# ROB 510 Exam-I Solutions

Winter 2022 (Prof. Gregg)

## Problem 1:

(a) True. Since the determinant of a matrix equals to the product of its eigenvalues and determinant of $R$ is 1.

(b) True. Another way to think about this is $R_1 = R_{ba}$ and $R_2 = R_{bc}$. We know that $R_{ca} = R_{cb}R_{ba} = R_2^T R_1$ because $R_2^T = R_2^{-1}$ by orthogonality.

(c) False. Orthogonal 3x3 matrices with determinant equal -1 represent reflections rather than rotations.

(d) False. $\omega$ can be arbitrary if $R = I$.

## Problem 2:

(a) True as discussed on Piazza. From Lemma 2.3, we can write $\hat{w}^2 = ww^T - \|w\|^2 I$. Let $w \times v = \hat{w}v$ and we know $v = -w \times q$, then $(I - e^{\hat{w}\theta})(w \times v) = (I - e^{\hat{w}\theta})\hat{w}v = -(I - e^{\hat{w}\theta})\hat{w}^2 q = -(I - e^{\hat{w}\theta})(ww^T - \|w\|^2 I)q$. Note that $\|w\|^2 = 1$ by construction and $(I - e^{\hat{w}\theta}) = -\hat{w}\theta - \frac{\hat{w}^2\theta^2}{2!} - ....$ In addition, we know $\hat{w}w = w \times w = 0$. Hence, $(I - e^{\hat{w}\theta})\hat{w}v = -(I - e^{\hat{w}\theta})(ww^T - ||w||^2 I)q = -(I - e^{\hat{w}\theta})(-I)q = (I - e^{\hat{w}\theta})q$.

(b) False in the case of a screw motion.

(c) True. $\dot{q} = \hat{\omega}\left(\frac{\omega \times v}{\|\omega\|^2}\right) + v = \frac{\hat{\omega}^2 v}{\|\omega\|^2} + v = \frac{\omega\omega^T v}{\|\omega\|^2} - v + v = \omega\frac{\omega^T v}{\|\omega\|^2}$, since $\frac{\omega^T v}{\|\omega\|^2}$ is a scalar, the vector field at $q$ points directly along the screw axis (i.e., direction of $w$).

(d) False. The rotational case has $\|\omega\| = 1$ with non-zero $v$ in general, and the translational case has $\|v\| = 1$ with $\omega = 0$ in the unit twist. Hence only the translational unit twist has a norm equal to one.

## Problem 3:

(a) False as discussed in Q@36 on Piazza. This operation would transform a point expressed with respect to the current tool frame to the coordinates of the spatial frame. The problem statement would be true if we replaced the operation with $e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n}\bar{p}$ (not including $g_{st}(0)$).

(b) True. This is an extension of the property shown in Lecture 9, where we account for switching the order of the product of exponentials.

(c) False. Would need to use the twist $\hat{\xi}_{1,2}$ defined in the coordinates of frame $L_1$ (see Lecture 9).

(d) True. This is known as the product of exponential in the tool frame. Let $g = g_{st}(0)$, from $\xi_i' = Ad_g^{-1}\xi_i = g^{-1}\xi_i g$ and $e^{g^{-1}\hat{\xi}g} = g^{-1}e^{\hat{\xi}}g \Longrightarrow ge^{g^{-1}\hat{\xi}g} = e^{\hat{\xi}}g$, we can write:

$$
\begin{aligned}
g_{st}(\theta) &= e^{\hat{\xi_1}\theta_1} \cdots e^{\hat{\xi_n}\theta_n} g \\
&= e^{\hat{\xi_1}\theta_1} \cdots g e^{g^{-1}\hat{\xi_n} g\theta_n} \\
&= e^{\hat{\xi_1}\theta_1} \cdots g e^{g^{-1}\widehat{\xi_{n-1}} g\theta_{n-1}} e^{g^{-1}\hat{\xi_n} g\theta_n} \\
&= g e^{g^{-1}\hat{\xi_1} g\theta_1} \cdots e^{g^{-1}\widehat{\xi_{n-1}} g\theta_{n-1}} e^{g^{-1}\hat{\xi_n} g\theta_n} \\
&= g e^{\hat{\xi_1'}\theta_1} \cdots e^{\widehat{\xi_{n-1}'}\theta_{n-1}} e^{\hat{\xi_n'}\theta_n} \\
&= g_{st}(0) e^{\hat{\xi_1'}\theta_1} \cdots e^{\widehat{\xi_{n-1}'}\theta_{n-1}} e^{\hat{\xi_n'}\theta_n}
\end{aligned}
$$

## Problem 4:

(a) True. Let $e^1 \cdots e^6 = g_d g_{st}^{-1}(0) := g_1$, where $g_1$ is known. If we choose a point $p$ at the intersection of axes 3, 4, 5, and 6 (known by backtracking from the tool frame at $g_d$), then $e^3 e^4 e^5 e^6 p = p$ noting that the prismatic joint does not change alignment of the rotational axes. Then $e^1 e^2 p = g_1 p$, which fits the form of SP2.

(b) True. Geometrically, it does not matter if you rotate then translate along the same axis and vice versa. Therefore, $e^{\hat{\xi_3}\theta_3} e^{\hat{\xi_4}\theta_4} = e^{\hat{\xi_4}\theta_4} e^{\hat{\xi_3}\theta_3}$.

(c) False. Given the rest of the joints, we can pick a point $q$ not on the axis of $\xi_3$ and apply SP1 to find out $\theta_3$, which gives a unique solution.

(d) True. The possible solutions will not change if the inverse kinematics problem is solved correctly.

## Problem 5:

(a) True. The dot product of twist and wrench is the power, which is a scalar and is not dependent on the coordinate frame.

(b) True. $Ad_{e^{\hat{\xi_1}\theta} e^{\hat{\xi_2}\theta}} = Ad_{e^{\hat{\xi_1}\theta}} Ad_{e^{\hat{\xi_2}\theta}} = \begin{bmatrix} R_1 R_2 & R_1 \hat{p}_2 R_2 + \hat{p}_1 R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix}$.

(c) False. $F_p = \sum Ad^T_{g_{pc_i}^{-1}} F_{c_i}$ and the adjoint transformation varies with the location of the frame.

(d) False. The Jacobian always maps joint velocity to tool velocity. However, the Jacobian transpose maps tool wrench to joint torque. The other direction requires invertibility of the Jacobian, which isn't possible with $n < 6$. Even a pseudo-inverse would not yield a unique solution.

## Problem 6:

(a) The twists and reference configuration are defined as follows:

$$\xi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} -\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

$$\xi_4 = \begin{bmatrix} -\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \xi_5 = \begin{bmatrix} -\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix} \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_1 + l_2 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

$$\xi_6 = \begin{bmatrix} -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix} \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$g_{st}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 + l_2 + l_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) The formula for $v$ changes to $v = -\omega \times q + h\omega$ for pitch $h = 0.01/2\pi$. Hence, in this case,

$$\xi_6 = \begin{bmatrix} -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix} + (0.01/2\pi) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.01/2\pi \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(c) First we need to calculate $g_{s\ell_2}(\theta)$ through the product of exponentials. This starts by defining the reference configuration of frame $L_2$ as $g_{s\ell_2}(0) = I_{4\times 4}$ because the assumption of zero length for link 2 means its frame initially aligns with the spatial frame. We also need to calculate the following matrix exponentials:

$$e^{\hat{\xi}_1\theta_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{\hat{\xi}_2\theta_2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then,

$$g_{s\ell_2}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} g_{s\ell_2}(0) = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 & \theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we extract the twist coordinates following Proposition 2.9. In particular, we determine by inspection of the rotation matrix (or through Proposition 2.5) that $\omega = (0, 0, 1)^T$ with angle $\theta_2$. Then we build matrix $A$ to solve $Av = p$ for $v$:

$$A = (I - e^{\hat{\omega}\theta})\hat{\omega} + \omega\omega^T\theta_2 = \begin{bmatrix} \sin\theta_2 & \cos\theta_2 - 1 & 0 \\ 1 - \cos\theta_2 & \sin\theta_2 & 0 \\ 0 & 0 & \theta_2 \end{bmatrix} \implies v = A^{-1}p = A^{-1} \begin{bmatrix} 0 \\ \theta_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\theta_1 \\ \frac{1}{2}\theta_1 \cot\left(\frac{1}{2}\theta_2\right) \\ 0 \end{bmatrix}$$

Hence, $\xi(\theta_1, \theta_2) = \begin{bmatrix} \frac{1}{2}\theta_1 \\ \frac{1}{2}\theta_1 \cot\left(\frac{1}{2}\theta_2\right) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Note that multiplying $\xi$ by $\theta_2$ would not give a unit twist in general, but I only took off one point for that oversight. It can be verified that $e^{\hat{\xi}(\theta_1,\theta_2)\theta_2} = g_{s\ell_2}(\theta)$.

## Problem 7:

(a) To find the columns of $J^b_{st}$ we need to find the twist coordinates with respect to the current tool frame. The first one can be found by inspection: $\omega_3^\dagger = \omega_3 = (-1,0,0)^T$, $q_3^\dagger = (0,-l_2,0)^T$ implying

$$\xi_3^\dagger = (0,0,-l_2,-1,0,0)^T.$$

This could also have been obtained by $\xi_3^\dagger = \mathrm{Ad}^{-1}_{\left(e^{\hat{\xi}_3\theta_3}g_{st}(0)\right)}\xi_3$, where $\xi_3 = (0,-l_0,l_1,-1,0,0)^T$ and

$$g_{st}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1+l_2 \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Noting that $v_2 = (0,1,0)^T$ with respect to S, the change of coordinates can be found by simply accounting for the 3rd joint rotation (which changes the orientation of T relative to S): $v_2^\dagger = e^{-\hat{\omega}_3\theta_3}v_2 = (0,\cos\theta_3,\sin\theta_3)^T$, implying

$$\xi_2^\dagger = (0,\cos\theta_3,\sin\theta_3,0,0,0)^T.$$

This could also have been obtained by $\xi_2^\dagger = \mathrm{Ad}^{-1}_{\left(e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}g_{st}(0)\right)}\xi_2$, where $\xi_2 = (0,1,0,0,0,0)^T$.

Noting that $\xi_1 = (0,0,0,0,0,1)^T$ with respect to S, the change of coordinates is best handled with the Adjoint transformation:

$$\xi_1^\dagger = \mathrm{Ad}^{-1}_{\left(e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}g_{st}(0)\right)}\xi_1 = \left(-(l_1+\theta_2) - l_2\cos\theta_3, 0, 0, 0, -\sin\theta_3, \cos\theta_3\right)^T.$$

Hence,

$$J^b_{st}(\theta) = \begin{bmatrix} -(l_1+\theta_2) - l_2\cos\theta_3 & 0 & 0 \\ 0 & \cos\theta_3 & 0 \\ 0 & \sin\theta_3 & -l_2 \\ 0 & 0 & -1 \\ -\sin\theta_3 & 0 & 0 \\ \cos\theta_3 & 0 & 0 \end{bmatrix}$$

(b) No, because the columns of $J^b_{st}(\theta) \in \mathbb{R}^{6\times 3}$ cannot become linearly dependent for any $\theta$, so rank cannot drop below 3. This is a little tricky because when the elbow (joint 3) is pointing straight up or down, the end-effector loses the instantaneous direction of motion along the z-axis. However, this doesn't correspond to a rank drop because $J^b_{st}(\theta)$ already has rank below 6 by virtue of the 3-DOF configuration space.

## Problem 8:

(a) The problem statement gives us $\omega = (0, 1/\sqrt{2}, 1/\sqrt{2})^T$ and $\theta = \pi$. We use Rodrigues formula to calculate the rotation matrix associated with the equivalent axis and angle. First note that

$$\hat{\omega} = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 \end{bmatrix} \implies \hat{\omega}^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix}$$

Then,

$$R = e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta) = I + 2\hat{\omega}^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(b) We first need to calculate the rotation matrix in terms of the Euler angles $\alpha$, $\beta$, and $\gamma$ using elementary Euler rotation matrices:

$$\begin{aligned} R(\alpha,\beta,\gamma) = R_y(\alpha)R_z(\beta)R_x(\gamma) &= \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha\cos\beta & * & * \\ \sin\beta & \cos\beta\cos\gamma & -\cos\beta\sin\gamma \\ -\sin\alpha\cos\beta & * & * \end{bmatrix}, \end{aligned}$$

where $*$ represents terms that we do not need later.

Setting $R(\alpha,\beta,\gamma) = R$ from part (a), we can now solve for the Euler angles:

$r_{21} = \sin\beta = 0 \implies \beta = 0$ or $\pi$.

Then, $\gamma = \text{atan2}(-r_{23}/\cos\beta, r_{22}/\cos\beta)$ implies

$\gamma = \text{atan2}(-1/1, 0/1) = 3\pi/2 = -\pi/2$ when $\beta = 0$

or

$\gamma = \text{atan2}(-1/(-1), 0/(-1)) = \pi/2$ when $\beta = \pi$

Finally, $\alpha = \text{atan2}(-r_{31}/\cos\beta, r_{11}/\cos\beta)$ implies

$\alpha = \text{atan2}(0, -1/1) = \pi$ when $\beta = 0$

or

$\alpha = \text{atan2}(0, -1/(-1)) = 0$ when $\beta = \pi$.

Hence, the two possible solutions are $(\alpha,\beta,\gamma) = (\pi, 0, 3\pi/2)$ or $(\alpha,\beta,\gamma) = (0,\pi,\pi/2)$. Note $3\pi/2 = -\pi/2$ on the unit circle.