Lesson 19:

I. Stability

Robot Control

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Nonlinear Systems

Consider an n-order homogeneous nonlinear system of the form

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n)\\ \dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n)\\ \vdots \quad & \quad \vdots\\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n)\end{aligned}$$

which we write in vector form as

$$\dot{x} = f(x)$$

where $x = [x_1, x_2, \ldots, x_n]^T$ and $f = [f_1, f_2, \ldots, f_n]^T$. f is called a **Vector Field** on R^n.
Note: The vector field $f(x)$ is **Linear** if $f(x)$ is of the form

$$f(x) = Ax$$

Equilibrium Points

Definition

An **Equilibrium** or **Equilibrium Point** of the vector field $f(x)$ is a vector x_0 satisfying $f(x_0) = 0$.

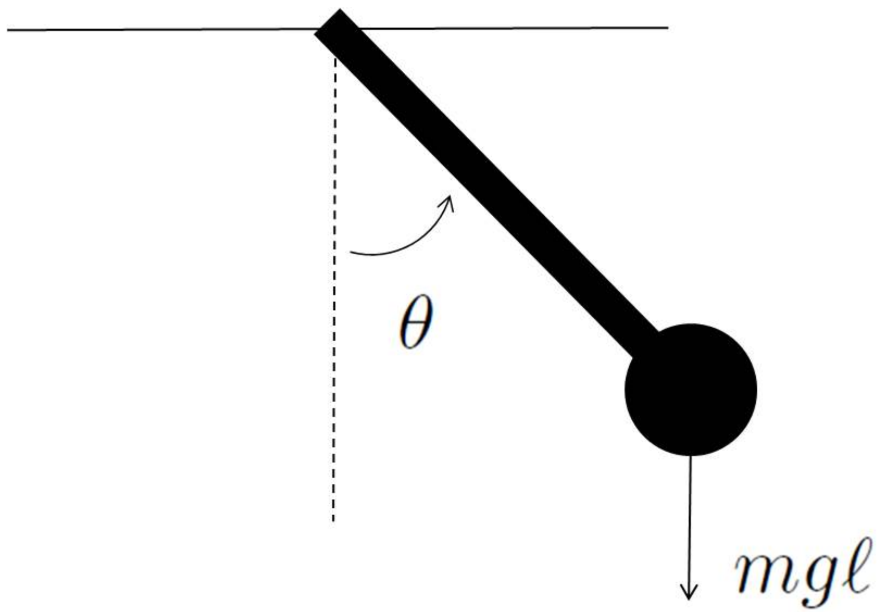
The reason that such an x_0 is called an equilibrium point is that if $x(0) = x_0$ is the initial condition for the differential equation $\dot{x} = f(x)$, then $x(t) = x_0$ is a solution of the differential equation.

Proof: If $x(t) = x_0$ then we have $\dot{x}(t) = 0 = f(x_0) = f(x(t))$ showing that $x(t) = x_0$ is a solution.

Other terms for equilibrium point are **Equilibrium Solution**, **Fixed Point**, **Critical Point**.

Simple Pendulum

Example



The equation for the simple pendulum is

$$m\ell^2\ddot{\theta} + mg\ell\sin(\theta) = 0$$

where m is the mass of the bob, g is the gravitational constant, and ℓ is the length of the pendulum. Dividing through by $m\ell^2$ we can write this as

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

Simple Pendulum

Given the equation for the simple pendulum

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

we can take as state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$ and write

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin(x_1) \end{aligned}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) \end{bmatrix}$$

Equilibrium Points of the Simple Pendulum

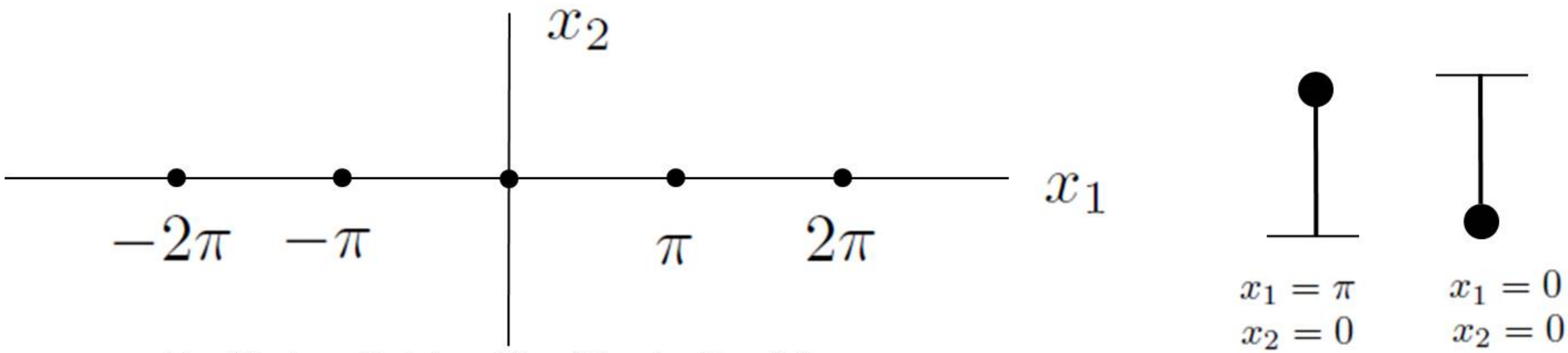
To find the equilibrium points of the simple pendulum equations, we set

$$\begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore we have the two equations

$$\begin{aligned} x_2 &= 0 \\ \sin(x_1) &= 0 \end{aligned}$$

which implies that $x_1 = 0, \pm\pi, \pm 2\pi, \ldots$ and $x_2 = 0$.



Equilibrium Points of the Simple Pendulum

Linearization of Nonlinear Systems

We will next discuss the linearization or linear approximation of nonlinear systems near equilibrium points. Such approximations often give useful information about the system for states near to the equilibrium.

Definition

Suppose $f(x) : R^n \to R^n$ is a vector field on R^n. The **Jacobian** or **Jacobian Matrix**, J of f is the $n \times n$ matrix of partial derivatives of $f(x)$,

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Example

Suppose a vector field $f(x)$ on R^3 is given as

$$f(x) = \begin{bmatrix} x_2 \\ x_3^2 \\ x_1 - \sin(x_2) \end{bmatrix}$$

Then the Jacobian of $f(x)$ is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2x_3 \\ 1 & -\cos(x_2) & 0 \end{bmatrix}$$

Note that the vector $[x_1, x_2, x_3]^T = [0, 0, 0]^T$ is an equilibrium of the vector field f.

Evaluating the Jacobian at this equilibrium point yields

$$J(0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Since the Jacobian is generally a function of x, it's value will be different at different equilibrium points. This will become important when we talk about the linear approximation of a nonlinear system near a given equilibrium point.

$\dot{x} = J(0)\, x$

Linearization

Definition

Suppose that $f(x_0) = 0$ and define $\delta x = x - x_0$. Then the **Linearization** or **Linear Approximation** of the system $\dot{x} = f(x)$ about (x_0) is the linear system

$$\delta\dot{x} = A\delta x$$

where A is the Jacobian of $f(x)$ evaluated at the equilibrium x_0.

In other words, the components a_{ij} of A are given by

$$a_{ij} = \frac{\partial f_i}{\partial x_j}(x_0)$$

Note that $\delta x = x$ when $x_0 = 0$ and so we will often simply write the linearized system as $\dot{x} = Ax$ even when $x_0 \neq 0$.

Linearization of the Simple Pendulum

The vector field defining the simple pendulum equation is

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) \end{bmatrix}$$

Therefore, the Jacobian of $f(x)$ is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos(x_1) & 0 \end{bmatrix}$$

Evaluating the Jacobian at the two equilibrium points $(0, 0)$ and $(\pi, 0)$ leads to the two linear approximations, each valid near the respective equilibrium points

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{bmatrix} x \quad ; \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} x$$

The Harmonic Oscillator

An important example of a periodic linear system is the **Harmonic Oscillator**

$$\ddot{\theta} + \omega^2\theta = 0$$

Setting $x_1 = \theta$ and $x_2 = \dot{\theta}$ we obtain the state equations

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} x$$

We note that the harmonic oscillator is identical to the linear approximation of the simple pendulum equation at the equilibrium $(0, 0)$ with $\omega^2 = \frac{g}{\ell}$.

Stability

Stability theory deals with the behavior of systems near an equilibrium. Intuitively, if the initial condition for a given (autonomous) differential equation is "close" to an equilibrium, the resulting trajectory can

- remain near the equilibrium (we call this behavior **stable**).
- converge to the equilibrium (we call this behavior **asymptotically stable**).
- diverge away from the equilibrium (we call this behavior **unstable**).

We consider an equation of the form

$$\dot{x} = f(x) \ ; \ x \in R^n \tag{1}$$

where the vector field $f(x)$ is continuously differentiable and we suppose that $\bar{x} = 0$ is an equilibrium point.

Stability Definitions

Definition

The equilibrium point $\bar{x} = 0$ is

1. **Stable** if for any $\epsilon > 0$ there exists

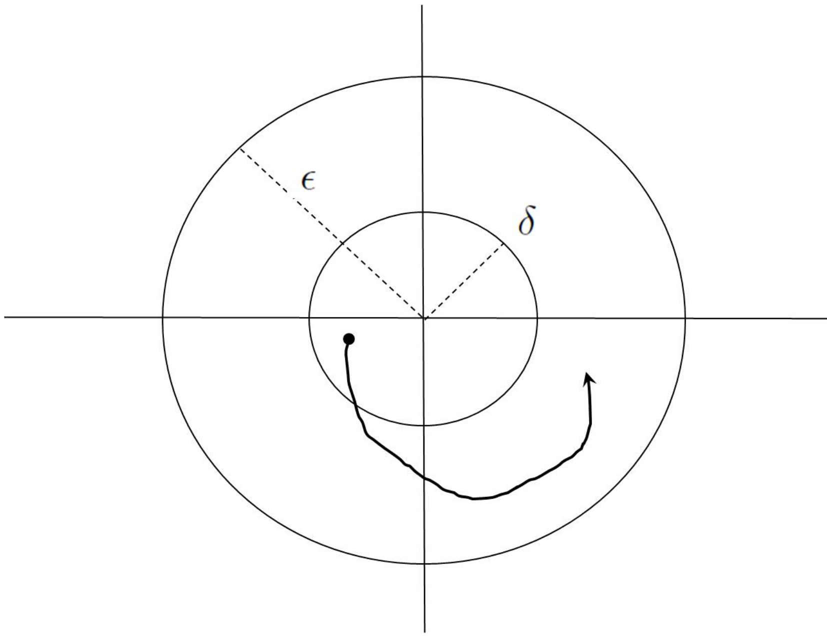
   $\delta > 0$ such that

   $||x(0)|| < \delta \Longrightarrow ||x(t)|| < \epsilon$, for all $t \geq 0$

2. **Asymptotically Stable** if it is stable and, in addition,

$$\lim_{t \to \infty} x(t) = 0$$

3. **Globally Asymptotically Stable** if it is asymptotically stable for **all** initial conditions $x(0) = x_0 \in R^n$.
4. **Unstable** if it is not stable.



Exponential Stability

Definition

The equilibrium $\bar{x} = 0$ is **Exponentially Stable** if trajectories satisfy

$$||x(t)|| \leq ke^{-\lambda t}||x_0|| \text{ for some positive constants } k > 0, \lambda > 0.$$

The exponential stability is local or global depending on whether or not the condition holds for all initial conditions or only for initial conditions in some region around the equilibrium.

Remark

For a linear system, exponential stability and asymptotic stability are the same thing since asymptotically stable solutions for linear systems contain exponential factors. However, the two notions are distinct for nonlinear systems. This is important.

Hyperbolic Case

Theorem

Consider the autonomous nonlinear system

$$\dot{x} = f(x)$$

and suppose that $\bar{x} = 0$ *is a hyperbolic fixed point. Recall that this means that the Jacobian* $A = \frac{\partial f}{\partial x}$ *has no eigenvalues with real part equal to zero. Let*

$$\dot{x} = Ax$$

be the linearization of the nonlinear system at $\bar{x} = 0$.

1. *If* A *has all eigenvalues with negative real part, then* $\bar{x} = 0$ *is exponentially stable for the nonlinear system.*
2. *If* A *has any eigenvalue with positive real part, then* $\bar{x} = 0$ *is unstable for the nonlinear system.*
3. *If* A *has any eigenvalue with zero real part, then* $\bar{x} = 0$ *may be stable, asymptotically stable, or unstable for the nonlinear system. However,* $\bar{x} = 0$ *cannot be exponentially stable for the nonlinear system.*

Examples

$J = \left[\frac{\partial}{\partial x}(-x^3)\right] = -3x^2$

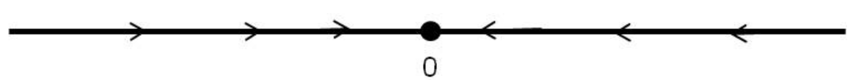
Consider the first-order autonomous nonlinear system

$J(0) = 0$

$$\dot{x} = -x^3$$

$\dot{x} = 0x = 0$

It is easy to see by sketching the phase portrait that $\bar{x} = 0$ is globally asymptotically stable. However it is not exponentially stable since the linearized system is $\dot{x} = 0$.



Phase Portrait for $\dot{x} = -x^3$

Examples

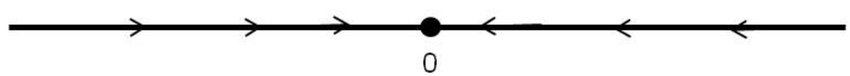
$J = \frac{d}{dx}(-x-x^3) = -1-3x^2$

Next, consider the first-order autonomous nonlinear system $J(0) = -1$

$$\dot{x} = -x - x^3$$

$\dot{x} = (-1)x$

In this case the linearized system is $\dot{x} = -x$ and so the equilibrium $\bar{x} = 0$ is exponentially stable. In fact $\bar{x} = 0$ is globally exponentially stable.



Phase Portrait for $\dot{x} = -x - x^3$

Examples

Now consider the first-order autonomous nonlinear system

$J(0) = -1$

$$\dot{x} = -x + x^3$$

In this case the linearized system is $\dot{x} = -x$ and so the equilibrium $\bar{x} = 0$ is exponentially stable. However, $\bar{x} = 0$ is only locally exponentially stable, not globally exponentially stable. $\bar{x} = 0$ is not even globally asymptotically stable.

Examples

Now consider the first-order autonomous nonlinear system

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Can you see why?

Examples

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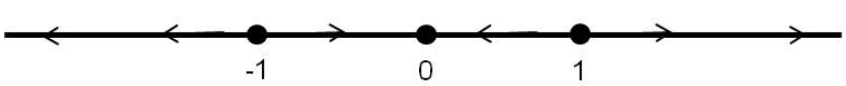
Can you see why?

There are two other equilibrium points at $x = \pm 1$, both of which are unstable.



Phase Portrait for $\dot{x} = -x + x^3$

Lyapunov Stability Theory

Remark

The good news is that for hyperbolic equilibrium points the linearization tells us everything we need to know to determine local stability properties of the nonlinear system.

However, the linear theory is not adequate to address the issue of global stability and it does not tell us much about the stability properties in the non-hyperbolic case.

Thus we need to develop additional techniques beyond simply computing eigenvalues of the linearization. The so-called **Second Method of Lyapunov** *is one such method that is a very powerful tool for analyzing nonlinear systems.*

Motivating Example

Lyapunov developed his methods as a generalization of the concept of energy dissipation in mechanical systems. To see what we mean, consider the velocity $v(t)$ of a particle of mass m acted on by a dissipative force that is proportional to the velocity (like friction). By Newton's Second Law we can write

$$m\dot{v} = -bv \tag{2}$$

where m and b are positive constants.

Now the kinetic energy of the particle $E(t)$ is given by $E = \frac{1}{2}mv^2$. If we compute the rate of change of energy for the particle we obtain

$$\dot{E}(t) = mv(t)\dot{v}(t) = -bv^2(t)$$

Motivating Example

Therefore, since $\dot{E} = -bv^2$ is always negative, it means that the kinetic energy of the particle is always decreasing. Since E itself is positive, it cannot decrease below zero. It follows that

$$E(t) \to 0 \qquad \text{as } t \to \infty$$

which implies that the velocity

$$v(t) \to 0 \qquad \text{as } t \to \infty$$

Thus we can conclude that the equilibrium $\bar{v} = 0$ of the equation (2) is asymptotically stable without computing eigenvalues.

Lyapunov recognized that the same conclusion could be drawn by using other functions, not just energy, provided they have similar properties as energy, namely, a scalar function of the state that is zero at the equilibrium, positive at other values of the state, and decreasing along trajectories of the system as $t \to \infty$.

Lyapunov Functions

Definition

Let $V(x) : R^n \to R$ be a differentiable scalar function. The **derivative of V along trajectories of (1)**, denoted $\dot{V}(x)$ is given by

$$\begin{aligned} \dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i \\ &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \end{aligned}$$

xdot=f(x) (1)

Other notation used for this:

$$\begin{aligned} \dot{V} &= \nabla V \cdot f \\ &= \frac{\partial V}{\partial x} f(x) \\ &= L_f V \end{aligned}$$

Stability in the Sense of Lyapunov

Theorem

Let $\bar{x} = 0$ be an equilibrium point of (1) and let V be a continuously differentiable scalar function such that

(a) $V(0) = 0$ *and* $V(x) > 0$ *for* $x \neq 0$. (positive definite function)

(b) $\dot{V}(x) \leq 0$ (negative semi-definite)

Then $\bar{x}$ is stable. Moreover, if

(c) $\dot{V} < 0$ $\quad$ Vdot(0) = 0 (negative definite function)

then $\bar{x}$ is asymptotically stable.

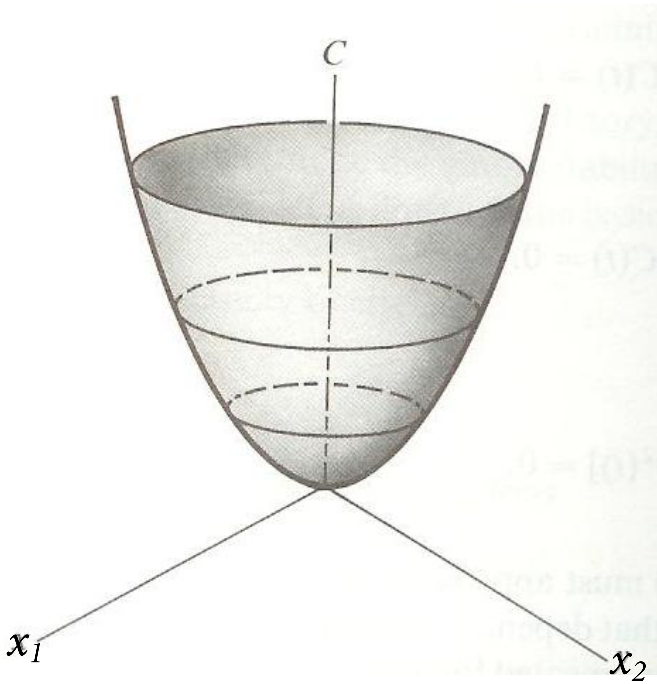
Example

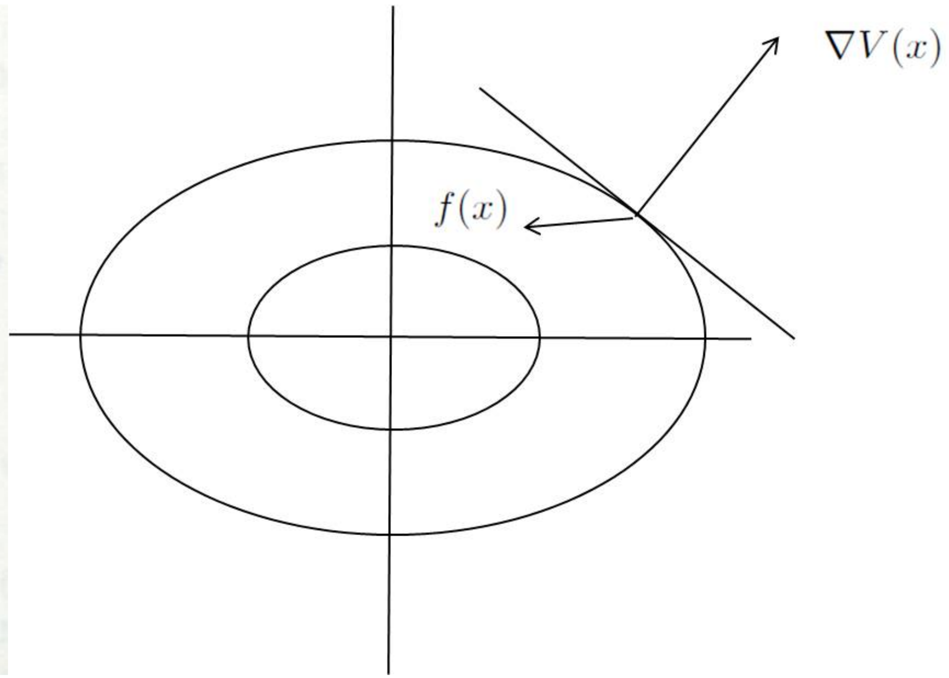
Consider the system $\dot{x} = -x$ and suppose $V = x^2$. Then $\dot{V} = 2x\dot{x} = -2x^2 < 0$. Therefore $\bar{x} = 0$ is asymptotically stable.

Lyapunov Function

The function $V(x)$ is called a **Lyapunov Function** for the system (1).

The intuitive idea behind the notion of Lyapunov stability is shown in the figure below. $\dot{V} = \nabla V \cdot f$ is the ‘dot’ product of the vector field f with the gradient of V. If this dot product is negative, then the vector field f must point inside the level curves of V as shown.



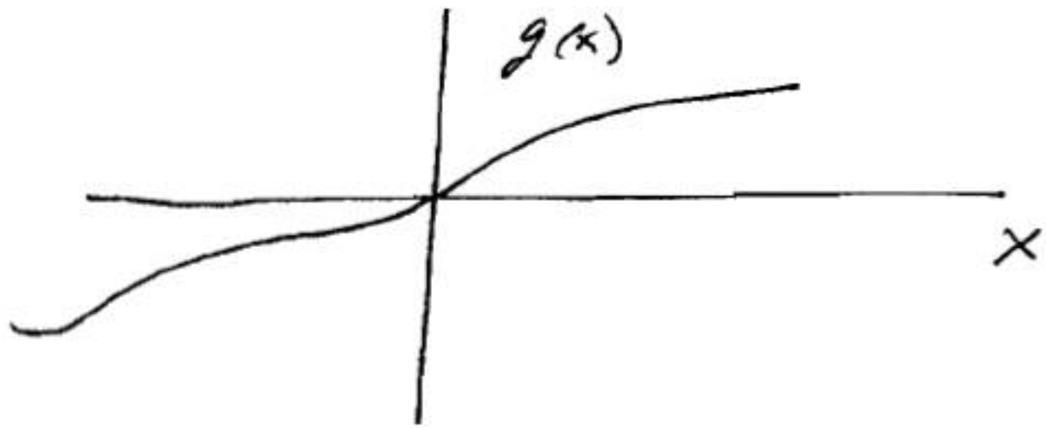


Example

Consider the first-order system

$$\dot{x} = -g(x)$$

where $g(x)$ is any nonlinear function with $g(0) = 0$ and $xg(x) > 0$ for $x \neq 0$.



Choose a Lyapunov function $V = \frac{1}{2}x^2$. Then

$$\dot{V} = x\dot{x} = -xg(x) < 0$$

Therefore, the origin $x = 0$ is asymptotically stable.

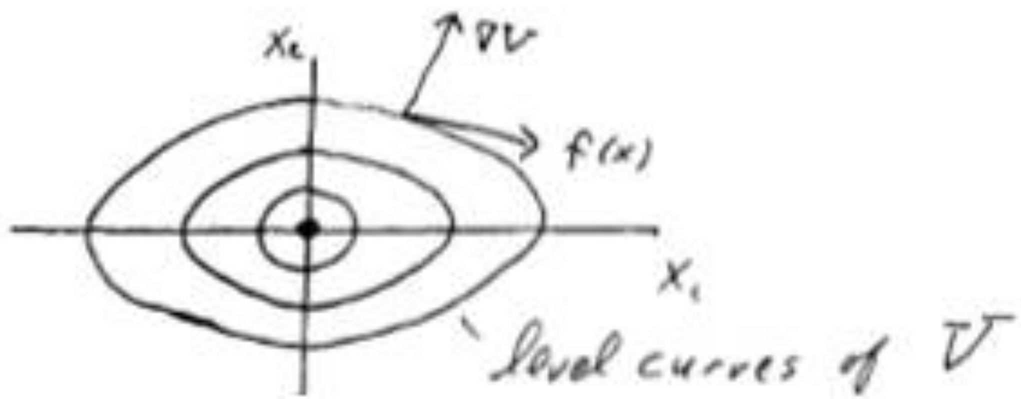
Simple Pendulum

We have already seen that if $E(t)$ is the total energy of the pendulum (kinetic plus potential)

$$E = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1-\cos(\theta))$$

$$\begin{aligned}\dot{E} &= m\ell^2\dot{\theta}\cdot\ddot{\theta} + mg\ell\sin(\theta)\dot{\theta} \\ &= (-mg\ell\sin(\theta) + mg\ell\sin(\theta))\dot{\theta} = 0\end{aligned}$$

Thus, $V = E$ satisfies $V(0) = 0$ and $V > 0$ near the equilibrium $(0,0)$ and is therefore a Lyapunov function for the simple pendulum equation and shows that the origin is stable.



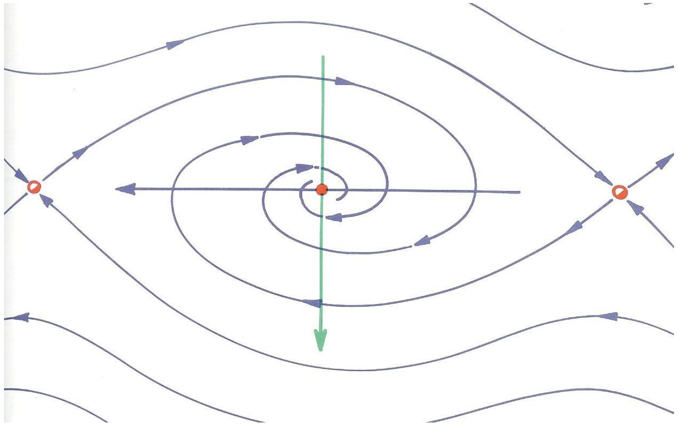
In fact, $\dot{V} = 0$ means that the vector field of the simple pendulum is tangent to the level curves of V, which are therefore trajectories of the system.

Note that the linearization of the simple pendulum at the origin is a center, which is not hypberbolic and so stability of the origin is not implied by the linear approximation.

Note also that the condition $\dot{V} \leq 0$ or $\dot{V} <$ is only a sufficient condition for **Local** stability or local asymptotic stability. Checking global stability required additional work.

Pendulum with Friction

Consider again the simple pendulum with friction



$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin(x_1) - Bx_2\end{aligned}$$

We know by examining the phase portrait above and also by physics that the origin is asymptotically stable since the friction will eventually dissipate all the energy and bring the pendulum to rest.

Pendulum with Friction

Let's examine the Lyapunov function

$$V = E = \frac{1}{2}m\ell^2 x_2^2 + mg\ell(1 - \cos(x_1))$$

Computing $\dot{V}$ along trajectories of the system gives

$$\dot{V} = -m\ell^2 B x_2^2 \leq 0$$

not negative definite because it doesn't include x_1.

Therefore we can conclude only that the origin is stable, not asymptotically stable.

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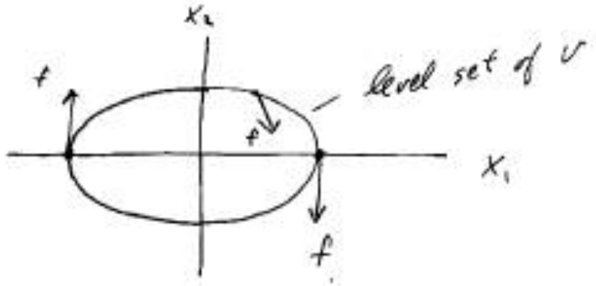


In this case, when the velocity $x_2^2 = 0$ the vector field is tangent to the energy level curves and so does not point inside the level set.

Lyapunov Functions

The previous example illustrates a couple of important principles

1. Lyapunov's Theorem gives only a sufficient condition for stability.
2. If a particular function V fails to show that the equilibrium is stable, it does not mean that the equilibrium is unstable. It may be that a different Lyapunov function will give a different conclusion.
3. Finding the right Lyapunov function can be difficult

Summary so Far

1. We can determine local stability properties for hyperbolic fixed points of nonlinear systems by examining the eigenvalues of the linear approximation.
   1.1 The fixed point is asymptotically stable for the nonlinear system if it is asymptotically stable for the linear systems (all eigenvalues have negative real part)
   1.2 The fixed point is unstable for the nonlinear system if it is unstable for the linear system. (one or more eigenvalues with positive real part)
   1.3 If any eigenvalue of the linear approximation has zero real part (critical eigenvalues) then the fixed point of the nonlinear system may be stable or unstable, i.e. the linearization is inconclusive.
2. Lyapunov functions can be used to determine stability in the case of critical eigenvalues

Basin of Attraction

Since stability is, in general, a local property, the question arises 'which initial conditions give rise to trajectories that converge to the equilibrium,'

Definition

The **Basin of Attraction** $\mathcal{B}$ of a fixed point $\bar{x}$ is defined as the set

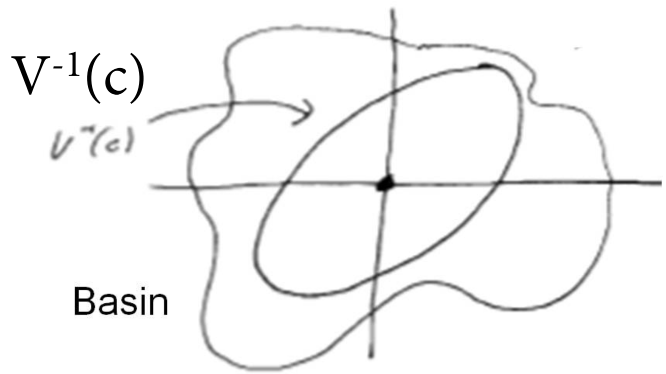
$$\mathcal{B} = \{y \in R^n \mid \lim_{t \to \infty} x(t, y) = \bar{x}\}$$

where $x(t, y)$ means the trajectory beginning at initial condition $y \in R^n$.

Other names for the basin of attraction are **Region of Attraction**, **Domain of Attraction**, or simply **Basin**. In general, the basin of attraction is difficult to determine.

Estimating the Basin of Attraction

Lyapunov functions can be used to provide estimates of the basin of attraction.



$\dot{V} < 0$ means that the interior of the level set is positively invariant. This means that any solution starting in this set converges to the equilibrium and so the set is contained within the basin of attraction.

Global Stability

Question

Under what conditions is the basin of attraction equal to the entire state space R^n?

This is called **Global Asymptotic Stability**.

Clearly, a necessary condition for global asymptotic stability is that $\bar{x}$ is the only equilibrium point.

It turns out that a simple additional condition on the function V is needed to ensure global asymptotic stability.

Definition

A scalar function $V : R^n \to R$ is **Radially Unbounded** if and only if

$$V(x) \to \infty \text{ as } ||x|| \to \infty$$

The Global Case

Theorem

Let $\bar{x} = 0$ be an equilibrium point for the autonomous nonlinear system $\dot{x} = f(x)$. Let $V : R^n \to R$ be a continuously differentiable function such that

$$V(0) = 0, \ V(x) > 0 \text{ for all } x \neq 0$$
$$V(x) \to \infty \text{ as } ||x|| \to \infty$$
$$\dot{V}(x) < 0 \text{ for all } x \neq 0$$

The $\bar{x} = 0$ is globally asymptotically stable.

The Invariance Principle

We next give an extension to Lyapunov's Theorem due to LaSalle that allows additional conclusions about the asymptotic behavior of the system. A weak form of LaSalle's theorem can be stated as

Theorem

Suppose $V : R^n \to R$ is a positive definite scalar function and suppose $\dot{V} \leq 0$ along trajectories of (1). If $\dot{V}$ does not vanish identically along trajectories of (1) other than the equilibrium solution, then $\bar{x} = 0$ is asymptotically stable.

xdot = f(x)   (1)

Example

Back to the simple pendulum with friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin(x_1) - Bx_2 \end{aligned}$$

We know that with

$$V = \frac{g}{\ell}(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

satisfies

$$\dot{V} \quad = \text{-}Bx_2^2 \leq 0$$

Suppose $\dot{V} \equiv 0$. Then $x_2 \equiv 0 \implies \dot{x}_2 \equiv 0$
$\implies \sin(x_1) \equiv 0 \implies x_1 = n\pi$. *we consider local stability with -pi < x1 < pi
Therefore, $\dot{V}$ vanishes identically only at the equilibrium point and so $\bar{x} = 0$ is asymptotically stable.

LaSalle's Theorem

We next present a stronger version of LaSalle's Theorem.

Theorem

Suppose V is a Lyapunov function satisfying $\dot{V} \leq 0$ along trajectories of $\dot{x} = f(x)$. Let E be the set of points where $\dot{V} = 0$ and let M be the largest invariant set contained in E. Then all trajectories of the system approach M as $t \to \infty$.

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Example

Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1(1 + x_2) \\ \dot{x}_2 &= x_1^2 \end{aligned}$$

Example

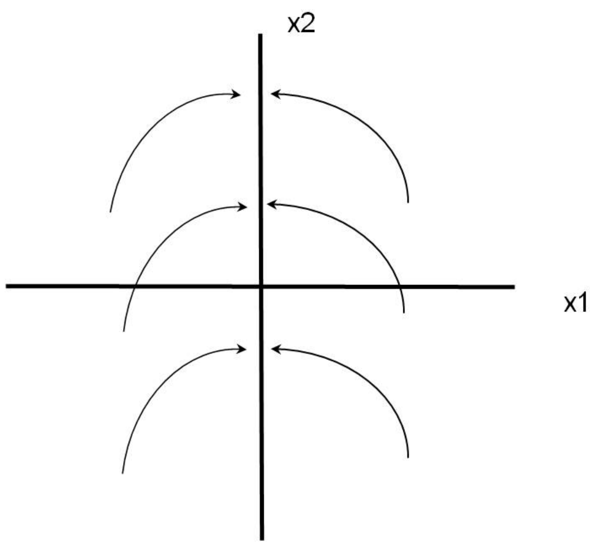
Let

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

Then

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= -x_1^2(1+x_2) + x_2x_1^2 = -x_1^2 \leq 0\end{aligned}$$

Therefore $E = \{x_1 = 0\}$, which is also an invariant set. Therefore, all trajectories approach E as $t \to \infty$.



Stability of Linear Systems

We have been using functions of the form

$$\mathcal{V} = x_1^2 + x_2^2 + \cdots + x_n^2$$

as potential Lyapunov functions in several examples. The 'sum of squares' or 'norm' function is called a **Quadratic Form**. A more general quadratic form can be written as

$$V = x^T P x \text{ where } x \in R^n \text{ and } P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

P is an $n \times n$ constant matrix. To qualify as a Lyapunov function, the matrix P should satisfy $x^T P x > 0$ for $x \neq 0$.

Positive Definite Matrices

Definition

An $n \times n$ matrix P is **Positive Definite** if and only if

$$x^T P x > 0 \text{ for } x \neq 0$$

The matrix P is

- **Positive Semi-definite** if $x^T P x \geq 0$
- **Negative Definite** if $x^T P x < 0$
- **Negative Semi-Definite** if $x^T P x \leq 0$
- P is **Indefinite** if it is otherwise not one of the above.

Positive Definite Matrices

Remark

- *The matrix P is positive definite if and only if all eigenvalues of P are positive.*
- *P is positive semi-definite if all eigenvalues of P are greater than or possibly equal to zero.*

Given an n-dimensional linear system

$$\dot{x} = Ax$$

let us try to find a quadratic Lyapunov function of the form

$$V = x^T P x$$

Stability of Linear Systems

Computing the derivative of $V = x^T P x$ along trajectories of $\dot{x} = Ax$ yields

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + P A) x \end{aligned}$$

Let's define $-Q = A^T P + P A$. Then we have

$$\begin{aligned} V &= x^T P x \\ \dot{V} &= -x^T Q x \end{aligned}$$

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$$\begin{aligned} V &= x^T P x \\ \dot{V} &= -x^T Q x \end{aligned}$$

Suppose both P and Q are positive definite. What can you conclude? The conclusion is that the linear system is (exponentially) stable.

The Lyapunov Equation

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where P is the unknown and A and Q are given. Suppose that we solve Equation (1) for P and find that P is positive-definite. What conclusions can you draw?

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The equation (1) is called a **Lyapunov Equation**.

Stability of Linear Systems

The leads to the following result to determine stability of linear systems

Theorem

(globally) asymptotically stable

The $n \times n$ matrix A is ~~stable~~ (i.e. all eigenvalues of A have negative real part) if and only if for any symmetric positive definite $n \times n$ matrix Q there is a unique symmetric positive definite matrix P satisfying the Lyapunov Equation

$$A^T P + PA + Q = 0$$

Remark

This gives a way to determine stability of linear systems by solving a system of linear algebraic equations; in other words, without computing matrix eigenvalues.

Example

Remark

The Matlab Control Systems Toolbox has the built-in function `lyap`*, which solves the Lyapunov equation.*

Matlab Script

$$>> \; A = [0 \; 1; \; -2 \; -3];$$
$$>> \; Q = [1 \; 0; \; 0 \; 1];$$
$$>> \; P = lyap(A, Q)$$
$$>> \; P =$$
$$\begin{matrix} 1.0000 & -0.5000 \\ -0.5000 & 0.5000 \end{matrix}$$
$$>> \; eig(P)$$

ans $=$

0.1910

1.3090