Gaussian filter $G \in \mathbb{R}^{k\times k}$, Seperating $G$ to two 1D Gaussian filters $G_y \in \mathbb{R}^{k\times 1}$ and $G_x \in \mathbb{R}^{1\times k}$

Let $x[n,m]$ represent the original image. Let $y[n,m]$ be the image after the filter.

When using 2D filter: $(x * G)[m,n] = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} G[k-i-1, k-j-1]\, x[m+i, n+j]$

Since $G = G_x G_y$ So

$$= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} G_x[k-i-1]\, G_y[k-j-1]\, x[m+i, n+j]$$

$$= \underbrace{\sum_{i=0}^{k-1} G_x[k-i-1]}_{\text{part ①}}\ \underbrace{\sum_{j=0}^{k-1} G_y[k-j-1]\, x[m+i, n+j]}_{\text{part ②}}$$

For part ②: $\sum_{j=0}^{k-1} G_y[k-j-1]\, x[m+i, n+j]$ $\quad i$ doesn't change here. So we can consider it as a 1D convolution.

$= (x[m+i][n] * G_y)$

Add part① to part②: $\sum_{i=0}^{k-1} G_x[k-i-1]\,(x[m+i][n] * G_y)$

$= G_x * x[m][n] * G_y$

$= x[m][n] * G_x * G_y$

So we prove $x[m,n] * G = x[m,n] * G_x * G_y$.

Also since $G_x * G_y = G_x G_y$

So $G_x * G_y = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{y^2}{2\sigma^2})$

$= \frac{1}{2\pi\sigma^2} \exp(-\frac{x^2+y^2}{2\sigma^2})$

which is equal to $G \in \mathbb{R}^{k\times k}$

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2.5

i. $k_x \in \mathbb{R}^{1\times 3}$: $I_x = I * k_x$ $\quad k_x$: $[1, 0, -1]$ $\quad$ Since it's convolution, So $k_x$ and $k_y$ need to be flipped.

$I_x = [I(x-1,y)\ \ I(x,y)\ \ I(x+1,y)] * [1, 0, -1]$

$= I(x+1,y) - I(x-1,y)$

ii. $k_y \in \mathbb{R}^{3\times 1}$: $I_y = I * k_y$ $\quad k_y$: $[1, 0, -1]^T$

$I_y = [I(x,y-1)\ \ I(x,y)\ \ I(x,y+1)]^T * [1, 0, -1]^T$

$= I(x,y+1) - I(x,y-1)$

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3.1 $k_x$: $[-1\ 0\ 1]$

$$G_s * k_x:\ \begin{bmatrix} 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 4 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix} * [-1\ 0\ 1] = \begin{bmatrix} 2 & 0 & -2 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{bmatrix} \div 2 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} = S_x$$