Course: ME 567: Robot Kinematics and Dynamics
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Midterm Review
(not comprehensive, but hopefully helpful)

For twist $\hat{\xi} = \widehat{\begin{bmatrix} v \\ w \end{bmatrix}} \in \mathfrak{se}(3)$, we have:

- Elements of SE(3): $g = e^{\hat{\xi}\theta}$
- Translation: $w = 0$, $e^{\hat{\xi}\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$
- Rotation: $w \neq 0, ||w|| = 1, v = -w \times q$, $e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})(w \times v) \\ 0 & 1 \end{bmatrix}$
- Screw motion: $v = -w \times q + hw$, $e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})(w \times v) + ww^T v\theta \\ 0 & 1 \end{bmatrix}$
- Adjoint transformation: $\mathrm{Ad}_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$
- Instantaneous spatial or body velocity: $\xi^s = (\hat{V}^s)^\vee = (\dot{g}g^{-1})^\vee = (g\hat{V}^b g^{-1})^\vee = \mathrm{Ad}_g \xi^b$
- Transformation between tool and spatial frames for a fixed rotbot configuration: $g_{st}(\theta) = e^{\hat{\xi}\theta} g_{st}(0)$

Inverse Kinematics

- SP1: Rotation about a single axis: $e^{\hat{\xi}\theta} p = q$
- SP2: Rotation about two subsequent intersecting axes: $e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = q$
- SP3: Rotation to a given distance: $||q - e^{\hat{\xi}\theta} p|| = \delta$

Planar rigid body transformations: A transformation $g = (p, R) \in SE(2)$ consists of a translation $p \in \mathbb{R}^2$ and 2×2 rotation matrix $R \in \mathrm{SO}(2)$. A twist $\hat{\xi} \in \mathfrak{se}(2)$ is represented by a matrix $\hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix}$, where $\exp(\hat{\xi}) \in \mathrm{SE}(2)$.

- The dimensions of $\hat{\xi}$ and its elements $\hat{w}$ and v are:

$$\hat{\xi} \in \mathbb{R}^{3\times 3}, \hat{w} \in \mathbb{R}^{2\times 2}, v \in \mathbb{R}^2$$

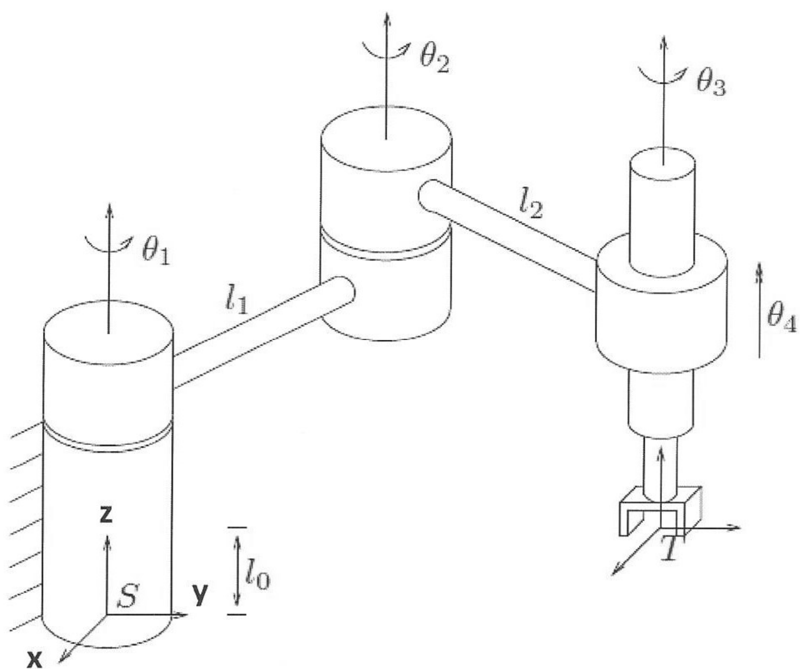
· $\hat{w}$ is defined as:

$$\hat{w} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} \Rightarrow \hat{w} \in \mathfrak{so}(2)$$

· The twist coordinates $\xi = (\hat{\xi})^\vee$ are:

$$w = (\hat{w})^\vee \in \mathbb{R}^1, v \in \mathbb{R}^2$$
$$\Rightarrow \xi = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{R}^3$$

Consider the following 4-DOF manipulator:



For the third joint, write the twist coordinates ξ_3 in the reference configuration (defined with the arm fully extended along y-axis). Now write the twist coordinates ξ_3' in the general configuration θ (hint: just use trigonometry). Finally, give the specific transformation that maps ξ_3 to ξ_3'. You do not need to give expressions for other twist coordinates or their matrix exponentials, symbolic representations such as $e^{\hat{\xi}_1\theta_1}$ are fine.

Answer:

In ref. config,

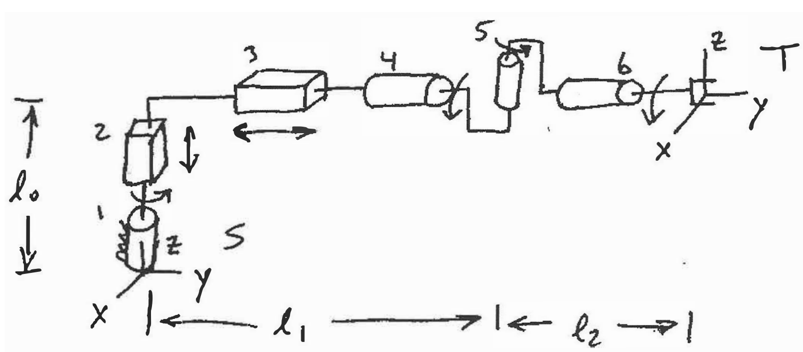
$$\xi_3 = \begin{bmatrix} -w_3 \times q_3 \\ w_3 \end{bmatrix} = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Leftarrow w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix}$$

In configuration θ,

$$q_3' = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix} \text{ and } w_3' = w_3$$

$$\Rightarrow \xi_3' = \begin{bmatrix} -w_3' \times q_3' \\ w_3' \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Finally, $\xi_3' = \mathrm{Ad}_{(e^{\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_2\theta_2})} \cdot \xi_3$ (change of twist coordinates via Adjoint transformation).



(a) For the manipulator shown above, find the twist coordinates ξ_i and $g_{st}(0)$ for the product of exponentials. *Hint:* Assume $l_0 = l_1 = 0$ at the reference configuration, and assume the axes of joints 4, 5, and 6 intersects.

Answer:

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = q_2 = q_3 = q_4 = q_5 = q_6$$

$$w_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, w_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$w_i \times q_i = 0 \text{ for } i = 1, 4, 5, 6 \text{ (only applicable to rotary joints)}$$

$$\Rightarrow \xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \xi_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \xi_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \xi_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$g_{st}(0) = \begin{bmatrix} & & & 0 \\ & I_{3\times 3} & & l_2 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Determine $\exp(\hat{\xi}_i \theta_i)$ for joints 3 and 5.

Answer:

$$e^{\hat{\xi}_3 \theta_3} = \begin{bmatrix} & & & 0 \\ & I_{3\times 3} & & \theta_3 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_5 \theta_5} = \begin{bmatrix} \cos\theta_5 & -\sin\theta_5 & 0 & 0 \\ \sin\theta_5 & \cos\theta_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Without calculating the manipulator Jacobian J_{st}, give an example of a singular configuration and justify your answer.

Answer:

$\theta_5 = 0$ would correspond to singular config. because joints 4 and 6 would have exact same axis of rotation, i.e., loss of uniqueness.

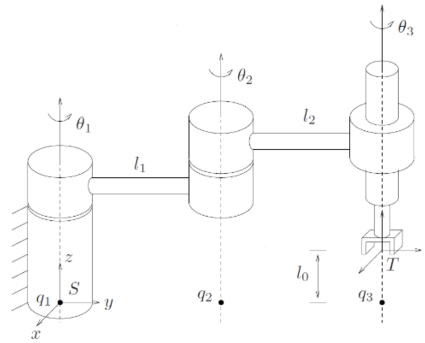


Figure 3.12: SCARA manipulator in its reference configuration.

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_4\theta_4} g_{st}(0) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & x \\ \sin\phi & \cos\phi & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} =: g_d \tag{3.38}$$

<u>Find θ_4</u>

We see that θ_4 solely affects the height (z coordinate) of the tool frame. Therefore

$$\theta_4 = z - l_0$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} = g_d g_{st}^{-1}(0) e^{-\hat{\xi}_4\theta_4} =: g_1. \tag{3.39}$$

<u>Find θ_2</u>

Strategy is to isolate θ_2 by defining points p on ξ_3 and q on ξ_1 and using SP3.

$$\begin{aligned} \|e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p - q\| &= \|e^{\hat{\xi}_1\theta_1}(e^{\hat{\xi}_2\theta_2} p - q)\| \\ &= \|e^{\hat{\xi}_2\theta_2} p - q\| = \|g_1 p - q\| =: \delta. \end{aligned} \tag{3.40}$$

<u>Find θ_1</u>

Strategy is to isolate $\theta_1$by defining point p' on ξ_3 and solving using SP1

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p' = e^{\hat{\xi}_1\theta_1}\left(e^{\hat{\xi}_2\theta_2} p'\right) = g_1 p'.$$

<u>Find θ_3</u>

Use SP1.

$$e^{\hat{\xi}_3\theta_3} = e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_d g_{st}^{-1}(\theta) e^{-\hat{\xi}_4\theta_4}. \tag{3.41}$$