ROB 510 Exam-II Solutions

Winter 2022 (Prof. Gregg)

Problem 1:

(a) True, recall ω is orthogonal to $\hat{w}q$. Therefore, $\omega^T(w \times q) = 0$.

(b) True. Body velocity is independent of inertial frame, thus $V_{ab}^b = V_{cb}^b$. From Lemma 2.16 in MLS, we have $V_{cb}^b = -Ad_{g_{bc}} V_{bc}^b$.

(c) False, $Q = S^1 \times S^1 = \mathbb{T}^2$, a torus.

(d) True, this is the generalization of what we saw in class with a square matrix $J(q)$. Also recall that the rank of $J(q)^T J(q)$ is the same as $J(q)$.

Problem 2:

(a) False, it only contains Coriolis and centrifugal terms.

(b) True, this is due to the recursively cyclic property. The bottom-right 2x2 block, $M_b(q)$, can only depend on q_4. In fact, the bottom-right scalar in $M_b(q)$ is constant!

(c) False, the prismatic-joint robots in Homework 4 Problem 5 and 6 have bounded inertia matrices.

(d) True. Changing the location of the spatial frame by a fixed amount corresponds to adding constant offsets to global Cartesian coordinates, and these constant offsets get eliminated by the derivatives in the Euler-Lagrange equations.

Problem 3:

(a) True, which can be seen through linearization. In particular, the Jacobian evaluated at the origin is $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, and both eigenvalues have negative real parts. Note that because the linear system is asymptotically stable, it is also exponentially stable. Hence, the nonlinear system is locally exponentially stable.

(b) True, this set would be positively invariant if $\dot{V} < 0$ everywhere in the set except the origin, where $V(0) = 0$. Positively invariant level-sets of Lyapunov functions are used to estimate the basin of attraction.

(c) False, Lyapunov's second method only provides a sufficient condition; nothing about stability can be concluded if we are simply unsuccessful at finding a valid Lyapunov function.

(d) True. This is proven with total mechanical energy as the storage function, by showing that its time-derivative is equal to $\dot{q}_2^T u$ as in Problem 6b of the Winter 2021 final exam.

Problem 4:

(a) True, we proved this result when gravitational torques are zero.

(b) False, this matrix is skew-symmetric but generally non-zero. A quadratic form with this skew-symmetric matrix becomes zero.

(c) True. Since q^d is constant in $\tilde{q} = q - q^d$, we have $\dot{\tilde{q}} = \dot{q}$ and $\ddot{\tilde{q}} = \ddot{q}$. Then $r = \dot{q} + \Lambda\tilde{q}$ and $\dot{r} = \ddot{q} + \Lambda\dot{q}$. The rest follows from the closed-loop dynamics $M(q)\dot{r} + C(q, \dot{q})r + Kr = 0$.

(d) False, only the desired inertia/mass matrix $\bar{M}(q)$ must be inverted in the control law.

Problem 5:

(a) False, if the feedback linearization is local to subset $U \subset \mathbb{R}^n$, then it is possible that $L_g T_n(x)$ becomes singular outside of U.

(b) False, a block on ice has an inertial impedance which means only hybrid impedance and velocity control is possible.

(c) True. Although the zero off-diagonal terms do not satisfy the strongly inertially coupled property, this property is only required for the non-collocated case of partial feedback linearization. The collocated case has no such requirement.

(d) True. Taking the first derivative of y will produce $\ddot{q}$ by chain rule. Then plugging in the second-order robot dynamics will expose the control input u.

Problem 6:

(a) $\xi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\xi_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\xi_3 = \begin{bmatrix} -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix} \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ L \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\xi_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $g_{st}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) We first calculate the four matrix exponentials based on the twist coordinates above:

$$e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{\hat{\xi}_2 \theta_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$e^{\hat{\xi}_3 \theta_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 & L\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 & L(1-\cos\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{\hat{\xi}_4 \theta_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, $g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} e^{\hat{\xi}_4 \theta_4} g_{st}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 & \theta_1 + \sin\theta_3 (L + \theta_4) \\ 0 & \sin\theta_3 & \cos\theta_3 & L - L\cos\theta_3 + \theta_2 - \theta_4 \cos\theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) We first must determine the Body Jacobian for each point mass. We begin by defining the reference configuration of each point mass reference frame: $g_{sh}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $g_{st}(0) = I_{4\times 4}$. Using these reference configurations and the twist definitions, we can calculate each Body Jacobian:

$$
\begin{aligned}
J_h^b(\theta) &= [\mathrm{Ad}^{-1}_{e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}g_{sh}(0)}\xi_1, \mathrm{Ad}^{-1}_{e^{\hat{\xi}_2\theta_2}g_{sh}(0)}\xi_2, 0, 0] \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
J_t^b(\theta) &= [\mathrm{Ad}^{-1}_{e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}g_{st}(0)}\xi_1, \mathrm{Ad}^{-1}_{e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}g_{st}(0)}\xi_2, \mathrm{Ad}^{-1}_{e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}g_{st}(0)}\xi_3, \mathrm{Ad}^{-1}_{e^{\hat{\xi}_4\theta_4}g_{st}(0)}\xi_4] \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ \cos\theta_3 & \sin\theta_3 & L+\theta_4 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Note it is OK if you accounted for the orientation of the hip mass frame (depending on θ_3) with $J_h^b(\theta) = [\mathrm{Ad}^{-1}_{e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}g_{sh}(0)}\xi_1, \mathrm{Ad}^{-1}_{e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}g_{sh}(0)}\xi_2, \mathrm{Ad}^{-1}_{e^{\hat{\xi}_3\theta_3}g_{sh}(0)}\xi_3, 0]$; we will get the same inertia/mass matrix in the end because the point mass coincides with the third DOF. We next define the the generalized inertia matrix for each link (i.e., each point mass): $\mathcal{M}_h = M_h \begin{bmatrix} I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}$, $\mathcal{M}_t = M_t \begin{bmatrix} I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}$. Finally, the inertia/mass matrix is

$$
\begin{aligned}
M(\theta) &= (J_h^b(\theta))^T \mathcal{M}_h J_h^b(\theta) + (J_t^b(\theta))^T \mathcal{M}_t J_t^b(\theta) \\
&= \begin{bmatrix} M_h + M_t & 0 & M_t\cos\theta_3(L+\theta_4) & M_t\sin\theta_3 \\ 0 & M_h + M_t & M_t\sin\theta_3(L+\theta_4) & -M_t\cos\theta_3 \\ M_t\cos\theta_3(L+\theta_4) & M_t\sin\theta_3(L+\theta_4) & M_t(L+\theta_4)^2 & 0 \\ M_t\sin\theta_3 & -M_t\cos\theta_3 & 0 & M_t \end{bmatrix}
\end{aligned}
$$

(d) The potential energy comprises the gravitational energy for both masses and the elastic energy $(1/2)K\theta_4^2$. The gravitational energy can be computed via forward kinematics of each point mass. Consider the position of the hip and toe at the reference configuration: $p_h(0) = [0, 0, L, 1]^T$, $p_t(0) = [0, 0, 0, 1]^T$. Then,

$$
p_h(\theta) = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p_h(0) = \begin{bmatrix} 0 \\ \theta_1 \\ L+\theta_2 \\ 1 \end{bmatrix}, \quad p_t(\theta) = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}p_t(0) = \begin{bmatrix} 0 \\ \theta_1 + (L+\theta_4)\sin\theta_3 \\ L+\theta_2-(L+\theta_4)\cos\theta_3 \\ 1 \end{bmatrix}.
$$

Finally, we extract the z-component of each position vector to determine the height for each mass's potential energy, resulting in the total potential energy

$$
\begin{aligned}
P(\theta) &= [0, 0, 1, 0] \cdot g(M_h p_h(\theta) + M_t p_t(\theta)) + (1/2)K\theta_4^2 \\
&= gM_h(L+\theta_2) + gM_t(L+\theta_2-(L+\theta_4)\cos\theta_3) + (1/2)K\theta_4^2.
\end{aligned}
$$

Problem 7:

(a) $f(x) = \begin{bmatrix} x_3 \\ x_4 \\ -(mg\ell/J_1)\cos x_1 \\ 0 \end{bmatrix}$, $g(x) = \begin{bmatrix} 0 \\ 0 \\ -1/J_1 \\ 1/J_2 \end{bmatrix}$

(b) First note that $\frac{\partial g}{\partial x} = 0$ and $\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (mg\ell/J_1)\sin x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then,

$$\begin{aligned}
\mathrm{ad}_f g &= [f, g] = 0 \cdot f - \frac{\partial f}{\partial x} \cdot g = \begin{bmatrix} 1/J_1 \\ -1/J_2 \\ 0 \\ 0 \end{bmatrix} \\
\mathrm{ad}_f^2 g &= [f, \mathrm{ad}_f g] = 0 \cdot f - \frac{\partial f}{\partial x} \cdot \mathrm{ad}_f g = \begin{bmatrix} 0 \\ 0 \\ -(mg\ell/J_1^2)\sin x_1 \\ 0 \end{bmatrix} \\
\mathrm{ad}_f^3 g &= [f, \mathrm{ad}_f^2 g] = \frac{\partial \mathrm{ad}_f^2 g}{\partial x} \cdot f - \frac{\partial f}{\partial x} \cdot \mathrm{ad}_f^2 g = \begin{bmatrix} (mg\ell/J_1^2)\sin x_1 \\ 0 \\ -(mg\ell/J_1^2)x_3 \cos x_1 \\ 0 \end{bmatrix}
\end{aligned}$$

(c) The vector fields are linearly independent for $U = \{x | 0 < x_1 < \pi\}$ (upward configurations) or $U = \{x | -\pi < x_1 < 0\}$ (downward configurations). Either or both answers is fine. Note that we don't care about the third row in $\mathrm{ad}_f^3 g$ being zero because that row is already covered by vector g.

(d) Yes. For notational convenience, let $\chi_1 = g$, $\chi_2 = \mathrm{ad}_f g$, and $\chi_3 = \mathrm{ad}_f^2 g$. Because χ_1 and χ_2 are constant, we immediately have $[\chi_1, \chi_2] = [\chi_2, \chi_1] = 0$ which is trivially contained in the distribution. Although χ_3 is non-zero,

$$[\chi_1, \chi_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(mg\ell/J_1^2)\cos x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1/J_1 \\ 1/J_2 \end{bmatrix} - 0 = 0 = [\chi_3, \chi_1].$$

Moreover,

$$[\chi_2, \chi_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(mg\ell/J_1^2)\cos x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/J_1 \\ -1/J_2 \\ 0 \\ 0 \end{bmatrix} - 0 = \begin{bmatrix} 0 \\ 0 \\ -(mg\ell/J_1^3)\cos x_1 \\ 0 \end{bmatrix} = \chi_3 \cdot \mathrm{cotan}(x_1)/J_1 = -[\chi_3, \chi_2].$$

Hence, all possible Lie brackets of vector fields in the distribution are contained in the distribution; it is therefore involutive. Parts (c) and (d) prove the system is locally feedback linearizable. See SHV page 465 for the associated coordinate transformation and feedback linearizing control law.

Problem 8:

(a) FALSE. We stated in class that Barbalat's Lemma could be used to prove the tracking error state $e \to 0$, but we can only guarantee boundedness of the parameter error state. Stating this would be sufficient, but we can also try to use LaSalle: Suppose $\dot{V} \equiv 0 \implies e \equiv 0 \implies B\hat{M}^{-1}Y(q, \dot{q}, \ddot{q})\tilde{\Theta} = 0$ and $\dot{\tilde{\Theta}} = 0$. Hence, $\tilde{\Theta}$

is indeed constant but not necessarily zero. This is because $Y(q, \dot{q}, \ddot{q}) \in \mathbb{R}^{n \times \ell}$, where the DOF n is less than the number of parameters ℓ, so $B\hat{M}^{-1}Y(q, \dot{q}, \ddot{q})\tilde{\Theta} = 0$ does not imply $\tilde{\Theta} = 0$.

(b) TRUE. The actuator inertias could be lumped with the other constants in the diagonal elements, and each lumped constant could be one of the adapted parameters in adaptive inverse dynamics. The important thing is these unknown constants appear linearly in the dynamic equations of motion, allowing separation into a regressor matrix and parameter vector using the linearity-in-the-parameters property.

(c) TRUE. This is seen in the class example of the robot pushing a mass/inertia M with robot force u in the presence of an environmental reaction force F: $M\ddot{x} = u - F$. The only way to modify the inertia term in front of $\ddot{x}$ is to use feedback from force or acceleration. If we alternatively choose $u = cv$ for some constant c and an auxiliary input v, then $(M/c)\ddot{x} = v - F/c$ has modified the environment force, so inertia hasn't truly been modified.