Switching Functions

If we talk about the truth table for three variables → eight rows / 8 combinations. for each one of the combinations the function value is 1 or 0.

A B C D.

$\{0,1\}^2 = \{(0,0),(0,1),(1,0),(1,1)\}$

Cartesian product. $B_2^n = \underbrace{B_2 \times B_2 \times B_2 \times \cdots \times B_2}_{n \text{ times}}$

- $f(x_1, x_2, \ldots, x_n)$ is a mapping from $B_2^n \to B_2$ → this is the domain of the function
- f can be specified by many equivalent expressions or by tables of combinations (truth tables)
- Elementary functions:
  - A minterm m_i is an AND term of n literals either a variable or its complement
    the position in the table
  - A maxterm M_i is an OR term of n literals

  Ex: 4 variables A, B, C, D number of variables is 4 domain has 16 points → So we must have 16 min term function. and 16 maxterm function.
  - $m_5(A,B,C,D) = A'BC'D$ (0101) 0 → complement, 1 → not complement
  - $M_5(A,B,C,D) = A + B' + C + D'$ (0101) 0 → not complement, 1 → complement
- $m_i = 1$ for exactly one combination of variables, and 0 for all others
- $M_i = 0$ for exactly one combination of variables, and 1 for all others
- $m_i = M_i'$

If ABCD is (0101) the (A'BC'D) is (1111) → "1"
else combination of ABCD $m_n(A,B,C,D)$ → "0"

duality 和 complement 比较.

dual: replacing "and" with "or" ~
replacing "0" with "1" ~
complement: take the dual
for the variables, you compliment variables.

若 input 信号的改变,最终 output 没变.
- No zero delay, it can not express causality.
- have a transition. but at the end the output doesn't change. it's not zero delay, that's no delay. (Because No causality)
- In some part of the circuit they got blocked.
So the transition did not have a causal effect on the output if input change but output doesn't change → no delay
Each falling down dominoes is a transition

minterm · 会使得所对应的 input variable 最终 结果是 1. (input 0 的 implement 为 1)
· input 组合和index不相对应的 minterm 结果一定是 0. 即 只有一一对应的结果是 1.
maxterm · 会使得所对应的 input variable 最终 结果是 0 (input 1 的 implement 为 0)
· input 组合和index不相对应的 maxterm 结果一定是 1. 即 只有一一对应的结果是 0.
minterm 和 maxterm 之间是 implement 的关系: 即 minterm′ = max term.

- 一个 switch function 可以表达成多个 minterm 相加的形式: 所有 input 的组合都能使 switch function 结果为 1
- 如果有一个 input 组合,其对应的 minterm 并不在 switch function Canonical sum-of-production 表达当中,则这个 input 组合所对应的 switch function 的 output 为 0.
- Maxterm 则完全相反.

Canonical Forms

- Canonical Sum-of-Products (SOP) — conjure up "or", uniquness, conjure up "and"
  - Also known as Disjunctive Normal Form (DNF)
  - Sum of minterms (those for which $f = 1$)
  - Shorthand: $\Sigma(\ldots)$
- Canonical Product-of-Sums (POS)
  - Also known as Conjunctive Normal Form (CNF)
  - Product of maxterms (those for which $f = 0$)
  - Shorthand: $\prod(\ldots)$

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  - Product of maxterms (those for which $f = 0$)
  - Shorthand: $\prod(\ldots)$ the order matters in defining what the mid term is

Decimal	xyz	f
0	000	1
1	001	0
2	010	1
3	011	1
4	100	0
5	101	0
6	110	1
7	111	1

row 0 row 2 row 3 row 6 row 7 is where the function is equal to 1. All other rows "0"

$f(x,y,z) = \sum_{x,y,z}(0,2,3,6,7)$ f=1

$f(x,y,z) = \underset{m_0}{x'y'z'} + \underset{m_2}{x'yz'} + \underset{m_3}{x'yz} + \underset{m_6}{xyz'} + \underset{m_7}{xyz}$

$f(x,y,z) = \prod_{x,y,z}(1,4,5)$ 3 different way to express same thing

$f(x,y,z) = \underset{M_1}{(x+y+z')}\underset{M_4}{(x'+y+z)}\underset{M_5}{(x'+y+z')}$

for these two expressions they are the same

- Σ和Π都能表示 switch function. 并且出现在Σ中的input组合的 index不会出现在Π中.
- $\Sigma m(\{a\}) = \prod M(\{b\})$
  $a \cup b = \{0, \ldots, 2^{n}-1\}$ variable 数量.
  $a \cap b = \{\phi\}$
  注意,什么时候相等 什么时候 implement 之后相等.
- $\Sigma m(\{a\}) = \overline{\prod M(\{a\})}$ proof — index 的 set
  $\overline{M_{a_1} \cdot M_{a_2} \cdots M_{a_n}} = \overline{M_{a_1}} + \overline{M_{a_2}} + \cdots + \overline{M_{a_n}}$
  $= m_{a_1} + m_{a_2} + \cdots + m_{a_n}$
  $= \Sigma m(\{a\})$

In-Class Exercise

Express $f(a,b,c) = ab + ac + bc$ in canonical SOP and POS forms

$= ab(c+c') + ac(b+b') + bc(a+a')$

$= abc + abc' + ab'c + a'bc$

$= \Sigma(7,6,5,3)$ — 假设 我们 input 是 (011) index 3. • 若 function 用 Canonical SOP 表示 则 0+0+0+1 = 1

$= \Pi(0,1,2,4)$ — • 若 function 用 Canonical POS 表示 则 因为 index 3 不在 (0,1,2,4) 中, 所以 maxterm 都不和 index 3 相对应 结果是 1 1·1·1·1

$= (a+b+c)(a+b+c')(a+b'+c)(a'+b+c)$

- 在 canonical sum-of-production 中. 要想补全所有 variable 相乘 要用 $(x+x')=1$, 因为 $xy \cdot 1$ 还是 xy.

$m_3: a'bc$ $M_3: a+b'+c'$
$m_5: ab'c$ $M_5: a'+b+c'$
$m_6: abc'$ $M_6: a'+b'+c$
$m_7: abc$ $M_7: a'+b'+c'$

$\Sigma(3,5,6,7) = a'bc + ab'c + abc' + abc$
$\Pi(3,5,6,7) = (a+b'+c')(a'+b+c')(a'+b'+c)(a'+b'+c')$
若一个 input 组合是 3.5.6.7 中的一个,则 Σ(3.5.6.7) 肯定是 1. 则 Π(3.5.6.7) 肯定是 0.

Boole's (Shannon's) Expansion Theorem

- Decomposition of a switching function (arbitrary function) of n variables into functions of $n-1$ variables
  - $f(x_1, x_2, \ldots, x_n) = x_1' f(0, x_2, \ldots, x_n) + x_1 f(1, x_2, \ldots, x_n)$ — n variables; we put a variable outside; n-1 variables
  - $f(x_1, x_2, \ldots, x_n) = [x_1 + f(0, x_2, \ldots, x_n)][x_1' + f(1, x_2, \ldots, x_n)]$ duality
- Proof?

Perfect induction

$x_1 = 0$: LHS $= f(0, x_2 \cdots x_n)$ same RHS $= 0' \cdot f(0, x_2 \cdots x_n) + 0 \cdot f(1, \cdots x_n)$
$= f(0, x_2 \cdots x_n)$.

$x_1 = 1$ use the same way to proof.

该 Thm 意在 展开一个 switch Function. 将一个由 n 个 variable 组成的 function 变成 1个 variable 和一个 (n-1) 个 variable 的新 switch 方程组合.

$f(x_1 \cdots x_n) = x_1' f(0 \cdots x_n) + x_1 f(1 \cdots x_n)$

duality (2个变 $\cdot \to +$, $0 \to 1$)

$[x_1' + f(1 \cdots x_n)][x_1 + f(0 \cdots x_n)]$

Boole's (Shannon's) Expansion Theorem

- Decomposition of a switching function of n variables into functions of $n-1$ variables
  - $f(x_1, x_2, \ldots, x_n) = x_1' f(0, x_2, \ldots, x_n) + x_1 f(1, x_2, \ldots, x_n)$
  - $f(x_1, x_2, \ldots, x_n) = [x_1 + f(0, x_2, \ldots, x_n)][x_1' + f(1, x_2, \ldots, x_n)]$
- The functions resulting from fixing x are referred to as co-factors
  - $f(0, x_2, \ldots, x_n)$ is the negative cofactor of f wrt (with respect to.) x_1
  - $f(1, x_2, \ldots, x_n)$ is the positive cofactor of f wrt x_1
- Notation:
  - $f_{x_1'} = f(0, x_2, \ldots, x_n)$
  - $f_{x_1} = f(1, x_2, \ldots, x_n)$

  just notation

  like $x^2 + y - 2z \big|_{x=1} = 1 + y - 2z$

Boole's (Shannon's) Expansion Theorem

- Decomposition of a switching function of n variables into functions of $n-1$ variables
  - $f(x_1, x_2, \ldots, x_n) = x_1' f_{x_1'} + x_1 f_{x_1}$
  - $f(x_1, x_2, \ldots, x_n) = [x_1 + f_{x_1'}][x_1' + f_{x_1}]$
- Repeated application yields canonical forms

$f(x_1, x_2) = x_1' f(0, x_2) + x_1 f(1, x_2)$

$f(x_1, x_2) = x_1'[x_2' f(0,0) + x_2 f(0,1)] + x_1[x_2' f(1,0) + x_2 f(1,1)]$

指当我们 input (0,0) 时 switch function 最终的 output.

$f(x_1, x_2) = \boxed{f(0,0)} x_1' x_2' + f(0,1) x_1' x_2 + f(1,0) x_1 x_2' + f(1,1) x_1 x_2$

$f(x_1, x_2) = \boxed{f_{x_1' x_2'}} m_0 + f_{x_1' x_2} m_1 + f_{x_1 x_2'} m_2 + f_{x_1 x_2} m_3$ minterm 0

we can express it as

$f(x_1, x_2) = \boxed{a_0} m_0 + a_1 m_1 + a_2 m_2 + a_3 m_3$

a_0 is the function value for minterm zero (m_0)

Switching Functions of 2 Variables

$$f(x, y) = a_0 m_0 + a_1 m_1 + a_2 m_2 + a_3 m_3$$

four constant

$a_3 a_2 a_1 a_0$	$f(x,y)$	Name	Symbol	Unique?

basically say is m_0 in function or not
is m_1 in function or not

Switching Functions of 2 Variables

$$f(x, y) = a_0 m_0 + a_1 m_1 + a_2 m_2 + a_3 m_3$$

$a_3 a_2 a_1 a_0$	$f(x,y)$	Name	Symbol	Unique?
0000	0	Inconsistency		
0001	$x'y'$	NOR	$x \downarrow y$	
0010	$x'y$	Inhibition		
0011	x'	NOT		
0100	xy'	Inhibition		
0101	y'	NOT		
0110	$x'y + xy'$	XOR	$x \oplus y$	
0111	$x' + y'$	NAND	$x \uparrow y$	
1000	xy	AND	$x \cdot y = x \wedge y = xy$	
1001	$xy + x'y'$	XNOR / EQV	$x \odot y$	
1010	y	Transfer		
1011	$x' + y$	Implication	$x \to y$	
1100	x	Transfer		
1101	$x + y'$	Implication	$y \to x$	
1110	$x + y$	OR	$x + y = x \vee y$	
1111	1	Tautology		

The number of functions of two variables is 16.
The number of functions of n variables is 2^{2^n}

Functional Properties

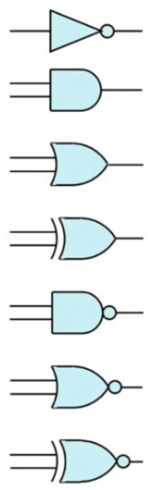
- The canonical SOP (POS) form is unique, subject to commutativity
- Two functions are logically equivalent iff their canonical SOP (POS) form are identical 判断两个 switch function 相等的方法.
- The canonical SOP (POS) form contains 2^n coefficients each of which can be either 0 or 1. Thus, there are 2^{2^n} switching functions of n variables

若有 n 个 variable, 则有 2^n 个 minterm 和 max term. 且 switch function 也因此会用 2^n 种 input. 即 $a_0 m_0 + a_1 m_1 + \cdots + a_{2^n-1} m_{2^n-1}$. 但该 switch function 最终的表达形式. $a_0 \cdots a_{2^n-1}$ 都有可能是 "0" 或 "1" 所以 switch function 的可能性是 2^{2^n}.

If only have one variable

it should be $f(x) = 1$
$f(x) = 0$
$f(x) = x$
$f(x) = x'$

Logic Gates



NOT	x	x'	
AND	x, y	xy	
OR	x, y	$x + y$	
XOR	x, y	$x \oplus y = x'y + xy'$	input 不相同，则 output 为"1"
NAND	x, y	$(xy)' = x' + y'$	通过 De Morgan's rule 对 and 和 or 的整体表达进行 implementation.
NOR	x, y	$(x + y)' = x'y'$	
XNOR	x, y	$x \odot y = x'y' + xy$	input 相同，则 output 为"1"