SOME NONTRIVIAL SECONDARY ADAMS DIFFERENTIALS ON THE FOURTH LINE

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Abstract. Let $p \geq 5$ be an odd prime. Using the correspondence between secondary Adams differentials and secondary algebraic Novikov differentials, we compute four families of nontrivial secondary differentials on the fourth line of the Adams spectral sequence. We also recover all secondary differentials on the first three lines of the Adams spectral sequence.

1. Introduction

The Adams spectral sequence (ASS) is one of the most useful tools to compute the stable homotopy groups of the sphere $\pi_*(S)$. The ASS has E_2-page $Ext^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, \mathbb{F}_p)$, where $\mathcal{A}_*$ is the dual mod p Steenrod algebra.

In this paper, we always assume p is an odd prime. Then we have

$$\mathcal{A}_* = P[t_1, t_2, \cdots] \otimes E[\tau_0, \tau_1, \tau_2, \cdots]$$

where $P[t_1, t_2, \cdots]$ is a polynomial algebra with coefficients in $\mathbb{F}_p$, $E[\tau_0, \tau_1, \tau_2, \cdots]$ is an exterior algebra with coefficients in $\mathbb{F}_p$.

The Adams E_2-page $Ext^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, \mathbb{F}_p)$ can be computed via another spectral sequence, called the Cartan-Eilenberg spectral sequence (CESS), which has E_2-page

$$H^{s,t}(P; Q^k) := Cotor^{s,t}_P(\mathbb{F}_p, Q^k) \Longrightarrow Ext^{s+k,t+k}_{\mathcal{A}_*}(\mathbb{F}_p, \mathbb{F}_p)$$

Here we denote $P = P[t_1, t_2, \cdots] \subset \mathcal{A}_*$, $Q = \mathbb{F}_p[q_0, q_1, q_2, \cdots]$, and Q^k is the degree k part of Q. When p is odd, the CESS collapses from E_2-page without nontrivial extensions. If $z \in Ext^{s+k,t+k}_{\mathcal{A}_*}(\mathbb{F}_p, \mathbb{F}_p)$ is detected in the CESS by $x \in H^{s,t}(P; Q^k)$, we say z has Cartan-Eilenberg filtration s.

The Adams-Novikov spectral sequence (ANSS) is another useful tool for computing $\pi_*(S)$. The ANSS has E_2-page $Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$, where BP denotes the Brown-Peterson spectrum. We have

$$BP_* := \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots], \quad BP_*BP = BP_*[t_1, t_2, \cdots]$$

where $\mathbb{Z}_{(p)}$ denotes the integers localized at p.

The Adams-Novikov E_2-page can be computed via the algebraic Novikov spectral sequence (algNSS) with E_2-page

$$Ext^{s,t}_{BP_*BP}(BP_*, I^k/I^{k+1}) \Longrightarrow Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$$

where I denote the ideal $(p, v_1, v_2, \cdots)$ of BP_*. Here, we have reindexed the pages to align with the notations in Gheorghe-Wang-Xu [3] and Isaksen-Wang-Xu [4].

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When p is odd, the E_2-terms of the CESS and the algNSS coincide. We have

$$Ext^{s,t}_{BP_*BP}(BP_*, I^k/I^{k+1}) \cong H^{s,t}(P; Q^k) \tag{1.1}$$

Then, we have the following diagram of spectral sequences.

$$\begin{tikzcd}
Ext^{s,t}_{BP_*BP}(BP_*, I^k/I^{k+1}) \cong H^{s,t}(P; Q^k) \arrow[r, Rightarrow, "CESS", "\cong"'] \arrow[d, Rightarrow, "algNSS"'] & Ext^{s+k,t+k}_{\mathcal{A}_*}(\mathbb{F}_p, \mathbb{F}_p) \arrow[d, Rightarrow, "ASS"] \\
Ext^{s,t}_{BP_*BP}(BP_*, BP_*) \arrow[r, Rightarrow, "ANSS"] & \pi_{t-s}(S)
\end{tikzcd}$$

In practice, the main difficulty of computing with the ASS is that the Adams differentials d_r^{Adams}'s are difficult to be determined in general. On the other hand, the algebraic Novikov differentials d_r^{alg}'s are much easier to be computed. This is because the entire construction of the algNSS is purely algebraic. Computing d_r^{alg}'s does not require any topological background knowledge. It turns out that when $r = 2$, there is a direct correspondence between d_2^{Adams}'s and d_2^{alg}'s.

Theorem 1.1 (Novikov [10], Andrews-Miller [2, 8])**.** *Suppose z is an element in $Ext^{s+k,t+k}_{\mathcal{A}_*}(\mathbb{F}_p, \mathbb{F}_p)$ with Cartan-Eilenberg filtration s, then $d_2^{Adams}(z)$ has higher Cartan-Eilenberg filtration.*

Moreover, if z is detected in the CESS by $x \in Ext^{s,t}_{BP_*BP}(BP_*, I^k/I^{k+1})$, then $d_2^{Adams}(z)$ is detected by $d_2^{alg}(x) \in Ext^{s+1,t}_{BP_*BP}(BP_*, I^{k+1}/I^{k+2})$.

Let $p \geq 5$. A complete list of generators together with their $d_2^{Adams}(z)$ has been determined for the first three lines of the Adams E_2-page, i.e. $Ext^{s,t}_{\mathcal{A}_*}(\mathbb{F}_p, \mathbb{F}_p)$ with $s = 1, 2, 3$ (see [1, 5, 9, 12, 13, 14]). Meanwhile, only partial results are known for the fourth line $Ext^{4,*}_{\mathcal{A}_*}(\mathbb{F}_p, \mathbb{F}_p)$ (see, for example [15]).

In this paper, we analyze d_2^{Adams}'s of elements on the fourth line of the Adams spectral sequence by computing their corresponding d_2^{alg}'s. We also further simplify the computations using the May spectral sequence [6]. Our main result is the following.

Theorem 3.4. *There are nontrivial secondary Adams differentials given as follows:*

1. $d_2^{Adams}(h_{4,i}h_{3,i}g_i) = a_0 b_{4,i-1}h_{3,i}g_i$, *for* $i \geq 1$.
2. $d_2^{Adams}(h_{4,i}h_{3,i+1}k_{i+2}) = a_0 b_{4,i-1}h_{3,i+1}k_{i+2}$, *for* $i \geq 1$.
3. $d_2^{Adams}(h_{4,i}g_i h_{i+3}) = a_0 b_{4,i-1}g_i h_{i+3}$, *for* $i \geq 1$.
4. $d_2^{Adams}(h_{3,i}h_{2,i+1}k_i) = a_0 b_{3,i-1}h_{2,i+1}k_i$, *for* $i \geq 1$.

It is straightforward to verify that these four families of elements are indecomposable, i.e., they can not be write as products of elements from the first three lines. Consequently, one can not deduce the differentials simply via Leibniz rule.

From our point of view, the practical computational strategy here is possibly more interesting than the result itself. To further demonstrate this, in Section 4, we use the same strategy to recover all secondary Adams differentials on the first three lines.

Previously, the nontrivial Adams differentials on the third line were computed in [13] using matrix Massey products [7]. Comparatively, our computational strategy has the following advantages: (i) Our strategy is applicable to general elements without restrictions. Our computations can be easily adapted to analyze other

d_2^{Adams}'s of interest. On the contrary, the matrix Massey product method could fail when the relevant indeterminacy is nontrivial; (ii) Our computations of the algebraic Novikov differentials are routine and purely algebraic. Such computations are comparatively simpler than the previous ones using matrix Massey products.

Organization of the paper. In Section 2, we review the basic setup of the algebraic Novikov spectral sequence. In Section 3, we compute relevant algebraic Novikov differentials and prove Theorem 3.4. In Section 4, we use the same computational strategy to recover the secondary Adams differentials on the first three lines.

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2. The algebraic Novikov spectral sequence

In this section, we review the basic algebraic definitions and computational results related to the Hopf algebroid (BP_*, BP_*BP). We will also recall the basic setup of the algebraic Novikov spectral sequence.

2.1. **Hopf algebroids.**

Definition 2.2. A *Hopf algebroid* over a commutative ring K is a pair (A, Γ) of commutative K-algebras with the following structure maps

$$\begin{gathered}\text{left unit map } \eta_L : A \to \Gamma\\ \text{right unit map } \eta_R : A \to \Gamma\\ \text{coproduct map } \Delta : \Gamma \to \Gamma \otimes_A \Gamma\\ \text{counit map } \varepsilon : \Gamma \to A\\ \text{conjugation map } c : \Gamma \to \Gamma\end{gathered}$$

such that for any other commutative K-algebra B, the two sets of K-homomorphisms $\mathrm{Hom}_K(A, B)$ and $\mathrm{Hom}_K(\Gamma, B)$ are the objects and morphisms of a groupoid.

Recall that we have

$$BP_* := \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots], \quad BP_*BP = BP_*[t_1, t_2, \cdots]$$

where $|v_n| = |t_n| = 2(p^n - 1)$. Throughout this paper, we denote $v_0 = p$ and $t_0 = 1$.

The pair (BP_*, BP_*BP) has a Hopf algebroid structure.

The left unit map $\eta_L : BP_* \to BP_*BP$ is determined by

$$\eta_L(v_n) = v_n \tag{2.1}$$

The right unit map $\eta_R : BP_* \to BP_*BP$ satisfies

$$\eta_R(v_n) = v_n + v_{n-1}t_1^{p^{n-1}} + \cdots + v_1 t_{n-1}^p + pt_n \mod I^p \tag{2.2}$$

where we let I denote the ideal $(p, v_1, v_2, \cdots)$ of BP_*.

The coproduct map $\Delta : BP_*BP \to BP_*BP \otimes_{BP_*} BP_*BP$ is determined by

$$\Delta(t_n) = \sum_{k=0}^{n} t_{n-k} \otimes t_k^{p^{n-k}} - \sum_{k=1}^{n-1} v_k b_{n-k,k-1} \tag{2.3}$$

where $b_{i,j} = \sum_{k=1}^{p-1} \binom{p}{k}/p \, (t_i^{p^j})^k \otimes (t_i^{p^j})^{p-k}$.

The counit map $\varepsilon : BP_*BP \to BP_*$ is determined by

$$\varepsilon(v_n) = v_n, \quad \varepsilon(t_n) = 0. \tag{2.4}$$

The Steenrod algebra provides another important example of Hopf algebroids.

Let $\mathcal{A}_*$ denote the dual mod p Steenrod algebra for an odd prime p, we have

$$\mathcal{A}_* = P[t_1, t_2, \cdots] \otimes E[\tau_0, \tau_1, \tau_2, \cdots]$$

where $P[t_1, t_2, \cdots]$ is a polynomial algebra with coefficients in $\mathbb{F}_p$, $E[\tau_0, \tau_1, \tau_2, \cdots]$ is an exterior algebra with coefficients in $\mathbb{F}_p$. For the internal degrees, we have $|t_n| = 2(p^n - 1), |\tau_n| = 2p^n - 1$.

One can show $\mathcal{A}_*$ is a Hopf algebra over $\mathbb{F}_p$. In particular, $(\mathbb{F}_p, \mathcal{A}_*)$ has a Hopf algebroid structure, where the coproduct map $\Delta : \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ is given by:

$$\Delta t_n = \sum_{i=0}^{n} t_i \otimes t_{n-i}^{p^i}, (t_0 = 1), \quad \Delta \tau_n = \sum_{i=0}^{n} \tau_i \otimes t_{n-i}^{p^i} + 1 \otimes \tau_n \tag{2.5}$$

2.3. **cobar complexes.**

Definition 2.4. Let (A, Γ) be a Hopf algebroid. A *right* Γ*-comodule* M is a right A-module M together with a right A-linear map $\psi : M \to M \otimes_A \Gamma$ which is counitary and coassociative, i.e., the following diagrams commute.

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[dr, equal] & M \otimes_A \Gamma \arrow[d, "M\otimes\varepsilon"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\psi"'] & M \otimes_A \Gamma \arrow[d, "M\otimes\Delta"] \\ M \otimes_A \Gamma \arrow[r, "\psi\otimes\Gamma"'] & M \otimes_A \Gamma \otimes_A \Gamma \end{tikzcd}$$

Left Γ-comodules are defined similarly.

Given a right Γ-comodule M, we construct the cobar complex of M as

$$\Omega_\Gamma^{s,*}(M) = M \otimes_A \overline{\Gamma}^{\otimes s}$$

where $\overline{\Gamma}$ is the augmentation ideal of $\varepsilon : \Gamma \to A$. The differentials $d : \Omega_\Gamma^{s,*}(M) \to \Omega_\Gamma^{s+1,*}(M)$ are given by

$$\begin{aligned} d(m \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_s) &= -(\psi(m) - m \otimes 1) \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_s \\ &- \sum_{i=1}^{s} (-1)^{\lambda_{i,j_i}} m \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes (\sum_{j_i} x'_{i,j_i} \otimes x''_{i,j_i}) \otimes x_{i+1} \otimes \cdots \otimes x_s \end{aligned} \tag{2.6}$$

where

$$\sum_{j_i} x'_{i,j_i} \otimes x''_{i,j_i} = \Delta(x_i) - 1 \otimes x_i - x_i \otimes 1$$

$$\lambda_{i,j_i} = i + |x_1| + \cdots + |x_{i-1}| + |x'_{i,j_i}|$$

The cohomology of $\Omega_\Gamma^{s,*}(M)$ is $Ext_\Gamma^{s,*}(A, M)$ (see [11, Section A1.2]).

2.5. **constructing the algebraic Novikov spectral sequence.** Let I be the ideal of BP_* generated by $(p, v_1, v_2, \cdots)$. Then I is an invariant ideal as a BP_*BP-comodule, in other words, we have $I \cdot BP_*BP = BP_*BP \cdot I$.

Consider the filtration

$$BP_* = I^0 \supset I^1 \supset I^2 \supset I^3 \supset \cdots \supset I^k \supset I^{k+1} \supset \cdots$$

Given $y \in BP_*$, we write $y = ap^{k_0}v_1^{k_1}v_2^{k_2}\cdots$, where $a \in \mathbb{Z}_{(p)}$ is invertible. We define the length of y to be $l(y) = k_0 + k_1 + k_2 + \cdots$. Then $y \in I^k$ if and only if $l(y) \geq k$. Therefore, we have

$$\bigoplus_{k \geq 0} I^k/I^{k+1} = P[q_0, q_1, q_2, \cdots]$$

is the polynomial algebra on $q_0, q_1, q_2, \cdots$, where q_i corresponds to v_i, I^k/I^{k+1} corresponds to homogeneous polynomials of degree k.

The short exact sequence

$$0 \to I^{k+1} \to I^k \to I^k/I^{k+1} \to 0 \tag{2.7}$$

induces a BP_*BP-comodule structure on I^k/I^{k+1}. The structure map

$$\psi : I^k/I^{k+1} \to I^k/I^{k+1} \otimes_{BP_*} BP_*BP$$

is induced from the right unit map (2.2). More explicitly, we have

$$\psi(q_k) = q_k + q_{k-1}t_1^{p^{k-1}} + \cdots + q_1 t_{k-1}^p + q_0 t_k \tag{2.8}$$

From (2.7), we get short exact sequences of cobar complexes

$$0 \to \Omega^{*,*}_{BP_*BP}(I^{k+1}) \to \Omega^{*,*}_{BP_*BP}(I^k) \to \Omega^{*,*}_{BP_*BP}(I^k/I^{k+1}) \to 0$$

which induces an associated spectral sequence by standard algebraic arguments.

Proposition 2.6. *There is a spectral sequence, called the algebraic Novikov spectral sequence, with E_2-page*

$$E_2^{s,t,k} = Ext^{s,t}_{BP_*BP}(BP_*, I^k/I^{k+1})$$

and differentials

$$d_r^{alg} : E_r^{s,t,k} \to E_r^{s+1,t,k+r-1}$$

that converges to the E_2-terms $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$ of the Adams-Novikov spectral sequence.

Remark 2.7. Our notations here is different from the ones used in [2]. We have regraded the spectral sequence to better align with the notations in [3, 4] .

3. Secondary Adams differentials on the fourth line

In this section, we prove our main result Theorem 3.4. Using Theorem 1.1, we determine these Adams differentials by computing their corresponding algebraic Novikov differentials.

To simplify computations, we work with the May spectral sequence representatives of the Adams E_2-terms. We recall the basic results of the May spectral sequence (MSS) below. More detailed discussions can be found in, for example, [11, Section 3.2].

The May spectral sequence (MSS) is a useful spectral sequence which converges to $Ext_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p, \mathbb{F}_p)$. Let $p \geq 3$, the May spectral sequence has the following form

$$(E_r^{s,t,M}, d_r) \Longrightarrow Ext_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p, \mathbb{F}_p), \quad d_r : E_r^{s,t,M} \to E_r^{s+1,t,M-r}$$

where the last degree M is called the May degree. The E_1-page of the May spectral sequence is isomorphic to

$$E[h_{i,j} | i \geq 1, j \geq 0] \otimes P[b_{i,j} | i \geq 1, j \geq 0] \otimes P[a_i | i \geq 0]$$

where

$$h_{i,j} = [t_i^{p^j}] \in E_1^{1,2(p^i-1)p^j,2i-1}$$

$$b_{i,j} = [\sum_{k=1}^{p-1} \binom{p}{k} / p \ (t_i^{p^j})^k \otimes (t_i^{p^j})^{p-k} \ ] \in E_1^{2,2(p^i-1)p^{j+1},p(2i-1)}$$

$$a_i = [\tau_i] \in E_1^{1,2(p^i-1)+1,2i+1}$$

In general, we could use the following strategy to compute secondary Adams differentials:

(1) For an element x in the Adams E_2-page, we find the MSS representative of x.
(2) We deduce the MSS leading term of x', where x' is the algebraic Novikov E_2-term that corresponds to x.
(3) We determine the MSS leading term of the algebraic Novikov differential $d_2^{alg}(x')$.
(4) We find the element y in the Adams E_2-page that corresponds to $d_2^{alg}(x')$. Then we have $d_2^{Adams}(x) = y$.

In particular, we will use the following table for the four family of Adams E_2-terms in Theorem 3.4.

Adams E_2-terms	Representation in MSS	algNov representation
$h_{4,i}h_{3,i}g_i$	$h_{4,i}h_{3,i}h_{2,i}h_{1,i}$	$t_4^{p^i} \otimes t_3^{p^i} \otimes t_2^{p^i} \otimes t_1^{p^i}$
$h_{4,i}h_{3,i+1}k_{i+2}$	$h_{4,i}h_{3,i+1}h_{2,i+2}h_{1,i+3}$	$t_4^{p^i} \otimes t_3^{p^{i+1}} \otimes t_2^{p^{i+2}} \otimes t_1^{p^{i+3}}$
$h_{4,i}g_ih_{i+3}$	$h_{4,i}h_{2,i}h_{1,i}h_{1,i+3}$	$t_4^{p^i} \otimes t_2^{p^i} \otimes t_1^{p^i} \otimes t_1^{p^{i+3}}$
$h_{3,i}h_{2,i+1}k_i$	$h_{3,i}h_{2,i+1}h_{2,i}h_{1,i+1}$	$t_3^{p^i} \otimes t_2^{p^{i+1}} \otimes t_2^{p^i} \otimes t_1^{p^{i+1}}$

Table 1. Representations of the four elements

Now we start the actual computations.

Lemma 3.1. *We have the following results for the differential d in the cobar complex $\Omega_{BP_*BP}^{*,*}(BP_*)$ (2.6).*

(1) $d(t_1^{p^i}) = pb_{1,i-1} \mod I^2$, *for $i \geq 1$.*
(2) $d(t_2^{p^i}) = t_1^{p^i} \otimes t_1^{p^{i+1}} + pb_{2,i-1} + A_i \mod I^2$, *for $i \geq 1$, where A_i consists of terms in I/I^2 with May degree lower than that of $pb_{2,i-1}$.*
(3) $d(t_3^{p^i}) = t_2^{p^i} \otimes t_1^{p^{i+2}} + t_1^{p^i} \otimes t_2^{p^{i+1}} + pb_{3,i-1} + B_i \mod I^2$, *for $i \geq 1$, where B_i consists of terms in I/I^2 with May degree lower than that of $pb_{3,i-1}$.*

(4) $d(t_4^{p^i}) = t_3^{p^i} \otimes t_1^{p^{i+3}} + t_2^{p^i} \otimes t_2^{p^{i+2}} + t_1^{p^i} \otimes t_3^{p^{i+1}} + pb_{4,i-1} + C_i \mod I^2$, *for* $i \geq 1$, *where* C_i *consists of terms in* I/I^2 *with May degree lower than that of* $pb_{4,i-1}$.

Proof. (1) For $i \geq 1$, we have

$$\begin{aligned} d(t_1^{p^i}) &= \Delta(t_1^{p^i}) - 1 \otimes t_1^{p^i} - t_1^{p^i} \otimes 1 \\ &= (1 \otimes t_1 + t_1 \otimes 1)^{p^i} - 1 \otimes t_1^{p^i} - t_1^{p^i} \otimes 1 \\ &= \sum_{j=1}^{p^i-1} \binom{p^i}{j} t_1^j \otimes t_1^{p^i-j} \\ &= \sum_{k=1}^{p-1} \binom{p^i}{kp^{i-1}} t_1^{kp^{i-1}} \otimes t_1^{(p-k)p^{i-1}} \quad \mathrm{mod} I^2 \\ &= \sum_{k=1}^{p-1} \binom{p}{k} t_1^{kp^{i-1}} \otimes t_1^{(p-k)p^{i-1}} \quad \mathrm{mod} I^2 \\ &= pb_{1,i-1} \quad \mathrm{mod} I^2 \end{aligned} \tag{3.1}$$

(2) For $i \geq 1$, we have

$$\begin{aligned} d(t_2^{p^i}) &= \Delta(t_2^{p^i}) - 1 \otimes t_2^{p^i} - t_2^{p^i} \otimes 1 \\ &= (t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 - v_1 b_{1,0})^{p^i} - 1 \otimes t_2^{p^i} - t_2^{p^i} \otimes 1 \\ &= (t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2)^{p^i} - 1 \otimes t_2^{p^i} - t_2^{p^i} \otimes 1 \quad \mathrm{mod} I^2 \\ &= t_1^{p^i} \otimes t_1^{p^{i+1}} + \sum C_{n,k} \quad \mathrm{mod} I^2 \end{aligned} \tag{3.2}$$

where we require $0 \leq k, n-k, p-n < p$, and denote

$$\begin{aligned} C_{n,k} &:= \binom{p^i}{np^{i-1}} \binom{np^{i-1}}{kp^{i-1}} (t_2 \otimes 1)^{kp^{i-1}} (1 \otimes t_2)^{(n-k)p^{i-1}} (t_1 \otimes t_1^p)^{(p-n)p^{i-1}} \\ &= \binom{p^i}{np^{i-1}} \binom{np^{i-1}}{kp^{i-1}} t_2^{kp^{i-1}} t_1^{(p-n)p^{i-1}} \otimes t_2^{(n-k)p^{i-1}} t_1^{(p-n)p^i} \end{aligned} \tag{3.3}$$

The May degree of $C_{n,k}$ is

$$M(C_{n,k}) = 3k + (p-n) + 3(n-k) + (p-n) = 3n + 2(p-n) = 2p + n$$

The May degree $M(C_{n,k})$ is maximized when $n = p$. Hence, we can write

$$\begin{aligned} d(t_2^{p^i}) &= t_1^{p^i} \otimes t_1^{p^{i+1}} + \sum C_{p,k} + A_i \quad \mathrm{mod} I^2 \\ &= t_1^{p^i} \otimes t_1^{p^{i+1}} + \sum_{k=1}^{p-1} \binom{p^i}{kp^{i-1}} t_2^{kp^{i-1}} \otimes t_2^{(p-k)p^{i-1}} + A_i \quad \mathrm{mod} I^2 \\ &= t_1^{p^i} \otimes t_1^{p^{i+1}} + pb_{2,i-1} + A_i \quad \mathrm{mod} I^2 \end{aligned} \tag{3.4}$$

where A_i consists of terms in I/I^2 with May degree lower than that of $pb_{2,i-1}$.

(3) and (4) can also be computed directly using the same ideas. □

Proposition 3.2. *Let* x *be an element in the* E_2*-page of the algebraic Novikov spectral sequence such that* x *has May filtration leading term* $t_4^{p^i} \otimes t_3^{p^i} \otimes t_2^{p^i} \otimes t_1^{p^i}$, $i \geq 1$, *then* $d_2^{alg}(x)$ *has May filtration leading term* $pb_{4,i-1} \otimes t_3^{p^i} \otimes t_2^{p^i} \otimes t_1^{p^i}$.

Proof. We let d denote the differential in the cobar complex $\Omega^{*,*}_{BP_*BP}(BP_*)$ (2.6).

Using Lemma 3.1 and Leibniz rule, we have

$$\begin{aligned} d(t_4^{p^i}\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes t_1^{p^i}) &= d(t_4^{p^i})\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes t_1^{p^i} - t_4^{p^i}\otimes d(t_3^{p^i})\otimes t_2^{p^i}\otimes t_1^{p^i} \\ &\quad + t_4^{p^i}\otimes t_3^{p^i}\otimes d(t_2^{p^i})\otimes t_1^{p^i} - t_4^{p^i}\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes d(t_1^{p^i}) \\ &\equiv R + pb_{4,i-1}\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes t_1^{p^i} + L \mod I^2 \end{aligned} \tag{3.5}$$

where we denote

$$\begin{aligned} R &= (t_3^{p^i}\otimes t_1^{p^{i+3}} + t_2^{p^i}\otimes t_2^{p^{i+2}} + t_1^{p^i}\otimes t_3^{p^{i+1}})\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes t_1^{p^i} \\ &\quad - t_4^{p^i}\otimes (t_2^{p^i}\otimes t_1^{p^{i+2}} + t_1^{p^i}\otimes t_2^{p^{i+1}})\otimes t_2^{p^i}\otimes t_1^{p^i} + t_4^{p^i}\otimes t_3^{p^i}\otimes t_1^{p^i}\otimes t_1^{p^{i+1}}\otimes t_1^{p^i} \end{aligned} \tag{3.6}$$

and L consists of terms in I/I^2 with May degree lower than $M(pb_{4,i-1}\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes t_1^{p^i}) = 7p+10$.

By assumption, x survives to the E_2-page of the algebraic Novikov spectral sequence. Hence, we have

$$d(x)\equiv 0 \mod I$$

We can write

$$x = t_4^{p^i}\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes t_1^{p^i} - \sum_r y_r$$

where $y_r\in\Omega^{4,*}_{BP_*BP}(BP_*)$, $M(y_r) < M(t_4^{p^i}\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes t_1^{p^i}) = 7+5+3+1 = 16$ for each r, and that

$$\sum_r d(y_r)\equiv d(t_4^{p^i}\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes t_1^{p^i})\equiv R \mod I$$

For notation simplicity, we denote

$$d(y_r)\equiv \sum_s A_{r,s} + \sum_t B_{r,t} \mod I^2$$

where $A_{r,s}\in I^0/I, B_{r,t}\in I/I^2$.

The condition $M(y_r)\le 15$ puts a strong restriction on the form of y_r. One can conduct a tedious but straightforward check through all of the possible forms to show we always have $M(B_{r,t}) < M(pb_{4,i-1}\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes t_1^{p^i}) = 7p+10$. We can also briefly summarize the idea as follows. There are three different cases to consider:

(a) If $y_r = t_4^{p^k}\otimes A$ with $k\ge 1$, where A is formed by t_1, t_2, t_3 terms and that $M(A)\le 8$. Then we have $M(B_{r,t})\le M(pb_{4,k-1}\otimes A) = 7p+1+M(A)\le 7p+9 < 7p+10$.

(b) If $y_r = t_4\otimes A$, where A is formed by t_1, t_2, t_3 terms and that $M(A)\le 8$. Note $d(t_4) = t_3\otimes t_1^{p^3} + t_2\otimes t_2^{p^2} + t_1\otimes t_3^p - v_1b_{3,0} - v_2b_{2,1} - v_3b_{1,2}$, and that $M(b_{i,j}) = p(2i-1)\le 5p$ with $i\le 3$. It is also not difficult to observe that $M(B_{r,t}) < 7p+10$.

(c) If y_r is formed by t_1, t_2, t_3 terms. One can also check that $M(B_{r,t}) < 7p+10$ using similar ideas.

Hence, after modding out lower May filtration terms, we could write

$$d(x) = d(t_4^{p^i}\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes t_1^{p^i}) - \sum_r d(y_r)\equiv pb_{4,i-1}\otimes t_3^{p^i}\otimes t_2^{p^i}\otimes t_1^{p^i} \mod I^2$$

This completes the proof. $\square$

Similar to Proposition 3.2, we can compute the following differentials.

Proposition 3.3. *We have the following secondary algebraic Novikov differentials.*

(1) $d_2^{alg}(t_4^{p^i} \otimes t_3^{p^{i+1}} \otimes t_2^{p^{i+2}} \otimes t_1^{p^{i+3}}) = pb_{4,i-1} \otimes t_3^{p^{i+1}} \otimes t_2^{p^{i+2}} \otimes t_1^{p^{i+3}}$, *for* $i \geq 1$.
(2) $d_2^{alg}(t_4^{p^i} \otimes t_2^{p^i} \otimes t_1^{p^i} \otimes t_1^{p^{i+3}}) = pb_{4,i-1} \otimes t_2^{p^i} \otimes t_1^{p^i} \otimes t_1^{p^{i+3}}$, *for* $i \geq 1$.
(3) $d_2^{alg}(t_3^{p^i} \otimes t_2^{p^{i+1}} \otimes t_2^{p^i} \otimes t_1^{p^{i+1}}) = pb_{3,i-1} \otimes t_2^{p^{i+1}} \otimes t_2^{p^i} \otimes t_1^{p^{i+1}}$, *for* $i \geq 1$.

Here, the equations hold after modding out lower May filtration terms.

Proof. These results can be computed directly analogous to Proposition 3.2. □

Theorem 3.4. *There are nontrivial secondary Adams differentials given as follows:*

(1) $d_2^{Adams}(h_{4,i}h_{3,i}g_i) = a_0b_{4,i-1}h_{3,i}g_i$, *for* $i \geq 1$.
(2) $d_2^{Adams}(h_{4,i}h_{3,i+1}k_{i+2}) = a_0b_{4,i-1}h_{3,i+1}k_{i+2}$, *for* $i \geq 1$.
(3) $d_2^{Adams}(h_{4,i}g_ih_{i+3}) = a_0b_{4,i-1}g_ih_{i+3}$, *for* $i \geq 1$.
(4) $d_2^{Adams}(h_{3,i}h_{2,i+1}k_i) = a_0b_{3,i-1}h_{2,i+1}k_i$, *for* $i \geq 1$.

Proof. These results can be directly deduced from Propositions 3.2 and 3.3. Moreover, these differentials are all nontrivial. We can take $a_0b_{4,i-1}h_{3,i}g_i$ as an example to show $a_0b_{4,i-1}h_{3,i}g_i \neq 0 \in Ext_{\mathcal{A}_*}^{6,*}$. The other three cases are similar.

Note $a_0b_{4,i-1}h_{3,i}g_i$ has May spectral sequence representative

$$a_0b_{4,i-1}h_{3,i}h_{2,i}h_{1,i} \in E_1^{6,t,M}$$

Here the inner degree is

$$t = 1 + qp^i((1 + p + p^2 + p^3) + (1 + p + p^2) + (1 + p) + 1)$$

where we denote $q = 2(p-1)$. Let x be an element in $E_1^{5,t,*}$. Inspection of degrees shows x must be $a_0h_{4,i}h_{3,i}h_{2,i}h_{1,i}$. Then $M(x) < M(a_0b_{4,i-1}h_{3,i}h_{2,i}h_{1,i})$. Hence $a_0b_{4,i-1}h_{3,i}h_{2,i}h_{1,i}$ can not be the image of any May differential $d_r : E_r^{5,t,M+r} \to E_r^{6,t,M}$, $r \geq 1$. This completes the proof. □

It is worth pointing out that Zhong-Hong-Zhao [15] also computed two other nontrivial differentials on the fourth line.

Theorem 3.5 (Zhong-Hong-Zhao [15])**.** *On the fourth line* $Ext_{\mathcal{A}_*}^{4,*}(\mathbb{F}_p, \mathbb{F}_p)$ *of the Adams spectral sequence, there exist two nontrivial secondary Adams differentials given as follows:*

(1) $d_2^{Adams}(h_{3,i}g_ih_{2,i-1}) = a_0b_{3,i-1}g_ih_{2,i-1}$ *for* $i \geq 2$.
(2) $d_2^{Adams}(h_{3,i}k_{i+1}h_{2,i+2}) = a_0b_{3,i-1}k_{i+1}h_{2,i+2}$ *for* $i \geq 1$.

Their result can be recovered from computing the following corresponding algebraic Novikov differentials.

Proposition 3.6. *We have the following secondary algebraic Novikov differentials. Here, the equations hold after modding out lower May filtration terms.*

(1) $d_2^{alg}(t_3^{p^i} \otimes t_2^{p^i} \otimes t_1^{p^i} \otimes t_2^{p^{i-1}}) = pb_{3,i-1} \otimes t_2^{p^i} \otimes t_1^{p^i} \otimes t_2^{p^{i-1}}$, *for* $i \geq 2$.
(2) $d_2^{alg}(t_3^{p^i} \otimes t_2^{p^{i+1}} \otimes t_1^{p^{i+2}} \otimes t_2^{p^{i+2}}) = pb_{3,i-1} \otimes t_2^{p^{i+1}} \otimes t_1^{p^{i+2}} \otimes t_2^{p^{i+2}}$, *for* $i \geq 1$.

Proof. These results can be computed directly analogous to Proposition 3.2. □

Our computations here are comparatively simpler than the original computations in [15] using matrix Massey products.

4. Secondary Adams differentials on the first three lines

In this section, we use the strategy explained in Section 3 to recover secondary Adams differentials on the first three lines.

The generators for the first two lines of the Adams spectral sequence was determined by Liulevicius in [5]. We summarize them in the following table.

Generator	Representation in MSS	Inner Degree	Range of indices
a_0	a_0	1	
h_i	$h_{1,i}$	qp^i	$i \geq 0$
a_1h_0	$a_1h_{1,0}$	$2q+1$	
a_0^2	a_0^2	2	
a_0h_i	$a_0h_{1,i}$	qp^i+1	$i \geq 1$
g_i	$h_{2,i}h_{1,i}$	$q(2p^i+p^{i+1})$	$i \geq 0$
k_i	$h_{2,i}h_{1,i+1}$	$q(p^i+2p^{i+1})$	$i \geq 0$
b_i	$b_{1,i}$	qp^{i+1}	$i \geq 0$
h_ih_j	$h_{1,i}h_{1,j}$	$q(p^i+p^j)$	$j-2 \geq i \geq 0$

Table 2. A $\mathbb{F}_p$-basis of $Ext^{1,*}_{\mathcal{A}_*}$ and $Ext^{2,*}_{\mathcal{A}_*}$

For odd primes, Aikawa [1] determined a basis for $Ext^{3,*}_{\mathcal{A}_*}$ using Λ-algebra. For $p \geq 5$, Wang [13] determined the May spectral sequence representatives of the generators. The result is summarized in the following table.

Generator	MSS Representation	Inner Degree	Range of indices
$h_ih_jh_k$	$h_{1,i}h_{1,j}h_{1,k}$	$q(p^i+p^j+p^k)$	$k-4 \geq j-2 \geq i \geq 0$
$a_0h_ih_j$	$a_0h_{1,i}h_{1,j}$	$q(p^i+p^j)+1$	$j-2 \geq i \geq 1$
$a_0^2h_i$	$a_0^2h_{1,i}$	qp^i+2	$i \geq 1$
a_0^3	a_0^3	3	
b_ih_j	$b_{1,i}h_{1,j}$	$q(p^{i+1}+p^j)$	$i,j \geq 0, j \neq i+2$
a_0b_i	$a_0b_{1,i}$	$qp^{i+1}+1$	$i \geq 1$
g_ih_j	$h_{2,i}h_{1,i}h_{1,j}$	$q(2p^i+p^{i+1}+p^j)$	$j \neq i+2, i, i-1,$ and $i,j \geq 0$
g_ia_0	$h_{2,i}h_{1,i}a_0$	$q(2p^i+p^{i+1})+1$	$i \geq 1$
k_ih_j	$h_{2,i}h_{1,i+1}h_{1,j}$	$q(p^i+2p^{i+1}+p^j)$	$j \neq i+2, i \pm 1, i,$ and $i,j \geq 0$
k_ia_0	$h_{2,i}h_{1,i+1}a_0$	$q(p^i+2p^{i+1})+1$	$i \geq 1$
$a_1h_0h_j$	$a_1h_{1,0}h_{1,j}$	$q(2+p^j)+1$	$j \geq 2$
$h_{3,i}g_i$	$h_{3,i}h_{2,i}h_{1,i}$	$q(3p^i+2p^{i+1}+p^{i+2})$	$i \geq 0$
a_2k_0	$a_2h_{2,0}h_{1,1}$	$q(2+3p)+1$	
$h_{2,i}g_{i+1}$	$h_{2,i}h_{2,i+1}h_{1,i+1}$	$q(p^i+3p^{i+1}+p^{i+2})$	$i \geq 0$
a_1g_0	$a_1h_{2,0}h_{1,0}$	$q(3+p)+1$	

$h_{3,i}h_{i+2}h_i$	$h_{3,i}h_{1,i+2}h_{1,i}$	$q(2p^i + p^{i+1} + 2p^{i+2})$	$i \geq 0$
$h_{3,i}k_{i+1}$	$h_{3,i}h_{2,i+1}h_{1,i+2}$	$q(p^i + 2p^{i+1} + 3p^{i+2})$	$i \geq 0$
$a_1^2 h_0$	$a_1^2 h_{1,0}$	$3q + 2$	
$b_{2,i}h_{i+1}$	$b_{2,i}h_{1,i+1}$	$q(2p^{i+1} + p^{i+2})$	$i \geq 0$
$b_{2,i}h_{i+2}$	$b_{2,i}h_{1,i+2}$	$q(p^{i+1} + 2p^{i+2})$	$i \geq 0$

TABLE 3. A $\mathbb{F}_p$-basis of $Ext_{\mathcal{A}_*}^{3,*}$

We can compute d_2^{Adams} for the basis elements in Table 2 via computing d_2^{alg} of their corresponding elements. For simplicity, we only list the nontrivial d_2^{alg} differentials here.

Proposition 4.1. *Let p be an odd prime. Amongst the elements in the algebraic Novikov spectral sequence that corresponds to the first and second line basis listed in Table 2, all nontrivial d_2^{alg}'s are summarized as follows. Here, the equations hold after modding out lower May filtration terms.*

1. $d_2^{alg}(t_1^{p^i}) = pb_{1,i-1}$, *for* $i > 0$.
2. $d_2^{alg}(pt_1^{p^i}) = p^2 b_{1,i-1}$, $i \geq 1$.
3. $d_2^{alg}(t_2^{p^i} \otimes t_1^{p^i}) = pb_{2,i-1} \otimes t_1^{p^i}$, $i \geq 1$.
4. $d_2^{alg}(t_2 \otimes t_1) = -v_1 b_{1,0} \otimes t_1$.
5. $d_2^{alg}(t_2^{p^i} \otimes t_1^{p^{i+1}}) = pb_{2,i-1} \otimes t_1^{p^{i+1}}$, $i \geq 1$.
6. $d_2^{alg}(t_1^{p^i} \otimes t_1^{p^j}) = pb_{1,i-1} \otimes t_1^{p^j} - t_1^{p^i} \otimes pb_{1,j-1}$, $j - 2 \geq i \geq 1$.

Proof. Analogous to Proposition 3.2, all of the results are computed directly from the construction (2.6) of the cobar complex. $\square$

Then, we can recover the d_2^{Adams} results on the first two lines directly from Proposition 4.1.

Theorem 4.2 (Liulevicius[5], Shimada-Yamanoshita [12], Miller-Ravenel-Wilson [9], Zhao-Wang [14])**.** *Amongst the first and second line basis in Table 2, all nontrivial Adams d_2 differentials can be summarized as follows.*

1. $d_2^{Adams}(h_i) = a_0 b_{i-1}$, $i \geq 1$.
2. $d_2^{Adams}(a_0 h_i) = a_0^2 b_{i-1}$, $i \geq 1$.
3. $d_2^{Adams}(g_i) = a_0 b_{2,i-1} h_i$, $i \geq 1$.
4. $d_2^{Adams}(g_0) = -a_1 b_0 h_0$.
5. $d_2^{Adams}(k_i) = a_0 b_{2,i-1} h_{i+1}$, $i \geq 1$.
6. $d_2^{Adams}(h_i h_j) = a_0 b_{i-1} h_j - h_i a_0 b_{j-1}$, $j - 2 \geq i \geq 1$.

Similarly, we can compute d_2^{Adams} for the third line basis via computing d_2^{alg} of their corresponding elements. For simplicity, we only list the nontrivial differentials here.

Proposition 4.3. *Let $p \geq 5$ be an odd prime. Amongst the elements in the algebraic Novikov spectral sequence that corresponds to the third line basis listed in Table 3, all nontrivial d_2^{alg}'s are summarized as follows. Here, the equations hold after modding out lower May filtration terms.*

(1) $d_2^{alg}(t_1^{p^i} \otimes t_1^{p^j} \otimes t_1^{p^k}) = pb_{1,i-1} \otimes t_1^{p^j} \otimes t_1^{p^k} - t_1^{p^i} \otimes pb_{1,j-1} \otimes t_1^{p^k} + t_1^{p^i} \otimes t_1^{p^j} \otimes pb_{1,k-1}$, *for* $k-4 \geq j-2 \geq i \geq 1$.
(2) $d_2^{alg}(pt_1^{p^i} \otimes t_1^{p^j}) = p^2 b_{1,i-1} \otimes t_1^{p^j} - p^2 t_1^{p^i} \otimes b_{1,j-1}$, *for* $j-2 \geq i \geq 1$.
(3) $d_2^{alg}(p^2 t_1^{p^i}) = p^3 b_{1,i-1}$, *for* $i \geq 1$.
(4) $d_2^{alg}(b_{1,i} \otimes t_1^{p^j}) = pb_{1,i} b_{1,j-1}$, *for* $i \geq 0, j \geq 1, j \neq i+2$.
(5) $d_2^{alg}(t_2^{p^i} \otimes t_1^{p^i} \otimes t_1^{p^j}) = pb_{2,i-1} \otimes t_1^{p^i} \otimes t_1^{p^j}$, *for* $i, j \geq 1, j \neq i+2, i, i-1$.
(6) $d_2^{alg}(t_2 \otimes t_1 \otimes t_1^{p^j}) = -v_1 b_{1,0} \otimes t_1 \otimes t_1^{p^j} + t_2 \otimes t_1 \otimes pb_{1,j-1}$, *for* $j > 0, j \neq 2$.
(7) $d_2^{alg}(pt_2^{p^i} \otimes t_1^{p^i}) = p^2 b_{2,i-1} \otimes t_1^{p^i}$, *for* $i \geq 1$.
(8) $d_2^{alg}(t_2^{p^i} \otimes t_1^{p^{i+1}} \otimes t_1^{p^j}) = pb_{2,i-1} \otimes t_1^{p^{i+1}} \otimes t_1^{p^j}$, *for* $i, j \geq 1, j \neq i+2, i \pm 1, i$.
(9) $d_2^{alg}(pt_2^{p^i} \otimes t_1^{p^{i+1}}) = p^2 b_{2,i-1} \otimes t_1^{p^{i+1}}$, *for* $i \geq 1$.
(10) $d_2^{alg}(t_3^{p^i} \otimes t_2^{p^i} \otimes t_1^{p^i}) = pb_{3,i-1} \otimes t_2^{p^i} \otimes t_1^{p^i}$, *for* $i \geq 1$.
(11) $d_2^{alg}(t_3 \otimes t_2 \otimes t_1) = -v_1 b_{2,0} \otimes t_2 \otimes t_1$.
(12) $d_2^{alg}(t_2^{p^i} \otimes t_2^{p^{i+1}} \otimes t_1^{p^{i+1}}) = pb_{2,i-1} \otimes t_2^{p^{i+1}} \otimes t_1^{p^{i+1}} - t_2^{p^i} \otimes pb_{2,i} \otimes t_1^{p^{i+1}}$, *for* $i \geq 1$.
(13) $d_2^{alg}(v_1 t_2 \otimes t_1) = -v_1^2 b_{1,0} \otimes t_1$.
(14) $d_2^{alg}(t_3^{p^i} \otimes t_1^{p^{i+2}} \otimes t_1^{p^i}) = pb_{3,i-1} \otimes t_1^{p^{i+2}} \otimes t_1^{p^i}$, *for* $i \geq 1$.
(15) $d_2^{alg}(t_3 \otimes t_1^{p^2} \otimes t_1) = -v_1 b_{2,0} \otimes t_1^{p^2} \otimes t_1$.
(16) $d_2^{alg}(t_3^{p^i} \otimes t_2^{p^{i+1}} \otimes t_1^{p^{i+2}}) = pb_{3,i-1} \otimes t_2^{p^{i+1}} \otimes t_1^{p^{i+2}}$, *for* $i \geq 1$.

Then, we can recover the following result directly from Proposition 4.3.

Theorem 4.4 (Wang [13]). *Let $p \geq 5$ be an odd prime. Amongst the third line basis in Table 3, all nontrivial Adams d_2 differentials can be summarized as follows.*

(1) $d_2^{Adams}(h_i h_j h_k) = a_0 b_{i-1} h_j h_k - a_0 h_i b_{j-1} h_k + a_0 h_i h_j b_{k-1}$, $k-4 \geq j-2 \geq i \geq 1$.
(2) $d_2^{Adams}(a_0 h_i h_j) = a_0^2 b_{i-1} h_j - a_0^2 h_i b_{j-1}$, $j-2 \geq i \geq 1$.
(3) $d_2^{Adams}(a_0^2 h_i) = a_0^3 b_{i-1}$, $i \geq 1$.
(4) $d_2^{Adams}(b_i h_j) = a_0 b_i b_{j-1}$, $i \geq 0, j \geq 1, j \neq i+2$.
(5) $d_2^{Adams}(g_i h_j) = a_0 b_{2,i-1} h_i h_j$, $i, j \geq 1, j \neq i+2, i, i-1$.
(6) $d_2^{Adams}(g_0 h_j) = -a_1 b_0 h_0 h_j + a_0 g_0 b_{j-1}$, $j > 0, j \neq 2$.
(7) $d_2^{Adams}(g_i a_0) = a_0^2 b_{2,i-1} h_i$, $i \geq 1$.
(8) $d_2^{Adams}(k_i h_j) = a_0 b_{2,i-1} h_{i+1} h_j$, $i, j \geq 1, j \neq i+2, i \pm 1, i$.
(9) $d_2^{Adams}(k_i a_0) = a_0^2 b_{2,i-1} h_{i+1}$, $i \geq 1$.
(10) $d_2^{Adams}(h_{3,i} g_i) = a_0 b_{3,i-1} g_i$, $i \geq 1$.
(11) $d_2^{Adams}(h_{3,0} g_0) = -a_1 b_{2,0} g_0$.
(12) $d_2^{Adams}(h_{2,i} g_{i+1}) = a_0 b_{2,i-1} g_{i+1} - a_0 h_{2,i} k_i$, $i \geq 1$.
(13) $d_2^{Adams}(a_1 g_0) = -a_1^2 b_0 h_0$.
(14) $d_2^{Adams}(h_{3,i} h_{i+2} h_i) = a_0 b_{3,i-1} h_{i+2} h_i$, $i \geq 1$.
(15) $d_2^{Adams}(h_{3,0} h_2 h_0) = -a_1 b_{2,0} h_2 h_0$.
(16) $d_2^{Adams}(h_{3,i} k_{i+1}) = a_0 b_{3,i-1} k_{i+1}$, $i \geq 1$.

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