**Course:** MECH 567: Robot Kinematics & Dynamics
**Instructor:** Daniel Bruder, PhD

**Assigned Date:** 03/07/2024
**Due:** 11:59 PM on Thursday, 03/28/2024
*Submit by uploading on canvas*

# Homework 4

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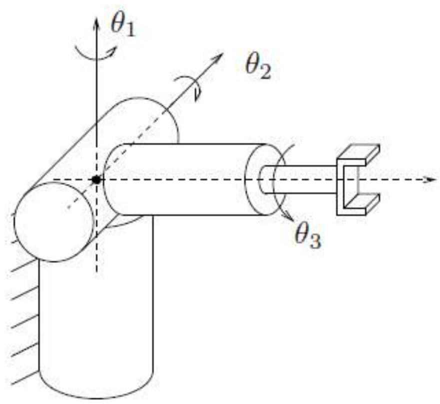
## Problem 1

For manipulator (i) in Figure 3.23 (MLS book), find the spatial Jacobian. Please find this by hand; do not use Mathematica or the equivalent. Use a spatial frame with its origin at the intersection of $\xi_1$ and $\xi_2$ and principal axes parallel to those used in the book; (i.e., z vertical, y to the right, x out of the page).

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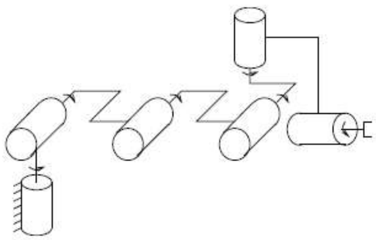
## Problem 2

Suppose that each of the four manipulators in Figure 3.24 of MLS experiences the following wrench on their end-effectors (Given in body coordinates): $F_B = [\, f_x, f_y, f_z, \tau_x, \tau_y, \tau_z \,]^T$. Use the values of q, ω, gst(0) and the resulting twists ξ as shown in the solutions of homework 2, problem 3, and $\theta = [0, \pi/2, 0, 0, 0, 0]^T$ for each manipulator.

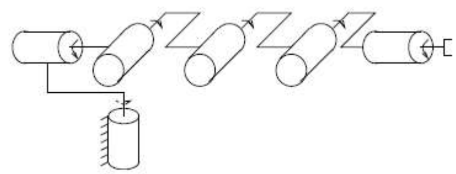
(a) Find the joint torque array $\tau$ that would be necessary to resist the wrench $F_B$ for each manipulator. Note: in this context, “resist the wrench” means to exert the appropriate joint torque to keep the manipulator in equilibrium, i.e., sum of forces = 0.

(b) If not all of six terms of $F_B$ appear in the solution of $\tau$, explain why (your explanation should be general, not for each manipulator individually).

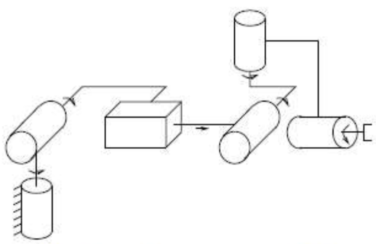
Note: Problem 2 allows the use of Mathematica. You are also permitted to use the "Screws" and "RobotLinks" packages, but specifically not the BodyJacobian and SpatialJacobian functions (or any other function that finds the Jacobian directly). Copy and paste your code in the homework.



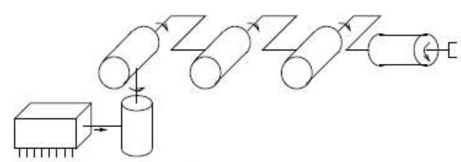
(i) Elbow manipulator



(ii) Inverse elbow manipulator



(iii) Stanford manipulator



(iv) Rhino robot

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## Problem 3

Euler angles can be used to represent rotations via the product of exponentials formula. If we think of *(α, β, γ)* as joint angles of a robot manipulator, then we can find the singularities of a Euler angle parameterization by calculating the Jacobian of the "forward kinematics", where we are concerned only with the rotation portion of the forward kinematics map. Use this point of view to find singularities for the following class of Euler angles: i) ZYZ Sequence, and ii) ZXY sequence.

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## Problem 4

Recall for a particle with kinetic energy, $K = \frac{1}{2}m\dot{x}^2$ the **momentum** is defined as

$$p = m\dot{x} = \frac{dK}{d\dot{x}}$$

Therefore, for a mechanical system with generalized coordinates $q_1,\dots,q_n$, we define the **generalized momentum** $p_k$ as

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

where $L$ is the Lagrangian of the system. With $K = \frac{1}{2}\dot{q}^\top D(q)\,\dot{q}$ and $L = K - V$ prove that

$$\sum_{k=1}^{n} \dot{q}_k\, p_k = 2K$$

## Problem 5

There is another formulation of the equations of motion of a mechanical system that is useful, the so-called **Hamiltonian** formulation. Define the Hamiltonian function $H$ by

$$H = \sum_{k=1}^{n} \dot{q}_k \, p_k - L$$

(a) Show that $H = K + V$.

(b) Using the Euler-Lagrange equations, derive Hamilton's equations.

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$
$$\dot{p}_k = -\frac{\partial H}{\partial q_k} + \tau_k$$

where $\tau_k$ is the input generalized force.

## Problem 6

For two-link manipulator of Figure 1 compute Hamiltonian equations in vector form. Note that Hamilton's equations are a system of first order differential equations as opposed to a second order system given by Lagrange's equations.

As derived in SHV, the inertia matrix $D(q)$ is given as $D(q) = \begin{bmatrix} d_{11}(q) & d_{12}(q) \\ d_{12}(q) & d_{22} \end{bmatrix}$. The total potential energy is $V(q) = V_1(q) + V_2(q)$.

Note: You do not have to plug in the actual expressions for $d_{ij}(q)$, $V_1(q)$, or $V_2(q)$, but feel free to use Mathematica to see how it turns out.

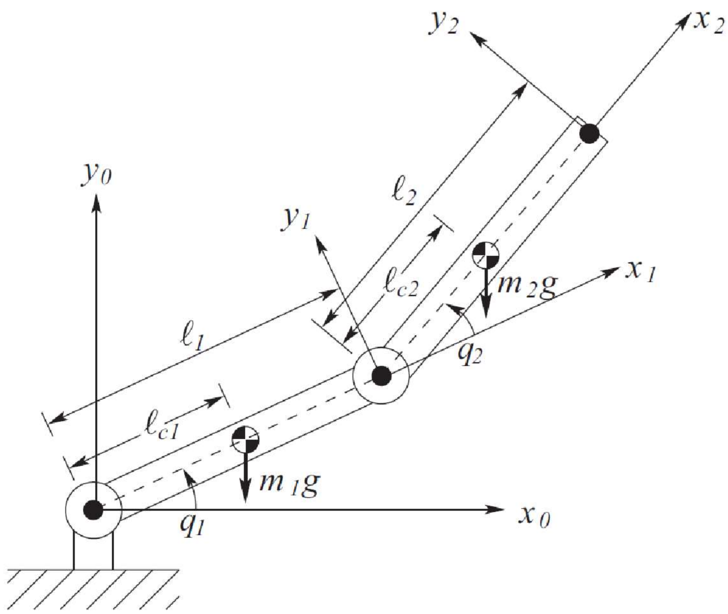


Figure 1: Two-link revolute joint arm. The rotational joint motion introduces dynamic coupling between the joints. (Figure 6.9 in SHV)