Lesson 3:

I. Uses of Rotation Matrices (MLS 2.1)

II. Parameterizations of 3D rotations

  A. Euler Angles (MLS 2.3)

  B. Exponential Coordinates (MLS 2.2)

  C. Quaternions (MLS 2.3) — if time permits

I. Uses of Rotation Matrices

① Representing the configuration (orientation) of a rigid body relative to a fixed frame.

i.e., $Q = SO(3)$

Examples: satellites, spherical wrist joint

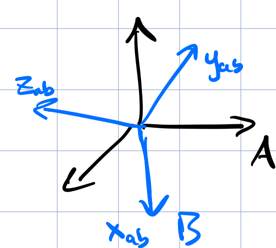
* Every orientation can be uniquely identified with some $R \in SO(3)$

② Transforming coordinates of a point from one reference frame to another (or, rotating a point relative to a fixed frame)

Example: $q_b = [x_b, y_b, z_b]^T \leftarrow$ coords of q in B

Then coords of q in A are given by

$$q_a = R_{ab} \cdot q_b = \begin{bmatrix} | & | & | \\ x_{ab} & y_{ab} & z_{ab} \\ | & | & | \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix}$$



$$= \vec{x}_{ab}\, x_b + \vec{y}_{ab}\, y_b + \vec{z}_{ab}\, z_b$$

i.e., a point q can be expressed in the coords. of any frame, and $R_{ab}: \mathbb{R}^3 \to \mathbb{R}^3$ rotates the coords of q from frame B to frame A.

Other properties of Rotation Matrices:

- Rot. Matrices can act on vectors:

If $v_b = q_b - p_b$, then

$$R_{ab} \cdot v_b = R_{ab} q_b - R_{ab} p_b = q_a - p_a = v_a$$

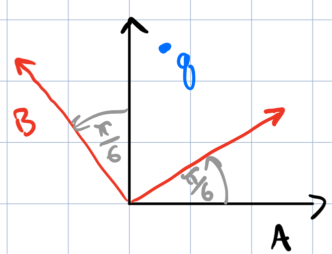
- Rotation matrices can be combined

$$R_{ac} = R_{ab} \cdot R_{bc}$$ transforms coords from C->A

$$R_{ac} q_c = R_{ab} \cdot (R_{bc} \cdot q_c) = R_{ab} q_b = q_a$$

Exercise:

https://join.iclicker.com/MMAW



The coordinates of q in frame B are $[2,2]^T$

What are the coordinates of q in frame A?

(a.) $[\sqrt{3}+1,\ -1+\sqrt{3}]^T$

(b.) $[\sqrt{3}-1,\ 1+\sqrt{3}]^T$

(c.) $\left[\frac{\sqrt{3}-1}{2},\ \frac{1+\sqrt{3}}{2}\right]^T$

$$R_{ab} = \begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$q_a = R_{ab}\, q_b = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} - 1 \\ 1 + \sqrt{3} \end{bmatrix}$$

II. Parameterizations of 3D rotations

A. Euler Angles Parameterizations (3-dim, local)

$$R(\underbrace{\alpha, \beta, \gamma}_{\text{Euler Angles}})$$

Can describe the orientation of coord. frame B relative to another frame A by:

0.) Start frame B coincident with frame A
1.) Rotate B about its z-axis by angle α
2.) Rotate B about its <u>new</u> y-axis (y') by β
3.) Rotate B about its <u>newer</u> z-axis (z'') by γ

This yields net orientation $R_{ab}(\alpha, \beta, \gamma)$ parameterized by ZYZ Euler angle triple.

- Could also use ZYX (yaw, pitch, roll)
- Net orientation is obtained by the composition of elementary rotations about x, y, and/or z axes.
- This is intrinsic Euler angles vs. extrinsic (fixed/world axes)

Elementary rotations:

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \qquad R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Net orientation obtained from the ZYZ Euler angles is given by:

$$R_{ab}(\alpha, \beta, \gamma) = R_z(\alpha) \cdot R_y(\beta) \cdot R_z(\gamma)$$

see Eq. 2.19 in MLS for closed form of this

This provides a local parameterization of $SO(3)$, i.e. given a rotation $R \in SO(3)$, $\exists$ a triple of ZYZ - Euler angles characterizing matrix $R = [r_{ij}]$

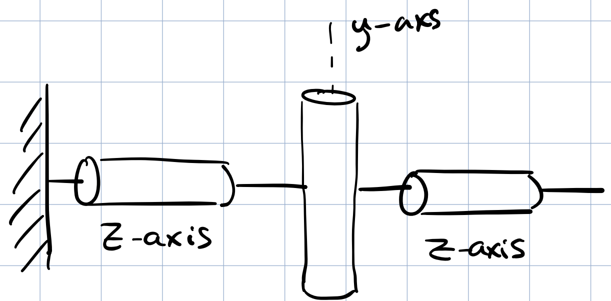
For $\sin\beta \neq 0$,

$$\beta = \operatorname{atan2}\left(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)$$

$$\alpha = \operatorname{atan2}\left(\frac{r_{23}}{\sin\beta}, \frac{r_{13}}{\sin\beta}\right)$$

$$\gamma = \operatorname{atan2}\left(\frac{r_{32}}{\sin\beta}, \frac{-r_{31}}{\sin\beta}\right)$$

* This is not global because at $\beta = 0 \pm k\pi$, we have a <u>singularity</u> where we lose uniqueness of the parameterization.



e.g. for $R = I$, we have ZYZ Euler angles $(\alpha, 0, -\alpha)$ for any $\alpha \in S^1$

- For ZYX convention

$$R_{ab}(\psi, \theta, \phi) = R_z(\psi) R_y(\theta) R_x(\phi)$$

the singularity resides at $\theta = \frac{\pi}{2}$ (airplane vertical)

Topological fact: Singularities cannot be eliminated in any 3-dim representation of $SO(3)$.

B. Exponential Coordinates of Rotation (3-dim, local)

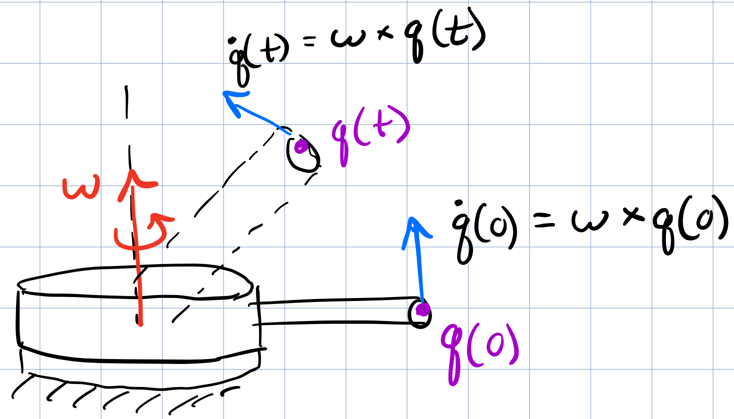
$$R(\omega, \theta)$$

axis of rotation

angle of rotation

Let $\omega \in \mathbb{R}^3$, $\|\omega\| = 1$ $\Rightarrow$ const. unit "velocity" of rotation

$\theta \in [0, 2\pi)$



LTI diff. eq.

matrix version of cross product with ω

$$\dot{q}(t) = \omega \times q(t) = \hat{\omega} \cdot q(t) \in \mathbb{R}^3$$

Solve:

$$q(t) = \left(e^{\hat{\omega} t}\right) q(0)$$

* There exists unique rotation from $g(0)$ to $g(t)$ s.t. for a unit velocity over θ units of time, we have

$$\boxed{R(\omega, \theta) = e^{\hat{\omega} \cdot \theta}}$$

Now interpret θ as angle of rotation independent of velocity/path to get to this orientation.

*** Did not cover quaternions in class, but included here in the posted notes for those interested ***

C. Quaternions (4-dim, global) (see MLS 2.3)

$$R(Q)$$

$\uparrow$ unit quaternion $\in \mathbb{R}^4$

- A <u>quaternion</u> can be thought of as a "higher dimensional complex number"

$$Q = \underbrace{q_0}_{\substack{\text{scalar}\\\text{component}\\\text{("real")}}} + \underbrace{q_1 i + q_2 j + q_3 k}_{\substack{\text{vector}\\\text{component}\\\text{("imaginary")}}} = (q_0, \vec{q})$$

where

$$i \cdot i = j \cdot j = k \cdot k = i \cdot j \cdot k = -1$$

<u>Conjugate</u>: $Q^* = (q_0, -\vec{q})$

<u>Magnitude</u>: $\|Q\| = \sqrt{Q \cdot Q^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$

<u>Multiplication</u>: $Q \cdot P = (q_0 p_0 - \vec{q} \cdot \vec{p},\ q_0 \vec{p} + p_0 \vec{q} + \vec{q} \times \vec{p})$

- Given a Rotation matrix $R = e^{\hat{\omega}\theta}$, the associated quaternion is defined as:

$$Q = \left(\cos\left(\theta/2\right),\ \omega \sin\left(\frac{\theta}{2}\right)\right)$$

- Given a unit Quaternion $Q = (q_0, \vec{q})$ the corresponding rotation matrix can be extracted

$$\theta = 2\cos^{-1} q_0 \qquad \omega = \begin{cases} \dfrac{\vec{q}}{\sin(\theta/2)} & \text{if } \theta \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and $R = e^{\hat{\omega}\theta}$

- Rotations can be composed through quaternion multiplication:

$$Q_{ac} = Q_{ab} \cdot Q_{bc}$$

$\Rightarrow$ Can do computations w/ quats without converting to rotation matrices (as is the case with other parameterizations)

<u>Note:</u>

Quaternions do not suffer from singularities.