# ECE240 Introduction to Linear Dynamical Systems
# Lecture 5: LTI System Solution

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# Outline

# Homogeneous System

System without input:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{A} \in \mathbb{R}^{n\times n}.$$

We seek a function $\mathbf{\Phi}(t)$ such that

$$\dot{\mathbf{\Phi}}(t) = \mathbf{A}\,\mathbf{\Phi}(t), \quad \mathbf{\Phi}(0) = \mathbf{I},$$

so that the solution is

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\,\mathbf{x}_0.$$

Goal: Find $\mathbf{\Phi}(t)$ explicitly $\Rightarrow$ matrix exponential $e^{\mathbf{A}t}$.

# Series Definition of the Matrix Exponential

Define:

$$e^{\mathbf{A}t} := \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} = I + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \cdots$$

Differentiate term by term:

$$\frac{d}{dt} e^{\mathbf{A}t} = \sum_{k=1}^{\infty} \frac{\mathbf{A}^k t^{k-1}}{(k-1)!} = \mathbf{A} \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} = \mathbf{A}\, e^{\mathbf{A}t}.$$

Initial condition: $e^{\mathbf{A}\times 0} = \mathbf{I}$. Hence, $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ satisfies

$$\dot{\mathbf{\Phi}}(t) = \mathbf{A}\mathbf{\Phi}(t), \quad \mathbf{\Phi}(0) = \mathbf{I}.$$

# Various Definitions of the Matrix Exponential

*(Optional – for reference only)*

| Definition Type | Expression | Key Idea / Use |
|---|---|---|
| Series | $e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$ | Fundamental definition; always convergent. |
| Differential Equation | $\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t),\ \mathbf{X}(0) = \mathbf{I}$ | Defines the state transition operator in linear systems. |
| Limit | $e^{\mathbf{A}} = \lim_{n\to\infty} \left(\mathbf{I} + \frac{\mathbf{A}}{n}\right)^n$ | Connects to continuous compounding and growth. |
| Diagonalization (special case) | $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1},\ e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}$ | Applies when $\mathbf{A}$ is diagonalizable; shows modal evolution. |
| Jordan Form (general case) | $\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1},\ e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{J}t}\mathbf{P}^{-1}$ | Handles non-diagonalizable $\mathbf{A}$ using Jordan blocks. |
| Laplace Transform | $e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$ | Used in control and system theory for transition matrices. |

# Solution for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

Since $\boldsymbol{\Phi}(t) = e^{\mathbf{A}t}$ satisfies the matrix differential equation, the state trajectory is

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)}.$$

Verification:

$$\dot{\mathbf{x}}(t) = \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{I} \times \mathbf{x}(0),$$

where $e^{\mathbf{A}t}$ = **state transition matrix**: describes how the state evolves continuously with time.

# Semigroup Property of the Matrix Exponential

- **Property:**

$$e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}\, e^{\mathbf{A}t_2}$$

- The family $\{e^{\mathbf{A}t} : t \geq 0\}$ forms a **one-parameter semigroup** of linear operators. If $t \in \mathbb{R}$, it forms a **one-parameter group**.
- **Implications:** $e^{\mathbf{A}\times 0} = \mathbf{I}$, $\quad (e^{\mathbf{A}t})^{-1} = e^{-\mathbf{A}t}$, $\quad (e^{\mathbf{A}t})^{n} = e^{\mathbf{A}nt}$.
- **In linear systems:** for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$,

$$\mathbf{x}(t_1 + t_2) = e^{\mathbf{A}t_1} e^{\mathbf{A}t_2}\, \mathbf{x}(0)$$

  The state transition over two intervals composes naturally.
- **Caution:** $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}} e^{\mathbf{B}}$ only if $\mathbf{AB} = \mathbf{BA}$.
- **Example:**

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad e^{\mathbf{A}t} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$ is the rotation matrix with angle $t$

  Rotating by $t_1$ then $t_2$ equals rotating by $t_1 + t_2$.

# Modal Derivation

Consider $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)\mathbf{x}$, let $\mathbf{x}(t) = \mathbf{V}\mathbf{y}(t) \implies \dot{\mathbf{y}}(t) = \mathbf{\Lambda}\mathbf{y}(t)$ (decoupled dynamics: $\dot{y}_i(t) = \lambda_i y_i(t)$.)

Hence,

$$\mathbf{y}(t) = e^{\mathbf{\Lambda}t}\mathbf{y}(0) \implies \mathbf{x}(t) = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}\mathbf{x}(0) = e^{\mathbf{A}t}\mathbf{x}(0),$$

where $e^{\mathbf{\Lambda}t} = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$.

- Vector $\mathbf{y}(0) = \mathbf{V}^{-1}\mathbf{x}(0)$ collects the coordinates of the initial state expressed in the eigenbasis. Thus,

$$\mathbf{x}(t) = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{y}(0) = \sum_{i=1}^{n} \underbrace{y_i(0)e^{\lambda_i t}\mathbf{v}_i}_{\mathbf{x}_i^{\text{mode}}(t)}.$$

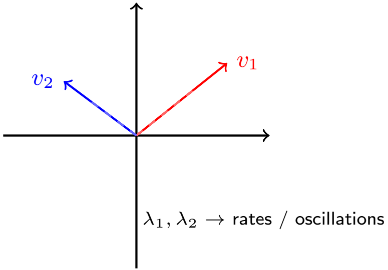
- The full trajectory is a superposition of these independent modes

$$\mathbf{x}_i^{\text{mode}}(t) = y_i(0)e^{\lambda_i t}\mathbf{v}_i,\ \forall i = 1, \ldots, n.$$

# Geometric and Physical Interpretation

In $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$:

- Each eigenvector $\mathbf{v}_i$ defines an invariant direction in state space.
- Along $\mathbf{v}_i$, the trajectory scales as $e^{\lambda_i t}$, where eigenvalue $\lambda_i$ controls time behavior.
- 1) eigenvectors $\mathbf{v}_i \Rightarrow$ *mode shape/direction*; 2) eigenvalues $\lambda_i \Rightarrow$ *growth, decay, or oscillation rate*.




**Eigen-decomposition decouples dynamics into independent exponential modes.**

# Dynamic Implications of Eigenvalues

| Eigenvalue $\lambda_i$ | Time behavior | Interpretation |
|---|---|---|
| $\lambda_i < 0$ (real) | $e^{\lambda_i t}$ decays | Stable (energy dissipates) |
| $\lambda_i > 0$ (real) | $e^{\lambda_i t}$ grows | Unstable (diverges) |
| $\lambda_i = 0$ | Constant | Marginally stable |
| $\lambda_i = \alpha \pm j\beta$ | $e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$ | Oscillation (freq. $\beta$, damping $\alpha$) |

System stability:

$$\text{Stable} \iff \mathrm{Re}(\lambda_i) < 0,\ \forall i$$

Geometric view:

- Each eigenvector $\mathbf{v}_i$: direction of independent motion.
- Each eigenvalue $\lambda_i$: exponential rate or frequency of that motion.

# 2×2 Linear System

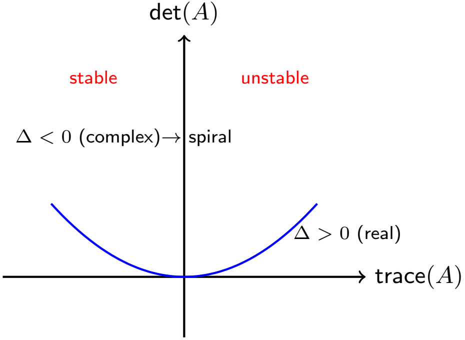
Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \text{characteristic polynomial}: \ \lambda^2 - (\mathrm{tr}\mathbf{A})\lambda + \det(\mathbf{A}) = 0 \implies$$

$$\text{two eigenvalues}: \ \lambda_{1,2} = \frac{\mathrm{tr}(\mathbf{A}) \pm \sqrt{(\mathrm{tr}\mathbf{A})^2 - 4\det(\mathbf{A})}}{2}$$

Interpretation:

- $\mathrm{tr}(\mathbf{A}) \to$ shifts eigenvalues left/right
- $\det(\mathbf{A}) \to$ sets their product (sign $\Rightarrow$ type of equilibrium)
- Discriminant $\Delta = (\mathrm{tr}\mathbf{A})^2 - 4\det(\mathbf{A})$ controls whether eigenvalues are real or complex



# Stability Classification (2×2 System)

For $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with

$$\lambda^2 - (\mathrm{tr}\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

| Type | $\mathrm{tr}(\mathbf{A})$ | $\det(\mathbf{A})$ | $\Delta = (\mathrm{tr}\mathbf{A})^2 - 4\det(\mathbf{A})$ | Behavior |
|---|---|---|---|---|
| Stable node | $< 0$ | $> 0$ | $> 0$ | Real and negative eigenvalues |
| Stable spiral | $< 0$ | $> 0$ | $< 0$ | Complex, Re $< 0$ |
| Center | $= 0$ | $> 0$ | $< 0$ | Purely imaginary (oscillatory) |
| Unstable node/spiral | $> 0$ | $> 0$ | any | Divergent |
| Saddle point | any | $< 0$ | $> 0$ | One eigenvalue $> 0$, one $< 0$ |

Key:

- $\mathrm{tr}(\mathbf{A}) < 0 \rightarrow$ contraction (stable)
- $\mathrm{tr}(\mathbf{A}) > 0 \rightarrow$ expansion (unstable)
- $\det(\mathbf{A}) < 0 \rightarrow$ opposite-sign eigenvalues $\Rightarrow$ saddle

# Trace and Determinant in Linear Dynamical Systems

Homogeneous system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{A} \in \mathbb{R}^{n \times n}$$

Solution: $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$

If $\mathbf{A}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$: then

$$\mathrm{trace}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i, \qquad \det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$$

Interpretation:

- $\mathrm{trace}(\mathbf{A}) \to$ sum of eigenvalue real parts $\to$ net expansion or decay rate
- $\det(\mathbf{A}) \to$ product of eigenvalues $\to$ overall scaling or rotation sign

*Together they give key insight into system stability.*

# Jacobian Determinant of the Matrix Exponential

- Consider the linear system: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad \Rightarrow \quad \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$.
- The Jacobian matrix of the transformation $\mathbf{x}(0) \mapsto \mathbf{x}(t)$ is:

$$\mathbf{J}(t) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(0)} = e^{\mathbf{A}t}$$

- Hence, the **Jacobian determinant** is:

$$\boxed{\det\left(e^{\mathbf{A}t}\right) = e^{t \times \mathrm{tr}(\mathbf{A})}}$$

- Example:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathrm{tr}(\mathbf{A}) = 1 \Rightarrow \det(e^{\mathbf{A}t}) = e^{t}$$

**Interpretation:**

- $\mathrm{tr}(\mathbf{A})$ is the *divergence* of the vector field $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, i.e., total rate of local volume expansion.
- $|\det(e^{\mathbf{A}t})|$ measures the *volume scaling* in state space.

# Physical Insights

**Physical meaning:**

$$\begin{cases} \mathrm{tr}(\mathbf{A}) = 0 & \Rightarrow \text{trajectories preserve volume (e.g., rotations).} \\ \mathrm{tr}(\mathbf{A}) < 0 & \Rightarrow \text{trajectories contract (stable flow).} \\ \mathrm{tr}(\mathbf{A}) > 0 & \Rightarrow \text{trajectories expand apart (unstable flow)} \end{cases}$$

**Why it matters in dynamical systems:**

- Divergence describes whether energy or volume in the state space is conserved or dissipated.
- It's a local stability indicator: if divergence is negative, nearby trajectories converge.
- It connects linear algebra (trace), system stability (eigenvalues), and geometry (volume flow).
- **divergence** $\leftrightharpoons$ **trace** $\leftrightharpoons$ **determinant** $\leftrightharpoons$ **volume change** $\leftrightharpoons$ **stability**

# Extension: Solution to Forced System (1/2)

Given LTI System $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0$

**Step 1:** Guess solution with time-varying constant

$$\mathbf{x}(t) = e^{\mathbf{A}t}\,\mathbf{c}(t).$$

**Step 2:** Differentiate and substitute:

$$\dot{\mathbf{x}} = \mathbf{A}e^{\mathbf{A}t}\mathbf{c}(t) + e^{\mathbf{A}t}\dot{\mathbf{c}}(t) = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t) \Rightarrow e^{\mathbf{A}t}\dot{\mathbf{c}}(t) = \mathbf{B}\mathbf{u}(t) \Rightarrow \dot{\mathbf{c}}(t) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

**Step 3:** Integrate both sides from $0$ to $t$:

$$\int_0^t \dot{\mathbf{c}}(\tau)\,d\tau = \int_0^t e^{-\mathbf{A}\tau}\,\mathbf{B}\,\mathbf{u}(\tau)\,d\tau \implies \mathbf{c}(t) = \mathbf{c}(0) + \int_0^t e^{-\mathbf{A}\tau}\,\mathbf{B}\,\mathbf{u}(\tau)\,d\tau.$$

Initial condition: $\quad \mathbf{c}(0) = \mathbf{x}(0) = \mathbf{x}_0 \quad \Rightarrow$

$$\mathbf{c}(t) = \mathbf{x}_0 + \int_0^t e^{-\mathbf{A}\tau}\,\mathbf{B}\,\mathbf{u}(\tau)\,d\tau$$

*Intuition:* Multiply by $e^{-\mathbf{A}t}$ to *undo* the natural dynamics and isolate the input's effect in the moving frame.

# Extension: Solution to Forced System (2/2)

**Step 4:** Substitute back, we get **variation of constants formula**:

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau.}$$

*Interpretation:*

- First term $\rightarrow$ homogeneous (natural) response.
- Second term $\rightarrow$ particular (forced) response.
- The integral is a **convolution**:

  $$\mathbf{x}_{\text{forced}}(t) = (e^{\mathbf{A}t}\mathbf{B}) * \mathbf{u}(t),$$

  showing how past inputs are "filtered" by the system's impulse response.
- The state transition matrix $e^{\mathbf{A}t}$ governs both natural evolution and forced response.