

# THE $p$ -LOCAL COHOMOLOGY OF $BPU_n$ IN DIMENSIONS LESS THAN $2p + 5$

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ABSTRACT. Let  $PU_n$  denote the projective unitary group of rank  $n$  and  $BPU_n$  be its classifying space. For an odd prime  $p$ , we extend previous results to a complete description of  $H^s(BPU_n; \mathbb{Z})_{(p)}$  for  $s < 2p + 5$  by showing that the  $p$ -primary subgroups of  $H^s(BPU_n; \mathbb{Z})$  are trivial for  $s = 2p + 3$  and  $s = 2p + 4$ .

## 1. INTRODUCTION

Let  $U_n$  denote the group of  $n \times n$  unitary matrices. The unit circle  $S^1$  can be viewed as the normal subgroup of scalar matrices of  $U_n$ . We let  $PU_n$  denote the quotient group of  $U_n$  by  $S^1$ , and  $BPU_n$  be the classifying space of  $PU_n$ . In this paper we consider the cohomology of  $BPU_n$  with coefficients in  $\mathbb{Z}_{(p)}$ , where  $p > 2$  is a prime number.

**A review of the literature.** The ordinary and generalized cohomology of  $BPU_n$  for special  $n$  has been the subject of various works such as Kono-Mimura [15], Kameko-Yagita [14], Toda [18], and Vavpetič-Viruel [20]. Vezzosi [21] and Vistoli [22] studied the Chow ring of the classifying space (in the sense of Totaro [19]) of  $BPGL_3(\mathbb{C})$  and  $BPGL_p(\mathbb{C})$  for  $p$  an odd prime, respectively. Much of their results applies to the ordinary cohomology of  $BPU_p$ .

None of the works above dealt with  $H^*(BPU_n; \mathbb{Z})$ , the ordinary cohomology of  $BPU_n$  with coefficients in  $\mathbb{Z}$ , for  $n$  not a prime number. The first named author considered  $H^*(BPU_n; \mathbb{Z})$ , as well as the Chow ring of  $BPGL_n(\mathbb{C})$  for an arbitrary  $n$  in [10], [12] and [13]. In particular, in [12], the first named author determined the ring structure of  $H^*(BPU_n; \mathbb{Z})$  in dimensions less than or equal to 10.

Other related works include Duan [6], in which the integral cohomology of  $PU_n$  is fully determined, and Crowley-Gu [5], in which the image of the canonical map  $H^*(BPU_n; \mathbb{Z}) \rightarrow H^*(BU_n; \mathbb{Z})$  is studied.

The cohomology of  $BPU_n$  plays significant roles in the study of the topological period-index problem ([1], [2], [9] and [11]), and in the study of anomalies in partial physics ([4], [8]).

**Notations.** Throughout the rest of this paper,  $H^*(-)$  denotes  $H^*(-; \mathbb{Z})$ . For an abelian group  $A$  and a prime number  $p$ , let  $A_{(p)}$  be the localization of  $A$  at  $p$ , and let  ${}_pA$  denotes the  $p$ -primary subgroup of  $A$ , i.e., the subgroup of  $A$  of all torsion elements with torsion order a power of  $p$ . In particular, we have a canonical isomorphism  ${}_pH^*(-) \cong {}_p[H^*(-)_{(p)}]$ , and we will not distinguish the two throughout this paper. Tensor products of  $\mathbb{Z}_{(p)}$ -modules are always taken over  $\mathbb{Z}_{(p)}$ .

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**The main theorem and some remarks.** We review a basic fact on the cohomology of  $BPU_n$ . Consider the short exact sequence of Lie groups

$$1 \rightarrow \mathbb{Z}/n \rightarrow SU_n \rightarrow PSU_n \simeq PU_n \rightarrow 1,$$

which induces a fiber sequence of their classifying spaces

$$(1.1) \quad B(\mathbb{Z}/n) \rightarrow BSU_n \rightarrow BPU_n$$

The space  $B(\mathbb{Z}/n) = K(\mathbb{Z}/n, 1)$  is  $\mathbb{Q}$ -locally contractible. Then, from the fiber sequence (1.1) we deduce that the canonical map  $BSU_n \rightarrow BPU_n$  is a  $p$ -local equivalence and induces an isomorphism of  $\mathbb{Q}$ -algebras

$$(1.2) \quad H^*(BPU_n; \mathbb{Q}) \cong H^*(BSU_n; \mathbb{Q}) = \mathbb{Q}[c_2, c_3, \dots, c_n], \quad |c_i| = 2i.$$

If  $p \nmid n$ , the space  $B(\mathbb{Z}/n)$  is  $p$ -locally contractible and we have an isomorphism of  $\mathbb{Z}_{(p)}$ -algebras

$$(1.3) \quad H^*(BPU_n)_{(p)} \cong H^*(BSU_n)_{(p)} = \mathbb{Z}_{(p)}[c_2, c_3, \dots, c_n], \quad \text{for } p \nmid n.$$

We conclude that the  $p$ -local cohomology of  $BPU_n$  is torsion-free in this case.

Therefore, to determine the  $\mathbb{Z}_{(p)}$ -module structure of  $H^s(BPU_n)_{(p)}$ , it suffices to consider the  $p$ -primary subgroup  ${}_pH^s(BPU_n)_{(p)}$  for  $p \mid n$ .

*Remark 1.1.* In the case of Chow rings, Vezzosi [21] proved the stronger result that all torsion classes in the Chow ring of  $BPGL_n(\mathbb{C})$  are  $n$ -torsion. In principal, the proof of this result should work for the case of the integral cohomology of  $BPU_n$  as well, but we do not pursue it in this paper.

The main theorem of this paper is as follows.

**Theorem 1.** *Let  $p > 2$  be a prime number,  $r > 0$  an integer, and  $n = p^r m$  for a positive integer  $m$  co-prime to  $p$ . Then the  $p$ -primary subgroup of  $H^s(BPU_n)$  in dimensions less than  $2p + 5$  is as follows:*

$${}_pH^s(BPU_n) \cong \begin{cases} \mathbb{Z}/p^r, & s = 3, \\ \mathbb{Z}/p, & s = 2p + 2, \\ 0, & s < 2p + 5, \quad s \neq 3, 2p + 2. \end{cases}$$

*Remark 1.2.* By the discussion preceding Remark 1.1, Theorem 1 completely determines  $H^s(BPU_n)_{(p)}$  for  $0 \leq s < 2p + 5$ , for both  $p \mid n$  and  $p \nmid n$ .

For  $s \leq 3$ , the groups  ${}_pH^s(BPU_n)$  are well known and are part of Theorem 1.1 of [12]. For  $3 < s < 2p + 2$ , they are given in Theorem 1.2 of [12]. Therefore, what remains to show is

$$(1.4) \quad {}_pH^{2p+2}(BPU_n) \cong \mathbb{Z}/p, \quad {}_pH^{2p+3}(BPU_n) = {}_pH^{2p+4}(BPU_n) = 0.$$

*Remark 1.3.* For  $p = 2$ , it was shown by the first named author [12] that the 2-torsion subgroup of  $H^s(BPU_n)$  in dimension  $s = 2p + 3 = 7$  is  $\mathbb{Z}/2$  if  $n \equiv 2 \pmod{4}$ , and is 0 otherwise. In particular, Theorem 1 does not generalize to the case  $p = 2$ .

*Remark 1.4.* For  $p = 3$ , (1.4) follows immediately from the computation in [12] of  $H^s(BPU_n)$  in dimensions 8, 9 and 10.

**Organization of the paper.** In Section 2, we discuss some preliminary results of the Serre spectral sequence  ${}^UE$  associated to the fiber sequence  $U : BU_n \rightarrow BPU_n \rightarrow K(\mathbb{Z}, 3)$ . This will be our main tool for computing the  $p$ -local cohomology of  $BPU_n$ . We will also show that (1.4) can be deduced from Theorem 1.2 of [12] and Proposition 2.10, which says that certain chain complex  $\mathcal{M}$  constructed from the differentials in  ${}^UE$  is exact.

In Section 3, we prove Proposition 2.10. The proof is based on the explicit computation of some relevant differentials in  ${}^UE$ . This section finishes our proof of Theorem 1.

## 2. THE SPECTRAL SEQUENCES

Let  $p \geq 3$  be a prime number. By the discussions in Section 1, especially (1.3) and Remark 1.1, we assume  $p \nmid n$  for the rest of this paper.

**The Serre spectral sequence  ${}^UE$ .** We follow the strategy employed in [12] to compute the cohomology of  $BPU_n$ . The short exact sequence of Lie groups

$$1 \rightarrow S^1 \rightarrow U_n \rightarrow PU_n \rightarrow 1$$

induces a fiber sequence of their classifying spaces

$$BS^1 \rightarrow BU_n \rightarrow BPU_n.$$

Notice that  $BS^1$  is of the homotopy type of the Eilenberg-Mac Lane space  $K(\mathbb{Z}, 2)$  and indeed we obtain another fiber sequence

$$(2.1) \quad U : BU_n \rightarrow BPU_n \xrightarrow{\chi} K(\mathbb{Z}, 3).$$

*Remark 2.1.* In general, it is not always possible to obtain a fiber sequence of the form  $F \rightarrow E \rightarrow B$  from a fiber sequence  $\Omega B \rightarrow F \rightarrow E$ . See Ganea [7] for more.

We will use the Serre spectral sequence associated to the last fiber sequence to compute the cohomology of  $BPU_n$ . For notational convenience, we denote this spectral sequence by  ${}^UE$ . The  $E_2$  page of  ${}^UE$  has the form

$${}^UE_2^{s,t} = H^s(K(\mathbb{Z}, 3); H^t(BU_n)) \implies H^{s+t}(BPU_n).$$

In principle, Cartan and Serre [3] determined the cohomology of  $K(A, n)$  for all finitely generated abelian groups  $A$ . Also see Tamanoi [17] for a nice treatment.

We summarize the  $p$ -local cohomology of  $K(\mathbb{Z}, 3)$  in low dimensions as follows.

**Proposition 2.2.** *Let  $p > 2$  be a prime. In degrees up to  $2p + 5$ , we have*

$$(2.2) \quad H^s(K(\mathbb{Z}, 3))_{(p)} = \begin{cases} \mathbb{Z}_{(p)}, & s = 0, 3, \\ \mathbb{Z}/p, & s = 2p + 2, 2p + 5, \\ 0, & s < 2p + 5, s \neq 0, 3, 2p + 2. \end{cases}$$

where  $x_1, y_{p,0}, x_1y_{p,0}$  are generators on degree  $3, 2p + 2, 2p + 5$  respectively.

Here we use the same notations for the generators as in [12]. Sometimes we abuse notations and let  $x_1, y_{p,0}$  denote  $\chi^*(x_1), \chi^*(y_{p,0})$ , where  $\chi : BPU_n \rightarrow K(\mathbb{Z}, 3)$  is defined in (2.1). For instance, we have

**Proposition 2.3** (Theorem 1.2, [12]). *Let  $p$  be a prime. In  $H^{2p+2}(BPU_n)$ , we have  $y_{p,0} \neq 0$  of order  $p$  when  $p \nmid n$ , and  $y_{p,0} = 0$  otherwise. Furthermore, the  $p$ -torsion subgroup of  $H^k(BPU_n)$  is 0 for  $3 < k < 2p + 2$ .*

Also recall

$$(2.3) \quad H^*(BU_n) = \mathbb{Z}[c_1, c_2, \dots, c_n], \quad |c_i| = 2i.$$

In particular,  $H^*(BU_n)$  is torsion-free. We have

$${}^U E_2^{s,t} \cong H^s(K(\mathbb{Z}, 3)) \otimes H^t(BU_n).$$

**The auxiliary fiber sequences and spectral sequences.** To determine some of the differentials in  ${}^U E$ , we consider two more fiber sequences.

Let  $T^n$  be the maximal torus of  $U^n$  with the inclusion denoted by

$$\psi : T^n \rightarrow U_n.$$

Passing to quotients over  $S^1$ , we have another inclusion of maximal torus

$$\psi' : PT^n \rightarrow PU_n.$$

The quotient map  $T^n \rightarrow PT^n$  fits in an exact sequence of Lie groups

$$1 \rightarrow S^1 \rightarrow T^n \rightarrow PT^n \rightarrow 1,$$

which induces a fiber sequence

$$T : BT^n \rightarrow BPT^n \rightarrow K(\mathbb{Z}, 3).$$

Notice that we have

$$(2.4) \quad H^*(BT^n) = \mathbb{Z}[v_1, v_2, \dots, v_n], \quad |v_i| = 2.$$

The next fiber sequence is simply the path fibration for the space  $K(\mathbb{Z}, 3)$

$$K : K(\mathbb{Z}, 2) \rightarrow * \rightarrow K(\mathbb{Z}, 3)$$

where  $*$  denotes a contractible space. We denote their associated Serre spectral sequences as  ${}^T E$  and  ${}^K E$  respectively.

We denote the corresponding differentials of  ${}^U E$ ,  ${}^T E$ , and  ${}^K E$  by  ${}^U d_*^{*,*}$ ,  ${}^T d_*^{*,*}$ , and  ${}^K d_*^{*,*}$ , respectively, if there are risks of ambiguity. Otherwise, we simply denote the differentials by  $d_*^{*,*}$ .

These fiber sequences fit into the following homotopy commutative diagram:

$$(2.5) \quad \begin{array}{ccccccc} K : & & K(\mathbb{Z}, 2) & \longrightarrow & * & \longrightarrow & K(\mathbb{Z}, 3) \\ & \downarrow \Phi & \downarrow B\varphi & & \downarrow & & \downarrow = \\ T : & & BT^n & \longrightarrow & BPT^n & \longrightarrow & K(\mathbb{Z}, 3) \\ & \downarrow \Psi & \downarrow B\psi & & \downarrow B\psi' & & \downarrow = \\ U : & & BU_n & \longrightarrow & BPU_n & \longrightarrow & K(\mathbb{Z}, 3) \end{array}$$

Here, the map  $B\varphi : K(\mathbb{Z}, 2) \simeq BS^1 \rightarrow BT^n$  is the de-loop of the diagonal map  $S^1 \rightarrow T^n$ . The induced homomorphism between cohomology rings is as follows:

$$B\varphi^* : H^*(BT^n) = \mathbb{Z}[v_1, v_2, \dots, v_n] \rightarrow H^*(BS^1) = \mathbb{Z}[v], \quad v_i \mapsto v.$$

The map  $B\psi : BT^n \rightarrow BU_n$  induces the injective ring homomorphism

$$\begin{aligned} B\psi^* : H^*(BU_n) = \mathbb{Z}[c_1, \dots, c_n] &\rightarrow H^*(BT^n) = \mathbb{Z}[v_1, \dots, v_n], \\ c_i &\mapsto \sigma_i(v_1, \dots, v_n), \end{aligned}$$

where  $\sigma_j(t_1, t_2, \dots, t_n)$  be the  $j$ th elementary symmetric polynomial in variables  $t_1, t_2, \dots, t_n$ :

$$\begin{aligned}
 (2.6) \quad & \sigma_0(t_1, t_2, \dots, t_n) = 1, \\
 & \sigma_1(t_1, t_2, \dots, t_n) = t_1 + t_2 + \dots + t_n, \\
 & \sigma_2(t_1, t_2, \dots, t_n) = \sum_{i < j} t_i t_j, \\
 & \vdots \\
 & \sigma_p(t_1, t_2, \dots, t_n) = t_1 t_2 \dots t_n.
 \end{aligned}$$

We will use the associated maps of spectral sequences to compute the differentials in  ${}^U E$ . This is possible because we have a good understanding of the corresponding differentials in  ${}^T E$  and  ${}^K E$ . In particular, we have the following results.

**Proposition 2.4** ([12], Corollary 2.16). *The higher differentials of  ${}^K E_*^{*,*}$  satisfy*

$$\begin{aligned}
 d_3(v) &= x_1, \\
 d_{2p-1}(x_1 v^{l p^e - 1}) &= v^{l p^e - 1 - (p-1)} y_{p,0}, \quad e > 0, \quad \gcd(l, p) = 1, \\
 d_r(x_1) &= d_r(y_{p,0}) = 0, \quad \text{for all } r,
 \end{aligned}$$

and the Leibniz rule.

*Remark 2.5.* Proposition 2.4 is a special case of Corollary 2.16, [12]. Here, we take the opportunity to correct a typo in the original Corollary 2.16, [12], where the condition  $k \geq e$  should be replaced by  $e > k$ .

**Proposition 2.6** ([12], Proposition 3.2). *The differential  ${}^T d_r^{*,*}$ , is partially determined as follows:*

$$(2.7) \quad {}^T d_r^{*,2t}(v_i^t \xi) = (B\pi_i)^*({}^K d_r^{*,2t}(v^t \xi)),$$

where  $\xi \in {}^T E_r^{*,0}$ , a quotient group of Proposition 3.2  $H^*(K(\mathbb{Z}, 3))$ , and  $\pi_i : T^n \rightarrow S^1$  is the projection of the  $i$ th diagonal entry. In plain words,  ${}^T d_r^{*,2t}(v_i^t \xi)$  is simply  ${}^K d_r^{*,2t}(v^t \xi)$  with  $v$  replaced by  $v_i$ .

*Remark 2.7.* Here we correct another typo in the original Proposition 3.2 in [12], in which “ $\xi \in {}^T E_r^{0,*}$ ” should be replaced by “ $\xi \in {}^T E_r^{*,0}$ ”.

**Proposition 2.8** ([12], Proposition 3.3). (1) *The differential  ${}^T d_3^{0,t}$  is given by the “formal divergence”*

$$\nabla = \sum_{i=1}^n (\partial/\partial v_i) : H^t(BT^n; R) \rightarrow H^{t-2}(BT^n; R),$$

in such a way that  ${}^T d_3^{0,*} = \nabla(-) \cdot x_1$ . For any ground ring  $R = \mathbb{Z}$  or  $\mathbb{Z}/m$  for any integer  $m$ .

(2) *The spectral sequence degenerates at  ${}^T E_4^{0,*}$ . Indeed, we have  ${}^T E_\infty^{0,*} = {}^T E_4^{0,*} = \text{Ker } {}^T d_3^{0,*} = \mathbb{Z}[v_1 - v_n, \dots, v_{n-1} - v_n]$ .*

**Corollary 2.9** ([12], Corollary 3.4).

$${}^U d_3^{0,*}(c_k) = \nabla(c_k)x_1 = (n - k + 1)c_{k-1}x_1.$$

**Computations in the spectral sequence  ${}^U E$ .** To study  ${}_p H^*(BPU_n)$ , it suffices to look at the  $p$ -localized spectral sequence, where the  $E_2$  page becomes

$$(2.8) \quad ({}^U E_2^{s,t})_{(p)} = H^s(K(\mathbb{Z}, 3))_{(p)} \otimes H^t(BU_n) = H^s(K(\mathbb{Z}, 3)) \otimes H^t(BU_n)_{(p)}.$$

By abuse of notation, for the rest of this paper, we let  ${}^U E$ ,  ${}^T E$  and  ${}^K E$  denote the corresponding  $p$ -localized Serre spectral sequences.

By Proposition 2.2 and (2.3), in the range  $s \leq 2p + 5$ , the only cases in which  ${}^U E_2^{s,t}$  could be nonzero are when  $s = 0, 3, 2p + 2, 2p + 5$  and  $t$  is even.

To simplify the notations, we let

$$M^0 = {}^U E_2^{0,2p+2}, \quad M^1 = {}^U E_2^{3,2p}, \quad M^2 = {}^U E_2^{2p+2,2}, \quad M^3 = {}^U E_2^{2p+5,0}.$$

Inspection of the degrees shows that  ${}^U E_*^{3,2p}$  could only receive  $d_3$  differential and support  $d_{2p-1}$  differential. Similarly,  ${}^U E_*^{2p+2,2}$  could only receive  $d_{2p-1}$  differential and support  $d_3$  differential. In addition, all  $d_2$ 's are trivial and therefore we have  ${}^U E_2^{*,*} = {}^U E_3^{*,*}$ .

We let  $\delta^0$  be the map

$$\delta^0 : M^0 = {}^U E_3^{0,2p+2} \xrightarrow{d_3} {}^U E_3^{3,2p} = M^1.$$

We let  $\delta^1$  be the composition

$$\delta^1 : M^1 = {}^U E_3^{3,2p} \rightarrow {}^U E_3^{3,2p} / \text{Im } d_3 = {}^U E_{2p-1}^{3,2p} \xrightarrow{d_{2p-1}} {}^U E_{2p-1}^{2p+2,2} = \text{Ker } d_3 \subset M^2.$$

We let  $\delta^2$  be the map

$$\delta^2 : M^2 = {}^U E_3^{2p+2,2} \xrightarrow{d_3} {}^U E_3^{2p+5,0} = M^3.$$

One immediately sees that

$$M^0 \xrightarrow{\delta^0} M^1 \xrightarrow{\delta^1} M^2 \xrightarrow{\delta^2} M^3$$

is a chain complex of  $\mathbb{Z}_{(p)}$ -modules, which we denote by  $\mathcal{M}$ . We will show later that Theorem 1 is a consequence of the following

**Proposition 2.10.** *The chain complex  $\mathcal{M}$  is exact.*

*Proof of Theorem 1 assuming Proposition 2.10.* First, we prove

$${}_p H^{2p+2}(BPU_n) \cong \mathbb{Z}/p.$$

By Proposition 2.3,  $y_{p,0} \in {}^U E_2^{2p+2,0}$  survives to a nonzero element in  $H^{2p+2}(BPU_n)$  of order  $p$ . Therefore, we have

$${}^U E_\infty^{2p+2,0} = {}^U E_2^{2p+2,0} \cong \mathbb{Z}/p.$$

Since the only nontrivial entries in  ${}^U E_2^{*,*}$  of total degree  $2p + 2$  are  ${}^U E_2^{2p+2,0}$  and  ${}^U E_2^{0,2p+2}$ , we have a short exact sequence of  $\mathbb{Z}_{(p)}$ -modules

$$0 \rightarrow {}^U E_\infty^{2p+2,0} \rightarrow H^{2p+2}(BPU_n)_{(p)} \rightarrow {}^U E_\infty^{0,2p+2} \rightarrow 0.$$

Since  ${}^U E_\infty^{0,2p+2} \subset {}^U E_2^{0,2p+2}$  is a free  $\mathbb{Z}_{(p)}$ -module, the above short exact sequence splits and we have

$$H^{2p+2}(BPU_n)_{(p)} \cong {}^U E_\infty^{2p+2,0} \oplus {}^U E_\infty^{0,2p+2},$$

from which we deduce

$${}_p H^{2p+2}(BPU_n) \cong {}^U E_\infty^{2p+2,0} \cong \mathbb{Z}/p.$$

Next, we prove

$${}_p H^{2p+3}(BPU_n) = H^{2p+3}(BPU_n)_{(p)} = 0.$$

The exactness of  $\mathcal{M}$  at  $M^1$  implies  ${}^U E_\infty^{3,2p} = 0$ . On the other hand,  ${}^U E_2^{3,2p}$  is the only nontrivial entry in  ${}^U E_2^{*,*}$  of total degree  $2p+3$ . Hence, we have

$${}_p H^{2p+3}(BPU_n) \subset H^{2p+3}(BPU_n)_{(p)} = {}^U E_\infty^{3,2p} = 0.$$

Finally, we prove

$${}_p H^{2p+4}(BPU_n) = 0.$$

The exactness of  $\mathcal{M}$  at  $M^2$  implies  ${}^U E_\infty^{2p+2,2} = 0$ . Since  ${}^U E_2^{0,2p+4}$  and  ${}^U E_2^{2p+2,2}$  are the only nontrivial entries in  ${}^U E_2^{*,*}$  of total degree  $2p+4$ , we have

$$H^{2p+4}(BPU_n)_{(p)} \cong {}^U E_\infty^{0,2p+4},$$

which is torsion-free. In particular, we have  ${}_p H^{2p+4}(BPU_n) = 0$ . □

The proof of Proposition 2.10 occupies Section 3.

### 3. THE PROOF OF PROPOSITION 2.10

From (2.8), we can write out the  $\mathbb{Z}_{(p)}$ -modules  $M^0, M^1, M^2, M^3$  more explicitly:

$$M^0 = H^0(K(\mathbb{Z}, 3)) \otimes H^{2p+2}(BU_n)_{(p)} \cong H^{2p+2}(BU_n)_{(p)}$$

is the free  $\mathbb{Z}_{(p)}$ -module generated by monomials in  $c_1, \dots, c_{p+1}$  in dimension  $2p+2$ , and

$$M^1 = H^3(K(\mathbb{Z}, 3)) \otimes H^{2p}(BU_n)_{(p)} \cong H^{2p}(BU_n)_{(p)}$$

is the free  $\mathbb{Z}_{(p)}$ -module generated by elements of the form  $cx_1$  where  $c$  is a monomial in  $c_1, \dots, c_p$  in dimension  $2p$ . Furthermore, we have

$$M^2 = H^{2p+2}(K(\mathbb{Z}, 3)) \otimes H^2(BU_n)_{(p)} = \mathbb{Z}_{(p)}\{c_1 y_{p,0}\}/p \cong \mathbb{Z}/p$$

and

$$M^3 = H^{2p+5}(K(\mathbb{Z}, 3)) \otimes H^0(BU_n)_{(p)} = \mathbb{Z}_{(p)}\{x_1 y_{p,0}\}/p \cong \mathbb{Z}/p.$$

**The exactness of  $\mathcal{M}$  at  $M^2$ .** To determine the image of  $\delta^1$ , we need the following

**Lemma 3.1.** *Let  $p$  be a prime number,  $n > 0$  be a multiple of  $p$ . Then  $p \nmid \binom{n-1}{p-1}$ .*

*Proof.* This follows from the simple observation that the expression

$$\binom{n-1}{p-1} = \frac{(n-1)(n-2) \cdots (n-p+1)}{(p-1)(p-2) \cdots 1}$$

is a multiple of elements of  $\mathbb{Z}_{(p)}$  co-prime to  $p$ . □

**Lemma 3.2.** *In the spectral sequence  ${}^T E$ , we have*

$$(3.1) \quad \begin{cases} v_n^k x_1 \in \text{Im } {}^T d_3, & 0 \leq k \leq p-2 \text{ or } k = p, \\ {}^T d_{2p-1}^{3,*}(v_n^{p-1} x_1) = y_{p,0}. \end{cases}$$

*Proof.* When  $p \nmid k+1$ , the first formula in Proposition 2.4 together with Proposition 2.6 imply that

$$v_n^k x_1 = \frac{1}{k+1} {}^T d_3(v_n^{k+1})$$

is in the image of  ${}^T d_3$ . This completes the proof for the case  $0 \leq k \leq p-2$  or  $k = p$ .

The remaining case is proved by applying the second formula in Proposition 2.4, taking  $e = l = 1$ , and then Proposition 2.6.  $\square$

**Lemma 3.3.** *The map  $\delta^1 : M^1 \rightarrow M^2 \cong \mathbb{Z}/p$  is surjective.*

*Proof.* Recall the morphism of fiber sequences  $\Psi$  introduced in (2.5), and the induced morphism  $\Psi^* : {}^U E \rightarrow {}^T E$  of spectral sequences.

For  $1 \leq i \leq n$ , let  $v'_i = v_i - v_n$ . It follows from (2) of Proposition 2.8 that the  $v'_i$ 's are permanent cycles. To determine the value of  $\delta^1$  at  $c_p x_1 \in M^1$ , we have

$$\begin{aligned} & \Psi^* \delta^1(c_p x_1) \\ &= \Psi^* {}^U d_{2p-1}^{3,2p}(c_p x_1) = {}^T d_{2p-1}^{3,2p} \Psi^*(c_p x_1) \\ &= {}^T d_{2p-1}^{3,2p} \left( \sum_{n \geq i_1 > i_2 > \dots > i_p \geq 1} v_{i_1} v_{i_2} \dots v_{i_p} x_1 \right) \\ (3.2) \quad &= {}^T d_{2p-1}^{3,2p} \left( \sum_{n \geq i_1 > i_2 > \dots > i_p \geq 1} (v'_{i_1} + v_n)(v'_{i_2} + v_n) \dots (v'_{i_p} + v_n) x_1 \right) \\ &= {}^T d_{2p-1}^{3,2p} \left( \sum_{n \geq i_1 > i_2 > \dots > i_p \geq 1} \sum_{j=0}^p \sigma_j(v'_{i_1}, \dots, v'_{i_p}) v_n^{p-j} x_1 \right). \end{aligned}$$

where  $\Psi^* : {}^U E \rightarrow {}^T E$  is the morphism of spectral sequences induced by the inclusions of maximal tori  $T^n \rightarrow U_n$  and  $PT^n \rightarrow PU_n$ , as introduced in (2.5), and  $\sigma_i$  the elementary symmetric polynomials in  $p$  variables, as in (2.6).

By Lemma 3.2, we simplify (3.2) and obtain

$$(3.3) \quad \Psi^* \delta^1(c_p x_1) = {}^T d_{2p-1} \left( \sum_{n \geq i_1 > i_2 > \dots > i_p \geq 1} \sigma_1(v'_{i_1}, \dots, v'_{i_p}) v_n^{p-1} x_1 \right).$$

To proceed, we evaluate the expression

$$\sum_{n \geq i_1 > i_2 > \dots > i_p \geq 1} \sigma_1(t_{i_1}, \dots, t_{i_p})$$

for variables  $t_i$ ,  $1 \leq i \leq n$ . Since it is multi-linear and symmetric in the variables  $t_1, \dots, t_n$ , we have

$$\sum_{n \geq i_1 > i_2 > \dots > i_p \geq 1} \sigma_1(t_{i_1}, \dots, t_{i_p}) = \lambda \sum_{i=1}^n t_i$$

for some  $\lambda \in \mathbb{Z}$ . Taking the substitution  $t_1 = \dots = t_n = 1$  and comparing both sides of the above, we obtain

$$\lambda = \frac{p}{n} \binom{n}{p} = \binom{n-1}{p-1}$$

and

$$(3.4) \quad \sum_{n \geq i_1 > i_2 > \dots > i_p \geq 1} \sigma_1(t_{i_1}, \dots, t_{i_p}) = \binom{n-1}{p-1} \sum_{i=1}^n t_i.$$



We resume the computation of  $\Psi^* \delta^1(c_p x_1)$  started in (3.3):

$$\begin{aligned}
 & \Psi^* \delta^1(c_p x_1) \\
 &= {}^T d_{2p-1} \left( \binom{n-1}{p-1} \sum_{i=1}^n v'_i v_n^{p-1} x_1 \right) \quad (\text{by (3.4)}) \\
 (3.5) \quad &= \binom{n-1}{p-1} \sum_{i=1}^n v'_i y_{p,0} \quad (\text{since } v'_i \text{'s are permanent cocycles}) \\
 &= \binom{n-1}{p-1} \sum_{i=1}^n v_i y_{p,0} \quad (\text{since } y_{p,0} \text{ is } p\text{-torsion}) \\
 &= \Psi^* \left( \binom{n-1}{p-1} c_1 y_{p,0} \right).
 \end{aligned}$$

By the injectivity of

$$\Psi^* : M^2 = {}^U E_2^{2p+2,2} \rightarrow {}^T E_2^{2p+2,2}$$

and Lemma 3.1, we have

$$\delta^1(c_p x_1) = \binom{n-1}{p-1} c_1 y_{p,0} \neq 0$$

and we conclude.  $\square$

**Lemma 3.4.** *The chain complex  $\mathcal{M}$  is exact at  $M^2$ .*

*Proof.* By Lemma 3.3, and the fact that  $\mathcal{M}$  is a chain complex, we have  $\delta^2 = 0$  and the lemma follows.

Alternatively, one may compute  $\delta^2 = d_3^{2p+2,2}$  directly with Corollary 2.9 and obtain the same result.  $\square$

**The exactness of  $\mathcal{M}$  at  $M^1$ .** Recall that the  $\mathbb{Z}_{(p)}$ -module  $M^1$  is freely generated by elements of the form  $cx_1$  for

$$c \in S' := \{c_1^{i_1} c_2^{i_2} \cdots c_p^{i_p} \mid i_k \geq 0, \sum_k k i_k = p\}.$$

Indeed,  $S'$  is simply the set of monomials in  $c_1, c_2, \dots, c_n$  in  $H^{2p}(BU_n)$ . We define a total ordering  $\mathfrak{D}$  on monomials in  $c_1, c_2, \dots, c_n$  as follows. We assert

$$c_1^{i_1} c_2^{i_2} \cdots c_p^{i_p} > c_1^{j_1} c_2^{j_2} \cdots c_p^{j_p}$$

if and only if

- (1) there is at least one  $k$  such that  $i_k \neq j_k$ , and
- (2) for the smallest such  $k$ , we have  $i_k > j_k$ .

Let  $S := S' - \{c_p\}$ . Then  $\mathfrak{D}$  defines total orderings on  $S$ ,  $S'$  and  $S'x_1$  as well. To compare  $cx_1, c'x_1 \in S'x_1$ , we assert  $cx_1 > c'x_1$  if and only if  $c > c'$ .

Let  $L$  be the  $\mathbb{Z}_{(p)}$ -submodule of  $H^{2p}(BU_n)_{(p)}$  spanned by  $S$ . We define a  $\mathbb{Z}_{(p)}$ -linear map

$$\tau : L \rightarrow M^0 = H^{2p+2}(BU_n)_{(p)}$$

as follows. Each element in  $S$  is of the form  $c_1^{i_1} c_2^{i_2} \cdots c_k^{i_k}$  such that  $k < p$  and  $i_k > 0$ , and we define

$$\tau(c_1^{i_1} c_2^{i_2} \cdots c_k^{i_k}) := (c_1^{i_1} c_2^{i_2} \cdots c_{k-1}^{i_{k-1}})(c_k^{i_k-1} c_{k+1}).$$

**Lemma 3.5.** *Let  $\bar{\tau} : L/pL \rightarrow M^0/pM^0$  and  $\bar{\delta}^0 : M^0/pM^0 \rightarrow M^1/pM^1$  denote the mod  $p$  reductions of  $\tau$  and  $\delta^0$ , respectively. Then the image of the composition*

$$L/pL \xrightarrow{\bar{\tau}} M^0/pM^0 \xrightarrow{\bar{\delta}^0} M^1/pM^1$$

*is  $Lx_1/pLx_1$ . In particular, we have*

$$(3.6) \quad \text{Im } \delta^0 \tau \subset W := Lx_1 + (pc_p x_1) \subset M^1.$$

*Proof.* Consider the  $\mathbb{Z}_{(p)}$ -basis  $S, S'x_1$  for  $L$  and  $M^1$ , respectively, both in the descending order with respect to the ordering  $\mathfrak{D}$ . Notice that  $c_p x_1$  is the smallest element in  $S'$ . We study the  $(N+1) \times N$  matrix  $A$  of the map

$$\delta^0 \tau : L \rightarrow M^1$$

with respect to these basis, where  $N$  is the cardinality of  $S$ .

Consider an arbitrary element

$$c := c_1^{i_1} \cdots c_k^{i_k} \in S$$

with  $k < p$  and  $i_k > 0$ . By Corollary 2.9 and the Leibniz's formula, we have

$$\begin{aligned} \delta^0 \tau(c) &= \delta^0(c_1^{i_1} \cdots c_{k-1}^{i_{k-1}} c_k^{i_k-1} c_{k+1}) \\ &= \begin{cases} (n-k)cx_1 + ni_1 c_1^{i_1-1} c_2^{i_2} \cdots c_k^{i_k-1} c_{k+1}x_1 + (\text{higher order terms}), & i_1 > 0, \\ (n-k)cx_1 + (\text{higher order terms}), & i_1 = 0. \end{cases} \end{aligned}$$

In both cases, we have

$$\delta^0 \tau(c) \equiv (n-k)cx_1 + (\text{higher order terms}) \pmod{p}.$$

Therefore, the matrix  $A$  satisfies

$$A \equiv \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & * & \vdots \\ 0 & 0 & \ddots & * \\ 0 & \cdots & 0 & \lambda_N \\ 0 & 0 & \cdots & 0 \end{pmatrix} \pmod{p},$$

where the  $\lambda_i$ 's are of the form  $n-k$  for  $k < p$ , which are invertible in  $\mathbb{Z}_{(p)}$ , and we have verified that the image of the composition

$$L/pL \xrightarrow{\bar{\tau}} M^0/pM^0 \xrightarrow{\bar{\delta}^0} M^1/pM^1$$

is  $Lx_1/pLx_1$ . The equation (3.6) follows from the above and the fact

$$M^1 = Lx_1 + (c_p x_1).$$

□

**Lemma 3.6.** *Consider the  $\mathbb{Z}_{(p)}$ -submodule  $V = \tau(L) + (c_1 c_p - c_{p+1})$  of  $M^0$ . We have  $\delta^0(V) \subset W$  where*

$$W := Lx_1 + (pc_p x_1) \subset M^1$$

*is the  $\mathbb{Z}_{(p)}$ -submodule of  $M^1$  defined in Lemma 3.5.*

*Proof.* By Lemma 3.5 we have  $\delta^0(\tau(L)) \subset W$ . On the other hand, we have

$$(3.7) \quad \delta^0(c_1 c_p - c_{p+1}) = (n-p+1)c_1 c_{p-1} x_1 + pc_p x_1 \in W,$$

and we conclude.

□

**Lemma 3.7.** *The chain complex  $\mathcal{M}$  is exact at  $M^1$ .*

*Proof.* By Lemma 3.6, the restriction of  $\delta^0$  to  $V$  has image in  $W$ . Therefore, we write  $\delta_V^0 := \delta^0|_V : V \rightarrow W$  and consider its mod  $p$  reduction

$$\bar{\delta}_V^0 : V/pV \rightarrow W/pW = Lx_1/pLx_1 + (pc_px_1)/(p^2c_px_1).$$

By Lemma 3.5, we have  $Lx_1/pLx_1 \subset \text{Im } \bar{\delta}_V^0|_V$ .

By  $Lx_1/pLx_1 \subset \text{Im } \bar{\delta}_V^0$  and (3.7), we have  $[pc_px_1] \in \text{Im } \bar{\delta}_V^0$ , where  $[pc_px_1]$  is the class in  $W/pW$  represented by  $pc_px_1$ . Therefore,  $\bar{\delta}_V^0 : V/pV \rightarrow W/pW$  is surjective. By Nakayama's lemma in commutative algebra (Theorem 2.2, Chapter 1, [16]),  $\delta_V^0 : V \rightarrow W$  is surjective.

Therefore, we have

$$(3.8) \quad \text{Im } \delta^0 \supset \text{Im } \delta_V^0 = W = Lx_1 + (pc_px_1).$$

On the other hand, we have  $\text{Ker } \delta^1 \supset \text{Im } \delta^0$ , and therefore  $\text{Ker } \delta^1 \supset W$ . Now, by Lemma 3.3, we have

$$\mathbb{Z}/p \cong M^1/(L + (pc_px_1)) = M^1/W \rightarrow M^1/\text{Ker } \delta^1 \cong \mathbb{Z}/p,$$

where the arrow is the tautological quotient map, which is surjective. Therefore, the above composition is a bijection. It follows that we have

$$(3.9) \quad W = \text{Ker } \delta^1 \supset \text{Im } \delta^0,$$

and the lemma follows from (3.8) and (3.9).  $\square$

Lemma 3.7 and Lemma 3.4 complete the proof of Proposition 2.10.

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## REFERENCES

- [1] Benjamin Antieau and Ben Williams. The period-index problem for twisted topological K-theory. *Geometry & Topology*, 18(2):1115–1148, 2014.
- [2] Benjamin Antieau and Ben Williams. The topological period-index problem over 6-complexes. *Journal of Topology*, 7(3):617–640, 2014.
- [3] Henri Cartan and Jean-Pierre Serre. Séminaire Henri Cartan vol. 7. pages 283–288, 1954–1955.
- [4] Clay Cordova, Daniel Freed, Ho Tat Lam, and Nathan Seiberg. Anomalies in the space of coupling constants and their dynamical applications II. *SciPost Physics Proceedings*, 8(1), 2020.
- [5] Diarmuid Crowley and Xing Gu. On  $H^*(BPU_n; \mathbb{Z})$  and Weyl group invariants. *arXiv preprint arXiv:2103.03523*, 2021.
- [6] Haibao Duan. The cohomology and K-theory of the projective unitary groups  $PU(n)$ . *arXiv preprint arXiv:1710.09222*, 2017.
- [7] Tudor Ganea. Induced fibrations and cofibrations. *Transactions of the American Mathematical Society*, 127(3):442–459, 1967.

- [8] Iñaki García-Etxebarria and Miguel Montero. Dai-Freed anomalies in particle physics. *Journal of High Energy Physics*, 2019(8):3, 2019.
- [9] Xing Gu. The topological period-index problem over 8-complexes, I. *Journal of Topology*, 12(4):1368–1395, 2019.
- [10] Xing Gu. An almost polynomial subring of the Chow ring and cohomology of  $BPGL_n$ . *arXiv preprint arXiv:2012.00199*, 2020.
- [11] Xing Gu. The topological period-index problem over 8-complexes, II. *Proceedings of the American Mathematical Society*, 148:4541–4545, 2020.
- [12] Xing Gu. On the cohomology of the classifying spaces of projective unitary groups. *Journal of Topology and Analysis*, 13(02):535–573, 2021.
- [13] Xing Gu. Some torsion classes in the Chow ring and cohomology of  $BPGL_n$ . *Journal of the London Mathematical Society*, 103(1):127–160, 2021.
- [14] Masaki Kameko and Nobuaki Yagita. The Brown-Peterson cohomology of the classifying spaces of the projective unitary groups  $PU(p)$  and exceptional Lie groups. *Transactions of the American Mathematical Society*, 360(5):2265–2284, 2008.
- [15] Akira Kono and Mamoru Mimura. On the cohomology of the classifying spaces of  $PSU(4n+2)$  and  $PO(4n+2)$ . *Publications of the Research Institute for Mathematical Sciences*, 10(3):691–720, 1975.
- [16] Hideyuki Matsumura. *Commutative ring theory*. Cambridge university press, 1989.
- [17] Hirotaka Tamanoi.  $Q$ -subalgebras, Milnor basis, and cohomology of Eilenberg-MacLane spaces. *Journal of Pure and Applied Algebra*, 137(2):153–198, 1999.
- [18] Hiroshi Toda et al. Cohomology of classifying spaces. In *Homotopy theory and related topics*, pages 75–108. Mathematical Society of Japan, 1987.
- [19] Burt Totaro. The Chow ring of a classifying space. In *Proceedings of symposia in pure mathematics*, volume 67, pages 249–284. Providence, RI; American Mathematical Society; 1998, 1999.
- [20] Aleš Vavpetič and Antonio Viruel. On the mod  $p$  cohomology of  $BPU(p)$ . *Transactions of the American Mathematical Society*, pages 4517–4532, 2005.
- [21] Gabriele Vezzosi. On the Chow ring of the classifying stack of  $PGL_{3,\mathbb{C}}$ . *Journal für die reine und angewandte Mathematik*, 2000(523):1–54, 2000.
- [22] Angelo Vistoli. On the cohomology and the Chow ring of the classifying space of  $PGL_p$ . *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2007(610):181–227, 2007.

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