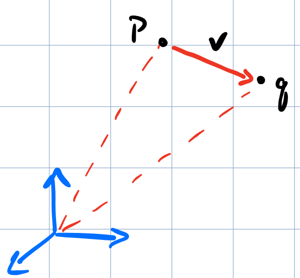
Lesson 2

I. Rigid Body Transformations (MLS 2.1)

II. Rotational Motion (MLS 2.2)

I. Rigid Body Transformations (RBTs)

Points vs. vectors



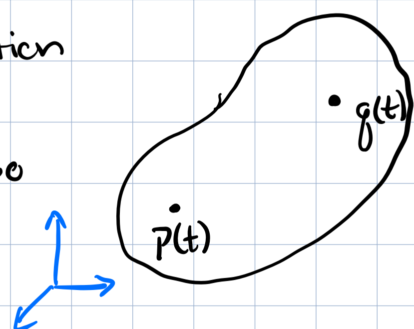
location specified relative to some coordinat frame

$$v = q - p$$

Has a direction and magnitude, but not fixed in space

Both represented by 3-tuples of numbers ($\mathbb{R}^3$), but they are conceptually different.

Def: A <u>rigid body</u> is a collection of points such that the distance between any two points remains fixed.

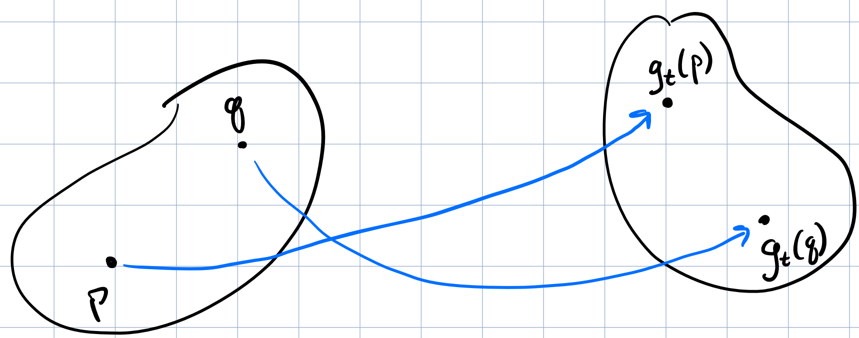


$$\|p(t) - q(t)\| = \|p(0) - q(0)\| = \text{const.}$$

A rigid body can be described as a subset of $\mathbb{R}^3$

$$\mathcal{O} \subset \mathbb{R}^3$$

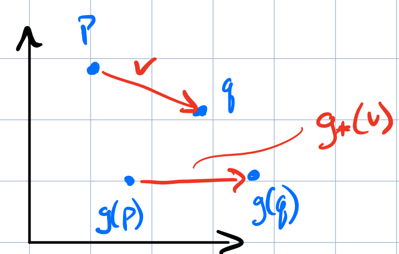
A rigid motion can be represented by a cts. family of mappings, $g_t : \mathcal{O} \to \mathbb{R}^3$



A rigid motion on points $g : \mathcal{O} \to \mathbb{R}$ has a corresponding action on vectors $g_* : \mathbb{R}^3 \to \mathbb{R}^3$

$$v = q - p$$

$$g_*(v) = g(q) - g(p)$$



Def: A mapping $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid body transformation if it satisfies the following properties:

① Length is preserved

"for all/every"

$$\| g(p) - g(q) \| = \| p - q \| \quad \forall \text{ points } p, q \in \mathbb{R}^3$$

② The cross product is preserved

$$g_*(v \times w) = g_*(v) \times g_*(w) \quad \forall \text{ vectors } v, w \in \mathbb{R}^3$$

$\Rightarrow$ reflections are NOT RBTs

① & ② ⟹ Inner product is preserved by RBT

$$v^T w = g_*(v)^T g_*(w) \qquad \forall \text{ vectors } v, w \in \mathbb{R}^3$$

⟹ Orthogonal vectors are transformed to orthogonal vectors.

Q: Is $g(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} x$ a RBT?

Ⓐ Yes

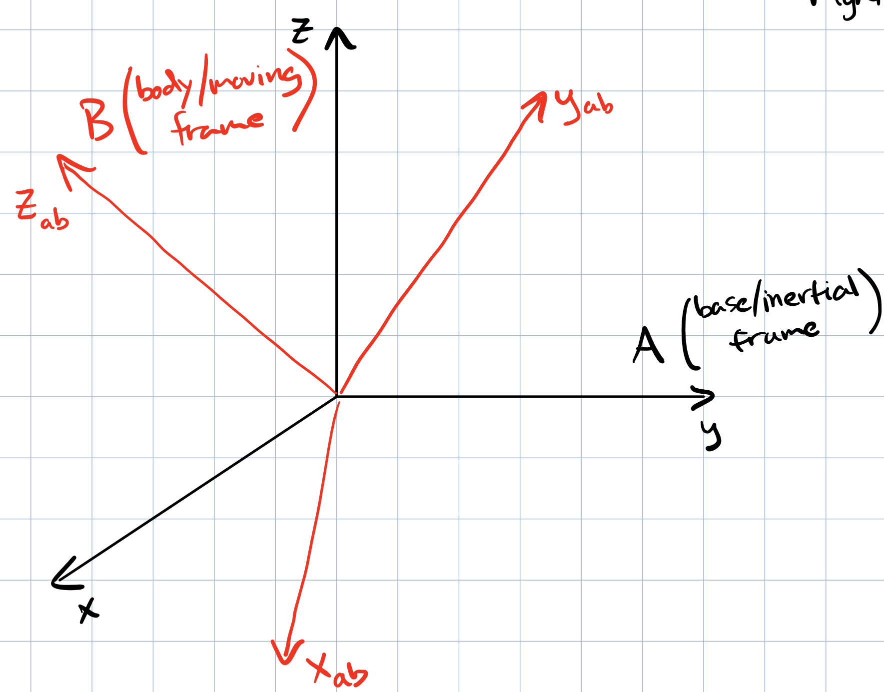
Ⓑ No

Because it doesn't preserve length.

Ⓒ unsure

II. Rotational Motion

* All frame are right-handed



Rotation Matrix

$$R_{ab} = \underbrace{\begin{bmatrix} | & | & | \\ x_{ab} & y_{ab} & z_{ab} \\ | & | & | \end{bmatrix}}_{\text{axes of } B \text{ expressed w.r.t } A}$$

R_{ab} describes the orientation of B relative to A.

<u>Note</u>: Another notation convention R^b_a (Spong book)

<u>Def</u>: Special Orthogonal Set (in 3D)

$$SO(3) = \{ R \in \mathbb{R}^{3\times3} \mid \underbrace{R^T R = I = RR^T}_{\text{colums are orthogonal}} \text{ and } \underbrace{\det(R) = 1}_{\text{"special" excludes reflections}} \}$$

i.e. $R^T = R^{-1}$ from orthogonality

and $\det(R_{ab}) = x_{ab}^T (y_{ab} \times z_{ab}) = x_{ab}^T x_{ab} = 1$

- More general case: $SO(n)$ for any integer $n \geq 2$ forms the rotation <u>group</u> of $\mathbb{R}^n$ vector space.

Def: A set G together w/ a binary operator $\circ$ defined on elements of G is a group iff:

① Closure:

If $g_1, g_2 \in G$ then $g_1 \circ g_2 \in G$

② Identity:

$\exists$ an identity element $e \in G$ s.t.

$g \circ e = g = e \circ g \quad \forall g \in G$

③ Inverse:

For each $g \in G$, $\exists$ a unique inverse $g^{-1} \in G$

s.t. $g \circ g^{-1} = e = g^{-1} \circ g$

④ Associativity

If $g_1, g_2, g_3 \in G$, then $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$

Ex: $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$ are groups

"regular multiplication"

Q: Is $(\mathbb{R}, \cdot)$ a group

A. Yes

B. No

C. Unsure

Because 0 does not have an inverse:

$\frac{1}{0} \notin \mathbb{R}$

Claim: $SO(3)$ is a group under matrix multiplication

Proof:

① For any $R_1, R_2 \in SO(3)$, is $R_1 R_2 \in SO(3)$?

a) $(R_1 R_2)^T (R_1 R_2) = R_2^T \underbrace{(R_1^T R_1)}_{I} R_2 = \underbrace{R_2^T R_2}_{I} = I$ ✓

· same process works for $(R_1 R_2)(R_1 R_2)^T$

b) $\det(R_1 R_2) = \det(R_1)\det(R_2) = 1 \cdot 1 = 1$ ✓

② Identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

③ Inverse $R^{-1} = R^T$ by orthogonality ✓

④ Associativity by matrix multiplication ✓ $\square$

Def: A group $(G, \circ)$ is abelian if every $g_1, g_2 \in G$ commutes under $\circ$, i.e. $g_1 \circ g_2 = g_2 \circ g_1$

Ex: $SO(2)$ is abelian because $R \in SO(2)$ is uniquely parameterized by a single axis and angle $\theta \in S^1$:

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Ex: $SO(n)$ for $n > 2$ is non-abelian

↳ we'll show later that 3D rotations are parameterized by multiple rotation axes and angles where the order matters.

Two Uses of Rotation Matrices

① Representing the configuration (orientation) of a rigid body that is free to rotate relative to a fixed frame.

② Transform coordinates of a point from one reference frame to another, i.e., rotating a point relative to a fixed frame.