ECE253/CSE208 Introduction to Information Theory

Lecture 4: Convexity and Inequalities

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- Chap 2 of *Elements of Information Theory (2nd Edition) by Thomas Cover & Joy Thomas*

Convex Functions

Definition (Convexity of functions)

- A function $f(x)$ is *convex* over an interval (a, b) if and only if

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

holds for any $x_1, x_2 \in (a, b)$ and $\lambda \in [0, 1]$.

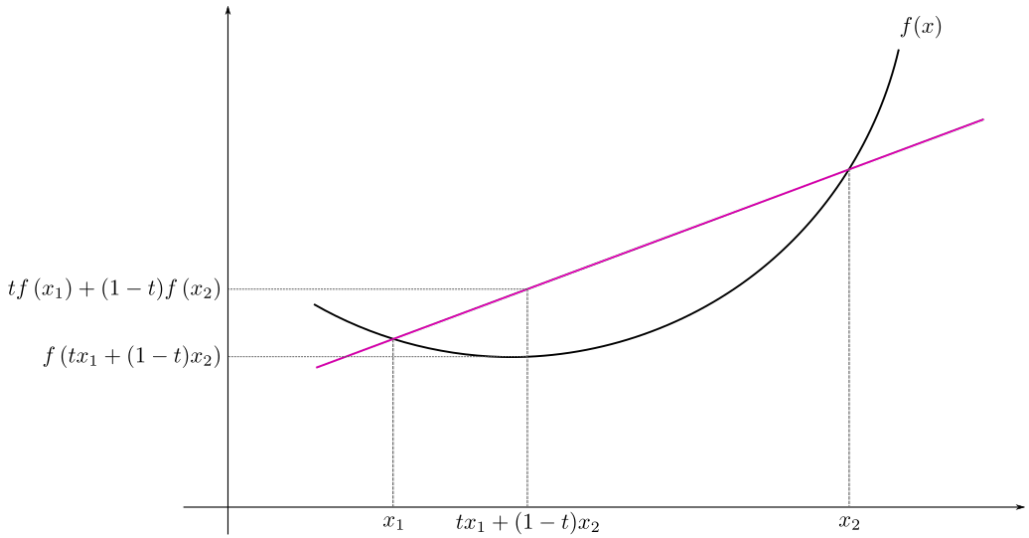


Figure: Any *chord* of a convex function is always above the function itself: The 0th-order condition of convexity.

Convex Functions

Definition (Convexity of functions)

- A function $f(\cdot)$ that is *differentiable everywhere* in (a, b) is *convex* if and only if

$$f(y) \geq f(x) + f'(x)(y - x)$$

for any $x, y \in (a, b)$.

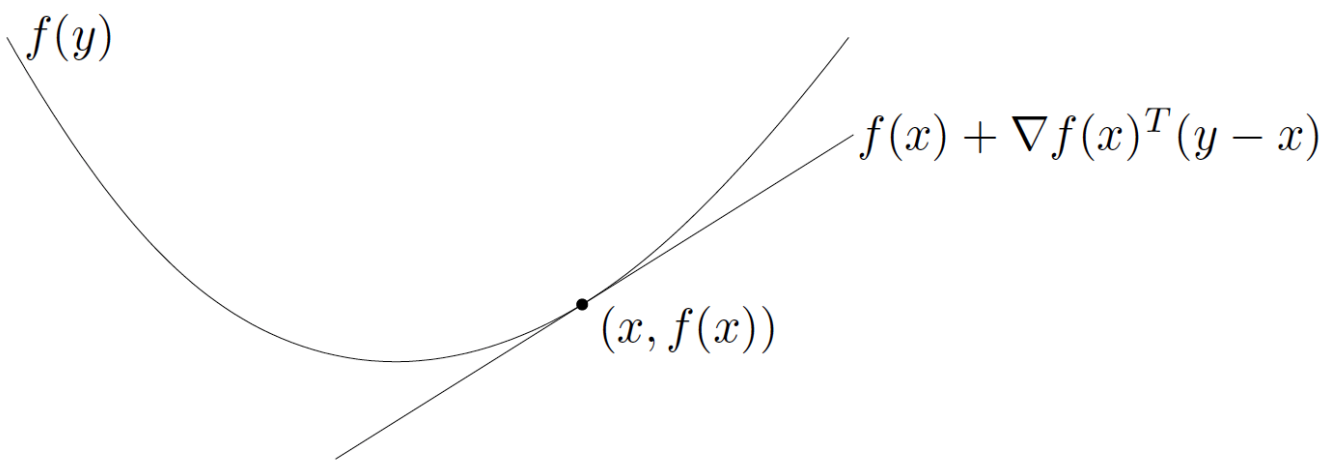


Figure: Tangent lines are always global under-estimator of the function: The 1st-order condition of convexity.

Convex Functions

Definition (Convexity of functions)

- A function $f(x)$ that is *twice differentiable* over (a, b) is *convex* if and only if

$$f''(x) \geq 0$$

for any $x \in (a, b)$.

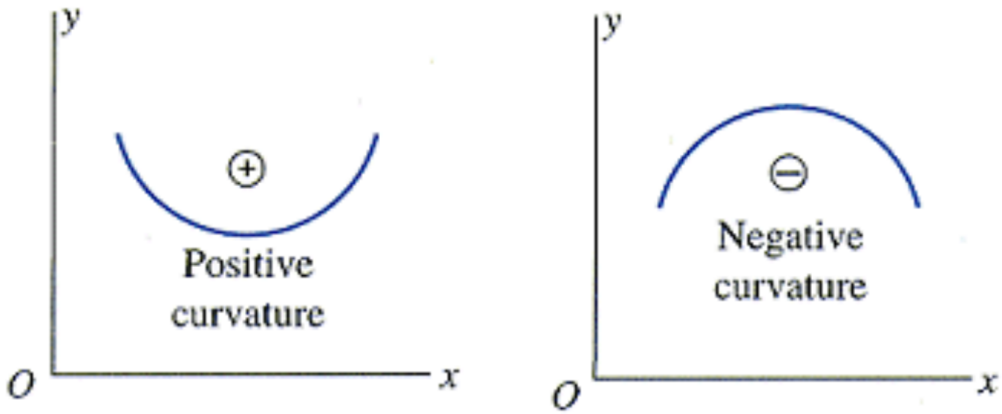


Figure: Convex functions always curve upward (positive curvature).: The 2nd-order condition of convexity.

Examples. Convex functions: $ax + b, |x|, x^2, x^4, e^{\pm x}, x \log x$.

If $f(x)$ is *convex*, then $-f(x)$ is *concave*. Affine functions are both convex and concave.

Convexity in High-dimensional Spaces

Definition (Convex function)

A function $f(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ is *convex* iif for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have one of the following:

- 0th-order condition: $f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$ for any $\lambda \in [0, 1]$.
- 1st-order condition: $f(\mathbf{y}) \geq f(\mathbf{x}) + \bigtriangledown f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x})$
- 2nd-order condition: $\mathbf{H}_f := \bigtriangledown^2 f(\mathbf{x}) \succeq \mathbf{0}$; i.e., the *Hessian* matrix is positive semi-definite (all eigenvalues are nonnegative), where $\mathbf{H}_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

- Strictly convex: if strict inequality always holds when $\mathbf{x} \neq \mathbf{y}$ and $\lambda \in (0, 1)$.
- Strongly convex: $\mathbf{H}_f \succeq a\mathbf{I}$ for some constant $a > 0$ (the Hessian is positive definite).
- Geometrically, strict/strong convexity implies that the function has no flat part and curves upward everywhere $\implies$ unique minimizer.

Convexity-preserving Operations

restriction to a line: f convex $\iff$ $f(\mathbf{x}_0 + t\mathbf{h})$ convex in t for all $\mathbf{x}_0, \mathbf{h}$

positive weighted sum: $\{f_i(\mathbf{x})\}_{i=1}^n$ convex and $w_i \geq 0 \implies$ $\sum_{i=1}^n w_i f_i(\mathbf{x})$ convex

integrals: for every $\mathbf{y} \in \mathcal{A}$, $w(\mathbf{y}) \geq 0$ and $f(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{x}$
$\implies$ $g(\mathbf{x}) := \int_{\mathcal{A}} w(\mathbf{y}) f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$ is convex in $\mathbf{x}$

pointwise maximum: f_1, f_2 convex $\implies$ $\max\{f_1(\mathbf{x}), f_2(\mathbf{x})\}$ convex

pointwise supremum: f_α convex $\implies$ $\sup_{\alpha \in \mathcal{A}}\{f_\alpha\}$ convex

composition with affine mapping: f convex $\implies$ $g(\mathbf{x}) := f(\mathbf{A}\mathbf{x} + \mathbf{b})$ is convex in $\mathbf{x}$.

Convex Sets

Definition (Convex sets)

A set $S \subseteq \mathbb{R}^n$ is convex if and only if $\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in S, \forall\ \mathbf{x}_1, \mathbf{x}_2 \in S, \lambda \in [0,1]$.

Geometrically, a convex set contains line segment between any two points in the set.

Convex sets are solid body without holes and curve outward.

examples (one convex, two nonconvex sets)

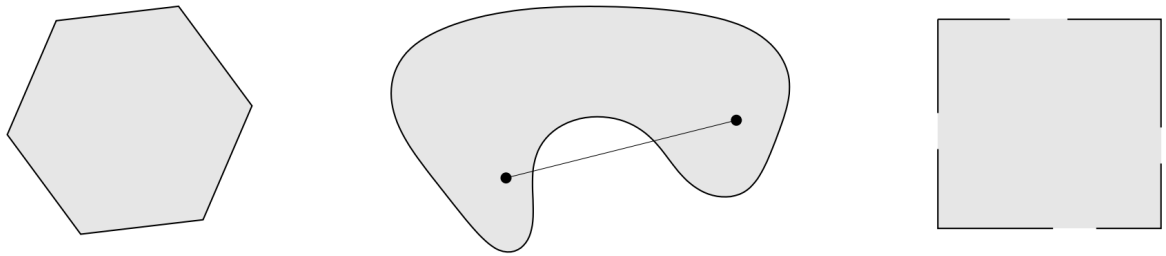


Figure: Convex and nonconvex sets (source: Stephen Boyd, Stanford).

Epigraph and Sublevel Set

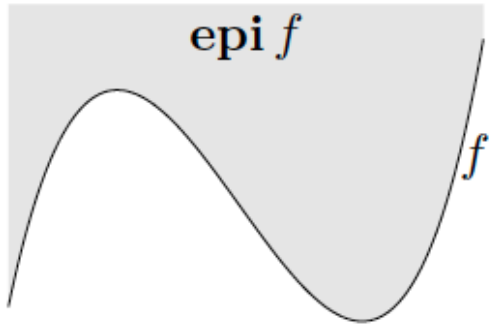
- α-**sublevel set** of $f : \mathbb{R}^n \to \mathbb{R}$:

$$C_\alpha = \{\mathbf{x} \in \mathrm{dom} f : f(\mathbf{x}) \leq \alpha\}$$

sublevel sets of convex functions are convex (converse is false)

- **epigraph** of $f : \mathbb{R}^n \to \mathbb{R}$:

$$\mathrm{epi} f = \{(\mathbf{x}, t) \in \mathbb{R}^{n+1} : \mathbf{x} \in \mathrm{dom} f,\ f(\mathbf{x}) \leq t\}$$



f is a convex function $\iff$ $\mathrm{epi} f$ is a convex set

Convex Optimization Problem

Convex optimization problem in standard form:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & \mathbf{a}_i^\top \mathbf{x} = b_i, \quad i = 1, \ldots, p \end{aligned}$$

- $f_0, f_1, \ldots, f_m$ are convex
- equality constraints are affine (alternatively $\mathbf{Ax} = \mathbf{b}$)

Important properties:

1. For convex problems, any local solution is also global.
2. If $f_0()$ is strictly convex, the minimizer is unique.
3. The optimal set X_{opt} is convex.

Jensen's Inequality

Lemma (Jenson's Inequality)

If X is a random variable and $f(\cdot)$ is a convex function, then

$$\boxed{E(f(X)) \geq f(E(X))}$$

Moreover, if $f(X)$ is strictly convex, equality implies $X = \mathrm{E}(X)$ with probability 1.

Proof of Jensen's Inequality

1) For a *two-point distribution* $X \in \{x_1, x_2\}$, the convexity of $f(\cdot) \implies$

$$p_1 f(x_1) + p_2 f(x_2) \geq f(p_1 x_1 + p_2 x_2).$$

2) Proof by *induction*: Assume Jenson's inequality holds for a $(k-1)$-mass point distribution. To show the inequality holds for a k-mass point distribution, define $p'_i = \frac{p_i}{1-p_k}$ for all $i = 1, 2, ..., k-1$:

$$\begin{aligned}
\sum_{i=1}^{k} p_i f(x_i) &= p_k f(x_k) + (1-p_k) \sum_{i=1}^{k-1} p'_i f(x_i) \\
&\geq p_k f(x_k) + (1-p_k) f\left(\sum_{i=1}^{k-1} p'_i x_i \right) \\
&\geq f\left(p_k x_k + (1-p_k) \sum_{i=1}^{k-1} p'_i x_i \right) = f\left(\sum_{i=1}^{k} p_i x_i \right)
\end{aligned}$$

where the 1st inequality is from the induction while the 2nd inequality is due to the convexity of $f(\cdot)$.

Log-sum Inequality

Lemma (Log-sum inequality)

For nonnegative numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$*, we have*

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i \right) \log \left(\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \right)$$

with equality iif $\frac{a_i}{b_i} = \text{constant}$.

Proof: $f(t) = t \log t$ is convex $(f''(t) > 0) \implies \sum \lambda_i f(t_i) \geq f(\sum \lambda_i t_i)$;
Setting $\lambda_i = \frac{b_i}{\sum b_i}$ and $t_i = \frac{a_i}{b_i}$ yields the log-sum inequality.

Example: When $n = 2 \implies a_1 \log \frac{a_1}{b_1} + a_2 \log \frac{a_2}{b_2} \geq (a_1 + a_2) \log \left(\frac{a_1 + a_2}{b_1 + b_2} \right)$.

Gibbs' Inequality (Information Inequality)

Theorem (Gibbs' Inequality)

Let $p(x)$ and $q(x)$ be two probability mass functions. Then, $D(p\|q) \geq 0$ with equality if and only if $p(x) = q(x)$ for all x.

Proof: $-D(p\|q) = \sum_{x\in\mathcal{X}} p(x) \log \frac{q(x)}{p(x)} \leq \log\left(\sum_{x\in\mathcal{X}} p(x) \frac{q(x)}{p(x)}\right) = \log\left(\sum_{x\in\mathcal{X}} q(x)\right) = 0.$

Corollary (Nonnegativity of mutual information)

For any two RVs X and Y, we have $I(X;Y) \geq 0$ with equality iff X and Y are independent.

Corollary (Conditional mutual Information)

$$I(X;Y|Z) \geq 0$$

with equality iff X and Y are conditionally independent given Z, which is denoted as $(X \perp\!\!\!\perp Y) \mid Z$.

Gibbs' Inequality (cont'd)

Theorem (Conditioning reduces entropy (information cannot hurt))

$H(X|Y) \leq H(X)$ *with equality iif* $X \perp\!\!\!\perp Y$.

- Intuitively, knowing Y can only reduce the uncertainty in X.
- Caveat: This is true only in the *average* sense. That is, $H(X|Y=y) > H(X)$ can happen for a specific y.

Theorem (Independence bound on entropy)

Let RVs $\{X_i\}_{i=1}^n$ *be drawn from the PMF* $p(x_1, \ldots, x_n)$. *Then,*

$H(X_1, \ldots, X_n) \leq \sum_{i=1}^n H(X_i)$

with equality iff $\{X_i\}_{i=1}^n$ *are independent.*

Proof: By chain rule: $H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^n H(X_i|X_{i-1}, \ldots, X_1) \leq \sum_{i=1}^n H(X_i)$.

Maximum Entropy

Theorem (Uniform distribution has the maximum entropy)

$$H(X) \leq \log |\mathcal{X}|$$

where $|\mathcal{X}|$ is the cardinality of the set $\mathcal{X}$ (i.e., the number of elements in the set) with equality iff X has a uniform distribution over $\mathcal{X}$.

Proof: Let $u(x) = \frac{1}{|\mathcal{X}|}$ be the uniform PMF, and $p(x)$ be the PMF of X over $\mathcal{X}$, respectively. Then, $D(p\|u) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{u(x)} = \log |\mathcal{X}| - H(X) \geq 0.$

Convexity of Information Measures

Theorem (Concavity of entropy)

$H(\mathbf{p})$ is a concave function of $\mathbf{p}$, where $\mathbf{p} := [p_1, \ldots, p_k]$ denotes an arbitrary k-point discrete probability mass function.

Proof 1: Let $X_i \sim \mathbf{p}_i (i = 1, 2)$ be two RVs defined on the same set. For any $\lambda \in [0, 1]$, define

$$\theta = \begin{cases} 1, & \text{with prob } \lambda \\ 2, & \text{with prob } 1 - \lambda. \end{cases}$$

Hence, the new RV $X_\theta \sim \lambda \mathbf{p}_1 + (1 - \lambda)\mathbf{p}_2$.

$\because H(X_\theta) \geq H(X_\theta|\theta) \implies H(\lambda \mathbf{p}_1 + (1 - \lambda)\mathbf{p}_2) \geq \lambda H(\mathbf{p}_1) + (1 - \lambda) H(\mathbf{p}_2)$.

Proof 2: $H(\mathbf{p}) = -\sum_i p_i \log(p_i)$. Note that each term in the sum is convex and only depends on one p_i. Hence, $H(\mathbf{p})$ is concave.

Proof 3: $H(\mathbf{p}) = \log |\mathcal{X}| - D(\mathbf{p}||\mathbf{u})$, where $\mathbf{u}$ is the uniform distribution. Since $D(\mathbf{p}||\mathbf{u})$ is convex in $\mathbf{p}$, $H(\mathbf{p})$ is concave.

Convexity of Information Measures (cont'd)

Theorem (Convexity of relative entropy)

$D(p||q)$ is convex in the pair (p, q). That is, if (p_1, q_1) and (p_2, q_2) are two pairs of PMFs. Then, $D(\lambda p_1 + (1-\lambda)p_2 \| \lambda q_1 + (1-\lambda)q_2) \leq \lambda D(p_1 \| q_1) + (1-\lambda) D(p_2 \| q_2)$ holds for any $\lambda \in [0, 1]$.

Proof: By log-sum inequality, we have

$$[\lambda p_1(x) + (1-\lambda)p_2(x)] \log \left(\frac{\lambda p_1(x) + (1-\lambda)p_2(x)}{\lambda q_1(x) + (1-\lambda)q_2(x)} \right)$$
$$\leq \lambda p_1(x) \log \frac{p_1(x)}{q_1(x)} + (1-\lambda) p_2(x) \log \frac{p_2(x)}{q_2(x)}$$

Summing this over all $x \in \mathcal{X}$ completes the proof.

Convexity of Information Measures (cont'd)

Theorem (Convexity of mutual information)

Let $(X, Y) \sim p(x, y) = p(x)p(y|x)$. $I(X;Y)$ *is a concave function of* $p(x)$ *for fixed* $p(y|x)$ *and a convex function of* $p(y|x)$ *for fixed* $p(x)$.

Proof:[1] part i) $I(X;Y) = H(Y) - H(Y|X) = H(Y) - \sum_x p(x)H(Y|X=x)$

- $p(y) = \sum_x p(x)p(y|x) \implies p(y)$ is linear in $p(x)$ for fixed $p(y|x)$.
- $H(Y)$ is concave in $p(y)$ and hence in $p(x)$.
- The second term in linear in $p(x)$. Hence, the difference is concave in $p(x)$.

part ii)

- $p(x, y) = p(x)p(y|x)$ is linear in $p(y|x)$ for fixed $p(x)$.
- $I(X;Y) = D(p(x, y)||p(x)p(y))$ is convex in $p(x, y)$, and hence convex in $p(y|x)$.
- Regarding the capacity of conveying information, mixing two channels is always worse than using the two channels separately.

[1] See alternative proofs in `https://vvvvalvalval.github.io/posts/2019-12-18_2-proofs-in-Information-Theory-channel-convexity-of-mutual-information.html`

Data Processing Inequality (DPI)

Definition

Random variables X, Y, Z are said to form a Markov chain in the $X \to Y \to Z$ if the joint PMF can be written as $p(x, y, z) = p(x)p(y|x)p(z|y)$.

Note: $p(x, y, z) = p(x)p(y, z|x) = p(x)p(y|x)p(z|y, x) = p(x)p(y|x)p(z|y)$.

Corollary

$X \to Y \to Z$ *iff* X *and* Z *are conditionally independent given Y, denoted as* $(X \perp\!\!\!\perp Z) \mid Y$. *Markovity implies conditional independence since*

$$p(x, z|y) = \frac{p(x, y, z)}{p(y)} = \frac{p(x, y)p(z|y)}{p(y)} = p(x|y)p(z|y).$$

Corollary ($X \leftrightarrow Y \leftrightarrow Z$)

$X \to Y \to Z \Leftrightarrow Z \to Y \to X$.

DPI

Theorem (Data Processing Inequality (DPI))

If $X \to Y \to Z$ *then* $I(X;Y) \geq I(X;Z)$ *with equality iif* $I(X;Y|Z) = 0$ *(i.e.,* $X \to Z \to Y$*).*

Proof: By the chain rule, we expand mutual information in two different ways:

$$I(X;Y,Z) = I(X;Y) + I(X;Z|Y) = I(X;Z) + I(X;Y|Z)$$

$\because (X \perp\!\!\!\perp Z) \mid Y \implies I(X;Z|Y) = 0 \implies I(X;Y) \geq I(X;Z).$

Corollary

1. *If* $X \to Y \to Z$*, then* $H(X|Z) \geq H(X|Y)$*.*
2. $I(X;Y) \geq I(X;g(Y))$ *for any function* $g(\cdot)$*; i.e., post-processing decreases the info.*
3. *If* $X_1 \to X_2 \to \cdots \to X_n$*, for any* i,j,k,l *such that* $1 \leq i \leq j \leq k \leq l \leq n$*, we have* $I(X_i;X_l) \leq I(X_j;X_k)$*.*

Proof:

1. $H(X) - H(X|Z) = \boxed{I(X;Z) \leq I(X;Y)} = H(X) - H(X|Y).$

DPI (cont'd)

Corollary

If $X \to Y \to Z$, then $I(X;Y|Z) \leq I(X;Y)$.

Generally, it is possible to have $I(X;Y|Z) > I(X;Y)$; see the following example when X, Y, Z do not form a Markov chain.

Example

Consider $Z = X + Y$ for two i.i.d. $X, Y \sim \text{Bern}(0.5)$. Find $I(X;Y|Z)$.

$$\begin{aligned} I(X;Y|Z) &= H(X|Z) - H(X|Y,Z) \\ &= \Pr(Z=0)H(X|Z=0) + \Pr(Z=2)H(X|Z=2) + \Pr(Z=1)H(X|Z=1) \\ &= [\Pr(\mathrm{X}=1,\mathrm{Y}=0) + \Pr(\mathrm{Y}=1,\mathrm{X}=0)] \times H(\mathrm{X}|\mathrm{Z}=1) \\ &= 2 \times \frac{1}{4} \times H(1/2,1/2) = 0.5 > I(X;Y) = 0 \end{aligned}$$

Statistic: Function of Samples

- A statistic is any quantity computed from values in a sample which is considered for a statistical purpose; i.e., statistic is a function of a sample; e.g., sample mean and sample variance. We estimate things all the time. Estimation help us understand the behavior of a large population with a small sample.
- Example 1: we want to know the average height of students in a school district with a population of 10,000. We take a sample of 100 students and find the sample mean, which is a statistic (estimator).
- Example 2: we survey the people in a neighborhood about their preference for a candidate. Using this limited amount of data, we can estimate the likeliness of a candidate to win an election in a larger area.

Sufficient Statistics (SS)

- How do we know if every bit of information in data has been explored to estimate the parameter?
- A statistic is sufficient w.r.t. a statistical model and its associated unknown parameter if "no other statistic that can be calculated from the same sample provides any additional information as to the value of the parameter."

Given a family of distributions $\{f_\theta(x)\}$, let X be a sample drawn from a distribution in this family, and $T(X)$ be any statistic. Hence, $\theta \to X \to T(X) \Rightarrow I(\theta; X) \geq I(\theta; T(X))$.

Definition (Sufficient Statistic)

A function $T(X)$ is a sufficient statistic relative to the family $\{f_\theta(x)\}$ if X is independent of θ given $T(X)$ for any distribution on θ ($\theta \perp\!\!\!\perp X \mid T(X)$); i.e., $\theta \to T(X) \to X$ forms a Markov chain, and

$$\boxed{I(\theta; X) = I(\theta; T(X)).}$$

Sufficient Statistics (cont'd)

Implication

1. Given a set X of i.i.d. data conditioned on an unknown parameter θ, a sufficient statistic is a function $T(X)$ whose value contains all the information needed to compute any estimate of θ.
2. A statistics is sufficient for θ if it contains all information in X about θ: Once we know $T(X)$, the remaining randomness in X does not depend on θ.

Definition (Sufficient Statistic)

Let X be a random sample from a distribution with parameter θ. $T(X)$ is a sufficient statistic for θ if the conditional probability $\Pr(X \mid T(X) = k)$ does not depend on θ.

Note that from Bayesian's view, we also have $\Pr(\theta \mid X, T(X) = k) = \Pr(\theta \mid T(X) = k)$

Examples of Sufficient Statistic

Example

Given i.i.d. $X_1, \ldots, X_n \sim \text{Bern}(\theta)$. Let $X := (X_1, \ldots, X_n)$, a sufficient statistic of θ is $T(X) = \sum_{i=1}^n X_i$.

Proof: We need to show that $\Pr(X)$ is independent of θ given $T(X)$.

$\Pr\left((X_1, ..., X_n) = (x_1, ..., x_n)\right) = \prod_{i=1}^n \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum x_i}(1-\theta)^{(n-\sum x_i)}$.

$T(X) \sim B(n, \theta) \implies \Pr(T(X) = k) = \binom{n}{k}\theta^k(1-\theta)^{n-k}$. Hence, we get

$$\Pr\left((X_1, ..., X_n) = (x_1, ..., x_n) \middle| \sum_{i=1}^n X_i = k\right) = \begin{cases} 0, & \text{if } \sum_{i=1}^n x_i \neq k \\ \binom{n}{k}^{-1}, & \text{if } \sum_{i=1}^n x_i = k \end{cases}$$

Example 2: For $X_i \sim \text{Unif}(\theta, \theta+1)$, a SS for θ is: $T(X) = (\max\{X_i\}, \min\{X_i\})$.

Factorization Theorem[2]

Theorem (Fisher–Neyman factorization theorem)

A statistic $T(X)$ is sufficient for θ if and only if functions g and h can be found such that the probability mass or density function $f_\theta(\mathbf{x})$ can be factorized as

$$f_\theta(\mathbf{x}) = h(\mathbf{x}) g_\theta(T(\mathbf{x})),$$

where $h(\cdot)$ does not depend on θ while $g_\theta(\cdot)$ depends on $\mathbf{x}$ only through $T(\mathbf{x})$.

- The Bernoulli distribution:

$$f_\theta(\mathbf{x}) = \theta^{\sum x_i}(1-\theta)^{(n-\sum x_i)} = \underbrace{\theta^{T(\mathbf{x})}(1-\theta)^{(n-T(\mathbf{x}))}}_{g_\theta(T(\mathbf{x}))} \times \underbrace{1}_{h(\mathbf{x})}$$

- The uniform distribution $\mathrm{Unif}(\theta, \theta+1)$:

$$f_\theta(\mathbf{x}) = \prod_{i=1}^{n} \mathbb{1}_{\{\theta \le X_i \le \theta+1\}} = \underbrace{\mathbb{1}_{\{\theta \le \min\{X_i\} \le \max\{X_i\} \le \theta+1\}}}_{g_\theta(T(\mathbf{x}))} \times \underbrace{1}_{h(\mathbf{x})}$$

[2] https://bookdown.org/jkang37/stat205b-notes/lecture04.html

Minimal Sufficient Statistic

- Sufficient statistic always exists and it is not unique (infinitely many). Any one-to-one function of a sufficient statistic is a sufficient statistic.
- Data reduction: Typically we want the dimension of the sufficient statistic to be as small as possible since lower dimensional statistics are easier to understand/use for inference than higher dimensional ones.
- In other words, we are interested in finding a statistic that achieves the most data reduction while retaining all the information about θ: **Minimal sufficient statistic**.
- A statistic $T(X)$ is a minimal sufficient statistic if it is a function of every other sufficient statistic $U(X)$.
- Minimal sufficient statistic is a statistic that has the smallest mutual information with X while having the largest mutual information with θ. That is,

$$\boxed{\begin{aligned} &T(X) \in \arg\min_{U(X)} I(X; U(X)) \\ &\text{s.t. } I(\theta; U(X)) = \max_{T'(X)} I(\theta; T'(X)) \end{aligned}}$$

- Regarding Markov chain, we have $\theta \to T(X) \to U(X) \to X$.

Minimal Sufficient Statistic (cont'd)[3]

- Minimal sufficient statistic most efficiently captures (i.e., maximally compresses) the information about θ in the sample.
- Minimal sufficient statistic is not unique (may not even exist). Any one-to-one function of a minimal sufficient statistic is also a minimal sufficient statistic.

Theorem (Lehmann-Scheffé Theorem for Minimal Sufficiency)

A statistic $T(\cdot)$ is minimal sufficient for θ if the following property holds: For every two sample points $\mathbf{x}$ and $\mathbf{y}$, $\frac{f_\theta(\mathbf{x})}{f_\theta(\mathbf{y})}$ is independent of $\theta \iff T(\mathbf{x}) = T(\mathbf{y})$.

[3] https://bookdown.org/egarpor/inference/point-est-min-sufficient.html

Fano's Inequality

Consider a Markov chain $X \to Y \to \hat{X}$. In the context of communication:

- Send X through a noisy channel that possibly yields the received symbol $Y \neq X$.
- Recover X by post-processing Y to get an estimate $\hat{X} = g(Y)$ for some function g.
- Define the probability of error as $P_e := \Pr(\hat{X} \neq X)$.

Fano's inequality: We may estimate X with small P_e when $H(X|Y)$ is small.

For any estimator $\hat{X}$ such that $X \to Y \to \hat{X}$, define $P_e = \Pr(X \neq \hat{X})$, we have

$$H(P_e) + P_e \log |\mathcal{X}| \geq H(X|\hat{X}) \geq H(X|Y) \implies P_e \geq \frac{H(X|Y) - 1}{\log |\mathcal{X}|}.$$

If $\hat{X} \in \mathcal{X}$, we then have a slightly better upper bound for $H(X|Y)$:

$$H(P_e) + P_e \log(|\mathcal{X}| - 1) \geq H(X|Y)^a.$$

[a] $H(X|Y)$ was called the 'equivocation', which corresponds to the effect of channel noise. So, the inequality relates the equivocation (information measure) to the probability of error (traditional measure).

Fano's Inequality (cont'd)

$$\boxed{H(X|\hat{X}) \leq H\left(P_e\right) + P_e \log(|\mathcal{X}| - 1) \leq H\left(P_e\right) + P_e \log |\mathcal{X}|}$$

Intuition & implication:

- $H(X|\hat{X})$ is the remaining uncertainty about X when its estimate $\hat{X}$ is known. Intuitively, if P_e is small, $H(X|\hat{X})$ should also be small.
- Coding view: $H(X|\hat{X})$ quantify how many bits need for measuring X from $\hat{X}$. Robert Fano established an upper bound of it. First, use $H(P_e)$ bits to show if $\hat{X} = X$. In case that they differ (with prob P_e), we need $\log |\mathcal{X}|$ bits, or $\log(|\mathcal{X}| - 1)$ bits if $\hat{X} \in \mathcal{X}$, to describe X in the worst case.
- Fano's inequality provides a useful way of lower bounding the error of a predictor. We can use it to characterize when a perfect reconstruction of the sent code is impossible, i.e., P_e is bounded away from zero. This is useful to prove converse results of coding theorems.

Permissible Region of Fano's Inequality

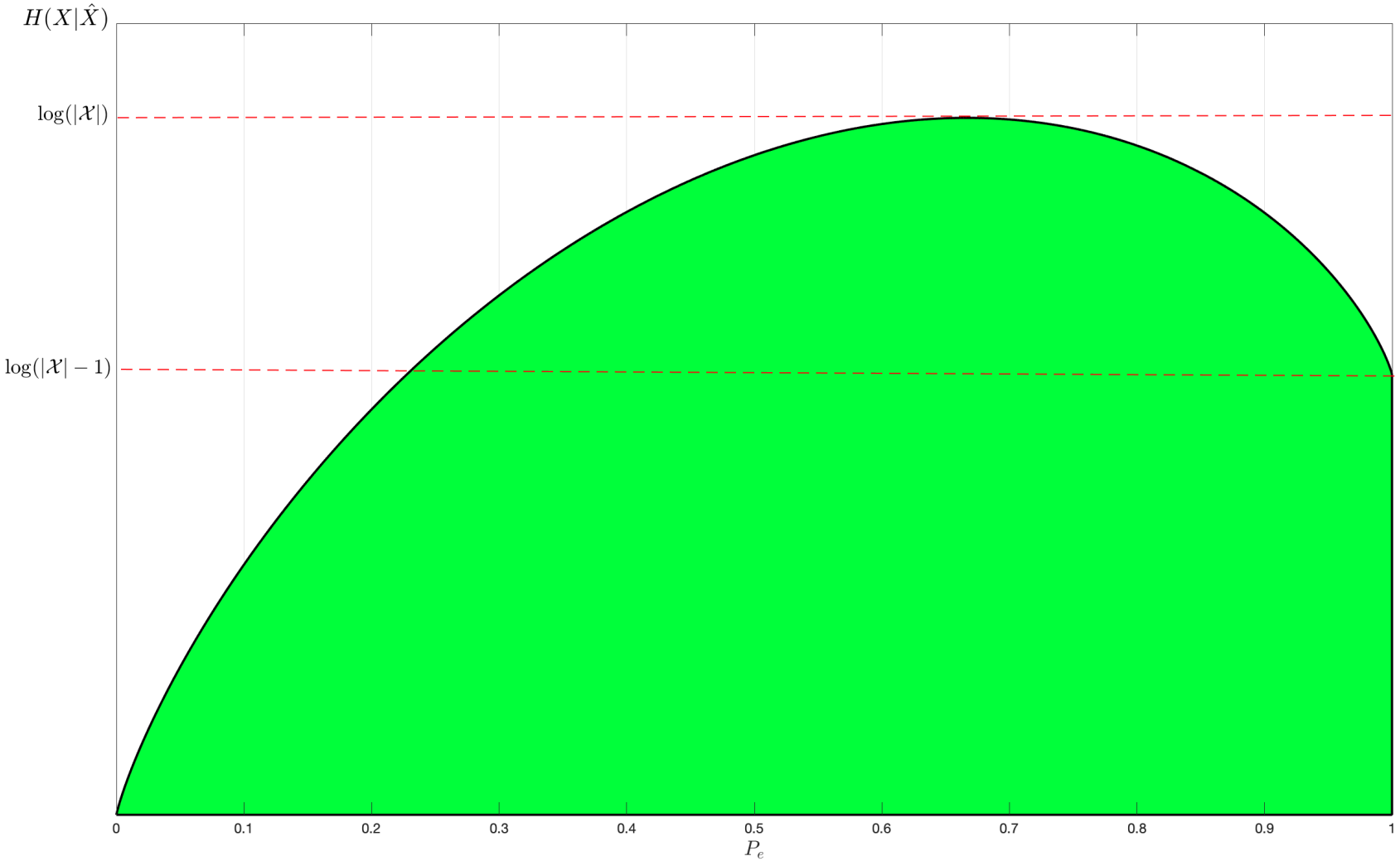


Figure: The boundary (blue curve) of the permissible region (green area) is given by the function: $H(X|\hat{X}) = H(P_e) + P_e \log(|\mathcal{X}| - 1)$.

Fano's Inequality Proof

Proof: Define the mis-classification event $E = \mathbb{1}_{\{\hat{X} \neq X\}}$. By the chain rule, we have

$$\begin{aligned} H(E, X|\hat{X}) &= H(X|\hat{X}) + \underbrace{H(E|X, \hat{X})}_{=0} \\ &= \underbrace{H(E|\hat{X})}_{\leq H(P_e)} + \underbrace{H(X|E, \hat{X})}_{\leq P_e \log |\mathcal{X}|} \\ &\leq H(P_e) + \Pr(E = 0) \underbrace{H(X|\hat{X}, E = 0)}_{=0} + \Pr(E = 1) H(X|\hat{X}, E = 1) \\ &\leq H(P_e) + P_e \log |\mathcal{X}| \end{aligned}$$

$$\implies H(X|Y) \leq H(X|\hat{X}) \leq H(P_e) + P_e \log |\mathcal{X}|.$$

If $X, \hat{X} \in \mathcal{X} \implies H(X|\hat{X}, E = 1) \leq P_e \log(|\mathcal{X}| - 1)$.

Corollaries of Fano's Inequality

Corollary (Alternative form of Fano's inequality)

Consider a Markov chain $X \to Y \to \hat{X}$. If X is uniform on $\{1, 2, \ldots, M\}$, then

$$P_e \geq 1 - \frac{I(X;\hat{X}) + 1}{\log M}.$$

Proof: Plugging $H(X|\hat{X}) = H(X) - I(X;\hat{X}) = \log M - I(X;\hat{X})$ and $H(P_e) \leq 1$ into the Fano's ineq, the result follows immediately.

The intuition: $\log M$ is the prior uncertainty of X while $I(X;\hat{X})$ represents how much information $\hat{X}$ reveals about X. To have a small P_e, the information revealed should be close to the prior uncertainty.

Corollary

For any two RVs X and Y, let $p = \Pr(X \neq Y)$. We have $H(p) + p \log |\mathcal{X}| \geq H(X|Y)$.

Proof: Let $\hat{X} = Y$ in Fano's inequality.

Fano's Inequality is Sharp

Example

Let $X \in \{1, 2, \ldots, m\}$ and $p_1 \geq p_2 \geq \cdots \geq p_m$. Then the best guess of X is $\hat{X} = 1$ and the resulting probability of error is $P_e = 1 - p_1$. Fano's inequality becomes

$$H(P_e) + P_e \log(m-1) \geq H(X).$$

The PMF $(p_1, p_2, \ldots, p_m) = \left(1 - P_e, \frac{P_e}{m-1}, \ldots, \frac{P_e}{m-1}\right)$ achieves the lower bound with equality. To see this,

$$\begin{aligned} H(X) &= -(1-P_e)\log(1-P_e) - (m-1) \times \frac{P_e}{m-1} \log \frac{P_e}{m-1} \\ &= -(1-P_e)\log(1-P_e) - P_e \log P_e + P_e \log(m-1) \\ &= H(P_e) + P_e \log(m-1). \end{aligned}$$

Applications of Fano's Inequality

As an information-theoretic tool, Fano's inequality is not only ubiquitous in studies of communications, but has been applied extensively in statistical inference, hypothesis testing, learning and optimization problems, etc; see, e.g.,


Jonathan Scarlett and Volkan Cevher (2021). *An Introductory Guide to Fano's Inequality with Applications in Statistical Estimation*.


Maxim Raginsky and Alexander Rakhlin (2011). *Information-Based Complexity, Feedback and Dynamics in Convex Programming*.


Alekh Agarwal et al (2011). *Information-theoretic lower bounds on the oracle complexity of stochastic convex optimization*.

Entropy and Probability of Errors

Lemma

Consider two independent RVs defined over the same set: $X \sim p(x), X' \sim r(x')$. Then,

$$\Pr(X = X') \geq \max\left\{2^{-H(p,r)}, 2^{-H(r,p)}\right\}.$$

Proof: $2^{-H(p,r)} = 2^{\mathrm{E}_p(\log_2 r(X))} \leq \mathrm{E}_p\left(2^{\log_2 r(X)}\right) = \sum_{x\in\mathcal{X}} p(x)r(x) = \Pr(X = X')$.

Swapping p and r in the cross entropy, we have $2^{-H(r,p)} \leq \Pr(X = X')$.

Corollary

Let X, X' are i.i.d. with entropy $H(X)$. Then,

$$\Pr(X = X') \geq 2^{-H(X)},$$

with equality iif X has a uniform distribution.

Thank You!

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