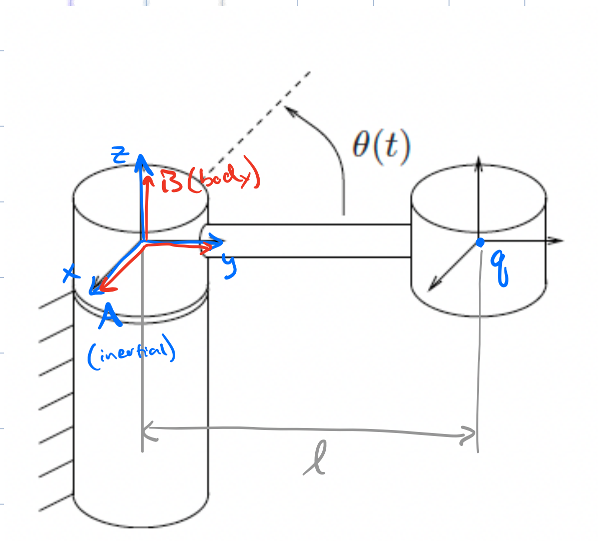
Warm-up problem: https://join.iclicker.com/MMAW



What is the velocity of q in frame A, V_{q_a}?

(a.) $\begin{bmatrix} -\ell\dot{\theta}\cos\theta \\ -\ell\dot{\theta}\sin\theta \\ 0 \end{bmatrix}$

(b.) $\begin{bmatrix} -\ell\sin\theta \\ \ell\cos\theta \\ 0 \end{bmatrix}$

(c.) $\begin{bmatrix} 0 \\ 0 \\ \ell\dot{\theta} \end{bmatrix}$

(d.) None of the above

Solution:

$$q_a = R_{ab}(t)\, q_b$$

$$R_{ab}(t) = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad q_b = \begin{bmatrix} 0 \\ \ell \\ 0 \end{bmatrix}$$

$$V_{q_a} = \dot{R}_{ab}(t)\, q_b$$

$$= \begin{bmatrix} -\dot{\theta}\sin\theta(t) & -\dot{\theta}\cos\theta(t) & 0 \\ \dot{\theta}\cos\theta(t) & -\dot{\theta}\sin\theta(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \ell \\ 0 \end{bmatrix}$$

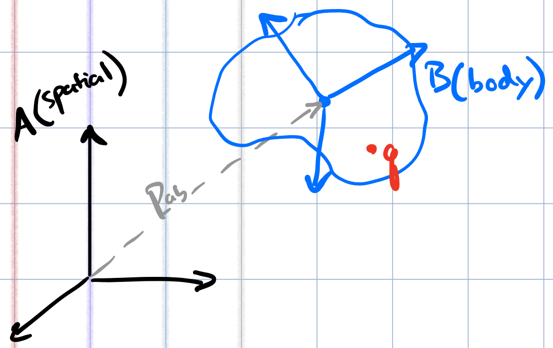
$$= \begin{bmatrix} -\ell\dot{\theta}\cos\theta(t) \\ -\ell\dot{\theta}\sin\theta(t) \\ 0 \end{bmatrix}$$

Lesson 12:

I. General Velocity (MLS 2.4)

II. Wrenches (MLS 2.5)

I. General Velocity



Let $g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix} \in SE(3)$

which describes path of body frame B wrt spatial frame A.

Consider a point on body q:

$$\bar{q}_a(t) = g_{ab}(t)\,\bar{q}_b$$

$$V_{q_a}(t) = \dot{g}_{ab}(t)\,\bar{q}_b = \underbrace{\dot{g}_{ab}\,g_{ab}^{-1}}_{\text{Let's look at this matrix}}\,\bar{q}_a(t)$$

$$\dot{g}_{ab}\,g_{ab}^{-1} = \begin{bmatrix} \dot{R}_{ab} & \dot{p}_{ab} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R_{ab}^T & -R_{ab}^T p_{ab} \\ 0 & 1 \end{bmatrix}$$

skew symmetric mtx

vector in $\mathbb{R}^3$

$$= \begin{bmatrix} \dot{R}_{ab} R_{ab}^T & -\dot{R}_{ab} R_{ab}^T p_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \qquad (4\times 4)$$

this has the form of a twist $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in se(3)$

Def: Spatial Velocity

$$\hat{V}^s_{ab} := \dot{g}_{ab}\, g^{-1}_{ab} \in se(3)$$

with coords $\quad V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \begin{bmatrix} -\dot{R}_{ab} R^T_{ab} p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R^T_{ab})^\vee \end{bmatrix} \in \mathbb{R}^6$

- Can compute velocity of a point:

  Consider (homogeneous) point q attached to rigid body.

$$\bar{q}_a(t) = g_{ab}(t)\,\bar{q}_b \quad \in \mathbb{R}^4$$

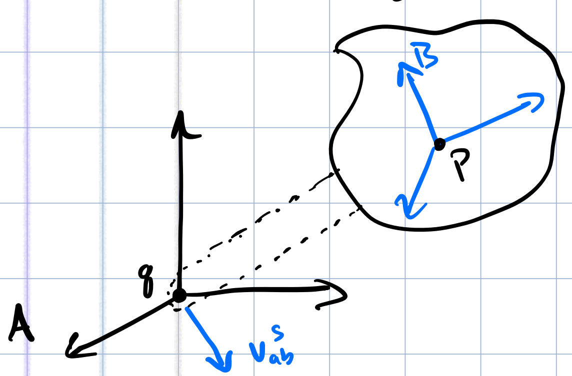
$$\Rightarrow V_{q_a}(t) := \hat{V}^s_{ab}\,\bar{q}_a$$

rot. velocity

trans. vel.

$$= \begin{bmatrix} \omega^s_{ab} \times q_a + v^s_{ab} \\ 0 \end{bmatrix}$$

Note:
- Angular component ω^s_{ab} is inst. spatial angular velocity of body viewed in spatial frame A.
- Linear component v^s_{ab} is not the velocity of the origin of B, but rather the velocity of a point attached to the body traveling through the origin of A at time t.



To find vel. of origin of B (point p)

$$\bar{V}_{p_a} = \hat{V}^s_{ab}\,\bar{p}_a$$

If q is origin of A, ie. $\bar{q}_a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, then

$$\bar{V}_{q_a} = \hat{V}^s_{ab} \bar{q}_a = \begin{bmatrix} v^s_{ab} \\ 0 \end{bmatrix}$$

- Intuition from <u>screw</u> associated w/ twist $\hat{V}^s_{ab}$, which gives the inst. axis, pitch, and magnitude of rigid velocity of B, relative to A.

Def: Body Velocity

$$\hat{V}^b_{ab} := g^{-1}_{ab} \dot{g}_{ab} = \begin{bmatrix} R^T_{ab} \dot{R}_{ab} & R^T_{ab} \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \in se(3)$$

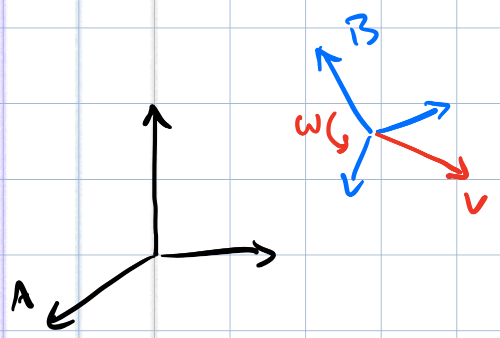
with coords $V^b_{ab} := \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix} = \begin{bmatrix} R^T_{ab} \dot{p}_{ab} \\ (R_{ab}^T \dot{R}_{ab})^\vee \end{bmatrix} \in \mathbb{R}^6$

- Applied to a point q:

$$\bar{V}_{q_b} := g^{-1}_{ab} \bar{V}_{q_a} = (g^{-1}_{ab} \underbrace{\dot{g}_{ab}) \bar{q}_b}_{V_{q_a}} = \hat{V}^b_{ab} \bar{q}_b$$

$$= \begin{bmatrix} \omega^b_{ab} \times q_b + v^b_{ab} \\ 0 \end{bmatrix}$$

Notes: • Linear component v_{ab}^b is the velocity of the origin of frame B w.r.t. A as viewed from B.



• Angular component ω_{ab}^b is inst. angular velocity in coords. of B.

* not velocity of body relative to B, which is always zero!

• Body velocity is independent of inertial frame.

$$V_{ab}^b = V_{cb}^b = V_{db}^b$$

• Relate spatial & body velocities by similarity transformation:

$$\hat{V}_{ab}^s := \dot{g}_{ab}\, g_{ab}^{-1} = g_{ab}\, \hat{V}_{ab}^b\, g_{ab}^{-1}$$

• Twist coords

$$V_{ab}^s = Ad_{g_{ab}} V_{ab}^b \qquad \text{where recall}$$

$$Ad_{g_{ab}} = \begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Lemma 2.16: $V_{ab}^b = -V_{ba}^s$ (recall $g_{ab}^{-1} = g_{ba}$)

$$V_{ab}^b = -Ad_{g_{ba}} V_{ba}^b$$

II. Wrenches

Force acting on a R.B. has linear (force) components and angular (moment) components acting on a point.

<u>Def</u>: Wrench $F = \begin{bmatrix} f \\ \tau \end{bmatrix} \in \mathbb{R}^6$, where $f, \tau \in \mathbb{R}^3$

- wrenches are defined in coords of a frame, e.g.

  $F_b = \begin{bmatrix} f_b \\ \tau_b \end{bmatrix}$ is wrench applied to the origin of B, given in coords. of B.

- Wrenches combine naturally with velocity twists to define instantaneous work (i.e. power). E.g. if A is inertial frame and B is body frame then:

$$\delta W = V_{ab}^b \cdot F_b = (V_{ab}^b)^T F_b = v_{ab}^b \cdot f_b + \omega_{ab}^b \cdot \tau_b$$

$$\Rightarrow W = \int_{t_1}^{t_2} V_{ab}^b \cdot F_b \, dt$$

- Wrenches are "equivalent" if they generate the same work over any RBT.

  $\Rightarrow$ generate the same instantaneous work.