Lesson 16:

I. Lagrange's Equations (cont'd) (MLS 4.2)

II. Inertial Properties of Rigid Bodies (MLS 4.2.2)

III. Lagrangian of Open-Chain Robot (MLS 4.3.1)

I. Lagrange's Equations (cont'd)

- Recall:

$$\underbrace{L(q,\dot q)}_{\text{Langrangian}} := \underbrace{K(q,\dot q)}_{\text{kinetic energy}} - \underbrace{P(q)}_{\text{potential energy}}$$

The EOM of a mechanical system is given by:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = \Upsilon$$

- Generalized momentum: $p_i = \dfrac{\partial L}{\partial \dot q_i}$

$$\Rightarrow \frac{d}{dt} p_i = \underbrace{\frac{\partial L}{\partial q_i}}_{\text{conservative forces and kinetic terms}} + \Upsilon_i ,$$

if RHS is zero, then conservation of momentum (p_i = const.)

- Preview of HW 4: Hamilton's Method uses

$$x = \begin{bmatrix} q \\ p \end{bmatrix} \in \mathbb{R}^{2n}$$

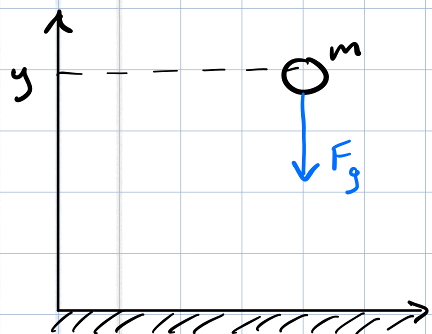
with set of 2n 1st order ODEs

Def: The *Hamiltonian* function $H(q,p) \in \mathbb{R}^1$ given by

$$H(q,p) = \underbrace{K(q,p)}_{\text{kinetic energy}} + \underbrace{P(q)}_{\text{potential energy}}$$

$\longrightarrow$ See HW 4 for Hamilton's equations.

Ex: Falling mass



Find equation of motion (EOM):

- Recall Newton's law

$$p = m \cdot v \overset{\frac{d}{dt}}{\Longrightarrow} \dot{p} = ma = F$$

$$m\ddot{y} = F_g = -mg$$

$$\ddot{y} = -g \checkmark$$

- Now let's try with E-L:

$$K = \frac{1}{2} m \dot{y}^2 \qquad P = mgy$$

$$\Rightarrow L = K - P = \frac{1}{2} m\dot{y}^2 - mgy$$

E-L: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dt}(m\dot{y}) - (-mg) = 0$$

$$m\ddot{y} + mg = 0$$

$$\ddot{y} = -g \checkmark$$

* Same process for multi-body kinematic chains, but need to derive K and P first.

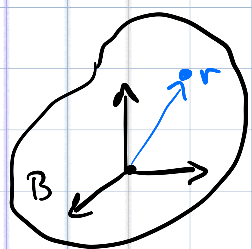
II. Inertial Properties of Rigid Bodies

$\mathbb{V} \subset \mathbb{R}^3$ is volume occupied by a body.

For position $r = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{V}$, density $\rho(r)$ (mass distribution)

mass $\quad m = \int_{\mathbb{V}} \rho(r)\, d\mathbb{V} = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \rho(x,y,z)\, dz\, dy\, dx$

center of mass (COM) $\quad \bar{r} = \frac{1}{m} \int_{\mathbb{V}} \rho(r)\, r\, d\mathbb{V} \in \mathbb{R}^3$



Let $r \in \mathbb{R}^3$ be coords. of a body point relative to the body frame.

Inertia Tensor (constant 3x3)

$$\mathcal{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ & I_{yy} & I_{yz} \\ & & I_{zz} \end{bmatrix} = \mathcal{I}^T$$

$$= -\int_{\mathbb{V}} \rho(r)\, \hat{r}^2\, d\mathbb{V}$$

where $\hat{r}^2 = \begin{bmatrix} -(z^2+y^2) & xy & xz \\ & -(x^2+z^2) & yz \\ & & -(x^2+y^2) \end{bmatrix} = (\hat{r}^2)^T$

Then,

$$I_{xx} = -\int_V \rho(r)(-z^2 - y^2)\, dx\,dy\,dz \quad \left(\text{Moment of Inertia}\right)$$

$$= \int_V \rho(r)(z^2+y^2)\, dx\,dy\,dz$$

$\vdots$ ($\tau_x = I_{xx}\,\omega_x$ for ang. vel. ω_x about x-axis, if x-axis is principal axis.)

$$I_{xy} = -\int_V \rho(r)\, xy\, dx\,dy\,dz \quad \left(\text{Product of Inertia}\right)$$

$\vdots$

<u>Note</u>: Inertia tensor is real, symmetric
$\Rightarrow$ diagonalizable $\Rightarrow$ can find principal axes.

- Let $v^b \in \mathbb{R}^3$ be linear component and $\omega^b \in \mathbb{R}^3$ be the angular component of the body velocity coords:

$$V^b = \begin{bmatrix} v^b \\ \omega^b \end{bmatrix} = \begin{bmatrix} R^T \dot{p} \\ (R^T \dot{R})^\vee \end{bmatrix} \in \mathbb{R}^6 \longleftrightarrow \hat{V}^b = g^{-1}\dot{g} \in se(3)$$

from path $g(t) = (R(t), p(t)) \in SE(3)$

• Then kinetic energy

$$K = \underbrace{\frac{1}{2} m \|v^b\|^2}_{\text{linear}} + \underbrace{\frac{1}{2} (\omega^b)^T \mathcal{I} \omega^b}_{\text{angular}}$$

$$= \frac{1}{2} (\mathcal{V}^b)^T \underbrace{\begin{bmatrix} mI_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & \mathcal{I} \end{bmatrix}}_{\mathcal{M}} \mathcal{V}^b$$

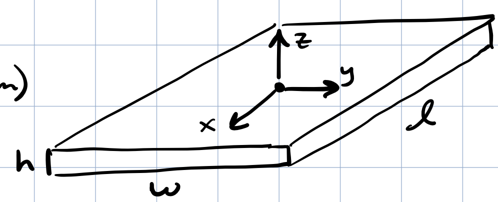
$$= \frac{1}{2} (\mathcal{V}^b)^T \mathcal{M} \mathcal{V}^b$$

where $\mathcal{M} \in \mathbb{R}^{6\times6}$ is the generalized inertia matrix of rigid body.

- real, symmetric, constant, and positive definite
  $(x^T \mathcal{M} x > 0, \ \forall x \neq 0)$

Ex: Homogeneous bar

$\rho = \frac{m}{\ell w h}$ (uniform)



(frame is at center of bar)

Moments of inertia:

$$I_{xx} = \int_{\mathcal{V}} \rho (y^2 + z^2) d\mathcal{V} = \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{w}{2}}^{\frac{w}{2}} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} (y^2 + z^2) \, dx \, dy \, dz$$

$$= \rho \ell (w^3 h + h^3 w)/12$$

$$= \frac{m}{12} (w^2 + h^2)$$

I_{yy} and I_{zz} are similar...

Products of inertia:

$$I_{xy} = -\rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{w}{2}}^{\frac{w}{2}} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} x\, y \; dx\, dy\, dz$$

$$= -\rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{w}{2}}^{\frac{w}{2}} \underbrace{\left(\frac{1}{2} x^2 y \Big|_{x=-\frac{\ell}{2}}^{x=\frac{\ell}{2}} \right)}_{=0} dy\, dz = 0 = I_{xz} = I_{yz}$$

* Products of inertia are alway zero when axes are principal (here, located at COM and aligned with the dimensions of body)

$$\Rightarrow \mathcal{I} = \begin{bmatrix} \frac{m}{12}(w^2+h^2) & 0 & 0 \\ 0 & \frac{m}{12}(\ell^2+h^2) & 0 \\ 0 & 0 & \frac{m}{12}(\ell^2+w^2) \end{bmatrix}$$

$$\Rightarrow M = \begin{bmatrix} m I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & \mathcal{I} \end{bmatrix} = m \begin{bmatrix} I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & \frac{1}{12}\begin{bmatrix} w^2+h^2 & 0 & 0 \\ 0 & \ell^2+h^2 & 0 \\ 0 & 0 & \ell^2+w^2 \end{bmatrix} \end{bmatrix}$$

III. Langrangian of Open-Chain Robot (n-DOF / n-links)

- Obtain KE by summing the KE of each link.