ECE240 Introduction to Linear Dynamical Systems

Lecture 10: Laplace Transform

Dr. Yu Zhang

ECE Department, UC Santa Cruz

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Outline

Laplace Transform: Definition

For a signal $f(t)$ defined for $t \geq 0$, the Laplace transform is

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt, \qquad s \in \mathbb{C}.$$

- Converts a time-domain function to a complex-frequency representation.
- Converges for $\mathrm{Re}(s)$ in the *region of convergence* (ROC).

Common Laplace Transform Properties

- Linearity:

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s).$$

- Time scaling:

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right), \quad a > 0.$$

- Exponential shift:

$$\mathcal{L}\{e^{ct} f(t)\} = F(s - c).$$

- Derivative:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

Useful Laplace Transform Table (I)

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad \mathrm{Re}(s) > 0$
$t^n, \ n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad \mathrm{Re}(s) > 0$
e^{at}	$\frac{1}{s-a}, \quad \mathrm{Re}(s) > \mathrm{Re}(a)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \quad \mathrm{Re}(s) > \mathrm{Re}(a)$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}, \quad \mathrm{Re}(s) > 0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}, \quad \mathrm{Re}(s) > 0$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}, \quad \mathrm{Re}(s) > \mathrm{Re}(a)$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}, \quad \mathrm{Re}(s) > \mathrm{Re}(a)$

Useful Laplace Transform Table (II)

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
$u(t)$ (unit step)	$\frac{1}{s}, \quad \mathrm{Re}(s) > 0$
$u(t-T)$ (delayed step)	$\frac{e^{-Ts}}{s}, \quad \mathrm{Re}(s) > 0$
$t\,u(t)$	$\frac{1}{s^2}, \quad \mathrm{Re}(s) > 0$
$(t-T)u(t-T)$	$\frac{e^{-Ts}}{s^2}, \quad \mathrm{Re}(s) > 0$
$\delta(t)$ (impulse)	1
$\delta(t-T)$	e^{-Ts}
$f(t-T)u(t-T)$	$e^{-Ts}F(s)$ (time shift)

Inverse Laplace Transform

The *inverse Laplace transform* of a function $F(s)$ is defined by the **Bromwich integral** (a.k.a. inverse Laplace integral):

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\gamma - j\infty}^{\gamma + j\infty} F(s)\, e^{st}\, ds,$$

where:

- γ is a real constant chosen so that the vertical line $\text{Re}(s) = \gamma$ lies to the *right* of all singularities (poles) of $F(s)$;
- the integration path is a vertical line in the complex plane (parallel to the imaginary axis), called the *Bromwich contour*;
- the integral converges for all $t \geq 0$ when $F(s)$ is of exponential order.

In practice, we use: 1) Tables of standard transforms; 2) Partial fraction expansion; and 3) Polynomial / residue methods for rational $F(s)$.

Partial Fraction Expansion

Suppose

$$F(s) = \frac{P(s)}{Q(s)},$$

with $Q(s)$ factorizable into linear or quadratic terms.

If $Q(s) = \prod_k (s - \lambda_k)$, then

$$F(s) = \sum_k \frac{A_k}{s - \lambda_k}.$$

Inverse transform:

$$\mathcal{L}^{-1}\left\{\frac{A_k}{s - \lambda_k}\right\} = A_k e^{\lambda_k t}.$$

Example: Simple Inverse Transform

Given

$$F(s) = \frac{3}{s+2} + \frac{1}{s^2+4},$$

apply known transforms:

$$\mathcal{L}^{-1}\left(\frac{3}{s+2}\right) = 3e^{-2t}, \qquad \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \tfrac{1}{2}\sin(2t).$$

Thus

$$f(t) = 3e^{-2t} + \tfrac{1}{2}\sin(2t).$$

Laplace Transform of Matrix-Valued Function

Given a matrix-valued function $\mathbf{z}(t) : \mathbb{R}_+ \to \mathbb{R}^{p\times q}$, the Laplace transform is defined as

$$\mathbf{Z}(s) = \mathcal{L}\{\mathbf{z}(t)\} = \int_0^\infty e^{-st}\mathbf{z}(t)\, dt$$

with $\mathbf{Z} : D \subseteq \mathbb{C} \to \mathbb{C}^{p\times q}$, where D is the *domain* or *region of convergence* of $\mathbf{Z}$.

- D includes at least $\{\, s \mid \mathrm{Re}(s) > a \,\}$, where a satisfies
  $|z_{ij}(t)| \leq \alpha e^{at}, \quad t \geq 0,\ i = 1, \ldots, p,\ j = 1, \ldots, q.$
- **Integral applied entrywise**.

Derivative Property

$$\mathcal{L}(\dot{\mathbf{z}}) = s\,\mathbf{Z}(s) - \mathbf{z}(0)$$

Proof (via integrate by parts):

$$\begin{aligned}
\mathcal{L}(\dot{\mathbf{z}})(s) &= \int_0^\infty e^{-st}\,\dot{\mathbf{z}}(t)\,dt \\
&= e^{-st}\,\mathbf{z}(t)\Big|_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st}\,\mathbf{z}(t)\,dt \\
&= s\,\mathbf{Z}(s) - \mathbf{z}(0).
\end{aligned}$$

Resolvent and State Transition Matrix (1/2)

Take Laplace transform of both sides of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$:

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) \implies \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0).$$

Inverse transform:

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{x}(0).$$

- $(s\mathbf{I} - \mathbf{A})^{-1}$ is called the *resolvent* of $\mathbf{A}$.
- Poles occur at eigenvalues of $\mathbf{A}$.
- State-transition matrix:

$$\mathbf{\Phi}(t) = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right].$$

- Solution:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0).$$

Resolvent and State Transition Matrix (2/2)

- Series expansion:

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{s}\left(\mathbf{I}-\frac{\mathbf{A}}{s}\right)^{-1} = \frac{1}{s}\left(\mathbf{I}+\frac{\mathbf{A}}{s}+\frac{\mathbf{A}^2}{s^2}+\cdots\right).$$

- Inverse Laplace transform gives

$$\Phi(t) = \mathbf{I} + t\mathbf{A} + \frac{(t\mathbf{A})^2}{2!} + \frac{(t\mathbf{A})^3}{3!} + \cdots = e^{t\mathbf{A}}.$$

- Thus, the resolvent $(s\mathbf{I}-\mathbf{A})^{-1}$ and the state-transition matrix $\Phi(t) = e^{t\mathbf{A}}$ form a Laplace transform pair:

$$\boxed{e^{t\mathbf{A}} = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right]}$$

Example 1: Computing $e^{t\mathbf{A}}$ for a Skew-symmetric Matrix

For

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix},$$

the resolvent is

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ -\frac{1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}.$$

The state transition matrix is

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

a rotation matrix by angle $-t$.

Example 2: Computing $e^{\mathbf{A}}$ for a Nilpotent Matrix

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We have

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\left((s\mathbf{I} - \mathbf{A})^{-1}\right) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \implies e^{\mathbf{A}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ (setting } t = 1)$$

Check using the power series:

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \cdots = \mathbf{I} + \mathbf{A},$$

since $\mathbf{A}^k = \mathbf{0}, \ \forall k \geq 2$.

Matrix Exponential Solution

The exact solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0)$.

- Generalization of scalar $x(t) = e^{at}x(0)$.
- $e^{t\mathbf{A}}$ is always nonsingular, with $(e^{t\mathbf{A}})^{-1} = e^{-t\mathbf{A}}$.
- $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}\, e^{\mathbf{B}}$ if $\mathbf{AB} = \mathbf{BA}$, i.e., $\mathbf{A}$ and $\mathbf{B}$ commute.

Time Transfer Property

- For any τ and t, the exact update is

$$\mathbf{x}(\tau + t) = e^{t\mathbf{A}}\mathbf{x}(\tau) = \left(\mathbf{I} + t\mathbf{A} + \frac{(t\mathbf{A})^2}{2!} + \frac{(t\mathbf{A})^3}{3!} + \cdots\right)\mathbf{x}(\tau).$$

  Interpretation: $e^{t\mathbf{A}}$ propagates state t seconds forward in time (backward if $t < 0$).

- Forward Euler approximation is a first-order numerical procedure for solving ODEs.

  For small t:

$$\mathbf{x}(\tau + t) \approx (\mathbf{I} + t\mathbf{A})\mathbf{x}(\tau).$$

Eigenvalues of $\mathbf{A}$ and Poles of the Resolvent

The (i, j) entry of the resolvent $(s\mathbf{I} - \mathbf{A})^{-1}$ can be written via Cramer's rule as

$$\frac{(-1)^{i+j} \det \boldsymbol{\Delta}_{ij}}{\det(s\mathbf{I} - \mathbf{A})},$$

where $\boldsymbol{\Delta}_{ij}$ is $s\mathbf{I} - \mathbf{A}$ with the jth row and ith column removed.

- $\det \boldsymbol{\Delta}_{ij}$ is a polynomial of degree less than n. Therefore each entry of the resolvent has the form
$$\frac{f_{ij}(s)}{\mathcal{X}(s)},$$
where $f_{ij}(s)$ has degree $< n$ and $\mathcal{X} = \det(s\mathbf{I} - \mathbf{A})$ is the characteristic polynomial.
- Poles of the resolvent entries must be eigenvalues of $\mathbf{A}$.
- But not every eigenvalue appears in every entry (cancellations between $\det \boldsymbol{\Delta}_{ij}$ and $\mathcal{X}(s)$ may occur).

Example: Eigenvalues vs. Poles of Resolvent Entries

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \mathcal{X}(s) = \det(s\mathbf{I} - \mathbf{A}) = (s-1)(s-2).$$

Then

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix}, \qquad (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \dfrac{1}{s-1} & 0 \\ 0 & \dfrac{1}{s-2} \end{bmatrix}.$$

Using Cramer's rule for the $(1,1)$ entry:

$$\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]_{11} = \frac{\det \mathbf{\Delta}_{11}}{\det(s\mathbf{I} - \mathbf{A})} = \frac{s-2}{(s-1)(s-2)} = \frac{1}{s-1}.$$

- Eigenvalues of $\mathbf{A}$ are 1 and 2.
- The $(1,1)$ entry has only a pole at $s = 1$; the factor $(s-2)$ cancels out.
- Similarly, the $(2,2)$ entry has only a pole at $s = 2$.

Qualitative Behavior of $\mathbf{x}(t)$

Suppose $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, with $\mathbf{x}(t) \in \mathbb{R}^n$.

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0), \qquad \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0).$$

The ith component has form

$$X_i(s) = \frac{a_i(s)}{\mathcal{X}(s)},$$

where $a_i(s)$ is a polynomial with degree $< n$ and $\mathcal{X}(s) = \det(s\mathbf{I} - \mathbf{A})$.

Thus, all poles of $X_i(s)$ are eigenvalues of $\mathbf{A}$ (but not necessarily the other way around, as shown before).

Case 1: Distinct Eigenvalues

Assume eigenvalues λ_j are distinct, so $X_i(s)$ has no repeated poles.

Then

$$x_i(t) = \sum_{j=1}^{n} \beta_{ij} e^{\lambda_j t},$$

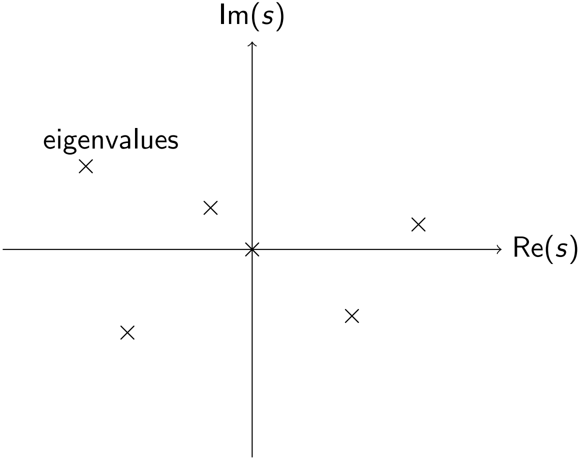
where coefficients β_{ij} depend linearly on $\mathbf{x}(0)$.

Eigenvalues determine the possible qualitative behavior of $\mathbf{x}(t)$:

- Real λ gives exponential decay or growth: $e^{\lambda t}$.
- Complex $\lambda = \sigma + j\omega$ gives decaying/growing sinusoids: $e^{\sigma t} \cos(\omega t + \phi)$.

Growth/Oscillation Rates from Eigenvalues

- $\mathrm{Re}(\lambda_j)$ gives exponential growth rate (if > 0) or decay rate (if < 0).
- $\mathrm{Im}(\lambda_j)$ gives frequency of oscillation (if $\neq 0$).



Case 2: Repeated Eigenvalues

If $\mathbf{A}$ has repeated eigenvalues, then $X_i(s)$ can have repeated poles.

Let the distinct eigenvalues be $\lambda_1, \ldots, \lambda_r$ with multiplicities $n_1, \ldots, n_r$ (so $n_1 + \cdots + n_r = n$).

Then

$$x_i(t) = \sum_{j=1}^{r} p_{ij}(t)\, e^{\lambda_j t},$$

where $p_{ij}(t)$ is a polynomial of degree $< n_j$ and depends linearly on $\mathbf{x}(0)$.

Example: Repeated Eigenvalues and Polynomial Terms (1/2)

Consider

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

a 2×2 Jordan block with repeated eigenvalue $\lambda = 2$.

Compute the resolvent:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s-2 & -1 \\ 0 & s-2 \end{bmatrix}, \qquad (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{s-2} & \frac{1}{(s-2)^2} \\ 0 & \frac{1}{s-2} \end{bmatrix}.$$

Taking inverse Laplace transform:

$$\boldsymbol{\Phi}(t) = e^{t\mathbf{A}} = e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Example: Repeated Eigenvalues and Polynomial Terms (2/2)

Thus, for any initial state $\mathbf{x}(0)$,

$$\mathbf{x}(t) = e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}(0).$$

Key observation: The repeated eigenvalue $\lambda = 2$ produces a term of the form te^{2t}, which is the signature of a repeated pole and a nontrivial Jordan block.

Region of Convergence for $(s\mathbf{I} - \mathbf{A})^{-1}$

For the homogeneous LDS $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, we have $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$. The characteristic polynomial is $\mathcal{X}(s) = \det(s\mathbf{I} - \mathbf{A})$, with roots $\lambda_1, \ldots, \lambda_n$.

The resolvent $(s\mathbf{I} - \mathbf{A})^{-1}$ converges for

$$\mathrm{Re}(s) > \max_j \mathrm{Re}(\lambda_j).$$

- If all eigenvalues satisfy $\mathrm{Re}(\lambda_j) < 0$, the ROC includes the entire right-half-plane and the system is stable.
- If any eigenvalue satisfies $\mathrm{Re}(\lambda_j) > 0$, the ROC shifts right and the system is unstable.

ROC directly reflects growth/decay rates of $e^{t\mathbf{A}}$.

Transfer Function and Impulse Response

For the LDS

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u},$$

taking Laplace transforms yields

$$\mathbf{Y}(s) = \left(\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)\mathbf{U}(s).$$

The transfer function is

$$\mathbf{G}(s) := \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

Impulse response is the inverse Laplace transform:

$$\mathbf{h}(t) = \mathcal{L}^{-1}\{\mathbf{G}(s)\} = \mathbf{C}e^{t\mathbf{A}}\mathbf{B} + \mathbf{D}\delta(t).$$

$\mathbf{G}(s)$ and $\mathbf{h}(t)$ are a Laplace transform pair.

Forced LDS and Convolution Representation

Consider the input–output LDS

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t).$$

Taking the Laplace transform (zero initial condition shown for clarity):

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s).$$

Inverse transform gives

$$\mathbf{x}(t) = \int_0^t e^{(t-\tau)\mathbf{A}}\,\mathbf{B}\,\mathbf{u}(\tau)\,d\tau.$$

Using $\mathcal{L}(f * g) = F(s)G(s)$:

$$\mathbf{x}(t) = e^{t\mathbf{A}} * \mathbf{B}\mathbf{u}(t),$$

so the state is a **convolution** of the matrix exponential with the input.