HOMOTOPY PRO-NILPOTENT STRUCTURED RING SPECTRA AND TOPOLOGICAL QUILLEN LOCALIZATION

YU ZHANG

ABSTRACT. The aim of this paper is to show that homotopy pro-nilpotent structured ring spectra are TQ-local, where structured ring spectra are described as algebras over a spectral operad $\mathcal{O}$. Here, TQ is short for topological Quillen homology, which is weakly equivalent to $\mathcal{O}$-algebra stabilization. An $\mathcal{O}$-algebra is called homotopy pro-nilpotent if it is equivalent to a limit of nilpotent $\mathcal{O}$-algebras. Our result provides new positive evidence to a conjecture by Francis-Gaisgory on Koszul duality for general operads. As an application, we simultaneously extend the previously known 0-connected and nilpotent TQ-Whitehead theorems to a homotopy pro-nilpotent TQ-Whitehead theorem.

1. INTRODUCTION

Spectra play a key role in the development of modern algebraic topology. Lots of important examples of spectra, such as Eilenberg–Mac Lane spectra, bordism spectra and complex (or real) K-theory spectra, are equipped with natural algebraic structures. However, the algebraic structures on spectra are often more general than their classical analogs, such as commutative rings. Spectra equipped with generalized algebraic structures are called structured ring spectra.

We can formalize our definition of structured ring spectra as follows. Let $\mathcal{R}$ be any commutative monoid in the category of symmetric spectra of simplicial sets. In other words, let $\mathcal{R}$ be any commutative ring spectrum. Structured ring spectra are spectra with extra algebraic structures that can be described as algebras over an operad $\mathcal{O}$ in symmetric spectra, or more generally, in $\mathsf{Mod}_{\mathcal{R}}$. Here, we let $(\mathsf{Mod}_{\mathcal{R}}, \wedge, \mathcal{R})$ denote the symmetric monoidal category of $\mathcal{R}$-modules. For a fixed operad $\mathcal{O}$, denote by $\mathsf{Alg}_{\mathcal{O}}$ the category of $\mathcal{O}$-algebras. For readers not familiar with operads, [6, 18, 19, 29, 33, 37] are some useful references. In this paper we work with reduced operads $\mathcal{O}$ (i.e., such that $\mathcal{O}[0] = *$, where $*$ denotes the trivial $\mathcal{R}$-module); algebras over $\mathcal{O}$ are then called non-unital. This includes the examples of non-unital E_n algebra spectra.

Topological Quillen homology [3, 5], or TQ-homology, is the topological analog of André–Quillen homology [7, 20, 36] in the setting of non-unital structured ring spectra. We recall the precise definition of TQ-homology below.

Fix an operad $\mathcal{O}$ as above. Let operad $\tau_1 \mathcal{O}$ be the natural truncation of $\mathcal{O}$ above level 1. In particular, $\tau_1 \mathcal{O}[1] = \mathcal{O}[1]$ and $\tau_1 \mathcal{O}[k] = *$ for $k \geq 0, k \neq 1$. Then there is a canonical truncation map $\mathcal{O} \to \tau_1 \mathcal{O}$ in the category of operads. We can factor the truncation map as $\mathcal{O} \to J \to \tau_1 \mathcal{O}$, a cofibration followed by a weak equivalence with respect to the projective model structure of operads, see [25, Definition 5.47, 7.10] for more details.

The map $\mathcal{O} \to J$ induces the corresponding change of operads adjunction

(1.1) $$\mathsf{Alg}_{\mathcal{O}} \underset{U}{\overset{Q}{\rightleftarrows}} \mathsf{Alg}_J$$

with left adjoint on top, where $Q(X) = J \circ_{\mathcal{O}} (X)$ and U denotes the forgetful functor. Here $\circ$ denotes composition product, $J \circ_{\mathcal{O}} (X) = coeq(J \circ \mathcal{O} \circ X \rightrightarrows J \circ X)$, see, for example, [24, Definition 2.8].

Convention 1.1. *Throughout this paper, we work with the positive flat stable model structure (see, for example [25, 7.15]) on $\mathsf{Alg}_{\mathcal{O}}$ and Alg_J unless otherwise specified. A map of $\mathcal{O}$-algebras is called a (co)fibration if it is so with respect to the positive flat stable model structure on $\mathsf{Alg}_{\mathcal{O}}$. Similarly, an $\mathcal{O}$-algebra is called (co)fibrant if it is so with respect to the positive flat stable model structure on $\mathsf{Alg}_{\mathcal{O}}$.*

Remark 1.2. The category Alg_J of J-algebras is Quillen equivalent to the category of $\mathcal{O}[1]$-modules [25, 7.21]. One can think of J as a fattened-up version of $\tau_1 \mathcal{O}$. The advantage of working with J instead of $\tau_1 \mathcal{O}$ will become clear after we introduce the TQ-completion construction (Definition 2.1). See Remark 2.2.

Definition 1.3. Let X be an $\mathcal{O}$-algebra. The *topological Quillen homology* (or TQ-homology, for short) of X is

$$\mathsf{TQ}(X) := \mathsf{R}U(\mathsf{L}Q(X))$$

the $\mathcal{O}$-algebra defined via the indicated composite of total right and left derived functors. Note the forgetful functor U preserves all weak equivalences. Therefore, if X is cofibrant, then $\mathsf{TQ}(X) \simeq UQ(X)$ and the unit of the (Q, U) adjunction in (1.1) is the TQ-Hurewicz map $X \to UQX$ of the form $X \to \mathsf{TQ}(X)$.

TQ-homology has been shown to enjoy several properties analogous to the ordinary homology of spaces; see, for instance, [3, 4, 25]. Furthermore, it turns out that TQ-homology is weakly equivalent to stabilization $\Omega^{\infty}\Sigma^{\infty}$ in the category of $\mathcal{O}$-algebras [4, 11, 30, 35]. One can think of the adjunction (1.1) as an analog of suspension spectrum and infinite loop space adjunction $(\Sigma^{\infty}, \Omega^{\infty})$.

Let $K := QU$ denote the comonad associated to the adjunction (Q, U). Then the image of Q lands in the category of K-coalgebras. Moreover, there is an associated adjunction of ∞-categories [11, 1.3]

(1.2) $$\mathsf{Alg}_{\mathcal{O}} \rightleftarrows \mathsf{coAlg}_K$$

where the left adjoint is Q.

Francis-Gaitsgory [17] studied analogous phenomena in terms of koszul duality of general operads. They made a conjecture, which we can rephrase in terms of structured ring spectra as follows.

The Francis-Gaitsgory Conjecture [17, 3.4.5]**.** *Adjunction* (1.2) *induces an equivalence of homotopy categories after restricting $\mathsf{Alg}_{\mathcal{O}}$ to the full subcategory of homotopy pro-nilpotent $\mathcal{O}$-algebras.*

We recall relevant definitions below.

Definition 1.4. Let X be an $\mathcal{O}$-algebra and $M \geq 2$. We say that X is *M-nilpotent* if all the M-ary and higher operations $\mathcal{O}[t] \wedge X^{\wedge t} \to X$ of X are trivial (i.e., if these maps factor through the trivial $\mathcal{R}$-module $*$ for each $t \geq M$). An $\mathcal{O}$-algebra is called *nilpotent* if it is M-nilpotent for some $M \geq 2$. An $\mathcal{O}$-algebra is *homotopy*

pro-nilpotent if it is weakly equivalent to the homotopy limit of a small diagram of nilpotent $\mathcal{O}$-algebras.

Some special cases of the Francis-Gaitsgory conjecture have been proved.

If $\mathcal{O}$ is truncated, meaning there exists some large enough n such that $\mathcal{O}[k] = *$ for all $k \geq n$, then the conjecture has been proved by Heuts [27, 6.9]. In this special case, all $\mathcal{O}$-algebras are nilpotent.

For general operad $\mathcal{O}$, Ching-Harper [11, 1.2] proved that adjunction (1.2) induces an equivalence of homotopy categories after restricting to 0-connected objects on both sides, under the assumption that $\mathcal{R}$ and $\mathcal{O}[k]$ for each k are all (-1)-connected. Here we say an $\mathcal{O}$-algebra X is 0-connected if the homotopy groups $\pi_k X$ of the underlying spectra is trivial for all $k \leq 0$.

If an $\mathcal{O}$-algebra X is 0-connected, then X is homotopy pro-nilpotent. This is because the homotopy completion tower of X converges strongly to X [25, 1.12]. Hence, the result of Ching-Harper partially solves the Francis-Gaitsgory conjecture. The general question for homotopy pro-nilpotent objects remains open. This is the reason why the main result of [1] has 0-connected assumptions.

Remark 1.5. In particular, if one take $\mathcal{O}$ to be an E_n operad in $\mathsf{Mod}_{\mathcal{R}}$, the result of Ching-Harper [11] is related to the Koszul duality between E_n-algebras and E_n-coalgebras, see also [2, 12, 19, 32].

The unit of adjunction (1.2) is shown [11] to be weakly equivalent to TQ-completion (Definition 2.1), which is an analog of Bousfield-Kan completion [9] of spaces. Hence, for general operad $\mathcal{O}$, to prove the "unit side" of the Francis-Gaitsgory conjecture amounts to prove for each (cofibrant) homotopy pro-nilpotent $\mathcal{O}$-algebra X, the TQ-completion map $X \to X_{\mathsf{TQ}}^{\wedge}$ is a weak equivalence of $\mathcal{O}$-algebras. The following are results in this direction.

(1) The result of Ching-Harper [11] implies for each 0-connected $\mathcal{O}$-algebra X, $X \to X_{\mathsf{TQ}}^{\wedge}$ is a weak equivalence. Here $\mathcal{R}$ and $\mathcal{O}$ are assumed to be (-1)-connected.

(2) Ching-Harper [10] proved for nilpotent $\mathcal{O}$-algebra X, X is a retract of $X_{\mathsf{TQ}}^{\wedge}$ in the homotopy category of $\mathcal{O}$-algebras.

(3) Schonsheck [39] proved that if X is the homotopy fiber of a fibration $E \to B$ of $\mathcal{O}$-algebras where both E, B are 0-connected, then $X \to X_{\mathsf{TQ}}^{\wedge}$ is a weak equivalence. Here $\mathcal{R}$ and $\mathcal{O}$ are assumed to be (-1)-connected.

However, none of the known results could work for arbitrary homotopy pro-nilpotent $\mathcal{O}$-algebras. In this paper, we take a different approach and work with TQ-localization (Definition 2.6) in place of TQ-completion. Our main result is the following.

Theorem 1.6. *Let X be a homotopy pro-nilpotent $\mathcal{O}$-algebra, then arbitrary fibrant replacement of X in $\mathsf{Alg}_{\mathcal{O}}$ is TQ-local.*

Remark 1.7. The appearance of fibrant replacement is due to our definition (Definition 2.6) that TQ-local $\mathcal{O}$-algebras are required to be fibrant (with respect to the positive flat stable model structure, see Convention 1.1). If such X is already fibrant, then X is TQ-local.

Our result provides positive evidence to the Francis-Gaitsgory conjecture for arbitrary homotopy pro-nilpotent $\mathcal{O}$-algebra X. Indeed, $X \to X_{\mathsf{TQ}}^{\wedge}$ is a weak equivalence if and only if (1) X is TQ-local, and (2) $X \to X_{\mathsf{TQ}}^{\wedge}$ is a TQ-homology equivalence (Proposition 3.6). We have proved the first part for homotopy pro-nilpotent $\mathcal{O}$-algebras, only the second half remains.

As an application of the main result, we obtain the following homotopy pro-nilpotent TQ-Whitehead theorem that simultaneously extends the previously known 0-connected and nilpotent TQ-Whitehead theorems [10, 25].

Theorem 1.8. *A map $X \to Y$ between homotopy pro-nilpotent $\mathcal{O}$-algebras is a weak equivalence if and only if it is a TQ-homology equivalence.*

There are lots of important examples of $\mathcal{O}$-algebras that are homotopy pro-nilpotent but are not nilpotent nor 0-connected. For example, in the context of Goodwillie calculus, the Taylor tower of the identity functor on $\mathsf{Alg}_{\mathcal{O}}$ always converges to homotopy pro-nilpotent $\mathcal{O}$-algebras [25, 1.14]. But those $\mathcal{O}$-algebras are not nilpotent nor 0-connected in general. See also [13, 30, 35, 40] for related discussions.

Organization of the paper. In Section 2, we review the basic setup for TQ-completion and TQ-localization. We also recall the $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion construction, which will play a key role in our proof of the main (Theorem 1.6).

In Section 3, we prove Theorem 1.6 and 1.8. Along the way, we also discuss the relation between TQ-completion and TQ-localization (Proposition 3.6).

Assumptions on the operad $\mathcal{O}$. We work in the category $\mathsf{Alg}_{\mathcal{O}}$ of algebras over an operad $\mathcal{O}$ in $\mathsf{Mod}_{\mathcal{R}}$, the category of $\mathcal{R}$-modules, where $\mathcal{R}$ is a commutative monoid in the category of symmetric spectra. Throughout this paper, we assume that $\mathcal{O}[0] = *$. We also make a technical assumption that the natural maps $\mathcal{R} \to \mathcal{O}[1]$ and $* \to \mathcal{O}[n]$ are flat stable cofibrations in $\mathcal{R}$-modules for each $n \geq 0$; see, for instance, [11, 2.1, 6.12]. This is the same cofibrancy condition that also appears in [11, 25]. This assumption does not limit the usage of our main result since, up to weak equivalence, any operad $\mathcal{O}$ can be replaced by one that satisfies such conditions. We do not need connectivity assumptions on $\mathcal{R}$ and $\mathcal{O}$.

2. TQ-COMPLETION AND TQ-LOCALIZATION

In this section, we review the definitions of TQ-completion and $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion. We also recall the definitions of TQ-localization and the TQ-local homotopy theory on $\mathsf{Alg}_{\mathcal{O}}$.

We first recall the TQ-completion construction [25].

Let Z be a cofibrant $\mathcal{O}$-algebra. Consider the cosimplicial resolution of Z with respect to TQ-homology of the form

$$(2.1) \qquad Z \longrightarrow (UQ)Z \rightrightarrows (UQ)^2 Z \Rrightarrow (UQ)^3 Z \cdots$$

in $\mathsf{Alg}_{\mathcal{O}}$, denoted $Z \to \mathbf{C}(Z)$, with coface maps obtained by iterating the TQ-Hurewicz map $\mathrm{id} \to UQ$ (Definition 1.3) and codegeneracy maps built from the counit map of the adjunction (Q, U) in the usual way. Taking the homotopy limit (over Δ) gives a map [11, 25] of the form

$$(2.2) \qquad Z \to Z_{\mathsf{TQ}}^{\wedge} = \mathrm{holim}_{\Delta}\, \mathbf{C}(Z) \simeq \mathrm{holim}_{\Delta}\, \widetilde{\mathbf{C}(Z)}$$

in $\mathsf{Alg}_{\mathcal{O}}$, where $\widetilde{\mathbf{C}(Z)}$ denotes any functorial fibrant replacement functor $\widetilde{(-)}$ on $\mathsf{Alg}_{\mathcal{O}}$ (obtained, for instance, by running the small object argument with respect to the generating acyclic cofibrations in $\mathsf{Alg}_{\mathcal{O}}$) applied to the cosimplicial diagram $\mathbf{C}(Z)$.

Definition 2.1. Let Z be a cofibrant $\mathcal{O}$-algebra. The TQ-*completion* of Z is the map $Z \to Z_{\mathsf{TQ}}^{\wedge}$ of $\mathcal{O}$-algebras constructed above.

Remark 2.2. The construction of J guarantees that both U and Q preserve cofibrant objects [25, 5.49]. Hence, $UQ(Z) \simeq \mathsf{TQ}(Z)$, $(UQ)^2(Z) \simeq (\mathsf{TQ})^2(Z)$, etc. This shows the TQ-completion construction is homotopically well defined; weakly equivalent cofibrant $\mathcal{O}$-algebras have weakly equivalent TQ-completions.

Next, we recall the $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion construction from [10]. This construction is very similar to the TQ-completion construction. However, it is only defined for M-nilpotent $\mathcal{O}$-algebras.

For each $n \geq 1$, let $\tau_n \mathcal{O}$ denote the operad associated to $\mathcal{O}$ where

$$(2.3) \qquad (\tau_n \mathcal{O})[t] := \begin{cases} \mathcal{O}[t] & \text{for } t \leq n \\ * & \text{otherwise} \end{cases}$$

and consider the associated commutative diagram of operad maps [10]

$$(2.4)$$

where the upper horizontal maps are cofibrations of operads, the left-hand and bottom horizontal maps are the natural truncations, and the vertical maps are weak equivalences of operads; for notational simplicity, here we take $J = J_1$. The corresponding change of operad adjunctions have the form

$$(2.5) \qquad \mathsf{Alg}_{\mathcal{O}} \underset{V_n}{\overset{R_n}{\rightleftarrows}} \mathsf{Alg}_{J_n} \underset{U_n}{\overset{Q_n}{\rightleftarrows}} \mathsf{Alg}_J \qquad \mathsf{Alg}_{\mathcal{O}} \underset{U}{\overset{Q}{\rightleftarrows}} \mathsf{Alg}_J$$

with left adjoints on top, where $R_n = J_n \circ_{\mathcal{O}} (-)$, $Q_n = J \circ_{J_n} (-)$, $Q = J \circ_{\mathcal{O}} (-)$, and V_n, U_n, U denote the indicated forgetful functors; in particular, the adjunction on the right is the composite of the adjunctions on the left.

Let $n \geq 1$ and define $M := n + 1$. Let X be a cofibrant J_n-algebra and consider the cosimplicial resolution of X with respect to $\mathsf{TQ}|_{\mathrm{Nil}_M}$-homology of the form

$$(2.6) \qquad X \longrightarrow (U_n Q_n) X \rightrightarrows (U_n Q_n)^2 X \Rrightarrow (U_n Q_n)^3 X \ \cdots$$

in Alg_{J_n}, denoted $X \to \mathbf{N}(X)$, with coface maps obtained by iterating the $\mathsf{TQ}|_{\mathrm{Nil}_M}$-Hurewicz map $\mathrm{id} \to U_n Q_n$ and codegeneracy maps built from the counit map of the adjunction (Q_n, U_n) in the usual way. Applying the forgetful functor V_n gives the diagram $V_n X \to V_n \mathbf{N}(X)$ of the form

$$(2.7) \qquad V_n X \longrightarrow V_n(U_n Q_n) X \rightrightarrows V_n(U_n Q_n)^2 X \Rrightarrow V_n(U_n Q_n)^3 X \cdots$$

in $\mathsf{Alg}_{\mathcal{O}}$. Taking the homotopy limit (over Δ) gives a map of the form

$$(2.8) \qquad V_n X \to X^{\wedge}_{\mathsf{TQ}|_{\mathrm{Nil}_M}} = \mathrm{holim}_{\Delta} V_n \mathbf{N}(X) \simeq \mathrm{holim}_{\Delta} \widetilde{V_n \mathbf{N}(X)}$$

in $\mathsf{Alg}_{\mathcal{O}}$, where $\widetilde{V_n \mathbf{N}(X)}$ denotes any functorial fibrant replacement functor $\widetilde{(-)}$ on $\mathsf{Alg}_{\mathcal{O}}$ applied to the cosimplicial diagram $V_n \mathbf{N}(X)$.

Definition 2.3. Let Z be an M-nilpotent $\mathcal{O}$-algebra. Choose a cofibrant J_n-algebra X such that Z is weakly equivalent to $V_n X$ as $\mathcal{O}$-algebras. The $\mathsf{TQ}|_{\mathrm{Nil}_M}$-*completion* of Z is defined as the map $V_n X \to X^{\wedge}_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$.

Remark 2.4. The existence of X in Definition 2.3 is explained in [10] (see the discussion following [10, Proposition 2.8]). Moreover, we can make the choice of X to be functorial, although we do not need the extra property in this paper.

Next, we recall the definition of TQ-localization, as well as the TQ-local homotopy theory constructed in [26].

Definition 2.5. Let $f \colon A \to B$ be a map in $\mathsf{Alg}_{\mathcal{O}}$. We say that f is a

- TQ-*equivalence* if f induces a weak equivalence $\mathsf{TQ}(A) \simeq \mathsf{TQ}(B)$ on TQ-homology.
- *strong cofibration* if f is a cofibration between cofibrant objects.
- TQ-*acyclic strong cofibration* if f is a strong cofibration which is also a TQ-equivalence.
- *weak* TQ-*fibration* if f has the right lifting property with respect to every TQ-acyclic strong cofibration.

Definition 2.6. An $\mathcal{O}$-algebra X is called TQ-*local* if (i) X is fibrant in $\mathsf{Alg}_{\mathcal{O}}$, and (ii) every TQ-acyclic strong cofibration $A \to B$ induces a weak equivalence

$$\mathbf{Hom}(A, X) \xleftarrow{\simeq} \mathbf{Hom}(B, X)$$

on mapping spaces in sSet; here we are using the simplicial model structure on $\mathsf{Alg}_{\mathcal{O}}$ (see, for instance, [11, 16, 21, 22, 25]). The TQ-*localization* of X is a map $l \colon X \to L_{\mathsf{TQ}}(X)$ in $\mathsf{Alg}_{\mathcal{O}}$ such that (i) l is a TQ-equivalence, and (ii) $L_{\mathsf{TQ}}(X)$ is TQ-local.

Proposition 2.7. [26, 5.14] *The category* $\mathsf{Alg}_{\mathcal{O}}$ *with the three distinguished classes of maps (i)* TQ-*equivalences, (ii) weak* TQ-*fibrations, and (iii) cofibrations (Convention 1.1), has the structure of a (left) semi-model category in the sense of Goerss-Hopkins [21, 1.1.6].*

For us, the main difference of working with the semi-model structure compared to full model structures is that (1) we often need to work with strong cofibrations instead of arbitrary cofibrations, and (2) the factorization axiom for the semi-model structure only provides functorial fibrant replacements for cofibrant objects.

Remark 2.8. The TQ-local homotopy theory only results in a semi-model structure instead of a full model structure because the model structure on $\mathsf{Alg}_{\mathcal{O}}$ (recall Convention 1.1) is almost never left proper, in general (e.g., associative ring spectra are not left proper); see, for instance, [38, 2.10].

The following proposition will be useful for detecting TQ-local $\mathcal{O}$-algebras.

Proposition 2.9. [26, 5.16] *An $\mathcal{O}$-algebra X is TQ-local if and only if the map $X \to *$ is a weak TQ-fibration.*

Consequently, the functorial factorization of TQ-local semi-model structure gives functorial TQ-localization for cofibrant $\mathcal{O}$-algebras [26, 5.17].

3. Homotopy pro-nilpotent $\mathcal{O}$-algebras are TQ-local

In this section, we discuss the relation between TQ-completion and TQ-localization (Proposition 3.6). After that, we will use similar strategy to study $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion and show that fibrant nilpotent $\mathcal{O}$-algebras are TQ-local (Proposition 3.8). Then, we can prove the main result (Theorem 1.6). As an application, we will also discuss the homotopy pro-nilpotent TQ-Whitehead theorem (Theorem 1.8).

TQ-localization enjoys most nice properties possessed by general (left) Bousfield localizations. However, we do want to be careful since the TQ-local structure (Proposition 2.7) is only a semi-model structure instead of a full model structure. We list some useful properties below. Some good references for general localization techniques include [8, 9, 14, 15, 28, 34, 41].

Proposition 3.1. *(1) A map $X \to Y$ between TQ-local $\mathcal{O}$-algebras is a weak equivalence if and only if it is a TQ-homology equivalence.*

(2) If X and Y are fibrant $\mathcal{O}$-algebras that are weakly equivalent, then X is TQ-local if and only if Y is TQ-local.

(3) The homotopy limit of a small diagram of TQ-local $\mathcal{O}$-algebras is TQ-local.

Proof. (1) and (2) are standard facts about localization; see, for instance, Hirschhorn [28, 3.2.13, 3.2.2]. (3) is also a standard result for left Bousfield localization. We spell out the details here to show the proof still works when the TQ-local homotopy theory only has a semi-model structure.

Note the TQ-local semi-model structure has strictly less fibrations compared to the original model structure on $\mathsf{Alg}_{\mathcal{O}}$. It follows from (1) that the homotopy limit in $\mathsf{Alg}_{\mathcal{O}}$ of a small diagram of TQ-local $\mathcal{O}$-algebras is weakly equivalent to its homotopy limit calculated in the TQ-local semi-model structure. Moreover, the diagram is already objectwise fibrant with respect to the TQ-local semi-model structure by Proposition 2.9. Hence, the result follows from the fibrancy property of homotopy limits in a homotopy theory (in this case, in the TQ-local homotopy theory); see, for instance, Hirschhorn [28, 18.5.2], together with Ching-Harper [11, 8.9] for a discussion of homotopy limits in the context of $\mathcal{O}$-algebras. $\square$

For instance, let $f : X \to Y$ be a map between TQ-local $\mathcal{O}$-algebras. It follows from Proposition 3.1 (3) that the homotopy fiber of f is also TQ-local. This is not

expected to be true, in general, if we replace "TQ-local" with "TQ-complete" (Definition 3.5), and is one of the reasons why TQ-localization is often better behaved than TQ-completion. See [39] for related discussions.

The following proposition gives our first examples of TQ-local $\mathcal{O}$-algebras.

Proposition 3.2. *Let Y be a fibrant object in Alg_J. Then $UY \in \mathsf{Alg}_{\mathcal{O}}$ is TQ-local. Here, U is the right adjoint of adjunction (1.1).*

Proof. By proposition 2.9, it suffices to show $UY \to *$ has the right lifting property with respect to every TQ-acyclic strong cofibration $i : A \to B$. Using the (Q, U) adjunction (1.1), it is equivalent to show $Y \to *$ has the right lifting property with respect to $Qi : QA \to QB$ in Alg_J. This is certainly true since Y is fibrant and $Qi : QA \to QB$ is an acyclic cofibration in Alg_J. $\square$

The following generalization of Proposition 3.2 will be used in our proof of Propositions 3.4 and 3.7.

Proposition 3.3. *Let Y be any object in Alg_J, then every fibrant replacement of UY in $\mathsf{Alg}_{\mathcal{O}}$ is TQ-local.*

Proof. Note different fibrant replacements of an object are always related by a zig-zag of weak equivalences. By Proposition 3.1 (2), it suffices to prove one particular fibrant replacement of UY in $\mathsf{Alg}_{\mathcal{O}}$ is TQ-local. Let Y' be a fibrant replacement of Y in Alg_J. Then UY' is a fibrant replacement of UY. Now the result follows from Proposition 3.2. $\square$

Proposition 3.4. *Let Z be a cofibrant $\mathcal{O}$-algebra. Then the TQ-completion $Z_{\mathsf{TQ}}^{\wedge}$ of Z is TQ-local.*

Proof. We claim that the Δ-shaped diagram $\widetilde{\mathbf{C}(Z)}$ in (2.2) is objectwise TQ-local; i.e., that $\widetilde{\mathbf{C}(Z)^s}$ is TQ-local for each $s \geq 0$. Then we can conclude the homotopy limit $Z_{\mathsf{TQ}}^{\wedge}$ is TQ-local by Proposition 3.1 (3).

To prove the claimed property, consider $Y := \widetilde{Q(UQ)^s Z} \in \mathsf{Alg}_J$, then $UY = (UQ)^{s+1} Z$. Hence, the fibrant replacement $\widetilde{UY} = \widetilde{\mathbf{C}(Z)^s}$ is TQ-local by Proposition 3.3. $\square$

We now discuss the connection between TQ-completion and TQ-localization.

Definition 3.5. Let X be a cofibrant $\mathcal{O}$-algebra. We say X is TQ-*good* if the TQ-completion map $X \to X_{\mathsf{TQ}}^{\wedge}$ is a TQ-equivalence. We say X is TQ-*complete* if the TQ-completion map $X \to X_{\mathsf{TQ}}^{\wedge}$ is a weak equivalence.

Proposition 3.6. *Let X be a cofibrant $\mathcal{O}$-algebra. Then X is TQ-complete if and only if (1) X is TQ-good, and (2) the fibrant replacements of X are TQ-local.*

Proof. The "if direction" follows from Proposition 3.1(1) and 3.4. The "only if direction" follows from Definition 3.5 and Proposition 3.1(2), 3.4. $\square$

In the Introduction, we mentioned that the "unit side" of the Francis-Gaitsgory conjecture amounts to prove for each cofibrant homotopy pro-nilpotent $\mathcal{O}$-algebra X that X is TQ-complete. So far, none of the known results could work for all general homotopy pro-nilpotent objects. In Theorem 1.6, we can prove all homotopy pro-nilpotent objects have TQ-local fibrant replacements. By Proposition

3.6, the remaining open question is that whether cofibrant homotopy pro-nilpotent $\mathcal{O}$-algebras are TQ-good.

We can use similar strategy to show $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion also results in TQ-local $\mathcal{O}$-algebras.

Proposition 3.7. *Let X be a cofibrant J_n-algebra. Then $X^\wedge_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ constructed from $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion is TQ-local.*

Proof. This is similar to the proof of Proposition 3.4. The key observation is that the Δ-shaped diagram $\widetilde{V_n\mathbf{N}(X)}$ in (2.8) is objectwise TQ-local. $\qquad\square$

Now we can prove nilpotent $\mathcal{O}$-algebras are TQ-local up to fibrant replacements.

Proposition 3.8. *Let Z be a nilpotent $\mathcal{O}$-algebra. Then arbitrary fibrant replacement of X in $\mathsf{Alg}_{\mathcal{O}}$ is TQ-local.*

Proof. Let Z be M-nilpotent for some $M \geq 2$. Choose a cofibrant J_n-algebra X such that Z is weakly equivalent to $V_n X$ as $\mathcal{O}$-algebras (Remark 2.4). It is proved in Ching-Harper [10, 2.12] that the map $V_n X \to X^\wedge_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ is a weak equivalence. Since X is weakly equivalent to $V_n X$ and $V_n X$ is weakly equivalent to the TQ-local $\mathcal{O}$-algebra $X^\wedge_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ (Proposition 3.7), the result follows from Proposition 3.1(2). $\quad\square$

Proof of Theorem 1.6. By definition, the homotopy pro-nilpotent $\mathcal{O}$-algebra X is weakly equivalent to the homotopy limit of a small diagram of nilpotent $\mathcal{O}$-algebras. By taking objectwise fibrant replacements for the small diagram, X is weakly equivalent to the homotopy limit of a small diagram of TQ-local $\mathcal{O}$-algebras (Proposition 3.8). Now the result follows from Proposition 3.1(2)(3). $\qquad\square$
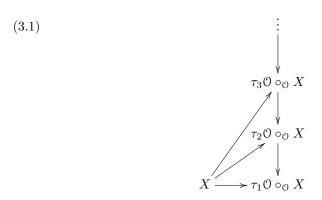
As a corollary, we obtain the homotopy pro-nilpotent TQ-Whitehead theorem.

Proof of Theorem 1.8. This follows from Theorem 1.6 and Proposition 3.1(1). $\quad\square$

Previously, TQ-Whitehead theorems have been established for 0-connected and nilpotent $\mathcal{O}$-algebras separately [10, 25]. However, if one consider a map $X \to Y$ from a 0-connected $\mathcal{O}$-algebra to a nilpotent $\mathcal{O}$-algebra, then none of those TQ-Whitehead theorems could apply. Now, TQ-Whitehead theorem becomes applicable to $X \to Y$ since 0-connected $\mathcal{O}$-algebras and nilpotent $\mathcal{O}$-algebras are all homotopy pro-nilpotent [25, 1.12].

We also want to point out that Goodwillie calculus [23, 31] provides a class of naturally occurring examples that are homotopy pro-nilpotent but are, in general, not 0-connected nor nilpotent.

As explained in [25, 1.14], up to weak equivalence, the Taylor tower for a cofibrant $\mathcal{O}$-algebra X has the following form:

(3.1)

$$
\begin{array}{c}
\vdots \\
\downarrow \\
\tau_3\mathcal{O} \circ_{\mathcal{O}} X \\
\nearrow \qquad \downarrow \\
\tau_2\mathcal{O} \circ_{\mathcal{O}} X \\
\nearrow \qquad \downarrow \\
X \longrightarrow \tau_1\mathcal{O} \circ_{\mathcal{O}} X
\end{array}
$$

where $\tau_k\mathcal{O}$ is the operad defined in (2.3). By definition, $\tau_k\mathcal{O} \circ_{\mathcal{O}} X$ regarded as an $\mathcal{O}$-algebra is $(k+1)$-nilpotent. Therefore, the Taylor tower of the identity functor on $\mathsf{Alg}_{\mathcal{O}}$ always converges to homotopy pro-nilpotent $\mathcal{O}$-algebras. Also see [13, 30, 35, 40] for related discussions.

References

[1] A. Amabel. Poincare/Koszul duality for general operads. *arXiv preprint arXiv:1910.09076*, 2019.

[2] D. Ayala and J. Francis. Zero-pointed manifolds. *J. Inst. Math. Jussieu*, 20(3):785–858, 2021.

[3] M. Basterra. André-Quillen cohomology of commutative S-algebras. *J. Pure Appl. Algebra*, 144(2):111–143, 1999.

[4] M. Basterra and M. A. Mandell. Homology and cohomology of E_∞ ring spectra. *Math. Z.*, 249(4):903–944, 2005.

[5] M. Basterra and M. A. Mandell. Homology of E_n ring spectra and iterated THH. *Algebr. Geom. Topol.*, 11(2):939–981, 2011.

[6] C. Berger and I. Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003.

[7] D. Blanc, M. W. Johnson, and J. M. Turner. Higher homotopy operations and André-Quillen cohomology. *Adv. Math.*, 230(2):777–817, 2012.

[8] A. K. Bousfield. The localization of spaces with respect to homology. *Topology*, 14:133–150, 1975.

[9] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972.

[10] M. Ching and J. E. Harper. A nilpotent Whitehead theorem for TQ-homology of structured ring spectra. *Tbilisi Math. J.*, 11:69–79, 2018.

[11] M. Ching and J. E. Harper. Derived Koszul duality and TQ-homology completion of structured ring spectra. *Adv. Math.*, 341:118–187, 2019.

[12] M. Ching and P. Salvatore. Koszul duality for topological En-operads. *arXiv preprint arXiv:2002.03878*, 2020.

[13] D. Clark. On the Goodwillie derivatives of the identity in structured ring spectra. *arXiv preprint arXiv:2004.02812*, 2020.

[14] E. Dror Farjoun. *Cellular spaces, null spaces and homotopy localization*, volume 1622 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

[15] W. G. Dwyer. Localizations. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 3–28. Kluwer Acad. Publ., Dordrecht, 2004.

[16] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[17] J. Francis and D. Gaitsgory. Chiral Koszul duality. *Selecta Math. (N.S.)*, 18(1):27–87, 2012.

[18] B. Fresse. Lie theory of formal groups over an operad. *J. Algebra*, 202(2):455–511, 1998.

[19] B. Fresse. Koszul duality of E_n-operads. *Selecta Math. (N.S.)*, 17(2):363–434, 2011.

[20] P. G. Goerss. On the André-Quillen cohomology of commutative $\mathbf{F}_2$-algebras. *Astérisque*, 186:169, 1990.

[21] P. G. Goerss and M. J. Hopkins. Moduli problems for structured ring spectra. *preprint*, 2005. Available at `http://hopf.math.purdue.edu`.

[22] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[23] T. G. Goodwillie. Calculus. III. Taylor series. *Geom. Topol.*, 7:645–711, 2003.

[24] J. E. Harper. Bar constructions and Quillen homology of modules over operads. *Algebr. Geom. Topol.*, 10(1):87–136, 2010.

[25] J. E. Harper and K. Hess. Homotopy completion and topological Quillen homology of structured ring spectra. *Geom. Topol.*, 17(3):1325–1416, 2013.

[26] J. E. Harper and Y. Zhang. Topological Quillen localization of structured ring spectra. *Tbilisi Math. J.*, 12:67–89, 2019.

[27] G. Heuts. Goodwillie Approximations to Higher Categories. *Mem. Amer. Math. Soc.*, 272(1333), 2021.

[28] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[29] I. Kriz and J. P. May. Operads, algebras, modules and motives. *Astérisque*, (233):iv+145pp, 1995.

[30] N. J. Kuhn. Localization of André-Quillen-Goodwillie towers, and the periodic homology of infinite loopspaces. *Adv. Math.*, 201(2):318–378, 2006.

[31] N. J. Kuhn. Goodwillie towers and chromatic homotopy: an overview. In *Proceedings of the Nishida Fest (Kinosaki 2003)*, volume 10 of *Geom. Topol. Monogr.*, pages 245–279. Geom. Topol. Publ., Coventry, 2007.

[32] J. Lurie. Higher algebra. , 2012. Available at: `http://www.math.harvard.edu/~lurie/`.

[33] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin, 1972. Lectures Notes in Mathematics, Vol. 271.

[34] J. P. May and K. Ponto. *More concise algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories.

[35] L. A. Pereira. Goodwillie calculus in the category of algebras over a spectral operad. , 2013. Available at: `http://math.mit.edu/~luisalex/`.

[36] D. Quillen. On the (co-) homology of commutative rings. In *Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968)*, pages 65–87. Amer. Math. Soc., Providence, R.I., 1970.

[37] C. Rezk. *Spaces of Algebra Structures and Cohomology of Operads*. PhD thesis, MIT, 1996. Available at `http://www.math.uiuc.edu/~rezk/`.

[38] C. Rezk. Every homotopy theory of simplicial algebras admits a proper model. *Topology Appl.*, 119(1):65–94, 2002.

[39] N. Schonsheck. Fibration theorems for TQ-completion of structured ring spectra. *arXiv preprint arXiv:2002.00038*, 2020.

[40] N. Schonsheck. TQ-completion and the Taylor tower of the identity functor. *arXiv preprint arXiv:2011.00570*, 2020.

[41] D. Sullivan. *Geometric topology. Part I*. Massachusetts Institute of Technology, Cambridge, Mass., 1971. Localization, periodicity, and Galois symmetry, Revised version.

Department of Mathematics, Nankai University, 94 Weijin Road, Nankai District, Tianjin, P.R.China 300071

Email address: `zhang.4841@osu.edu`