Lesson 18:

I. Representations of EOM

II. Properties of EOM (SHV 6.5)

I. Representations of EOM

- Recall, EOM in vector form:

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = \tau \qquad \left(\begin{array}{l}\text{2nd order}\\ \text{n-dim ODE}\end{array}\right)$$

- Can also write EOM is state space form

Define state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \in \mathbb{R}^{2n}$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ M^{-1}(x_1)\left(\tau - C(x_1,x_2)x_2 - G(x_1)\right) \end{bmatrix} \qquad \left(\begin{array}{l}\text{1st order}\\ \text{2n-dim ODE}\end{array}\right)$$

inertia/mass mtx is pos. def. $\Rightarrow$ invertible

- Can also write as "control affine" $\dot{x} = f(x) + g(x)\cdot\tau$

where $f(x) = \begin{bmatrix} x_2 \\ -M^{-1}(x_1)\left(C(x_1,x_2)x_2 + G(x_1)\right) \end{bmatrix} \in \mathbb{R}^{2n}$ "drift vector field"

$g(x) = \begin{bmatrix} 0 \\ M^{-1}(x_1) \end{bmatrix} \in \mathbb{R}^{2n\times n}$ "control vector fields"

II. Properties of EOM

Note : $q = \theta$ in SHV and elsewhere

$D(q) = M(\theta)$ in SHV and elsewhere

① Inertia/mass matrix $M(\theta)$ is symmetric and positive definite.

- Symmetry by def: $M(\theta) = \sum_{i=1}^{n} J_i^T(\theta) \underbrace{\mathcal{M}_i}_{\text{symmetric}} J_i(\theta)$

- Positive def. by $K = \frac{1}{2} \dot{\theta}^T M(\theta) \dot{\theta} > 0 \quad \forall \dot{\theta} \neq 0$

  note : $[\cdot] > 0$ means positive definite

  $[\cdot] \geq 0$ means positive semi-definite

② Inertia/mass matrix is bounded above and below for revolute-joint robots.

- For fixed θ, let $0 < \lambda_1(\theta) \leq \cdots \leq \lambda_n(\theta) < \infty$ be the n ordered eigenvalues of $M(\theta) > 0$

$\Rightarrow \lambda_1(\theta) I_{n \times n} \leq M(\theta) \leq \lambda_n(\theta) I_{n \times n}$

where

$\lambda_i(\theta) I \leq M(\theta)$ means $\underbrace{\left(M(\theta) - \lambda_i(\theta) I\right) \geq 0}_{\text{pos. semi-def.}}$

- For revolute joints, $M(\theta)$ terms contain constants, cosines and sines of θ and hence bounded.

$\Rightarrow \exists$ constants λ_{min} and λ_{max} s.t.

$$0 < \lambda_{min} I \leq M(\theta) \leq \lambda_{max} I < \infty \qquad \forall \theta \in Q$$

config space

* useful for robust control

③ Skew - Symmetry Property (Prop. 6.1 in SHV)

$$(\dot{M}-2C)^T = -(\dot{M}-2C)$$

i.e. $\dot{M}-2C \in so(n)$, skew-symmetric matrix.

Pf: (see SHV pg. 228)

* Useful in Lyapunov & passivity analysis.

④ Passivity Property

A robotic system is input-output passive with input τ and output $\dot{\theta}$, i.e. $\exists$ constant $\beta \geq 0$ s.t.

$$\underbrace{\int_0^T \underbrace{\dot{\theta}^T(t)\,\tau(t)}_{\text{power}}\,dt}_{\substack{\text{work or energy} \\ \text{change over } [0,T]}} \geq -\beta \quad , \forall\, T \geq 0$$

Means that the energy dissipated by robot has a lower bound.

Pf: Define total energy $E(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} + P(\theta) \geq 0$

In context of passivity, we call this a "storage function"

$$\Rightarrow \dot{E} = \dot{\theta}^T \underbrace{M(\theta)\ddot{\theta}} + \frac{1}{2}\dot{\theta}^T \dot{M}(\theta)\dot{\theta} + \dot{\theta}^T \frac{\partial P}{\partial \theta}$$

$$= \dot{\theta}^T \left[\tau - C\dot{\theta} - \cancel{G} \right] + \frac{1}{2}\dot{\theta}^T \dot{M}(\theta)\dot{\theta} + \cancel{\dot{\theta}^T \frac{\partial P}{\partial \theta}}$$

G

$$= \dot{\theta}^T \tau + \underbrace{\frac{1}{2}\dot{\theta}^T \underbrace{\left[\dot{M}(\theta) - 2C(\theta,\dot{\theta}) \right]}_{\text{skew-symmetric}} \dot{\theta}}_{=0\text{, because skew-sym.}}$$

$$= \dot{\theta}^T \tau \quad \text{(power)}$$

$\Rightarrow$ System cannot generate/dissipate energy "internally"

$$\Rightarrow \int_0^T \dot{\theta}^T(t)\, \tau(t)\, dt = E(T) - E(0) \quad \text{by fundamental thm. of calculus.}$$

$$\geq -E(0) \quad \text{because } E(T) \geq 0$$

$\Rightarrow$ robots are input-output passive, $\beta = E(0)$

$\square$

* Useful property for passivity-based control and proving stability — though (input-output) alone does not imply stability.

⑤ Linearity in the Parameters

$\exists$ function $\underline{Y}(\theta, \dot{\theta}, \ddot{\theta}) \in \mathbb{R}^{n \times \ell}$ (regressor) and a parameter $\underline{\Theta} \in \mathbb{R}^{\ell}$ s.t. robot dynamics

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) = \underline{Y}(\theta, \dot{\theta}, \ddot{\theta})\, \underline{\Theta}$$

$\underline{Y}$ contains $\dot{\theta}, \ddot{\theta}$, and trig functions of θ (all nonlinearities)

$\underline{\Theta}$ contains mass, inertia, length constants (in combinations)

- Number of parameters ℓ is not unique but has an upper bound of $10n$ for an n-link robot.

  mass, ≤ 6 inertia terms, 3 COM coords for each link.

- However, robot geometry results in fewer parameter dependencies by lumping parameters together in EoM.

  $\rightarrow$ found by inspection, e.g. see SHV 230-231

* Use least-squares regression for system ID given dataset $(\theta(t), \dot{\theta}(t), \ddot{\theta}(t))$.

* Also useful for adaptive control.