HOMOTOPY PRO-NILPOTENT STRUCTURED RING SPECTRA AND TOPOLOGICAL QUILLEN LOCALIZATION

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Abstract. The aim of this short paper is to show that the homotopy limit of any diagram of nilpotent structured ring spectra is TQ-local, where structured ring spectra are described as algebras over a spectral operad $\mathcal{O}$; in particular, every homotopy pro-nilpotent structured ring spectrum is TQ-local. Here, TQ is short for topological Quillen homology, which is weakly equivalent to $\mathcal{O}$-algebra stabilization. As an application, we simultaneously extend the previously known connected and nilpotent TQ-Whitehead theorems to a homotopy pro-nilpotent TQ-Whitehead theorem. We also compare TQ-localization with TQ-completion and show that TQ-local $\mathcal{O}$-algebras that are TQ-good are TQ-complete. Finally, we show that every (-1)-connected $\mathcal{O}$-algebra with a principally refined Postnikov tower is TQ-local, provided that $\mathcal{O}$ is (-1)-connected.

1. Introduction

Structured ring spectra are spectra with extra algebraic structure encoded by the action of an operad $\mathcal{O}$ in the closed symmetric monoidal category of modules $(\mathsf{Mod}_{\mathcal{R}}, \wedge, \mathcal{R})$ over a commutative ring spectrum $\mathcal{R}$. For a fixed operad $\mathcal{O}$, denote by $\mathsf{Alg}_{\mathcal{O}}$ the category of $\mathcal{O}$-algebras. In this paper we are working with reduced operads $\mathcal{O}$ (i.e., such that $\mathcal{O}[0] = *$, where $*$ denotes the trivial $\mathcal{R}$-module); such $\mathcal{O}$-algebras are non-unital. Similar to the ordinary homology of spaces, a precisely analogous notion of homology for $\mathcal{O}$-algebras is topological Quillen (TQ) homology [4, 6, 11, 21, 27, 37, 35]; useful introductions to notions of homology for algebraic structures appear, for instance, in [20, 24, 41, 43, 44]. It turns out that TQ-homology is weakly equivalent to stabilization $\Omega^\infty\Sigma^\infty$ in $\mathcal{O}$-algebras [5, 28, 42, 45, 46] under appropriate connectivity conditions—this is because TQ is 1-excisive and agrees to order 1 with the identity functor on $\mathcal{O}$-algebras (via the TQ-Hurewicz map $\mathrm{id} \to \mathsf{TQ}$) in the sense of Goodwillie [25, 1.2]; see, for instance, [11, 28, 36].

The TQ-local homotopy theory for $\mathcal{O}$-algebras is established in [29], where the upshot is that: if X is a cofibrant $\mathcal{O}$-algebra, then its weak TQ-fibrant replacement $X \to L_{\mathsf{TQ}}(X)$ is the TQ-localization of X. By construction, the comparison map $X \to L_{\mathsf{TQ}}(X)$ is a cofibration that is also a TQ-equivalence such that $L_{\mathsf{TQ}}(X)$ is TQ-local (Defintion 2.2). Intuitively, the TQ-localization $L_{\mathsf{TQ}}(X)$ can be thought of as "the part of X that TQ-homology sees".

In this short paper we attack the following question: When is the comparison map $X \to L_{\mathsf{TQ}}(X)$ a weak equivalence? In other words, when is an $\mathcal{O}$-algebra X already TQ-local? For instance, we know from [13] that every connected $\mathcal{O}$-algebra is TQ-complete and hence $X \simeq L_{\mathsf{TQ}}(X)$, but we also know from [28] that every connected $\mathcal{O}$-algebra is the homotopy limit of a tower of nilpotent $\mathcal{O}$-algebras and hence X is homotopy pro-nilpotent (Definition 1.2); here, $\mathcal{R}, \mathcal{O}$ were assumed to be (-1)-connected.

This leads us to one of the motivations of our work: what amounts to the "first half" of a conjecture of Francis-Gaitsgory [18, 3.4.5] that (i) the natural map comparing X with its TQ-completion should be a weak equivalence for every homotopy pro-nilpotent $\mathcal{O}$-algebra X. Our main result, Theorem 1.8, is that (i) is true in general, provided that in the comparison map we replace "TQ-completion" with "TQ-localization". Our strategy of attack is to leverage the TQ-local homotopy theory of $\mathcal{O}$-algebras in [29] with the fact, proved in [12], that M-nilpotent $\mathcal{O}$-algebras are $\mathsf{TQ}|_{\mathrm{Nil}_M}$-complete.

Definition 1.1. Let X be an $\mathcal{O}$-algebra and $M \geq 2$. We say that X is *M-nilpotent* if all the M-ary and higher operations $\mathcal{O}[t] \wedge X^{\wedge t} \to X$ of X are trivial (i.e., if these maps factor through the trivial $\mathcal{R}$-module $*$ for each $t \geq M$).

Definition 1.2. An $\mathcal{O}$-algebra is *nilpotent* (resp. *homotopy pro-nilpotent*) if it is M-nilpotent for some $M \geq 2$ (resp. if it is weakly equivalent to the homotopy limit of a tower of nilpotent $\mathcal{O}$-algebras).

It is worth pointing out that homotopy pro-nilpotent $\mathcal{O}$-algebras need not be nilpotent; the following describes a large class of such $\mathcal{O}$-algebras.

Proposition 1.3. *If X is a connected $\mathcal{O}$-algebra and $\mathcal{O}, \mathcal{R}$ are (-1)-connected, then X is homotopy pro-nilpotent.*

Proof. This is proved in Harper-Hess [28, 1.12] by showing that the homotopy completion tower of X converges strongly to X. □

Proposition 1.4 (TQ-local Whitehead theorem)**.** *A map $X \to Y$ between TQ-local $\mathcal{O}$-algebras is a weak equivalence if and only if it is a TQ-homology equivalence.*

Proof. This follows from the definition of TQ-local $\mathcal{O}$-algebras (Definition 2.2); see, for instance, Hirschhorn [32, 3.2.13]. □

Proposition 1.5 (Preservation of the TQ-local property: Homotopy limits)**.** *The homotopy limit of a small diagram of TQ-local $\mathcal{O}$-algebras is TQ-local.*

Proof. This follows from the definition of TQ-local $\mathcal{O}$-algebras; see, for instance, Dror Farjoun [14, 1.A.8, 1.G] and Hirschhorn [32, 19.4.4]. Here is another, essentially equivalent, proof: It follows from Proposition 1.4 that the homotopy limit in $\mathsf{Alg}_{\mathcal{O}}$ of a small diagram of TQ-local $\mathcal{O}$-algebras is weakly equivalent to its homotopy limit calculated in the TQ-local homotopy theory [29, 5.14]; hence, verifying that the homotopy limit in $\mathsf{Alg}_{\mathcal{O}}$ is TQ-local reduces to the usual fibrancy property of homotopy limits in a homotopy theory (in this case, in the TQ-local homotopy theory); see, for instance, Hirschhorn [32, 18.5.2], together with Ching-Harper [13, 8.9] for a discussion of homotopy limits in the context of $\mathcal{O}$-algebras. □

For instance, consider any pullback diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "p"] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$. It follows from Proposition 1.5 that if B, C, D are TQ-local and p is a fibration, then A is TQ-local. Taking $C = *$, for instance, shows that TQ-local $\mathcal{O}$-algebras play nicely with fibration sequences; this is not expected to be true, in general, if we replace "TQ-local" with "TQ-complete" and is one of the reasons why TQ-localization is often better behaved than TQ-completion (at the expense of a much larger construction).

Proposition 1.6. *If X is an M-nilpotent $\mathcal{O}$-algebra (resp. Z is an $\mathcal{O}$-algebra) for some $M \geq 2$, then its $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion $X^\wedge_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ (resp. TQ-completion $Z^\wedge_{\mathsf{TQ}}$) is TQ-local.*

Proof. By Proposition 1.5, it suffices to verify that $X^\wedge_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ (resp. $Z^\wedge_{\mathsf{TQ}}$) is the homotopy limit of a small diagram of TQ-local $\mathcal{O}$-algebras—we defer the proof of this to Section 2 (see Propositions 2.10 and 2.9, respectively). □

Proposition 1.7. *Let $M \geq 2$.*

(a) *If X is an M-nilpotent $\mathcal{O}$-algebra, then the natural map $X \simeq X^\wedge_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ is a weak equivalence.*
(b) *If Z is a connected $\mathcal{O}$-algebra and $\mathcal{O}, \mathcal{R}$ are (-1)-connected, then the natural map $Z \simeq Z^\wedge_{\mathsf{TQ}}$ is a weak equivalence.*

Proof. Part (a) is proved in Ching-Harper [12, 2.12] and part (b) is proved in Ching-Harper [13, 1.2]. □

The following theorem is the main result of this paper. Our proof is embarrassingly simple—one of the significant advantages of having available the TQ-local homotopy theory framework to work within and organize our arguments in—it follows immediately from the above propositions, together with verifying that $\mathsf{TQ}|_{\mathrm{Nil}_M}$-resolutions have TQ-local fibrant replacements in $\mathsf{Alg}_{\mathcal{O}}$; we defer the proof of this to Section 2 (see Proposition 2.10).

Theorem 1.8 (Homotopy pro-nilpotent TQ-localization theorem)**.** *Let X be a fibrant $\mathcal{O}$-algebra.*

(a) *If X is nilpotent, then X is TQ-local.*
(b) *If X is homotopy pro-nilpotent, then X is TQ-local.*
(c) *If X is the homotopy limit of any small diagram of nilpotent $\mathcal{O}$-algebras, then X is TQ-local.*
(d) *If X is connected and $\mathcal{O}, \mathcal{R}$ are (-1)-connected, then X is TQ-local.*

Remark 1.9. It is worth pointing out (Proposition 2.7) that if some fibrant replacement of an $\mathcal{O}$-algebra X is TQ-local, then every fibrant replacment of X is TQ-local.

Proof of Theorem 1.8. Part (a) follows from Propositions 1.6 and 1.7. Parts (b) and (c) follow from part (a), together with Proposition 1.5. Part (d) follows from part (c), together with Proposition 1.3; alternately, it follows from Propositions 1.6 and 1.7. □

As an application of the main result, we obtain the following homotopy pro-nilpotent TQ-Whitehead theorem that simultaneously extends the previously known connected and nilpotent TQ-Whitehead theorems.

Theorem 1.10 (Homotopy pro-nilpotent TQ-Whitehead theorem)**.** *A map $X \to Y$ between homotopy pro-nilpotent $\mathcal{O}$-algebras is a weak equivalence if and only if it is a TQ-homology equivalence; more generally, this remains true if X, Y are homotopy limits of small diagrams of nilpotent $\mathcal{O}$-algebras.*

Proof. This follows from Theorem 1.8, together with Proposition 1.4. □

We also compare TQ-localization with TQ-completion and show that TQ-local $\mathcal{O}$-algebras that are TQ-good are TQ-complete (Theorem 3.1). Finally we show that $\mathcal{O}$-algebras with a principally refined Postnikov tower are TQ-local, provided that mild connectivity assumptions are satisfied; here is the theorem, whose proof we defer to Section 4.

Theorem 1.11. *Let X be a fibrant $\mathcal{O}$-algebra. If $X, \mathcal{O}, \mathcal{R}$ are (-1)-connected and X has a principally refined Postnikov tower, then X is TQ-local.*

1.12. **Conventions and notations.** We work in the category $\mathsf{Alg}_{\mathcal{O}}$ of algebras over an operad $\mathcal{O}$ in $\mathsf{Mod}_{\mathcal{R}}$, the category of $\mathcal{R}$-modules, where $\mathcal{R}$ is a commutative monoid in the category of symmetric spectra [33, 47]. Throughout this paper, we assume that: (i) $\mathcal{O}$ is reduced, meaning that $\mathcal{O}[0] = *$ (i.e., $\mathcal{O}$-algebras are non-unital) and (ii) the natural maps $\mathcal{R} \to \mathcal{O}[1]$ and $* \to \mathcal{O}[n]$ are flat stable cofibrations [28] in $\mathcal{R}$-modules for each $n \geq 0$; see, for instance, [13, 2.1, 6.12]. Unless otherwise specified, we work with the positive flat stable model structure on $\mathsf{Alg}_{\mathcal{O}}$ [28, 47]. To keep this paper appropriately concise, we freely use notation from [13, 28, 29]. Finally, "connected" is short for "0-connected".

Acknowledgments. The author would like to thank John E. Harper and Niko Schonsheck for enlightening discussions and helpful suggestions, and Michael Ching and Mark W. Johnson for helpful conversations. The author is grateful to Oscar Randal-Williams for detailed and helpful critical comments on an early draft of this paper. The author was supported in part by the Simons Foundation: Collaboration Grants for Mathematicians #638247.

2. TQ-local $\mathcal{O}$-algebras and $\mathsf{TQ}|_{\mathrm{Nil}_M}$-resolutions

The purpose of this section is to recall the definition of TQ-local $\mathcal{O}$-algebras, discuss their basic properties, and finally to verify that TQ-resolutions of $\mathcal{O}$-algebras and $\mathsf{TQ}|_{\mathrm{Nil}_M}$-resolutions of M-nilpotent $\mathcal{O}$-algebras have TQ-local fibrant replacements in $\mathsf{Alg}_{\mathcal{O}}$ (Propositions 2.9 and 2.10).

TQ-homology and its relative form, topological Andre-Quillen (TAQ) homology, first introduced in [4] for commutative ring spectra (see also [3, 5, 6, 21, 35, 38, 39, 45]), are defined as derived indecomposables of $\mathcal{O}$-algebras analogous to Quillen homology of commutative algebras [1, 44]; see also [19, 24, 20, 41]. More precisely, factoring the canonical truncation map $\mathcal{O} \to \tau_1\mathcal{O}$ (see [28]) in the category of operads as $\mathcal{O} \to J \to \tau_1\mathcal{O}$, a cofibration followed by a weak equivalence, we get the corresponding change of operads adjunction

$$\mathsf{Alg}_{\mathcal{O}} \underset{U}{\overset{Q}{\rightleftarrows}} \mathsf{Alg}_J \tag{1}$$

with left adjoint on top, where $Q(X) = J \circ_{\mathcal{O}} (X)$ and U denotes the forgetful functor (or less concisely, the "forget along the map $\mathcal{O} \to J$ functor").

Definition 2.1. Let X be an $\mathcal{O}$-algebra. The *topological Quillen homology* (or TQ-homology, for short) of X is

$$\mathsf{TQ}(X) := \mathsf{R}U(\mathsf{L}Q(X))$$

the $\mathcal{O}$-algebra defined via the indicated composite of total right and left derived functors. If X is cofibrant, then $\mathsf{TQ}(X) \simeq UQ(X)$ and the unit of the (Q, U) adjunction in (1) is the TQ-Hurewicz map $X \to UQX$ of the form $X \to \mathsf{TQ}(X)$.

TQ-homology has been shown to enjoy several properties analogous to the ordinary homology of spaces; see, for instance, [4, 5, 11, 13, 28, 36]. Furthermore, it turns out that TQ-homology is weakly equivalent to stabilization in the category of $\mathcal{O}$-algebras, provided that $\mathcal{O}, \mathcal{R}$ are (-1)-connected; see, for instance, [5, 36, 42, 45, 46]; a simple proof using Goodwillie's functor calculus [25] is given in [28, 1.14].

Localization has been proved to be a useful method in algebraic topology. The idea is to focus on the information one cares about and ignore everything else, hence simplifying problems accordingly. See [2, 9, 10, 16, 31, 48] for a discussion of localization methods. Also see, for useful developments and ideas, [8, 14, 16, 32, 34, 40].

In this paper we consider localization of $\mathcal{O}$-algebras with respect to TQ-homology. Recall the following from [29].

Definition 2.2. Let $f\colon A \to B$ be a map in $\mathsf{Alg}_{\mathcal{O}}$. We say that f is a TQ-*equivalence* if f induces a weak equivalence $\mathsf{TQ}(A) \simeq \mathsf{TQ}(B)$ on TQ-homology. We say that f is a TQ-*acyclic strong cofibration* if f is a cofibration between cofibrant objects which is also a TQ-equivalence. An $\mathcal{O}$-algebra X is called TQ-*local* if (i) X is fibrant in $\mathsf{Alg}_{\mathcal{O}}$ and (ii) every TQ-acyclic strong cofibration $A \to B$ induces a weak equivalence

$$\mathbf{Hom}(A, X) \xleftarrow{\simeq} \mathbf{Hom}(B, X)$$

on mapping spaces in sSet; here we are using the simplicial model structure on $\mathsf{Alg}_{\mathcal{O}}$ (see, for instance, [13, 17, 22, 23, 28]).

It is useful to note that the collection of TQ-local $\mathcal{O}$-algebras is exactly the collection of weak TQ-fibrant objects in the TQ-local homotopy theory [29]. Intuitively, a TQ-local object in $\mathsf{Alg}_{\mathcal{O}}$ is an $\mathcal{O}$-algebra X which only sees TQ-homology information. The following proposition will be useful for detecting TQ-local $\mathcal{O}$-algebras.

Proposition 2.3. *Let X be a fibrant $\mathcal{O}$-algebra. Then X is TQ-local if and only if $X \to *$ satisfies the right lifting property with respect to every TQ-acyclic strong cofibration $A \to B$ in $\mathsf{Alg}_{\mathcal{O}}$.*

Proof. This is proved in [29, 3.12]. □

Proposition 2.4. *Let Y be a fibrant object in Alg_J. Then $UY \in \mathsf{Alg}_{\mathcal{O}}$ is TQ-local.*

Proof. This follows from Proposition 2.3 by using the (Q, U) adjunction (1). □

Next we observe that the TQ-local property is preserved by weak equivalences between fibrant $\mathcal{O}$-algebras.

Proposition 2.5. *Let $X \to Y$ be a weak equivalence between fibrant objects in $\mathsf{Alg}_{\mathcal{O}}$. Then X is TQ-local if and only if Y is TQ-local.*

Proof. Let $A \to B$ be a TQ-acyclic strong cofibration. Consider the commutative diagram of mapping spaces of the form

$$\begin{tikzcd} \mathbf{Hom}(A,X) \arrow[d, "\sim"] & \mathbf{Hom}(B,X) \arrow[l] \arrow[d, "\sim"] \\ \mathbf{Hom}(A,Y) & \mathbf{Hom}(B,Y) \arrow[l] \end{tikzcd}$$

in sSet. Since the vertical maps are weak equivalences, it follows that the top map is a weak equivalence if and only if the bottom map is a weak equivalence. □

This observation generalizes as follows.

Proposition 2.6. *Consider any weak equivalence $X \to Y$ in $\mathsf{Alg}_{\mathcal{O}}$. Let X', Y' be fibrant replacements of X, Y, respectively, in $\mathsf{Alg}_{\mathcal{O}}$. Then X' is TQ-local if and only if Y' is TQ-local.*

Proof. By assumption, the comparison map $X \to X'$ is an acyclic cofibration and Y' is fibrant. Then it follows immediately (via lifting) that there exists a map ξ that makes the diagram

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "\sim"] & Y \arrow[d, "\sim"] \\ X' \arrow[r, dotted, "\xi"] & Y' \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$ commute. By Proposition 2.5, X' is TQ-local if and only if Y' is TQ-local since ξ is a weak equivalence between fibrant objects. □

Proposition 2.7. *Let X be an $\mathcal{O}$-algebra and suppose X', X'' are fibrant replacements of X in $\mathsf{Alg}_{\mathcal{O}}$. Then X' is TQ-local if and only if X'' is TQ-local.*

Proof. This follows from Proposition 2.6. □

The following generalization of Proposition 2.4 will be used in our proof of Propositions 2.9 and 2.10.

Proposition 2.8. *Let Y be a (not necessarily fibrant) object in Alg_J, then every fibrant replacement of UY in $\mathsf{Alg}_{\mathcal{O}}$ is TQ-local.*

Proof. Let $UY \to \widetilde{UY}$ be a fibrant replacement of UY in $\mathsf{Alg}_{\mathcal{O}}$; in particular, $UY \to \widetilde{UY}$ is an acyclic cofibration. Let Y' be a fibrant replacement of Y in Alg_J. Then it follows immediately (via lifting) that there exists a map ξ that makes the diagram

$$\begin{tikzcd} UY \arrow[r, "\sim"] \arrow[d, "\sim"'] & UY' \arrow[d] \\ \widetilde{UY} \arrow[r] \arrow[ur, dotted, "\xi"] & * \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$ commute. Since UY' is TQ-local (Proposition 2.4) and ξ is a weak equivalence between fibrant objects, $\widetilde{UY}$ is TQ-local by Proposition 2.5. □

Now we are in a good position to prove Propositions 2.9 and 2.10. First recall the TQ-completion construction. Let Z be a cofibrant $\mathcal{O}$-algebra and consider the cosimplicial resolution of Z with respect to TQ-homology of the form

$$Z \longrightarrow (UQ)Z \rightrightarrows (UQ)^2 Z \rightrightarrows (UQ)^3 Z \cdots$$

in $\mathsf{Alg}_{\mathcal{O}}$, denoted $Z \to \mathbf{C}(Z)$, with coface maps obtained by iterating the TQ-Hurewicz map $\mathrm{id} \to UQ$ (Definition 2.1) and codegeneracy maps built from the counit map of the adjunction (Q, U) in the usual way. Taking the homotopy limit (over Δ) gives the TQ-completion map [13, 28] of the form

$$Z \to Z^{\wedge}_{\mathsf{TQ}} = \mathrm{holim}_{\Delta}\, \mathbf{C}(Z) \simeq \mathrm{holim}_{\Delta}\, \widetilde{\mathbf{C}(Z)} \tag{2}$$

in $\mathsf{Alg}_{\mathcal{O}}$, where $\widetilde{\mathbf{C}(Z)}$ denotes any functorial fibrant replacement functor $\widetilde{(-)}$ on $\mathsf{Alg}_{\mathcal{O}}$ (obtained, for instance, by running the small object argument with respect to the generating acyclic cofibrations in $\mathsf{Alg}_{\mathcal{O}}$) applied to the cosimplicial $\mathcal{O}$-algebra $\mathbf{C}(Z)$.

The following proposition verifies that $\widetilde{\mathbf{C}(Z)}$ is an objectwise TQ-local diagram—this amounts to checking the weak TQ-fibrancy condition in the TQ-local homotopy theory [29, 5.14] using the above technical propositions.

Proposition 2.9. *If Z is an $\mathcal{O}$-algebra, then the TQ-completion $Z^{\wedge}_{\mathsf{TQ}}$ of Z is the homotopy limit of a small diagram of TQ-local $\mathcal{O}$-algebras.*

Proof. It suffices to consider the case when Z is a cofibrant $\mathcal{O}$-algebra; e.g., otherwise, work with its cofibrant replacement in $\mathsf{Alg}_{\mathcal{O}}$. We want to show that the Δ-shaped diagram $\widetilde{\mathbf{C}(Z)}$ in (2) is objectwise TQ-local; i.e., that $\widetilde{\mathbf{C}(Z)}^s$ is TQ-local for each $s \geq 0$. This follows from Proposition 2.8. In more detail: Consider the case $s = 0$. Let $Y := QZ \in \mathsf{Alg}_J$. Then $UY = (UQ)Z$, hence it suffices to verify that $\widetilde{UY}$ is TQ-local which is true by Proposition 2.8. Similarly, consider the case $s \geq 1$. Let $Y := Q(UQ)^s Z \in \mathsf{Alg}_J$. Then $UY = (UQ)^{s+1} Z$, hence it suffices to verify that $\widetilde{UY}$ is TQ-local and Proposition 2.8 completes the proof. □

Recall the $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion construction from [12]. For each $n \geq 1$, $\tau_n \mathcal{O}$ is the operad associated to $\mathcal{O}$ where

$$(\tau_n \mathcal{O})[t] := \begin{cases} \mathcal{O}[t] & \text{for } t \leq n \\ * & \text{otherwise} \end{cases}$$

and consider the associated commutative diagram of operad maps [12]

$$\begin{tikzcd} \mathcal{O} \arrow[r] \arrow[dr, bend right] & J_n \arrow[r] \arrow[d, "\sim"] & J_1 \arrow[r, equal] \arrow[d, "\sim"] & J \\ & \tau_n \mathcal{O} \arrow[r] & \tau_1 \mathcal{O} & \end{tikzcd}$$

where the upper horizontal maps are cofibrations of operads, the left-hand and bottom horizontal maps are the natural truncations, and the vertical maps are weak equivalences of operads; for notational simplicity, here we take $J = J_1$. The corresponding change of operad adjunctions have the form

$$\mathsf{Alg}_{\mathcal{O}} \underset{V_n}{\overset{R_n}{\rightleftarrows}} \mathsf{Alg}_{J_n} \underset{U_n}{\overset{Q_n}{\rightleftarrows}} \mathsf{Alg}_J \qquad \mathsf{Alg}_{\mathcal{O}} \underset{U}{\overset{Q}{\rightleftarrows}} \mathsf{Alg}_J$$

with left adjoints on top, where $R_n = J_n \circ_{\mathcal{O}} (-)$, $Q_n = J \circ_{J_n} (-)$, $Q = J \circ_{\mathcal{O}} (-)$, and V_n, U_n, U denote the indicated forgetful functors; in particular, the adjunction on the right is the composite of the adjunctions on the left.

Next recall the $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion construction. Let $n \geq 1$ and define $M := n+1$. Let X be a cofibrant J_n-algebra and consider the cosimplicial resolution of X with respect to $\mathsf{TQ}|_{\mathrm{Nil}_M}$-homology of the form

$$X \longrightarrow (U_nQ_n)X \rightrightarrows (U_nQ_n)^2X \rightrightarrows (U_nQ_n)^3X \ \cdots$$

in Alg_{J_n}, denoted $X \to \mathbf{N}(X)$, with coface maps obtained by iterating the $\mathsf{TQ}|_{\mathrm{Nil}_M}$-Hurewicz map $\mathrm{id} \to U_nQ_n$ and codegeneracy maps built from the counit map of the adjunction (Q_n, U_n) in the usual way; see Ching-Harper [11]. Applying the forgetful functor V_n gives the diagram $V_nX \to V_n\mathbf{N}(X)$ of the form

$$V_nX \longrightarrow V_n(U_nQ_n)X \rightrightarrows V_n(U_nQ_n)^2X \rightrightarrows V_n(U_nQ_n)^3X \cdots$$

in $\mathsf{Alg}_{\mathcal{O}}$. Taking the homotopy limit (over Δ) gives the $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion map of the form

$$X \to X^{\wedge}_{\mathsf{TQ}|_{\mathrm{Nil}_M}} = \mathrm{holim}_\Delta\, V_n\mathbf{N}(X) \simeq \mathrm{holim}_\Delta\, \widetilde{V_n\mathbf{N}(X)} \tag{3}$$

in $\mathsf{Alg}_{\mathcal{O}}$, where $\widetilde{V_n\mathbf{N}(X)}$ denotes any functorial fibrant replacement functor $\widetilde{(-)}$ on $\mathsf{Alg}_{\mathcal{O}}$ applied to the cosimplicial $\mathcal{O}$-algebra $V_n\mathbf{N}(X)$.

Proposition 2.10. *If X is a J_n-algebra, then the $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion $X^{\wedge}_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ of X is the homotopy limit of a small diagram of TQ-local $\mathcal{O}$-algebras.*

Proof. It suffices to consider the case when X is a cofibrant J_n-algebra; e.g., otherwise, work with its cofibrant replacement in Alg_{J_n}. We want to show that the Δ-shaped diagram $\widetilde{V_n\mathbf{N}(X)}$ in (3) is objectwise TQ-local; i.e., that $\widetilde{V_n\mathbf{N}(X)}^s$ is TQ-local for each $s \geq 0$. This follows from Proposition 2.8. In more detail: Consider the case $s = 0$. Let $Y := Q_nX \in \mathsf{Alg}_J$, then $UY = V_n(U_nQ_n)X$ and hence it suffices to verify that $\widetilde{UY}$ is TQ-local; this is true by Proposition 2.8. Similarly, consider the case $s \geq 1$. Let $Y := Q_n(U_nQ_n)^sX \in \mathsf{Alg}_J$. Then $UY = V_n(U_nQ_n)^{s+1}X$ and hence it suffices to verify that $\widetilde{UY}$ is TQ-local; this is true by Proposition 2.8 which completes the proof. □

3. Comparing TQ-localization with TQ-completion

In this section we discuss the relation between TQ-localization and TQ-completion. Let X be a cofibrant $\mathcal{O}$-algebra. One can construct the TQ-localization map $X \to L_{\mathsf{TQ}}(X)$ (see [29]) by running the small object argument. By construction, $L_{\mathsf{TQ}}(X)$ is TQ-local and the TQ-localization map $X \to L_{\mathsf{TQ}}(X)$ is a TQ-acyclic strong cofibration (Definition 2.2).

The TQ-completion map $X \to X^{\wedge}_{\mathsf{TQ}}$ can be thought of as an approximation of the TQ-localization map. For instance, we know that the TQ-completion $X^{\wedge}_{\mathsf{TQ}}$ of X is always TQ-local (Proposition 1.6).

Theorem 3.1 (Recognizing when TQ-local $\mathcal{O}$-algebras are TQ-complete)**.** *Let X be a cofibrant $\mathcal{O}$-algebra. Then the TQ-completion map $c\colon X \to X^{\wedge}_{\mathsf{TQ}}$ factors through the TQ-localization map $l\colon X \to L_{\mathsf{TQ}}(X)$ via a commutative diagram of the form*

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "l"'] & X^{\wedge}_{\mathsf{TQ}} \arrow[d] \\ L_{\mathsf{TQ}}(X) \arrow[r] \arrow[ur, dotted, "\xi"] & * \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$. Furthermore, if X is TQ-local, then the following are equivalent:

(i) *The natural map $X \to X^{\wedge}_{\mathsf{TQ}}$ is a TQ-equivalence; i.e., X is TQ-*good.
(ii) *The natural map $X \simeq X^{\wedge}_{\mathsf{TQ}}$ is a weak equivalence; i.e., X is TQ-complete.*
(iii) *The comparison map ξ is a weak equivalence.*

Proof. This is analogous to the Bousfield-Kan completion of spaces [10]. Since $X^{\wedge}_{\mathsf{TQ}}$ is TQ-local and $l\colon X \to L_{\mathsf{TQ}}(X)$ is a TQ-acyclic strong cofibration, there exists a lift ξ that makes the diagram commute (Proposition 2.3) in $\mathsf{Alg}_{\mathcal{O}}$. Suppose X is TQ-local, then l is a TQ-equivalence between TQ-local objects, hence a weak equivalence by the TQ-local Whitehead theorem (Proposition 1.4). Therefore ξ is a weak equivalence if and only if c is a weak equivalence. This verifies (ii) $\Leftrightarrow$ (iii). Since $X, X^{\wedge}_{\mathsf{TQ}}$ are TQ-local, c is a TQ-equivalence if and only if c is a weak equivalence by the TQ-local Whitehead theorem. This verifies (i) $\Leftrightarrow$ (ii). ☐

It is worth pointing out the following two propositions.

Proposition 3.2. *A map $f\colon X \to Y$ between $\mathcal{O}$-algebras is a TQ-homology equivalence if and only if the induced map $f^{\wedge}_{\mathsf{TQ}}\colon X^{\wedge}_{\mathsf{TQ}} \to Y^{\wedge}_{\mathsf{TQ}}$ is a weak equivalence.*

Proof. This is proved by arguing exactly as in [10, I.5], but here is the basic idea: The "if" direction is proved using retract argument and the "only if" direction is because holim_{Δ} preserves weak equivalences. ☐

Proposition 3.3. *Let X be an $\mathcal{O}$-algebra, then the following are equivalent:*

(i) *X is TQ-*good.
(ii) *$X^{\wedge}_{\mathsf{TQ}}$ is TQ-complete.*
(iii) *$X^{\wedge}_{\mathsf{TQ}}$ is TQ-good.*

Proof. This follows from exactly the same argument as in [10, I.5]. ☐

4. Postnikov towers and TQ-localization

In this section we assume that $\mathcal{O}, \mathcal{R}$ are (-1)-connected. We show that a (-1)-connected $\mathcal{O}$-algebra is TQ-local if it has a principally refined Postnikov tower.

Proposition 4.1. *Let X be a (-1)-connected cofibrant $\mathcal{O}$-algebra. Then there exists a coaugmented tower $\{X\} \to \{X_n\}$ (the Postnikov tower of X) of the form*

$$\begin{tikzcd} X \arrow[d] \arrow[dr, bend left] \arrow[drr, bend left] \arrow[drrr, bend left] & & & & \\ * = X_{-1} & X_0 \arrow[l] & X_1 \arrow[l] & X_2 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$ such that for each $n \geq -1$:

(a) *X_n is a cofibrant and fibrant $\mathcal{O}$-algebra.*

(b) *the structure map* $X \to X_n$ *is* $(n+1)$*-connected and* $\pi_k X_n = *$ *for all* $k \geq n+1$.
(c) *the structure map* $X_{n+1} \to X_n$ *is a fibration.*

Proof. The Postnikov tower can be constructed using small object arguments analogous to the arguments in [11, 15, 17, 47]. In more detail: Let I_n be the set of n-connected generating cofibrations in $\mathsf{Alg}_{\mathcal{O}}$ and let J be the set of generating acyclic cofibrations in $\mathsf{Alg}_{\mathcal{O}}$ (see, for instance, [26]). Start by setting $X_{-1} = *$. For each $n \geq 0$, we inductively run the small object argument with respect to $I_{n+1} \bigcup J$ to factor the map $X \to X_{n-1}$ in $\mathsf{Alg}_{\mathcal{O}}$ as $X \to X_n \to X_{n-1}$. Then $X \to X_n$ is a cofibration, $X_n \to X_{n-1}$ is a fibration and $\pi_k X_n = *$ for all $k \geq n+1$. By assumption, $\mathcal{O}, \mathcal{R}$ are (-1)-connected, hence $X \to X_n$ is $(n+1)$-connected by the small object argument construction. $\square$

Analogous to the definition for spaces, principal Postnikov towers and principally refined Postnikov towers are defined as follows.

Definition 4.2. Let X be a (-1)-connected $\mathcal{O}$-algebra. We say that a Postnikov tower $\{X_n\}$ of X is *principal* if for each $n \geq 0$, the structure map $X_n \to X_{n-1}$ fits into a homotopy pullback diagram of the left-hand form

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & *^{\mathrm{fat}} \arrow[d] \\ X_{n-1} \arrow[r] & K(\pi_n X, n+1) \end{tikzcd} \qquad \begin{tikzcd} M_i \arrow[r] \arrow[d] & *^{\mathrm{fat}} \arrow[d] \\ M_{i-1} \arrow[r] & K(G_i, n+1) \end{tikzcd} \tag{4}$$

in $\mathsf{Alg}_{\mathcal{O}}$, where $*^{\mathrm{fat}}$ is an $\mathcal{O}$-algebra that is weakly equivalent to $*$ (i.e., a "fat point" in $\mathsf{Alg}_{\mathcal{O}}$) and $K(\pi_n X, n+1)$ is an object in $\mathsf{Alg}_{\mathcal{O}}$ with $\pi_n X$ as the only nontrivial homotopy group concentrated at level $n+1$.

We say that $\{X_n\}$ is *principally refined* if, for each $n \geq 0$, the structure map $X_n \to X_{n-1}$ can be factored as a finite composite $X_n = M_{t_n} \to \cdots \to M_2 \to M_1 \to M_0 = X_{n-1}$ of maps such that, for each $t_n \geq i \geq 1$, the map $M_i \to M_{i-1}$ fits into a homotopy pullback diagram of the right-hand form (4) in $\mathsf{Alg}_{\mathcal{O}}$, where the G_i's are abelian groups and $K(G_i, n+1)$ is an object in $\mathsf{Alg}_{\mathcal{O}}$ with G_i as the only nontrivial homotopy group concentrated at level $n+1$. In particular, every principal Postnikov tower is principally refined.

Proof of Theorem 1.11. We know that every 0-connected fibrant $\mathcal{O}$-algebra is TQ-local (Theorem 1.8), hence, in particular, each Eilenberg-MacLane object $K(G_i, n+1)$ appearing in the principally refined Postnikov tower of X has TQ-local fibrant replacements in $\mathsf{Alg}_{\mathcal{O}}$. By inducting up the principally refined Postnikov tower, it follows that each X_n is TQ-local (Proposition 1.5). Since X is the homotopy limit of its Postnikov tower $\{X_n\}$, which is objectwise TQ-local, it follows that X is TQ-local (Proposition 1.5) which completes the proof. $\square$

We provide some examples of (-1)-connected algebras which admit principally refined Postnikov towers.

(i) Let X be a cofibrant 0-connected $\mathcal{O}$-algebra. Analogous to results in [4, 7, 15, 30], one can show that the Postnikov tower of X is principal.
(ii) Consider ΩX for any 0-connected cofibrant $\mathcal{O}$-algebra X. Since the loop functor Ω commutes with homotopy pullbacks in $\mathsf{Alg}_{\mathcal{O}}$, ΩX has a principal Postnikov tower by applying Ω to the principal Postnikov tower of X.

(iii) Consider UY for any (-1)-connected cofibrant J-algebra Y. The category Alg_J is Quillen equivalent to $\mathsf{Alg}_{\tau_1 \mathcal{O}} \cong \mathsf{Mod}_{\mathcal{O}[1]}$ [28, 7.21], hence the homotopy category of Alg_J is stable. Therefore, the Postnikov tower of Y in Alg_J is already principal. Applying forgetful functor U induces principal Postnikov tower for UY in $\mathsf{Alg}_{\mathcal{O}}$.

(iv) One can construct additional examples by pulling back quotient towers along cocellular maps as described in [40, 3.3].

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