THE SECONDARY PERIODIC ELEMENT β_{p^2/p^2-1} AND ITS APPLICATIONS

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Abstract. In this paper, we prove that β_{p^2/p^2-1} survives to E_∞ in the Adams-Novikov spectral sequence for all $p \geqslant 5$. From the Thom map $\Phi : Ext^{s,t}_{BP_*BP}(BP_*, BP_*) \longrightarrow Ext^{s,t}_{A}(\mathbb{Z}/p, \mathbb{Z}/p)$, we also see that h_0h_3 survives to E_∞ in the classical Adams spectral sequence. As an application, we prove that $\beta^p_{p/p}$ is divisible by β_1.

1. Introduction

Let $p \geqslant 5$ be an odd prime. The Adams-Novikov spectral sequence (ANSS) based on the Brown-Peterson spectrum BP is one of the most powerful tools to compute the p-component of the stable homotopy groups of spheres $\pi_*(S^0)$ (*cf.* [1, 6, 10, 22]).

The E_2-term of the ANSS is $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$, which has been extensively studied in low dimensions. For $s = 1$, $Ext^{1,*}_{BP_*BP}(BP_*, BP_*)$ is generated by $\alpha_{sp^n/n+1}$ for $n \geqslant 0$, $p \nmid s \geqslant 1$, where $\alpha_{sp^n/n+1}$ has order p^{n+1} (*cf.* [12, 10]). For $s = 2$, $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$ is the direct sum of cyclic groups generated by $\beta_{sp^n/j,i+1}$ for suitable (n, s, j, i) (*cf.* [10, 22, 23]), $\beta_{sp^n/j,i+1}$ has order p^{i+1}. For $s \geqslant 3$, only partial results of $Ext^{s,*}_{BP_*BP}(BP_*, BP_*)$ are known (*cf.* [11]).

In order to compute the stable homotopy groups of the sphere, we still need to know which elements of the E_2-page could survive to the E_∞-page of the ANSS. It is known that each element $\alpha_{sp^n/n+1}$ is a permanent cycle in the ANSS which represents an element of $\mathrm{Im}J$ with the same order. But we are far from fully determining which elements of the $\beta_{sp^n/j,i+1}$ family could survive to E_∞.

Let $\beta_{sp^n/j}$ denote $\beta_{sp^n/j,1}$. H. Toda [26, 27] proved that $\alpha_1\beta_1^p$ is zero in $\pi_*(S^0)$. This relation supports a non-trivial Adams-Novikov differential called the Toda differential

$$d_{2p-1}(\beta_{p/p}) = a \cdot \alpha_1\beta_1^p \neq 0 \tag{1.1}$$

where a is a non-zero scalar *mod* p. Hence $\beta_{p/p}$ could not survive the ANSS.

Based on the Toda differential (1.1), D. Ravenel [19] generalized the result and proved that there are nontrivial differentials

$$d_{2p-1}(\beta_{p^n/p^n}) \equiv a \cdot \alpha_1\beta^p_{p^{n-1}/p^{n-1}}, \quad \mathrm{mod} \quad \ker\ \beta_1^{p(p^{n-1}-1)/(p-1)}$$

for $n \geqslant 1$. Consequently, β_{p^n/p^n} also can not survive to E_∞ in the ANSS. From this one can see that only $\beta_{sp^n/j} \in H^2(BP_*)$ for $s \geqslant 2$, $1 \leqslant j \leqslant p^n$ or $s = 1$, $1 \leqslant j \leqslant p^n - 1$ might survive to E_∞ in the ANSS. The following are some known results in this area:

Oka [13] proved that for $s = 1$, $1 \leqslant j \leqslant p-1$ or $s \geqslant 2$, $1 \leqslant j \leqslant p$, $\beta_{sp/j}$ is a permanent cycle in the ANSS.

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Oka [15] proved that for $s = 1$, $1 \leqslant j \leqslant 2p-2$ or $s \geqslant 2$, $1 \leqslant j \leqslant 2p$, $\beta_{sp^2/j}$ is a permanent cycle in the ANSS.

Later Oka [17, 18] generalized the result to $n \geqslant 2$, i.e. for $n \geqslant 2$; $s = 1$, $1 \leqslant j \leqslant 2^{n-1}(p-1)$ or $s \geqslant 2$, $1 \leqslant j \leqslant 2^{n-1}p$, $\beta_{sp^n/j}$ survives to E_∞ in the ANSS.

Shimomura [25] proved that for $s \geqslant 1$, $1 \leqslant j \leqslant p^2-2$, $\beta_{sp^2/j}$ survives to E_∞ in the ANSS.

In this paper, we prove:

Theorem A *Let $p \geqslant 5$ be an odd prime. Then β_{p^2/p^2-1} is a permanent cycle in the Adams-Novikov spectral sequence.*

Let M be the *mod* p Moore spectrum and $M(1, p^n-1)$ be the cofiber of the map $v_1^{p^n-1}$

$$\Sigma^* M \xrightarrow{v_1^{p^n-1}} M \longrightarrow M(1, p^n-1).$$

D. Ravenel ([24] Theorem 7.12) claimed that if $M(1,p^n-1)$ is a ring spectrum and β_{p^n/p^n-1} is a permanent cycle, then $\beta_{sp^n/j}$ is a permanent cycle for all $s \geqslant 1$, $j \leqslant p^n-1$.

Between the ANSS and the classical Adams spectral sequence (ASS), there is the Thom reduction map

$$\Phi : Ext^*_{BP_*BP}(BP_*, BP_*) \longrightarrow Ext^*_A(\mathbb{Z}/p, \mathbb{Z}/p)$$

such that $\Phi(\beta_{p^n/p^n-1}) = h_0 h_{n+1}$. Thus

Corollary B *Let $p \geqslant 5$ be an odd prime. Then h_0h_3 is a permanent cycle in the classical Adams spectral sequence.*

In [3] R. Cohen and P. Goerss claimed the existence of h_0h_{n+1} in the classical ASS. One can see that the existence of h_0h_{n+1} in ASS is equivalent to the existence of β_{p^n/p^n-1} in the Adams-Novikov spectral sequence. But N. Minami found a fatal error in their proof, so it is still an open problem in odd primary stable homotopy theory. Due to its extreme importance, M. Hovey [4] listed the convergence of h_0h_{n+1} as one of the major open problems in algebraic topology.

Consider the ANSS for the Moore spectrum $Ext^{*,*}_{BP_*BP}(BP_*, BP_*(M)) \Longrightarrow \pi_*(M)$. From the Toda differential, one can see that in the ANSS for the Moore spectrum

$$d_{2p-1}(h_{n+2}) = v_1\beta^p_{p^n/p^n}, \qquad d_{2p-1}(v_1h_{n+2}) = v_1^2\beta^p_{p^n/p^n}.$$

Applying the connecting homomorphism $\delta : Ext^{1,*}_{BP_*BP}(BP_*, BP_*(M)) \longrightarrow Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$ induced by the cofiber sequence

$$S^0 \xrightarrow{p} S^0 \longrightarrow M$$

one get an Adams differential in the ANSS for sphere

$$d_{2p-1}(\beta_{p^{n+1}/p^{n+1}-1}) = \alpha_2\beta^p_{p^n/p^n}.$$

In Section 6, we prove that $\beta^p_{p/p}$ is divisible by β_1, i.e. $\beta^p_{p/p} = \beta_1\mathfrak{g}$. Note $\alpha_2\beta_1 = 0$, this provides another perspective for understanding why we could have

$$d_{2p-1}(\beta_{p^2/p^2-1}) = \alpha_2\beta^p_{p/p} = 0 \text{ in } Ext^{2p+1,*}_{BP_*BP}(BP_*, BP_*).$$

in Theorem A.

Based on the analysis of $\beta^p_{p/p}$, we conjecture that:

Conjecture C *For $n < p-1$, $\beta^p_{p^n/p^n}$ is divisible by β_1 and*

$$\begin{aligned}
&\beta^p_{p/p} = \beta_1 h_{11} b_{20}^{p-3}\gamma_2\\
&\beta^p_{p^2/p^2} = \beta_1 h_{21} h_{11} b_{30}^{p-4}\delta_3\\
&\cdots\\
&\beta^p_{p^i/p^i} = \beta_1 h_{i,1} h_{i-1,1}\cdots h_{11} b_{i+1,0}^{p-i-2}\alpha^{(i+2)}_{i+1}\\
&\cdots\\
&\beta^p_{p^{p-2}/p^{p-2}} = \beta_1 h_{p-2,1} h_{p-3,1}\cdots h_{11}\alpha^{(p)}_{p-1}
\end{aligned}$$

where $\alpha^{(i+2)}_{i+1}$ is the $i+2$-th Greek letter elements. Thus $\alpha_2\beta^p_{p^n/p^n} = \alpha_2\beta_1\mathfrak{g} = 0$ for $n < p-1$.

For $n \geqslant p-1$, $\beta^p_{p^n/p^n}$ is not divisible by β_1 and $\alpha_2\beta^p_{p^n/p^n}$ might be non-zero. This implies that $\beta_{p^{n+1}/p^{n+1}-1}$ does not survives to E_∞ in the ANSS when $n \geqslant p-1$.

This paper is arranged as follows. In section 2 we recall the construction of the topological small descent spectral sequence (TSDSS) and the small descent spectral sequence (SDSS), where the SDSS is a spectral sequence that converges to $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$ started from the Ext groups of a complex with p-cells. Then we describe the E_1-terms of the SDSS in the form of Generator, total degree $t-s$ and $t-s$ *mod* $pq-2$ and range of index. This gives a method to compute the E_2-page of the ANSS with specialized $t-s$. In section 3 we compute the Adams-Novikov E_2-term $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$ subject to $t-s = q(p^3+1)-3$ by the SDSS. In section 4, a non-trivial Adams-Novikov differential $d_{2p-1}(h_{20}b_{11}\gamma_s) = \alpha_1\beta_1^p h_{20}\gamma_s$ is proved. From which we prove our main theorem by showing that $d_r(\beta_{p^2/p^2-1}) = 0$ in section 5. At last, in section 6, we prove that $\beta^p_{p/p}$ is divisible by β_1 and give our conjecture.

2. The small descent spectral sequence and the ABC Theorem

In 1985, D. Ravenel [20, 21, 22, 23] introduced the *method of infinite descent* and used it to compute the first thousand stems of the stable homotopy groups of spheres at the prime 5. This method is an approach to finding the E_2-term of the ANSS by the spectral sequence referred as the *small descent spectral sequence* (SDSS).

Hereafter we set that $q = 2p-2$. Let $T(n)$ be the Ranevel spectrum (*cf.* [22] Section 5, Chapter 6) characterized by

$$BP_*(T(n)) = BP_*[t_1, t_2, \cdots, t_n].$$

Then we have the following diagram

$$S^0 = T(0) \longrightarrow T(1) \longrightarrow T(2) \longrightarrow \cdots \longrightarrow T(n) \longrightarrow \cdots \longrightarrow BP,$$

where S^0 denote the sphere spectrum localized at an odd prime $p \geqslant 5$. Let $T(0)_{p-1}$ and $T(0)_{p-2}$ denote the $q(p-1)$ and $q(p-2)$ skeletons of $T(1)$ respectively, they are denoted by X and $\overline{X}$ for simple. Then

$$X = S^0 \cup_{\alpha_1} e^q \cup \cdots \cup_{\alpha_1} e^{(p-2)q} \cup_{\alpha_1} e^{(p-1)q} \qquad \text{and} \qquad \overline{X} = S^0 \cup_{\alpha_1} e^q \cup \cdots \cup_{\alpha_1} e^{(p-2)q}.$$

The BP-homologies of them are

$$BP_*(X) = BP_*[t_1]/\langle t_1^p\rangle \qquad \text{and} \qquad BP_*(\overline{X}) = BP_*[t_1]/\langle t_1^{p-1}\rangle.$$

From the definition above we get the following cofibre sequences

$$S^0 \xrightarrow{i'} X \xrightarrow{j'} \Sigma^q\overline{X} \xrightarrow{k'} S^1, \tag{2.1}$$

$$\overline{X} \xrightarrow{i''} X \xrightarrow{j''} S^{(p-1)q} \xrightarrow{k''} \Sigma\overline{X}, \tag{2.2}$$

and the short exact sequences of BP-homologies

$$0 \longrightarrow BP_*(S^0) \xrightarrow{i'_*} BP_*(X) \xrightarrow{j'_*} BP_*(\Sigma^q\overline{X}) \longrightarrow 0, \tag{2.3}$$

$$0 \longrightarrow BP_*(\overline{X}) \xrightarrow{i''_*} BP_*(X) \xrightarrow{j''_*} BP_*(S^{(p-1)q}) \longrightarrow 0. \tag{2.4}$$

Put (2.3) and (2.4) together, one has the following long exact sequence

$$0 \longrightarrow BP_*(S^0) \longrightarrow BP_*(X) \longrightarrow BP_*(\Sigma^q X) \longrightarrow BP_*(\Sigma^{pq} X) \longrightarrow \cdots . \tag{2.5}$$

Put (2.1) and (2.2) together, one has the following Adams diagram of cofibres

$$\begin{array}{ccccccccc} S^0 & \longleftarrow & \Sigma^{q-1}\overline{X} & \longleftarrow & S^{pq-2} & \longleftarrow & \Sigma^{(p+1)q-3}\overline{X} & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X & & \Sigma^{q-1}X & & \Sigma^{pq-2}X & & \Sigma^{(p+1)q-3}X. & & \end{array} \tag{2.6}$$

Thus one has:

Proposition 2.1 *(Ravenel [22] 7.4.2 Proposition) Let X be as above. Then*

(a) *There is a spectral sequence converging to $Ext^{s+u,*}_{BP_*BP}(BP_*, BP_*(S^0))$ with E_1-term*

$$\begin{aligned} E_1^{s,t,u} =& Ext^{s,t}_{BP_*BP}(BP_*, BP_*(X)) \otimes E[\alpha_1] \otimes P[\beta_1], \quad \text{where} \\ & E_1^{s,t,0} = Ext^{s,t}_{BP_*BP}(BP_*, BP_*(X)), \qquad \alpha_1 \in E_1^{0,q,1}, \qquad \beta_1 \in E_1^{0,qp,2} \end{aligned}$$

and $d_r : E_r^{s,t,u} \longrightarrow E_r^{s-r+1,t,u+r}$. Where $E[-]$ denotes the exterior algebra and $P[-]$ denotes the polynomial algebra on the indicated generators. This spectral sequence is referred as the small descent spectral sequence (SDSS).

(b) *There is a spectral sequence converging to $\pi_*(S^0)$ with E_1-term*

$$\begin{aligned} E_1^{s,t} =& \pi_*(X) \otimes E[\alpha_1] \otimes P[\beta_1], \quad \text{where} \\ & E_1^{0,t} = \pi_t(X), \qquad \alpha_1 \in E_1^{1,q}, \qquad \beta_1 \in E_1^{2,pq} \end{aligned}$$

and $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$. This spectral sequence is referred as the topological small descent spectral sequence (TSDSS).

The above two spectral sequences produce the 0-line and the 1-line $Ext^{0,*}_{BP_*BP}(BP_*, BP_*(S^0))$, $Ext^{1,*}_{BP_*BP}(BP_*, BP_*(S^0))$ or the corresponding elements in $\pi_*(S^0)$ by $Ext^{0,*}_{BP_*BP}(BP_*, BP_*(X))$ and $Ext^{1,*}_{BP_*BP}(BP_*, BP_*(X))$. $Ext^{s,*}_{BP_*BP}(BP_*, BP_*(S^0))$ ($s \geqslant 2$) or the corresponding elements in $\pi_*(S^0)$ is produced by $Ext^{s,*}_{BP_*BP}(BP_*, BP_*(X))$ ($s \geqslant 2$) as described as the following ABC Theorem.

ABC Theorem *([23] 7.5.1 ABC Theorem) For $p > 2$ and $t - s < q(p^3 + p - 1) - 3$, $s \geqslant 2$*

$$Ext^{s,t}_{BP_*BP}(BP_*, BP_*(X)) = Ext^{s-2,*}_{BP_*BP}(BP_*, BP_*(X) \otimes E_1^2) = A \oplus B \oplus C,$$

where A is the $\mathbb{Z}/p$-vector space spanned by

$$\begin{aligned} A =& \{\beta_{ip},\ \beta_{ip+1} | i \leqslant p-1\} \cup \left\{\beta_{p^2/p^2-j} | 0 \leqslant j \leqslant p-1\right\}, \\ B =& R \otimes \{\gamma_i | i \geqslant 2\} \end{aligned}$$

where

$$R = P[b_{20}^p] \otimes E[h_{20}] \otimes \mathbb{Z}/p\left\{\left\{b_{11}^k | 0 \leqslant k \leqslant p-1\right\} \cup \left\{h_{11}b_{20}^k | 0 \leqslant k \leqslant p-2\right\}\right\},$$

and

$$C^{s,t} = \bigoplus_{i\geqslant 0} R^{s+2i,t+i(p^2-1)q}.$$

From the generators of R and [20] 4.11, 4.12 Theorem, we read the generators of C as follows: Let $i = jp + m$. Then from $R^{s+2\underline{i},t+\underline{i}(p^2-1)q} \subset C^{s,t}$ we have:

(1) $b_{20}^{(j+1)p} \in R^{2(p-m)+2(\underline{jp+m}),t+(\underline{jp+m})(p^2-1)q} \subset C^{2(p-m),t}$, which is represented by

$$b_{20}^{p-m-1} u_{jp+m}$$

for $p-1 \geqslant m \geqslant 1$. From which we get

$$b_{20}^{p-m-1} u_{jp+m} \otimes E[h_{20}] \otimes \left\{ b_{11}^k | 0 \leqslant k \leqslant p-1 \right\} \cup \left\{ h_{11} b_{20}^k | 0 \leqslant k \leqslant p-2 \right\},$$

where

$$u_{jp+m} \in C^{2,q\left[(j+1)p^2+(j+m+1)p+m\right]}.$$

(2) $b_{11}^k b_{20}^{jp} \in R^{2(k-m)+2(\underline{jp+m}),t+(\underline{jp+m})(p^2-1)q} \subset C^{2(k-m),t}$, which is represented by

$$b_{11}^{k-m-1} \beta_{(j+1)p/p-m}$$

for $p-1 \geqslant k \geqslant m+1 \geqslant 1$. From which we get

$$b_{11}^{k-m-1} \beta_{(j+1)p/p-m} \otimes E[h_{20}],$$

where

$$\beta_{(j+1)p/p-m} \in C^{2,q\left[(j+1)p^2+jp+m\right]}.$$

- Especially $h_{20} b_{11}^{p-1} b_{20}^{jp} \in R^{3+2(\underline{jp+p-2}),t+(\underline{jp+p-2})(p^2-1)q} \subset C^{3,t}$ is represented by $h_{11}\beta_{(j+1)p/1,2}$, which is an element of order p^2.

(3) $h_{11} b_{20}^k b_{20}^{jp} \in R^{2(k-m)+1+2(\underline{jp+m}),t+(\underline{jp+m})(p^2-1)q} \subset C^{2(k-m)+1,t}$, which is represented by

$$b_{20}^{k-m-1} \eta_{jp+m+1}$$

for $p-2 \geqslant k \geqslant m+1 \geqslant 1$, where

$$\eta_{jp+m+1} = h_{11} u_{jp+m} \in C^{3,q\left[(j+1)p^2+(j+m+2)p+m\right]}.$$

(4) $h_{20} h_{11} b_{20}^k b_{20}^{jp} \in R^{2(k-m+1)+2(\underline{jp+m}),t+(\underline{jp+m})(p^2-1)q} \subset C^{2(k-m+1)t}$, which is represented by

$$b_{20}^{k-m} \beta_{jp+m+2}$$

for $p-2 \geqslant k \geqslant m \geqslant 0$, where

$$\beta_{jp+m+2} \in C^{2,q\left[jp^2+(j+m+2)p+m+1\right]}.$$

- Especially $h_{20} h_{11} b_{20}^{p-2} b_{20}^{jp} \in R^{2+2(\underline{jp+p-2}),t+(\underline{jp+p-2})(p^2-1)q} \subset C^{2,t}$ is represented by $\beta_{(j+1)p/1,2}$, which is an element of order p^2.

From the ABC Theorem above, we know that $Ext_{BP_*BP}^{s,t}(BP_*, BP_*(X))$ for $s \geqslant 2$, $t-s < q(p^3+p-1)-3$ is the $\mathbb{Z}_{(p)}$-module generated by the following generators, here the generators are listed as generators, total degree $t-s$ and $t-s$ *mod* $pq-2$, range of index; where $pq-2 = 2p^2-2p-2$ is the total degree of $\beta_1 \in E_1^{0,qp,2}$ in the SDSS.

Generators of A	$t-s$ and $t-s$ *mod* $pq-2$	Range of index
β_{ip}	$q[ip^2+ip-1]-2$	
	$\equiv 2(i-1)p+2i$	if $i \leqslant p-2$
	$\equiv 0$	if $i=p-1$
β_{ip+1}	$q[ip^2+(i+1)p]-2$	
	$\equiv 2ip+2i$	if $i \leqslant p-2$
	$\equiv \underline{2p}_{2p}$	if $i=p-1$
β_{p^2/p^2-j}	$q[p^3+j]-2$	
	$\equiv \underline{2(j+1)p-2j}_{2p}$	if $j \leqslant p-2$
	$\equiv 4$	if $j=p-1$

Generators of B	$t-s$ and $t-s$ mod $pq-2$	Range of index
$h_{11}b_{20}^k\gamma_i$	$q[(i+k)p^2+(i+k)p+i-2]-2k-4$	for $2 \leqslant i \leqslant p-1, 0 \leqslant k \leqslant p-2, 2 \leqslant i+k \leqslant p-1$
	$\equiv 2(k+2i-2)p$	if $k+2i \leqslant p$
	$\equiv 2(k+2i-p-1)p+2$	if $k+2i > p$
$h_{20}h_{11}b_{2,0}^k\gamma_i$	$q[(i+k)p^2+(i+k+1)p+i-1]-2k-5$	for $2 \leqslant i \leqslant p-1, 0 \leqslant k \leqslant p-2, 2 \leqslant i+k \leqslant p-1$
	$\equiv 2(k+2i-1)p-1$	if $k+2i < p$
	$\equiv 2(k+2i-p)p+1$	if $k+2i \geqslant p$
$b_{11}^k\gamma_i$	$q[(i+k)p^2+(i-1)p+i-2]-2k-3$	for $2 \leqslant i \leqslant p-1, 0 \leqslant k \leqslant p-1, 2 \leqslant i+k \leqslant p-1$
	$\equiv 2(k+2i-2)p-2k-1$	if $k+2i \leqslant p+1$
	$\equiv 1$	if $k=0,\ 2i=p+1$
	$\equiv \underline{2(k+2i-p-1)p-2k+1}_{4p-3}$	if $k+2i \geqslant p+2$
$h_{20}b_{11}^k\gamma_i$	$q[(i+k)p^2+ip+i-1]-2k-4$	for $2 \leqslant i \leqslant p-1, 0 \leqslant k \leqslant p-1, 2 \leqslant i+k \leqslant p-1$
	$\equiv 2(k+2i-1)p-2(k+1)$	if $k+2i \leqslant p$
	$\equiv \underline{2(k+2i-p)p-2k}_{2p}$	if $k+2i > p$

Generators of C	$t-s$ and $t-s$ mod $pq-2$	Range of index
$b_{11}^{k}b_{20}^{p-m-1}u_{jp+m}$	$q[(p-m+j+k+1)p^2+jp+m]-2(p-m+k)$	for $1\leqslant m<p, 0\leqslant j\leqslant p-2, 0\leqslant k<p$ and $j+k<m$
	$\equiv 2(j+k+1)p+2(j-k+1)$	
$h_{20}b_{11}^{k}b_{2,0}^{p-m-1}u_{jp+m}$	$q[(p-m+j+k+1)p^2+(j+1)p+m+1]-2(p-m+k)-1$	for $1\leqslant m<p, 0\leqslant j\leqslant p-2, 0\leqslant k<p$ and $j+k<m, j+k\leqslant p-3$
	$\equiv 2(j+k+2)p+2(j-k+1)-1$	if $j+k\leqslant p-4$ or $j+k=p-3, 2j<p-5$
	$\equiv 2(j-k+2)p-1$	if $j+k=p-3, 2j\geqslant p-5$
$h_{11}b_{20}^{k+p-m-1}u_{jp+m}$	$q[(p-m+j+k+1)p^2+(j+k+1)p+m]-2(p-m+k)-1$	for $1\leqslant m<p, 0\leqslant j\leqslant p-2, 0\leqslant k\leqslant p-2$ and $j+k<m, j+k\leqslant p-3$
	$\equiv 2(j+k+2)p+2(j-p)+3$	
$h_{20}h_{11}b_{2,0}^{k+p-m-1}u_{jp+m}$	$q[(p-m+j+k+1)p^2+(j+k+2)p+m]+2(m-k-2)$	for $1\leqslant m<p, 0\leqslant j\leqslant p-2, 0\leqslant k\leqslant p-2$ and $j+k<m, j+k\leqslant p-3$
	$\equiv 2(j+k+2)p+2j+2$	if $j+k\leqslant p-4$
	$\equiv 2j+4$	if $j+k=p-3$
$b_{11}^{k-m-1}\beta_{(j+1)p/p-m}$	$q[(j+k-m)p^2+jp+m]-(2k-2m)$	for $1\leqslant m+1\leqslant k<p, 0\leqslant j\leqslant p-2$
	$\equiv 2(j+k)p+2(j-k)$	if $j+k\leqslant p-2$ or $j+k=p-1$, $2j<p-1$
	$\underline{\equiv 2(j+k-p+1)p+2(j-k+1)}_{2p}$	if $j+k\geqslant p$ or $j+k=p-1$, $2j\geq p-1$
$h_{20}b_{11}^{k-m-1}\beta_{(j+1)p/p-m}$	$q[(j+k-m)p^2+(j+1)p+m+1]-(2k-2m+1)$	for $1\leqslant m+1\leqslant k<p, 0\leqslant j\leqslant p-2$
	$\underline{\equiv 2(j+k+1)p+2(j-k)-1}_{4p-3}$	if $j+k\leqslant p-3$ or $j+k=p-2$, $2j\leqslant p-3$
	$\equiv 2(j+k-p+2)p+2(j-k)+1$	if $j+k>p-2$ or $j+k=p-2$, $2j>p-3$

continue to next page. . .

Generators of C	$t-s$ and $t-s$ mod $pq-2$	Range of index
$h_{1,1}\beta_{(j+1)p/1,2}$	$q[(j+1)p^2+(j+2)p-1]-3$	for $0\leqslant j\leqslant p-2$
	$\equiv 2jp+2(j+1)+1$	if $j\leqslant p-3$
	$\equiv 1$	if $j=p-2$
$b_{2,0}^{k-m-1}\eta_{jp+m+1}$	$q[(j+k-m)p^2+(j+k+1)p+m]-(2k-2m+1)$	for $1\leqslant m+1\leqslant k\leqslant p-2, 0\leqslant j\leqslant p-2$
	$\equiv 2(j+k)p+2j+1$	if $j+k\leqslant p-2$
	$\equiv \underline{2(j+k-p+2)p+2(j-p)+3}_{4p-3}$	if $j+k>p-2$
$b_{2,0}^{k-m}\beta_{jp+m+2}$	$q[(j+k-m)p^2+(j+k+2)p+m+1]-2(k-m+1)$	for $0\leqslant m\leqslant k\leqslant p-2, 0\leqslant j\leqslant p-2$
	$\equiv \underline{2(j+k+1)p+2j}_{2p}$	if $j+k\leqslant p-3$
	$\equiv 2(j+k-p+3)p+2(j-p)+2$	if $j+k>p-3$
	$\equiv 0$	if $j=k=p-2$
$\beta_{(j+1)p/1,2}$	$q[(j+1)p^2+(j+1)p-1]-2$	for $0\leqslant j\leqslant p-2$
	$\equiv 2jp+2(j+1)$	if $j\leqslant p-3$
	$\equiv 0$	if $j=p-2$

Let us take $h_{11}b_{20}^k\gamma_i$ from the B-family as an example to illustrate the calculation.

The total degree of $h_{11}b_{20}^k\gamma_i$ is

$$q[(i+k)p^2+(i+k)p+i-2]-2k-4=2(i+k)p^3-2(k+2)p-2(i+k)$$

for $2\leqslant i\leqslant p-1, 0\leqslant k\leqslant p-2$. To ensure that the total degree of $h_{11}b_{20}^k\gamma_i$ is less than $q(p^3+p-1)-3$, we need $i+k<p$. Straightforward computation shows

$$2(i+k)p^3-2(k+2)p-2(i+k)\equiv 2(k+2i-2)p \quad mod \quad pq-2$$

Notice that $2(k+2i-2)p>pq-2$ if $k+2i>p$, the total degree of $h_{11}b_{20}^k\gamma_i$ is

$$2(k+2i-2)p-(pq-2)=2(k+2i-p-1)p+2 \quad mod \ \ pq-2$$

if $k+2i>p$.

Remark. The Adams-Novikov spectral sequence for the spectrum X collapses from E_2-term $Ext_{BP_*BP}^{s,t}(BP_*,BP_*(X))$ in the range $t-s<q(p^3+p-1)-3$, since there are no elements with filtration $>2p$. Thus we actually get the homotopy groups $\pi_{t-s}(X)$ in this range.

3. The ANSS E_2-term $Ext_{BP_*BP}^{s,t}(BP_*,BP_*)$ at $t-s=q(p^3+1)-3$

Consider the Adams-Novikov differential $d_r: E_r^{s,t}\to E_r^{s+r,t+r-1}$ in the ANSS. From the total degree of β_{p^2/p^2-1}, we know that $d_r(\beta_{p^2/p^2-1})\in Ext_{BP_*BP}^{s,t}(BP_*,BP_*)$ such that $t-s=q(p^3+1)-3$. The SDSS $E_1^{s,t,u}$ converges to $Ext_{BP_*BP}^{s+u,t}(BP_*,BP_*)$. Fix $t-s-u=q(p^3+1)-3$, we have:

Lemma 3.1. *Fix $t-s-u=q(p^3+1)-3$, the E_1-term $E_1^{s,t,u}$ of the SDSS is the $\mathbb{Z}/p$-module generated by the following 8 generators:*

$$\mathfrak{g}_1=\alpha_1\beta_1^{p^2-1}\beta_2\in E_1^{2,*,2p^2-1}; \qquad \mathfrak{g}_2=\beta_1^{p^2-p}h_{20}\beta_{p/p}\in E_1^{3,*,2p^2-2p};$$

$$\mathfrak{g}_3=\alpha_1\beta_1^{\frac{p^2-2p-1}{2}}h_{2,0}\gamma_{\frac{p+1}{2}}\in E_1^{4,*,p^2-2p}; \qquad \mathfrak{g}_4=\beta_1^{\frac{p^2-6p+1}{2}}b_{11}^2\gamma_{\frac{p+1}{2}}\in E_1^{7,*,p^2-6p+1};$$

$$\mathfrak{g}_5=\alpha_1\beta_1^{mp-\frac{p-1}{2}}b_{11}^{\frac{p-1}{2}-m}\beta_{(\frac{p+1}{2})p/p-m}\in E_1^{p+1-2m,*,*}; \qquad \mathfrak{g}_6=\beta_1^{p-1}\eta_{(p-3)p+3}\in E_1^{3,*,2p-2};$$

$$\mathfrak{g}_7=\alpha_1\beta_{(p-1)p+1}\in E_1^{2,q(p^3+1),1}; \qquad \mathfrak{g}_8=\alpha_1\beta_{p^2/p^2}\in E_1^{2,q(p^3+1),1}.$$

Proof. Fix $t-s-u=q(p^3+1)-3$. From the ABC Theorem, we know that the generators of the E_1-terms in the SDSS are of the form $W=\beta_1^k w$ or $W=\alpha_1\beta_1^k w$, where w is an element listed in the ABC Theorem.

1. If a generator of $E_1^{s,t,u}$ is of the form $W=\beta_1^k w$, then the total degree of $\beta_1^p w$ is $q(p^3+1)-3$ and the total degree of w is $q(p^3+1)-3$ modulo the total degree of β_1 which is $t-u=qp-2$. Note that

$$q(p^3+1)-3\equiv 4p-3 \qquad mod\ qp-2,$$

we list all the generators whose total degree might be $4p-3$ mod $qp-2$, which are marked with underline and subscript $4p-3$ in the table for ABC Theorem.

$b_{11}^k\gamma_i$	at $k=2$ and $i=(p+1)/2$;
$h_{20}b_{11}^{k-m-1}\beta_{(j+1)p/p-m}$	at $k=1$ and $j=0$;
$b_{20}^{k-m-1}\eta_{jp+m+1}$	at $k=3$ and $j=p-3$.

From which we get the following generators in $E_1^{s,t,u}$:

$$b_{11}^2\gamma_{\frac{p+1}{2}} \Longrightarrow \mathfrak{g}_4 = \beta_1^{\frac{p^2-6p+1}{2}} b_{11}^2\gamma_{\frac{p+1}{2}} \in E_1^{7,*,p^2-6p+1};$$
$$h_{20}\beta_{p/p} \Longrightarrow \mathfrak{g}_2 = \beta_1^{p^2-p}h_{20}\beta_{p/p} \in E_1^{3,*,2p^2-2p};$$
$$\eta_{(p-3)p+3} \Longrightarrow \mathfrak{g}_6 = \beta_1^{p-1}\eta_{(p-3)p+3} \in E_1^{3,*,2p-2}.$$

2. If a generator of $E_1^{s,t,u}$ is of the form $W = \alpha_1\beta_1^k w_1$, then from the total degree of α_1 being $t-u = 2p-3$ we see that the total degree of w_1 is $2p$ modulo $qp-2$. Similarly we can find all such w_1's, which are marked with underline and subscript $2p$ in the ABC Theorem:

$$\beta_{(p-1)p+1}; \qquad \beta_{p^2/p^2}; \qquad h_{20}\gamma_{\frac{p+1}{2}}; \qquad b_{11}^{\frac{p-1}{2}-m}\beta_{(\frac{p+1}{2})p/p-m}; \qquad \beta_2.$$

From which we get the following generators in $E_1^{s,t,u}$:

$$\mathfrak{g}_7 = \alpha_1\beta_{(p-1)p+1}; \qquad \mathfrak{g}_8 = \alpha_1\beta_{p^2/p^2};$$
$$\mathfrak{g}_3 = \alpha_1\beta_1^{\frac{p^2-2p-1}{2}} h_{2,0}\gamma_{\frac{p+1}{2}}; \qquad \mathfrak{g}_5 = \alpha_1\beta_1^{mp-\frac{p-1}{2}} b_{11}^{\frac{p-1}{2}-m}\beta_{(\frac{p+1}{2})p/p-m};$$
$$\mathfrak{g}_1 = \alpha_1\beta_1^{p^2-1}\beta_2.$$

Computing the filtration of the corresponding generators, we get the lemma. □

Remark: The method in proving Lemma 3.1 is a general method in computing the E_1-term $E_1^{s,t,u}$ of the SDSS with specialized $t-s-u$.

Theorem 3.2. *Fix $t-s = q(p^3+1)-3$, the Adams-Novikov E_2-term $Ext_{BP_*BP}^{s,t}(BP_*, BP_*)$ is the $\mathbb{Z}/p$-module generated by the following 6 elements*

$$\mathfrak{g}_1 = \alpha_1\beta_1^{p^2-1}\beta_2 \in Ext_{BP_*BP}^{2p^2+1,*};$$
$$\mathfrak{g}_3 = \alpha_1\beta_1^{\frac{p^2-2p-1}{2}} h_{2,0}\gamma_{\frac{p+1}{2}} \in Ext_{BP_*BP}^{p^2-2p+4,*}; \qquad \mathfrak{g}_4 = \beta_1^{\frac{p^2-6p+1}{2}} b_{11}^2\gamma_{\frac{p+1}{2}} \in Ext_{BP_*BP}^{p^2-6p+8,*};$$
$$\mathfrak{g}_6 = \beta_1^{p-1}\eta_{(p-3)p+3} \in Ext_{BP_*BP}^{2p+1,*};$$
$$\mathfrak{g}_7 = \alpha_1\beta_{(p-1)p+1} \in Ext^{3,q(p^3+1)}; \qquad \mathfrak{g}_8 = \alpha_1\beta_{p^2/p^2} \in Ext^{3,q(p^3+1)}.$$

Proof. Following D. Ravenel [22] page 287, we compute in the cobar complex of $N_0^2 = BP_*/(p^\infty, v_1^\infty)$

$$\begin{aligned}
d\left(\frac{v_2^{jp}}{pv_1^p}(t_2 - t_1^{p+1})\right) &= \frac{v_2^{jp}}{pv_1^p}t_1^p \otimes t_1 + \frac{v_2^{jp}}{pv_1^{p-1}}b_{10},\\
-d\left(\frac{v_2^{jp+1}}{pv_1^{p+1}}t_1\right) &= -\frac{v_2^{jp}}{pv_1^p}t_1^p \otimes t_1 - j\frac{v_2^{(j-1)p+1}}{pv_1}t_1^{p^2} \otimes t_1 + \frac{v_2^{jp}}{pv_1}t_1 \otimes t_1,\\
d\left(j\frac{v_2^{(j-1)p}v_3}{pv_1}t_1\right) &= j\frac{v_2^{(j-1)p+1}}{pv_1}t_1^{p^2} \otimes t_1 - j\frac{v_2^{jp}}{pv_1}t_1 \otimes t_1,\\
-(j-1)/2d\left(\frac{v_2^{jp}}{pv_1}t_1^2\right) &= (j-1)\frac{v_2^{jp}}{pv_1}t_1 \otimes t_1.
\end{aligned}$$

Straightforward calculation shows that the coboundary of

$$\frac{v_2^{jp}}{pv_1^p}t_2 - \frac{v_2^{jp}}{pv_1^p}t_1^{p+1} - \frac{v_2^{jp+1}}{pv_1^{p+1}}t_1 + j\frac{v_2^{(j-1)p}v_3}{pv_1}t_1 - (j-1)/2\frac{v_2^{jp}}{pv_1}t_1^2$$

is $\dfrac{v_2^{jp}}{pv_1^{p-1}}b_{10}$. Then from $\delta\delta\left(\dfrac{v_2^{jp}}{pv_1^{p}}\right) = \beta_{jp/p}$, we get a differential in the SDSS

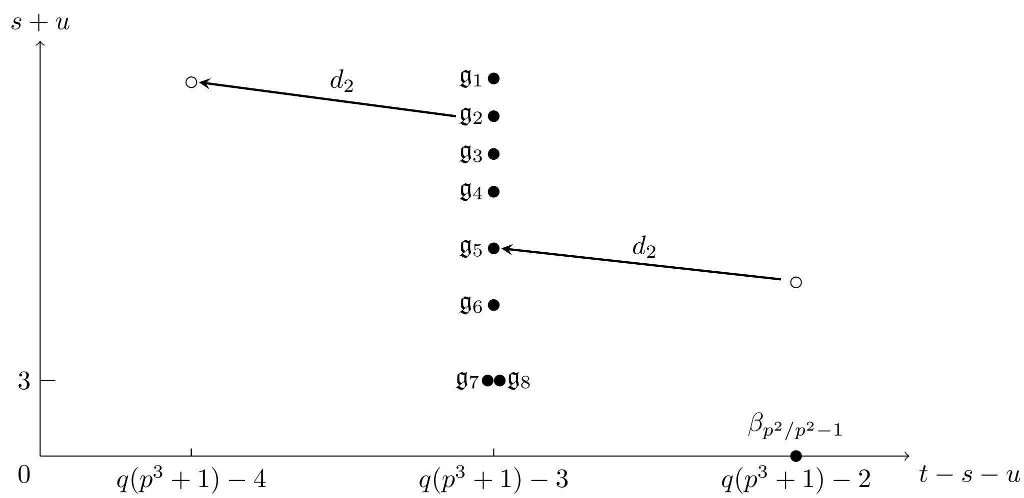
$$d_2(h_{20}\beta_{jp/p}) = \beta_1\beta_{jp/p-1}.$$

Similarly, we have

$$d_2(h_{20}\beta_{jp/i}) = \beta_1\beta_{jp/i-1} \qquad \text{for } 2 \leqslant i \leqslant p. \tag{3.1}$$

Applying formula (3.1), we get the following differentials in the SDSS

$$d_2(\mathfrak{g}_2) = d_2(\beta_1^{p^2-p}h_{20}\beta_{p/p}) = \beta_1^{p^2-p+1}\beta_{p/p-1},$$
$$d_2(\alpha_1\beta_1^{mp-\frac{p-1}{2}-1}b_{11}^{\frac{p-1}{2}-m}h_{20}\beta_{(\frac{p+1}{2})p/p-m+1}) = \alpha_1\beta_1^{mp-\frac{p-1}{2}}b_{11}^{\frac{p-1}{2}-m}\beta_{(\frac{p+1}{2})p/p-m} = \mathfrak{g}_5$$



The theorem follows. ∎

4. A differential in the ANSS

This section is armed at showing that

$$d_{2p-1}(h_{20}b_{11}\gamma_s) = \alpha_1\beta_1^p h_{20}\gamma_s \tag{4.1}$$

in the ANSS, which will be used in proving Theorem A in section 5.

We begin from showing that $\pi_{q(p^2+2p+2)-2}(V(2)) = 0$. From which we show that the Toda bracket $\langle\alpha_1\beta_1, p, \gamma_s\rangle = 0$ and the Toda bracket $\langle\alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle$ is well defined. Then from the relation

$$\langle\alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle = \alpha_1\beta_1^{p-1}h_{20}\gamma_s = \beta_{p/p-1}\gamma_s$$

in $\pi_*(S^0)$ and $d(h_{20}b_{11}) = \beta_1\beta_{p/p-1}$, we get the desired differential in the ANSS.

Let $p \geqslant 5$ be an odd prime and $V(2)$ be the Smith-Toda spectrum characterized by

$$BP_*(V(2)) = BP_*/I_3$$

where I_3 is the invariant ideal of $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_i, \cdots]$ generated by p, v_1 and v_2. To compute the homotopy groups of $V(2)$, one has the ANSS $\{E_r^{s,t}V(2), d_r\}$ that converges to $\pi_*(V(2))$. The E_2-page of this spectral sequence is

$$E_2^{s,t}V(2) = Ext_{BP_*BP}^{s,t}(BP_*, BP_*(V(2)))$$

Let

$$\Gamma = BP_*/I_3 \otimes_{BP_*} BP_*BP \otimes_{BP_*} BP_*/I_3 = BP_*/I_3[t_1, t_2, \cdots].$$

Then $(BP_*/I_3, \Gamma)$ is a Hopf algebroid, and its structure map is deduced from that of $(BP_*, BP_*(BP))$. By a change of ring theorem, one sees that

$$Ext^{s,t}_{BP_*BP}(BP_*, BP_*(V(2))) = Ext^{s,t}_{\Gamma}(BP_*, BP_*/I_3) \Longrightarrow \pi_*(V(2))$$

Lemma 4.1. *The $q(p^2+2p+2)-2$ dimensional stable homology group of $V(2)$ is trivial, i.e.,*

$$\pi_{q(p^2+2p+2)-2}(V(2)) = 0.$$

Proof. Fix $t-s = q(p^2+2p+2)-2$, we know that the Adams-Novikov E_2-term

$$Ext^{s,s+q(p^2+2p+2)-2}_{BP_*BP}(BP_*, BP_*(V(2))) = Ext^{s,s+q(p^2+2p+2)-2}_{\Gamma}(BP_*, BP_*/I_3)$$

converges to $\pi_{q(p^2+2p+2)-2}(V(2))$. We will prove that $\pi_{q(p^2+2p+2)-2}(V(2)) = 0$ by showing that $Ext^{s,s+q(p^2+2p+2)-2}_{BP_*BP}(BP_*, BP_*(V(2))) = 0$.

In the cobar complex $C^s_\Gamma BP_*/I_3$, the inner degree of v_i, $|v_i| = |t_i| \geqslant q(p^3+p^2+p+1)$ for $i \geqslant 4$. It follows that in the range $t-s \leqslant q(p^3+p^2+p+1)-1$,

$$Ext^{s,t}_{BP_*BP}(BP_*, BP_*/I_3) = Ext^{s,t}_{\Gamma}(BP_*, BP_*/I_3) = Ext^{s,t}_{\Gamma'}(BP_*, BP_*/I_3).$$

where $\Gamma' = \mathbb{Z}/p[v_3][t_1,t_2,t_3]$. From $\eta_R(v_3) \equiv v_3 \ mod \ I_3$, we see that

$$Ext^{s,*}_{\mathbb{Z}/p[v_3][t_1,t_2,t_3]}(BP_*, BP_*/I_3) \cong Ext^{s,*}_{\mathbb{Z}/p[t_1,t_2,t_3]}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes \mathbb{Z}/p[v_3].$$

To compute the Ext groups $Ext^*_{\mathbb{Z}/p[t_1,t_2,t_3]}(\mathbb{Z}/p, \mathbb{Z}/p)$, we can use the modified May spectral sequence (MSS) introduced in [7, 8, 9, 23].

There is the May spectral sequence $\{E_r^{s,t,*}, \delta_r\}$ that converges to $Ext^{s,t}_{\mathbb{Z}/p[t_1,t_2,t_3]}(\mathbb{Z}/p, \mathbb{Z}/p)$. The E_1-term of this spectral sequence is

$$E_1^{*,*,*} = E[h_{ij} | 0 \leqslant j, i = 1,2,3] \otimes P[b_{ij} | 0 \leqslant j, i = 1,2,3], \tag{4.2}$$

where

$$h_{ij} \in E_1^{1, q(1+p+\cdots+p^{i-1})p^j, 2i-1} \qquad \text{and} \qquad b_{ij} \in E_1^{2, q(1+p+\cdots+p^{i-1})p^{j+1}, p(2i-1)}.$$

The first May differential is given by

$$\delta_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j} h_{k,j} \quad \text{and} \quad \delta_1(b_{i,j}) = 0. \tag{4.3}$$

For the reason of the total degree, to compute $Ext^{s,s+(q(p^2+2p+2)-2)}_{BP_*BP}(BP_*, BP_*/I_3)$ we only need to consider the sub-module generated by $h_{30}, h_{20}, h_{10}, h_{21}, h_{11}, h_{12}$ and b_{20}, b_{10}, b_{11}, i. e. the subcomplex

$$E[h_{ij} | 1 \leqslant i, i+j \leqslant 3] \otimes E[b_{20}, b_{11}] \otimes P[b_{10}].$$

From (4.3), we know that within $t-s \leqslant q(p^2+2p+2)-2$ the May's E_2-term

$$E_2^{s,*,*} = H^{s,*,*}(E_1^{s,*,*}, \delta_1) = H^{*,*,*}(E[h_{ij} | 0 \leqslant j, i+j \leqslant 3], \delta_1) \otimes E[b_{20}, b_{11}] \otimes P[b_{10}].$$

H. Toda in [28] computed the cohomology of $(E[h_{ij} | 0 \leqslant j, i+j \leqslant 3], \delta_1)$. Here we only jot down the even dimensional elements within that range.

$$\begin{array}{llll} h_{20}h_{10}, & q(p+2)-2; & h_{20}h_{11}, & q(2p+1)-2; \\ h_{12}h_{10}, & q(p^2+1)-2; & h_{21}h_{11}, & q(p^2+2p)-2. \end{array}$$

Thus within $t-s \leqslant q(p^2+2p+2)-2$, the even dimensional May's E_2-term $E_2^{s,t,*}$ is a submodule of

$$\mathbb{Z}/p\{1, h_{20}h_{10}, h_{20}h_{11}, h_{12}h_{10}, h_{21}h_{11}\} \otimes E[b_{20}, b_{11}] \otimes P[b_{10}].$$

Suppose we have a generator y in $Ext^{s,s+q(p^2+2p+2)-2}_{\mathbb{Z}/p[v_3][t_1,t_2,t_3]}(BP_*, BP_*/I_3)$. Then y is the form of x or v_3x where x is an even dimensional generator in $H^*(E[h_{ij}|i+j \leqslant 3]) \otimes E[b_{20}, b_{11}] \otimes P[b_{10}]$.

(1) If $y = v_3x$, then $x \in E_2^{s,t,*}$ subject to $t - s = q(p+1) - 2$. An easy computation shows that the corresponding E_2-term is zero.

(2) If $y = x$, then $x \in E_2^{s,t,*}$ subject to $t - s = q(p^2+2p+2) - 2$. Similarly, from

$$q(p^2 + 2p + 2) - 2 \equiv 6p - 2 \qquad mod \quad qp - 2$$

we compute that the total degree $t - s$ mod $qp - 2$ of the generators in

$$\mathbb{Z}/p\{1, h_{20}h_{10}, h_{20}h_{11}, h_{12}h_{10}, h_{21}h_{11}\} \otimes [b_{20}, b_{11}]$$

and find none of them is $6p - 2$. Thus the corresponding E_2-term is zero.

The Lemma follows. □

It is easily showed that the following theorem holds from the lemma above.

Theorem 4.2. *For $p \geqslant 5$, $s \geqslant 1$, the Toda bracket $\langle \alpha_1\beta_1, p, \gamma_s\rangle = 0$.*

Proof. Let $\widetilde{v}_3$ be the composition of the following maps

$$S^{q(p^2+p+1)} \xrightarrow{\widetilde{i}} \Sigma^{q(p^2+p+1)}V(2) \xrightarrow{v_3} V(2),$$

where the first map is inclusion to the bottom cell.

It is known that $\widetilde{v}_3$ is an order p element in $\pi_{q(p^2+p+1)}(V(2))$. Thus the Toda bracket $\langle \alpha_1\beta_1, p, \widetilde{v}_3\rangle$ is well defined and $\langle \alpha_1\beta_1, p, \widetilde{v}_3\rangle \in \pi_{q(p^2+2p+2)-2}(V(2)) = 0$. It follows that the Toda bracket $\langle \alpha_1\beta_1, p, \widetilde{v}_3\rangle = 0$.

Let $\widetilde{j} : V(2) \longrightarrow S^{q(p+2)+3}$ be the collapsing lower cells map from $V(2)$, then $\gamma_s = \widetilde{v}_3 \cdot v_3^{s-1} \cdot \widetilde{j}$. As a result,

$$\langle \alpha_1\beta_1, p, \gamma_s\rangle = \langle \alpha_1\beta_1, p, \widetilde{v}_3 \cdot v_3^{s-1} \cdot \widetilde{j}\rangle = \langle \alpha_1\beta_1, p, \widetilde{v}_3\rangle \cdot v_3^{s-1} \cdot \widetilde{j} = 0$$

because $\langle \alpha_1\beta_1, p, \widetilde{v}_3\rangle = 0 \in \pi_{q(p^2+2p+2)-2}V(2) = 0$. □

Proposition 4.3. *(see also* [22] *7.5.11) Let $p \geqslant 5$ be an odd prime. Then in $\pi_*(S^0)$, the Toda bracket $\langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle$ is well defined and*

$$\alpha_1\beta_1^{p-1}h_{20}\gamma_s = \langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle = \beta_{p/p-1}\gamma_s.$$

Proof. From $\langle \beta_1^{p-1}, \alpha_1\beta_1, p\rangle = 0$, $\langle \alpha_1\beta_1, p, \alpha_1\rangle = 0$, $\langle \alpha_1, \alpha_1\beta_1, p\rangle = 0$ and $\langle \alpha_1\beta_1, p, \gamma_s\rangle = 0$, we know that the following 4-fold Toda bracket is well defined and

$$\beta_{p/p-1} = \langle \beta_1^{p-1}, \alpha_1\beta_1, p, \alpha_1\rangle; \qquad \alpha_1h_{20}\gamma_s = \langle \alpha_1, \alpha_1\beta_1, p, \gamma_s\rangle.$$

On the other hand, one has

$$\begin{aligned}
\beta_1^{p-1}\alpha_1h_{20}\gamma_s &= \beta_1^{p-1}\langle \alpha_1, \alpha_1\beta_1, p, \gamma_s\rangle \\
&= \langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle \\
&= \alpha_1\langle \beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle \\
&= \langle \beta_1^{p-1}, \alpha_1\beta_1, p, \alpha_1\gamma_s\rangle \\
&= \langle \beta_1^{p-1}, \alpha_1\beta_1, p, \alpha_1\rangle \cdot \gamma_s \\
&= \beta_{p/p-1}\gamma_s
\end{aligned}$$

The proposition follows. □

Theorem 4.4. *Let $p \geqslant 5$ be an odd prime and $2 \leqslant s \leqslant p-2$. Then in the ANSS, we have the following Adams-Novikov differential*

$$d_{2p-1}(h_{2,0}b_{1,1}\gamma_s) = \alpha_1\beta_1^p h_{2,0}\gamma_s.$$

Proof. Note that $b_{11} = \beta_{p/p}$. Then from (3.1) one has the differential in the small descent spectral sequence

$$d_2(h_{20}b_{11}) = \beta_1\beta_{p/p-1},$$

which could be read as $d(h_{20}\beta_{p/p}) = \beta_1\beta_{p/p-1}$ and $d(h_{20}\beta_{p/p}\gamma_s) = \beta_1\beta_{p/p-1}\gamma_s$ in the cobar complex of BP_* or equivalently the first Adams-Novikov differential in the ANSS. Then from the relation $\beta_{p/p-1}\gamma_s = \alpha_1\beta_1^{p-1}h_{20}\gamma_s$ in $\pi_*(S^0)$ and $\beta_{p/p-1}\gamma_s = 0$ in $Ext^{5,*}_{BP_*BP}(BP_*, BP_*)$, we get the Adams differential in the ANSS

$$d_{2p-1}(h_{2,0}b_{1,1}\gamma_s) = \beta_1 \cdot \beta_1^{p-1}\alpha_1 h_{20}\gamma_s = \alpha_1\beta_1^p h_{20}\gamma_s.$$

The theorem follows. □

5. The proof of Theorem A

In this section we prove our main theorem by showing that β_{p^2/p^2-1} survives to E_∞ in the ANSS. Note that β_{p^2/p^2-1} has too low dimension to be the target of an Adams-Novikov differential, we will do this by showing that all the Adams-Novikov differentials $d_r(\beta_{p^2/p^2-1})$ are trivial.

Lemma 5.1. *Let $p \geqslant 5$ and $i \not\equiv 0$ mod p. In the ANSS, one has the following Adams-Novikov differential*

$$d_{2p-1}(\eta_i) = \beta_1^p\beta_{i+1}$$

Proof. Recall from [22] 7.3.11 Theorem (e), in the SDSS

$$E_1 = Ext^{s,t}_{BP_*BP}(BP_*, BP_*(X^{p^2-1})) \otimes E[h_{11}] \otimes P[b_{11}] \Longrightarrow Ext^{s,t}_{BP_*BP}(BP_*, BP_*(X)),$$

where $BP_*(X^{p^2-1}) = BP_*[t_1]/\langle t_1^{p^2}\rangle$ (*cf.* [22] 7.3.8 Theorem), one has $d_2(h_{20}\mu_{i-1}) = ib_{11}\beta_{i+1}$. And from its definition we know that $\eta_i = h_{11}\mu_{i-1}$ is represented by

$$\delta\delta\left(\frac{v_2^{p+i-1}t_2 + v_2^i t_2^p - v_2^i t_1^{p^2+p} - v_2^{i-1}v_3 t_1^p}{pv_1}\right)$$

(*cf.* [22] p.288) which is also denoted by $\delta\delta\left(\frac{v_2^{p+i}}{pv_1}\zeta_2\right)$ in [10, 29]. In the cobar complex of $N_0^2 = BP_*/(p^\infty, v_1^\infty)$, a straightforward computation shows that the coboundary of

$$\frac{v_2^i(t_3 - t_1t_2^p - t_2t_1^{p^2} + t_1^{p^2+p+1}) + v_2^{p+i-1}(t_1t_2 - t_1^{p+2}) - v_2^{i-1}v_3(t_2 - t_1^{p+1})}{pv_1}$$
$$+\frac{2v_2^{p+i}}{(p+i)p^2v_1}t_1 - \frac{v_2^{p+i}}{(p+i)pv_1^2}t_1^2$$

is $\dfrac{(v_2^{p+i-1}t_2 + v_2^i t_2^p - v_2^i t_1^{p^2+p} - v_2^{i-1}v_3t_1^p) \otimes t_1}{pv_1} + \dfrac{v_2^{i+1}}{pv_1}b_{11}$. This shows that in $Ext^{2,*}_{BP_*BP}(BP_*, N_0^2)$ the cohomology class

$$\left[\frac{(v_2^{p+i-1}t_2 + v_2^i t_2^p - v_2^i t_1^{p^2+p} - v_2^{i-1}v_3t_1^p) \otimes t_1}{pv_1}\right] = -\left[\frac{v_2^{i+1}}{pv_1}b_{11}\right].$$

Applying the connecting homomorphism $\delta\delta$, we get $\alpha_1\eta_i = \beta_{i+1}\beta_{p/p}$.

From $\alpha_1\eta_i = \beta_{i+1}\beta_{p/p}$ and the Toda differential, one has:

$$\alpha_1 d_{2p-1}(\eta_i) = d_{2p-1}(\alpha_1\eta_i) = d_{2p-1}(\beta_{i+1}\beta_{p/p}) = \alpha_1\beta_1^p\beta_{i+1}$$

The lemma follows from $\alpha_1 d_{2p-1}(\eta_i) = \alpha_1\beta_1^p\beta_{i+1}$. □

Proof of Theorem A From $\beta_{p^2/p^2-1} \in Ext^{2,q(p^3+1)}_{BP_*BP}(BP_*, BP_*)$, we know that $d_r(\beta_{p^2/p^2-1}) \in Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$ subject to $t-s = q(p^3+1)-3$. From Theorem 3.2 we know that the corresponding $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$ is the $\mathbb{Z}/p$-module generated by $\mathfrak{g}_1, \mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_6$ and $\mathfrak{g}_7, \mathfrak{g}_8$.

$\mathfrak{g}_7 = \alpha_1\beta_{(p-1)p+1}$ and $\mathfrak{g}_8 = \alpha_1\beta_{p^2/p^2}$ have too low dimension to be the target of $d_r(\beta_{p^2/p^2-1})$.

From the Toda differential $d_{2p-1}(b_{11}) = \alpha_1\beta_1^p$ we have

$$d_{2p-1}(\beta_1^{p^2-p-1}b_{11}\beta_2) = \alpha_1\beta_1^{p^2-1}\beta_2 = \mathfrak{g}_1$$

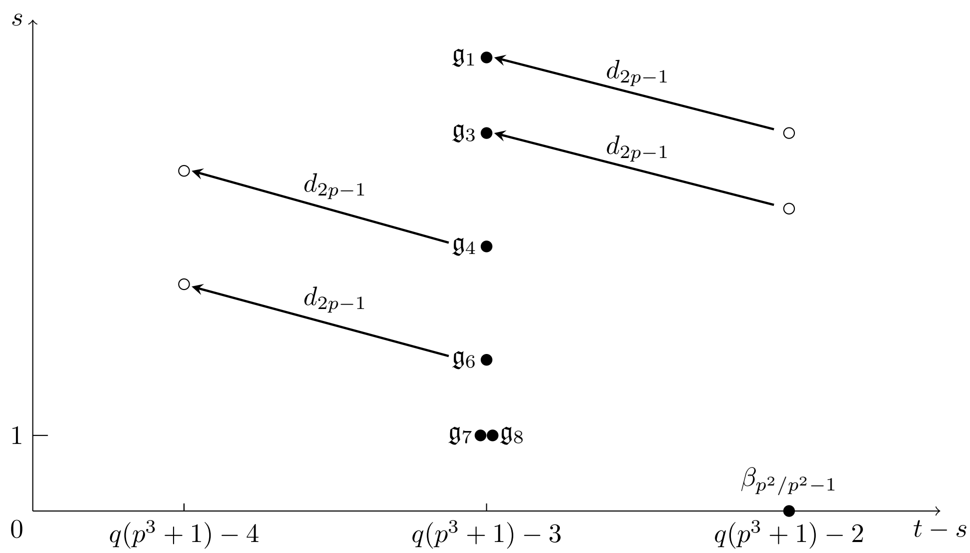
$$d_{2p-1}(\mathfrak{g}_4) = d_{2p-1}(\beta_1^{\frac{p^2-6p+1}{2}}b_{11}^2\gamma_{\frac{p+1}{2}}) = 2\alpha_1\beta_1^{\frac{p^2-4p+1}{2}}b_{11}\gamma_{\frac{p+1}{2}}.$$

From $d_{2p-1}(h_{20}b_{11}\gamma_s) = \alpha_1\beta_1^p h_{20}\gamma_s$ (*cf.* Theorem 4.4), we have

$$d_{2p-1}\left(\beta_1^{\frac{p^2-4p-1}{2}}h_{20}b_{11}\gamma_{\frac{p+1}{2}}\right) = \alpha_1\beta_1^{\frac{p^2-2p-1}{2}}h_{20}\gamma_{\frac{p+1}{2}} = \mathfrak{g}_3.$$

From Lemma 5.1, we have

$$d_{2p-1}(\mathfrak{g}_6) = d_{2p-1}(\beta_1^{p-1}\eta_{(p-3)p+3}) = \beta_1^{2p-1}\beta_{(p-3)p+4}.$$



Theorem A follows. □

6. A conjecture

Consider the cofiber sequence

$$S^0 \xrightarrow{p} S^0 \longrightarrow M$$

and the induced short exact sequence of BP-homologies

$$0 \longrightarrow BP_*(S^0) \xrightarrow{p} BP_*(S^0) \longrightarrow BP_*(M) \longrightarrow 0,$$

which induces a long exact sequence of Ext groups

$$\begin{tikzcd}
\cdots \arrow[r] & Ext^{1,t}(BP_*(S^0)) \arrow[r] \arrow[d, "d_{2p-1}"] & Ext^{1,t}(BP_*(S^0)) \arrow[r] \arrow[d, "d_{2p-1}"] & Ext^{1,t}(BP_*(M)) \arrow[r, "\delta"] \arrow[d, "d_{2p-1}"] & Ext^{2,t}(BP_*(S^0)) \arrow[r] \arrow[d, "d_{2p-1}"] & \cdots \\
\cdots \arrow[r] & Ext^{2p,*}(BP_*(S^0)) \arrow[r] & Ext^{2p,*}(BP_*(S^0)) \arrow[r] & Ext^{2p,*}(BP_*(M)) \arrow[r, "\delta"] & Ext^{2p+1,*}(BP_*(S^0)) \arrow[r] & \cdots .
\end{tikzcd}$$

For the connecting homomorphism δ, one has

$$\delta(h_{i+2}) = \beta_{p^{i+1}/p^{i+1}}, \qquad \delta(v_1 h_{i+2}) = \beta_{p^{i+1}/p^{i+1}-1} \qquad \text{and} \qquad \delta(v_1^i) = i\alpha_i.$$

From the Toda differential $d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p$, one can get a non-trivial differential in the ANSS for the Moore spectrum M

$$d_{2p-1}(h_2) = v_1\beta_1^p.$$

Then from the relation $h_{i+1}\beta_{p/p}^{p^i} = h_{i+2}\beta_1^{p^i}$ (*cf.* [19] and [22] 6.4.7), we get the following Adams-Novikov differential by induction

$$\begin{aligned}
d_{2p-1}(h_{i+2})\beta_1^{p^i} = d_{2p-1}(h_{i+2}\beta_1^{p^i}) &= d_{2p-1}(h_{i+1}\beta_{p/p}^{p^i}) \\
&= d_{2p-1}(h_{i+1})\beta_{p/p}^{p^i} \\
&= v_1\beta_{p^{i-1}/p^{i-1}}^p\beta_{p/p}^{p^i} \\
&= v_1(\beta_{p^{i-1}/p^{i-1}}\beta_{p/p}^{p^{i-1}})^p \\
&= v_1\beta_{p^i/p^i}^p\beta_1^{p^i},
\end{aligned}$$

which implies $d_{2p-1}(h_{i+2}) = v_1\beta_{p^i/p^i}^p$ in the ANSS for the Moore spectrum M. Then from the convergence of v_1 in the ANSS for the Moore spectrum one has

$$d_{2p-1}(v_1h_{i+2}) = v_1^2\beta_{p^i/p^i}^p$$

Applying the connecting homomorphism δ, we have the Adams-Novikov differential for the sphere

$$d_{2p-1}(\beta_{p^{i+1}/p^{i+1}-1}) = d_{2p-1}(\delta(v_1h_{i+2})) = \delta(d_{2p-1}(v_1h_{i+2})) = \delta(v_1^2\beta_{p^i/p^i}^p) = 2\alpha_2\beta_{p^i/p^i}^p. \tag{5.2}$$

So one can prove the non-existence of $\beta_{p^{i+1}/p^{i+1}-1}$ from the non-triviality of

$$\alpha_2\beta_{p^i/p^i}^p \neq 0 \in Ext_{BP_*BP}^{2p+1,*}(BP_*, BP_*).$$

(1) $\beta_{p/p-1}$ exists and $\alpha_2\beta_1^p = 0$ because $\alpha_2\beta_1 = 0$.
(2) β_{p^2/p^2-1} exists, this implies $\alpha_2\beta_{p/p}^p = 0$.

As we know that $\beta_{p/p}^p \neq 0$ in $Ext_{BP_*BP}^{2p,qp^3}(BP_*, BP_*)$ [19, 22]. But we could not find its representative element b_{11}^p in $Ext_{BP_*BP}^{2p,qp^3}(BP_*, BP_*(X))$ (*cf.* [22] 7.3.12 (b) and the ABC Theorem) because of the differential in the SDSS.

$$d(h_{11}b_{20}^{p-1}) = b_{11}^p$$

(1) At the prime $p = 5$, β_1x_{952} converges to $\beta_{5/5}^5$, where $x_{952} = h_{11}b_{20}^{p-3}\gamma_2$. This implies $\alpha_2\beta_{5/5}^5 = \alpha_2\beta_1x_{952} = 0$ (*cf.* [22] 7.5.5 stem 990) because $\alpha_2\beta_1 = 0$.

(2) At the prime $p \geqslant 5$, we compute $Ext^{2p,qp^3}_{BP_*BP}(BP_*, BP_*)$ by the SDSS. The E_1-term

$$E_1^{s,t,u} = Ext^{s,*}_{BP_*BP}(BP_*, BP_*(X)) \otimes E[\alpha_1] \otimes P[\beta_1]$$

subject to $s + u = 2p$, $t = qp^3$ is the $\mathbb{Z}/p$ module generated by

$$\beta_1 h_{11} b_{20}^{p-3} \gamma_2, \qquad \alpha_1 \beta_1 b_{20}^{p-3} \eta_p, \qquad \alpha_1 \beta_1^{\frac{p-1}{2}} h_{20} b_{11}^{\frac{p-5}{2}} b_{20} \mu_{\frac{p-3}{2}p+p-2}.$$

In any case, we can conclude $\beta^p_{p/p}$ is divisible by β_1. Here we believe that it is $\beta_1 h_{11} b_{20}^{p-3} \gamma_2$ converges to $\beta^p_{p/p}$. So we have conjectures for the behavior of $\beta^p_{p^i/p^i}$ in general as summarized in Conjecture C.

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