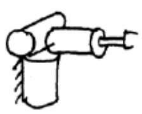
Course: MECH 567: Robot Kinematics & Dynamics
Instructor: Daniel Bruder, PhD

Homework 4

Problem 1



$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad e^{\hat{\xi}_2 \theta_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta_2 & s\theta_2 & 0 \\ 0 & -s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_3 \theta_3} = \begin{bmatrix} c\theta_3 & 0 & s\theta_3 & 0 \\ 0 & 1 & 0 & 0 \\ -s\theta_3 & 0 & c\theta_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xi_1' = \xi_1$$

$$\xi_2' = Ad_{(e^1)} \xi_2 \qquad Ad_{(e^1)} = \begin{bmatrix} R_z(\theta_1) & 0 \\ 0 & R_z(\theta_1) \end{bmatrix}$$

$$\xi_2' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -c\theta_1 \\ -s\theta_1 \\ 0 \end{bmatrix}$$

$$\xi_3' = Ad_{(e^1 e^2)} \xi_3 \qquad Ad_{(e^1 e^2)} = \begin{bmatrix} R_z(\theta_1) R_x(\theta_2) & 0 \\ 0 & R_z(\theta_1) R_{-x}(\theta_2) \end{bmatrix}$$

$$\xi_3' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ R_z(\theta_1) \begin{bmatrix} 0 \\ c\theta_2 \\ -s\theta_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -s\theta_1 c\theta_2 \\ c\theta_1 c\theta_2 \\ -s\theta_2 \end{bmatrix}$$

$$J^s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -c\theta_1 & -s\theta_1 c\theta_2 \\ 0 & -s\theta_1 & c\theta_1 c\theta_2 \\ 1 & 0 & -s\theta_2 \end{bmatrix}$$

Problem 2

(a) See next pages!

(b) Some components of the force FB do not appear in τ because they map to the nullspace of $(J^B)^T$. In other words, these forces do not induce torque on the joints because they are resisted by the structure of the links directly.

---

i) Elbow

In[650]:=
```
ClearAll["Global`*"]
Needs["Screws`", "C://Users/            /Desktop//Screws.m"]
xi1 = {0, 0, 0, 0, 0, 1};
xi2 = {0, 0, 0, -1, 0, 0};
xi3 = {0, 0, l1, -1, 0, 0};
xi4 = {0, 0, l1 + l2, -1, 0, 0};
xi5 = {l1 + l2, 0, 0, 0, 0, 1};
xi6 = {0, 0, 0, 0, 1, 0};


MatrixForm[e1 = TwistExp[xi1, (0)]];
MatrixForm[e2 = TwistExp[xi2, (Pi/2)]];
MatrixForm[e3 = TwistExp[xi3, (0)]];
MatrixForm[e4 = TwistExp[xi4, (0)]];
MatrixForm[e5 = TwistExp[xi5, (0)]];
MatrixForm[e6 = TwistExp[xi6, (0)]];
MatrixForm[gst0 = {{1, 0, 0, 0}, {0, 1, 0, l1 + l2 + l3}, {0, 0, 1, 0}, {0, 0, 0, 1}}];

MatrixForm[gst = e1.e2.e3.e4.e5.e6.gst0 // Simplify];

g1 = e1.e2.e3.e4.e5.e6.gst0;
g2 = e2.e3.e4.e5.e6.gst0;
g3 = e3.e4.e5.e6.gst0;
g4 = e4.e5.e6.gst0;
g5 = e5.e6.gst0;
g6 = e6.gst0;
Ad1 = RigidAdjoint[g1];
Ad2 = RigidAdjoint[g2];
Ad3 = RigidAdjoint[g3];
Ad4 = RigidAdjoint[g4];
Ad5 = RigidAdjoint[g5];
Ad6 = RigidAdjoint[g6];

Ad1 = Inverse[Ad1];
Ad2 = Inverse[Ad2];
Ad3 = Inverse[Ad3];
Ad4 = Inverse[Ad4];
Ad5 = Inverse[Ad5];
Ad6 = Inverse[Ad6];

xi1t = Ad1.xi1;
xi2t = Ad2.xi2;
xi3t = Ad3.xi3;
xi4t = Ad4.xi4;
xi5t = Ad5.xi5;
xi6t = Ad6.xi6;
Jb = {xi1t, xi2t, xi3t, xi4t, xi5t, xi6t};
Fb = {fx, fy, fz, taux, tauy, tauz};
MatrixForm[Jb = Transpose[Jb]]
Tau = MatrixForm[-(Transpose[Jb]).Fb]
```

Out[692]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -l3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -l1-l2-l3 & -l2-l3 & -l3 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Out[693]//MatrixForm=

$$\begin{pmatrix} tauy \\ fz\,(l1+l2+l3)+taux \\ fz\,(l2+l3)+taux \\ fz\,l3+taux \\ fx\,l3-tauz \\ -tauy \end{pmatrix}$$

ii) Inverse Elbow

In[694]:=
```
ClearAll["Global`*"]
Needs["Screws`", "C://Users            //Desktop//Screws.m"]
xi1 = {0, 0, 0, 0, 0, 1};
xi2 = {0, 0, 0, 0, 1, 0};
xi3 = {0, 0, 0, -1, 0, 0};
xi4 = {0, 0, l1, -1, 0, 0};
xi5 = {0, 0, l1 + l2, -1, 0, 0};
xi6 = {0, 0, 0, 0, 1, 0};


MatrixForm[e1 = TwistExp[xi1, (0)]];
MatrixForm[e2 = TwistExp[xi2, (Pi / 2)]];
MatrixForm[e3 = TwistExp[xi3, (0)]];
MatrixForm[e4 = TwistExp[xi4, (0)]];
MatrixForm[e5 = TwistExp[xi5, (0)]];
MatrixForm[e6 = TwistExp[xi6, (0)]];
MatrixForm[gst0 = {{1, 0, 0, 0}, {0, 1, 0, l1 + l2 + l3}, {0, 0, 1, 0}, {0, 0, 0, 1}}];

MatrixForm[gst = e1.e2.e3.e4.e5.e6.gst0 // Simplify];



g1 = e1.e2.e3.e4.e5.e6.gst0;
g2 = e2.e3.e4.e5.e6.gst0;
g3 = e3.e4.e5.e6.gst0;
g4 = e4.e5.e6.gst0;
g5 = e5.e6.gst0;
g6 = e6.gst0;
Ad1 = RigidAdjoint[g1];
Ad2 = RigidAdjoint[g2];
Ad3 = RigidAdjoint[g3];
Ad4 = RigidAdjoint[g4];
Ad5 = RigidAdjoint[g5];
Ad6 = RigidAdjoint[g6];

Ad1 = Inverse[Ad1];
Ad2 = Inverse[Ad2];
Ad3 = Inverse[Ad3];
Ad4 = Inverse[Ad4];
Ad5 = Inverse[Ad5];
Ad6 = Inverse[Ad6];

xi1t = Ad1.xi1;
xi2t = Ad2.xi2;
xi3t = Ad3.xi3;
xi4t = Ad4.xi4;
xi5t = Ad5.xi5;
xi6t = Ad6.xi6;
Jb = {xi1t, xi2t, xi3t, xi4t, xi5t, xi6t};
Fb = {fx, fy, fz, taux, tauy, tauz};
```

```
MatrixForm[Jb = Transpose[Jb]]
Tau = MatrixForm[-(Transpose[Jb]).Fb]
```

Out[736]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -l1-l2-l3 & 0 & -l1-l2-l3 & -l2-l3 & -l3 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[737]//MatrixForm=

$$\begin{pmatrix} fz\,(l1+l2+l3)+taux \\ -tauy \\ fz\,(l1+l2+l3)+taux \\ fz\,(l2+l3)+taux \\ fz\,l3+taux \\ -tauy \end{pmatrix}$$

iii) Stanford

```
In[738]:= ClearAll["Global`*"]
Needs["Screws`", "C://Users//          //Desktop//Screws.m"]
xi1 = {0, 0, 0, 0, 0, 1};
xi2 = {0, 0, 0, -1, 0, 0};
xi3 = {0, 1, 0, 0, 0, 0};
xi4 = {0, 0, l1, -1, 0, 0};
xi5 = {l1, 0, 0, 0, 0, 1};
xi6 = {0, 0, 0, 0, 1, 0};


MatrixForm[e1 = TwistExp[xi1, (0)]];
MatrixForm[e2 = TwistExp[xi2, (Pi / 2)]];
MatrixForm[e3 = TwistExp[xi3, (0)]];
MatrixForm[e4 = TwistExp[xi4, (0)]];
MatrixForm[e5 = TwistExp[xi5, (0)]];
MatrixForm[e6 = TwistExp[xi6, (0)]];
MatrixForm[gst0 = {{1, 0, 0, 0}, {0, 1, 0, l1 + l2}, {0, 0, 1, 0}, {0, 0, 0, 1}}];

MatrixForm[gst = e1.e2.e3.e4.e5.e6.gst0 // Simplify];



g1 = e1.e2.e3.e4.e5.e6.gst0;
g2 = e2.e3.e4.e5.e6.gst0;
g3 = e3.e4.e5.e6.gst0;
g4 = e4.e5.e6.gst0;
g5 = e5.e6.gst0;
g6 = e6.gst0;
Ad1 = RigidAdjoint[g1];
Ad2 = RigidAdjoint[g2];
Ad3 = RigidAdjoint[g3];
Ad4 = RigidAdjoint[g4];
Ad5 = RigidAdjoint[g5];
Ad6 = RigidAdjoint[g6];

Ad1 = Inverse[Ad1];
Ad2 = Inverse[Ad2];
Ad3 = Inverse[Ad3];
Ad4 = Inverse[Ad4];
Ad5 = Inverse[Ad5];
Ad6 = Inverse[Ad6];

xi1t = Ad1.xi1;
xi2t = Ad2.xi2;
xi3t = Ad3.xi3;
xi4t = Ad4.xi4;
xi5t = Ad5.xi5;
xi6t = Ad6.xi6;
MatrixForm[Jb = {xi1t, xi2t, xi3t, xi4t, xi5t, xi6t}];
Fb = {fx, fy, fz, taux, tauy, tauz};
```

```
MatrixForm[Jb = Transpose[Jb]]
Tau = MatrixForm[-(Transpose[Jb]).Fb]
```

Out[780]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -l2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -l1-l2 & 0 & -l2 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Out[781]//MatrixForm=

$$\begin{pmatrix} tauy \\ fz\,(l1+l2)+taux \\ -fy \\ fz\,l2+taux \\ fx\,l2-tauz \\ -tauy \end{pmatrix}$$

iv) Rhino

In[870]:=
```
ClearAll["Global`*"]
Needs["Screws`", "C://Users            //Desktop//Screws.m"]
xi1 = {0, 1, 0, 0, 0, 0};
xi2 = {0, 0, 0, 0, 0, 1};
xi3 = {0, 0, 0, -1, 0, 0};
xi4 = {0, 0, l1, -1, 0, 0};
xi5 = {0, 0, l1 + l2, -1, 0, 0};
xi6 = {0, 0, 0, 0, 1, 0};


MatrixForm[e1 = TwistExp[xi1, (0)]];
MatrixForm[e2 = TwistExp[xi2, (Pi / 2)]];
MatrixForm[e3 = TwistExp[xi3, (0)]];
MatrixForm[e4 = TwistExp[xi4, (0)]];
MatrixForm[e5 = TwistExp[xi5, (0)]];
MatrixForm[e6 = TwistExp[xi6, (0)]];
MatrixForm[gst0 = {{1, 0, 0, 0}, {0, 1, 0, l1 + l2 + l3}, {0, 0, 1, 0}, {0, 0, 0, 1}}];

MatrixForm[gst = e1.e2.e3.e4.e5.e6.gst0 // Simplify];



g1 = e1.e2.e3.e4.e5.e6.gst0;
g2 = e2.e3.e4.e5.e6.gst0;
g3 = e3.e4.e5.e6.gst0;
g4 = e4.e5.e6.gst0;
g5 = e5.e6.gst0;
g6 = e6.gst0;
Ad1 = RigidAdjoint[g1];
Ad2 = RigidAdjoint[g2];
Ad3 = RigidAdjoint[g3];
Ad4 = RigidAdjoint[g4];
Ad5 = RigidAdjoint[g5];
Ad6 = RigidAdjoint[g6];

Ad1 = Inverse[Ad1];
Ad2 = Inverse[Ad2];
Ad3 = Inverse[Ad3];
Ad4 = Inverse[Ad4];
Ad5 = Inverse[Ad5];
Ad6 = Inverse[Ad6];

xi1t = Ad1.xi1;
xi2t = Ad2.xi2;
xi3t = Ad3.xi3;
xi4t = Ad4.xi4;
xi5t = Ad5.xi5;
xi6t = Ad6.xi6;
Jb = {xi1t, xi2t, xi3t, xi4t, xi5t, xi6t};
Fb = {fx, fy, fz, taux, tauy, tauz};
```

```
MatrixForm[Jb = Transpose[Jb]]
Tau = MatrixForm[-(Transpose[Jb]).Fb]
```

Out[912]//MatrixForm=

$$\begin{pmatrix} 1 & -l1-l2-l3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -l1-l2-l3 & -l2-l3 & -l3 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[913]//MatrixForm=

$$\begin{pmatrix} -fx \\ -fx\,(-l1-l2-l3)-tauz \\ -fz\,(-l1-l2-l3)+taux \\ -fz\,(-l2-l3)+taux \\ fz\,l3+taux \\ -tauy \end{pmatrix}$$

Problem 3

ZYZ

$$\xi_1 = \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \quad \xi_2 = \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix} \quad \xi_3 = \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}$$

$$e^1 = \begin{bmatrix} c\alpha & -s\alpha & 0 & 0 \\ s\alpha & c\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad e^2 = \begin{bmatrix} c\beta & 0 & s\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s\beta & 0 & c\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xi_2' = \begin{bmatrix}0\\0\\0\\-s\alpha\\c\alpha\\0\end{bmatrix} \qquad \xi_3' = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s\beta \\ 0 \\ c\beta \end{bmatrix} = \begin{bmatrix}0\\0\\0\\s\beta c\alpha\\s\alpha s\beta\\c\beta\end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -s\alpha & c\alpha s\beta \\ 0 & c\alpha & s\alpha s\beta \\ 1 & 0 & c\beta \end{bmatrix}$$

Singularity when $\beta = n\pi$ (columns 1, 3 degenerate)

ZXY:

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -c\alpha & -s\alpha c\beta \\ 0 & -s\alpha & c\alpha c\beta \\ 1 & 0 & -s\beta \end{bmatrix}$$

Singularity when $\beta = \pi/2 + n\pi$
(columns 1, 3 degenerate)

Problem 4

$K = \frac{1}{2} m\dot{x}^2$: Kinetic Energy

$p = m\dot{x} = \frac{dK}{d\dot{x}}$: Momentum

For a mechanical system with generalized coordinates $q_1,\dots,q_n$.

Generalized momentum: $p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial K}{\partial \dot{q}_k} - \cancelto{0}{\frac{\partial V}{\partial \dot{q}_k}}$

Kinetic Energy: $K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$

Lagrangian: $L = K - V$

$$\sum_{k=1}^{n} \dot{q}_k p_k = \dot{q}^T p = \dot{q}^T \frac{\partial}{\partial \dot{q}}[\frac{1}{2}\dot{q}^T D(q)\dot{q}] = \dot{q}^T D(q)\dot{q} = 2K$$

Problem 5

(a)

$$H = \sum_{k=1}^{n} \dot{q}_k p_k - L = \dot{q}^T p - L = \dot{q}^T \frac{\partial L}{\partial \dot{q}} - L \qquad\qquad p = \frac{\partial L}{\partial \dot{q}}$$

$$L = K - V = \frac{1}{2}\dot{q}^T D(q)\dot{q} - V(q)$$

$$\Downarrow$$

$$H = \dot{q}^T \left(\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}\dot{q}^T D(q)\dot{q} - V(q)\right)\right) - \frac{1}{2}\dot{q}^T D(q)\dot{q} + V(q)$$

$$H = \dot{q}^T \left(\frac{1}{2}\left(\left(\dot{q}^T D(q)\right)^T + D(q)\dot{q}\right)\right) - \frac{1}{2}\dot{q}^T D(q)\dot{q} + V(q)$$

$$H = \dot{q}^T D(q)\dot{q} - \frac{1}{2}\dot{q}^T D(q)\dot{q} + V(q) = \frac{1}{2}\dot{q}^T D(q)\dot{q} + V(q) = K + V \quad \text{Total Energy}$$

An alternative way of doing this: from Problem 4 we know

$$\sum_{k=1}^{n} \dot{q}_k p_k = 2K \ ,$$

and $L = K - V$. Therefore,

$$H = 2K - K + V = K + V.$$

(b)

E-L Equations

$$\frac{d}{dt}\frac{\partial}{\partial \dot q_k}L-\frac{\partial L}{\partial q_k}=\tau_k \qquad k = 1,\ldots,n$$

$$H(q,p)=\sum_{e=1}^{n}\dot q_e p_e - L = \Big(\dot q(q,p)\Big)^T p - L(q,\dot q(q,p))$$

From E-L

- $$\frac{\partial L}{\partial q_k}=\frac{d}{dt}\underbrace{\frac{\partial}{\partial \dot q_k}L}_{p_k}-\tau_k=\frac{d}{dt}p_k-\tau_k$$

- $$\frac{\partial L\big(q,\dot q(q,p)\big)}{\partial q_k}=\frac{\partial L}{\partial q_k}+\cancel{\left(\frac{\partial L}{\partial \dot q}\right)^T\frac{\partial \dot q}{\partial q_k}}=\cancel{\left(\frac{\partial \dot q}{\partial q_k}\right)^T p}-\frac{\partial H}{\partial q_k}\Rightarrow\frac{\partial L}{\partial q_k}=-\frac{\partial H}{\partial q_k}$$

  $$\Downarrow$$

  $$-\frac{\partial H}{\partial q_k}=\dot p_k-\tau_k\Rightarrow\boxed{\dot p_k=-\frac{\partial H}{\partial q_k}+\tau_k}$$

- $$\frac{\partial H}{\partial p_k}=\frac{\partial}{\partial p_k}\Big(\sum_{e=1}^{n}\dot q_e p_e\Big)-\frac{\partial L}{\partial p_k}=\dot q_k+\sum_{e=1}^{n}\frac{\partial \dot q_e}{\partial p_k}p_e-\Big(\frac{\partial K}{\partial p_k}-\cancelto{0}{\frac{\partial V}{\partial p_k}}\Big)$$

  only k-th component is $\neq 0$

- $$\frac{\partial H}{\partial p_k}=\dot q_k+\sum_{e=1}^{n}\frac{\partial \dot q_e}{\partial p_k}p_e-\sum_{e=1}^{n}\frac{\partial K}{\partial \dot q_e}\frac{\partial \dot q_e}{\partial p_k}=\dot q_k+\sum_{e=1}^{n}\frac{\partial \dot q_e}{\partial p_k}p_e-\sum_{e=1}^{n}p_e\frac{\partial \dot q_e}{\partial p_k}=\dot q_k$$

  $$\Rightarrow\boxed{\dot q_k=\frac{\partial H}{\partial p_k}}$$

Problem 6

From the textbook (SHV) pages 186-188.

$$K = \frac{1}{2}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}^T D(q_1, q_2)\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}p_1\\ p_2\end{bmatrix}^T D^{-1}(q_1, q_2)\begin{bmatrix}p_1\\ p_2\end{bmatrix}$$

Where $D(q_1, q_2) = \begin{bmatrix} d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}$

$$H = K + V$$
$$\frac{\partial H}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} = p = \frac{\partial K}{\partial \dot{q}} = D(q)\dot{q} \Rightarrow \dot{q} = D^{-1}(q)p$$

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \begin{bmatrix}\frac{\partial H}{\partial p_1}\\ \frac{\partial H}{\partial p_2}\end{bmatrix} = D^{-1}(q)\begin{bmatrix}p_1\\ p_2\end{bmatrix}$$
$$\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\end{bmatrix} = \begin{bmatrix}-\frac{\partial H}{\partial q_1} + \tau_1\\ -\frac{\partial H}{\partial q_2} + \tau_2\end{bmatrix} = -\begin{bmatrix}\frac{\partial K(q,p)}{\partial q_1} + \frac{\partial V}{\partial q_1} - \tau_1\\ \frac{\partial K(q,p)}{\partial q_2} + \frac{\partial V}{\partial q_2} - \tau_2\end{bmatrix}$$

See the next page for the extra (optional) step of calculating out these equations based on the robot's dynamics terms.

In[4]:=
```
ClearAll["Global`*"];
Needs["Screws`", "C://Mathematica//Screws.m"]
Needs["RobotLinks`", "C://Mathematica//RobotLinks.m"]
```

In[7]:=
```
q = {{q1[t]}, {q2[t]}};
p = {{p1[t]}, {p2[t]}};
d11 = m1 * lc1^2 + m2 * (l1^2 + lc2^2 + 2 * l1 * lc2 * Cos[q2[t]]) + I1 + I2;
d12 = m2 * (lc2^2 + l1 * lc2 * Cos[q2[t]]) + I2;
d22 = m2 * lc2^2 + I2;
MatrixForm[Dq = {{d11, d12}, {d12, d22}}]
```

Out[12]//MatrixForm=

$$\begin{pmatrix} I1 + I2 + lc1^2\, m1 + m2\, (l1^2 + lc2^2 + 2\, l1\, lc2\, \mathrm{Cos}[q2[t]]) & I2 + m2\, (lc2^2 + l1\, lc2\, \mathrm{Cos}[q2[t]]) \\ I2 + m2\, (lc2^2 + l1\, lc2\, \mathrm{Cos}[q2[t]]) & I2 + lc2^2\, m2 \end{pmatrix}$$

In[13]:=
```
MatrixForm[DqInverse = Inverse[Dq] // Simplify]
```

Out[13]//MatrixForm=

$$\begin{pmatrix} \frac{I2+lc2^2\, m2}{(I1+lc1^2\, m1+l1^2\, m2)\,(I2+lc2^2\, m2)-l1^2\, lc2^2\, m2^2\, \mathrm{Cos}[q2[t]]^2} & -\frac{I2+lc2^2\, m2+l1\, lc2\, m2\, \mathrm{Cos}[q2[t]]}{(I1+lc1^2\, m1+l1^2\, m2)\,(I2+lc2^2\, m2)-l1^2\, lc2^2\, m2^2\, \mathrm{Cos}[q2[t]]^2} \\ -\frac{I2+lc2^2\, m2+l1\, lc2\, m2\, \mathrm{Cos}[q2[t]]}{(I1+lc1^2\, m1+l1^2\, m2)\,(I2+lc2^2\, m2)-l1^2\, lc2^2\, m2^2\, \mathrm{Cos}[q2[t]]^2} & \frac{I1+I2+lc1^2\, m1+l1^2\, m2+lc2^2\, m2+2\, l1\, lc2\, m2\, \mathrm{Cos}[q2[t]]}{(I1+lc1^2\, m1+l1^2\, m2)\,(I2+lc2^2\, m2)-l1^2\, lc2^2\, m2^2\, \mathrm{Cos}[q2[t]]^2} \end{pmatrix}$$

In[19]:=
```
(*Kinetic energy*)
K = First[First[1/2 * Transpose[p].DqInverse.p // Simplify]]
```

Out[19]=

$$\Big((I2 + lc2^2\, m2)\, p1[t]^2 - 2\,(I2 + lc2^2\, m2 + l1\, lc2\, m2\, \mathrm{Cos}[q2[t]])\, p1[t] \times p2[t] + (I1 + I2 + lc1^2\, m1 + l1^2\, m2 + lc2^2\, m2 + 2\, l1\, lc2\, m2\, \mathrm{Cos}[q2[t]])\, p2[t]^2\Big) \Big/ \Big(2\,(I1 + lc1^2\, m1 + l1^2\, m2)\,(I2 + lc2^2\, m2) - 2\, l1^2\, lc2^2\, m2^2\, \mathrm{Cos}[q2[t]]^2\Big)$$

In[20]:=
```
MatrixForm[DKDq  =  Transpose[{D[K, Transpose[q]]}] // FullSimplify]
```

Out[20]//MatrixForm=

$$\begin{pmatrix} 0 \\ -\frac{4\, l1\, lc2\, m2\, (-(I2+lc2^2\, m2)\,(p1[t]-p2[t])+l1\, lc2\, m2\, \mathrm{Cos}[q2[t]]\, p2[t])\,((I1+lc1^2\, m1+l1^2\, m2)\, p2[t]+l1\, lc2\, m2\, \mathrm{Cos}[q2[t]]\,(-p1[t]+p2[t]))}{(2\, I2\,(I1+lc1^2\, m1)+2\,(I2\, l1^2+lc2^2\,(I1+lc1^2\, m1))\, m2+l1^2\, lc2^2\, m2^2-l1^2\, lc2^2\, m2^2\, \mathrm{Cos}[2\, q2[t]])^2} \end{pmatrix}$$

In[21]:=
```
PE = m1 * g * lc1 * Sin[q1[t]] + m2 * g * (l1 * Sin[q1[t]] + lc2 * Sin[q1[t] + q2[t]]);
MatrixForm[DPEDq = Transpose[{D[PE, Transpose[q]]}] // FullSimplify]
```

Out[22]//MatrixForm=

$$\begin{pmatrix} g\,((lc1\, m1 + l1\, m2)\, \mathrm{Cos}[q1[t]] + lc2\, m2\, \mathrm{Cos}[q1[t] + q2[t]]) \\ g\, lc2\, m2\, \mathrm{Cos}[q1[t] + q2[t]] \end{pmatrix}$$

In[23]:=
```
MatrixForm[DKDq + DPEDq]
```

Out[23]//MatrixForm=

$$\begin{pmatrix} g\,((lc1\, m1 + l1\, m2)\, \mathrm{Cos}[q1[t]] + lc2\, m2\, \mathrm{Cos}[q1[t] + q2[t]]) \\ g\, lc2\, m2\, \mathrm{Cos}[q1[t] + q2[t]] - \frac{4\, l1\, lc2\, m2\, (-(I2+lc2^2\, m2)\,(p1[t]-p2[t])+l1\, lc2\, m2\, \mathrm{Cos}[q2[t]]\, p2[t])\,((I1+lc1^2\, m1+l1^2\, m2)\, p2[t}{(2\, I2\,(I1+lc1^2\, m1)+2\,(I2\, l1^2+lc2^2\,(I1+lc1^2\, m1))\, m2+l1^2\, lc2^2\, m2^2-l1^2} \end{pmatrix}$$