Lesson 10:

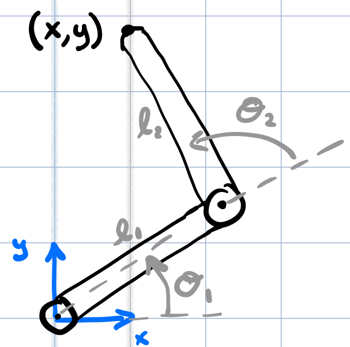
I. Inverse Kinematics

II. Paden - Kahan Subproblems

I. Inverse Kinematics (IK)

Problem: Given a forward kinematic map $g_{st} : Q \to SE(3)$ and desired tool config. $g_d \in SE(3)$, solve $g_{st}(\theta) = g_d$ for θ.

Ex: Planar Arm

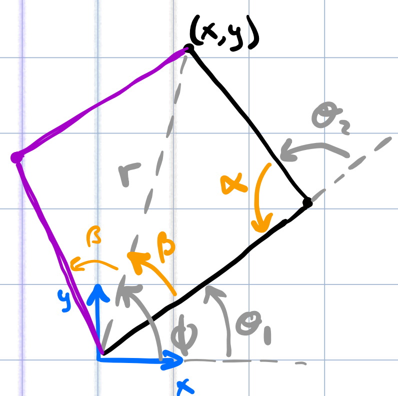


Forward: $x(\theta) = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1+\theta_2)$

$y(\theta) = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1+\theta_2)$

IK: Given desired x, y find θ

Idea: Rewrite in polar coordinates



$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \phi = \operatorname{atan2}(y, x)$$

Law of cosines: $\alpha = \cos^{-1}\left(\dfrac{\ell_1^2 + \ell_2^2 - r^2}{2\ell_1\ell_2}\right)$

Step 1: $\theta_2 = \pi \pm \alpha$ (2 sol'ns for $\alpha \neq 0$)

Step 2: $\theta_1 = \phi \pm \beta$ where $\beta = \cos^{-1}\left(\dfrac{r^2 + \ell_1^2 - \ell_2^2}{2\ell_1 r}\right)$

sign must agree w/ sign chosen for θ_2 above

$\Rightarrow$ 2 solutions for (θ_1, θ_2)

* Here we divided IK into "subproblems"
  - Each subproblem can have zero, one, or mult. sol'ns
  - We take analytical approach (possible for most industrial robot arms)
  - But numerical approaches are becoming popular. (take ROB 511)

II. Paden - Kahan Subproblems

SP1: Rotation about a single axis

Let ξ be a zero-pitch unit twist (pure rotation) and (known) points $p, q \in \mathbb{R}^3$. Find θ s.t.

$$e^{\hat{\xi}\theta} \cdot \bar{p} = \bar{q}.$$

Solution (MLS, pg. 99) involves projections onto axis and orthogonal plane resulting in one or infinite (when $p = q$ lie on axis), or zero sol'ns.

SP2: Rotation about two subsequent, intersecting axes.

Let ξ_1, ξ_2 be zero-pitch unit twists w/ intersecting axes, and $p, q \in \mathbb{R}^3$ be two known points. Find θ_1, θ_2 s.t.

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdot \bar{p} = \bar{q}$$

Solution (MLS, pg. 101) involves dividing into two cases of SP1, resulting in two, one, or no solutions.

SP3: Rotation to a given distance

Let ξ be zero-pitch unit twist, and $p, q \in \mathbb{R}^3$ be two known points, and real $\delta > 0$.

Find θ s.t.

$$\| \bar{q} - e^{\hat{\xi}\theta} \cdot \bar{p} \| = \delta$$

Solution (MLS, pg. 102) involves projections & law of cosines, resulting in two, one, or no solutions.

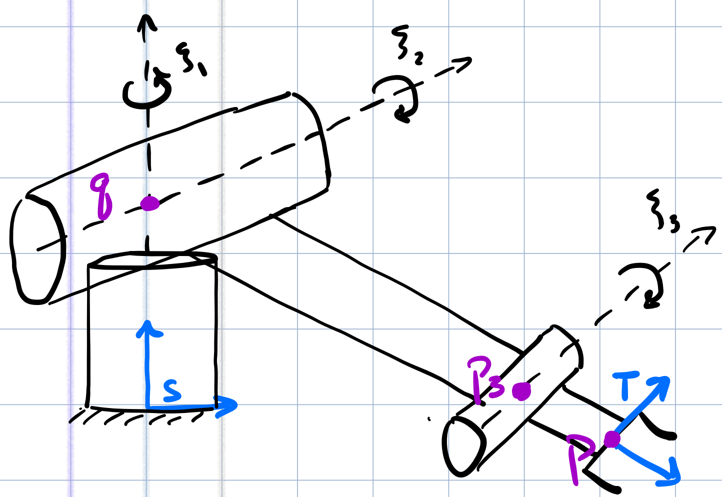
* See MLS for solutions and additional subproblems.

Big Idea: Apply kinematic equations to special (known) points to simplify the IK problem so that it matches one of the PK subproblems.

*Useful trick:

Recall that $\exp(\hat{\xi}\theta) \cdot \bar{p} = \bar{p}$ if $\bar{p}$ is on the revolute axis of ξ.

Ex: 3-DOF arm



$$g_{st}(\theta) = \underbrace{e^{\hat{\xi}_1\theta_1}}_{e^1}\underbrace{e^{\hat{\xi}_2\theta_2}}_{e^2}\underbrace{e^{\hat{\xi}_3\theta_3}}_{e^3} g_{st}(0)$$

note: ξ_1 and ξ_2 intersect

$g_{st}(\theta) = g_d$ ← solve for θ to make this true.

$$e^1 e^2 e^3 g_{st}(0) = g_d$$

$$\Rightarrow e^1 e^2 e^3 = \overbrace{g_d \cdot g_{st}^{-1}(0) := g}^{\text{known}}$$

Choose p_3 on axis of ξ_3, then $\underbrace{g \cdot \bar{p}_3}_{\bar{q}_3 \text{ (known)}} = e^1 e^2 e^3 \bar{p}_3 = e^1 e^2 \bar{p}_3$

$\Rightarrow e^1 e^2 \bar{p}_3 = \bar{q}_3$ can be solved by SP2 for θ_1 and θ_2

Now solve for θ_3. Let $\bar{q}$ be intersecting point of ξ_1, ξ_2 and choose $\bar{p}$ not on the axis ξ_3. Then

$$\underbrace{\|g \cdot \bar{p} - \bar{q}\|}_{:= \delta} = \| e^1 e^2 e^3 \bar{p} - \bar{q} \|,$$

note that $e^1 e^2 \bar{q} = \bar{q}$

$$= \| e^1 e^2 e^3 \bar{p} - e^1 e^2 \bar{q} \|$$

$$= \| e^1 e^2 (e^3 \bar{p} - \bar{q}) \|$$

$$\delta = \| e^3 \bar{p} - \bar{q} \| \longrightarrow$$ Can solve SP3 for θ_3 ✓

Ex: Elbow Manipulator (MLS Fig. 3.11)

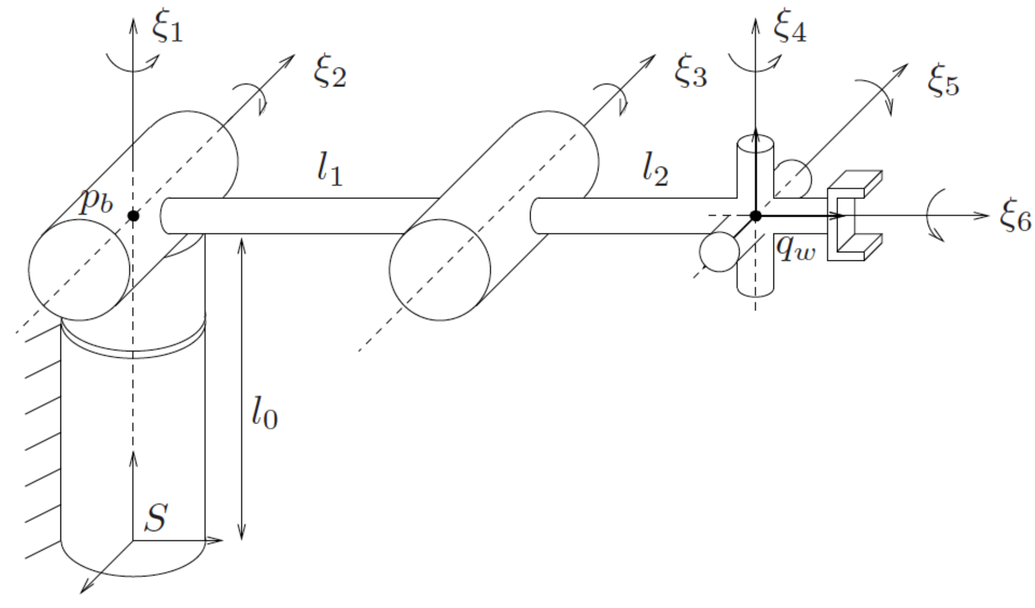


Figure 3.11: Elbow manipulator.

$$g_{st}(\theta) = e^1 e^2 e^3 e^4 e^5 e^6 \cdot g_{st}(0) = g_d$$

$$\Rightarrow e^1 e^2 e^3 e^4 e^5 e^6 = \underbrace{g_d \cdot g_{st}^{-1}(0) := g}_{\text{known}}$$