Lesson 7:

I. Screw Motion (MLS 2.3.3)

II. Forward Kinematics (MLS 3.2)

A. Denavit-Hartenberg Convention

B. Product of Exponentials (if time)

I. Screw Motion

(1) Rotate about some axis ω by an angle $\theta \in \mathbb{R}$

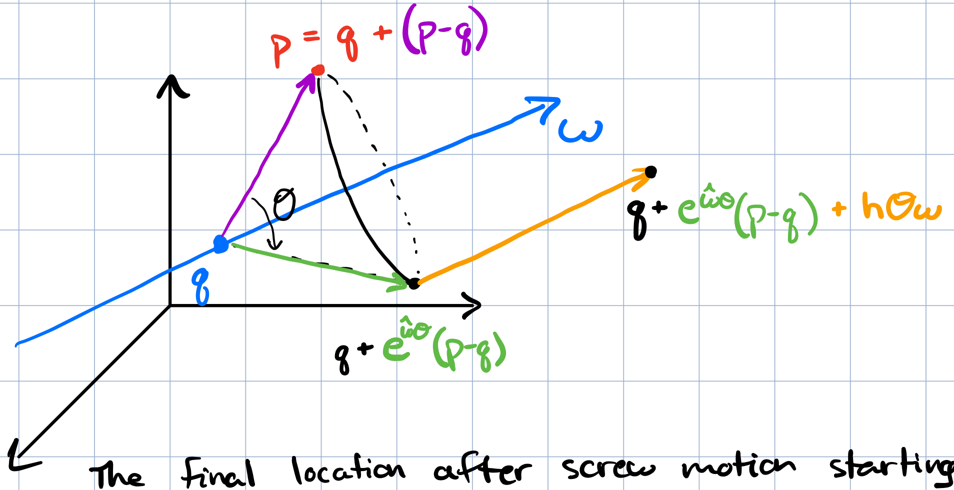
(2) Translate along same axis by a distance $d \in \mathbb{R}$

- Define pitch as the ratio of translation to rotation:

$$h := \frac{d}{\theta} \quad (\text{const.})$$

- Net translation after rotating by θ is $h \cdot \theta = d$
  - if $h = \infty$, then pure translation
  - if $h = 0$, then pure rotation

Q: How to compute RBT associated with a screw?



(assume $\|\omega\| = 1$)

The final location after screw motion starting from $p \in \mathbb{R}^3$ is

$$\bar{g} \cdot \bar{p} = \begin{bmatrix} q + e^{\hat{\omega}\theta}(p-q) + h\theta\omega \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix}}_{\text{RBT for the screw}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Recall for $\xi = (v, \omega)$, $e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T v \theta \\ 0 & 1 \end{bmatrix}$

Equate $\bar{g} = e^{\hat{\xi}\theta}$ and solve for v (ω is the same in both)

1) $v = -\omega \times q + h\omega$ for screw motions, assuming $\|\omega\| = 1$

2) $v = -\omega \times q$ for pure rotation $(h=0)$, $\|\omega\| = 1$

3) v is axis of translation with $\omega = 0$ $(h = \infty)$, $\|v\| = 1$

What about rotation & translation along <u>different</u> axes?

- Consider arbitrary (unit) twist $\xi = (v, \omega)$, $\|\omega\| = 1$
- <u>Claim</u>: ξ describes a screw motion for some q and h.

  i.e. $\forall \xi \in se(3)$, $\exists q \in \mathbb{R}^3$, $h \in \mathbb{R}$ such that $v = -\omega \times q + h\omega$

This is true!

$q = \omega \times v$ $h = \omega^T v$ (verify for yourself)

<u>Thm 2.11</u>: (Chasles) Every rigid body motion can be realized by a rotation about an axis combined with a translation parallel to that axis (i.e. screw motion)

- The screw associated with an arbitrary twist (not necessarily $\|\omega\|=1$) $\xi=(v,\omega)$ has

  - **Magnitude**: $$M=\begin{cases}\|\omega\| & \text{if } \omega\neq 0\\ \|v\| & \text{if } \omega=0\end{cases}$$

  - **Pitch**: $$h=\frac{\omega^T v}{\|\omega\|^2}$$

  - **Axis**: $$\ell=\begin{cases}\overset{\swarrow q}{\frac{\omega\times v}{\|\omega\|^2}}+\lambda\omega & \text{for } \lambda\in\mathbb{R}, \text{ if } \omega\neq 0\\ \lambda v & \text{for } \lambda\in\mathbb{R}, \text{ if } \omega=0\end{cases}$$

Note on some conventions:

- A unit twist defines R.B. motion with either $\|\omega\|=1$ (otherwise) or $\|v\|=1$ (pure translation), not necessarily $\|\xi\|=1$

- Convenient to use unit twists when modeling robots because they have joints w/ angles θ_i measured by encoders.

- Can express non-unit twists as product of unit twist and $\theta\in\mathbb{R}$

$$\xi=\begin{bmatrix}v\\ \omega\end{bmatrix}=\xi_{unit}\theta=\begin{bmatrix}v_u\\ \omega_u\end{bmatrix}\theta,$$

where $\xi_{unit}=\begin{bmatrix}v_u\\ \omega_u\end{bmatrix}$ and only $\|\omega_u\|=1$ unless pure translation (when $\|v_u\|=1$)

II. Forward Kinematics

Goal: Map $\underset{\substack{\uparrow\\ \text{configuration}\\ \text{space}}}{Q} \longrightarrow \underset{\substack{\uparrow\\ \text{workspace}}}{W} \subset SE(3)$ in general

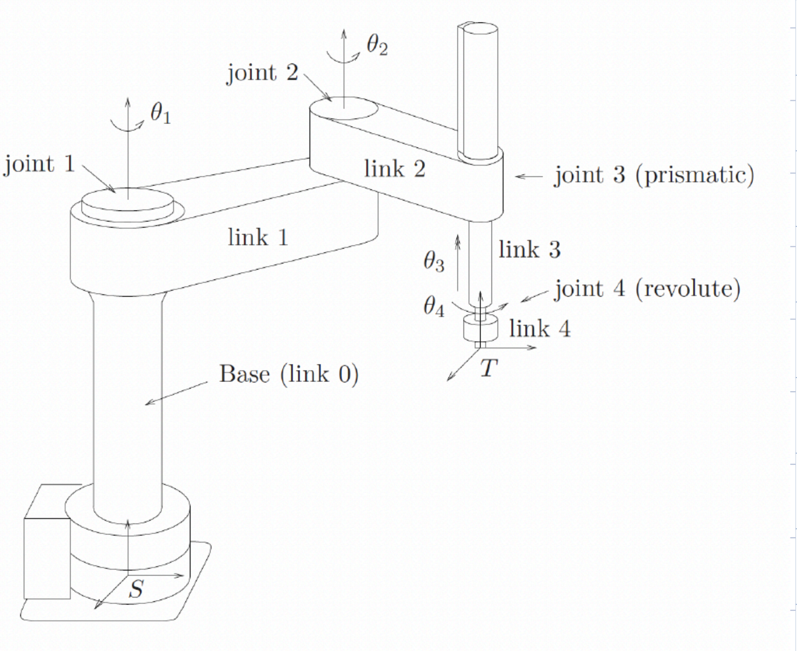


Figure 3.1: Numbering conventions for an AdeptOne robot.

Labeling conventions:

- Number joints from 1 to n for n-DOF robot.
  - prismatic, revolute, helical (rare)
- Joint $i \in \{1, \dots, n\}$ connects links $i-1$ to i

Ex: SCARA robot

- $Q = S^1 \times S^1 \times \mathbb{R}^1 \times S^1 = \overset{\text{"3 torus"}}{\mathbb{T}^3} \times \mathbb{R}$
- Configuration vector: $\theta = [\theta_1, \theta_2, \theta_3, \theta_4]^T \in Q$
- Base/spatial frame S attached to link 0
- Tool frame T attached to link 4 (end effector)
- Forward Kinematics: Given set of "angles" $\theta \in Q$, determine the config. of tool frame T relative to base frame S

$$g_{st}(\theta) \in SE(3)$$

$$g_{st} : Q \rightarrow W \subset SE(3)$$

A. Denavit-Hartenberg

If $g_{\ell-1,\ell}(\theta_i)$ is the transformation between adjacent link frames L_{i-1}, L_i defined at their respective joints,

$$g_{st}(\theta) = g_{s,\ell_1}(\theta_1) \cdot g_{\ell_1,\ell_2}(\theta)_2 \cdots g_{\ell_{n-1},\ell_n}(\theta_n) \cdot g_{\ell_n,t}$$

const. matrix: fixed relative transformation between L_n & T

Each transformation g_{ℓ_{i-1},ℓ_i} has the form:

$$g_{\ell_{i-1},\ell_i} = \begin{bmatrix} \cos\phi_i & -\sin\phi_i \cos\alpha_i & \sin\phi_i \sin\alpha_i & a_i \cos\phi_i \\ \sin\phi_i & \cos\phi_i \cos\alpha_i & -\cos\phi_i \sin\alpha_i & a_i \sin\phi_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where α_i, a_i, d_i, ϕ_i are called "DH-parameters"

$= \theta_i$ for prismatic joints.

$= \theta_i$ for revolute joints

*Somewhat unintuitive formulaic process for defining the DH parameters for each link (see MLS and SHV for details)

B. Product of Exponentials (next time)