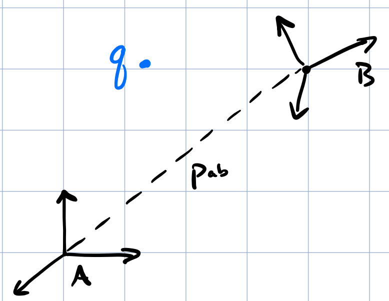
Lesson 5:

I. General Rigid Motion in $\mathbb{R}^3$ (MLS 2.3)

II. Homogeneous Representation (MLS 2.3.1)

III. Exponential Coordinates for Rigid Motion (MLS 2.3.2)

I. General Rigid Motion in $\mathbb{R}^3$



- Let $p_{ab} \in \mathbb{R}^3$ be a position vector

  $R_{ab} \in SO(3)$ be orientation of B relative to frame A.

- Configuration of B relative to A is (p_{ab}, R_{ab})

Def: Special Euclidean Group or "Big SE(3)"

$$SE(3) := \{(p, R) \mid p \in \mathbb{R}^3, R \in SO(3)\} = \mathbb{R}^3 \times SO(3)$$

note: This is in 3D, but can also define for $n \geq 2$

- Let $g = (p, R) \in SE(3)$, by abuse of notation we write $g(q)$ to denote the action of a rigid transformation on a point q.

- Given a point q_b,

$$q_a = p_{ab} + R_{ab} q_b$$

- For vector $v = s - r$ where $s, r \in \mathbb{R}^3$

$$g_*(v) = g(s) - g(r) = Rs - Rr = R(s-r) = Rv$$

$\Rightarrow$ general RBT of vectors is just rotation.

Prop 2.7: Elements of SE(3) represent rigid motion.

Proof: Let $g \in SE(3)$

1.) Is length preserved?

Because rotations are length preserving

$$\|g(p_1) - g(p_2)\| = \|Rp_1 - Rp_2\| \overset{\downarrow}{=} \|p_1 - p_2\| \checkmark$$

for all points p_1, p_2

2.) Is cross product preserved?

$$g_*(v) \times g_*(w) = Rv \times Rw = R(v \times w) = g_*(v \times w)$$

for all vectors v, w $\checkmark$

$\square$

In summary, SO(3) represents rigid rotations.
SE(3) represents general rigid motions.

II. Homogeneous Representation of SE(3)

Point $\bar{q} = \begin{bmatrix} q \\ 1 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix}$ e.g. $\bar{\emptyset} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ for origin

vector $\bar{v} = \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$

Rules of Syntax

1.) Sums and differences of vectors are vectors.
2.) Sum of a vector and a point is a point.
3.) Difference between two points is a vector.
4.) Sum of points is meaningless (never see a 2 in 4^{th} row)

- Can now represent $g_{ab} = (p_{ab}, R_{ab}) \in SE(3)$ as a linear transformations (i.e. a matrix)

$$\bar{q}_a = \begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} q_b + p_{ab} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & p_{ab} \\ \emptyset_{1\times3} & 1 \end{bmatrix}}_{\substack{4\times4 \text{ matrix } \bar{g}_{ab} \\ \text{homogeneous representation} \\ \text{of } g_{ab} \in SE(3)}} \begin{bmatrix} q_b \\ 1 \end{bmatrix} = \bar{g}_{ab} \cdot \bar{q}_b$$

$[0 \ 0 \ 0]$

note: Slight abuse of notation, $\bar{g}_{ab} \in SE(3)$

Why use this?

- Allows linear operator, analagous to $SO(3)$
- Allows composition

$$\bar{g}_{ac} = \bar{g}_{ab} \cdot \bar{g}_{bc} = \left[\begin{array}{c|c} R_{ab} \cdot R_{bc} & R_{ab} p_{bc} + p_{ab} \\ \hline \emptyset_{1\times3} & 1 \end{array}\right]$$

- Is a group
  1.) $\bar{g}_1 \cdot \bar{g}_2 \in SE(3)$ for $\bar{g}_1, \bar{g}_2 \in SE(3)$ (closure)

  2.) $I_{4\times4} \in SE(3)$ (Identity)

3.) $\bar{g}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \in SE(3)$ for $\bar{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in SE(3)$

(Inverse)

4.) Associativity by matrix mult. (Associativity) □

- Works w/ vectors

$$\bar{g}_* \cdot \bar{v} = \bar{g} \cdot \bar{s} - \bar{g}\bar{r}$$

where $\bar{s}, \bar{r}$ are points in $\mathbb{R}^3$ and $\bar{v} = \bar{s} - \bar{r}$

$$= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} Rv \\ 0 \end{bmatrix}$$

← equivalent to just rotation, as expected

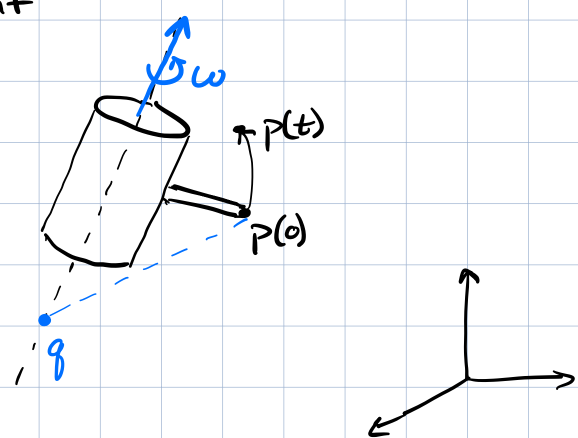
III. Exponential Coords for General Rigid Motion (Twist)

<u>Goal</u>: Generalize exp. map to SE(3) which will give us exp. coords. for general 3D motion.

① <u>Case 1</u>: Rotational Joint

$\omega \in \mathbb{R}^3$, $\|\omega\| = 1$

$q \in \mathbb{R}^3$ is a point anywhere on the axis of rotation



Assuming unit velocity

$$\dot{p}(t) = \omega \times (p(t) - q) = \hat{\omega} \cdot (p(t) - q)$$

<u>Def</u>: Twist $\quad \hat{\xi} = \underbrace{\begin{bmatrix} \hat{\omega} & v \\ 0_{1\times 3} & 0 \end{bmatrix}}_{4\times 4}$ where $v = -\omega \times q$
$= -\hat{\omega} q$

Then in homogeneous coords:

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0_{1\times 3} & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \iff \dot{\bar{p}} = \hat{\xi} \bar{p}$$

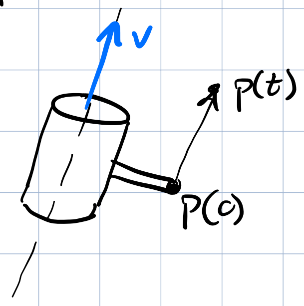
(Linear Time Invariant Diff eq.)

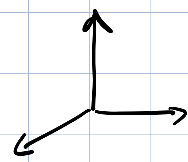
$$\Rightarrow \bar{p}(t) = \exp(\hat{\xi} \cdot t) \cdot \bar{p}(0)$$

$$\Rightarrow \bar{p}(\theta) = \underbrace{\exp(\hat{\xi}\theta)}_{\text{will derive finite form later}} \cdot \bar{p}(0)$$

Because $\|\omega\| = \dot{\theta} = 1$
$\Rightarrow \theta = t$

② Case 2: Translational joint





$$\dot{p}(t) = v$$

Let twist $\hat{\xi} = \begin{bmatrix} 0_{3\times3} & v \\ 0_{1\times3} & 0 \end{bmatrix}$

$$\dot{\bar{p}} = \hat{\xi} \cdot \bar{p} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

(LTI diff eq.)

$$\Rightarrow \bar{p}(t) = \exp(\hat{\xi} t)\, \bar{p}(0)$$

Assume $\|v\| = 1$

$$\Rightarrow \bar{p}(d) = \exp(\hat{\xi} \cdot d) \cdot \bar{p}(0)$$, for distance d.

* Twist $\hat{\xi} \in \mathbb{R}^{4\times4}$ is a generalization of $\hat{\omega} \in so(3)$

<u>Def</u>: "Little" se(3)

$$se(3) = \{ (v, \hat{\omega}) \mid v \in \mathbb{R}^3, \hat{\omega} \in so(3) \}$$

In homogeneous coords. elements of se(3) are twists $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in se(3)$

Then, $\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}^{\vee} = \underbrace{\begin{bmatrix} v \\ \omega \end{bmatrix}}_{\xi} \in \mathbb{R}^6$ and $\begin{bmatrix} v \\ \omega \end{bmatrix}^{\wedge} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$

ξ are the twist coords. of a twist $\hat{\xi} \in se(3)$

If rotation $v = -\omega \times q$ for any q on axis ω

If translation $\omega = 0$ and v is axis of translation.