ECE240 Introduction to Linear Dynamical Systems
Lecture 3: Subspace and Rank

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Outline

1. Subspaces and Dimension
   - Direct Sum and Dimension
   - Range/Row/Null Subspaces
2. Applications of Rank
   - Controllability and Observability
   - Solution to System of Linear Equations

Vectors and Linear Independence

- A vector in $\mathbb{R}^n$:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

- **Linear independence**: $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ are linearly independent if

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \quad \Leftrightarrow \quad c_1 = \cdots = c_k = 0.$$

- **Span**: all possible linear combinations of a set of vectors.
- The dimension of the span $=$ maximum number of linearly independent vectors.

Space vs Subspace

Concept	Space (Vector Space)	Subspace
Definition	A set of vectors closed under vector addition and scalar multiplication, satisfying all vector space axioms.	A subset of a vector space that is itself a vector space under the same operations.
Notation	Usually denoted V or $\mathbb{R}^n$.	Denoted $W \subseteq V$, meaning W lies inside V.
Containment	The full environment containing all possible linear combinations of its basis vectors.	A smaller linear region inside a space (line, plane, or hyperplane through the origin).
Example in $\mathbb{R}^3$	$V = \mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$.	$W = \{(x, y, z) : z = 0\}$ (a plane) or $W = \{t[1, 2, 3]^\top : t \in \mathbb{R}\}$ (a line).
Geometric Meaning	The "ambient" space containing all possible vectors or states.	A directionally constrained subset, e.g., range space, null space, or equilibrium manifold.
Zero Vector Requirement	Always contains the zero vector.	Must include the zero vector to qualify as a subspace.
Example in Linear Systems	State space: $\mathbf{x}(t) \in \mathbb{R}^n$.	Range$(\mathbf{A})$ = reachable outputs $y = Ax$; Null$(\mathbf{A})$ = steady-state directions $Ax = 0$.

Table: Comparison between a vector space and a subspace. Every subspace is a vector space, but not every subset of a vector space is a subspace (e.g., the plane $z = 1$ in $\mathbb{R}^3$ fails because it does not pass through the origin).

Direct Sum of Subspaces

Definition

Let U, W be subspaces of a vector space V. We say that V is the **direct sum** of U and W, written

$$V = U \oplus W,$$

if:

1. Every $v \in V$ can be written as $v = u + w$ with $u \in U$, $w \in W$, and
2. The representation is unique, which is equivalent to $U \cap W = \{0\}$.

Example in $\mathbb{R}^2$

Let $U = \text{span}\{(1,0)\}$ (the x-axis), $W = \text{span}\{(0,1)\}$ (the y-axis). Then

$$\mathbb{R}^2 = U \oplus W,$$

since every (a, b) has a unique decomposition $(a, 0) + (0, b)$.

Dimension of a Subspace

Definition: Let V be a vector space and $W \subseteq V$ be a subspace. The **dimension** of W is

$$\dim(W) = \text{the number of vectors in any basis of } W$$

or equivalently,

$$\dim(W) = \text{the maximal number of linearly independent vectors in } W.$$

Basic properties:

- $0 \leq \dim(W) \leq \dim(V)$
- $\dim(W) = 0 \iff W = \{0\}$
- $\dim(W) = \dim(V) \iff W = V$
- $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$
- $\dim(U \oplus W) = \dim(U) + \dim(W)$

Examples and Geometric Intuition

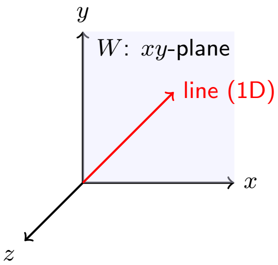
Example 1: In $\mathbb{R}^3$, the set

$$W = \{(x, y, 0) : x, y \in \mathbb{R}\}$$

is a 2-dimensional subspace (the xy-plane), with basis $\{(1,0,0),(0,1,0)\}$.

Geometric view:

- Line through the origin $\rightarrow$ 1D subspace
- Plane through the origin $\rightarrow$ 2D subspace
- Whole $\mathbb{R}^3$ $\rightarrow$ 3D space



Range and Null Spaces

Given $\mathbf{A} \in \mathbb{R}^{m \times n}$:

- **Range space (column space)** $=$ span of columns of $\mathbf{A}$:

$$\mathcal{R}(\mathbf{A}) = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \mathbf{A}\mathbf{x},\ \mathbf{x} \in \mathbb{R}^n\}.$$

- **Row space** $=$ span of rows of $\mathbf{A}$:

$$\text{Row}(\mathbf{A}) = \mathcal{R}(\mathbf{A}^\top)$$

- **Null space (kernel)** $=$ set of inputs mapped to zero:

$$\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{0}\}.$$

Fundamental Theorem of Linear Algebra:

- $\mathbb{R}^n = \mathcal{R}(\mathbf{A}^\top) \oplus \mathcal{N}(\mathbf{A})$.
- $\mathbb{R}^m = \mathcal{R}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^\top)$.

Dimension and Rank

Rank–Nullity Theorem: For any matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$,

$$\dim(\mathsf{Range}(\mathbf{A})) + \dim(\mathsf{Null}(\mathbf{A})) = n$$

Interpretation:

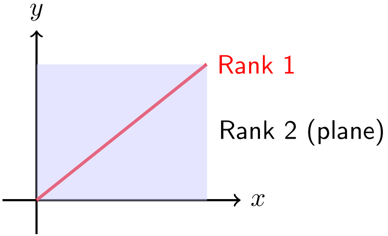
- $\mathsf{Range}(\mathbf{A})$: all possible outputs of $\mathbf{Ax}$
- $\mathsf{Null}(\mathbf{A})$: all inputs $\mathbf{x}$ mapped to zero
- Their dimensions together equal the total degrees of freedom in $\mathbb{R}^n$

Example: If $\mathbf{A}$ is 4×3 and $\mathrm{rank}(\mathbf{A}) = 2$, then

$$\dim(\mathsf{Range}(\mathbf{A})) = 2, \quad \dim(\mathsf{Null}(\mathbf{A})) = 1$$

Rank of Matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$

- $\mathrm{rank}(\mathbf{A})$ is the number of linearly independent columns (or rows).
- $\mathrm{rank}(\mathbf{A})$ is the dimension of its range or row space.
- $\mathrm{rank}(\mathbf{A})$ is the number of independent pieces of information a matrix carries
- **Geometric view:** dimension of the subspace where outputs live
  - Rank $= 1$: all outputs lie on a line
  - Rank $= 2$: all outputs lie in a plane
  - Rank $= n$: full space is covered (full rank)
- **System meaning:** rank measures how much of a system we can control, observe, or model



Rank Equalities and Inequalities

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times p}$.

- **Basic bounds:**
$$\operatorname{rank}(\mathbf{A}) \le \min(m, n).$$
$$\operatorname{rank}(\mathbf{A} + \mathbf{B}) \le \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}).$$
- **Product rule (Sylvester's inequality):**
$$\operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}) - n \;\le\; \operatorname{rank}(\mathbf{A}\mathbf{B}) \;\le\; \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}.$$
- **Transpose:** $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}^\top)$
- **Rank–Nullity Theorem:** $\operatorname{rank}(\mathbf{A}) + \dim \mathcal{N}(\mathbf{A}) = n$
- **Block matrices:** If $\mathbf{A}, \mathbf{B}$ have the same number of rows:
$$\max\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\} \;\le\; \operatorname{rank}\begin{bmatrix}\mathbf{A} & \mathbf{B}\end{bmatrix} \;\le\; \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}).$$

Controllability and Observability

- **Controllability:** Can we drive $\mathbf{x}(t)$ to any state with inputs $\mathbf{u}(t)$?

$$\mathcal{C} = [\mathbf{B}, \mathbf{AB}, \dots, \mathbf{A}^{n-1}\mathbf{B}]$$

  System controllable iff $\mathrm{rank}(\mathcal{C}) = n$.

- **Observability:** Can we reconstruct $\mathbf{x}(t)$ from outputs $\mathbf{y}(t)$?

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

  System observable iff $\mathrm{rank}(\mathcal{O}) = n$.

$\mathbf{Ax} = \mathbf{b}$: When Does a Solution Exist?

Rank tells us whether $\mathbf{Ax} = \mathbf{b}$ has a solution. **Key idea:**

$$\mathbf{Ax} = \mathbf{b} \text{ has a solution } \iff \mathbf{b} \in \mathrm{Range}(\mathbf{A})$$

Geometric view: $\mathbf{A}$ maps $\mathbb{R}^n \to \mathbb{R}^m$. Only vectors $\mathbf{b}$ in the column space of $\mathbf{A}$ are reachable.

$$\boxed{\textit{Unique} \iff \textit{Null}(\mathbf{A}) = \{\mathbf{0}\}, \quad \textit{Existence} \iff \mathbf{b} \in \mathrm{Range}(\mathbf{A}).}$$

Rank condition	Shape of $\mathbf{A}$	Existence?	Unique?	Type of system
$\mathrm{rank}(\mathbf{A}) = n = m$	Square, full rank	Always	Yes	Invertible
$\mathrm{rank}(\mathbf{A}) = n < m$	Tall, full col rank	Iff $\mathbf{b} \in \mathrm{Range}(\mathbf{A})$	Yes	Overdetermined
$\mathrm{rank}(\mathbf{A}) = m < n$	Wide, full row rank	Always	No (infinite)	Underdetermined
$\mathrm{rank}(\mathbf{A}) < \min(m, n)$	Rank deficient	Depends on $\mathbf{b}$	No (none or inf.)	Inconsistent/degenerate

If no exact solution: find least-squares $\mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_2 \quad \Rightarrow \quad \mathbf{x}^* = \mathbf{A}^\dagger \mathbf{b}$