ECE253/CSE208 Introduction to Information Theory

Lecture 14: AWGN Channel

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- Chap 9 of *Elements of Information Theory (2nd Edition) by Thomas Cover & Joy Thomas*.

White Gaussian Noise Channel (AWGN)

A Gaussian channel is of the form $Y_i = X_i + Z_i$ where X_i is input and $Z_i \sim (0, N)$ is the additive noise.

Theorem (AWGN channel capacity)

The information capacity of a Gaussian channel with power constraint P and noise variance N is $\boxed{C = \max_{f(x):\mathbb{E}(X^2)\leq P} I(X;Y) = \frac{1}{2}\log\left(1+\frac{P}{N}\right)}$ *bits per transmission.*

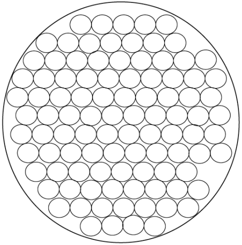
Proof:

$$
\begin{aligned}
I(X;Y) = h(Y) - h(Y|X) = h(Y) - h(Z) & \quad (1) \\
\leq \frac{1}{2}\log[2\pi e \mathrm{Var}(Y)] - \frac{1}{2}\log(2\pi e N) & \quad (2) \\
\leq \frac{1}{2}\log[2\pi e(P+N)] - \frac{1}{2}\log(2\pi e N) = \frac{1}{2}\log(1+\frac{P}{N}), & \quad (3)
\end{aligned}
$$

where the maximum is attained when $X \sim \mathcal{N}(0, P)$ (hence, $Y \sim \mathcal{N}(0, P+N)$). Note that the capacity is infinite if we have zero-variance noise or unconstrained input.

Sphere Packing

The operational capacity (supremum of all achievable rates) of the Gaussian channel can be argued in a similar fashion as for discrete channels. The theorem can also be geometrically proved by sphere packing.



By LLN, as $n \to \infty$, we have the following

- Most received vectors $\{\mathbf{y}^n\}$ lie inside the n-dimensional sphere of radius $r_{\text{large}} = \sqrt{n(P+N)}$.
- A received vector $\mathbf{y}^n \sim \mathcal{N}(\mathbf{x}^n, N\mathbf{I})$ is contained in a sphere of radius $r_{\text{small}} = \sqrt{nN}$ around the sent codeword (with high probability).
- Decoding: assign everything in this sphere to the given sent codeword. An error happens only if $\mathbf{y}^n$ falls outside the sphere, which has low probability. Similarly, we can choose other codewords and their corresponding decoding spheres.
- Maximum number of non-overlapping spheres = Maximum number of codewords that can be reliably transmitted.

Sphere Packing (Cont'd)

Q: How many non-overlapping spheres can be packed into the large sphere?

A: The n-dimensional volume of a sphere with radius r is $V_n(r) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} r^n \implies$.

$$2^{nR} \leq \left(\frac{r_{\text{large}}}{r_{\text{small}}}\right)^n = \left(\frac{\sqrt{n(P+N)}}{\sqrt{nN}}\right)^n \implies R \leq \frac{1}{2}\log\left(1+\frac{P}{N}\right).$$

Band-limited AWGN

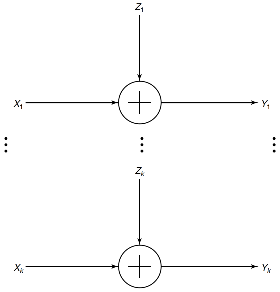
A more realistic channel model is the band-limited continuous AWGN $Y(t) = (X(t) + Z(t)) * h(t)$, where $h(t)$ is the impulse response of an ideal bandpass filter with $f_{\max} = W$. Consider the channel being used over the interval $[0, T]$. We have

- $2W$ samples per second (sufficient for reconstruction due to the *Nyquist-Shannon sampling theorem*).
- Signal energy per sample is $PT/2WT$.
- Noise variance per sample is $N_0WT/2WT$ [double-sided noise power spectral density is $N_0/2$].
- Channel capacity per sample is $C = \frac{1}{2}\log(1 + \frac{P}{N_0W})$.

Therefore, the capacity of a band-limited AWGN is

$$\boxed{C = W \log\left(1 + \frac{P}{N_0W}\right) \xrightarrow{W\to\infty} \frac{P}{N_0}\log_2 e \text{ bits per second}}$$

Parallel AWGN



We have a set of AWGN channels in parallel:

- channel output $Y_i = X_i + Z_j, \quad i = 1, \dots, k.$
- independent noise $Z_i \sim \mathcal{N}(0, N_i)$.
- coupling power budget: $E(\sum_i X_i^2) \le P$.
- mutual information:

$$
\begin{aligned}
I(X^k; Y^k) = h(Y^k) - h(Y^k|X^k) &= h(Y^k) - h(Z^k) \\
&\le \sum_i h(Y_i) - h(Z_i) \\
&\le \sum_i \frac{1}{2} \log\left(1 + \frac{P_i}{N_i}\right),
\end{aligned}
$$

where "$=$" is achieved by independent $X_i \sim \mathcal{N}(0, P_i)$.

Water-filling Optimal Resource Allocation

The channel capacity can be obtained by the following optimization problem:

$$\max_{P_i \geq 0} \quad \sum_{i=1}^{k} \frac{1}{2} \log(1 + P_i/N_i) \tag{4a}$$

$$\text{s.t.} \quad \sum_{i=1}^{k} P_i = P \tag{4b}$$

By Lagrangian relaxation, the optimal solution is obtained by the *water-filling* scheme: $P_i^* = (\nu - N_i)^+, \; i = 1, 2, \ldots k$, where ν satisfies $\sum_i (\nu - N_i)^+ = P$.

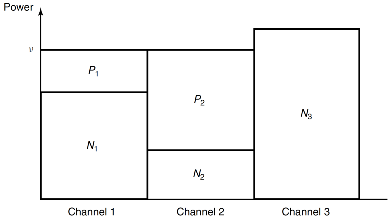


Figure: Water-filling solution: **allocate more power in less noisy channels**. $P_2 > P_1 > P_3 = 0$ and $P_1 + P_2 = P$. Channel 3 with the largest noise variance N_3 is the worst channel that gets no power allocation (i.e., no communication over this channel).

Colored Gaussian Noise

- Since the noise and the input are independent, the covariance matrix of the output is $\mathbf{K}_Y = \mathbf{K}_X + \mathbf{K}_Z$.
- Maximize $I(X^n; Y^n) = h(Y^n) - h(Z^n)$ boils down to the following problem:

$$\max_{\mathrm{Tr}(\mathbf{K}_X) = nP} \ \frac{1}{2} \log \Big((2\pi e)^n |\mathbf{K}_X + \mathbf{K}_Z| \Big).$$

- The optimal solution is obtained by *water-filling in the eigen-space* of $\mathbf{K}_Z$.
- Let $\mathbf{K}_Z = \mathbf{U}\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\mathbf{U}^\top$ denote its eigen-decomposition. Then, $\mathbf{K}_X^{\mathrm{opt}} = \mathbf{U}\mathrm{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)\mathbf{U}^\top$, where $\tilde{\lambda}_i = (\nu - \lambda_i)^+$ for $i = 1, \ldots, n$ and $\sum_i \tilde{\lambda}_i = nP$ (to find the water level ν).

Gaussian Channels with Feedback

- For a DMC, feedback does not increase the capacity, although it may help greatly in reducing the complexity of encoding or decoding.
- The same is true of memoryless AWGN channels. However, for channels with memory, where the noise is correlated across time, feedback does increase capacity.
- However, there is no simple explicit characterization of the capacity with feedback.

Capacity with feedback and the upper bounds.

$$\boxed{C_{n,\mathrm{FB}} = \max_{\mathrm{Tr}(\mathbf{K}_X) \le nP} \frac{1}{2n} \log \frac{|\mathbf{K}_{X+Z}|}{|\mathbf{K}_Z|} \le \min\{C_n + \frac{1}{2}, 2C_n\}.}$$

Thank You!

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