Lesson 8: Forward Kinematics (cont'd)

I. Product of Exponentials (MLS 3.2)

II. Mathematica Examples

I. Product of Exponentials

Recall if ξ is a twist, then RB motion along that twist is $g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)$

Step 1: Define the reference configuration of the kinematic chain $g_{st}(0)$, i.e. the RBT between T and S when config. vector $\Theta = \emptyset \leftarrow$ zero vector

- Defined based on coordinate system (angular conventions of Θ)
- Note: $g_{st}(\emptyset) \neq g_{\ell_n,t}$ from D-H

Step 2: Construct unit twist ξ_i corresponding to the motion for i^{th} joint for all $i \in \{1, \dots, n\}$ at the reference configuration $(\Theta = \emptyset)$:

- revolute

$$\xi_i = \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix}, \quad \text{where } \|\omega_i\| = 1 \text{ and } q_i \text{ is any point on axis } \lambda\omega_i$$

- prismatic

$$\xi_i = \begin{bmatrix} v_i \\ 0 \end{bmatrix}, \quad \text{where } \|v_i\| = 1$$

- helical joints as well

Step3: Generate all matrix exponentials $e^{\hat{\xi}_1\theta_1}, e^{\hat{\xi}_2\theta_2}, \ldots$ according to formula:

$$e^{\hat{\xi}_i\theta_i} = \begin{cases} \begin{bmatrix} e^{\hat{\omega}_i\theta_i} & (I - e^{\hat{\omega}_i\theta_i})(\omega_i \times v_i) + \omega_i\omega_i^T v_i\theta_i \\ 0 & 1 \end{bmatrix}, & \omega_i \neq 0 \\ \begin{bmatrix} I & v_i\theta_i \\ 0 & 1 \end{bmatrix}, & \omega_i = 0 \end{cases}$$

where $\xi_i = [\, v_i^T, \omega_i^T \,]^T$

Step4: Combine all the joint motions

$$g_{st}(\theta) = \underbrace{e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n}}_{\text{relative motion}} g_{st}(\emptyset)$$

$$= \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix} g_{st}(\emptyset)$$

$$= \underbrace{\begin{bmatrix} R_{st}(\theta) & p_{st}(\theta) \\ 0 & 1 \end{bmatrix}}_{\text{New config. of T relative to S given } \theta}$$

Example: SCARA (MLS Fig. 3.3)

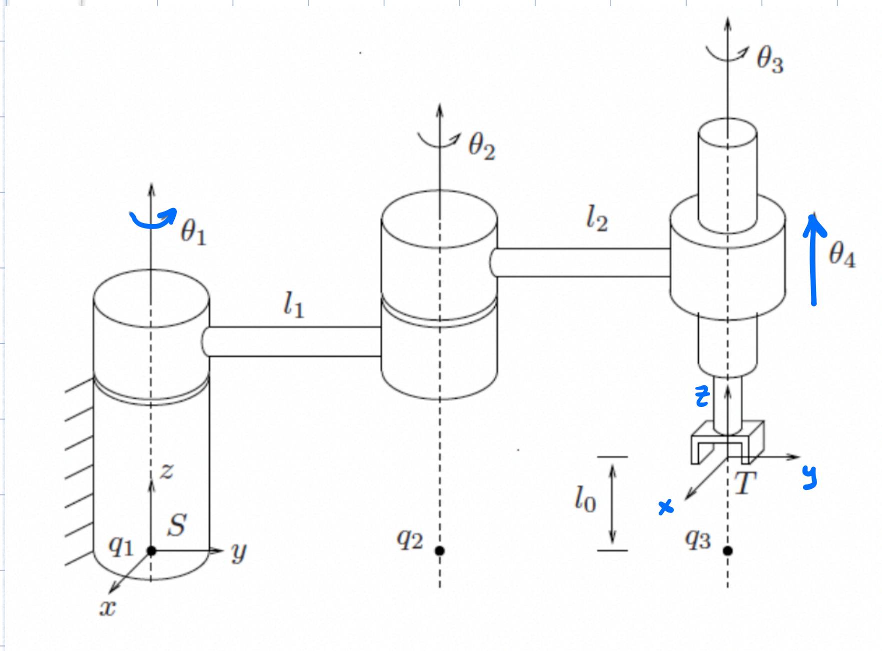

Figure 3.3: SCARA manipulator in its reference configuration.

Step 1: Ref. Config:

$$g_{st}(0) = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1+l_2 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

Step 2: Construct twists

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \omega_2 = \omega_3 \qquad \omega_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad q_2 = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} \qquad q_3 = \begin{bmatrix} 0 \\ l_1+l_2 \\ 0 \end{bmatrix}$$

$-\hat{\omega}_2 q_2$

• Revolute twists

$$\xi_1 = \begin{bmatrix} v_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\xi_3 = \begin{bmatrix} -\omega_3 \times q_3 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} l_1+l_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

• Prismatic twists

$$\xi_4 = \begin{bmatrix} v_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{bmatrix}$$

Step 3: Matrix exponentials

$$e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ etc...}$$

Step 4: Combine

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} e^{\hat{\xi}_4 \theta_4} g_{st}(0)$$

Note: Can also use POE on a homogeneous point on the robot:

Let $\bar{p}(0)$ be a reference location of a point on the i^{th} link of the robot

$$\bar{p}(\theta) = e^{\hat{\xi}_1 \theta_1} \cdots e^{\hat{\xi}_i \theta_i} \bar{p}(0)$$

Example: 2 dof manipulator

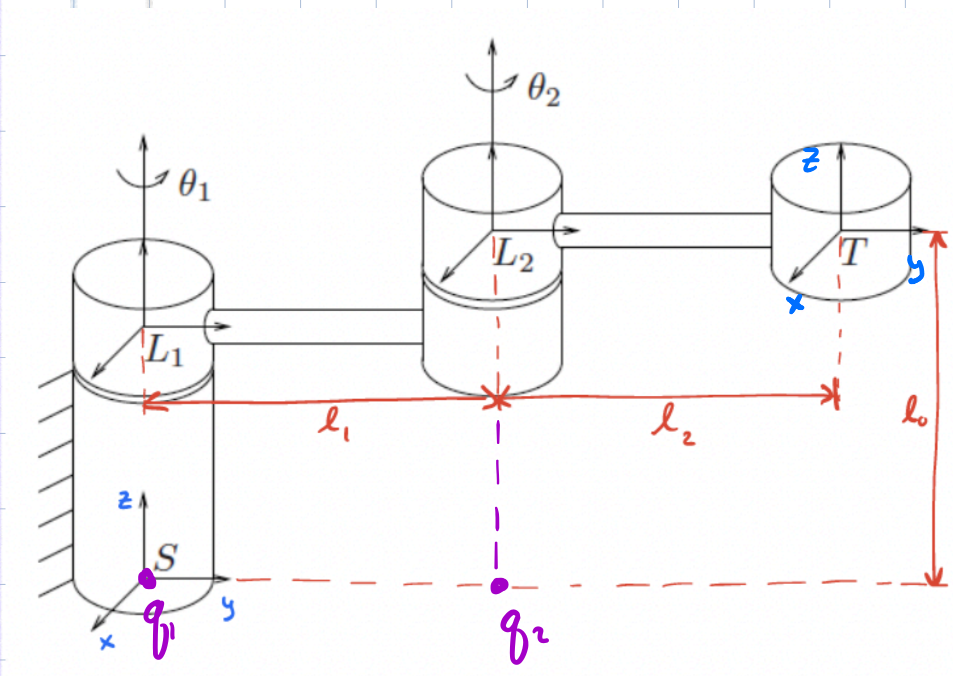

Figure 3.2: A two degree of freedom manipulator.

Step 1: What is $g_{st}(0)$?

$$g_{st}(0) = \begin{bmatrix} I & \begin{matrix} 0 \\ \ell_1+\ell_2 \\ \ell_0 \end{matrix} \\ 0 & 1 \end{bmatrix}$$

Step 2: What are the twists ξ_1, ξ_2?

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(a.) $\begin{cases} \xi_1 = [0, \ell_1, 0, 0, 0, 1]^T \\ \xi_2 = [0, \ell_1+\ell_2, 0, 0, 0, 1]^T \end{cases}$

(b.) $\begin{cases} \xi_1 = [-\ell_1, 0, 0, 0, 0, 1] \\ \xi_2 = [0, -\ell_1-\ell_2, 0, 0, 0, 1] \end{cases}$

(c.) $\begin{cases} \xi_1 = [\ell_1, 0, 0, 0, 0, 1]^T \\ \xi_2 = [\ell_1+\ell_2, 0, 0, 0, 0, 1]^T \end{cases}$

(d) None of these

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} \ell_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Steps 3 & 4 : Follow procedure...

II. Mathematica Examples

A. SCARA

B. Prosthetic Leg