ECE253/CSE208 Introduction to Information Theory

Lecture 11: Channel Coding Theorem & Separation Principle

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- Chap 7 of *Elements of Information Theory (2nd Edition) by Thomas Cover & Joy Thomas*

Communication Diagram

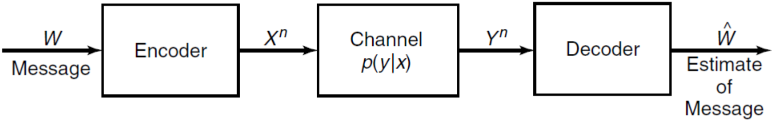


Figure: A diagram showing how a message is communicated through a noisy channel.

- Essentially, the communication system represents a Markov chain:

$$W \to X^n \to Y^n \to \hat{W}.$$

- Here, the encoder/decoder block represents a joint source-channel encoder/decoder.

Shannon's Second Theorem

- Reliable (virtually error-free) communication is possible at rates up to the capacity.
- Channel capacity is the sharp threshold between reliable and unreliable communication.

Theorem (Channel Coding Theorem)

For a DMC, all rates below capacity C are achievable. Specifically, for every rate $R < C$, there exists a sequence of $(2^{nR}, n)$ codes with maximum probability of error $\lambda_{\max}^{(n)} \to 0$ as $n \to \infty$. Conversely, any sequence of $(2^{nR}, n)$ codes with $\lambda_{\max}^{(n)} \to 0$ must have $R \leq C$.

Let $P_e^{(n)}$ denote the average probability of error, and A be a given finite non-negative constant. The weak and strong versions of the converse statement are given as follows.

- *Weak converse:* $P_e^{(n)} \geq 1 - \frac{1}{nR} - \frac{C}{R} \implies$ If $R > C$, $P_e^{(n)}$ is bounded away from zero as $n \to \infty$.
- *Strong converse:* $P_e^{(n)} \geq 1 - \frac{4A}{n(R-C)^2} - e^{\frac{-n(R-C)}{2}} \implies$ If $R > C$, $P_e^{(n)} \xrightarrow{n\to\infty} 1$.

Discrete Channel and Its Extension

A few definitions are needed for the proof of the channel coding theorem.

Definition

A discrete channel, denoted by $(\mathcal{X}, p(y|x), \mathcal{Y})$ consists of two finite sets $\mathcal{X}, \mathcal{Y}$ and a collection of probability mass functions $p(y|x)$. Assume $p(y|x) \geq 0$ for all (x, y). For all x, $\sum_y p(y|x) = 1$. Note that (x, y) is the input-output pair of the channel.

Definition

The n-th extension of the DMC is $(\mathcal{X}^n, p(y^n|x^n), \mathcal{Y}^n)$, where $p(y_k|x^k, y^{k-1}) = p(y_k|x_k)$ for $k = 1, 2, \ldots, n$.

For a channel without feedback, we have

$$p(x_k|x^{k-1}, y^{k-1}) = p(x_k|x_{k-1}) \Rightarrow p(y^n|x^n) = \prod_{i=1}^{n} p(y_i|x_i).$$

(M, n) Code and Code Rate

Definition ((M, n) code)

An (M, n) code for the channel $(\mathcal{X}, p(y|x), \mathcal{Y})$ consists of the following:

1. Message $W \in \{1, 2, \dots, M\} \triangleq \mathcal{M}$, where M is the size of the message set.
2. An encoding function: $X^n : \mathcal{M} \to \mathcal{X}^n$ yields the codebook $\mathcal{C} = [x^n(1), \dots, x^n(M)]$.
3. A deterministic decoding function: $g : \mathcal{Y}^n \to \mathcal{M}$ yields an estimate $\hat{W}$.

Definition

The rate R of an (M, n) code is $R = \frac{\log M}{n}$ bits per transmission.

- **Code rate R is the number of info bits conveyed per channel use.** If we only consider channel coding, then for every $k \triangleq \log M$ bits of useful information, the coder generates a total of n bits of data, of which $n - k$ are redundant for error detection/correction. Hence, the rate R quantifies the coder's efficiency.
- For notational simplicity, we write $(2^{nR}, n)$ codes to mean $(\lceil 2^{nR} \rceil, n)$ codes.

Probability of Error

Definition

- The **conditional** probability of error is

$$\boxed{\lambda_i := \Pr\left(g(Y^n) \neq i \mid X^n = x^n(i)\right) = \sum_{y^n} p(y^n|x^n(i)) \times \mathbb{1}(g(y^n) \neq i)}$$

- The **maximum** probability of error $\lambda_{\max}^{(n)}$ for an (M, n) code is

$$\boxed{\lambda_{\max}^{(n)} = \max_{i \in \{1,\ldots,M\}} \lambda_i.}$$

- The **average** probability of error is

$$\boxed{P_e^{(n)} = \frac{1}{M} \sum_{i=1}^{M} \lambda_i.}$$

Clearly, we have $P_e^{(n)} \leq \lambda_{\max}^{(n)}$. If the message W is chosen uniformly over $\mathcal{M}$ and $X^n = x^n(w)$, then $P_e^{(n)} = \Pr(W \neq g(Y^n))$.

Achievable Rate

Definition

A rate R is said to be achievable if there exists a sequence of $(\lceil 2^{nR} \rceil, n)$ codes such that $\lambda_{\max}^{(n)} \xrightarrow{n\to\infty} 0$.

Definition

The capacity of a channel is the supremum[a] of all achievable rates.

[a] In terms of sets, the *maximum* is the largest member of the set while the *supremum* is the smallest upper bound of the set. supremum = maximum for compact sets.

Joint Typicality Decoding

Roughly speaking, we decode as $g(Y^n) \mapsto w$, if $X^n(w)$ is *jointly typical* with Y^n.
Recall the following

$$A_\epsilon^{(n)} = \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \left| -\frac{1}{n} \log p(X^n) - H(X) \right| < \epsilon, \quad \left| -\frac{1}{n} \log p(Y^n) - H(Y) \right| < \epsilon, \right.$$
$$\left. \left| -\frac{1}{n} \log p(X^n, Y^n) - H(X,Y) \right| < \epsilon \right\}.$$

Theorem (Joint ARP)

1. $\Pr\left((X^n, Y^n) \in A_\epsilon^{(n)} \right) \to 1$ *as* $n \to \infty$.
2. $(1-\epsilon) 2^{n(H(X,Y)-\epsilon)} \le |A_\epsilon^{(n)}| \le 2^{n(H(X,Y)+\epsilon)}$.
3. *If* $(\tilde{X}^n, \tilde{Y}^n) \sim p(x^n)p(y^n)$, *then*

$$(1-\epsilon) 2^{-n(I(X;Y)+3\epsilon)} \le \Pr\left((\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)} \right) \le 2^{-n(I(X;Y)-3\epsilon)},$$

where both lower bounds in parts 2 and 3 hold for n sufficiently large.

Intuitive Proof of the Theorem

- Typical sets $|X^n| \approx 2^{nH(X)}$ and $|Y^n| \approx 2^{nH(Y)}$.
- Only about $2^{nH(X,Y)}$ paris are joint typical (not all pairs of typical X^n and typical Y^n are jointly typical):
- The probability of any randomly chosen pair is jointly typical is about

$$\frac{2^{nH(X,Y)}}{2^{n(H(X)+H(Y))}} = 2^{-nI(X;Y)}.$$

- This implies that there are about $2^{nI(X;Y)}$ distinguishable input signals X^n.
- If the number of possible input codewords is 2^{nR} with $R \leq I(X;Y) - \epsilon$, then $P_e^{(n)} = 2^{nR} \times 2^{-nI(X;Y)} \leq 2^{-n\epsilon} \to 0$ as $n \to \infty$.

Intuitive Proof: Sphere Packing

- For large block lengths, every channel looks like a noisy typewriter channel.
- For each (typical) input X sequence, there are about $2^{nH(Y|X)}$ possible Y sequences (all of them equally likely). we have about $2^{nH(Y)}$ typical Y sequences.
- Hence, the total number of disjoint sets we can afford is $\frac{2^{nH(Y)}}{2^{nH(Y|X)}} = 2^{nI(X;Y)}$ and C is no greater than $I(X;Y)$ (maximized over $p(x)$).

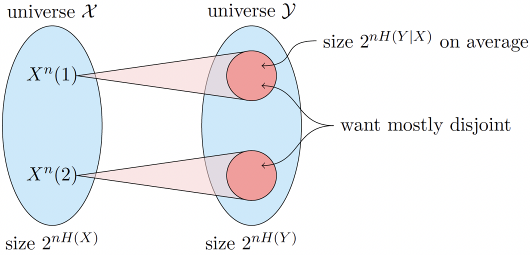


Figure: How many distinguishable input sequences X^n can produce disjoint sequences at the output? Figure credit to V. Guruswami's lecture note.

Proof Outline

1. At the transmitter, use **random coding**.
2. $W \in \{1, 2, \ldots, 2^{nR}\}$ has a uniform distribution.
3. At the receiver, use **jointly typical decoding** for Y^n to find $X^n(w)$.
   We will bound two types of error:
   - Type-1 error: $X^n(w)$ is not jointly typical with Y^n; and
   - Type-2 error: find a sequence $\tilde{X}^n(\hat{w})$ is jointly typical with Y^n, but $\hat{w} \neq w$.
4. Use joint AEP to prove achievability (direct part) and Fano's inequality for the converse statement.

Proof of the Channel Coding Theorem

On the sender side, do the following:

1. Randomly generate a $\left(2^{nR}, n\right)$ code according to a fixed $p(x)$. Specifically, we generate 2^{nR} codewords independently according to the distribution $p\left(x^n\right) = \prod_{i=1}^{n} p\left(x_i\right)$. Collect codewords as the rows of the codebook:

$$\mathcal{C} = \begin{bmatrix} x_1(1) & x_2(1) & \dots & x_n(1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1\left(2^{nR}\right) & x_2\left(2^{nR}\right) & \cdots & x_n\left(2^{nR}\right) \end{bmatrix}.$$

   The codebook $\mathcal{C}$ is known to both the encoder and the decoder.

   Each entry is i.i.d. $\sim p(x)$. Thus, the probability of a particular code $\mathcal{C}$ is given by

$$\Pr(\mathcal{C}) = \prod_{w=1}^{2^{nR}} \prod_{i=1}^{n} p\left(x_i(w)\right).$$

2. Uniformly choose a message W: $\Pr(W = w) = 2^{-nR},\ w = 1, 2, \dots, 2^{nR}$.

Proof of the Channel Coding Theorem (cont'd)

On the receiver side, do the following:

1. Obtain a sequence Y^n according to $\Pr\left(y^n|x^n(w)\right) = \prod_{i=1}^n p\left(y_i|x_i(w)\right)$.
2. Guess which message was sent. For the *jointly typical decoding*, the receiver declares:
   - **index** $\hat{W}$ **was sent** if $(X^n(\hat{W}), Y^n)$ is jointly typical, and there is no other message W' such that $(X^n(W'), Y^n)$ is jointly typical.
   - **an error** if no such $\hat{W}$ or more than one such.
3. Calculate the probability of errors. Let $\mathcal{E} = \left\{\hat{W}(Y^n) \neq W\right\}$ denote the error event.

$$P_e^{(n)} = \Pr(\mathcal{E}) = \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_w(\mathcal{C}) = \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_1(\mathcal{C}) = \Pr(\mathcal{E} \mid W = 1).$$

Note that this is the probability of error averaged over all codebooks and codewords.

Proof of the Channel Coding Theorem (cont'd)

4. For $i \in \{1, 2 \ldots, 2^{nR}\}$, let $E_i \triangleq \left\{(X^n(i), Y^n) \in A_\epsilon^{(n)}\right\}$ denote the event that the i-th codeword and Y^n are jointly typical. WLOG, assume that Y^n is the received sequence by sending $X^n(1)$ over the channel.

$$\Pr(\mathcal{E}|W=1) = \Pr\left(E_1^c \cup E_2 \cup E_3 \cup \cdots \cup E_{2^{nR}} | W=1\right) \tag{1}$$

$$\leq \underbrace{\Pr\left(E_1^c | W=1\right)}_{\text{type-I error}} + \sum_{i=2}^{2^{nR}} \underbrace{\Pr\left(E_i | W=1\right)}_{\text{type-II error}} \tag{2}$$

By the joint AEP, we have

- For the type-I error, $\Pr\left(E_1^c | W=1\right) \leq \epsilon$ for n sufficiently large.
- For the type-II error, $\Pr\left(E_i | W=1\right) \leq 2^{-n(I(X;Y)-3\epsilon)}$, $\forall i \neq 1$. Note that for any $i \neq 1$, Y^n and $X^n(i)$ are independent.

Proof of the Channel Coding Theorem (cont'd)

Therefore, we have

$$\Pr(\mathcal{E}) = \Pr(\mathcal{E}|W=1) \leq \Pr\left(E_1^c|W=1\right) + \sum_{i=2}^{2^{nR}} \Pr\left(E_i|W=1\right) \tag{3}$$

$$\leq \epsilon + (2^{nR}-1) \times 2^{-n(I(X;Y)-3\epsilon)} \tag{4}$$

$$\leq \epsilon + 2^{-n(I(X;Y)-3\epsilon-R)} \tag{5}$$

$$\leq 2\epsilon, \tag{6}$$

if n is sufficiently large and $R < I(X;Y) - 3\epsilon$.

Hence, if $R < I(X;Y)$, we can choose ϵ and n so that the average probability of error is less than 2ϵ.

Proof of the Channel Coding Theorem (cont'd)

We can strengthen the conclusion by a series of code selections.

1. In the proof, set $p(x) = p^*(x)$, which is the optimal input distribution achieving the capacity. Then the condition $R < I(X;Y)$ becomes $R < C$.
2. Get rid of the averaging over codebooks. Since the average probability of error over codebooks is less than 2ϵ, there exists at least one codebook $\mathcal{C}^*$ such that
$$\Pr\left(\mathcal{E}|\mathcal{C}^*\right) = \frac{1}{2^{nR}} \sum_{i=1}^{2^{nR}} \lambda_i\left(\mathcal{C}^*\right) \le 2\epsilon. \tag{7}$$
We can find $\mathcal{C}^*$ by exhaustive search over a total of $|\mathcal{X}|^{Mn}$ possible codebooks.
3. Relate $P_e^{(n)}$ to $\lambda_{\max}^{(n)}$: Discard the worst half of the codewords in $\mathcal{C}^*$.
Due to (7), we know that at least half the $\lambda_i(\mathcal{C}^*)$ are less than 4ϵ. If we keep this half of the codewords and discard the remaining half, we get $\lambda_{\max}^{(n)} \le 4\epsilon$ while the rate changes from R to $R - \frac{1}{n}$ (negligible for large n).

Zero-error Codes

The outline of the proof of the converse is most clearly motivated by going through the argument when absolutely no errors are allowed. We now prove that $P_e^{(n)} = 0$ implies that $R \leq C$.

Proof:

$$
\begin{aligned}
nR = H(W) &= \underbrace{H\left(W|Y^n\right)}_{=0} + I\left(W;Y^n\right) \\
&= I\left(W;Y^n\right) \leq I\left(X^n;Y^n\right) \leq \sum_{i=1}^{n} I\left(X_i;Y_i\right) \leq nC
\end{aligned}
$$

Lemma (Fano's inequality)

For a DMC with a codebook $\mathcal{C}$ and the input message W uniformly distributed over 2^{nR}, we have $H(W|\hat{W}) \leq 1 + P_e^{(n)} nR$.

Proof of the Converse Statement

Lemma

For a DMC of capacity C, we have $I(X^n; Y^n) \leq nC$ for all $p(x^n)$.

Proof:

$$
\begin{aligned}
I(X^n; Y^n) = H(Y^n) - H(Y^n|X^n) = H(Y^n) - \sum_{i=1}^{n} H(Y_i|Y_1, \ldots, Y_{i-1}, X^n) \\
\leq \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|X_i) \\
= \sum_{i=1}^{n} I(X_i; Y_i) \leq nC
\end{aligned}
$$

$$
\begin{aligned}
nR = H(W) &= H(W|\hat{W}) + I(W; \hat{W}) \\
&\leq H(W|\hat{W}) + I(X^n; Y^n) \\
&\leq 1 + P_e^{(n)} nR + nC \implies
\end{aligned}
$$

$P_e^{(n)} \geq 1 - \frac{C}{R} - \frac{1}{nR} \implies$ If $R > C$, then $P_e^{(n)}$ is bounded away from 0 as $n \to \infty$.

Feedback Capacity

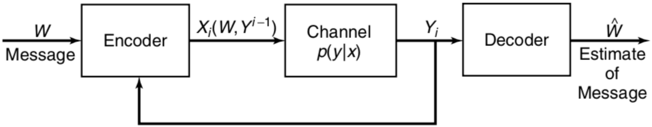


Figure: Discrete memoryless channel (DMC) with feedback.

For a DMC, feedback may help simplify encoding and decoding (e.g., for BEC), but will not increase the channel capacity.

Theorem (Feedback does not increase the channel capacity for a DMC)

For a DMC, $C_{FB} = C = \max_{p(x)} I(X;Y)$.

Proof: Clearly, $C_{FB} \geq C$. We need to show $C_{FB} \leq C$.

$$\begin{aligned} nR = H(W|\hat{W}) + I(W;\hat{W}) &\leq 1 + P_e^{(n)} nR + I(W;\hat{W}) \\ &\leq 1 + P_e^{(n)} nR + \boxed{I(W;Y^n)} \quad \text{[by DPI]} \end{aligned}$$

Proof (cont'd)

$$
\begin{aligned}
I(W;Y^n) &= H(Y^n) - H(Y^n|W) = H(Y^n) - \sum_{i=1}^{n} H(Y_i|Y_1,\dots,Y_{i-1},W) \\
&= H(Y^n) - \sum_{i=1}^{n} H\left(Y_i|Y_1,\dots,Y_{i-1},W,X_i(W,Y^{i-1})\right) \quad \text{[due to feedback]} \\
&\le \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|X_i) = \sum_{i=1}^{n} I(X_i;Y_i) \le nC \implies
\end{aligned}
$$

$$
nR \le 1 + P_e^{(n)} nR + nC \implies \boxed{R \le \frac{1}{n} + P_e^{(n)} R + C.}
$$

Finally, taking $n \to \infty$ and $P_e^{(n)} \to 0$, we get $R \le C$.

Remark

- $C_{\text{FB}} = C$: **A higher rate with feedback cannot be achieved for a DMC**.
- The availability of feedback often makes coding simpler.
- In general, if the channel has memory, feedback can increase the capacity.

Communication beyond Capacity[1]

Theorem (Communication with error)

If a probability of bit error P_b is acceptable, rates up to $R(P_b)$ are achievable, where

$$R(P_b) = \frac{C}{1 - H(P_b)}.$$

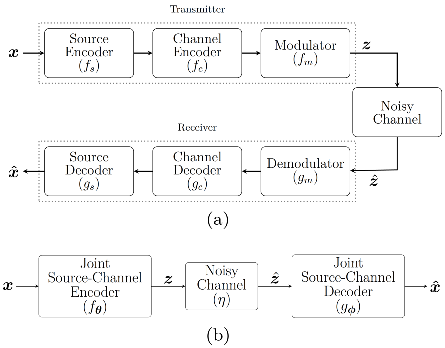
For any P_b, rates greater than $R(P_b)$ are not achievable.

[1] Page 162 of the book "Information Theory, Inference and Learning Algorithms" by David J. MacKay

Joint or Separate Coding?

Consider transmitting digitized speech or music across a DMC. Two options:

1. Compress the speech into its most efficient representation and then utilize the suitable channel code for transmission.
2. Design a code to directly map the speech samples into the channel input.
3. It is not immediately evident that we are not sacrificing anything by employing the two-stage (tandem) method, as data compression is independent of the channel, and the channel coding is unrelated to the source distribution.



Source-Channel Separation Theorem

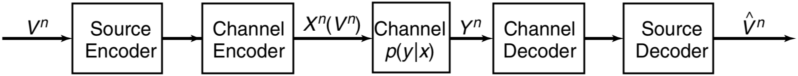


Figure: Separate coding scheme: Source coding reduces redundancy for data compression) while channel coding introduces structured redundancy for error detection/correction.

- **Source Coding Theorem**: If source symbols are compressed to $R_s > H$ information bits/source symbol, lossless compression is possible.
- **Channel Coding Theorem**: As long as $R_c < C$ information bits are transmitted per channel use, error-free transmission is possible.
- **Source-Channel Separation Theorem**: Under certain conditions, the separate design of source coding and channel coding is asymptotically optimal ($n \to \infty$).
- By using the two-stage procedure, we can send a source with entropy H reliably through a channel with capacity C provided $H < C$. Essentially, we have

$$\boxed{H < R_s \leq R_c < C}$$

Joint Source-Channel Coding Theorem (JSCC)

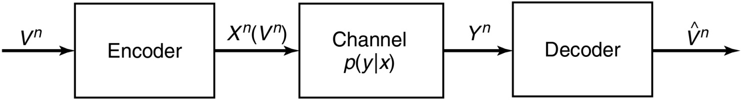


Figure: Joint source-channel coding scheme: Send the sequence of symbols $V^n = \{V_1, \ldots, V_n\}$ over the channel for the decoder $g(\cdot)$ to reconstruct the sequence. The probability of error is defined as $\Pr\left(V^n \neq \hat{V}^n\right) = \sum_{\{y^n, v^n\}} p(v^n)\, p(y^n \mid x^n(v^n))\, I(g(y^n) \neq v^n)$, where $I(\cdot)$ is the indicator function.

Theorem (A source with entropy rate H can be sent reliably over DMC iif $H < C$.)

If $V_1, V_2, \ldots, V_n$ is a finite alphabet stochastic process that satisfies the AEP and $H(\mathcal{V}) < C$, there exists a source-channel code with probability of error $\Pr(\hat{V}^n \neq V^n) \to 0$. Conversely, for any stationary stochastic process, if $H(\mathcal{V}) > C$, the probability of error is bounded away from zero, and it is not possible to send the process over the channel with an arbitrarily low probability of error.

Proof. Similar to the aforementioned proof, we can use typicality coding for *achievability* and Fano's ineq for *converse*; see details on page 220-221 of Cover's book.

JSCC (cont'd)

- JSCC theorem establishes the limit of achievable performance (i.e., upper bound on the maximum achievable transmission rate) when source coding and channel coding are done together.
- JSCC considers the characteristics of both the source and the channel.

Finite alphabet stochastic processes (a.k.a. finite-state stochastic process: a sequence of random events where the outcomes come from a finite set of symbols). Examples satisfying the AEP:

1. A sequence of i.i.d. random variables
2. A stationary irreducible Markov chain
3. Any stationary ergodic process (time average = ensemble average) [*Shannon-McMillan-Breiman Theorem*].

JSCC and Separation Theorem

Engineering implications: Two-stage can be as good as the single-stage.

- Asymptotic optimality can be achieved by separating source and channel coding.
- Design source codes for the most efficient representation of the data.
- *Separately and independently* design channel codes appropriate for the channel.

Caveat: Two-stage is not always optimal.

- The separation theorem applies only to point-to-point memoryless sources and channels. For general source/channel models, joint optimization is needed to achieve the optimum performance.
- Sending English text over an erasure channel: corrupted bits are difficult to decode.
- Redundancy in the source is suited to the channel: speech for the human ear.
- Multiuser channels.

JSCC and Separation Theorem (cont'd)

More works and insights on this topic:

- S. Vembu, S. Verdu and Y. Steinberg (1995), *The source-channel separation theorem revisited*.
- D. Gunduz, E. Erkip, A. Goldsmith and H. V. Poor (2009), *Source and Channel Coding for Correlated Sources Over Multiuser Channels*.
- K. Khezeli and J. Chen (2016), *A Source-Channel Separation Theorem With Application to the Source Broadcast Problem*.
- Yury Polyanskiy and Yihong Wu (2022), *Information Theory From Coding to Learning (Section 19.7)*.
- Deniz Gunduz (2019), *Joint Source and Channel Coding: Fundamental Bounds and Connections to Machine Learning*.
- Yuval Kochman (2020), *Some fundamental bounds in joint source-channel coding*.
- Po-Ning Chen (2019), *Lossless joint source-channel coding and Shannon's separation principle*.

Thank You!

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