A CORRESPONDENCE BETWEEN HIGHER ADAMS DIFFERENTIALS AND HIGHER ALGEBRAIC NOVIKOV DIFFERENTIALS AT ODD PRIMES

Abstract. This paper studies the higher differentials of the classical Adams spectral sequence at odd primes. In particular, we follow the "cofiber of τ philosophy" of Gheorghe, Isaksen, Wang, and Xu to show that higher Adams differentials agree with their corresponding higher algebraic Novikov differentials in a certain range.

1. Introduction

The computation of the stable homotopy groups of the sphere $\pi_*(S^0)$ is one of the most important problems in homotopy theory. In recent years, a major breakthrough in this area is the work of Isaksen-Wang-Xu [8], which successfully extended the computation of the 2-primary components of $\pi_*(S^0)$ to dimension 90. Their computation is based on the "cofiber of τ philosophy" developed in Gheorghe-Wang-Xu [5]. The key insight of [5, 8] is that one can compute difficult higher Adams differentials from higher algebraic Novikov differentials using motivic homotopy theory. Although the method developed in [5] holds for arbitrary primes, it has only been extensively exploited for 2-primary computations. In this paper, we apply the "cofiber of τ philosophy" to odd-primary computations.

Some classical spectral sequences. Throughout this paper, we let p denote a fixed odd prime. The Adams spectral sequence (ASS) and the Adams-Novikov spectral sequence (ANSS) are two of the most powerful tools to compute the p-primary component of $\pi_*(S^0)$.

The E_2-page of the ASS is $Ext_{\mathcal{A}_*}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$, where $\mathcal{A}_*$ is the dual mod p Steenrod algebra. We can write $\mathcal{A}_* = \mathbb{F}_p[t_1, t_2, \cdots] \otimes E[\tau_0, \tau_1, \tau_2, \cdots]$, where $\mathbb{F}_p[t_1, t_2, \cdots]$ is a polynomial algebra with coefficients in $\mathbb{F}_p$, and $E[\tau_0, \tau_1, \tau_2, \cdots]$ is an exterior algebra with coefficients in $\mathbb{F}_p$. The differentials of the ASS have the form $d_r^{Adams}: E_r^{s,t}(S) \to E_r^{s+r,t+r-1}(S)$, $r \geq 2$.

The E_2-page of the ANSS is $Ext_{BP_*BP}^{*,*}(BP_*, BP_*)$, where BP denote the Brown-Peterson spectrum. We have $BP_* := \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$, and $BP_*BP = BP_*[t_1, t_2, \cdots]$, where $\mathbb{Z}_{(p)}$ denotes the integers localized at p. The inner degrees of the generators are $|v_n| = |t_n| = 2(p^n - 1)$.

The algebraic Novikov spectral sequence (algNSS) [9, 13] converges to the Adams-Novikov E_2-page. The E_2-page of the algNSS is $Ext_{P_*}^{s,t}(\mathbb{F}_p, I^k/I^{k+1})$, where I denotes the ideal $(p, v_1, v_2, \cdots) \subset BP_*$, and $P_* = BP_*BP/I = \mathbb{F}_p[t_1, t_2, \cdots]$ is the $\mathbb{F}_p$-coefficient polynomial algebra. The differentials have the form $d_r^{alg}: \bar{E}_r^{s,t,k} \to \bar{E}_r^{s+1,t,k+r-1}$, $r \geq 2$.

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Remark 1.1. Here, we have re-indexed the pages of the algNSS to align with the notations in [5, 8].

The E_2-page of the Adams spectral sequence can also be computed via a spectral sequence, called the Cartan-Eilenberg spectral sequence (CESS) [2, 15]. For odd prime p, the E_2-page of the CESS coincides with the E_2-page of the algNSS up to degree shifting [9].

These four spectral sequences fit into the following square [10].

$$\begin{tikzcd} Ext_{P_*}^{s,t}(\mathbb{F}_p, I^k/I^{k+1}) \arrow[r, Rightarrow, "CESS"] \arrow[d, Rightarrow, "algNSS"'] & Ext_{\mathcal{A}_*}^{s+k,t+k}(\mathbb{F}_p, \mathbb{F}_p) \arrow[d, Rightarrow, "ASS"] \\ Ext_{BP_*BP}^{s,t}(BP_*, BP_*) \arrow[r, Rightarrow, "ANSS"] & \pi_{t-s}(\hat{S}^0) \end{tikzcd} \tag{1.1}$$

The Adams differentials d_r^{Adams} and the Adams-Novikov differentials d_r^{AN} are difficult to compute in general. Such computations might require an understanding of complicated geometric behaviors. On the other hand, the algebraic Novikov differentials d_r^{alg} are much easier to compute. This is because the entire construction of the algNSS is purely algebraic.

It is long been observed that d_r^{alg} and d_r^{Adams} are closely related. It is desirable if we could determine d_r^{Adams} differentials based on d_r^{alg} computations. The following result realizes the idea for $r = 2$, while the problem for general $r > 2$ remains open.

Theorem 1.2 (Novikov [13], Andrews-Miller [1, 10])**.** *Suppose z is an element in $Ext_{\mathcal{A}_*}^{s+k,t+k}(\mathbb{F}_p, \mathbb{F}_p)$ which is detected in the CESS by $x \in Ext_{P_*}^{s,t}(\mathbb{F}_p, I^k/I^{k+1})$, then $d_2^{Adams}(z)$ is detected by $d_2^{alg}(x) \in Ext_{P_*}^{s+1,t}(\mathbb{F}_p, I^{k+1}/I^{k+2})$.*

Remark 1.3. Indeed, one can use this approach to find new nontrivial secondary Adams differentials. See [19] for some practical examples.

New developments using motivic homotopy theory. For general $r \geq 2$, Gheorghe-Wang-Xu [5] developed the "cofiber of τ philosophy" to compare d_r^{alg} with d_r^{Adams}. One key feather of their approach is the usage of motivic homotopy theory, which originated from the work of Morel and Veovodsky [12]. Motivic homotopy theory behaves very similarly to the classical homotopy theory. For example, there are motivic dual Steenrod algebras [16, 17] and motivic Adams spectral sequences (mASS) [4, 6].

The computational technique of [5] can be illustrated in the following diagram.

$$\begin{tikzcd} Ext_{\mathcal{A}^{\mathbb{C}}_{*,*}}^{*;*,*}(\mathbb{F}_p[\tau], \mathbb{F}_p) \arrow[d, Rightarrow, "mASS"] & Ext_{\mathcal{A}^{\mathbb{C}}_{*,*}}^{*;*,*}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau]) \arrow[l] \arrow[r] \arrow[d, Rightarrow, "mASS"] & Ext_{\mathcal{A}_*}^{*;*}(\mathbb{F}_p, \mathbb{F}_p) \arrow[d, Rightarrow, "ASS"] \\ \pi_{*,*}(\widehat{S^{0,0}}/\tau) & \pi_{*,*}(\widehat{S^{0,0}}) \arrow[l] \arrow[r] & \pi_*(\widehat{S^0}) \end{tikzcd} \tag{1.2}$$

They considered a map $\tau : \widehat{S^{0,-1}} \to \widehat{S^{0,0}}$ between p-completed motivic sphere spectra. We let $\widehat{S^{0,0}}/\tau$ denote the cofiber of the map τ. In diagram (1.2), the left column is the mASS for $\widehat{S^{0,0}}/\tau$, the middle column is the mASS for $\widehat{S^{0,0}}$, and the right column is the classical ASS. The top horizontal maps are maps of spectral sequences. Gheorghe-Wang-Xu [5] proved the left column is isomorphic to the algNSS for any prime p. Hence diagram (1.2) provides a zig-zag diagram to compare higher Adams differentials with their corresponding higher algebraic Novikov differentials.

For $p = 2$, Isaksen-Wang-Xu [8] used the strategy above to determine d_r^{Adams} based on computer generated d_r^{alg} data. This enabled them to successfully extend the computation of 2-primary stable homotopy groups from around 60 stems to around 90 stems.

Our main results. In this paper, we follow the "cofiber of τ philosophy" but focus on the case when p is an odd prime. For odd prime p, the motivic Steenrod algebra has a simpler form and the diagram (1.2) presents different features. Let us explain our notations and then state our main results.

Notation 1.4. *Let x be an element in the E_2-page of a certain spectral sequence. We let $[x]_k$ denote the homology class of x on the E_k-page, where $[x]_2$ is understood to be x. We say x can be lifted to the E_r-page if $[x]_r$ is well defined, i.e., if $d_i([x]_i) = 0$ for each $2 \leq i < r$.*

Theorem 1.5. *Let p be an odd prime. Let $z \in Ext_{\mathcal{A}_*}^{s+k,t+k}(\mathbb{F}_p, \mathbb{F}_p)$ be an element in the E_2-page of the ASS which is detected by $x \in Ext_{P_*}^{s,t}(\mathbb{F}_p, I^k/I^{k+1})$ with $s < 2p-2$. Let r be an integer such that $r + k \leq 2p - 2$ and z can be lifted to the E_r-page. Then we can write $d_r^{Adams}([z]_r) = [w]_r$, $d_r^{alg}([x]_r) = [y]_r$, where $w \in Ext_{\mathcal{A}_*}^{s+k+r,t+k+r-1}(\mathbb{F}_p, \mathbb{F}_p)$ can be detected by $y \in Ext_{P_*}^{s+1,t}(\mathbb{F}_p, I^{k+r-1}/I^{k+r})$.*

Theorem 1.5 shows, in a certain range, the higher Adams differentials agree with their corresponding higher algebraic Novikov differentials. It is worth pointing out that the result could fail for $r \geq 2p - 1 - k$. For example, let $p \geq 5$. The k-value for $b_n \in Ext_{\mathcal{A}_*}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$ is 0. It is proved that b_n has nontrivial Adams differential [14]

$$d_{2p-1}^{Adams}(b_n) = h_0 b_{n-1}^p. \tag{1.3}$$

However, the corresponding algebraic Novikov differential is trivial

$$d_{2p-1}^{alg}(b_n) = 0. \tag{1.4}$$

We also have the following result comparing the length of nontrivial algebraic Novikov differentials and Adams differentials.

Theorem 1.6. *Let p be an odd prime. Let $z \in Ext_{\mathcal{A}_*}^{s+k,t+k}(\mathbb{F}_p, \mathbb{F}_p)$ be an element detected by $x \in Ext_{P_*}^{s,t}(\mathbb{F}_p, I^k/I^{k+1})$ with $s < 2p-2$. If x is not a permanent cycle in the algNSS, then z is not a permanent cycle in the ASS. Moreover, let r (resp. r') be the largest number n such that x (resp. z) can be lifted to the E_n-page, then $r \geq r'$.*

Organization of the paper. In Section 2, we review some basic results in motivic homotopy theory as well as the "cofiber of τ" method of Gheorghe, Isaksen, Wang, and Xu to compare higher algebraic Novikov differentials with higher Adams differentials. In Section 3, we give the proofs for Theorem 1.5 and Theorem 1.6.

2. Comparing d_r^{Adams} with d_r^{alg}

Motivic homotopy theory originated out of the work of Voevodsky and Morel [12, 16, 17]. Motivic homotopy theory can be viewed as a successful application of abstract homotopy theory to algebraic geometry and number theory. From categorical and computational perspectives, the motivic stable homotopy category behaves very similarly to the classical stable homotopy category. For example, there are motivic spheres, motivic homotopy and homology groups, motivic Eilenberg-MacLane spectra, and motivic Steenrod algebras analogous to the classical ones. For readers not familiar with motivic homotopy theory, [4, 11, 18] are some helpful references.

In this paper, we choose to work over the field $\mathbb{C}$ of complex numbers and assume p is an odd prime. Under this setting, we have explicit formulas for the mod p motivic Eilenberg-MacLane spectra $H\mathbb{F}_p^{\text{mot}}$ and motivic dual Steenrod algebra $\mathcal{A}^{\mathbb{C}}_{*,*}$.

Proposition 2.1 ([16, 17]). *Let p be an odd prime. We have*

$$H\mathbb{F}_p^{mot}{}_{*,*} = \mathbb{F}_p[\tau],$$

where τ has bi-degree $(0,-1)$, and that

$$\mathcal{A}^{\mathbb{C}}_{*,*} = \mathbb{F}_p[\tau] \otimes \mathbb{F}_p[t_1, t_2, \cdots] \otimes E[\tau_0, \tau_1, \cdots],$$

where τ has bi-degree $(0,-1)$, t_i has bi-degree $(2(p^i-1), p^i-1)$, and τ_i has bi-degree $(2(p^i-1)+1, p^i-1)$.

Under this setting, $\mathcal{A}^{\mathbb{C}}_{*,*} \cong \mathbb{F}_p[\tau] \otimes_{\mathbb{F}_p} \mathcal{A}_*$ is just the classical dual Steenrod algebra tensoring the new coefficient.

There is a motivic analog of the classical Adams spectral sequence called the motivic Adams spectral sequence (mASS) (see [4, 6]).

Proposition 2.2. *There is a motivic Adams spectral sequence which converges to the bi-graded homotopy groups of the $H\mathbb{F}_p^{mot}$-completed motivic sphere $\widehat{S^{0,0}}$. The mASS has E_2-page*

$$E_2^{s,t,u}(S) = Ext^{s,t,u}_{\mathcal{A}^{\mathbb{C}}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau]) \Longrightarrow \pi_{t-s,u}(\widehat{S^{0,0}}),$$

and differentials

$$d_r : E_r^{s,t,u}(S) \to E_r^{s+r,t+r-1,u}(S).$$

For $x \in E_r^{s,t,u}(S)$, we call s its homological degree, t the inner degree, and u the motivic weight.

There is a topological realization functor $Re : SH^{\text{mot}} \to SH$ (see [3, 12]) from the motivic stable homotopy category SH^{mot} to the classical stable homotopy category SH. This functor maps the motivic sphere $S^{a,b}$ to classical sphere S^a and maps the motivic Eilenberg-MacLane spectra $H\mathbb{F}_p^{\text{mot}}$ to the classical Eilenberg-MacLane spectra $H\mathbb{F}_p$. For the other direction, there is also a constant embedding functor $C : SH \to SH^{\text{mot}}$. We have $Re \circ C = id$.

The functor Re induces a map ϕ from the mASS of the motivic sphere to the classical ASS of the classical sphere.

$$\begin{tikzcd} E_r^{s,t,u}(S) \arrow[r, "\phi_r"] \arrow[d, "d_r^{mAdams}"'] & E_r^{s,t}(S) \arrow[d, "d_r^{Adams}"] \\ E_r^{s+r,t+r-1,u}(S) \arrow[r, "\phi_r"] & E_r^{s+r,t+r-1}(S) \end{tikzcd} \tag{2.1}$$

The effect of ϕ_r is inverting τ (see [3, 4, 7]).

Proposition 2.3 ([4, 7])**.** *Let p be an odd prime. There is an isomorphism*

$$Ext^{s,t,*}_{\mathcal{A}^{\mathbb{C}}_{*,*}}(\mathbb{F}_p[\tau],\mathbb{F}_p[\tau]) \cong \mathbb{F}_p[\tau]\otimes_{\mathbb{F}_p} Ext^{s,t}_{\mathcal{A}_*}(\mathbb{F}_p,\mathbb{F}_p).$$

Moreover, after inverting τ, the mASS of the motivic sphere becomes isomorphic to the classical ASS tensored over $\mathbb{F}_p$ with $\mathbb{F}_p[\tau,\tau^{-1}]$.

The element τ can be lifted to a map $\tau : \widehat{S^{0,-1}} \to \widehat{S^{0,0}}$ between $H\mathbb{F}_p^{\mathrm{mot}}$-completed motivic spectra. We denote the associated cofiber sequence as

$$\widehat{S^{0,-1}} \xrightarrow{\tau} \widehat{S^{0,0}} \xrightarrow{i} \widehat{S^{0,0}}/\tau.$$

The mASS for $\widehat{S^{0,0}}/\tau$ has E_2-page $E_2^{s,t,u}(C\tau) = Ext^{s,t,u}_{\mathcal{A}^{\mathbb{C}}_{*,*}}(\mathbb{F}_p[\tau],\mathbb{F}_p)$. To avoid potential confusions with the differentials in the mASS for $\widehat{S^{0,0}}$, we denote the differentials in the mASS for $\widehat{S^{0,0}}/\tau$ as δ_r^{mAdams}. The map i induces a map ψ from the mASS for $\widehat{S^{0,0}}$ to the mASS for $\widehat{S^{0,0}}/\tau$.

$$\begin{tikzcd}
E_r^{s,t,u}(C\tau) \arrow[d, "\delta_r^{mAdams}"'] & E_r^{s,t,u}(S) \arrow[l, "\psi_r"'] \arrow[d, "d_r^{mAdams}"] \\
E_r^{s+r,t+r-1,u}(C\tau) & E_r^{s+r,t+r-1,u}(S) \arrow[l, "\psi_r"']
\end{tikzcd} \tag{2.2}$$

The effect of ψ is just sending τ to 0 (see [5, 8]).

Finally, Gheorghe-Wang-Xu [5] proved there is an isomorphism κ between the mASS for $\widehat{S^{0,0}}/\tau$ and the regraded algebraic Novikov spectral sequence.

$$\begin{tikzcd}
\bar{E}_r^{s+2u-t,2u,t-2u} \arrow[r, "\kappa_r", "\cong"'] \arrow[d, "d_r^{alg}"'] & E_r^{s,t,u}(C\tau) \arrow[d, "\delta_r^{Adams}"] \\
\bar{E}_r^{s+2u-t+1,2u,t-2u+r-1} \arrow[r, "\kappa_r", "\cong"'] & E_r^{s+r,t+r-1,u}(C\tau)
\end{tikzcd} \tag{2.3}$$

We can summarize these three comparison maps in the following diagram.

(2.4)

$$\begin{tikzcd}
\bar{E}_r^{s+2u-t,2u,t-2u} \arrow[r, "\kappa_r", "\cong"'] \arrow[d, "d_r^{alg}"'] & E_r^{s,t,u}(C\tau) \arrow[d, "\delta_r^{mAdams}"'] & E_r^{s,t,u}(S) \arrow[l, "\psi_r"'] \arrow[r, "\phi_r"] \arrow[d, "d_r^{mAdams}"'] & E_r^{s,t}(S) \arrow[d, "d_r^{Adams}"'] \\
\bar{E}_r^{s+2u-t+1,2u,t-2u+r-1} \arrow[r, "\kappa_r", "\cong"'] & E_r^{s+r,t+r-1,u}(C\tau) & E_r^{s+r,t+r-1,u}(S) \arrow[l, "\psi_r"'] \arrow[r, "\phi_r"] & E_r^{s+r,t+r-1}(S)
\end{tikzcd}$$

The diagram (2.4) provides a zig-zag way to compare higher Adams differentials d_r^{Adams} with their corresponding higher algebraic Novikov differentials d_r^{alg}.

3. Proof of Theorems 1.5, 1.6

We discuss several lemmas before we prove our main results.

Lemma 3.1. *Let p be an odd prime, denote $q = 2(p-1)$. Given $s,t \geq 0$, we denote $C_{s,t} = \{i \in \mathbb{Z} | i \equiv t \text{ mod } q,\ 0 \leq i \leq s,t\}$. Then we have the following direct sum decomposition of the classical Adams E_2-terms.*

$$Ext^{s,t}_{\mathcal{A}_*}(\mathbb{F}_p,\mathbb{F}_p) \cong \bigoplus_{i\in C_{s,t}} Ext^{s-i,t-i}_{P_*}(\mathbb{F}_p, I^i/I^{i+1})$$

Proof. For odd prime p, the Cartan-Eilenberg spectral sequence collapses from E_2-page with no nontrivial extensions [15, Theorem 4.4.3]. Hence we have

$$Ext_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p,\mathbb{F}_p) \cong \bigoplus_{i\in\mathbb{Z}} Ext_{P_*}^{s-i,t-i}(\mathbb{F}_p, I^i/I^{i+1}).$$

Note the inner degrees $|v_n| = |t_n| = 2(p^n - 1)$ are all multiples of q. In order for $Ext_{P_*}^{s-i,t-i}(\mathbb{F}_p, I^i/I^{i+1})$ to be nontrivial, we need $i, s-i, t-i \geq 0$, and that $t - i \equiv 0 \bmod q$. Hence i needs to be in the set $C_{s,t}$.

□

Remark 3.2. It is worth pointing out that the Adams differential d_2^{Adams} may not respect this decomposition.

Notation 3.3. *Let $z \in Ext_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p,\mathbb{F}_p)$ be an element in the Adams E_2-page. We let $\tilde{z}$ denote the element $1\otimes z \in \mathbb{F}_p[\tau]\otimes_{\mathbb{F}_p} Ext_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p,\mathbb{F}_p) \cong Ext_{\mathcal{A}_{*,*}^{\mathbb{C}}}^{s,t,*}(\mathbb{F}_p[\tau],\mathbb{F}_p[\tau])$ in the E_2-page of the mASS of the motivic sphere.*

Note we have $\phi_2(\tilde{z}) = z$. If z and $\tilde{z}$ can both be lifted to the E_r-pages of the respected spectral sequences for some r. Then $\phi_r([\tilde{z}]_r) = [z]_r$.

Lemma 3.4. *Let z be an element in $Ext_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p,\mathbb{F}_p)$ which is detected by $x \in Ext_{P_*}^{s-k,t-k}(\mathbb{F}_p, I^k/I^{k+1})$. Then $\tilde{z} \in Ext_{\mathcal{A}_{*,*}^{\mathbb{C}}}^{s,t,*}(\mathbb{F}_p[\tau],\mathbb{F}_p[\tau])$ has motivic weight $\frac{t-k}{2}$.*

Proof. For notation simplicity, we let $t(-)$ denote the inner degree and let $u(-)$ denote the motivic weight of an element in $Ext_{\mathcal{A}_{*,*}^{\mathbb{C}}}^{*,*,*}(\mathbb{F}_p[\tau],\mathbb{F}_p[\tau])$. By Proposition 2.1, we have $u(\tilde{t}_i) = \frac{1}{2}t(\tilde{t}_i)$ for $i \geq 1$, and $u(\tilde{\tau}_i) = \frac{1}{2}t(\tilde{\tau}_i) - \frac{1}{2}$ for $i \geq 0$.

Since z is detected by $x \in Ext_{P_*}^{s-k,t-k}(\mathbb{F}_p, I^k/I^{k+1})$, the number of τ_i's in the expression of z is just k. Hence we conclude $u(\tilde{z}) = \frac{1}{2}t(\tilde{z}) - \frac{1}{2}k = \frac{t-k}{2}$. □

Lemma 3.5. *Let $z \in Ext_{\mathcal{A}_*}^{s+k,t+k}(\mathbb{F}_p,\mathbb{F}_p)$ be an element which is detected by $x \in Ext_{P_*}^{s,t}(\mathbb{F}_p, I^k/I^{k+1})$. (i) If $\tilde{z}$ can be lifted to the E_r-page for some $r \geq 2$, then z can also be lifted to the E_r-page. (ii) Further assume $s < 2p-2$. Then z can be lifted to the E_r-page implies $\tilde{z}$ can also be lifted to the E_r-page.*

Proof. (i) By assumption, $d_i^{mAdams}([\tilde{z}]_i) = 0$ for each $2 \leq i < r$. One can inductively show that $d_i^{Adams}([z]_i) = 0$ (hence $[z]_{i+1}$ is well defined) for each $2 \leq i < r$ by commutativity of diagram (2.1).

(ii) Assume, for the sake of contradiction, that there exists $2 \leq r_1 < r$ such that $d_{r_1}^{mAdams}([\tilde{z}]_{r_1}) \neq 0$. We claim that there exists a nontrivial differential $d_j^{mAdams}([u]_j) \neq 0$, where $u \in Ext_{\mathcal{A}_{*,*}^{\mathbb{C}}}^{s+b,t+b,\frac{t}{2}-a}(\mathbb{F}_p[\tau],\mathbb{F}_p[\tau])$ is non-τ-divisible, $2 \leq j < r_1$, and $a > 0$.

We let $[v_1]_{r_1}$ denote $d_{r_1}^{mAdams}([\tilde{z}]_{r_1}) \neq 0$. By the commutativity of diagram (2.1), we have $\phi_{r_1}([v_1]_{r_1}) = d_{r_1}^{Adams}([z]_{r_1}) = 0$. So $[v_1]_{r_1}$ is τ-torsion. Let $a_1 > 0$ be the smallest integer such that $\tau^{a_1}[v_1]_{r_1} = 0$. Then there exists a nontrivial differential $d_{r_2}^{mAdams}([u_1]_{r_2}) = \tau^{a_1}[v_1]_{r_2} \neq 0$ with $2 \leq r_2 < r_1$. The element $[u_1]_{r_2}$ is not divisible by τ on the E_{r_2}-page. Otherwise, we have $d_{r_2}^{mAdams}([u_1]_{r_2}/\tau) = \tau^{a_1-1}[v_1]_{r_2}$, which contradicts the definition of a_1.

By Lemma 3.4, we have $[\tilde{z}]_{r_1} \in E_{r_1}^{s+k,t+k,\frac{t}{2}}(S)$. This implies

$$[v_1]_{r_1} = d_{r_1}^{mAdams}([\tilde{z}]_{r_1}) \in E_{r_1}^{s+k+r_1,t+k+r_1-1,\frac{t}{2}}(S).$$

Comparing the degrees, we have

$$\tau^{a_1}[v_1]_{r_2} \in E_{r_2}^{s+k+r_1,t+k+r_1-1,\frac{t}{2}-a_1}(S),\ [u_1]_{r_2} \in E_{r_2}^{s+k+r_1-r_2,t+k+r_1-r_2,\frac{t}{2}-a_1}(S).$$

If u_1 is non-τ-divisible, we can take $d_{r_2}^{mAdams}([u_1]_{r_2}) \neq 0$ as the claimed differential.

Otherwise, since $[u_1]_{r_2}$ is not divisible by τ on the E_{r_2}-page, we conclude u_1/τ does not lift to the E_{r_2}-page. Then there exists differential $d_{r_3}^{mAdams}([u_1/\tau]_{r_3}) = [v_2]_{r_3} \neq 0$ with $2 \leq r_3 < r_2$.

By the commutativity of diagram (2.1), we have

$$\phi_{r_3}([v_2]_{r_3}) = d_{r_3}^{Adams}(\phi_{r_3}([u_1/\tau]_{r_3})) = d_{r_3}^{Adams}(\phi_{r_3}([u_1]_{r_3})) = \phi_{r_3}(d_{r_3}^{mAdams}([u_1]_{r_3})),$$

and $d_{r_3}^{mAdams}([u_1]_{r_3}) = 0$ since u_1 can be lifted to the E_{r_2}-page. So $[v_2]_{r_3}$ is τ-torsion. Let $a_2 > 0$ be the smallest integer such that $\tau^{a_2}[v_2]_{r_3} = 0$. Then there exists a nontrivial differential $d_{r_4}^{mAdams}([u_2]_{r_4}) = \tau^{a_2}[v_2]_{r_4} \neq 0$ with $2 \leq r_4 < r_3$. The element $[u_2]_{r_4}$ is not divisible by τ on the E_{r_4}-page.

For the degrees, we have

$$\begin{aligned}
[u_1/\tau]_{r_3} &\in E_{r_3}^{s+k+r_1-r_2,t+k+r_1-r_2,\frac{t}{2}-(a_1-1)}(S),\\
[v_2]_{r_3} &\in E_{r_3}^{s+k+r_1-r_2+r_3,t+k+r_1-r_2+r_3-1,\frac{t}{2}-(a_1-1)}(S),\\
\tau^{a_2}[v_2]_{r_4} &\in E_{r_4}^{s+k+r_1-r_2+r_3,t+k+r_1-r_2+r_3-1,\frac{t}{2}-(a_1-1)-a_2}(S),\\
[u_2]_{r_4} &\in E_{r_4}^{s+k+r_1-r_2+r_3-r_4,t+k+r_1-r_2+r_3-r_4,\frac{t}{2}-(a_1-1)-a_2}(S).
\end{aligned}$$

If u_2 is non-τ-divisible, we can take $d_{r_4}^{mAdams}([u_2]_{r_4}) \neq 0$ as the claimed differential. Otherwise, we can repeat the process and obtain $u_3, u_4, \cdots$. Note there are only finitely many integers between 2 and r_1. After repeating this process several times, we will eventually obtain a desired nontrivial differential $d_j^{mAdams}([u]_j) \neq 0$, where $u \in Ext_{\mathcal{A}^{\mathbb{C}}_{*,*}}^{s+b,t+b,\frac{t}{2}-a}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau])$ is non-τ-divisible, $2 \leq j < r_1$, and $a > 0$.

By Proposition 2.3 and Lemma 3.1, we can write

$$u = \sum_{i \in C_{s+b,t+b}} \tau^{n_i} \tilde{u}_i,$$

where u_i can be detected by some $m_i \in Ext_{P_*}^{s+b-i,t+b-i}(\mathbb{F}_p, I^i/I^{i+1})$. Comparing the motivic weights using Lemma 3.4, we get $n_i = \frac{b-i+2a}{2} \geq 0$ for $\tilde{u}_i \neq 0$.

Note u is non-τ-divisible. This implies the $n_i = 0$ term is nontrivial. In particular, we have $b + 2a \in C_{s+b,t+b}$. This implies

$$b + 2a \leq s + b, \quad b + 2a \equiv t + b \bmod q. \tag{3.1}$$

Note $x \in Ext_{P_*}^{s,t}(\mathbb{F}_p, I^k/I^{k+1})$ implies $t \equiv 0 \bmod q$. In summary, we have

$$0 < 2a \leq s, \quad 2a \equiv 0 \bmod q. \tag{3.2}$$

Then $s \geq 2a \geq 2(p-1)$, this contradicts $s < 2p-2$. Thus we have proved (ii). □

Now we proceed to prove Theorem 1.5 and Theorem 1.6.

Proof of Theorem 1.5. Our strategy is to compare the differentials via diagram (2.4). As we will see, with the given assumptions, diagram (2.4) can be specialized to the following diagram.

$$\begin{tikzcd} {[x]_r} \arrow[r, "\kappa_r"] \arrow[d, "d_r^{alg}"'] & {[\tilde{z}]_r} \arrow[d, "\delta_r^{mAdams}"] & {[\tilde{z}]_r} \arrow[l, "\psi_r"'] \arrow[r, "\phi_r"] \arrow[d, "d_r^{mAdams}"] & {[z]_r} \arrow[d, "d_r^{Adams}"] \\ {[y]_r} \arrow[r, "\kappa_r"] & {[\tilde{w}]_r} & {[\tilde{w}]_r} \arrow[l, "\psi_r"'] \arrow[r, "\phi_r"] & {[w]_r} \end{tikzcd} \tag{3.3}$$

By Lemma 3.5, $[\tilde{z}]_r$ is well defined and we have $\phi_r([\tilde{z}]_r) = [z]_r$. By Lemma 3.4, we have $[\tilde{z}]_r \in E_r^{s+k,t+k,\frac{t}{2}}(S)$. Then we can write $d_r^{mAdams}([\tilde{z}]_r) = [u]_r$, where $u \in Ext_{\mathcal{A}^{\mathbb{C}}_{*,*}}^{s+k+r,t+k+r-1,\frac{t}{2}}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau])$. By Proposition 2.3 and Lemma 3.1, we can write

$$u = \sum_{i \in C_{s+k+r,t+k+r-1}} \tau^{n_i} \tilde{u}_i, \tag{3.4}$$

where u_i can be detected by some $m_i \in Ext_{P_*}^{s+k+r-i,t+k+r-1-i}(\mathbb{F}_p, I^i/I^{i+1})$. Comparing the motivic weights using Lemma 3.4, we get $n_i = \frac{k+r-1-i}{2}$ for $\tilde{u}_i \neq 0$.

Since $n_i \geq 0$, this forces $i \leq k+r-1$. By definition, $i \in C_{s+k+r,t+k+r-1}$ implies

$$i \geq 0, \quad i \equiv t+k+r-1 \mod q, \tag{3.5}$$

where we denote $q = 2(p-1)$. Moreover, since $x \in Ext_{P_*}^{s,t}(\mathbb{F}_p, I^k/I^{k+1})$, we have $t \equiv 0 \bmod q$. In summary, we have

$$0 \leq i \leq k+r-1, \quad i \equiv k+r-1 \mod q \tag{3.6}$$

for nontrivial $\tilde{u}_i$.

By assumptions, we have $0 < k+r-1 < q$. Then (3.6) forces $i = k+r-1$, the corresponding $n_i = 0$. So we can rewrite (3.4) as $u = \tilde{w}$, where we let w denote u_{k+r-1}. We also let y denote m_{k+r-1} detecting w.

We have $d_r^{mAdams}([\tilde{z}]_r) = [\tilde{w}]_r$. Note $\phi_r([\tilde{w}]_r) = [w]_r$. The commutativity of diagram (2.1) implies $d_r^{Adams}([z]_r) = [w]_r$.

Note $[\tilde{z}]_r$ and $[\tilde{w}]_r$ are non-τ-divisible, ψ sends $[\tilde{z}]_r$ and $[\tilde{w}]_r$ to the corresponding elements in $E_r^{*,*,*}(C\tau)$ of the same form, which we abuse the notation and still denote by $[\tilde{z}]_r$ and $[\tilde{w}]_r$ respectively. The commutativity of diagram (2.2) implies $\delta_r^{mAdams}([\tilde{z}]_r) = [\tilde{w}]_r$.

Finally, the the isomorphism κ associates $\tilde{z}$ with x and $\tilde{w}$ with y. Hence $\kappa_r([x]_r) = [\tilde{z}]_r$, $\kappa_r([y]_r) = [\tilde{w}]_r$, and $d_r^{alg}([x]_r) = [y]_r$.

Now we have completed diagram (3.3). The results of the theorem follow directly. □

Proof of Theorem 1.6. We prove the following equivalent statement: let $r \geq 2$ be an integer such that z could be lifted to the E_r-page (of the ASS), then x could also be lifted to the E_r-page (of the algNSS).

The statement could be proved inductively. By definition, x could be lifted to the E_2-page. Now assume we already know x could be lifted to the E_i-page with $2 \leq i < r$, we show $d_i^{alg}([x]_i) = 0$. Hence x could be lifted to the E_{i+1}-page.

We study the differential $d_i^{alg}([x]_i)$ via diagram (2.4). As we will see, with the given assumptions, diagram (2.4) can be specialized to the following diagram.

(3.7)
$$\begin{tikzcd}
{[x]_i} \arrow[r, "\kappa_i"] \arrow[d, "d_i^{alg}"'] & {[\tilde{z}]_i} \arrow[d, "\delta_i^{mAdams}"] & {[\tilde{z}]_i} \arrow[l, "\psi_i"'] \arrow[r, "\phi_i"] \arrow[d, "d_i^{mAdams}"] & {[z]_i} \arrow[d, "d_i^{Adams}"] \\
0 \arrow[r, "\kappa_i"] & 0 & 0 \arrow[l, "\psi_i"'] \arrow[r, "\phi_i"] & 0
\end{tikzcd}$$

By Lemma 3.5, $\tilde{z}$ can be lifted to the E_r-page of the mASS of the sphere. Hence $d_i^{mAdams}([\tilde{z}]_i) = 0$. Note $[\tilde{z}]_i$ is non-τ-divisible, ψ sends $[\tilde{z}]_i$ to the corresponding element in $E_r^{*,*,*}(C\tau)$ of the same form, which we abuse the notation and still denote by $[\tilde{z}]_i$. The commutativity of diagram (2.2) implies $\delta_i^{mAdams}([\tilde{z}]_i) = 0$. Finally, from the isomorphism κ we deduce $d_i^{alg}([x]_i) = 0$.

□

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