Principles

Dual system obtained by: $\mathbf{A} \leftrightarrow \mathbf{A}^T$, $\mathbf{B} \leftrightarrow \mathbf{C}^T$.

Controllability of (\mathbf{A}, \mathbf{B}) is observability of $(\mathbf{A}^T, \mathbf{C}^T)$.

Key

Controllable subspace equals orthogonal complement of unobservable subspace of the dual system.

Duality Table

Controllability	Observability
$\mathbf{C} = [\mathbf{B} \ \mathbf{AB} \ \cdots]$	$\mathbf{O} = [\mathbf{C}; \ \mathbf{CA}; \cdots]$
PBH: $[\lambda \mathbf{I} - \mathbf{A} \ \mathbf{B}]$	PBH: $\begin{bmatrix} \lambda \mathbf{I} - \mathbf{A} \\ \mathbf{C} \end{bmatrix}$
Stabilizability	Detectability
Reachable set	Observable subspace

Example: Observable but not Controllable

System

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Observability and Controllability

$$\mathbf{O} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{rank}(\mathbf{O}) = 2.$$

$$\mathbf{C}_{\text{ctr}} = [\mathbf{B} \ \mathbf{AB}] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{rank}(\mathbf{C}_{\text{ctr}}) = 1.$$

System is observable but not controllable.

Example: Controllable but not Observable

System

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Controllability and Observability

$$\mathbf{C}_{\text{ctr}} = [\mathbf{B} \ \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{rank}(\mathbf{C}_{\text{ctr}}) = 2.$$

$$\mathbf{O} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{rank}(\mathbf{O}) = 1.$$

System is controllable but not observable.

Summary

- Observability: $\text{rank}(\mathbf{O}) = n$.
- Detectability: unobservable modes stable.
- Controllability: $\text{rank}(\mathbf{C}) = n$.
- Stabilizability: uncontrollable modes stable.
- Minimum energy input: pseudoinverse of \mathbf{C}_t .
- Duality: controllability of (\mathbf{A}, \mathbf{B}) equals observability of $(\mathbf{A}^T, \mathbf{C}^T)$.

Continuous-Time Observability

The output equation is

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t).$$

Definition: The system is **observable** if knowledge of $\mathbf{y}(t)$ on $[0, T]$ uniquely determines $\mathbf{x}(0)$.

Observability Matrix (same as discrete-time)

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}.$$

The system is observable iff $\text{rank}(\mathbf{O}) = n$.

Observability Gramian and Equivalences

The output over $[0, T]$ contains

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0),$$

so reconstructability of $\mathbf{x}(0)$ depends on accumulated output energy.

Observability Gramian

$$\mathbf{W}_o(T) = \int_0^T e^{\mathbf{A}^\top \tau} \mathbf{C}^\top \mathbf{C} e^{\mathbf{A} \tau} d\tau.$$

The system is observable iff

$$\mathbf{W}_o(T) \succ 0 \quad \text{for some } T > 0.$$

Equivalence:

$$\text{rank}(\mathbf{O}) = n \quad \Longleftrightarrow \quad \mathbf{W}_o(T) \succ 0.$$

Continuous-Time Controllability

Consider the continuous-time LTI system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t).$$

Definition: The system is **controllable** if for any $\mathbf{x}(0)$ and any desired \mathbf{x}_{des} , there exists an input $\mathbf{u}(t)$ that drives

$$\mathbf{x}(T) = \mathbf{x}_{\text{des}} \quad \text{for some } T > 0.$$

Controllability Matrix (same as discrete-time)

$$\mathbf{C}_{\text{ctr}} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}.$$

The system is controllable iff $\text{rank}(\mathbf{C}_{\text{ctr}}) = n$.

Controllability Gramian

For $\mathbf{x}(0) = \mathbf{0}$, the solution is

$$\mathbf{x}(T) = \int_0^T e^{\mathbf{A}(T-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau.$$

Controllability Gramian

$$\mathbf{W}_c(T) = \int_0^T e^{\mathbf{A}\tau} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T \tau} d\tau.$$

The reachable set at time T is

$$\mathcal{R}(T) = \text{range}(\mathbf{W}_c(T)).$$

- $\mathbf{W}_c(T) \succ \mathbf{0}$ for some $T > 0 \iff$ system is controllable.
- Minimum-energy input satisfies $E_{\min} = \mathbf{x}_{\text{des}}^T \mathbf{W}_c(T)^{-1} \mathbf{x}_{\text{des}}$.

Key Takeaways: Discrete-Time vs. Continuous-Time LDS

Topic	Discrete-Time LDS	Continuous-Time LDS
Controllability Matrix	$\mathbf{C}_{\text{ctr}} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$	same \mathbf{C}_{ctr}
Observability Matrix	$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$	same \mathbf{O}
Controllability Gramian	$\mathbf{W}_c(t) = \sum_{k=0}^{t-1} \mathbf{A}^k \mathbf{B} \mathbf{B}^T (\mathbf{A}^T)^k$	$\mathbf{W}_c(T) = \int_0^T e^{\mathbf{A}\tau} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T \tau} d\tau$
Observability Gramian	$\mathbf{W}_o(t) = \sum_{k=0}^{t-1} (\mathbf{A}^T)^k \mathbf{C}^T \mathbf{C} \mathbf{A}^k$	$\mathbf{W}_o(T) = \int_0^T e^{\mathbf{A}^T \tau} \mathbf{C}^T \mathbf{C} e^{\mathbf{A}\tau} d\tau$
Rank Test	$\text{rank}(\mathbf{C}_{\text{ctr}}) = n$ and $\text{rank}(\mathbf{O}) = n$	same rank tests; or $\mathbf{W}_c(T) \succ \mathbf{0}$, $\mathbf{W}_o(T) \succ \mathbf{0}$