# ECE253/CSE208 Introduction to Information Theory

## Lecture 5: Asymptotic Equipartition Property (AEP)

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- Chap 3 of *Elements of Information Theory (2nd Edition) by Thomas Cover & Joy Thomas*
- Lecture 1 Typical Sequences of *Information Theory for Wireless Comms.* by Dr. Saif Mohammed

## Law of Large Numbers (LLN) in Information Theory

- LLN in probability — Sample mean converges to the true mean: $\bar{X} = \frac{1}{n}\sum_i X_i \xrightarrow[n\to\infty]{\text{i.p.}} \mathrm{E}(X)$.
- LLN in info theory — Sample entropy converges to the true entropy: $\bar{H}(X) = \frac{1}{n}\log\frac{1}{p(X_1,\ldots,X_n)} \xrightarrow[n\to\infty]{\text{i.p.}} H(X)$.

### Example

Consider i.i.d. $\{X_i\}_{i=1}^n \sim$ Bern($p$), then $p(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n p(x_i)$. For example, $p(1,0,1,1,0,1) = p^{\sum x_i} \times (1-p)^{n-\sum x_i} = p^4(1-p)^2$. Clearly, not all sequences are generated equally.

We will see that $p(X_1, X_2, \ldots, X_n)$ is close to $2^{-nH(X)}$ with high probability. That is, the probability $p(X_1, X_2, \ldots, X_n)$ assigned to an observed sequence is close to $2^{-nH(X)}$.

# Typical Sequences

**Almost all events are almost equally surprising.**

$$\Pr\Big\{(X_1, X_2, \ldots, X_n) : p(X_1, X_2, \ldots, X_n) = 2^{-n(H\pm\epsilon)}\Big\} \approx 1,$$

if $X_1, X_2, \ldots, X_n$ are i.i.d. $\sim p(x)$.

We can thus divide the set of all sequences into two classes:

1. **Typical set**, where the probability of each typical sequence is close to $2^{-nH(X)}$.
2. **Atypical set** that contains all the other sequences.
3. Typical set is primarily a theoretical tool that is defined to help prove some theorems, even though its concept is somehow counter-intuitive, as we will see later.

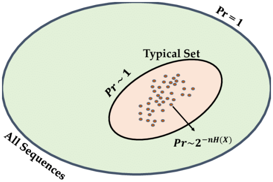



Figure: Typical sequences and typical set.

# Asymptotic Equipartition Property (AEP)

## Theorem (AEP: Empirical entropy converges to the true entropy.)

*If* $\{X_i\}_{i=1}^n \sim p(x)$ *are i.i.d., then* $-\frac{1}{n}\log p(X_1, X_2, \ldots, X_n) \xrightarrow{i.p.} H(X)$.

**Proof**: by the weak law of large numbers (WLLN), we have

$$-\frac{1}{n}\log p(X_1, X_2, \ldots, X_n) = -\frac{1}{n}\sum_i \log p(X_i) \xrightarrow{i.p.} -\mathrm{E}[\log p(X)] = H(X)$$

## Example (Sanity check of AEP)

Consider i.i.d. $\{X_i\}_{i=1}^n \sim$ Bern($p$), let $q = 1 - p$. We have

$p(x_1, \ldots, x_n) = p^{\sum_{i=1}^n x_i} \times q^{n - \sum_{i=1}^n x_i} \xrightarrow{i.p.} p^{np} q^{nq}$.

$H(X) = -p\log p - q\log q \implies -nH(X) = \log(p^{np}q^{nq})$.

This matches the AEP: $p(X^n) \xrightarrow{i.p.} 2^{-nH(X)}$.

**Q:** AEP is based on the assumption that $X^n$ are i.i.d. How about for non-iid case?

**A:** Entropy rate of stochastic processes; see the next lecture (Chap 4).

# Weakly Typical Sequences

Some sequences are "typical" in the sense that their information is about the same as the self-information expected. We define those typical sequences as follows:

**Definition ($\epsilon$-typical sequence and $\epsilon$-typical set)**

A sequence $(x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$ is an $\epsilon$-typical sequence with respect to $p(x)$ if

$$2^{-n(H(X)+\epsilon)} \leq p(x_1, x_2, \ldots, x_n) \leq 2^{-n(H(X)-\epsilon)}.$$

Further, a typical set $A_\epsilon^{(n)}$ is the set containing all $\epsilon$-typical sequences $(x_1, \ldots, x_n) \in \mathcal{X}^n$.

- Intuitively, we would like to assign shorter bit strings to the 'typical' sequences to reduce the expected length of the code. It turns out that as $n$ grows large, almost all the probability concentrates on the typical set.
- Typical set is made by all the sequences that are giving us an amount of information close to the average information of the source distribution.
- The least and most probable sequences give us less information than the average.
- Hence, AEP filters out a lot of highly unlikely sequences as well as a small number of highly likely sequences.

# Typical Sequences (cont'd)

## Example (Most likely sequence is often not in the typical set)

- For i.i.d. $X_i \sim \text{Bern}(0.9)$, $H(X) = 0.469$. The most likely sequence of outcome is the sequence of all 1's, $(1, 1, ..., 1)$.

$$-\frac{1}{n}\log_2 p\left((x_1, x_2, \ldots, x_n) = (1, 1, \ldots, 1)\right) = -\frac{1}{n}\log_2(0.9^n) = 0.152.$$

Hence, for small enough $\epsilon$, all-one sequence is not in the typical set.

- For Bernoulli RVs, the typical set consists of sequences with average numbers of 0's and 1's in $n$ independent trials. Because if a sequence has $np$ 1's and $nq$ 0's for $n$ trails, then $p(x_1, \ldots, x_n) = p^{np}q^{nq} \implies$

$$-\frac{1}{n}\log_2 p\left(x_1, x_2, \ldots, x_n\right) = -p\log p - q\log q = H(X).$$

If $p = 0.9, n = 10$, then the typical set consist of all sequences that have a single 0 in the entire sequence. If $p = 0.5$, then every possible binary sequences belong to the typical set.

# Typical Sequences and Set (cont'd)

**Example (The "typicality" is in the sense of *sample entropy close to the true entropy*, rather than "most likely")**

A computer program is used to generate a binary sequence of length 10 digits (i.i.d. $X_i \sim \text{Bern}(\frac{1}{3})$). One of the following four sequences is generated from the program. Which one is it?

(a) 0 0 0 0 0 0 0 0 0 0 0 0, $\Pr(\text{a}) = (2/3)^{12} = 7.7 \times 10^{-3}$

(b) 1 0 1 1 0 1 0 1 0 1 0 0, $\Pr(\text{b}) = (2/3)^{6} \times (1/3)^{6} = 1.2 \times 10^{-4}$

(c) 0 0 0 1 0 0 0 1 0 0 1 0, $\Pr(\text{c}) = (2/3)^{9} \times (1/3)^{3} = 9.6 \times 10^{-4}$

(d) 1 1 1 1 1 1 1 1 1 1 1 1, $\Pr(\text{d}) = (1/3)^{12} = 1.9 \times 10^{-6}$.

The answer is sequence (c), although sequence (a) has a higher probability of occurrence. An intuition based reasoning is that, since the source outputs are i.i.d., roughly $1/3$ of the 12 digits should be zero and $2/3$ should be one. This is in fact true as the length of the sequence is increasing. Those sequences are called "typical sequences".

# Typical Sequences and Set (cont'd)

- Consider a random source i.i.d. $X_i \sim \text{Bern}(p)$ generating a sequence of length $n$.
- There are $\binom{n}{np}$ independent sequences that have exactly $np$ ones and the probability of each such sequence is $p^{np}(1-p)^{n(1-p)}$.
- Approximate $\binom{n}{np}$ by using the Stirling's formula: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

$$
\begin{aligned}
\log \binom{n}{np} &= \log \left( \frac{n!}{(np)!(n-np)!} \right) \\
&\approx \log \left( \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi np}\left(\frac{np}{e}\right)^{np} \sqrt{2\pi n(1-p)}\left(\frac{n(1-p)}{e}\right)^{n(1-p)}} \right) \\
&= -\log\left(\sqrt{2\pi np(1-p)}\right) - n \times [p \log p + (1-p)\log(1-p)]
\end{aligned}
$$

$$
\implies \boxed{\binom{n}{np} \approx \frac{2^{nH(p)}}{\sqrt{2\pi np(1-p)}}}
$$

Hence, the number of such sequences increases as $2^{nH(p)}$, but it's a much smaller subset of all possible sequences.

# Properties of AEP

## Theorem

*Let $x^n := (x_1, x_2, \ldots, x_n)$, we have the following properties of the AEP:*

1. *If $x^n \in A_\epsilon^{(n)}$ then $H(X) - \epsilon \leq -\frac{1}{n} \log p(x^n) \leq H(X) + \epsilon$.*
2. $\Pr\{A_\epsilon^{(n)}\} > 1 - \epsilon$ *for $n$ sufficiently large.*
3. $\left|A_\epsilon^{(n)}\right| \leq 2^{n(H(X)+\epsilon)}$.
4. $\left|A_\epsilon^{(n)}\right| \geq (1-\epsilon)2^{n(H(X)-\epsilon)}$ *for $n$ sufficiently large.*

The above Theorem asserts that i) the typical set has probability nearly $1$; ii) all elements in it are nearly equiprobable; and iii) the size of the typical set is about $2^{nH(X)}$.

# Proof of AEP Properties

## Proof.

1. From the definition of $\epsilon$-typical sequences, if $(x_1, \cdots, x_n) \in A_\epsilon^{(n)}$, then we have

$$2^{-n(H(X)+\epsilon)} \leq p(x^n) \leq 2^{-n(H(X)-\epsilon)}$$

   Taking the $\log$ and dividing by $-n$ yields Property 1.

2. Since $-\frac{1}{n}\log p(X^n) \xrightarrow{\text{i.p.}} H(X)$, for any $\delta > 0$, $\exists\ n_0$ such that for all $n \geq n_0$,

$$\Pr\left(\left|-\frac{1}{n}\log p(X^n) - H(X)\right| < \epsilon\right) > 1 - \delta.$$

   Finally, we know $\left|-\frac{1}{n}\log p(X^n) - H(X)\right| < \epsilon$ holds iif $X^n$ is $\epsilon$-typical, so we can set $\delta := \epsilon$ to obtain Property 2.

# Proof of AEP Properties (cont.)

## Proof.

3. By Property 1, we have $2^{-n(H(X)+\epsilon)} \leq p(x^n),\ \forall x^n \in A_\epsilon^{(n)} \implies$

$$2^{-n(H(X)+\epsilon)} \left|A_\epsilon^{(n)}\right| \leq \sum_{x^n \in A_\epsilon^{(n)}} p(x^n) \leq \sum_{x^n \in \mathcal{X}^n} p(x^n) = 1,$$

   which proves Property 3.

4. By Property 2, for $n$ sufficiently large, we have $\Pr\{A_\epsilon^{(n)}\} > 1 - \epsilon$. Hence,

$$1 - \epsilon < \Pr\{A_\epsilon^{(n)}\} = \sum_{x^n \in A_\epsilon^{(n)}} p(x^n) \leq \sum_{x^n \in A_\epsilon^{(n)}} 2^{-n(H(X)-\epsilon)} = 2^{-n(H(X)-\epsilon)} \left|A_\epsilon^{(n)}\right|.$$

   The "$\leq$" follows from the upper bound of $p(x^n)$.

## Properties of AEP (cont'd)

For small $\epsilon$, we have $|A_\epsilon^{(n)}| \approx 2^{nH(X)}$. Thus, the fraction of sequences that are typical is

$$\rho_n := \frac{|A_\epsilon^{(n)}|}{|\mathcal{X}^n|} \approx \frac{2^{nH(X)}}{|\mathcal{X}|^n} = \frac{2^{nH(X)}}{2^{n\log|\mathcal{X}|}} = 2^{-n(\log|\mathcal{X}|-H(X))}.$$

- For non-uniform distribution: $H(X) < \log|\mathcal{X}|$, $\rho_n \to 0$ as $n \to \infty$.
- For uniform distribution: $H(X) = \log|\mathcal{X}| \to \rho_n = 1$, every sequence is typical.

**Everything outside the typical set has a negligible probability.**

- $|A_\epsilon^{(n)}|$ **is exponentially small fraction in** $n$. However, the typical sequences make up most of the probability because $\Pr\{A_\epsilon^{(n)}\} > 1-\epsilon$.
- In other words, the probability of a generated sequence being in the typical set is high, even though the number of elements in the typical set is much smaller than the total number of possible sequences.
- For $n$ sufficiently large, we can almost think of the sequence $X^n$ as being obtained by choosing a sequence from the weakly typical set according to the uniform distribution $\to$ "asymptotic equipartition".

# Strongly Typical Sequences

## Definition (Strongly Typical Sets)

The strongly typical set $T_\delta^{(n)}$ with respect to a distribution function $p(x)$ is the set of sequences $x^n \in \mathcal{X}^n$ such that

$$\sum_{x \in \mathcal{X}} \left| \frac{1}{n} N(x; x^n) - p(x) \right| \leq \delta,$$

where $N(x; x^n)$ is the number of occurrences of $x$ in the sequence $x^n$, and $\delta > 0$ is an arbitrarily small number. The sequences in $T_\delta^{(n)}$ are called strongly $\delta$-typical sequences.

## Theorem (Strong AEP)

*There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold*

1. *If $x^n \in T_\delta^{(n)}$, then $H(X) - \eta \leq -\frac{1}{n} \log p(x^n) \leq H(X) + \eta$.*
2. $\Pr\{T_\delta^{(n)}\} > 1 - \delta$ *for $n$ sufficiently large.*
3. $\left| T_\delta^{(n)} \right| \leq 2^{n(H(X)+\eta)}$.
4. $\left| T_\delta^{(n)} \right| \geq (1-\delta) 2^{n(H(X)-\eta)}$ *for $n$ sufficiently large.*

## Strong Typicality vs Weak Typicality

- Weak typicality (entropy typicality): empirical entropy $\approx$ true entropy.
- Strong typicality (letter typicality): empirical distribution $\approx$ true distribution.
- Strong typicality $\implies$ Weak typicality, but not vice versa.
- Strong typicality works only for finite alphabet, i.e., $|\mathcal{X}| < \infty$.

## High-probability Set

To this end, we know that the $A_\epsilon^{(n)}$ is a fairly small set that has most of the probability.

**Q**: Is it the smallest set with such a property?

### Definition

For $\delta > 0$, let $B_\delta^{(n)} \subset \mathcal{X}^n$ be the smallest set such that $\Pr\left(X^n \in B_\delta^{(n)}\right) \geq 1 - \delta$.

### Theorem

*Let $\delta < \frac{1}{2}$. For any $\delta' > 0$,*

$$\frac{1}{n} \log |B_\delta^{(n)}| > H - \delta'$$

*for $n$ sufficiently large.*

**Typical set vs High-probability set.**

For sufficiently large $n$ (depending on $\delta$ and $\delta'$), $B_\delta^{(n)}$ has at least $2^{n(H-\delta')}$ elements. The $\epsilon$-typical set $A_\epsilon^{(n)}$ has about $2^{n(H\pm\epsilon)}$ elements. Thus, $A_\epsilon^{(n)}$ and $B_\delta^{(n)}$ have roughly the same number of elements to first order in the exponent.

## Encoding for the Typical Set

The fact that the typical set has probability approaching $1$ as $n$ grows large means that we "only need" to care about encoding the sequences in the typical set.

The number of bits required to encode a set of size $\mathcal{S}$ is $\lceil \log |\mathcal{S}| \rceil$, where the ceiling operator $\lceil a \rceil$ outputs the smallest integer number no less than $a$.

Let i.i.d. $\{X_i\}_{i=1}^n \sim p(x)$. Consider the following scheme for coding $x^n \in \mathcal{X}^n$.

- First, consider a complete order of all the sequences in $A_\epsilon^{(n)}$ and its complement, according to a certain criterion (e.g., lexicographic order, "ABC, ACB, BAC, BCA, CAB, CBA").
- We use the first bit as an indicator to show if $x^n$ is typical, say, start with $0$ if the sequence is typical, otherwise start with $1$.

# Encoding for the Typical Set (cont'd)

- If $x^n \in A_\epsilon^{(n)}$, since $\left|A_\epsilon^{(n)}\right| \leq 2^{n(H(X)+\epsilon)}$, use $n(H(X)+\epsilon)+1$ bits for encoding (the additional $1$ bit is due to integrality),
- If $x^n \notin A_\epsilon^{(n)}$, use no more than $n \log |\mathcal{X}| + 1$ bits to encode it.

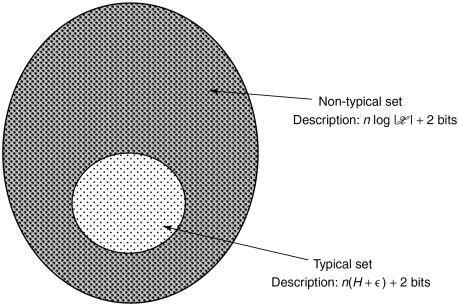



Figure: Encoding for the typical set.

## Consequences of AEP

Let $\ell(x^n)$ denote the length of the codeword (a binary string) corresponding to $x^n \in \mathcal{X}^n$. For a sufficiently large $n$, we have

$$\begin{aligned}
\mathrm{E}[\ell(x^n)] &\leq P\left(x^n \in A_\epsilon^{(n)}\right) \times (n(H+\epsilon)+2) + P\left(x^n \notin A_\epsilon^{(n)}\right) \times (n \log|\mathcal{X}| + 2) \\
&= 2 + P\left(x^n \in A_\epsilon^{(n)}\right) \times (n(H+\epsilon)) + P\left(x^n \notin A_\epsilon^{(n)}\right) \times (n \log|\mathcal{X}|) \\
&\leq 2 + n(H+\epsilon) + \epsilon n \log|\mathcal{X}| =: n(H+\tilde{\epsilon})
\end{aligned}$$

where $\tilde{\epsilon} = \epsilon(1+\log|\mathcal{X}|) + \frac{2}{n}$ can be arbitrarily small by appropriate choices of $\epsilon$ and $n$.

**Theorem ($H(X)$ bits are needed to encode $X^n$ per symbol on average)**

*Consider i.i.d. $\{X_i\}_{i=1}^n \sim p(x)$. Let $\epsilon > 0$, then there exists a code that maps sequences $x^n$ into binary strings, such that the mapping is one-to-one and $E\left[\frac{1}{n}\ell(x^n)\right] \leq H(x) + \epsilon$ for n sufficiently large.*

The above theorem explains the achievability part of the *Source Coding Theorem*: A sequence of symbols can be compressed to a binary string with an average of $H(X)$ bits per symbol. This further reinforces the interpretation of the entropy as the average information content of a random source.

## Jointly Typical Sequences

Two sequences $x^n$ and $y^n$ are jointly $\epsilon$-typical if

1. the pair $(x^n, y^n)$ is $\epsilon$-typical with respect to the joint distribution $p(x^n, y^n) = \prod_{i=1}^n p(x_i, y_i)$ (i.e., pairwise independence).
2. both $x^n$ and $y^n$ are $\epsilon$-typical w.r.t. their marginal distributions $p(x^n)$ and $p(y^n)$.

The set of all such pairs of sequences $(x^n, y^n)$ is denoted by

$$
\begin{aligned}
A_\epsilon^{(n)}(X,Y) = \Bigg\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : & \left| -\frac{1}{n} \log p(X^n) - H(X) \right| < \epsilon, \\
& \left| -\frac{1}{n} \log p(Y^n) - H(Y) \right| < \epsilon, \\
& \left| -\frac{1}{n} \log p(X^n, Y^n) - H(X,Y) \right| < \epsilon \Bigg\}
\end{aligned}
$$

# Joint AEP

**Theorem (See the proof on page 196–198 of Cover's book)**

*Let $(X^n, Y^n)$ be sequences of length $n$ drawn i.i.d. $\sim p(x^n, y^n) = \prod_{i=1}^n p(x_i, y_i)$. Then,*

1. $\Pr\left((X^n, Y^n) \in A_\epsilon^{(n)}\right) \to 1$ *as* $n \to \infty$.
2. $|A_\epsilon^{(n)}| \leq 2^{n(H(X,Y)+\epsilon)}$.
3. *If* $(\tilde{X}^n, \tilde{Y}^n) \sim p(x^n)p(y^n)$, *then*

$$(1-\epsilon)2^{-n(I(X;Y)+3\epsilon)} \leq \Pr\left((\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)}\right) \leq 2^{-n(I(X;Y)-3\epsilon)},$$

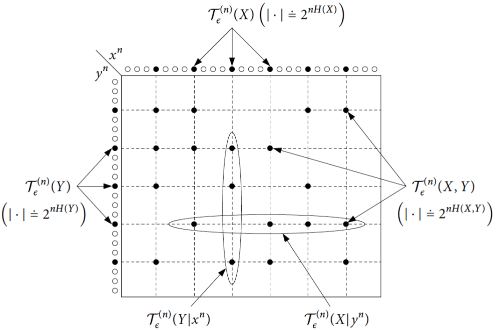
*where the upper bound holds for $n$ sufficiently large.*

**Implication**:

- Typical sets $|X^n| \approx 2^{nH(X)}$ and $|Y^n| \approx 2^{nH(Y)}$.
- Not all pair of typical $X^n$ and typical $Y^n$ are jointly typical: only about $2^{nH(X,Y)}$.
- Intuitive argument for joint typicality lemma: the probability of any randomly chosen pair is jointly typical is about $\frac{2^{nH(X,Y)}}{2^{n(H(X)+H(Y))}} = 2^{-nI(X;Y)}$

We use the joint AEP and random coding to prove the channel coding theorem (Chap 7).

# Joint AEP (cont'd)



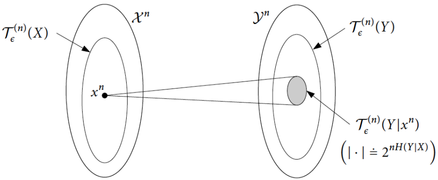




Figure: Source: Chapter 2 of Network Information Theory by El Gamal and Kim.

*Thank You!*

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