DETECTING NONTRIVIAL PRODUCTS IN THE STABLE HOMOTOPY RING OF SPHERES VIA THE THIRD MORAVA STABILIZER ALGEBRA

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Abstract. Let $p \geq 7$ be a prime number. Let $S(3)$ denote the third Morava stabilizer algebra. In recent years, Kato-Shimomura and Gu-Wang-Wu found several families of nontrivial products in the stable homotopy ring of spheres $\pi_*(S)$ using the cohomology of $S(3)$. In this paper, we determine all nontrivial products in $\pi_*(S)$ of the Greek letter family elements $\alpha_s, \beta_s, \gamma_s$ and Cohen's elements ζ_n that are detectable by $H^*(S(3))$. In particular, we show $\zeta_n\beta_1\gamma_s \neq 0 \in \pi_*(S)$, if $n \equiv 2 \bmod 3$, $s \not\equiv 0, \pm 1 \bmod p$.

1. Introduction

Computing the ring of stable homotopy groups of spheres $\pi_*(S)$ is one of the most fundamental problems in algebraic topology. Let p be an odd prime. The Adams-Novikov spectral sequence (ANSS) based on the Brown-Peterson spectrum BP is one of the most powerful tools to compute the p-component of $\pi_*(S)$ ([1, 10, 12, 19]). The E_2-page of the ANSS is $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$, which has been extensively studied in low dimensions.

For $s = 1$, $Ext^{1,*}_{BP_*BP}(BP_*, BP_*)$ is generated by $\alpha_{kp^n/n+1}$ for $n \geqslant 0$, $p \nmid k \geqslant 1$ ([16]).

For $s = 2$, $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$ is generated by $\beta_{kp^n/j,i+1}$ for suitable (n, k, j, i) ([12, 13]).

For $s = 3$, only partial results of $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$ are known (see, for example, [14, 15, 19]). Nonetheless, one can construct a family of linearly independent elements in $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$, which are denoted as $\gamma_{s_3/s_2,s_1}$ ([12]).

Based on the low dimensional computations of $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$, we obtain lots of nontrivial elements in $\pi_*(S)$. In particular, for $p \geq 7$, there are the Greek letter family elements, denoted as α_s, β_s, and γ_s with $s \geq 1$ [12, 16, 20, 21]. These three families of elements in $\pi_*(S)$ are represented by the corresponding elements of the same name in $Ext^{1,*}_{BP_*BP}(BP_*, BP_*)$, $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$, and $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$, respectively.

Using the Adams spectral sequence, Cohen [3] found another family of nontrivial elements $\zeta_n \in \pi_*(S)$ with $n \geq 1$. The representation of ζ_n in $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$ is also studied in [3] (also see [4]).

Nontrivial products on $\pi_*(S)$. The multiplication on $\pi_*(S)$ is graded commutative. To further understand the ring structure of $\pi_*(S)$, we need to determine whether the product of certain given elements is trivial. The main purpose of this paper is to find nontrivial products formed by the elements in $\{\alpha_s, \beta_s, \gamma_s, \zeta_s : s \geq 1\}$. To make sure these elements are all well defined, we assume $p \geq 7$ for the rest of the paper unless otherwise specified.

There have been plenty of results in this direction. Just to name a few:

(a) Aubry [2] shows that $\alpha_1\beta_2\gamma_2, \beta_1^r\beta_2\gamma_2 \neq 0$ if $r \leq p - 1$.

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(b) Lee-Ravenel [7] shows $\beta_1^{p^2-p-1} \neq 0$ for $p \geq 7$.
(c) Lee [8] shows: (1) $\beta_1^r\beta_s, \beta_1^{r-1}\beta_2\beta_{kp-1} \neq 0$ for $p \geq 5$, if $r, k \leq p-1$, $s < p^2 - p - 1$, and $s \not\equiv 0 \bmod p$; (2) $\beta_1^r\gamma_t, \beta_1^{r-1}\beta_2\gamma_t \neq 0$, if $r, t \leq p-1$; (3) $\alpha_1\beta_1^r\gamma_t \neq 0$, if $r \leq p-2$, $2 \leq t \leq p-1$; (4) $\beta_1^{p-1}\zeta_n \neq 0$.
(d) Liu-Liu [9] shows that $\alpha_1\beta_1^2\beta_2\gamma_s \neq 0$ if $4 < s < p$.
(e) Zhao-Wang-Zhong [24] shows that $\gamma_{p-1}\zeta_n \neq 0$ if $n \neq 4$.

Detection via $H^*(S(3))$. In recent years, Kato-Shimomura [6] developed the method to detect nontrivial products on $\pi_*(S)$ via $S(3)$. We can briefly recall their strategy as follows.

Let $S(3)$ denote the third Morava stabilizer algebra [17]. There is a natural map

$$\phi : Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \to Ext^{*,*}_{S(3)}(\mathbb{F}_p, \mathbb{F}_p) =: H^{*,*}(S(3)). \tag{1.1}$$

The cohomology $H^{*,*}(S(3))$ is computed in [4, 19, 23].

Given a product $x = x_1x_2\cdots x_n \in \pi_*(S)$. We let $y = y_1y_2\cdots y_n \in Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ denote the representative of x on the E_2-page of the ANSS. By routine algebraic arguments, one can compute $\phi(y) \in H^{*,*}(S(3))$. If $\phi(y) \neq 0$, then we have $y \neq 0 \in Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$. Then, for suitable examples, one can conclude y can not be killed by any Adams-Novikov differential by degree reasons. Therefore, $x \neq 0 \in \pi_*(S)$.

Using this strategy, Kato-Shimomura [6] shows: (1) $\alpha_1\gamma_s \neq 0$, if $s \not\equiv 0, \pm 1 \bmod p$; (2) $\beta_1\gamma_s \neq 0$, if $s \not\equiv 0, 1 \bmod p$; (3) $\beta_2\gamma_s \neq 0$, if $s \not\equiv 0, \pm 1 \bmod p$.

Analogously, Gu-Wang-Wu [4] shows $\zeta_n\gamma_s \neq 0$, if $n \not\equiv 1 \bmod 3$, $s \not\equiv 0, \pm 1 \bmod p$.

This new approach has the benefit that now we can study products with γ_s for arbitrary large s. On the contrary, the results we mentioned before require $s < p$.

Our main results. In this paper, we follow the "Detection via $H^*(S(3))$" method developed in [4, 6]. However, instead of a case-by-case study on specific examples, we exhaust the power of this method and enumerate all detectable products.

Our main results are the following.

Theorem 5.2. *Let $p \geq 7$ be a prime. Let $n \equiv 2$ mod 3, $s \not\equiv 0, \pm 1$ mod p. Then $\zeta_n\beta_1\gamma_s \neq 0 \in \pi_*(S)$.*

Theorem 5.3. *Let $p \geq 7$ be a prime. We consider the products in $\pi_*(S)$ where each factor belongs to $\{\alpha_s, \beta_s, \gamma_s, \zeta_s : s \geq 1\}$. Amongst all such products, only the following ones can be detected as nontrivial products using the comparison with $H^*S(3)$.*

i) $\alpha_1\beta_1$,
ii) $\alpha_1\beta_2$,
iii) $\beta_1\beta_2$,
iv) $\alpha_1\gamma_s$, *if $s \not\equiv 0, \pm 1$ mod p,*
v) $\beta_1\gamma_s$, *if $s \not\equiv 0, 1$ mod p,*
vi) $\beta_2\gamma_s$, *if $s \not\equiv 0, \pm 1$ mod p,*
vii) $\zeta_n\gamma_s$, *if $n \not\equiv 1$ mod 3, $s \not\equiv 0, \pm 1$ mod p,*
viii) $\zeta_n\beta_1$, *if $n \not\equiv 0$ mod 3,*
ix) $\zeta_n\beta_1\gamma_s$, *if $n \equiv 2$ mod 3, $s \not\equiv 0, \pm 1$ mod p.*

Remark 1.1. The non-triviality of i) ~ viii) have been determined by earlier works in [4, 6, 8, 12]. We single out the new result *ix)* as Theorem 5.2. Here, Theorem 5.3 gives all nontrivial products detectable by $H^*(S(3))$. It suggests that one would have to use different methods in order to find new nontrivial products of this form.

Organization of the paper. In Section 2, we review the algebraic constructions and structures that we need to use in this paper, including Hopf algebroids and cobar complexes. We will also review the definition and properties of the third Morava stabilizer algebra $S(3)$. In Section 3, we review the $\mathbb{F}_p$-algebra structure of $H^*S(3)$. In Section 4, we recall the constructions of the α, β, γ-family elements in the Adams-Novikov spectral sequence. Then we determine their images under the comparison map $\phi : Ext^*_{BP_*BP}(BP_*, BP_*) \to H^*(S(3))$. In Section 5, we use the $\mathbb{F}_p$-algebra structure of $H^*S(3)$ to detect nontrivial products in $Ext^*_{BP_*BP}(BP_*, BP_*)$. Then we prove Theorem 5.2 and Theorem 5.3.

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2. Hopf algebroids

This section recalls the basic definitions and constructions related to Hopf algebroids. In particular, we review the algebraic structures of the third Morava stabilizer algebra $S(3)$. We also review its relationship with the Hopf algebroid (BP_*, BP_*BP). This section serves as the algebraic foundation of the computations in this paper.

Definition 2.1. A *Hopf algebroid* over a commutative ring K is a pair (A, Γ) of commutative K-algebras with structure maps

$$\begin{aligned} \text{left unit map } \eta_L &: A \to \Gamma \\ \text{right unit map } \eta_R &: A \to \Gamma \\ \text{coproduct map } \Delta &: \Gamma \to \Gamma \otimes_A \Gamma \\ \text{counit map } \varepsilon &: \Gamma \to A \\ \text{conjugation map } c &: \Gamma \to \Gamma \end{aligned}$$

such that for any other commutative K-algebra B, the two sets $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(\Gamma, B)$ are the objects and morphisms of a groupoid.

2.2. **The Hopf algebroid (BP_*, BP_*BP).** An important example of Hopf algebroids is (BP_*, BP_*BP).

Recall that we have

$$BP_* := \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots], \quad BP_*BP = BP_*[t_1, t_2, \cdots] \tag{2.1}$$

We also have

$$H_*(BP) = \mathbb{Z}_{(p)}[m_1, m_2, \cdots] \tag{2.2}$$

where the inner degrees are $|v_n| = |t_n| = |m_n| = 2(p^n - 1)$. Throughout this paper, we denote $v_0 = p, m_0 = t_0 = 1$. The Hurewicz map induces an embedding

$$\begin{aligned} i : BP_* &\to H_*(BP) \\ v_n &\mapsto pm_n - \sum_{i=1}^{n-1} v_{n-i}^{p^i} m_i \end{aligned} \tag{2.3}$$

We can describe the structure maps of the Hopf algebroid (BP_*, BP_*BP) as follows [5, 12, 19].

The left unit and right unit maps $\eta_L, \eta_R : BP_* \to BP_*BP$ are determined by

$$\eta_L(v_n) = v_n, \quad \eta_R(m_n) = \sum_{i+j=n} m_i t_j^{p^i} \tag{2.4}$$

The coproduct map $\Delta : BP_*BP \to BP_*BP \otimes_{BP_*} BP_*BP$ is determined by

$$\sum_{i+j=n} m_i(\Delta t_j)^{p^i} = \sum_{i+j+k=n} m_i t_j^{p^i} \otimes t_k^{p^{i+j}} \tag{2.5}$$

The counit map $\varepsilon : BP_*BP \to BP_*$ is determined by

$$\varepsilon(v_n) = v_n, \quad \varepsilon(t_n) = 0. \tag{2.6}$$

The conjugation map $c : BP_*BP \to BP_*BP$ is determined by

$$\sum_{i+j+k=n} m_i t_j^{p^i} c(t_k)^{p^{i+j}} = m_n. \tag{2.7}$$

In practice, the following formulas [6] are also useful.

$$\eta_R(v_1) = v_1 + pt_1 \tag{2.8}$$

$$\eta_R(v_2) \equiv v_2 + v_1 t_1^p + pt_2 \mod (p^2, v_1^p) \tag{2.9}$$

$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1 \tag{2.10}$$

$$\Delta(t_2) = t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 - v_1 b_{1,0} \tag{2.11}$$

Notations 2.3. We let $b_{1,0}$ denote $\sum_{i=1}^{p-1} \binom{p}{i}/p \ t_1^i \otimes t_1^{p-i}$. In general, we denote $b_{i,j} = \frac{1}{p}[(\sum_{k=0}^{i} t_{i-k} \otimes t_k^{p^{i-k}})^{p^{j+1}} - \sum_{k=0}^{i} t_{i-k}^{p^{j+1}} \otimes t_k^{p^{i-k+j+1}}]$ for $i \geq 1$, $j \geq 0$. See [22] for related discussions.

2.4. **Morava stabilizer algebras.** Denote $K(n)_* := \mathbb{F}_p[v_n, v_n^{-1}]$. We can equip $K(n)_*$ with a BP_*-algebra structure via the ring homomorphism sending all v_i with $i \neq n$ to 0. Then we can define the Hopf algebra

$$\Sigma(n) := K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_* \tag{2.12}$$

As an algebra, one has [17]

$$\Sigma(n) \cong K(n)_*[t_1, t_2, \cdots]/(v_n t_i^{p^n} - v_n^{p^i} t_i : i > 0) \tag{2.13}$$

The coproduct structure of $\Sigma(n)$ is inherited from that of BP_*BP.

Moreover, one can prove [11]

$$Ext^*_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \cong Ext^*_{\Sigma(n)}(K(n)_*, K(n)_*) \tag{2.14}$$

Then, we define the Hopf algebra

$$S(n) = \Sigma(n) \otimes_{K(n)_*} \mathbb{F}_p \tag{2.15}$$

where $K(n)_*$ and $\Sigma(n)$ are here regarded as graded over $\mathbb{Z}/2(p^n - 1)$ and $\mathbb{F}_p$ is a $K(n)_*$-algebra via the map sending v_n to 1. We call $S(n)$ the n-th Morava stabilizer algebra. One can show [17]

$$Ext^*_{\Sigma(n)}(K(n)_*, K(n)_*) \otimes_{K(n)_*} \mathbb{F}_p \cong Ext^*_{S(n)}(\mathbb{F}_p, \mathbb{F}_p) =: H^*(S(n)) \tag{2.16}$$

For the purpose of this paper, from now on, we will only consider the case when $n = 3$. We have the following results.

Theorem 2.5 ([18]). *As an algebra, $S(3) \cong \mathbb{F}_p[t_1, t_2, ...]/(t_i^{p^3} - t_i)$ and the inner degrees are $|t_s| \equiv 2(p^s - 1)$ mod $2(p^3 - 1)$. The coproduct structure of $S(3)$ is that inherited from BP_*BP. In particular, for $s \leq 3$,*

$$\Delta(t_s) = \sum_{k=0}^{s} t_k \otimes t_{s-k}^{p^k} \tag{2.17}$$

and for $s > 3$

$$\Delta(t_s) = \sum_{k=0}^{s} t_k \otimes t_{s-k}^{p^k} - \tilde{b}_{s-3,2} \tag{2.18}$$

where $t_0 = 1$ and $\tilde{b}_{s-3,2}$ denotes the mod p reduction of $\frac{1}{p}[(\sum_{k=0}^{s-3} t_k \otimes t_{s-3-k}^{p^k})^{p^3} - \sum_{k=0}^{s-3} t_k^{p^3} \otimes t_{s-3-k}^{p^{k+3}}]$.

Notations 2.6. In general, we let $\tilde{b}_{i,j}$ denote the mod p reduction of $b_{i,j}$ in Notations 2.3.

2.7. **Cobar complexes.** Cobar complexes are helpful in computing certain Ext groups, such as $Ext^*_{BP_*BP}(BP_*, BP_*)$, $Ext^*_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$, and $Ext^*_{S(n)}(\mathbb{F}_p, \mathbb{F}_p)$. We now recall the relevant definitions and constructions.

Definition 2.8. Let (A, Γ) be a Hopf algebroid. A *right Γ-comodule M* is a right A-module M together with a right A-linear map $\psi : M \to M \otimes_A \Gamma$ which is counitary and coassociative, i.e., the following diagrams commute.

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[dr, equal] & M \otimes_A \Gamma \arrow[d, "M\otimes\varepsilon"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\psi"'] & M \otimes_A \Gamma \arrow[d, "M\otimes\Delta"] \\ M \otimes_A \Gamma \arrow[r, "\psi\otimes\Gamma"'] & M \otimes_A \Gamma \otimes_A \Gamma \end{tikzcd}$$

Left Γ-comodules are defined similarly.

Definition 2.9. Let (A, Γ) be a Hopf algebroid. Let M be a right Γ-comodule. The cobar complex $\Omega^{*,*}_\Gamma(M)$ is a cochain complex with

$$\Omega^{s,*}_\Gamma(M) = M \otimes_A \overline{\Gamma}^{\otimes s}$$

where $\overline{\Gamma}$ is the augmentation ideal of $\varepsilon : \Gamma \to A$. The differentials $d : \Omega^{s,*}_\Gamma(M) \to \Omega^{s+1,*}_\Gamma(M)$ are given by

$$\begin{aligned} d(m \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_s) &= -(\psi(m) - m \otimes 1) \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_s \\ &- \sum_{i=1}^{s} (-1)^{\lambda_{i,j_i}} m \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes (\sum_{j_i} x'_{i,j_i} \otimes x''_{i,j_i}) \otimes x_{i+1} \otimes \cdots \otimes x_s \end{aligned}$$

where we denote

$$\begin{aligned} \sum_{j_i} x'_{i,j_i} \otimes x''_{i,j_i} &= \Delta(x_i) - 1 \otimes x_i - x_i \otimes 1 \\ \lambda_{i,j_i} &= i + |x_1| + \cdots + |x_{i-1}| + |x'_{i,j_i}| \end{aligned}$$

Proposition 2.10 ([19] Section A1.2). *The cohomology of $\Omega^{s,*}_\Gamma(M)$ is $Ext^{s,*}_\Gamma(A, M)$. Moreover, if M is also a commutative associative A-algebra such that the structure map ψ is an algebra map, then $Ext^{s,*}_\Gamma(A, M)$ has a naturally induced product structure.*

3. The cohomology of $S(3)$

The cohomology $H^*S(3) := Ext^*_{S(3)}(\mathbb{F}_p, \mathbb{F}_p)$ of the Hopf algebra $S(3)$ has been extensively studied. For $p \geq 5$, Ravenel [18] computed the module structure of $H^*S(3)$. The algebra structure of $H^*S(3)$ is then computed by Yamaguchi in [23], though there are typos [4, Remark A.1]. Gu-Wang-Wu [4] recomputed the algebra structure of $H^*S(3)$ for $p \geq 7$ in a different way, using a carefully constructed May spectral sequence.

In this section, we recall the $\mathbb{F}_p$-algebra structure of $H^*S(3)$ determined in [4], we also take this opportunity to correct some typos in [4].

Theorem 3.1 ([4] Theorem 2.2). *Let $p \geq 7$ be a prime number. The Hopf algebra $S(3)$ can be given an increasing filtration by setting the May degrees as follows: (i) for $s = 1, 2, 3$, let $M(t_s^{p^j}) = 2s-1$, and (ii) for $s > 3$, $j \in \mathbb{Z}/3$, inductively define $M(t_s^{p^j}) = max\left\{M(t_k^{p^j}) + M(t_{s-k}^{p^{j+k}}),\ p \cdot M(t_{s-3}^{p^{j+2}}) \mid 0 < k < s\right\} + 1$. The filtration of $S(3)$ naturally induces a filtration of $\Omega^{*,*}_{S(3)}(\mathbb{F}_p)$. The associated May spectral sequence (MSS) converges to $H^{*,*}S(3)$. The MSS has E_1-page*

$$E_1^{*,*,*} = E[h_{i,j} | i \geq 1, j \in \mathbb{Z}/3] \otimes P[b_{i,j} | i \geq 1, j \in \mathbb{Z}/3] \tag{3.1}$$

and $d_r : E_r^{s,t,M} \to E_r^{s+1,t,M-r}$, where

$$\begin{aligned} h_{i,j} &= [t_i^{p^j}] \in E_1^{1,2(p^i-1)p^j,*} \\ b_{i,j} &= [\sum_{k=1}^{p-1} \binom{p}{k}/p\ (t_i^{p^j})^k \otimes (t_i^{p^j})^{p-k}\ ] \in E_1^{2,2(p^i-1)p^{j+1},*} \end{aligned} \tag{3.2}$$

Moreover, one has the following relations:

$$h_{i,j} \cdot h_{i_1,j_1} = -h_{i_1,j_1} \cdot h_{i,j}, \quad h_{i,j} \cdot b_{i_1,j_1} = b_{i_1,j_1} \cdot h_{i,j}, \quad b_{i,j} \cdot b_{i_1,j_1} = b_{i_1,j_1} \cdot b_{i,j}. \tag{3.3}$$

Proposition 3.2 ([4] Proposition 3.1). *Let $p \geq 7$ be a prime number. As a $\mathbb{F}_p$-module, $H^*S(3)$ is isomorphic to $E[\rho] \otimes M$, where $\rho \in H^1S(3)$, M is a $\mathbb{F}_p$-module with the following generators ($i \in \mathbb{Z}/3$):*

dim0: 1;
dim1: $h_{1,i}$;
dim2: $e_{4,i}$, g_i, k_i;
dim3: $e_{4,i}h_{1,i}$, $e_{4,i}h_{1,i+1}$, $g_ih_{1,i+1}$, μ_i, ν_i, ξ;
dim4: $e_{4,i}^2$, $e_{4,i}e_{4,i+1}$, $e_{4,i}g_{i+1}$, $e_{4,i}g_{i+2}$, $e_{4,i}k_i$, θ_i;
dim5: $e_{4,i}e_{4,i+1}h_{1,i+2}$, $(e_{4,i}e_{4,i+1}h_{1,i+2} = e_{4,i+1}e_{4,i+2}h_{1,i})$
$e_{4,i}^2h_{1,i+1}$, $e_{4,i}^2h_{1,i+2}$, $e_{4,i}\mu_{i+2}$, $e_{4,i}\nu_i$, η_i;
dim6: $e_{4,i}^2e_{4,i+1}$, $e_{4,i}^2e_{4,i+2}$, $e_{4,i}e_{4,i+1}g_{i+2}$;
dim7: $e_{4,i}e_{4,i+1}\mu_{i+2}$;
dim8: $e_{4,i}^2e_{4,i+2}g_{i+1}$, $(e_{4,i}^2e_{4,i+2}g_{i+1} = e_{4,i+1}^2e_{4,i}g_{i+2})$.

Here, the generators can be described via their MSS representatives as follows:

$h_{1,i} := t_1^{p^i}$	$\rho := h_{3,0} + h_{3,1} + h_{3,2}$
$e_{4,i} := h_{1,i}h_{3,i+1} + h_{2,i}h_{2,i+2} + h_{3,i}h_{1,i}$	$g_i := h_{2,i}h_{1,i}$
$k_i := h_{2,i}h_{1,i+1}$	$\mu_i = h_{3,i}h_{2,i}h_{1,i}$
$\nu_i := h_{3,i}h_{2,i+1}h_{1,i+2}$	$\xi = \sum_{i=0}^{2} h_{3,i}e_{3,i+1} + h_{2,0}h_{2,1}h_{2,2}$

$\theta_i = h_{3,i}h_{2,i+2}h_{2,i}h_{1,i}$	$\eta_i = h_{3,i}h_{3,i+1}h_{2,i+2}h_{2,i}h_{1,i}$

Table 1. MSS representation of generators of $H^*S(3)$

where we denote $h_{i,j} := t_i^{p^j}$ *for* $j \in \mathbb{Z}/3$, $e_{3,i} := h_{1,i}h_{2,i+1} + h_{2,i}h_{1,i+2}$ *for* $i \in \mathbb{Z}/3$.

Remark 3.3. We take the opportunity to correct a typo in [4], where they wrote $\xi = \sum h_{3,i+1}e_{3,i} + \sum h_{2,i}h_{2,i+1}h_{2,i+2}$. The formula is now corrected in Table 1.

The $\mathbb{F}_p$-algebra structure of $H^*S(3)$ is complicated. However, as we will see in Proposition 4.16, in this paper we only need to care about the product structure of the sub-algebra generated by the elements $\{\rho, h_{1,i}, e_{4,i}, g_i, k_i, \nu_i | i \in \mathbb{Z}/3\}$. Note for $x \in H^iS(3)$, $y \in H^jS(3)$, one can show $x{\cdot}y = (-1)^{ij}y{\cdot}x$. We record all nontrivial product relations of these generators as follows.

Proposition 3.4 ([4] Appendix A)**.** *Let $p \geq 7$ be a prime number. All nontrivial products amongst generators of $H^*S(3)$ in the set $\{\rho, h_{1,i}, e_{4,i}, g_i, k_i, \nu_i | i \in \mathbb{Z}/3\}$ can be listed as follows.*

dim3:

$e_{4,i} \cdot h_{1,i+2} = e_{4,i+2}h_{1,i}$ $\qquad k_i \cdot h_{1,i} = -g_ih_{1,i+1}$

dim4:

$e_{4,i} \cdot k_{i+1} = e_{4,i+1}g_{i+2}$ $\qquad \nu_i \cdot h_{1,i} = \frac{1}{3}e_{4,i+1}g_{i+2}$

$\nu_i \cdot h_{1,i+1} = \frac{2}{3}e_{4,i+2}g_{i+1} - \frac{1}{3}e_{4,i+1}k_{i+1} - \frac{1}{3}\rho g_{i+1}h_{1,i+2}$

dim5:

$e_{4,i}e_{4,i+1} \cdot h_{1,i} = e_{4,i}^2h_{1,i+1}$ $\qquad e_{4,i}e_{4,i+1} \cdot h_{1,i+1} = e_{4,i+1}^2h_{1,i}$

$\nu_i \cdot e_{4,i+1} = -e_{4,i+2}\mu_{i+1} + \frac{1}{3}\rho e_{4,i+2}g_{i+1} + \frac{1}{3}\rho e_{4,i+1}k_{i+1}$

$\nu_i \cdot k_i = \frac{1}{2}e_{4,i+1}^2h_{1,i+2}$ $\qquad \nu_ik_{i+2} = -\frac{1}{2}e_{4,i+2}^2h_{1,i}$

$\nu_i \cdot g_i = \frac{1}{6}e_{4,i}e_{4,i+1}h_{1,i+2}$

dim6:

$e_{4,i}h_{1,i} \cdot \nu_i = -\frac{1}{3}e_{4,i}e_{4,i+1}g_{i+2}$ $\qquad e_{4,i}h_{1,i+1} \cdot \nu_i = -\frac{1}{3}e_{4,i+2}e_{4,i}g_{i+1}$

$\nu_i \cdot \nu_{i+1} = \frac{1}{3}\rho e_{4,i+2}^2h_{1,i} - \frac{1}{6}e_{4,i+2}^2e_{4,i}$ $\qquad e_{4,i}k_i \cdot e_{4,i+2} = e_{4,i+1}e_{4,i+2}g_i$

$e_{4,i}^2 \cdot g_{i+1} = e_{4,i+1}e_{4,i+2}g_i$ $\qquad e_{4,i}^2 \cdot k_{i+1} = e_{4,i}e_{4,i+1}g_{i+2}$

dim7:

$e_{4,i}^2 \cdot \nu_i = \frac{2}{3}\rho e_{4,i}e_{4,i+1}g_{i+2} - e_{4,i}e_{4,i+1}\mu_{i+2}$

$e_{4,i}e_{4,i+1} \cdot \nu_i = -e_{4,i+2}e_{4,i}\mu_{i+1} + \frac{2}{3}\rho e_{4,i+2}e_{4,i}g_{i+1}$

dim8:

$e_{4,i}^2 \cdot e_{4,i+1}k_{i+1} = e_{4,i}^2e_{4,i+2}g_{i+1}$ $\qquad e_{4,i}^2h_{1,i+1} \cdot \nu_i = -\frac{1}{3}e_{4,i}^2e_{4,i+2}g_{i+1}$

dim9:

$e_{4,i}\nu_i \cdot e_{4,i}e_{4,i+1} = \frac{1}{3}\rho e_{4,i+2}e_{4,i}^2g_{i+1}$

Remark 3.5. We take this opportunity to correct a typo in [4]. The formula for $e_{4,i}e_{4,i+1} \cdot v_i$ in dimension 7 is now corrected here.

4. Representations of α, β, γ-family elements

In this section, we recall the constructions of the α, β, γ-family elements in the E_2-page $Ext^*_{BP_*BP}(BP_*, BP_*)$ of the Adams-Novikov spectral sequence. Then we determine their images under the comparison map $\phi : Ext^*_{BP_*BP}(BP_*, BP_*) \to H^*(S(3))$.

Note we can write ϕ as the composition of several maps. We have

$$\phi = Ext^*_{BP_*BP}(BP_*, BP_*) \xrightarrow{\eta} Ext^*_{BP_*BP}(BP_*, v_3^{-1}BP_*/I_3) \xrightarrow{\psi} H^*(S(3)) \tag{4.1}$$

with $I_3 = (p, v_1, v_2) \subset BP_*$ and $\psi = \psi_3\psi_2\psi_1$, where

$$\psi_1 : Ext^*_{BP_*BP}(BP_*, v_3^{-1}BP_*/I_3) \xrightarrow{\cong} Ext^*_{\Sigma(3)}(K(3)_*, K(3)_*) \tag{4.2}$$

$$\psi_2 : Ext^*_{\Sigma(3)}(K(3)_*, K(3)_*) \to Ext^*_{\Sigma(3)}(K(3)_*, K(3)_*) \otimes_{K(3)_*} \mathbb{F}_p \tag{4.3}$$

$$\psi_3 : Ext^*_{\Sigma(3)}(K(3)_*, K(3)_*) \otimes_{K(3)_*} \mathbb{F}_p \xrightarrow{\cong} Ext^*_{S(3)}(\mathbb{F}_p, \mathbb{F}_p) = H^*(S(3)) \tag{4.4}$$

We will first determine all nontrivial images of the α, β, γ-family elements under the map η. Then, we will determine all nontrivial images of the α, β, γ-family elements under the composition ϕ.

4.1. **α-family elements.** Let $n \geq 0$, $p \nmid s \geq 1$. Then one can check $v_1^{sp^n}$ lives in $Ext^0_{BP_*BP}(BP_*, BP_*/p^{n+1})$. We define

$$\alpha_{sp^n/n+1} := \delta_0(v_1^{sp^n}) \in Ext^1_{BP_*BP}(BP_*, BP_*) \tag{4.5}$$

where δ_0 is the boundary-homomorphism associated to the short exact sequence

$$0 \to \Omega_{BP_*BP}(BP_*) \xrightarrow{p^{n+1}} \Omega_{BP_*BP}(BP_*) \to \Omega_{BP_*BP}(BP_*/p^{n+1}) \to 0 \tag{4.6}$$

of cobar complexes (Definition 2.9). We often abbreviate $\alpha_{s/1}$ to α_s.

Theorem 4.2 ([16])**.** *Let p be an odd prime. Then $Ext^1_{BP_*BP}(BP_*, BP_*)$ is generated by $\alpha_{sp^n/n+1}$ for $n \geq 0$, $p \nmid s \geq 1$.*

In order to determine the image of η, we introduce the following notion.

Definition 4.3. Let $n \geq 1$. We define $I[n]$ to be the ideal of BP_* generated by monomials $p^i v_1^j v_2^k$ such that $i + j + k = n$. In particular, $I[1] = (p, v_1, v_2) = I_3 \subset BP_*$.

Lemma 4.4. *Let d denote the differential of the cobar complex $\Omega^*_{BP_*BP}(BP_*)$. Let $x \in I[n] \subset BP_* = \Omega^0_{BP_*BP}(BP_*)$ for some $n \geq 1$. Then $d(x) \in I[n] \cdot \Omega^1_{BP_*BP}(BP_*)$.*

Proof. This follows directly from Definition 2.9 and (2.8), (2.9). □

Proposition 4.5. *For the image of the α-family elements, we have*

- $\eta(\alpha_1) = -t_1$.
- $\eta(\alpha_{sp^n/n+1}) = 0$, *for any other* $\alpha_{sp^n/n+1}$.

Proof. Note $\alpha_1 = \delta_0(v_1) = -t_1$ follows directly from definition. We have $\eta(\alpha_1) = -t_1$.

For any other $\alpha_{sp^n/n+1}$, we have $\alpha_{sp^n/n+1} = \delta_0(v_1^{sp^n})$ with $v_1^{sp^n} \in I[n+2]$ since $sp^n \geq n+2$. Here, we abuse the notation and let $I[k]$ also denote the image of $I[k]$ under the projection $BP_* \to BP_*/p^{n+1}$. Let d denote the differential map of the cobar complex $\Omega_{BP_*BP}(BP_*)$. By Lemma 4.4, $d(v_1^{sp^n}) \in I[n+2] \cdot \Omega_{BP_*BP}(BP_*)$. Since $v_1^{sp^n}$ is a cocycle, $d(v_1^{sp^n})$ is a multiple of p^{n+1}. By definition, $\delta_0(v_1^{sp^n}) = 1/p^{n+1} d(v_1^{sp^n})$. Then $\alpha_{sp^n/n+1} \in I[1] \cdot \Omega_{BP_*BP}(BP_*) = I_3 \cdot \Omega_{BP_*BP}(BP_*)$. Therefore, $\eta(\alpha_{sp^n/n+1}) = 0$ after reduction module I_3. □

Notations 4.6. In this paper, we often abuse the notation and refer to the elements in $Ext_\Gamma^{s,*}(A, M)$ by their representatives in the associated cobar complex $\Omega_\Gamma^{s,*}(M)$ when no confusion arises. For example, here we let $-t_1$ denote the element in $Ext^1_{BP_*BP}(BP_*, v_3^{-1}BP_*/I_3)$ represented by $-t_1 \in \Omega_{BP_*BP}(v_3^{-1}BP_*/I_3)$.

4.7. **β-family elements.** Let $a_0 = 1$, $a_n = p^n + p^{n-1} - 1$ for $n \geq 1$. Define $x_n \in v_2^{-1}BP_*$ as

$$x_0 = v_2, \tag{4.7}$$

$$x_1 = x_0^p - v_1^p v_2^{-1} v_3, \tag{4.8}$$

$$x_2 = x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3, \tag{4.9}$$

$$x_n = x_{n-1}^p - 2v_1^{b_n} v_2^{p^n-p^{n-1}+1}, n \geq 3 \tag{4.10}$$

with $b_n = (p+1)(p^{n-1}-1)$ for $n > 1$. Now, if $s \geq 1$ and $p^i | j \leq a_{n-i}$ with $j \leq p^n$ if $s = 1$, then $x_n^s \in Ext^0_{BP_*BP}(BP_*, BP_*/(p^{i+1}, v_1^j))$. Define

$$\beta_{sp^n/j,i+1} := \delta'\delta''(x_n^s) \in Ext^2_{BP_*BP}(BP_*, BP_*) \tag{4.11}$$

where δ' (resp. δ'') is the boundary-homomorphism associated to E' (resp. E'')

$$E' : 0 \to \Omega(BP_*) \xrightarrow{p^{i+1}} \Omega(BP_*) \to \Omega(BP_*/p^{i+1}) \to 0 \tag{4.12}$$

$$E'' : 0 \to \Omega(BP_*/p^{i+1}) \xrightarrow{v_1^j} \Omega(BP_*/p^{i+1}) \to \Omega(BP_*/(p^{i+1}, v_1^j)) \to 0 \tag{4.13}$$

where we let $\Omega(-)$ denote $\Omega_{BP_*BP}(-)$. We often abbreviate $\beta_{sp^n/j,1}$ to $\beta_{sp^n/j}$ and $\beta_{sp^n/1}$ to β_{sp^n}.

Theorem 4.8 ([12, 13]). *Let p be an odd prime. $Ext^2_{BP_*BP}(BP_*, BP_*)$ is the direct sum of cyclic subgroups generated by $\beta_{sp^n/j,i+1}$ for $n \geq 0$, $p \nmid s \geq 1$, $j \geq 1$, $i \geq 0$, subject to*

(1) $j \leq p^n$, if $s = 1$,
(2) $p^i | j \leq a_{n-i}$, and
(3) $a_{n-i-1} < j$, if $p^{i+1} | j$.

Proposition 4.9. *For the image of the β-family elements, we have*

- $\eta(\beta_{p^n/p^n}) = -b_{1,n}$, *for $n \geq 1$.*
- $\eta(\beta_{p^n/p^n-1}) = t_1 \otimes t_1^{p^{n+1}}$, *for $n \geq 1$.*
- $\eta(\beta_1) = -b_{1,0}$.
- $\eta(\beta_2) = 2t_2 \otimes t_1^p + t_1 \otimes t_1^{2p}$.
- $\eta(\beta_{sp^n/j,i+1}) = 0$, *for any other $\beta_{sp^n/j,i+1}$.*

Proof. Direct observation shows $x_0 \in I[1]$, $x_1 \in I[p-1]$, $x_2 \in I[p^2-p]$, $x_n \in I[p^n - p^{n-1}]$ for $n \geq 3$. Here, we abuse the notation and let $I[k]$ also denote the images of $I[k]$ under the projection $BP_* \to BP_*/p^{i+1}$ as well as the projection $BP_*/p^{i+1} \to BP_*/(p^{i+1}, v_1^j)$.

Case 1. If $n \geq 1$, then $x_n^s \in I[sp^n - sp^{n-1}]$. From Theorem 4.8, we have

$$p^i \mid j \leq p^{n-i} + p^{n-i-1} - 1 \tag{4.14}$$

Case 1.1 If $i \geq 1$, $sp^n - sp^{n-1} \geq i + j + 2$ by (4.14).
Case 1.2 If $i = 0$ and $s \geq 2$, $sp^n - sp^{n-1} \geq i + j + 2$ by (4.14).
Case 1.3 If $i = 0$ and $s = 1$, we have $j \leq p^n$ by Theorem 4.8.
Case 1.3.1 If $j \leq p^n - 2$, we regard x_n as an element in $BP_*/(p, v_1^j)$, we have $x_n = v_2^{p^n} \in I[p^n]$. Then $p^n \geq i + j + 2 = j + 2$.

In summary, for all the cases we have mentioned above, we always have $x_n^s \in I[i+j+2]$. Then, using similar arguments as in Proposition 4.5, we can show $\delta'\delta''(x_n^s) \in I[1]$. Hence $\eta(\beta_{sp^n/j,i+1}) = 0$ in these cases.

Case 1.3.2 If $j = p^n - 1$ or $j = p^n$, $\eta(\beta_{p^n/j}) = \eta(\delta'\delta''(v_2^{p^n}))$. Direct computation following definitions give the claimed results.

Case 2. If $n = 0$, then from Theorem 4.8, $i = 0$ and $j = 1$. Moreover, $x_n^s = v_2^s$.
Case 2.1 If $s \geq 3$, then $x_n^s \in I[i + j + 2]$. Similarly, this forces $\eta(\beta_s) = 0$.
Case 2.2 If $s = 1$, or $s = 2$. Direct computation shows the claimed results.

□

4.10. **γ-family elements.** Let $s_1 = r_1 p^{e_1}$, $s_2 = r_2 p^{e_2}$, $s_3 = r_3 p^{e_3}$ with p^{e_i} being the largest power of p dividing s_i. For $1 \leq s_1 \leq p^{e_2}$, $1 \leq s_2 \leq p^{e_3}$, $1 \leq s_3$, one can show $v_3^{s_3}$ is a cycle in $\Omega_{BP_*BP}(BP_*/(p, v_1^{s_1}, v_2^{s_2}))$. Define [6, 12]

$$\gamma_{s_3/s_2,s_1} := \delta_0\delta_1\delta_2(v_3^{s_3}) \in Ext^3_{BP_*BP}(BP_*, BP_*) \tag{4.15}$$

where δ_0 (resp. δ_1, δ_2) is the boundary-homomorphism associated to E_0 (resp. E_1, E_2)

$$E_0 : 0 \to \Omega(BP_*) \xrightarrow{p} \Omega(BP_*) \to \Omega(BP_*/p) \to 0 \tag{4.16}$$

$$E_1 : 0 \to \Omega(BP_*/p) \xrightarrow{v_1^{s_1}} \Omega(BP_*/p) \to \Omega(BP_*/(p, v_1^{s_1})) \to 0 \tag{4.17}$$

$$E_2 : 0 \to \Omega(BP_*/(p, v_1^{s_1})) \xrightarrow{v_2^{s_2}} \Omega(BP_*/(p, v_1^{s_1})) \to \Omega(BP_*/(p, v_1^{s_1}, v_2^{s_2})) \to 0 \tag{4.18}$$

where we let $\Omega(-)$ denote $\Omega_{BP_*BP}(-)$. We often abbreviate $\gamma_{s_3/s_2,1}$ to γ_{s_3/s_2} and $\gamma_{s_3/1}$ to γ_{s_3}.

Theorem 4.11 ([12] Corollary 7.8)**.** *Using notations as above, we have $0 \neq \gamma_{s_3/s_2,s_1} \in Ext^3_{BP_*BP}(BP_*, BP_*)$ unless $s_1 < s_2 = p^{e_3} = s_3$. In fact, these elements are linearly independent.*

Proposition 4.12. *For the image of the γ-family elements, we have*

- $\eta(\gamma_{r_3}) = r_3(r_3 - 1)v_3^{r_3-2}(b_{2,0}t_1^{p^2} - t_2^p b_{1,1}) + \frac{r_3(r_3-1)}{2}v_3^{r_3-2}(b_{1,0} \otimes t_1^{2p^2} - 2t_1^p \otimes b_{1,1}(1 \otimes t_1^{p^2} + t_1^{p^2} \otimes 1)) - r_3(r_3 - 1)(r_3 - 2)v_3^{r_3-3}t_3 \otimes t_2^p \otimes t_1^{p^2}$.
- $\eta(\gamma_{r_3p^{e_3}/p^{e_3},p^{e_3}}) = r_3(r_3 - 1)v_3^{(r_3-2)p^{e_3}} b_{2,e_3} \otimes t_1^{p^{e_3+2}} - r_3(r_3 - 1)v_3^{(r_3-2)p^{e_3}} t_2^{p^{e_3+1}} \otimes b_{1,e_3+1} + \binom{r_3}{2}v_3^{(r_3-2)p^{e_3}} b_{1,e_3} \otimes t_1^{2p^{e_3+2}} - 2\binom{r_3}{2}v_3^{(r_3-2)p^{e_3}} t_1^{p^{e_3+1}} \otimes [b_{1,e_3+1}(1 \otimes t_1^{p^{e_3+2}})] - 2\binom{r_3}{2}v_3^{(r_3-2)p^{e_3}} t_1^{p^{e_3+1}} \otimes [b_{1,e_3+1}(t_1^{p^{e_3+2}} \otimes 1)]$, *for* $e_3 \geq 1$.
- $\eta(\gamma_{r_3p^{e_3}/p^{e_3},p^{e_3}-1}) = -r_3(r_3-1)v_3^{(r_3-2)p^{e_3}} t_1 \otimes t_2^{p^{e_3+1}} \otimes t_1^{p^{e_3+2}} - \binom{r_3}{2}v_3^{(r_3-2)p^{e_3}} t_1 \otimes t_1^{p^{e_3+1}} \otimes t_1^{2p^{e_3+2}}$, *for* $e_3 \geq 1$.
- $\eta(\gamma_{r_3p^{e_3}/p^{e_3}-1}) = r_3 v_3^{(r_3-1)p^{e_3}} b_{1,0} \otimes t_1^{p^{e_3+2}} - r_3 v_3^{(r_3-1)p^{e_3}} t_1^p \otimes b_{1,e_3+1}$, *for* $e_3 \geq 1$.
- $\eta(\gamma_{r_3p^{e_3}/p^{e_3}-2}) = -2r_3 v_3^{(r_3-1)p^{e_3}} t_2 \otimes t_1^p \otimes t_1^{p^{e_3+2}} - r_3 v_3^{(r_3-1)p^{e_3}} t_1 \otimes t_1^{2p} \otimes t_1^{p^{e_3+2}}$, *for* $e_3 \geq 1$.
- $\eta(\gamma_{r_3p^{e_3}/p^{e_3}-p^{e_2},p^{e_2}}) = r_3 v_3^{(r_3-1)p^{e_3}} b_{1,e_2} \otimes t_1^{p^{e_3+2}} - r_3 v_3^{(r_3-1)p^{e_3}} t_1^{p^{e_2+1}} \otimes b_{1,e_3+1}$, *for* $e_3 > e_2$.

- $\eta(\gamma_{r_3p^{e_3}/p^{e_3}-p^{e_2},p^{e_2}-1}) = r_3v_3^{(r_3-1)}t_1 \otimes t_1^{p^{e_2+1}} \otimes t_1^{p^{e_3+2}}$, *for* $e_3 > e_2$.
- $\eta(\gamma_{s_3/s_2,s_1}) = 0$, *for any other* $\gamma_{s_3/s_2,s_1}$.

Proof. The proof is very similar to the proofs of Propositions 4.5, 4.9.

Case 1. If $e_3 = 0$, then from $1 \le s_1 \le p^{e_2}$, $1 \le s_2 \le p^{e_3}$ we conclude $s_1 = s_2 = 1$. The image $\eta(\gamma_{r_3})$ is computed in [4] Lemma 4.1.

Case 2. If $e_3 \ge 1$. Note $1 \le s_1 \le p^{e_2}$, $1 \le s_2 \le p^{e_3}$ implies $e_1 \le e_2 \le e_3$.

Case 2.1 If $e_2 = e_3$, then $r_2 = 1$.

Case 2.1.1 If $r_3 = 1$, by Theorem 4.11, in this case $\gamma_{s_3/s_2,s_1} = 0$ unless $s_1 = s_2 = s_3 = p^{e_3}$. Direct computation shows $\eta(\gamma_{p^{e_3}/p^{e_3},p^{e_3}}) = 0$.

Case 2.1.2 If $r_3 \ge 2$, after reduction module $I[2]$, we have

$$\begin{aligned}\delta_1\delta_2(v_3^{r_3p^{e_3}}) &= r_3(r_3-1)v_3^{(r_3-2)p^{e_3}}v_1^{p^{e_3}-r_1p^{e_1}}t_2^{p^{e_3+1}} \otimes t_1^{p^{e_3+2}} \\ &\quad + \binom{r_3}{2}v_3^{(r_3-2)p^{e_3}}v_1^{p^{e_3}-r_1p^{e_1}}t_1^{p^{e_3+1}} \otimes t_1^{2p^{e_3+2}}.\end{aligned} \tag{4.19}$$

Case 2.1.2.1 If $p^{e_3} - r_1p^{e_1} \ge 2$, then $\delta_0\delta_1\delta_2(v_3^{r_3p^{e_3}}) \in I[1]$, hence $\eta(\gamma_{s_3/s_2,s_1}) = 0$.

Case 2.1.2.2 If $p^{e_3} = r_1p^{e_1}$ or $p^{e_3} = r_1p^{e_1}+1$, direct computation following (4.19) shows the claimed results.

Case 2.2 If $e_2 < e_3$.

Case 2.2.1 If $e_2 = 0$, then $1 \le s_1 \le p^{e_2}$ forces $r_1 = 1$ and $e_1 = 0$. After reduction module $I[2]$, we have

$$\begin{aligned}\delta_1\delta_2(v_3^{r_3p^{e_3}}) &= r_3(p^{e_3}-r_2)v_3^{(r_3-1)p^{e_3}}v_2^{(p^{e_3}-r_2-1)}t_1^{p} \otimes t_1^{p^{e_3+2}} \\ &\quad + r_3\binom{p^{e_3}-r_2}{2}v_3^{(r_3-1)p^{e_3}}v_2^{(p^{e_3}-r_2-2)}v_1t_1^{2p} \otimes t_1^{p^{e_3+2}}.\end{aligned} \tag{4.20}$$

Case 2.2.1.1 If $p^{e_3} - r_2 - 1 \ge 2$, then $\delta_0\delta_1\delta_2(v_3^{r_3p^{e_3}}) \in I[1]$, hence $\eta(\gamma_{s_3/s_2,s_1}) = 0$.

Case 2.2.1.2 If $r_2 = p^{e_3} - 1$ or $r_2 = p^{e_3} - 2$, direct computation following (4.20) shows the claimed results.

Case 2.2.2 If $e_2 \ge 1$, after reduction module $I[2]$, we have

$$\delta_1\delta_2(v_3^{r_3p^{e_3}}) = r_3(p^{e_3-e_2}-r_2)v_3^{(r_3-1)p^{e_3}}v_2^{(p^{e_3-e_2}-r_2-1)p^{e_2}}v_1^{p^{e_2}-r_1p^{e_1}}t_1^{p^{e_2+1}} \otimes t_1^{p^{e_3+2}} \tag{4.21}$$

The sum of powers of v_1, v_2 is $(p^{e_3-e_2}-r_2-1)p^{e_2} + p^{e_2} - r_1p^{e_1} = (p^{e_3-e_2}-r_2)p^{e_2} - r_1p^{e_1}$.

Case 2.2.2.1 If $(p^{e_3-e_2}-r_2)p^{e_2} - r_1p^{e_1} \ge 2$, then $\delta_0\delta_1\delta_2(v_3^{r_3p^{e_3}}) \in I[1]$, hence $\eta(\gamma_{s_3/s_2,s_1}) = 0$.

Case 2.2.2.2 If $(p^{e_3-e_2}-r_2)p^{e_2} - r_1p^{e_1} = 0$ (resp. $(p^{e_3-e_2}-r_2)p^{e_2} - r_1p^{e_1} = 1$), note $1 \le s_1 \le p^{e_2}$, $1 \le s_2 \le p^{e_3}$ this forces $p^{e_3-e_2} - r_2 = 1$ and $r_1p^{e_1} = p^{e_2}$ (resp. $p^{e_3-e_2} - r_2 = 1$ and $r_1p^{e_1} = p^{e_2} - 1$). Direct computation following (4.21) shows the claimed results.

□

Remark 4.13. Here the result for $\eta(\gamma_{r_3})$ differs from the formula in [4] by a negative sign because our definitions of the differential in the cobar complex (Definition 2.9) differ by a negative sign.

4.14. **Nontrivial images of ϕ.** In Propositions 4.5, 4.9, 4.12, we have determined the images of the α, β, γ-family elements under the map η. Recall from (4.1) that $\phi = \psi \circ \eta$. The direct computation followed by a comparison with $H^*S(3)$ (see Proposition 3.2) gives the images of the α, β, γ-family elements under the map ϕ.

Lemma 4.15. *The map* ψ *sends* $b_{1,n} \in Ext^*_{BP_*BP}(BP_*, v_3^{-1}BP_*/I_3)$ *to* $e_{4,n+1} \in H^*S(3)$ *for* $n \ge 0$.

Proof. On the level of cobar complexes, the effect of ψ is reduction mod p, sending all v_i with $i \neq 3$ to 0, and sending v_3 to 1. Hence we have $\psi(b_{1,n}) = \tilde{b}_{1,n}$.

By (2.18), in the cobar complex $\Omega^{*,*}_{S(3)}(\mathbb{F}_p)$, we have $d(t_4) = t_1 \otimes t_3^p + t_2 \otimes t_2^{p^2} + t_3 \otimes t_1^{p^3} - \tilde{b}_{1,2}$. Hence we have equivalent cohomology classes $[\tilde{b}_{1,2}] = [t_1 \otimes t_3^p + t_2 \otimes t_2^{p^2} + t_3 \otimes t_1^{p^3}] = e_{4,3}$. This implies $\psi(b_{1,2}) = e_{4,3}$.

Note if a is not a multiple of p, then $a^p \equiv a \bmod p$. Hence working over $\mathbb{F}_p$, we have $\tilde{b}_{1,n+1} = \tilde{b}^p_{1,n}$. Moreover, note $t_1^{p^3} = t_1 \in S(3)$, we have $b_{1,n+3} = b_{1,n}$. Similarly, one can show $e_{4,n+1} = e^p_{4,n}$ and $e_{4,n+3} = e_{4,n}$. Hence we conclude $\psi(b_{1,n}) = e_{4,n+1}$ for each $n \geq 0$. □

Proposition 4.16. *Under the comparison map $\phi : Ext^*_{BP_*BP}(BP_*, BP_*) \to H^*(S(3))$, all nonzero images of α, β, γ-family elements are listed as follows:*

$$\phi(\alpha_1) = -h_{1,0}, \tag{4.22}$$

$$\phi(\beta_{p^n/p^n}) = -e_{4,n+1} \tag{4.23}$$

$$\phi(\beta_1) = -e_{4,1} \tag{4.24}$$

$$\phi(\beta_2) = 2k_0, \tag{4.25}$$

$$\phi(\gamma_s) = -s(s^2-1)v_0 + s(s-1)\rho k_1 \quad \text{for } s \not\equiv 0, 1 \bmod p \tag{4.26}$$

$$\phi(\gamma_{sp^n/p^n,p^n}) = -3s(s-1)v_n + s(s-1)\rho k_{n+1} \quad \text{for } n \geq 1 \text{ and } s \not\equiv 0, 1 \bmod p \tag{4.27}$$

$$\phi(\gamma_{sp^n/p^n,p^n-1}) = s(s-1)g_0 h_{1,1} \quad \text{if } n \equiv 2 \bmod 3 \text{ and } s \not\equiv 0, 1 \bmod p \tag{4.28}$$

$$\phi(\gamma_{sp^n/p^n-2}) = 2sg_0 h_{1,1} \quad \text{if } n \equiv 1 \bmod 3 \tag{4.29}$$

Proof. The results for the α, β-family elements follow directly from Propositions 4.5, 4.9, Lemma 4.15, and Proposition 3.2.

The image $\phi(\gamma_s)$ is computed in Lemma 4.1 of [4]. Note our result listed here differs from the formula in [4] by a negative sign because our definitions of the differential in the cobar complex (Definition 2.9) differ by a negative sign. The image $\phi(\gamma_{sp^n/p^n,p^n})$ is computed similarly.

The computation of other γ-family elements is straightforward. □

5. Detection of nontrivial products via the cohomology of $S(3)$

In this section, we use the $\mathbb{F}_p$-algebra structure of $H^*S(3)$ to detect nontrivial products in $Ext^*_{BP_*BP}(BP_*, BP_*)$, as well as nontrivial products in $\pi_*(S)$.

Let $x_1, x_2, \cdots, x_m$ be a sequence of elements in $Ext^*_{BP_*BP}(BP_*, BP_*)$, where each x_i is one of the α, β, or γ-family elements as discussed in Section 4. In order to determine whether the product $x = x_1 x_2 \cdots x_m$ is trivial, we could detect it by mapping it into $H^*S(3)$. If $\phi(x) = \phi(x_1)\phi(x_2)\cdots\phi(x_m) \neq 0 \in H^*S(3)$, we can conclude $x \neq 0 \in Ext^*_{BP_*BP}(BP_*, BP_*)$. Using this strategy, we can enumerate all nontrivial products of this form which are detectable by $H^*S(3)$ as follows.

Proposition 5.1. *Let $p \geq 7$ be a prime. In $Ext^*_{BP_*BP}(BP_*, BP_*)$, all nontrivial products of the α, β, γ-family elements which are detectable by $H^*S(3)$ can be enumerated as follows.*

dim3: $\alpha_1\beta_1$
$\alpha_1\beta_2$
$\alpha_1\beta_{p^n/p^n}$
dim4: $\beta_1\beta_2$

	$\alpha_1\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p*
	$\alpha_1\gamma_{sp^n/p^n,p^n}$,	$s \not\equiv 0, 1$ *mod p and* $n \geq 1$
	$\beta_{p^n/p^n}\beta_{p^m/p^m}$	
	$\beta_2\beta_{p^n/p^n}$	$n \not\equiv 0$ *mod 3*
dim5:	$\beta_1\gamma_s$,	$s \not\equiv 0, 1$ *mod p*
	$\beta_2\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p*
	$\alpha_1\beta_{p^n/p^n}\beta_{p^m/p^m}$,	$m \equiv n \not\equiv 2$ *or* $m \not\equiv n$ *mod 3*
	$\beta_{p^n/p^n}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \not\equiv 1$ *mod 3*
	$\beta_2\gamma_{sp^n/p^n,p^n}$,	$s \not\equiv 0, 1$ *mod p and* $n \geq 1$
	$\beta_{p^n/p^n}\gamma_{sp^{3k+n}/p^{3k+n},p^{3k+n}}$,	$s \not\equiv 0, 1$ *mod p and* $n \geq 1$
	$\beta_{p^n/p^n}\gamma_{sp^{3k+n+1}/p^{3k+n+1},p^{3k+n+1}}$,	$s \not\equiv 0, 1$ *mod p*
dim6:	$\alpha_1\beta_{p^n/p^n}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \not\equiv 1$ *mod 3*
	$\alpha_1\beta_{p^n/p^n}\gamma_{sp^{3k+n}/p^{3k+n},p^{3k+n}}$,	$s \not\equiv 0, 1$ *mod p and* $n \not\equiv 1$ *mod 3,* $n \geq 1$
	$\alpha_1\beta_{p^n/p^n}\gamma_{sp^{3k+n+1}/p^{3k+n+1},p^{3k+n+1}}$,	$s \not\equiv 0, 1$ *mod p and* $n \not\equiv 0$ *mod 3*
	$\beta_{p^n/p^n}\beta_{p^m/p^m}\beta_{p^k/p^k}$	$n \not\equiv m \not\equiv k$ *mod 3*
	$\beta_2\beta_{p^n/p^n}\beta_{p^{3k+n}/p^{3k+n}}$,	$n \equiv 1$ *mod 3*
	$\beta_2\beta_{p^n/p^n}\beta_{p^{3k+n+1}/p^{3k+n+1}}$,	$n \equiv 1$ *mod 3*
	$\gamma_s\gamma_{s_1p^n/p^n,p^n}$,	$s \not\equiv 0, \pm 1,\ s_1 \not\equiv 0, 1$*mod p and* $n \not\equiv 0$ *mod 3*
	$\gamma_{sp^n,p^n/p^n}\gamma_{s_1p^{3k+n+1}/p^{3k+n+1},p^{3k+n+1}}$,	$s, s_1 \not\equiv 0, 1$ *mod p and* $n \geq 1$
dim7:	$\beta_{p^n/p^n}\beta_{p^{3k+n}/p^{3k+n}}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \equiv 2$ *mod 3*
	$\beta_{p^n/p^n}\beta_{p^{3k+n+1}/p^{3k+n+1}}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \equiv 2$ *mod 3*
	$\beta_{p^n/p^n}\beta_{p^{3k+n+2}/p^{3k+n+2}}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \equiv 0$ *mod 3*
	$\beta_{p^m/p^m}\beta_{p^k/p^k}\gamma_{sp^{n+1}/p^{n+1},p^{n+1}}$,	$s \not\equiv 0, 1$ *mod p and* $m \equiv k+2 \equiv n$ *mod 3*
	$\beta_{p^m/p^m}\beta_{p^k/p^k}\gamma_{sp^{n+1}/p^{n+1},p^{n+1}}$,	$s \not\equiv 0, 1$ *mod p and* $m \equiv k \equiv n$ *mod 3*
	$\beta_{p^m/p^m}\beta_{p^k/p^k}\gamma_{sp^{n+1}/p^{n+1},p^{n+1}}$,	$s \not\equiv 0, 1$ *mod p and* $m+2 \equiv k \equiv n$ *mod 3*
dim8:	$\alpha_1\beta_{p^n/p^n}\beta_{p^{3k+n+1}/p^{3k+n+1}}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \equiv 2$ *mod 3*
	$\alpha_1\beta_{p^m/p^m}\beta_{p^k/p^k}\gamma_{sp^{n+1}/p^{n+1},p^{n+1}}$,	$s \not\equiv 0, 1$ *mod p and* $m \equiv k+2 \equiv n$ *mod 3*
	$\beta_2\beta_{p^m/p^m}\beta_{p^k/p^k}\beta_{p^{n+1}/p^{n+1}}$,	$m \equiv k = n \equiv 1$ *mod 3*
dim9:	$\gamma_s\gamma_{s_1p^{3k+2}/p^{3k+2},p^{3k+2}}\gamma_{s_2p^n/p^n,p^n}$,	$s \not\equiv 0, \pm 1,\ s_1, s_2 \not\equiv 0, 1$ *mod p,* $n \equiv 1$ *mod 3*
	$\gamma_{sp^m/p^m,p^m}\gamma_{s_1p^n/p^n,p^n}\gamma_{s_2p^k/p^k,p^k}$,	$s_1, s_2 \not\equiv 0, 1$ *mod p,* $m-2 \equiv n-1 \equiv k \equiv 1$ *mod 3*

Proof. This follows from a tedious but straightforward computation using Proposition 4.16 and Proposition 3.4. □

Now we proceed to study nontrivial products in the stable homotopy ring of the sphere $\pi_*(S)$.

Let $p \geq 7$, $s \geq 1$. It is proved in [12, 16, 20, 21] that α_s, β_s, γ_s all represent nontrivial elements in $\pi_*(S)$.

Using the Adams spectral sequence, Cohen [3] also found another family of nontrivial elements $\zeta_n \in \pi_*(S)$, for $n \geq 1$. Cohen [3] shows that, in the Adams-Novikov spectral sequence, ζ_n is represented by $\alpha_1\beta_{p^n/p^n} + \alpha_1 x \in Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$, where $x = \sum\limits_{s,k,j} a_{s,k,j}\beta_{sp^k/j}$, $0 \leq a_{s,k,j} \leq p-1$, and $a_{1,n,p^n} = 0$. Moreover, [4] shows $s \geq 2$ by comparing inner degrees.

Our first main result of this paper is the following.

Theorem 5.2. *Let $p \geq 7$ be a prime. Let $n \equiv 2$ mod 3, $s \not\equiv 0, \pm 1$ mod p. Then $\zeta_n\beta_1\gamma_s \neq 0 \in \pi_*(S)$.*

Proof. The representation of $\zeta_n\beta_1\gamma_s$ in the E_2-page of the ANSS is $(\alpha_1\beta_{p^n/p^n} + \alpha_1 x)\beta_1\gamma_s \in Ext^{6,*}_{BP_*BP}(BP_*, BP_*)$. By Proposition 4.16, we have $\phi(x) = 0$. Furthermore, based on

Proposition 3.4, we have the following computations under the given assumptions on n, s.

$$\begin{aligned}\phi((\alpha_1\beta_{p^n/p^n}+\alpha_1 x)\beta_1\gamma_s) &= \phi(\alpha_1\beta_{p^n/p^n}\beta_1\gamma_s)\\ &= (-h_{1,0})(-e_{4,n+1})(-e_{4,1})(-s(s^2-1)v_0+s(s-1)\rho k_1)\\ &= s(s^2-1)h_{1,0}e_{4,n+1}e_{4,1}v_0 - s(s-1)h_{1,0}e_{4,n+1}e_{4,1}\rho k_1\\ &= s(s^2-1)h_{1,0}e_{4,n+1}e_{4,1}v_0\\ &= -\frac{s(s^2-1)}{3}e_{4,0}e_{4,1}g_2e_{4,1}\\ &\neq 0\end{aligned}$$

Hence $(\alpha_1\beta_{p^n/p^n}+\alpha_1 x)\beta_1\gamma_s \neq 0 \in Ext^{6,*}_{BP_*BP}(BP_*, BP_*)$. Note $\alpha_1\beta_{p^n/p^n}+\alpha_1 x, \beta_1, \gamma_s$ are all permanent cycles in the ANSS. We conclude their product is also a permanent cycle.

Notice that the differentials of the ANSS have the form $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$, $r \geq 2$. Meanwhile, the inner degrees of the elements in the ANSS are all multiples of $q = 2p-2$. Hence, the first possibly nontrivial differentials in the ANSS are d_{2p-1}'s. By degree reasons, $(\alpha_1\beta_{p^n/p^n}+\alpha_1 x)\beta_1\gamma_s \in Ext^{6,*}_{BP_*BP}(BP_*, BP_*)$ can not be the image of any differential. Therefore, $(\alpha_1\beta_{p^n/p^n}+\alpha_1 x)\beta_1\gamma_s$ represent nontrivial products $\zeta_n\beta_1\gamma_s \in \pi_*(S)$. □

We can use similar strategy to study other products in $\pi_*(S)$.

Theorem 5.3. *Let $p \geq 7$ be a prime. We consider the products in $\pi_*(S)$ where each factor belongs to $\{\alpha_s, \beta_s, \gamma_s, \zeta_s : s \geq 1\}$. Amongst all such products, only the following ones can be detected as nontrivial products using the comparison with $H^*S(3)$.*

i) $\alpha_1\beta_1$,
ii) $\alpha_1\beta_2$,
iii) $\beta_1\beta_2$,
iv) $\alpha_1\gamma_s$, *if* $s \not\equiv 0, \pm 1$ *mod p,*
v) $\beta_1\gamma_s$, *if* $s \not\equiv 0, 1$ *mod p,*
vi) $\beta_2\gamma_s$, *if* $s \not\equiv 0, \pm 1$ *mod p,*
vii) $\zeta_n\gamma_s$, *if* $n \not\equiv 1$ *mod 3,* $s \not\equiv 0, \pm 1$ *mod p,*
viii) $\zeta_n\beta_1$, *if* $n \not\equiv 0$ *mod 3,*
ix) $\zeta_n\beta_1\gamma_s$, *if* $n \equiv 2$ *mod 3,* $s \not\equiv 0, \pm 1$ *mod p.*

Proof. Let x be a product in $\pi_*(S)$ where each factor belongs to $\{\alpha_s, \beta_s, \gamma_s, \zeta_s : s \geq 1\}$. Let $y \in Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ be the representation of x on the Adams-Novikov E_2-page. If x can be detected as nontrivial by comparing with $H^*S(3)$, then we have $\phi(y) \neq 0 \in H^*(S(3))$. In Proposition 5.1, we have exhausted all possible forms of y. Through a tedious but straightforward check, we conclude x must have one of the nine forms listed in the theorem.

On the other hand, assume x is one of the products with the given forms listed in the theorem. Then we can show x is nontrivial analogous to the proof of Theorem 5.2. □

Hence we can say, we have exhausted the power of the "cohomology of $S(3)$" strategy in Theorem 5.3. To detect other nontrivial products in $\pi_*(S)$. One would have to use other methods.

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