ECE253/CSE208 Introduction to Information Theory

Lecture 2: Probability Theory Revisited

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- H. Stark and J. Woods, "Probability, Random Processes, and Estimation Theory for Engineers".
- A. Papoulis, "Probability, Random Variables, and Stochastic Processes".
- P. Cameron, "Notes on Probability".

Probability Space

Probability space is a three-tuple $(\Omega, \mathcal{F}, P)$:

- **Sample space** Ω: the set of all outcomes of a random experiment.
  - Ω may be finite ($\{H, T\}$), countably infinite ($\Omega = \{1,2,3,...\}$), or uncountably infinite ($[0,1]$).
- **Event space** $\mathcal{F}$: a set whose elements $A \in \mathcal{F}$ are subsets of Ω.
- **Probability function** P: satisfies three axioms:
  - $P(A) \geq 0$, $\forall A \in \mathcal{F}$.
  - $P(\Omega) = 1$.
  - If $A_1, A_2, ...$ are mutually exclusive events; i.e. $A_i \cap A_j = \emptyset$, $\forall i \neq j$, then

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i).$$

Properties of Probability

Consider events $A, B, C \subseteq \Omega$. Let $P(AB)$ denote $P(A \cap B)$.

- $A \subseteq B \Rightarrow P(A) \leq P(B)$.
- $P(AB) \leq \min\{P(A), P(B)\}$.
- $P(A \cup B) = P(A) + P(B) - P(AB)$.
- $P(\bar{A}) = 1 - P(A)$: $\bar{A} = \Omega \backslash A$.
- Complement rule (De Morgan's Law): $\overline{AB} = \bar{A} \cup \bar{B}$; $\overline{A \cup B} = \bar{A}\bar{B}$
- $P(AB) = P(A|B)P(B) = P(B|A)P(A)$.
- **Conditional probability**: $P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$ (Bayes' theorem).
- If A and B are independent, then $P(AB) = P(A)P(B)$.
- Distributive law: $(A \cup B) \cap C = (AC) \cup (BC)$.

Venn Diagram

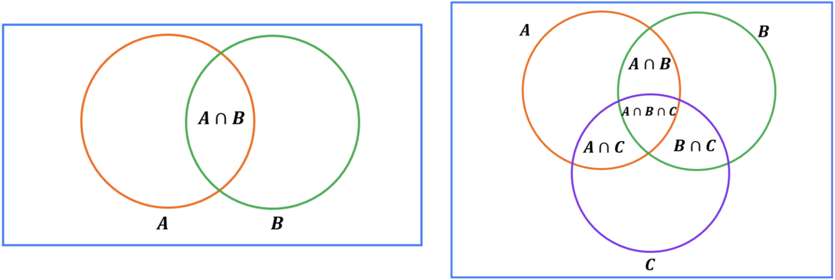


Figure: A Venn diagram is a diagram that shows the relationship between and among a finite collection of sets.

Conditional vis-à-vis Unconditional Probability

Q: $P(A|B) \lesseqqgtr P(A)$?

A: It can be any case in general.

Some special cases:

- If A and B are independent (denoted as $A \perp\!\!\!\perp B$), then $P(A|B) = P(A)$.
- If $AB = \emptyset$ and $0 < P(A), P(B) < 1$, then
  $P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0 < P(A) \Rightarrow P(A|B) < P(A)$.
- If $B \subset A$ and $P(A) < 1$, then
  $P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B)}{P(B)} = 1 > P(A) \Rightarrow P(A|B) > P(A)$.
- If $A \subseteq B$, and $0 < P(B) < 1$, then
  $P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} > P(A)$.

A Concrete Example

Take a standard deck of cards (52 cards without Jokers). Remove all black queens and kings. When picking a card, define 3 events:

- A: The card picked is a face card.
- B_1: The card picked is a heart.
- B_2: The card picked is a spade.

Question: Find the values of $P(A), P(A|B_1), P(A|B_2)$.

- A: 48 cards left with 8 face cards: $P(A) = \frac{1}{6}$.
- B_1: 13 cards of hearts left with 3 face cards: $P(A|B_1) = \frac{3}{13}$.
- B_2: 11 card of spades left with only 1 face card (the Jack of Spades): $P(A|B_2) = \frac{1}{11}$.

We have $P(A|B_2) < P(A) < P(A|B_1)$.

Monty Hall Problem (a.k.a. Three Doors Problem)[1]

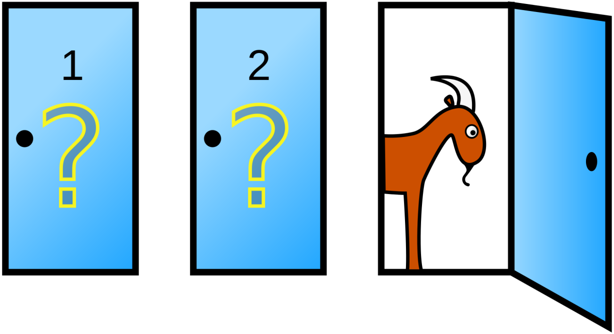
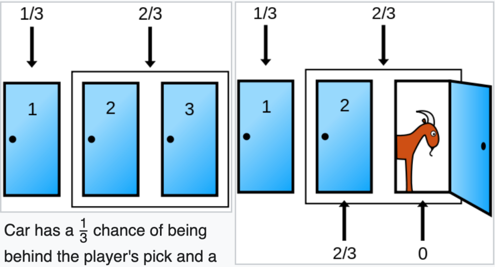


Figure: Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

[1] https://en.wikipedia.org/wiki/Monty_Hall_problem

Yes, switch!



Car has a $\frac{1}{3}$ chance of being behind the player's pick and a $\frac{2}{3}$ chance of being behind one of the other two doors.

The host opens a door, the odds for the two sets don't change but the odds move to 0 for the open door and $\frac{2}{3}$ for the closed door.

- Alternative: think of the complement about the answer $\frac{2}{3}$. The only way to get it wrong by switching is to have picked the correct door in the first place. The odds of picking the correct door first are $\frac{1}{3}$.
- **The more you know, the better your decision!**

Solution to Monty Hall Problem (cont'd)

Behind door 1	**Behind door 2**	**Behind door 3**	**Result if staying at door #1**	**Result if switching to the door offered**
Goat	Goat	**Car**	Wins goat	**Wins car**
Goat	**Car**	Goat	Wins goat	**Wins car**
Car	Goat	Goat	**Wins car**	Wins goat

- Most people come to the conclusion that switching does not matter because there are two unopened doors and one car and that it is a 50/50 choice.
- This would be true if the host opens a door randomly, but that is not the case; the door opened depends on the player's initial choice, so the assumption of independence does not hold.

Chain Rule for Conditional Probability

For 3 events, A, B and C, we have

$$P(ABC) = P(C|AB)P(AB) = P(C|AB)P(B|A)P(A).$$

This can be generalized to n events $A_1, A_2, ..., A_n$ (let $A_0 := \Omega$). We have

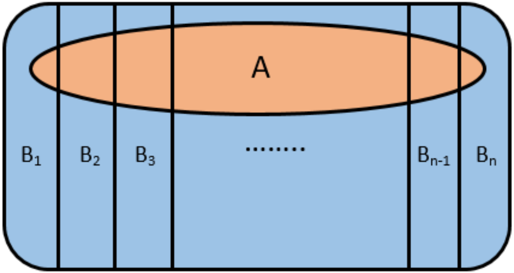
$$\begin{aligned} P(A_1 A_2 \cdots A_n) &= \prod_{i=1}^{n} P(A_i | A_1 A_2 \cdots A_{i-1}) \\ &= P(A_n | A_1 A_2 \cdots A_{n-1}) \times P(A_{n-1} | A_1 A_2 \cdots A_{n-2}) \times \cdots \\ &\qquad \times P(A_2 | A_1) \times P(A_1) \end{aligned}$$

Partition Theorem

Theorem (Partition Theorem, a.k.a. Law of Total Probability)

Let $B_1, B_2, ..., B_n$ form a **partition** *(i.e., $B_i B_j = \emptyset, \forall i \neq j$ and $\bigcup_i B_i = \Omega$) of the sample space Ω, and assume $P(B_i) \neq 0$ for all i. Then,*

$$\boxed{P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)}$$



The Birthday Paradox

Given n people named $p_1, p_2, ..., p_n$, what is the probability that at least two of them have the same birthday?

Solution 1:

Let A_2 be the event that p_2 has a different birthday from p_1. p_1 has their birthday on one of the days of the entire year. Hence, $P(A_2) = 1 - \frac{1}{365}$.

Let A_3 be the event that p_3 has a different birthday from p_2 and p_1. So,

$A_3|A_2$ denotes the event that p_3 has a different birthday from p_2 and p_1 given that p_2 and p_1 have different birthdays. We have $P(A_3|A_2) = 1 - \frac{2}{365}$.

A_2A_3 is the event that p_1, p_2, and p_3 have 3 different birthdays.

$$P(A_2A_3) = P(A_3|A_2)P(A_2) = \left(1 - \frac{2}{365}\right)\left(1 - \frac{1}{365}\right) = \frac{363 \times 364}{365^2} \approx 0.9918$$

$$\begin{aligned} P(A_3) &= P(A_3|A_2)P(A_2) + P(A_3|\bar{A}_2)P(\bar{A}_2) \\ &= \left(1 - \frac{2}{365}\right)\left(1 - \frac{1}{365}\right) + \left(1 - \frac{1}{365}\right) \times \frac{1}{365} \approx 0.9945 \end{aligned}$$

The Birthday Paradox (Cont'd)

Now, define a general A_i as the event that the birthday of p_i is not the same day as any of the birthdays of $p_1, p_2, ..., p_{i-1}$. We have $P(A_i|A_1A_2\cdots A_{i-1}) = 1 - \frac{i-1}{365}$.

The probability that all n people have different birthdays is

$$
\begin{aligned}
q_n := P(A_1A_2\cdots A_n) &= \prod_{i=1}^{n} P(A_i|A_1A_2\cdots A_{i-1}) \\
&= \left(1 - \frac{n-1}{365}\right) \times \left(1 - \frac{n-2}{365}\right) \times \cdots \times \left(1 - \frac{1}{365}\right)
\end{aligned}
$$

Note that $P(A_1) = 1 - \frac{0}{365} = 1$.

The complement event of $A_1A_2\cdots A_n$ is at least two people have the same birthday.

Hence, the probability we try to find is $b_n = 1 - q_n$, which is an increasing function in n.

$\Longrightarrow b_{23} = 0.507;\ b_{30} = 0.706;\ b_{40} = 0.891;\ b_{70} \approx 0.999.$

Solution 2:

To directly calculate q_n. The total number of all possibilities is 365^n since each person has 365 days to choose as his/her birthday. The number of cases that they all have different birthdays is 365 permute n: ${}^{365}P_n = \frac{365!}{(365-n)!}$. Hence, $q_n = \frac{365!}{(365-n)! \times 365^n}$.

Random Variables (RV)

Definition (Random Variable)

Let Ω be the sample space of an experiment, and $\mathbb{R}$ denote the set of real numbers. Then, a *random variable* $X : \Omega \mapsto \mathbb{R}$ associated with this experiment is a function that assigns each outcome in Ω to a real number. The range of X is denoted as $\mathrm{val}(X)$.

Example. Flip a coin 5 times. Let X denote the random variable for the number of times the coin came up heads. Then $X(\omega_0) = 3$, for the outcome $\omega_0 = \{\text{HHTHT}\}$.

Two different types of random variables that are often studied: **discrete** and **continuous**.

Random Variables (cont'd)

If X is a discrete random variable, we use the notation

$$\Pr(X = k) \coloneqq \Pr(\{\omega : X(\omega) = k\})$$

for the probability of the event $X = k$.

If X is a continuous random variable, we use the notation

$$\Pr(a \leq X \leq b) \coloneqq \Pr(\{\omega : a \leq X(\omega) \leq b\})$$

for the probability of the event $a \leq X \leq b$.

Cumulative Distribution Function (CDF)

Definition

Let X be a random variable associated with an experiment. Then,

$$\boxed{F_X(x) \coloneqq \Pr(X \leq x)}$$

is a *cumulative distribution function*.

Properties of CDFs:

(a) $0 \leq F_X(x) \leq 1$.

(b) $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$.

(c) F_X is nondecreasing, namely if $x \leq y$ then $F_X(x) \leq F_X(y)$.

(d) F_X is right-continuous, i.e. $\lim_{x \to a^+} F_X(x) = F_X(a)$.

CDF for Discrete and Continuous RVs

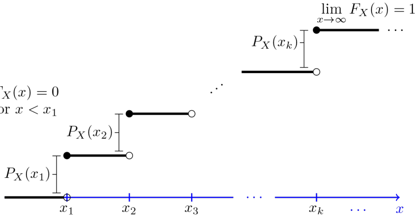


Figure: CDF of a discrete RV: staircase function.

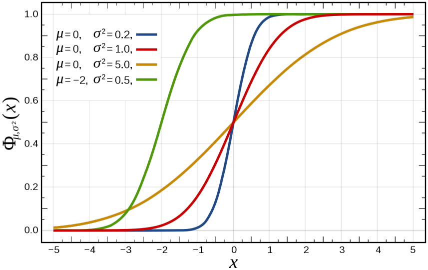


Figure: CDF of a continuous RV: continuous function.

PMF and PDF

For discrete random variables, we have the *probability mass function* (PMF):

$$p_X(x) \coloneqq \Pr(X = x),$$

where $\sum_{x \in \mathrm{val}(X)} p_X(x) = 1$.

For continuous random variables, we instead consider *probability density function* (PDF),

$$f_X(x) \coloneqq \frac{dF_X(x)}{dx} = F'_X(x),$$

provided that F_X is differentiable at x.[2] Notice that $f_X(x)$ and $\Pr(X = x)$ are two different concepts, which can be related by

$$\Pr(x \leq X \leq x + \Delta x) \approx f_X(x)\Delta x$$

and

$$\Pr(X \in A) = \int_{x \in A} f_X(x)dx,$$

where $A \subseteq \mathrm{val}(X)$.

[2]Note that F_X may not be everywhere differentiable even for continuous RV.

Common Distributions

Name of the probability distribution	Probability distribution function	Mean	Variance
Binomial distribution	$\Pr(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Geometric distribution	$\Pr(X=k) = (1-p)^{k-1} p$	$\frac{1}{p}$	$\frac{(1-p)}{p^2}$
Normal distribution	$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Uniform distribution (continuous)	$f(x \mid a, b) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential distribution	$f(x \mid \lambda) = \lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

Figure: PDF or PMF of commonly used random variables (Wiki).

Gaussian (Normal) Distribution

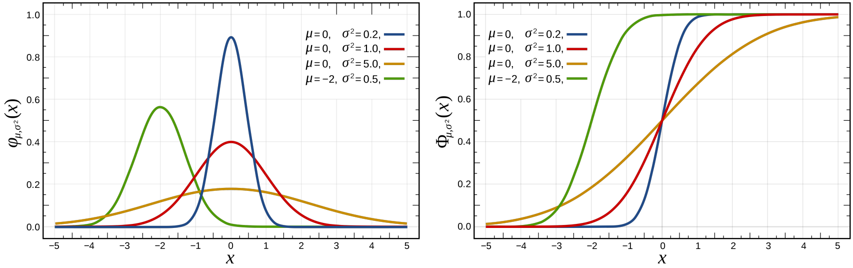


Figure: PDF and CDF of Gaussian distribution.

PDF of Gaussian distribution: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

CDF of standard Gaussian distribution ($\mu = 0$, $\sigma = 1$): $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$

PDF for multivariate Gaussian distribution:

$$\boxed{f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) := \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)}$$

where $\mathbf{x} = [x_1, \ldots, x_n]^\top$, $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_n]^\top \in \mathbb{R}^n$ and $\boldsymbol{\Sigma}$ is the $n \times n$ covariance matrix, of which the (i, j)-entry is $\mathrm{Cov}[X_i, X_j]$.

Expectation

Definition (Expectation)

Let $g : \mathbb{R} \to \mathbb{R}$ be a function and X be a random variable, discrete or continuous, then the *expectation* of the random variable $g(X)$ is

$$\mathrm{E}[g(X)] := \sum_{x \in \mathrm{val}(X)} g(x) p_X(x) \quad \text{or} \quad \mathrm{E}[g(X)] := \int_{x \in \mathrm{val}(X)} g(x) f_X(x) dx,$$

respectively.

Linearity of the expectation operator:

(a) $\mathrm{E}[ag(X) + bh(X)] = a\mathrm{E}[g(X)] + b\mathrm{E}[h(X)]$ for any constants a, b, and arbitrary functions $g(\cdot), h(\cdot)$.

(b) $X \perp\!\!\!\perp Y \Rightarrow \mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$.

The *indicator function*: $\mathbf{1}_A := \begin{cases} 1, & \text{if } A \text{ is true} \\ 0, & \text{otherwise.} \end{cases} \Rightarrow$

$\mathrm{E}[\mathbf{1}_A] = \Pr(A) \times 1 + \Pr(\bar{A}) \times 0 = \Pr(A)$, $F_X(x) = \mathrm{E}[\mathbf{1}_{\{X \leq x\}}]$.

Variance

Given a distribution of a random variable, we use the notion of variance to measure how concentrated that distribution is around the expectation (mean). Formally, we have

Definition (Variance)

$\mathrm{Var}[X] \coloneqq \mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$, where $\mathrm{E}[X^2]$ is the second moment of X.

The following can be derived immediately from the definition:

(a) $\mathrm{Var}[cX] = c^2 \mathrm{Var}[X]$.

(b) $\mathrm{Var}[c] = 0$ for any constant c.

(c) $\mathrm{Var}[aX \pm bY] = a^2 \mathrm{Var}[X] + b^2 \mathrm{Var}[Y] \pm 2ab \times \mathrm{Cov}[X, Y]$.

Examples of Expectation and Variance

Let $X \sim \exp(\lambda)$ whose density function is $f_X(x) = \lambda e^{-\lambda x}$. Find $\mathrm{E}[X]$ and $\mathrm{Var}[X]$.

Solution: From the definition of expectation and integration by parts, we have

$$\begin{aligned} E(X) &= \int_0^\infty x f_X(x)\,dx \\ &= \lambda \int_0^\infty x e^{-\lambda x}\,dx \\ &= -x e^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx \\ &= 0 + \frac{e^{-\lambda x}}{-\lambda}\Big|_0^\infty = \frac{1}{\lambda}\,. \end{aligned}$$

$$\begin{aligned} V(X) &= \int_0^\infty x^2 f_X(x)\,dx - \frac{1}{\lambda^2} \\ &= \lambda \int_0^\infty x^2 e^{-\lambda x}\,dx - \frac{1}{\lambda^2} \\ &= -x^2 e^{-\lambda x}\Big|_0^\infty + 2\int_0^\infty x e^{-\lambda x}\,dx - \frac{1}{\lambda^2} \\ &= -x^2 e^{-\lambda x}\Big|_0^\infty - \frac{2x e^{-\lambda x}}{\lambda}\Big|_0^\infty - \frac{2}{\lambda^2} e^{-\lambda x}\Big|_0^\infty - \frac{1}{\lambda^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}\,. \end{aligned}$$

Characteristic Function and Moment-generating Function

Definition (Characteristic Function)

The characteristic function of random variable X is defined as $\varphi_X : \mathbb{R} \to \mathbb{C}$:

$$\varphi_X(t) := \mathrm{E}[e^{itX}] = \int_{\mathbb{R}} e^{itx} dF_X(x) = \int_{\mathbb{R}} e^{itx} f_X(x) dx, \quad t \in \mathbb{R}$$

where $i = \sqrt{-1}$.

Definition (Moment-generating Function)

The moment-generating function of random variable X is defined as $M_X : \mathbb{R} \to \mathbb{R}$:

$$M_X(t) := \mathrm{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} f_X(x) dx, \quad t \in \mathbb{R}$$

(a) $\varphi_X(-it) = M_X(t)$. Characteristic function is the Fourier transform of the PDF with sign reversal in the complex exponential.

(b) $\mathrm{E}[X^n] = \frac{d^n M_X(t)}{dt^n}\Big|_{t=0}$.

(c) If $S_n = \sum_{i=1}^n a_i X_i$ for independent RVs $\{X_i\}_{i=1}^n$, then $M_{S_n}(t) = M_{X_1}(a_1 t) \times M_{X_2}(a_2 t) \times \cdots \times M_{X_n}(a_n t)$.

Multivariate Random Variables

Definition (Joint CDF)

Let X, Y be two random variables associated with an experiment. Then the *joint cumulative distribution function* of X and Y is

$$F_{XY} := \Pr(X \leq x, Y \leq y)$$

and the *marginal cumulative distribution function* of X is

$$F_X(x) := \lim_{y \to +\infty} \Pr(X \leq x, Y \leq y).$$

Similarly, the *joint* PMF $p_{XY}(x, y) := \Pr(X = x, Y = y)$
the *marginal* PMF of X, $p_X(x) = \sum_{y \in \mathrm{val}(Y)} \Pr(X = x, Y = y) \Leftarrow$ marginalisation.

The *joint* PDF of continuous X and Y is $f_{XY}(x, y) := \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$
the *marginal* PDF of X: $f_X(x) = \int_{y \in \mathrm{val}(Y)} f_{XY}(x, y) dy$.

Conditional Distribution

The above summation or integral operation is called *marginalization*. Note that

$$\iint_A f_{XY}(x,y)dxdy = \Pr\Big((x,y) \in A\Big).$$

Definition (Conditional Distribution for Discrete RVs)

Let X and Y be discrete random variables. The *conditional distribution* of Y given $X = x$ is

$$\boxed{p_{Y|X}(y \mid x) := \frac{p_{XY}(x,y)}{p_X(x)}},$$

provided that $p_X(x) \neq 0$.

We say that X and Y are *independent* if $p_{Y|X}(y \mid x) = p_Y(y)$.

Note that $X \perp\!\!\!\perp Y \Leftrightarrow p_{XY}(x,y) = p_X(x)p_Y(y)$.

Q: If $X \perp\!\!\!\perp Y$. For arbitrary functions $g(\cdot)$ and $h(\cdot)$, are $g(X) \perp\!\!\!\perp h(Y)$?

A: Yes.

Covariance & Correlation

Definition (Covariance and Correlation)

Let X and Y be two random variables. Their *covariance* is defined as

$$\mathrm{Cov}[X, Y] := \mathrm{E}[(X - \mathrm{E}[X])(Y - \mathrm{E}[Y])] = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y].$$

Their *correlation* is defined as

$$\mathrm{Corr}[X, Y] := \frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

1. $\mathrm{Corr}[X, Y] \in [-1, 1]$ is a measure of linear association between X and Y.
2. X and Y are uncorrelated if $\mathrm{Corr}[X, Y] = 0$.
3. $\mathrm{Corr}[X, Y] = \pm 1 \Leftrightarrow Y = aX + b$ for some constants a and b.
4. If $X \perp\!\!\!\perp Y \implies$ they are uncorrelated.
5. X and Y can be uncorrelated yet dependent due to a nonlinear relationship.

Example: $X \sim N(0, 1), Y = X^2 \implies \mathrm{Corr}[X, Y] = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y] = \mathrm{E}[X^3] = 0$ (all odd-order moments of X are equal to zero). Hence, X and Y are uncorrelated. But they are clearly dependent.

Example 1 of Covariance

Let X and Y be discrete random variables, with joint probability function $p_{X,Y}$ given by

$$p_{X,Y}(x, y) = \begin{cases} 1/2 & x = 3, y = 4 \\ 1/3 & x = 3, y = 6 \\ 1/6 & x = 5, y = 6 \\ 0 & \text{otherwise.} \end{cases}$$

Then $E(X) = (3)(1/2) + (3)(1/3) + (5)(1/6) = 10/3$, and $E(Y) = (4)(1/2) + (6)(1/3) + (6)(1/6) = 5$. Hence,

$$\begin{aligned} \text{Cov}(X, Y) &= E((X - 10/3)(Y - 5)) \\ &= (3 - 10/3)\,(4 - 5)/2 + (3 - 10/3)\,(6 - 5)/3 + (5 - 10/3)\,(6 - 5)/6 \\ &= 1/3. \blacksquare \end{aligned}$$

Example 2 of Covariance

Let X be any random variable with $\mathrm{Var}(X) > 0$. Let $Y = 3X$, and let $Z = -4X$. Then $\mu_Y = 3\mu_X$ and $\mu_Z = -4\mu_X$. Hence,

$$\begin{aligned} \mathrm{Cov}(X, Y) &= E((X - \mu_X)(Y - \mu_Y)) = E((X - \mu_X)(3X - 3\mu_X)) \\ &= 3\,E((X - \mu_X)^2) = 3\,\mathrm{Var}(X), \end{aligned}$$

while

$$\begin{aligned} \mathrm{Cov}(X, Z) &= E((X - \mu_X)(Z - \mu_Z)) = E((X - \mu_X)((-4)X - (-4)\mu_X)) \\ &= (-4)E((X - \mu_X)^2) = -4\,\mathrm{Var}(X). \end{aligned}$$

Note in particular that $\mathrm{Cov}(X, Y) > 0$, while $\mathrm{Cov}(X, Z) < 0$. Intuitively, this says that Y increases when X increases, whereas Z decreases when X increases. ∎

Conditional Expectation Definitions

1. The conditional expectation of a discrete RV X given an event A is defined as

$$\mathrm{E}[X|A] = \sum_x x \Pr[X = x|A] = \sum_x x \frac{\Pr[X = x \cap A]}{\Pr[A]} = \frac{\mathrm{E}[X\mathbf{1}_A]}{\Pr[A]}$$

2. The conditional expectation of a discrete RV Y given that $X = x$ is defined as

$$\mathrm{E}[Y|X = x] = \sum_y y \Pr[Y = y|X = x]$$

3. The conditional expectation of a continuous RV Y given that $X = x$ is defined as

$$\mathrm{E}[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X=x}(y) dy$$

Note that $h(x) = \mathrm{E}[Y|X = x]$ is a function depending on the particular observation x while $h(X) = \mathrm{E}[Y|X]$ is a random variable itself; i.e., $\mathrm{E}[Y|X](\omega) = \mathrm{E}[Y|X = X(\omega)]$.

Properties of Conditional Expectation[3]

Let $a, b \in \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and X, Y, Z be RVs. Then, we have

- $\mathrm{E}[aX + bY|Z] = a\mathrm{E}[X|Z] + b\mathrm{E}[Y|Z]$
- $\mathrm{E}[X|Y] \geq 0$ if $X \geq 0$
- $\mathrm{E}[X|Y] = \mathrm{E}[X]$ iif $X \perp\!\!\!\perp Y$
- $\mathrm{E}[g(X)|X] = g(X)$
- $\mathrm{E}[Xg(Y)|Y] = g(Y)\mathrm{E}[X|Y]$
- $\mathrm{E}[X|Y, g(Y)] = \mathrm{E}[X|Y]$
- $\mathrm{E}[X] = \mathrm{E}_Y\Big[\mathrm{E}[X|Y]\Big]$ /law of total expectation/
- $\mathrm{Var}[X] = \mathrm{E}_Y\Big[\mathrm{Var}(X|Y)\Big] + \mathrm{Var}_Y\Big[\mathrm{E}(X|Y)\Big]$ /law of total variance/
- For any function h, $\mathrm{E}[(X - \mathrm{E}[X|Y])^2] \leq \mathrm{E}[(X - h(Y))^2]$ and equality holds iif $h(Y) = \mathrm{E}[X|Y]$ /$\mathrm{E}[X|Y]$ is the function of Y that best approximates X in the sense of mean squared error/

[3] see more in `https://en.wikipedia.org/wiki/Conditional_expectation`

LLN and CLT

Theorem (Law of Large Numbers (LLN))

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables so that $\mathrm{E}[X_1] = \mathrm{E}[X_2] = \cdots = \mathrm{E}[X_n] < \infty$. Let

$$\bar{X}_n := \frac{X_1 + X_2 + \cdots + X_n}{n}$$

denote the sample mean of those n random variables. Then $\bar{X}_n \to \mathrm{E}[X_1]$ as $n \to \infty$ almost surely (a.s., strong law) and in probability (i.p., weak law).

Theorem (Central Limit Theorem (CLT))

Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables, and assume that $\mathrm{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2$, for all i. Let $S_n := X_1 + X_2 + \cdots + X_n$. Then, $\mathrm{E}[S_n] = n\mu$, $\mathrm{Var}[S_n] = n\sigma^2$ and we have the standardization of S_n,

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{\text{i.p.}} N(0,1) \text{ as } n \to \infty$$

where $N(0,1)$ denotes the standard normal random variable.

Convergence of Random Variables

Definition (Convergence of random variables)

The given sequence of random variables $X_1, X_2, \ldots$ converges to a random variable X:

1. In probability ($X_n \xrightarrow{P} X$) if for every $\epsilon > 0, \lim_{n\to\infty} \Pr\{|X_n - X| > \epsilon\} = 0$
2. Almost sure [a.k.a. convergence with probability 1] ($X_n \xrightarrow{a.s.} X$) if $\Pr\{\lim_{n\to\infty} X_n = X\} = 1$
3. In distribution ($X_n \xrightarrow{dist} X$) if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ for all continuity points x of $F_X(x)$
4. In r^{th}-order mean ($X_n \xrightarrow{L^r} X$) if $\lim_{n\to\infty} \mathrm{E}[|X_n - X|^r] = 0$
5. In mean square (special case when $r = 2$) if $\lim_{n\to\infty} \mathrm{E}[(X_n - X)^2] = 0$

Strong & Weak Convergence

Strong convergence: Convergence almost surely and convergence in r^{th}-order mean.

Weak convergence: Convergence in probability and convergence in distribution.

Their relationships are given as follows:

$$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{dist} X$$

$$X_n \xrightarrow{L^r} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{dist} X$$

$$X_n \xrightarrow{a.s.} X \not\Leftrightarrow X_n \xrightarrow{L^r} X$$

Concentration Inequalities

Concentration inequalities provide bounds on how a random variable deviates from some value (e.g., its expected value):

- Markov's inequality: If X is a nonnegative RV, then $\Pr(X \geq t) \leq \frac{\mathrm{E}(X)}{t}$, for any $t > 0$.

- Chernoff's inequality: If X is a nonnegative RV, then $\Pr(X \geq t) = \Pr(e^{aX} \geq e^{at}) \leq \frac{\mathrm{E}(e^{aX})}{e^{at}}$, for any $a > 0$.

- Chebyshev's inequality: $\Pr(|X - \mathrm{E}(X)| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}$, for any $t > 0$.

- Hoeffding's inequality: Consider the empirical mean $\bar{X}_n := \frac{1}{n}(X_1 + \cdots + X_n)$ for independent random variables $X_i \in [a_i, b_i]$ for all i. Then, $\Pr(|\bar{X}_n - \mathrm{E}(\bar{X}_n)| \geq t) \leq 2\exp\left(-\frac{2n^2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$, for any $t > 0$.

Relationships among Common Discrete Distributions

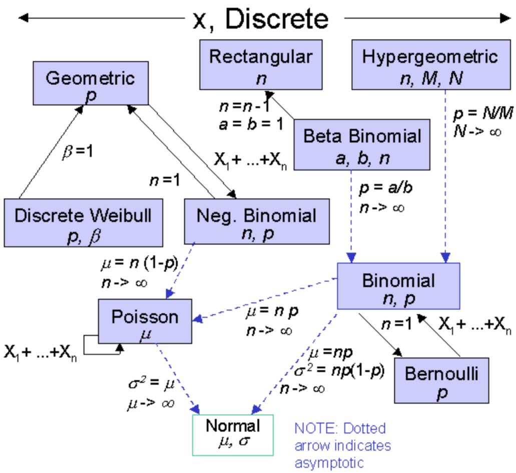


Figure: `https://statistical-engineering.com/relationships/`

Relationships among Common Continuous Distributions

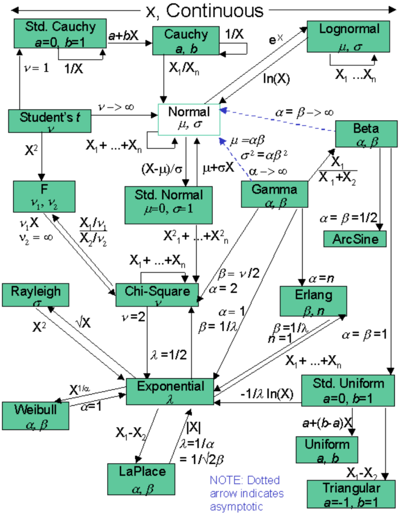


Figure: https://statistical-engineering.com/relationships/

Thank You!

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