Lesson 4: Exponential Coordinates for Rotation (MLS 2.2)

I. Skew Symmetric Matrices

II. Euler's Rotation Thm.

Recall from last time, the exponential parameterization

$$R(\omega, \theta) = e^{\hat{\omega}\theta}$$

axis of rotation (unit vector)

angle of rotation

Matrix version of the cross product by ω

I. Skew Symmetric Matrices

Def: A matrix M is skew-symmetric if $M^T = -M$

e.g.,

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

Note that all diagonal elements are zero

The cross product by a vector can be represented by a skew-symmetric matrix:

Let $a, b \in \mathbb{R}^3$, $a = [a_1\ a_2\ a_3]^T$, $b = [b_1\ b_2\ b_3]^T$

Recall

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{\hat{a} = -\hat{a}^T} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(skew-symmetric)

* Cross product by a vector is a linear operator.

- The vector space of all $n \times n$ skew-symmetric matrices is denoted $so(n)$ "Little $so(n)$"

$$so(n) = \{ S \in \mathbb{R}^{n \times n} : S^T = -S \}$$

"Big $SO(n)$"

* We will show an important connection to $SO(n)$, specifically for $n=3$.

- <u>Note</u>: $so(3)$ is a real vector space, this means it has closure over linear combinations of elements:

$$\hat{v} + \hat{w} = \widehat{(v+w)} \qquad \text{for } v, w \in \mathbb{R}^3$$

- <u>Notation</u>: $S \in so(3)$ then $S^\vee \in \mathbb{R}^3$

$\hat{\omega} \in so(3)$ then $(\hat{\omega})^\vee = \omega \in \mathbb{R}^3$

- Can uniquely represent any element $\hat{\omega} \in so(3)$ with $\omega \in \mathbb{R}$ by extracting elements of $\hat{\omega}$

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad \text{and} \quad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

- Consider any $S \in so(3)$ as $S = \hat{\omega} \cdot \theta$, $\|\omega\| = 1$

unit vector's skew-symmetric matrix, scaled by $\theta = \|S^\vee\|$

Equivalently $S^\vee = \omega\theta$, $\|\omega\| = 1$

- Components of vector $\omega \cdot \theta \in \mathbb{R}^3$ are the <u>exponential coordinates</u> of $R \in SO(3)$.

I. Euler's Rotation Theorem

<u>Thm:</u> Euler's Rotation Thm (MLS Thm. 2.6)

Any orientation $R \in SO(3)$ is equivalent to a rotation about a fixed axis $\omega \in \mathbb{R}^3$ (unit vector) by $\theta \in [0, 2\pi)$

*we want to prove this is true.

Recall:

$$e^{\hat{\omega}\theta} = I + \theta \cdot \hat{\omega} + \frac{\theta^2}{2!}\hat{\omega}^2 + \frac{\theta^3}{3!}\hat{\omega}^3 + \cdots$$

Let's exploiting properties of $so(3)$ to get a finite expression.

<u>Lemma 2.3:</u> Given $\hat{a} \in so(3)$, the following hold:

1) $\hat{a}^2 = \hat{a} \cdot \hat{a} = a \cdot a^T - \|a\|^2 I$, where $\|a\|^2 = a^T a$
2) $\hat{a}^3 = -\|a\|^2 \cdot \hat{a}$

$\Rightarrow$ Higher orders recursively, e.g., $\hat{a}^5 = \hat{a}^2 \cdot \hat{a}^3$

- Now let $a = \omega \cdot \theta$ where $\|\omega\| = 1$, then

$$\hat{a}^2 = \theta^2 \cdot \hat{\omega}^2 = \theta^2 \omega \cdot \omega^T - \theta^2 I$$

$$\hat{a}^3 = -\theta^2(\theta \cdot \hat{\omega}) = -\theta^3 \underline{\hat{\omega}}$$

$$\hat{a}^4 = \hat{a}^3 \hat{a} = -\theta^4 \underline{\hat{\omega}^2}$$

$$\hat{a}^5 = \hat{a}^3 \hat{a}^2 = (\underline{-\theta^3 \hat{\omega}})(\underline{\theta^2 \omega \cdot \omega^T} - \theta^2 I) = \theta^5 \cdot \underline{\hat{\omega}}$$

$$\vdots$$

- Then,

$$e^{\hat{\omega}\theta} = I + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin(\theta)} \hat{\omega} + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)}_{1-\cos(\theta)} \hat{\omega}^2$$

Rodrigues' Formula:

$$\boxed{e^{\hat{\omega}\theta} = I + \hat{\omega}\sin(\theta) + \hat{\omega}^2(1-\cos\theta)}$$

big $SO(3)$

Prop. 2.4: Given $\hat{\omega} \in so(3)$ and $\theta \in \mathbb{R}$, then $\exp(\hat{\omega} \cdot \theta) \in SO(3)$

Pf: 1.) Orthogonality

by Rodrigues

$$\left(e^{\hat{\omega}\theta}\right)^{-1} = e^{-\hat{\omega}\theta} = e^{\hat{\omega}^T\theta} = \left(e^{\hat{\omega}\theta}\right)^T \checkmark$$

2) Determinant $= 1$

Know $\det[e^{\hat{\omega}\theta}] = \pm 1$ by orthogonality

Know from Rodrigues that $\det[e^{\hat{\omega}\theta}] = 1$

Also know that determinant is a continuous function of matrix elements

$$\Rightarrow \det[e^{\hat{\omega}\theta}] = +1 \quad , \forall\theta \qquad \square$$

☆ Prop 2.4 shows that the <u>exponential</u> <u>map</u>:

$$\exp : so(3) \longrightarrow SO(3)$$

$$\hat{\omega}\cdot\theta \longmapsto R$$

- Geometrically, $\hat{\omega}$ represents axis of rotation, and exp. map "generates" the rotation by amount θ.
  - Equivalent axis representation
  - <u>Not</u> injective (one-to-one) because $\omega' = -\omega$ and $\theta' = 2\pi - \theta$ give the same rotation as $\omega\cdot\theta$

- Singularity at I because ω is arbitrary when $\theta = 0$.
  - Lose smooth dependence of ω as function of $R \in SO(3)$ at $R = I$

- Can show that exp. map is surjective (onto)

Prop 2.5: Given $R \in SO(3)$, $\exists\, \omega \in \mathbb{R}^3$, $\|\omega\| = 1$, and $\theta \in \mathbb{R}$ s.t. $R = \exp(\hat{\omega} \cdot \theta)$.

"there exists"

Pf: By construction.

Let $R = [r_{ij}]$, define $c_\theta = \cos\theta$, $s_\theta = \sin\theta$, $v_\theta = 1 - \cos\theta$

$$R = \exp(\hat{\omega}\theta) = I + \hat{\omega} s_\theta + \hat{\omega}^2 (1 - c_\theta)$$

$$= \begin{bmatrix} \omega_1^2 v_\theta + c_\theta, & \omega_1\omega_2 v_\theta - \omega_3 s_\theta, & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta, & \omega_2^2 v_\theta + c_\theta, & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta, & \omega_2\omega_3 v_\theta + \omega_1 s_\theta, & \omega_3^2 v_\theta + c_\theta \end{bmatrix}$$

Take the trace

$$\mathrm{tr}(R) = r_{11} + r_{22} + r_{33} = \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_{\|\omega\|^2 = 1} v_\theta + 3c_\theta$$

$$= 1 + 2c_\theta$$

$$\Longrightarrow \boxed{\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)}$$

Note $\cos^{-1}$ is only well defined $[-1,1]$. Turns out this is always satisfied.

- Can also have

$$\theta' = \theta \pm 2\pi k \quad \text{for } k \in \mathbb{Z}$$

or

$$\theta' = -\theta \pm 2\pi k$$

Now construct the axis ω:

- Use off-diagonal terms

$$r_{32} - r_{23} = 2\omega_1 s_\theta$$

$$r_{13} - r_{31} = 2\omega_2 s_\theta$$

$$r_{21} - r_{12} = 2\omega_3 s_\theta$$

If $\sin\theta \neq 0$, then $\omega = \frac{1}{2 s_\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$

If $\theta = 0$, then ω is arbitrary (singularity)

If $\theta = \pi$, see HW 1

<u>Note</u>: If $\theta' = 2\pi - \theta$, then $\omega' = -\omega$ $\square$

★ Prop 2.4 and Prop 2.5 prove Thm. 2.6 (Euler's)!

$\omega, \theta \rightarrow R$ or $R \rightarrow \omega, \theta$
(but not one-to-one)