Huntington Postulates

A Boolean Algebra is a set B with two binary operators $+$ and $\cdot$
And the equivalence relation $=$
that satisfies the following properties:

- Closure
  - with respect to $+$
  - with respect to $\cdot$
- Identity elements
  - 0 with respect to $+$
  - 1 with respect to $\cdot$
- Commutative
  - $x \cdot y = y \cdot x$
  - $x + y = y + x$
- Distributive
  - $\cdot$ is distributive over $+$
  - $+$ is distributive over $\cdot$
- Complements: $\forall x \in B, \exists x' \in B$ (called the complement of x) such that
  - $x + x' = 1$ and
  - $x \cdot x' = 0$
- There are at least 2 distinct elements in B

Formal Definition of Switching Algebra

- Base set: B_2={0, 1}
- One unary operation: NOT or COMPLEMENT: $(x', \bar{x}, \neg x)$
- Two binary operations: AND ($\cdot$, $\wedge$), OR ($+$, $\vee$)
- Postulates (axioms):

Postulate	Defines	A	B
P1	Switching Variables	$x = 0$ iff $x \neq 1$	$x = 1$ iff $x \neq 0$
P2	NOT	$0' = 1$	$1' = 0$
P3		$0 \cdot 0 = 0$	$1 + 1 = 1$
P4	AND / OR	$1 \cdot 1 = 1$	$0 + 0 = 0$
P5		$0 \cdot 1 = 1 \cdot 0 = 0$	$0 + 1 = 1 + 0 = 1$

- Duality: $0 \leftrightarrow 1, \cdot \leftrightarrow +$

Properties (Theorems) of Switching Algebra

- From postulates, we can derive many theorems which can be used to manipulate switching expressions
- Recursive definition of switching expressions:
  - Any switching constant or variable is a switching expression
  - If E and F are switching expressions, then so are E', F', $E \cdot F$, and $E + F$
- A literal is a variable x or its complement x'
- Theorems can be proved by:
  - Perfect induction (enumerating all possible combinations of variables)
  - Finite induction
  - Algebraic manipulation (using postulates and already proved theorems)
  - Use of duality

1) $x \cdot (x+y) = x \cdot x + x \cdot y = x + x \cdot y = x \cdot 1 + x \cdot y = x(1+y) = x$
$x + (xy) = x \cdot 1 + x \cdot y = x(1+y) = x$

(Some) Theorems

if you already prove version A. then you don't need to repeat perfect induction in version B.

x is switching variable. it could be 0 & 1

prove by duality

	A	Name	B
T1	$x \cdot 1 = x$	Identities	$x + 0 = x$
T2	$x \cdot 0 = 0$	Null Elements	$x + 1 = 1$
T3	$x \cdot x = x$	Idempotency	$x + x = x$
T4		Involution $(x')' = x$	
T5	$x \cdot x' = 0$	Complements	$x + x' = 1$
T6	$x \cdot y = y \cdot x$	Commutativity 交换率	$x + y = y + x$
T7	$\boxed{x \cdot (x+y) = x}$	Absorption	$x + (x \cdot y) = x$ +也有分配率.
T8	$\boxed{x \cdot (x'+y) = x \cdot y}$	No Name	$\boxed{x + (x' \cdot y) = x + y}$ $(x+x') \cdot (x+y) = 1 \cdot (x+y) = x+y$
T9	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	Associativity	$(x+y)+z = x+(y+z)$
T10	$x \cdot (y+z) = x \cdot y + x \cdot z$	Distributivity	$x + (y \cdot z) = (x+y) \cdot (x+z)$
T11	$\boxed{x} \cdot y + \boxed{x'} \cdot z + \boxed{y \cdot z} = x \cdot y + x' \cdot z$	Consensus	$(x+y) \cdot (x'+z) \cdot (y+z) = (x+y) \cdot (x'+z)$

no matter how many times you apply the same operation, it doesn't change the answer. (not true in algebra)

consensus of these two terms around x

this is redundant and not necessary.

$= x \cdot x' + xy$
$= 0 + xy$

only one of x. y. z. with / without compliment

x there called consensus variable

T12		De Morgan's	

there is duality in there, but something more

$$f(x_1, \ldots, x_n, 0, 1, \cdot, +)' = f(x_1', \ldots, x_n', 1, 0, +, \cdot)$$

$+ \to \cdot$, $x_i \to x_i'$, $0 \to 1$: this is duality in there.

① dualizing the expression $\begin{cases} 0 \to 1 \\ + \to \cdot \end{cases}$
② complementing the variables

absorption $\begin{cases} x(x+y) = x \\ x + xy = x \end{cases}$

No name $\begin{cases} x(x'+y) = xy \\ x + x'y = x+y \end{cases}$

consensus $\begin{cases} xy + x'z + \boxed{yz} \text{ redundant} \\ (x+y)(x'+z)\boxed{(y+z)} \text{ redundant} \end{cases}$

Proof by Perfect Induction

Prove the truth of the distributive law (theorem T10)

$$x \cdot (y+z) = x \cdot y + x \cdot z$$

	A	B
P3	$0 \cdot 0 = 0$	$1 + 1 = 1$
P4	$1 \cdot 1 = 1$	$0 + 0 = 0$
P5	$0 \cdot 1 = 1 \cdot 0 = 0$	$1 + 0 = 0 + 1 = 1$

xyz	$x \cdot y$	$x \cdot z$	$y+z$	$x \cdot (y+z)$	$x \cdot y + x \cdot z$	Using
000	0	0	0	0	0	P3A, P4B
001	0	0	1	0	0	P3A, P5A, P5B, P4B
010	0	0	1	0	0	P5A, P3A, P5B, P4B
011	0	0	1	0	0	P5A, P3B, P4B
100	0	0	0	0	0	P5A, P4B
101	0	1	1	1	1	P5A, P4A, P5B
110	1	0	1	1	1	P4A, P5A, P5B
111	1	1	1	1	1	P4A, P3B

Proof by Finite Induction

Prove that $(x_1 + x_2 + \ldots + x_n)' = x_1' \cdot x_2' \cdot \ldots \cdot x_n'$

- Basis: Establish truth for $n = 2$ by perfect induction:

$$\boxed{(x_1 + x_2)' = x_1' \cdot x_2'}$$

- Induction: Assume statement is true for $n = k, k \geq 2$ and prove its truth for $n = k + 1$

Induction Hypothesis: $\boxed{(x_1 + x_2 + \ldots + x_k)' = x_1' \cdot x_2' \cdot \ldots \cdot x_k'}$

$$\begin{aligned}(x_1 + x_2 + \ldots + x_k + x_{k+1})' &= [(x_1 + x_2 + \ldots + x_k) + x_{k+1}]' \\ &= (x_1 + x_2 + \ldots + x_k)' \cdot x_{k+1}' \\ &= x_1' \cdot x_2' \cdot \ldots \cdot x_k' \cdot x_{k+1}'\end{aligned}$$

$(x_1 + x_2 + \ldots + x_k + x_{k+1})' = [y + x_{k+1}]'$
$= y' \cdot x_{k+1}'$

Proof by Algebraic Manipulation

Prove the consensus theorem (T11)

$$xy + x'z + yz = xy + x'z$$

$$\begin{aligned}xy + x'z + yz &= xy + x'z + yz1 && \text{(Identity)} \\ &= xy + x'z + \underline{yz}(x + x') && \text{(Complement)} \\ &= xy1 + x'z1 + xyz + x'yz && \text{(Id., Dist., Assoc.)} \\ &= xy(1+z) + x'z(1+y) && \text{(Assoc., Dist.)} \\ &= xy1 + x'z1 && \text{(Null Element)} \\ &= xy + x'z && \text{(Identity)}\end{aligned}$$

it will not have change since "1" is identity for "and"

Additional Comments about Theorems

- The following properties are peculiar to switching algebra and are not true for the algebra of real numbers:
  - Idempotency $f+f=f$ $f \cdot f = f$ (its not 2f or f^2)
  - All properties involving complements
  - Distributivity of sum (or) over product (and) $x+(y \cdot z) = (x+y) \cdot (x+z)$
  $p \cdot (q+r) = pq + pr$ 可以展开式子
- Associativity allows the extension of the two binary operators AND and OR to three or more variables
- Simplification of switching expressions is facilitated by the absorption and consensus theorems 可以缩短式子
  $x(x+y) = x$ & $x + xy = x$ $(x+y)(x'+z)(y+z)$ & $xy + x'z + yz$
- The involution property and De Morgan's laws provide the rules for complementing switching expressions
  $\neg\neg x = x$ $(x+y)' = x'y'$