THE p-PRIMARY SUBGROUP OF THE COHOMOLOGY OF BPU_n IN DIMENSION $2p+6$

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ABSTRACT. Let PU_n denote the projective unitary group of rank n and BPU_n be its classifying space. For an odd prime p, we show that the p-primary subgroup of $H^{2p+6}(BPU_n;\mathbb{Z})$ is trivial.

1. INTRODUCTION

The purpose of this short paper is to study the integral cohomology of BPU_n. Here, U_n denotes the group of $n\times n$ unitary matrices. The projective unitary group PU_n is defined as the quotient group of U_n by S^1, where we identify S^1 with the normal subgroup of scalar matrices of U_n. Finally, BPU_n stands for the classifying space of PU_n.

The cohomology of BPU_n is a fundamental object in algebraic topology of general interests, and plays significant roles in the study of the period-index problem in algebraic geometry and algebraic topology (see, for example, [1], [2], [6] and [7]), as well as in the study of anormalies in partical physics (see, for example, [4], [5]).

The cohomology of BPU_n for special n has been studied by many researchers including Kameko-Yagita [10], Kono-Mimura [11], Kono-Yagita [12], Toda [14], and Vavpetič-Viruel [15], the only well-understood case being when $n = p$ is a prime number.

On the other hand, very little is understood about the cohomology of BPU_n for an arbitrary n, which is regarded as a very difficult problem. Indeed, none of the works above dealt with $H^*(BPU_n;\mathbb{Z})$, the ordinary cohomology of BPU_n with coefficients in $\mathbb{Z}$, for n not a prime number. Recently, a notable progress in this direction is made by Gu in [8], where the ring structure of $H^*(BPU_n;\mathbb{Z})$ in dimensions less than or equal to 10 for an arbitrary n is determined.

Before we discuss more computational results, let us first introduce some notations which are going to be used thoughout the rest of the paper.

Notations 1.1. *To simplify notations, we let $H^*(-)$ denote the integral cohomology $H^*(-;\mathbb{Z})$. Given an abelian group A and a prime number p, we let $A_{(p)}$ denote the localization of A at p, and let ${}_pA$ denote the p-primary subgroup of A. In other words, ${}_pA$ is the subgroup of A consisting of all torsion elements whose order is a power of p. A useful observation is that, there is a canonical isomorphism ${}_pH^*(-) \cong {}_p[H^*(-)_{(p)}]$. We will use these two interchangeably. Finally, when we*

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take tensor products of $\mathbb{Z}_{(p)}$-modules, the tensor products are always taken over $\mathbb{Z}_{(p)}$.

In the following, we discuss our strategy to study $H^*(BPU_n)$ for general n. First, we point out that the torsion-free part of $H^*(BPU_n)$ is completely known.

For a fixed positive integer n, there is a short exact sequence of Lie groups

$$1 \to \mathbb{Z}/n \to SU_n \to PSU_n \simeq PU_n \to 1,$$

which induces a fiber sequence of their classifying spaces

$$B(\mathbb{Z}/n) \to BSU_n \to BPU_n \tag{1.1}$$

Recall the cohomology of BSU_n is given by

$$H^*(BSU_n) = \mathbb{Z}[c_2, c_3, \cdots, c_n],\ |c_i| = 2i \tag{1.2}$$

Now, let p be a prime number such that $p \nmid n$, then the space $B(\mathbb{Z}/n)$ is p-locally contractible. From (1.1), we get

$$H^*(BPU_n; \mathbb{Z}_{(p)}) \cong H^*(BSU_n; \mathbb{Z}_{(p)}) \tag{1.3}$$

Since $\mathbb{Z}_{(p)}$ is a flat $\mathbb{Z}$-module, $H^*(-; \mathbb{Z}_{(p)}) \cong H^*(-)_{(p)}$. We have an isomorphism of $\mathbb{Z}_{(p)}$-algebras

$$H^*(BPU_n)_{(p)} \cong H^*(BSU_n)_{(p)} = \mathbb{Z}_{(p)}[c_2, c_3, \cdots, c_n], \tag{1.4}$$

Hence, we can conclude the rank of the torsion-free part of $H^s(BPU_n)$ is just the number of monomials in $c_2, c_3, \ldots, c_n$ in dimension s.

The remaining work is to determine the torsion part of $H^*(BPU_n)$. As a standard approach in algebraic topology, we work with one prime at a time. In other words, we fix a prime p, then study the p-primary subgroup ${}_pH^*(BPU_n)$. Note the result in (1.4) shows, if $p \nmid n$, then $H^*(BPU_n)_{(p)}$ is torsion-free. Hence,

$${}_pH^*(BPU_n) = {}_p[H^*(BPU_n)_{(p)}] = 0$$

Therefore, the only interesting cases happen when $p \mid n$.

The work in [9] gave a complete description of $H^s(BPU_n; \mathbb{Z})_{(p)}$ for $s < 2p + 5$ by showing that ${}_pH^s(BPU_n) = 0$ for $s = 2p + 3$ and $s = 2p + 4$ when p is an odd prime. In this paper, we extend the result in [9] by computing ${}_pH^s(BPU_n)$ for $s = 2p + 6$ for general n. Our main theorem is the following.

Theorem 1. *Let $p > 2$ be a prime number, n be a positive number. Then the p-primary subgroup of the cohomology of BPU_n is trivial in dimension $2p + 6$. In other words, we have*

$${}_pH^{2p+6}(BPU_n) = 0$$

Remark 1.2. When $p = 2$, the cohomology of BPU_n in dimension $2p + 6 = 10$ has been computed explicitly in [8], where the computation shows

$${}_2H^{10}(BPU_n) \cong \begin{cases} \mathbb{Z}/2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Hence the result in Theorem 1 does not hold for $p = 2$.

Remark 1.3. From the previous discussions, we have seen that the result in Theorem 1 holds trivially when $p \nmid n$. Hence, to prove the theorem, we only need to consider the case when $p \mid n$. See Notations 3.1.

Organization of the paper. In Section 2, we introduce the Serre spectral sequence that we use to compute the cohomology of BPU_n. We also recall some basic results of the differentials in the spectral sequence. In Section 3, we give explicit computations of all relevant differentials and prove Theorem 1.

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2. The spectral sequences

Our tool for computing the cohomology of BPU_n is the Serre spectral sequence ${}^UE$ (2.2). The spectral sequence ${}^UE$ also played a key role in the related computations of [8, 9]. In this section, we will recall the basic setup and computational results for ${}^UE$. To determine the differentials in ${}^UE$, we need to consider two auxiliary spectral sequences ${}^TE$ and ${}^KE$. The computations of the differentials in ${}^TE$ and ${}^KE$ will also be reviewed in this section.

2.1. **The Serre spectral sequence ${}^UE$.**

The short exact sequence of Lie groups

$$1 \to S^1 \to U_n \to PU_n \to 1$$

induces a fiber sequence of their classifying spaces

$$BS^1 \to BU_n \to BPU_n$$

Notice that BS^1 has the homotopy type of the Eilenberg-Mac Lane space $K(\mathbb{Z}, 2)$, there is an associated fiber sequence

$$U: \ BU_n \to BPU_n \to K(\mathbb{Z}, 3) \tag{2.1}$$

We will use the Serre spectral sequence associated to (2.1) to compute the cohomology of BPU_n. For notational convenience, we denote this spectral sequence by ${}^UE$. The E_2 page of ${}^UE$ has the form

$${}^UE_2^{s,t} = H^s(K(\mathbb{Z}, 3); H^t(BU_n)) \Longrightarrow H^{s+t}(BPU_n) \tag{2.2}$$

To carry out actual computations with this spectral sequence, we need to know the cohomology of $K(\mathbb{Z}, 3)$ and BU_n. As we will see in (3.1), since the purpose of this paper is to study the p-primary subgroup of $H^*(BPU_n)$ for a fixed prime p, it suffices to know the p-local cohomology of $K(\mathbb{Z}, 3)$.

We summarize the p-local cohomology of $K(\mathbb{Z}, 3)$ in low dimensions as follows. The original reference is [3], also see [13] for a nice treatment.

Proposition 2.2. *Let $p > 2$ be a prime. In degrees up to $2p + 7$, we have*

$$H^s(K(\mathbb{Z}, 3))_{(p)} = \begin{cases} \mathbb{Z}_{(p)}, & s = 0,\ 3, \\ \mathbb{Z}/p, & s = 2p+2,\ 2p+5, \\ 0, & s \le 2p+7, s \neq 0,\ 3,\ 2p+2,\ 2p+5. \end{cases} \tag{2.3}$$

where $x_1,\ y_{p,0},\ x_1y_{p,0}$ are generators on degree $3, 2p+2, 2p+5$ respectively.

Remark 2.3. Here the notations for the generators are taken from [8, 2.14].

Also recall

$$H^*(BU_n) = \mathbb{Z}[c_1, c_2, \dots, c_n],\ |c_i| = 2i \tag{2.4}$$

In particular, $H^*(BU_n)$ is torsion-free. We have

$${}^U E_2^{s,t} \cong H^s(K(\mathbb{Z},3)) \otimes H^t(BU_n) \tag{2.5}$$

2.4. **The auxiliary spectral sequences ${}^T E$ and ${}^K E$.** Instead of computing the differentials in ${}^U E$ directly, which is difficult in practice, our strategy is to compare ${}^U E$ with two auxiliary spectral sequences, which has simpler differential behaviors. We now introduce the two auxiliary fiber sequences and their associated Serre spectral sequences.

Let T^n be the maximal torus of U^n with the inclusion denoted by

$$\psi : T^n \to U_n.$$

Passing to quotients over S^1, we have another inclusion of maximal torus

$$\psi' : PT^n \to PU_n.$$

The quotient map $T^n \to PT^n$ fits into an exact sequence of Lie groups

$$1 \to S^1 \to T^n \to PT^n \to 1,$$

which induces another fiber sequence of their classifying spaces

$$T:\ BT^n \to BPT^n \to K(\mathbb{Z},3) \tag{2.6}$$

T is our first auxiliary fiber sequence.

We also consider the path fibration for $K(\mathbb{Z},3)$

$$K:\ K(\mathbb{Z},2) \simeq BS^1 \to * \to K(\mathbb{Z},3) \tag{2.7}$$

where $*$ denotes a contractible space. K is our second auxiliary fiber sequence.

These fiber sequences fit into the following homotopy commutative diagram:

$$\begin{tikzcd}
K: \arrow[d, "\Phi"] & BS^1 \arrow[r] \arrow[d, "B\varphi"] & * \arrow[r] \arrow[d] & K(\mathbb{Z},3) \arrow[d, "="] \\
T: \arrow[d, "\Psi"] & BT^n \arrow[r] \arrow[d, "B\psi"] & BPT^n \arrow[r] \arrow[d, "B\psi'"] & K(\mathbb{Z},3) \arrow[d, "="] \\
U: & BU_n \arrow[r] & BPU_n \arrow[r] & K(\mathbb{Z},3)
\end{tikzcd} \tag{2.8}$$

Here, the map $B\varphi : BS^1 \to BT^n$ is induced by the diagonal map $\varphi : S^1 \to T^n$.

We denote the Serre spectral sequences associated to U, T, and K as ${}^U E$, ${}^T E$ and ${}^K E$ respectively. We denote their corresponding differentials by ${}^U d_*^{*,*}$, ${}^T d_*^{*,*}$, and ${}^K d_*^{*,*}$ respectively. When the actural meaning is clear from the context, we also simply denote the differentials by $d_*^{*,*}$.

In this paper, we compute differentials in ${}^U E$ by comparing it with the differentials in ${}^T E$ and ${}^K E$. This is possible because: (1) we have explicit formulas for the maps between spectral sequences, and (2) we have a good understanding of the corresponding differentials in ${}^T E$ and ${}^K E$.

We first describe the comparison maps between ${}^U E$, ${}^T E$ and ${}^K E$.

Notice that we have

$$H^*(BT^n) = \mathbb{Z}[v_1, v_2, \dots, v_n],\ |v_i| = 2. \tag{2.9}$$

The induced homomorphism between cohomology rings is as follows:

$$B\varphi^*: H^*(BT^n) = \mathbb{Z}[v_1, v_2, \cdots, v_n] \to H^*(BS^1) = \mathbb{Z}[v],\ v_i \mapsto v.$$

The map $B\psi: BT^n \to BU_n$ induces the injective ring homomorphism

$$\begin{aligned} B\psi^*: H^*(BU_n) = \mathbb{Z}[c_1, \cdots, c_n] &\to H^*(BT^n) = \mathbb{Z}[v_1, \cdots, v_n], \\ c_i &\mapsto \sigma_i(v_1, \cdots, v_n), \end{aligned} \tag{2.10}$$

where $\sigma_i(t_1, t_2, \cdots, t_n)$ be the ith elementary symmetric polynomial in variables $t_1, t_2, \cdots, t_n$:

$$\begin{aligned} &\sigma_0(t_1, t_2, \cdots, t_n) = 1, \\ &\sigma_1(t_1, t_2, \cdots, t_n) = t_1 + t_2 + \cdots + t_n, \\ &\sigma_2(t_1, t_2, \cdots, t_n) = \sum_{i<j} t_i t_j, \\ &\vdots \\ &\sigma_n(t_1, t_2, \cdots, t_n) = t_1 t_2 \cdots t_n. \end{aligned} \tag{2.11}$$

We also recall some important propositions regarding the higher differentials in ${}^KE$ and ${}^TE$. The following result of differentials in ${}^KE$ is the starting point for relevant computations in ${}^TE$ and ${}^UE$.

Proposition 2.5. *The higher differentials of ${}^KE_*^{*,*}$ satisfy*

$$\begin{aligned} &d_3(v) = x_1, \\ &d_{2p-1}(x_1 v^{lp^e-1}) = v^{lp^e-1-(p-1)} y_{p,0}, \quad e > 0,\ \gcd(l,p) = 1, \\ &d_r(x_1) = d_r(y_{p,0}) = 0, \quad \textit{for all } r, \end{aligned}$$

and the Leibniz rule.

Remark 2.6. Proposition 2.5 is a special case of [8, Corollary 2.16]. Note there is a typo in the original reference, where the condition $k \geq e$ should be replaced by $e > k$.

By comparing with the differentials in ${}^KE$, one could obtain the following results on differentials in ${}^TE$.

Proposition 2.7 ([9], Lemma 3.1)**.** *In the spectral sequence ${}^TE$, we have*

$${}^Td_{2p-1}^{3,*}(v_n^k x_1) = 0$$

for $0 \leq k \leq p-2$ or $k = p$, and

$${}^Td_{2p-1}^{3,*}(v_n^{p-1} x_1) = y_{p,0}$$

Proposition 2.8 ([8], Proposition 3.3)**.** (1) *The differential ${}^Td_3^{0,t}$ is given by the "formal divergence"*

$$\nabla = \sum_{i=1}^{n} (\partial/\partial v_i): H^t(BT^n; R) \to H^{t-2}(BT^n; R),$$

in such a way that ${}^Td_3^{0,*} = \nabla(-) \cdot x_1$. For any ground ring $R = \mathbb{Z}$ or $\mathbb{Z}/m$ for any integer m.

(2) *The spectral sequence degenerates at ${}^TE_4^{0,*}$. Indeed, we have ${}^TE_\infty^{0,*} = {}^TE_4{}^{0,*} = \operatorname{Ker} {}^Td_3^{0,*} = \mathbb{Z}[v_1 - v_n, \cdots, v_{n-1} - v_n]$.*

The following is a useful corollary.

Corollary 2.9. *We have*

$$^Ud_3^{0,*}(c_k) = \nabla(c_k)x_1 = (n-k+1)c_{k-1}x_1$$

for $2 \leq k \leq n$*, and*

$$^Ud_3^{0,*}(c_1) = nx_1$$

Remark 2.10. Corollary 2.9 first appeared in [8, Corollary 3.4]. Here, we write out the result for c_1 separately since c_0 is not defined.

3. Computations in the spectral sequence ^UE

The purpose of this section is to prove Theorem 1 via explicit computations with the Serre spectral sequence ^UE. Noticing

$${}_pH^*(BPU_n) \cong {}_p[H^*(BPU_n)_{(p)}],$$

in order to study the p-primary subgroup of $H^*(BPU_n)$, it suffices to look at the p-localized spectral sequence, where the E_2 page becomes

$$(^UE_2^{s,t})_{(p)} = H^s(K(\mathbb{Z},3))_{(p)} \otimes H^t(BU_n) = H^s(K(\mathbb{Z},3)) \otimes H^t(BU_n)_{(p)}. \tag{3.1}$$

Notations 3.1. *For the rest of this paper, we fix a prime* $p \geq 3$ *and a positive integer* n *such that* $p \mid n$ *(see Remark 1.3). We let* $^UE, {}^TE$ *and* ^KE *denote the corresponding* p*-localized Serre spectral sequences.*

3.2. **Nontrivial elements of ^UE.** By Proposition 2.2 and equation (2.4), in the range $s \leq 2p+7$, $^UE_2^{s,t}$ could be nonzero only when (i) $s = 0, 3, 2p+2$, or $2p+5$, and (ii) $t \geq 0$ is even. Therefore, along the line $s+t = 2p+6$ of the E_∞-page, the only places where $^UE_\infty^{s,t}$ could possibly be nonzero are $^UE_\infty^{0,2p+6}$ and $^UE_\infty^{2p+2,4}$. Then the proof of Theorem 1 boils down to proving the following proposition.

Proposition 3.3. *None of the nontrivial elements in* $^UE_2^{2p+2,4}$ *could survive to the* E_∞*-page. In other words,* $^UE_\infty^{2p+2,4} = 0$.

Proof of Theorem 1 assuming Proposition 3.3. Let us first point out that, by the discussions following Theorem 1, we can feel free to assume $p \geq 3$ and $p \mid n$.

Now, using the Serre spectral sequence ^UE, we get a short exact sequence of $\mathbb{Z}_{(p)}$-modules

$$0 \to {}^UE_\infty^{2p+2,4} \to H^{2p+6}(BPU_n)_{(p)} \to {}^UE_\infty^{0,2p+6} \to 0 \tag{3.2}$$

From the isomorphism $^UE_2^{s,t} \cong H^s(K(\mathbb{Z},3)) \otimes H^t(BU_n)_{(p)}$, we get

$$^UE_2^{0,2p+6} = H^0(K(\mathbb{Z},3)) \otimes H^{2p+6}(BU_n)_{(p)} \cong H^{2p+6}(BU_n)_{(p)}$$

is the free $\mathbb{Z}_{(p)}$-module generated by monomials in $c_1, c_2, \ldots, c_n$ in dimension $2p+6$. Inspection of degrees shows that $^UE_*^{0,2p+6}$ can not receive any nontrivial differentials. Hence $^UE_\infty^{0,2p+6} \subset {}^UE_2^{0,2p+6}$ is a free $\mathbb{Z}_{(p)}$-module. Then the short exact sequence (3.2) splits and we get

$$H^{2p+6}(BPU_n)_{(p)} \cong {}^UE_\infty^{2p+2,4} \oplus {}^UE_\infty^{0,2p+6}$$

This implies

$${}_pH^{2p+6}(BPU_n)_{(p)} \subset {}^UE_\infty^{2p+2,4}$$

Now the result follows from Proposition 3.3. $\square$

3.4. **Inspection of ${}^UE_*^{2p+2,4}$.** Note the differentials in ${}^UE$ has the form

$$d_r : {}^UE_r^{s,t} \to {}^UE_r^{s+r,t-r+1}$$

Inspection of degrees shows that ${}^UE_*^{2p+2,4}$ can receive only the d_{2p-1} differential

$$d_{2p-1} : {}^UE_{2p-1}^{3,2p+2} \to {}^UE_{2p-1}^{2p+2,4}$$

and support the d_3 differential

$$d_3 : {}^UE_3^{2p+2,4} \to {}^UE_3^{2p+5,2}$$

By similar arguments, ${}^UE_*^{3,2p+2}$ can receive only the d_3 differential and support the d_{2p-1} differential.

To simplify the notations, we let

$$M^1 = {}^UE_2^{3,2p+2}, M^2 = {}^UE_2^{2p+2,4}, M^3 = {}^UE_2^{2p+5,2}$$

One simple observation is that, since ${}^UE_2$ is concentrated in even rows, all d_2 differentials are trivial. In particular, we also have

$$M^1 = {}^UE_3^{3,2p+2}, M^2 = {}^UE_3^{2p+2,4}, M^3 = {}^UE_3^{2p+5,2}$$

Moreover,

$$ {}^UE_{2p-1}^{2p+2,4} = {}^UE_{2p-2}^{2p+2,4} = \cdots = {}^UE_4^{2p+2,4} = Ker(d_3) \subset {}^UE_3^{2p+2,4} = M^2 \tag{3.3}$$

On the other hand,

$$ {}^UE_\infty^{2p+2,4} = \cdots = {}^UE_{2p}^{2p+2,4} = {}^UE_{2p-1}^{2p+2,4}/Im(d_{2p-1}) \tag{3.4}$$

Again, to simplify the notations, we let δ^1 denote the composition

$$\delta^1 : M^1 = {}^UE_3^{3,2p+2} \to {}^UE_3^{3,2p+2}/\operatorname{Im} d_3 = {}^UE_{2p-1}^{3,2p+2} \xrightarrow{d_{2p-1}} {}^UE_{2p-1}^{2p+2,4} \subset M^2$$

We let δ^2 denote the map

$$\delta^2 : M^2 = {}^UE_3^{2p+2,4} \xrightarrow{d_3} {}^UE_3^{2p+5,2} = M^3$$

Before we compute δ^1, δ^2, let us write down the explicit $\mathbb{Z}_{(p)}$-module structures of M^1, M^2, and M^3.

Using the isomorphism ${}^UE_2^{s,t} \cong H^s(K(\mathbb{Z},3)) \otimes H^t(BU_n)_{(p)}$, we get

$$M^1 = H^3(K(\mathbb{Z},3)) \otimes H^{2p+2}(BU_n)_{(p)} \cong H^{2p+2}(BU_n)_{(p)}$$

is the free $\mathbb{Z}_{(p)}$-module generated by elements of the form cx_1 where c is a monomial in $c_1, c_2, \dots, c_n$ in dimension $2p+2$.

We also have

$$M^2 = H^{2p+2}(K(\mathbb{Z},3)) \otimes H^4(BU_n)_{(p)} = \mathbb{Z}_{(p)}\{c_2y_{p,0}, c_1^2y_{p,0}\}/p \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$$

and

$$M^3 = H^{2p+5}(K(\mathbb{Z},3)) \otimes H^2(BU_n)_{(p)} = \mathbb{Z}_{(p)}\{c_1x_1y_{p,0}\}/p \cong \mathbb{Z}/p$$

Now, Proposition 3.3 could be proved using the following two lemmas.

Lemma 3.5. *As a subgroup of M^2, the kernel of $\delta^2 : M^2 \to M^3$ is generated by $c_1^2y_{p,0}$.*

Lemma 3.6. *The image of $\delta^1 : M^1 \to M^2$ contains the subgroup of M^2 generated by $c_1^2y_{p,0}$.*

Proof of Proposition 3.3 assuming Lemma 3.5 and 3.6. We have seen from (3.3) and (3.4) that ${}^{U}E_{2p-1}^{2p+2,4} = Ker(\delta^2)$ and ${}^{U}E_{\infty}^{2p+2,4} = {}^{U}E_{2p-1}^{2p+2,4}/Im(\delta^1)$. Lemma 3.5 together with 3.6 shows $Ker(\delta^2) \subset Im(\delta^1)$. Therefore, ${}^{U}E_{\infty}^{2p+2,4} = 0$. □

3.7. **The proofs of Lemma 3.5 and 3.6.** The rest of the paper is devoted to proving these two lemmas.

We first study the kernel of δ^2 and prove Lemma 3.5.

Proof of Lemma 3.5. Recall that

$$M^2 = \mathbb{Z}_{(p)}\{c_2 y_{p,0}, c_1^2 y_{p,0}\}/p \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$$

$$M^3 = \mathbb{Z}_{(p)}\{c_1 x_1 y_{p,0}\}/p \cong \mathbb{Z}/p$$

The map $\delta^2 : M^2 \xrightarrow{d_3} M^3$ is determined by its behavior on the generators.

By inspection of degrees, we have ${}^{U}d_3(y_{p,0}) = 0$. By Corollary 2.9 combined with the Leibniz rule, we know

$$\delta^2(c_2 y_{p,0}) = d_3(c_2 y_{p,0}) = (n-1)c_1 x_1 y_{p,0} \neq 0 \in M^3$$

$$\delta^2(c_1^2 y_{p,0}) = d_3(c_1^2 y_{p,0}) = 2nc_1 x_1 y_{p,0} = 0 \in M^3$$

Here, recall from Notation 3.1 that we assumed $p \mid n$.

Therefore, the kernel of δ^2 is generated by $c_1^2 y_{p,0}$. □

Now, we analyze the image of $\delta^1 : M^1 \to M^2$ and prove Lemma 3.6. The strategy is to find an explicit preimage of a nontrivial element in $\mathbb{Z}/p\{c_1^2 y_{p,0}\}$. We claim that

$$\delta^1(c_p c_1 x_1) = \binom{n-1}{p-1} c_1^2 y_{p,0}$$

Hence $c_p c_1 x_1 \in M^1$ could serve our purpose.

Proof of Lemma 3.6. We compute $\delta^1(c_p c_1 x_1)$ for the element $c_p c_1 x_1 \in M^1$. Instead of computing this differential directly, we first use the map $\Psi^* : {}^{U}E \to {}^{T}E$ of spectral sequences to consider the image of $\delta^1(c_p c_1 x_1)$ in ${}^{T}E$.

$$\begin{aligned} &\Psi^* \, {}^{U}d_{2p-1}(c_p c_1 x_1) = {}^{T}d_{2p-1}\Psi^*(c_p c_1 x_1) \\ &= {}^{T}d_{2p-1}[(\sum_{n \geq i_1 > i_2 > \cdots > i_p \geq 1} v_{i_1} v_{i_2} \cdots v_{i_p})(v_1 + v_2 + \cdots + v_n)x_1] \end{aligned} \tag{3.5}$$

To simplify the computation, we introduce the new elements $v_i' = v_i - v_n$ for $1 \leq i \leq n$. The advantage is that, by Proposition 2.8(2), the v_i''s are all permanent cocycles. Now, we use the v_i''s and the summation notation σ_i's defined in (2.11) to rewrite the result in (3.5).

$$
\begin{aligned}
&(\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1} v_{i_1}v_{i_2}\cdots v_{i_p})(v_1+v_2+\cdots+v_n)x_1\\
&=[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}(v'_{i_1}+v_n)(v'_{i_2}+v_n)\cdots(v'_{i_p}+v_n)](\sum_{k=1}^{n}v'_k+nv_n)x_1\\
&=[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sum_{j=0}^{p}\sigma_j(v'_{i_1},\cdots,v'_{i_p})v_n^{p-j}](\sum_{k=1}^{n}v'_k+nv_n)x_1\\
&=[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sum_{j=0}^{p}\sigma_j(v'_{i_1},\cdots,v'_{i_p})v_n^{p-j}][\sum_{k=1}^{n}v'_k]x_1\\
&\quad+n[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sum_{j=0}^{p}\sigma_j(v'_{i_1},\cdots,v'_{i_p})v_n^{p-j+1}]x_1
\end{aligned}
\tag{3.6}
$$

Now, using Proposition 2.7, we can continue the computations in (3.5) and (3.6)

$$
\begin{aligned}
&\Psi^{*}\,{}^{U}d_{2p-1}(c_pc_1x_1)\\
&={}^{T}d_{2p-1}\{[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sum_{j=0}^{p}\sigma_j(v'_{i_1},\ldots,v'_{i_p})v_n^{p-j}][\sum_{k=1}^{n}v'_k]x_1\}\\
&\quad+{}^{T}d_{2p-1}\{n[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sum_{j=0}^{p}\sigma_j(v'_{i_1},\ldots,v'_{i_p})v_n^{p-j+1}]x_1\}\\
&={}^{T}d_{2p-1}\{[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sigma_1(v'_{i_1},\ldots,v'_{i_p})][\sum_{k=1}^{n}v'_k]v_n^{p-1}x_1\}\\
&\quad+{}^{T}d_{2p-1}\{n[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sigma_2(v'_{i_1},\ldots,v'_{i_p})v_n^{p-1}]x_1\}\\
&=[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sigma_1(v'_{i_1},\ldots,v'_{i_p})][\sum_{k=1}^{n}v'_k]y_{p,0}\\
&\quad+n[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sigma_2(v'_{i_1},\ldots,v'_{i_p})]y_{p,0}
\end{aligned}
\tag{3.7}
$$

Here, we are using the fact that v'_i's are permanent cocycles. Noticing that $y_{p,0}$ is p-torsion and $p \mid n$, we can further simplify the result in (3.7)

$$
\begin{aligned}
&\Psi^{*}\ {}^{U}d_{2p-1}(c_p c_1 x_1)\\
&= [\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}(v'_{i_1}+v'_{i_2}+\cdots+v'_{i_p})][v'_1+v'_2+\cdots+v'_n]y_{p,0}\\
&= [\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}(v_{i_1}+v_{i_2}+\cdots+v_{i_p})][v_1+v_2+\cdots+v_n]y_{p,0}\\
&= [\binom{n-1}{p-1}\sum_{k=1}^{n}v_k](\sum_{k=1}^{n}v_k)y_{p,0} \qquad (3.8)\\
&= \binom{n-1}{p-1}(\sum_{k=1}^{n}v_k)^2 y_{p,0}\\
&= \Psi^{*}(\binom{n-1}{p-1}c_1^2 y_{p,0})
\end{aligned}
$$

Recall that we know the comparison map

$$\Psi^{*}: {}^{U}E_2^{2p+2,4} \to {}^{T}E_2^{2p+2,4}$$

is injective (2.10). We also know ${}^{U}E_{2p-1}^{2p+2,4}$ is a subgroup of ${}^{U}E_2^{2p+2,4}$ (3.3). Similar argument shows ${}^{T}E_{2p-1}^{2p+2,4}$ is a subgroup of ${}^{T}E_2^{2p+2,4}$. Hence the induced map

$$\Psi^{*}: {}^{U}E_{2p-1}^{2p+2,4} \to {}^{T}E_{2p-1}^{2p+2,4}$$

is also injective. Then (3.8) shows

$$\delta^1(c_p c_1 x_1) = {}^{U}d_{2p-1}(c_p c_1 x_1) = \binom{n-1}{p-1}c_1^2 y_{p,0}$$

Note $\binom{n-1}{p-1}$ is coprime to p, this shows the image of $\delta^1: M^1 \to M^2$ contains the subgroup of M^2 generated by $c_1^2 y_{p,0}$.

□

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