ECE253/CSE208 Introduction to Information Theory

Lecture 10: Channel Capacity

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- Chap 7 and Sections 10.7-10.8 of *Elements of Information Theory (2nd Ed) by Cover & Thomas*

Discrete Memoryless Channels (DMC)

We are seeking the answers to the following two core questions in information theory:

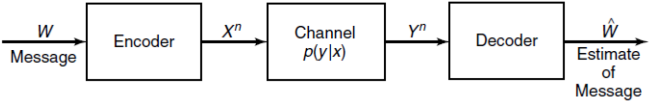
- The fundamental limit of data compression is $H(\mathcal{X})$: Chap 5
- The fundamental limit of data transmission: Chap 7

There is a duality between data compression and transmission. That is, we want to reduce redundancy as much as possible for data compression while adding controlled redundancy to combat errors for data transmission.

Definition

A **discrete channel** to be a system of an input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$ and a probability transition matrix $p(y|x)$ that expresses the probability of observing the output symbol y given that we send the symbol x. The channel is **memoryless** if the probability distribution of the output depends only on the input at the time and is conditionally independent of previous channel inputs or outputs.

Definitions of Channel Capacity



Definition

"Information" channel capacity of a DMC is defined as $\boxed{C = \max_{p(x)} I(X;Y)}$.

Definition

"Operational" channel capacity is the number of bits we can transmit reliably (with an arbitrarily low probability of error) per channel usage.

$C = \log$(number of identifiable inputs by passing through the channel with low error).

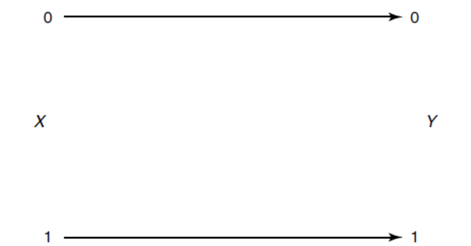
Shannon's second theorem (Noisy-channel coding theorem):

"Information" channel capacity = "operational" channel capacity

Noiseless Binary Channel

Example (Noiseless binary channel (NBC))

Operational-wise, we can transmit one bit without any error per channel use. Hence $C = 1$ bit. Meanwhile, we can calculate the *information* capacity as

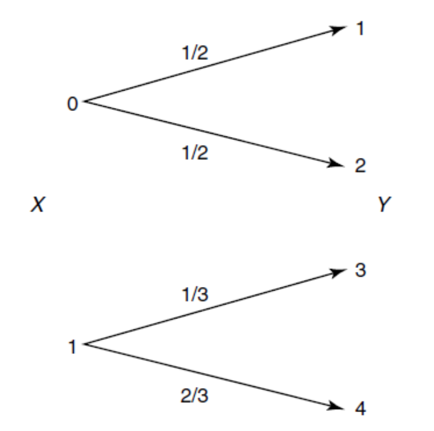


$$
\begin{aligned}
C &= \max_{p(x)} I(X;Y) \\
&= \max_{p(x)} \Big[H(X) - H(X|Y) \Big] \\
&= \max_{p(x)} H(X) \\
&= 1 \text{ bit} \qquad \text{if } p(x) = (1/2, 1/2)
\end{aligned}
$$

Noisy Channel with Non-overlapping Outputs

Example (Noisy channel with non-overlapping outputs)

Similar to the previous example, we can transmit one bit without any error per channel use. Hence $C = 1$ bit.



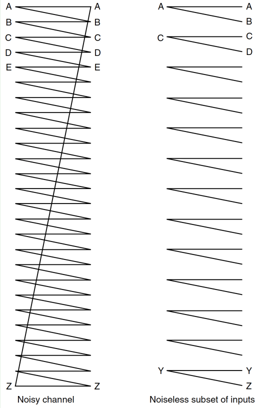
$$
\begin{aligned}
C &= \max_{p(x)} I(X;Y) \\
&= \max_{p(x)} \Big[H(X) - H(X|Y)\Big] \\
&= \max_{p(x)} H(X) \\
&= 1 \text{ bit} \qquad \text{if } p(x) = (1/2, 1/2)
\end{aligned}
$$

[Exercise: find the solution by an alternative way $C = \max_{p(x)} (H(Y) - H(Y|X))$.]

Noisy Typewriter

Example (Noisy Typewriter)

With probability 0.5, the channel input is received correctly or is transformed into the next letter. Clearly, no errors if we transmit every other symbols $\{A, C, \ldots, Z\} \to C = \log(13)$ bits.



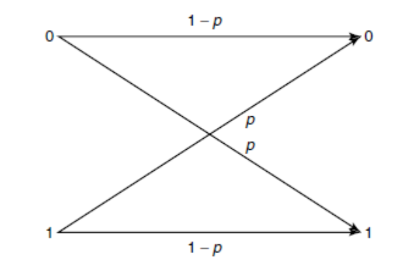
$$
\begin{aligned}
C = \max_{p(x)} I(X;Y) &= \max_{p(x)} \Big[H(Y) - H(Y|X) \Big] \\
&= \max_{p(x)} \Big[H(Y) - \sum p(x) H(Y|X=x) \Big] \\
&= \max_{p(x)} \big[H(Y) - 1 \big] = \log(13) \text{ bits}
\end{aligned}
$$

Interestingly, there are infinitely many input distributions that yield the capacity-achieving uniform distributed output, as long as the following conditions are satisfied:

$$
\begin{gathered}
\Pr(X = A) + \Pr(X = B) = 1/13 \\
\Pr(X = B) + \Pr(X = C) = 1/13 \\
\cdots \\
\Pr(X = Z) + \Pr(X = A) = 1/13
\end{gathered}
$$

Binary Symmetric Channel

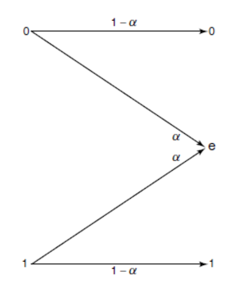
Example (Binary Symmetric Channel (BSC))



$$
\begin{aligned}
C &= \max_{p(x)} I(X;Y) \\
&= \max_{p(x)} \Big[H(Y) - H(Y|X) \Big] \\
&= \max_{p(x)} \Big[H(Y) - H(p) \Big] \qquad H(Y|X=i) = H(p),\ i = 0,1 \\
&= 1 - H(p) \text{ bits} \qquad \text{if } p(x) = (1/2, 1/2) \Rightarrow p(y) = (1/2, 1/2)
\end{aligned}
$$

Binary Erasure Channel

Example (Binary Erasure Channel (BEC))



$$
\begin{aligned}
C = \max_{p(x)} I(X;Y) &= \max_{p(x)} \Big[H(Y) - H(Y|X) \Big] \\
&= \max_{p(x)} \Big[H(Y) - H(\alpha) \Big] \\
&= \max_{\pi} \, (1-\alpha) H(\pi) \\
&= (1-\alpha) \text{ bits} \quad \text{when } \pi = 1/2
\end{aligned}
$$

Intuition: a fraction of α bits is lost in the channel, we can recover only $1-\alpha$ bits.

Define $p(X=1) := \pi$, we have

$$p(Y=0) = (1-\pi)(1-\alpha),\, p(Y=1) = \pi(1-\alpha),\, p(Y=e) = (1-\pi)\alpha + \pi\alpha = \alpha \implies$$

$$H(Y) = H\big((1-\pi)(1-\alpha), \pi(1-\alpha), \alpha\big) = H(\alpha) + (1-\alpha)H(\pi).$$

If bits are erased but the receiver is not notified (i.e. does not receive the output e), then the channel is a *deletion channel* whose capacity is an open problem!

Symmetric Channels

Example (Symmetric Channel (SC))

Given the channel, i.e., the doubly stochastic matrix collects all conditional probabilities:

$$p(y|x) = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}.$$

If all rows and columns are permutations of each other, then channel $p(y|x)$ is symmetric.

$$\begin{aligned} C &= \max_{p(x)} H(Y) - H(Y|X) \\ &= \max_{p(x)} H(Y) - H(\mathbf{r}) \quad /\mathbf{r} \text{ is any row of the above matrix}/ \\ &= \log|\mathcal{Y}| - H(\mathbf{r}) \quad /\text{achieved when output distribution is uniform}/ \end{aligned}$$

Weakly Symmetric Channel

Example (Weakly Symmetric Channel (WSC))

If all rows are permutations of each other, while all the column sums $\sum_x p(y|x)$ are equal to c (not necessarily 1), then the channel is called *weakly symmetric*; e.g.,

$$p(y|x) = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}.$$

Theorem

For a weakly symmetric channel, we have $C = \log|\mathcal{Y}| - H(\mathbf{r})$. This is achieved by a uniform distribution over $\mathcal{X}$.

uniform input $\rightarrow$ uniform output:

$$p(y) = \sum_x p(y|x)p(x) = \frac{1}{|\mathcal{X}|}\sum_x p(y|x) = \frac{c}{|\mathcal{X}|} = \frac{1}{|\mathcal{Y}|}$$

Computing Capacity of General Channels

Channel capacity has the following properties:

- Naive lower and upper bounds: $0 \le C \le \min\left(\log|\mathcal{X}|, \log|\mathcal{Y}|\right)$
- Given the channel ($p(y|x)$ is fixed), $I(X;Y)$ is a continuous concave function in $p(x)$

$$\boxed{C = \max_{p(x)\in\mathcal{P}} I(X;Y)}$$

where $\mathcal{P} = \{p(x) \mid 0 \le p(x) \le 1, \sum_x p(x) = 1\}$ denotes the probabilistic simplex.
Generally, no closed-form solution for the above convex optimization problem. But, we can leverage various algorithms to numerically evaluate the channel capacity C, e.g.,

1. Gradient search based algorithms
2. KKT conditions
3. Other iterative algorithms

Projected Gradient Ascent

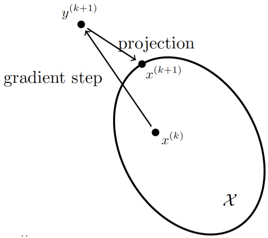
$$C = \max_{p(x)\in\mathcal{P}} I(X;Y) = \max_{p(x)\in\mathcal{P}} \sum_{x,y} p(x)p(y|x)\log\frac{p(x|y)}{p(x)} \tag{1}$$

$$= \max_{p(x)\in\mathcal{P}} \sum_{x,y} p(x)p(y|x)\log\frac{\frac{p(x)p(y|x)}{\sum_{x'} p(x')p(y|x')}}{p(x)} \tag{2}$$

Projected gradient ascent at the ℓ-th iteration:

$$p^{\ell+1}(x) = \mathsf{Proj}_{\mathcal{P}}\left[p^{\ell}(x) + \alpha^{\ell}\nabla f(p^{\ell}(x))\right],$$

where α^{ℓ} is the step size, and $\nabla f(\cdot)$ is the gradient of the objective function.



Blahut-Arimoto Algorithm[1]

Blahut-Arimoto algorithm: Treat $p(x)$ and $p(x|y)$ as independent variables (blocks).

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)} \sum_{x,y} p(x)p(y|x) \log \frac{p(x|y)}{p(x)} \tag{3}$$

$$= \max_{p(x)} \max_{q(x|y)} \sum_{x,y} p(x)p(y|x) \log \frac{q(x|y)}{p(x)} \tag{4}$$

For fixed $p(x)$, update $q(x|y)$ as

$$\boxed{q(x|y) = \frac{p(x)p(y|x)}{\sum_{x'} p(x')p(y|x')}.}$$

For fixed $q(x|y)$, update $p(x)$ as

$$\boxed{p(x) = \frac{\prod_y q(x|y)^{p(y|x)}}{\sum_{x'} \prod_y q(x'|y)^{p(y|x')}}.}$$

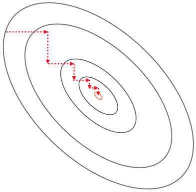


Figure: For block coordinate ascent, we only update one block at each iteration: $p^1(x) \to q^1(x|y) \to p^2(x) \to q^2(x|y) \to \cdots$

[1] S. Arimoto (1972), "An algorithm for computing the capacity of arbitrary discrete memoryless channels". R. Blahut (1972), "Computation of channel capacity and rate-distortion functions".

Two Lemmas[2]

We will show two lemmas to validate the Blahut-Arimoto algorithm. Let

$$J(p) \triangleq \sum_{x,y} p(x)p(y|x) \log \frac{p(x|y)}{p(x)} \tag{5}$$

$$J(p,q) \triangleq \sum_{x,y} p(x)p(y|x) \log \frac{q(x|y)}{p(x)} \tag{6}$$

Lemma

For any fixed p, $J(p,q) \leq J(p)$, *with equality iif* $q(x|y) = \frac{p(x)p(y|x)}{\sum_{x'} p(x')p(y|x')} = p(x|y)$.

Lemma

For any fixed $q(x|y)$, $J(p,q) \leq \log(\sum_x r(x))$, *where* $r(x) = \exp\left(\sum_y p(y|x) \log q(x|y)\right)$ *with equality iif* $p(x) = \frac{r(x)}{\sum_{x'} r(x')}$.

[2] http://ecse.rpi.edu/~pearlman/lec_notes/arimoto_2.pdf

Proof of Two Lemmas

Recall that

$$J(p) \triangleq \sum_{x,y} p(x)p(y|x) \log \frac{p(x|y)}{p(x)} \quad \text{and} \quad J(p,q) \triangleq \sum_{x,y} p(x)p(y|x) \log \frac{q(x|y)}{p(x)}.$$

Proof of Lemma 1:

$$J(p) - J(p,q) = \sum_{x,y} p(x)p(y|x) \log \left[\frac{p(x|y)}{q(x|y)}\right] = D(p(x|y)||q(x|y)) \geq 0$$

Proof of Lemma 2:

$$\begin{aligned} J(p,q) &= \sum_{x,y} p(x)p(y|x) \log \frac{q(x|y)}{p(x)} = \sum_{x,y} p(x)p(y|x) \log \frac{1}{p(x)} + \sum_{x,y} p(x)p(y|x) \log q(x|y) \\ &= \sum_{x} p(x) \log \left(\frac{\exp \sum_y p(y|x) \log q(x|y)}{p(x)}\right) = \sum_x p(x) \log \left(\frac{r(x)}{p(x)}\right) \\ &= \sum_x p(x) \log \left(\frac{\tilde{r}(x)}{p(x)} \times \sum_x r(x)\right), \quad /\tilde{r}(x) \triangleq \frac{r(x)}{\sum_x r(x)}/ \\ &= \sum_x p(x) \log \left(\sum_x r(x)\right) - D(p||\tilde{r}) \leq \log \left(\sum_x r(x)\right) \end{aligned}$$

with equality iif $p(x) = \tilde{r}(x)$.

Thank You!

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