Lesson 17:

I. Lagrangian of open-chain robot (MLS 3.1)

II. EOM of Open-chain robot (MLS 3.2)

I. Lagrangian of open-chain robot (n-DOF/n-links)

- Obtain KE by summing the KE of each link.
  - Define coord frame L_i at COM of the i^{th} link (not at the joint)
  - Then $g_{s\ell_i}(\theta) = \underbrace{e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_i\theta_i}}_{\text{defined as before}} \underbrace{g_{s\ell_i}(0)}_{\text{ref. config. of the } i^{th} \text{ link frame (at its COM)}}$

  - Link i COM velocity twist coords $V^b_{s\ell_i} = J^b_{s\ell_i}(\theta)\dot{\theta}$

    where each link's body Jacobian is

$$J^b_{s\ell_i}(\theta) = \left[\xi_1^{\dagger}, \xi_2^{\dagger}, \ldots, \xi_i^{\dagger}, 0, \ldots, 0 \right] \in \mathbb{R}^{6\times n}$$

and $\xi_j^{\dagger} = Ad^{-1}_{\left(e^{\hat{\xi}_j\theta_j} \cdots e^{\hat{\xi}_i\theta_i} g_{s\ell_i}(0)\right)} \xi_j$, for $1 \le j \le i$

this is the j^{th} instantaneous joint twist coords. w.r.t. the current config. of the i^{th} frame.

  - KE of each link is then

$$K_i(\theta, \dot{\theta}) = \frac{1}{2} (V^b_{s\ell_i})^T \underset{\substack{\uparrow \\ \text{generalized} \\ \text{inertia matrix}}}{\mathcal{M}_i} V^b_{s\ell_i} = \frac{1}{2} \dot{\theta}^T J_i^T(\theta) \mathcal{M}_i J_i(\theta) \dot{\theta}$$

$(J_i = J^b_{s\ell_i})$

$$\Rightarrow K(\theta,\dot\theta) = \sum_{i=1}^{n} K_i(\theta,\dot\theta) = \frac{1}{2}\dot\theta^T M(\theta)\dot\theta$$

where $\boxed{M(\theta) := \sum_{i=1}^{n} J_i^T(\theta)\mathcal{M}_i J_i(\theta)}$ $\in \mathbb{R}^{n\times n}$ is called the manipulator inertia/mass matrix.

(real, symmetric, pos. def)

$\Rightarrow$ invertible

- Obtain PE by summing the PE of each link
  - Includes:
    - Springs in parallel to joints
    - Gravitation PE (calculate height in the direction of gravity of COM of each link)
  - Recall $\bar{p}_{l_i}(\theta) = e^{\hat\xi_1\theta_1}\cdots e^{\hat\xi_i\theta_i}\bar{p}_{l_i}(0) \in \mathbb{R}^4$

    then, $h_i(\theta) = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}\bar{p}_{l_i}(\theta) \in \mathbb{R}^1$

    finally, $P_i(\theta) = m_i g h_i(\theta) \Longrightarrow P(\theta) = \sum_{i=1}^{n} m_i g h_i(\theta)$

    (assumes no elasticity)

- The Lagrangian is thus:

$$L(\theta,\dot\theta) = \frac{1}{2}\dot\theta^T M(\theta)\dot\theta - P(\theta)$$

$$= \left(\frac{1}{2}\sum_{i,j=1}^{n} M_{ij}(\theta)\dot\theta_i\dot\theta_j\right) - \left(\sum_{i=1}^{n} m_i g h_i(\theta)\right)$$

II. EOM of open-chain robot (n-DOF)

E-L: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i \quad , \text{ for } \quad i = 1, \dots, n$$

- First, compute $\frac{\partial L}{\partial \theta_i}$

$$\frac{\partial L}{\partial \theta_i} = \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial M_{jk}}{\partial \theta_i}\dot{\theta}_j \dot{\theta}_k - \frac{\partial P}{\partial \theta_i}$$

- Next, compute $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i}$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = \frac{d}{dt}\left(\sum_{j=1}^{n} M_{ij}(\theta)\dot{\theta}_j\right)$$

$$= \sum_{j=1}^{n}\left(M_{ij}(\theta)\ddot{\theta}_j + \underbrace{\dot{M}_{ij}(\theta)}_{\sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial \theta_k}\dot{\theta}_k}\dot{\theta}_j\right)$$

- Now, E-L equation is

$$\sum_{j=1}^{n} M_{ij}(\theta)\ddot{\theta}_j + \underbrace{\sum_{j,k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k}\dot{\theta}_j\dot{\theta}_k - \frac{1}{2}\frac{\partial M_{kj}}{\partial \theta_i}\dot{\theta}_k\dot{\theta}_j\right)}_{\sum_{j,k=1}^{n}\Gamma_{ijk}\dot{\theta}_j\dot{\theta}_k} + \frac{\partial P}{\partial \theta_i} = \tau_i$$

for $i = 1, \dots, n$

Γ_{ijk}: Christoffel symbols

• Christoffel symbols

$$\Gamma_{ijk} := \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i} \right)$$

• Ways to interpret middle term:

- Centrifugal terms $\sim \dot{\theta}_k^2 \; (j = k)$
- Coriolis terms $\sim \dot{\theta}_j \dot{\theta}_k \; (j \neq k)$

• Now let's express in vector form:

Def: The Coriolis Matrix $C(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n}$ defined by

$$\boxed{C_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^{n} \Gamma_{ijk} \dot{\theta}_k}$$

Then,

$$C(\theta, \dot{\theta})\dot{\theta} = \begin{bmatrix} \sum_{j,k=1}^{n} \Gamma_{1jk} \dot{\theta}_j \dot{\theta}_k \\ \vdots \\ \sum_{j,k=1}^{n} \Gamma_{njk} \dot{\theta}_j \dot{\theta}_k \end{bmatrix} \begin{array}{c} (i=1) \\ \vdots \\ (i=n) \end{array}$$

Def: Conservative/potential forces vector $\boxed{G(\theta) = \frac{\partial P}{\partial \theta}} \in \mathbb{R}^n$

• Then, EOM in vector form are

$$\boxed{M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) = \tau}$$

2nd-order n-dim ODE

- What about Υ? External / non-conservative forces

$$\Upsilon = \underbrace{B\tau}_{\text{actuators}} - \underbrace{\beta\dot{\theta}}_{\text{viscous friction}} - \underbrace{A^T(\theta)\,\lambda(\theta,\dot{\theta})}_{\text{contact constraint forces}} + \underbrace{J_{st}^b(\theta)^T F_b}_{\text{end-effector interaction}}$$

where

- $B \in \mathbb{R}^{n\times m}$ maps actuator torques $\tau \in \mathbb{R}^m$ to the coord. dynamics (n-dim)
  - Underactuated if $m<n$
  - Fully actuated if $m=n$

- β is a diagonal matrix of viscosities

- $A(\theta)$ is constraint matrix s.t. $A = \frac{\partial a}{\partial \theta}$ for constraint $a(\theta)=0$

- $\lambda(\theta,\dot{\theta})$ is a Lagrange multiplier (wrench) enforces constraint.
  - → MLS pg. 287 for derivation based on EoM