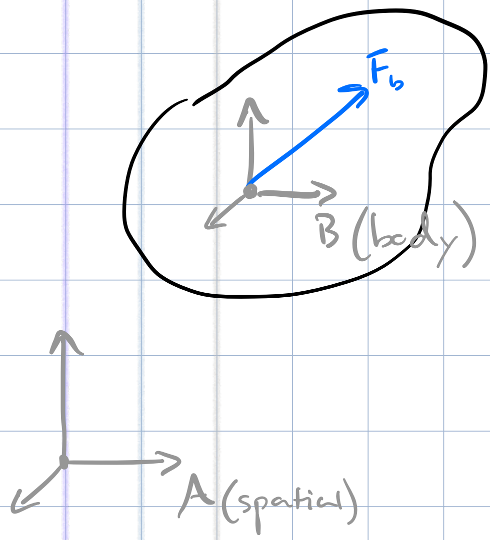
Lesson 13:

I. Wrenches (cont'd) (MLS 2.5)

II. Manipulator Jacobians (MLS 3.4)

Warm-up Problem: https://join.iclicker.com/MMAW



$$V_{ab}^b = \begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$F_b = \begin{bmatrix} f_b \\ \tau_b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

What is the instantaneous work done by F_b?

(a.) 1

(b.) 3

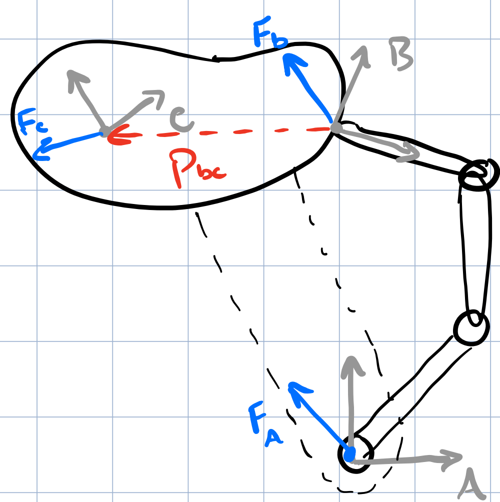
(c.) 0

(d.) None of the above

$$\delta W = V_{ab}^b \cdot F_b = V_{ab}^{b\,T} F_b = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0 \checkmark$$

I. Wrenches (cont'd)

Ex: Given wrench F_b applied origin of B, determine the equivalent wrench F_c applied to the origin of frame C:



$$\delta W = V_{ac}^b \cdot F_c = V_{ab}^b \cdot F_b$$

$$= \left(Ad_{g_{bc}} V_{ac}^b\right) \cdot F_b$$

$$= \left(Ad_{g_{bc}} V_{ac}^b\right)^T F_b$$

$$= V_{ac}^{b\,T} \underbrace{\left(Ad_{g_{bc}}^T F_b\right)}_{F_c}$$

$$= V_{ac}^b \cdot F_c \quad ✓$$

$$\Rightarrow F_c = Ad_{g_{bc}}^T F_b = \begin{bmatrix} R_{bc}^T \; (= R_{cb}) & 0 \\ -R_{bc}^T \hat{p}_{bc} & R_{bc}^T \end{bmatrix} \begin{bmatrix} f_b \\ \tau_b \end{bmatrix}$$

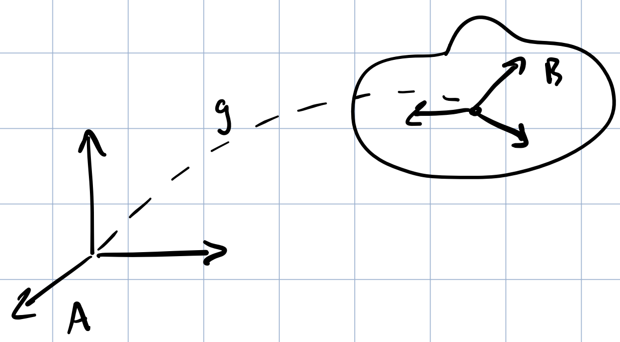
$$= \begin{bmatrix} R_{bc}^T f_b \\ -R_{bc}^T \hat{p}_{bc} f_b + R_{bc}^T \tau_b \end{bmatrix} = \begin{bmatrix} f_c \\ \tau_c \end{bmatrix}$$

$p_{bc} \times f_b$ is moment from the lever arm between frames B & C.

- Can also find an equivalent wrench F_A.

- Can add wrenches to get the net wrench if all wrenches are represented w.r.t. same frame.
  e.g. fingers grasping object.

- If define $g \in SE(3)$ to be config. of a rigid body w.r.t. inertial frame:



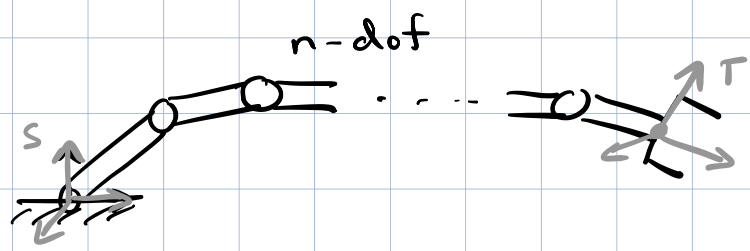
Define <u>body wrench</u> F^b and <u>spatial wrench</u> F^s, where

$$F^b = Ad_g^T F^s \quad \text{and} \quad \delta W = V^b \cdot F^b = V^s \cdot F^s$$

(equivalent wrenches)

II. Manipulator Jacobian

- How to relate joint velocity to end effector velocity?
- How to relate end effector wrenches to joint torques?



- Forward kinematic map $g_{st} : Q \to SE(3)$
- Given joint path $\theta(t) \in Q \Rightarrow$ end-effector path $g_{st}(\theta(t)) \in SE(3)$

- Instantaneous spatial velocity of end effector is given by the twist:

$$\hat{V}^s_{st} = \dot{g}_{st}(\theta)\, g_{st}^{-1}(\theta) \in se(3)$$

"chain rule"

$$= \left(\sum_{i=1}^{n} \frac{\partial g_{st}}{\partial \theta_i} \dot{\theta}_i \right) g_{st}^{-1}(\theta)$$

see "matrix calculus" handout on canvas

$$= \sum_{i=1}^{n} \left[\left(\frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1}(\theta) \right) \dot{\theta}_i \right]$$

$\Rightarrow$ Twist coords. written as $V^s_{st} = J^s_{st}(\theta)\, \dot{\theta} \in \mathbb{R}^6$

where $J^s_{st}(\theta) := \left[\left(\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1} \right)^\vee, \left(\frac{\partial g_{st}}{\partial \theta_2} g_{st}^{-1} \right)^\vee, \ldots, \left(\frac{\partial g_{st}}{\partial \theta_n} g_{st}^{-1} \right)^\vee \right] \in \mathbb{R}^{6 \times n}$

is the <u>spatial manipulator Jacobian</u>.

- At each joint config. θ, J^s_{st} maps joint velocities to end effector velocities:

$$J^s_{st}(\theta) : \mathbb{R}^n \to \mathbb{R}^6$$
$$\dot{\theta} \mapsto V^s_{st}$$

* Can obtain a more elegant formula for $J^s_{st}(\theta)$ using POE:

$$g_{st}(\theta) = \underbrace{e^{\hat{\xi}_1\theta_1}}_{e^1} \cdots \underbrace{e^{\hat{\xi}_n\theta_n}}_{e^n} g_{st}(0)$$

$$\begin{cases} \dfrac{\partial g_{st}}{\partial \theta_i} = e^1 \cdots e^{i-1} \underbrace{\dfrac{\partial}{\partial \theta_i}\left(e^{\hat{\xi}_i\theta_i}\right)}_{\hat{\xi}_i e^{\hat{\xi}_i\theta_i}} e^{i+1} \cdots e^n g_{st}(0) \\ \\ g_{st}^{-1} = g_{st}^{-1}(0)\, e^{-n} \cdots e^{-1} \end{cases}$$

$$\Rightarrow \frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1} = \left(e^1 \cdots e^{i-1} \hat{\xi}_i e^i e^{i+1} \cdots \cancel{e^n} \cancel{g_{st}(0)}\right)\left(\cancel{g_{st}^{-1}(0)} \cancel{e^{-n}} \cdots e^{-1}\right)$$

$$= \underbrace{e^1 \cdots e^{i-1}}_{g} \hat{\xi}_i \underbrace{e^{-(i-1)} \cdots e^{-1}}_{g^{-1}} := \hat{\xi}_i'$$

- This is a similarity transformation of $\hat{\xi}_i$ to new coords:

$$\left(\frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1}\right)^\vee = \mathrm{Ad}_{(\underbrace{e^1 \cdots e^{i-1}}_{g})} \xi_i := \xi_i'$$

$$\Rightarrow J^s_{st}(\theta) = \left[\xi_1, \xi_2', \ldots, \xi_n'\right] \quad \text{where } \xi_1 = \xi_1'$$

* The i^{th} column of spatial Jac. is the i^{th} joint twist coordinates transformed into current robot config.

(w.r.t. spatial frame)

<u>Note</u>: $J^s_{st}(\theta) = \left[\xi_1, \xi_2'(\theta_1), \xi_3'(\theta_1, \theta_2), \dots, \xi_n'(\theta_1, \dots, \theta_{n-1})\right]$

↪ Never depends of θ_n

<u>Def</u>: <u>Body Manipulator Jacobian</u> J^b_{st} satisfies

$$V^b_{st} = J^b_{st}(\theta)\,\dot{\theta}$$

where $J^b_{st}(\theta) = \left[\xi_1^{\dagger}, \dots, \xi_n^{\dagger}\right]$

where $\xi_i^{\dagger} = Ad^{-1}_{(e^{\hat{\xi}_i \theta_i} \cdots e^{\hat{\xi}_n \theta_n} g_{st}(0))}\, \xi_i$

- Columns $\xi_i^{\dagger}$ correspond to joint twist coords. wrt current config. of the tool frame.