Lesson 20:

I. Stability (cont'd)

II. Multivariable Control (SHV)

For later in class:

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Summary so Far

1. We can determine local stability properties for hyperbolic fixed points of nonlinear systems by examining the eigenvalues of the linear approximation.
   1.1 The fixed point is asymptotically stable for the nonlinear system if it is asymptotically stable for the linear systems (all eigenvalues have negative real part)
   1.2 The fixed point is unstable for the nonlinear system if it is unstable for the linear system. (one or more eigenvalues with positive real part)
   1.3 If any eigenvalue of the linear approximation has zero real part (critical eigenvalues) then the fixed point of the nonlinear system may be stable or unstable, i.e. the linearization is inconclusive.
2. Lyapunov functions can be used to determine stability in the case of critical eigenvalues

The Invariance Principle

We next give an extension to Lyapunov's Theorem due to LaSalle that allows additional conclusions about the asymptotic behavior of the system. A weak form of LaSalle's theorem can be stated as

Theorem

Suppose $V : R^n \to R$ is a positive definite scalar function and suppose $\dot{V} \leq 0$ along trajectories of (1). If $\dot{V}$ does not vanish identically along trajectories of (1) other than the equilibrium solution, then $\bar{x} = 0$ is asymptotically stable.

=0 over time

xdot = f(x) (1)

Example

Back to the simple pendulum with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin(x_1) - Bx_2\end{aligned}$$

We know that with

$$V = \frac{g}{\ell}(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

satisfies

$\dot{V} = \frac{\partial V}{\partial x} \dot{x} = \nabla V \cdot f(x)$

Not neg. def. because

$\dot{V}(x_1, 0) = 0$

could be anything

$$\dot{V} \quad = -Bx_2^2 \leq 0$$

Suppose $\dot{V} \equiv 0$. Then $x_2 \equiv 0 \implies \dot{x}_2 \equiv 0$
$\implies \sin(x_1) \equiv 0 \implies x_1 = n\pi$. *we consider local stability with -pi < x1 < pi
Therefore, $\dot{V}$ vanishes identically only at the equilibrium point and so $\bar{x} = 0$ is asymptotically stable.

LaSalle's Theorem

We next present a stronger version of LaSalle's Theorem.

Theorem

Suppose V *is a Lyapunov function satisfying* $\dot{V} \leq 0$ *along trajectories of* $\dot{x} = f(x)$. *Let* E *be the set of points where* $\dot{V} = 0$ *and let* M *be the largest invariant set contained in* E. *Then all trajectories of the system approach* M *as* $t \to \infty$.

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Example

Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1(1 + x_2) \\ \dot{x}_2 &= x_1^2 \end{aligned}$$

Example

Let

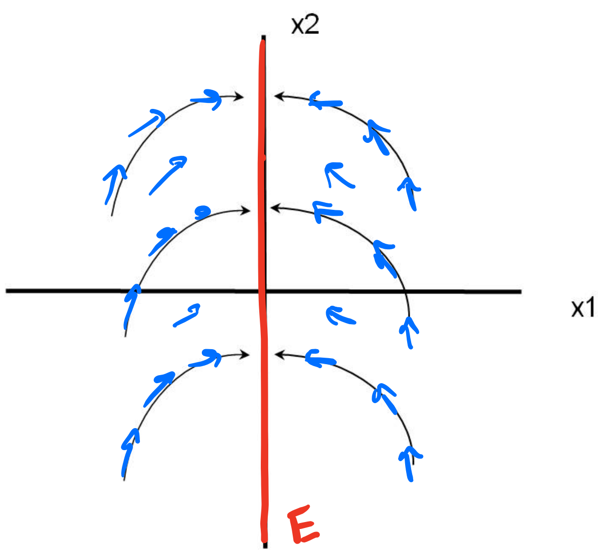
$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

Then

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= -x_1^2(1+x_2) + x_2x_1^2 = -x_1^2 \leq 0\end{aligned}$$

not neg. def. since doesn't depend on x_2.

Therefore $E = \{x_1 = 0\}$, which is also an invariant set. Therefore, all trajectories approach E as $t \to \infty$.



Stability of Linear Systems

We have been using functions of the form

$$\mathcal{V} = x_1^2 + x_2^2 + \cdots + x_n^2$$

as potential Lyapunov functions in several examples. The 'sum of squares' or 'norm' function is called a **Quadratic Form**. A more general quadratic form can be written as

$$V = x^T P x \text{ where } x \in R^n \text{ and } P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

P is an $n \times n$ constant matrix. To qualify as a Lyapunov function, the matrix P should satisfy $x^T P x > 0$ for $x \neq 0$.

Positive Definite Matrices

Definition

An $n \times n$ matrix P is **Positive Definite** if and only if

$$x^T P x > 0 \text{ for } x \neq 0$$

The matrix P is

- **Positive Semi-definite** if $x^T P x \geq 0$
- **Negative Definite** if $x^T P x < 0$
- **Negative Semi-Definite** if $x^T P x \leq 0$
- P is **Indefinite** if it is otherwise not one of the above.

Positive Definite Matrices

Remark

- *The matrix P is positive definite if and only if all eigenvalues of P are positive.*
- *P is positive semi-definite if all eigenvalues of P are greater than or possibly equal to zero.*

Given an n-dimensional linear system

$$\dot{x} = Ax$$

let us try to find a quadratic Lyapunov function of the form

$$V = x^T P x$$

Stability of Linear Systems

Computing the derivative of $V = x^T P x$ along trajectories of $\dot{x} = Ax$ yields

$$\begin{aligned}\dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + P A) x\end{aligned}$$

Let's define $-Q = A^T P + PA$. Then we have

$$\begin{aligned}V &= x^T P x \\ \dot{V} &= -x^T Q x\end{aligned}$$

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Suppose both P and Q are positive definite. What can you conclude?

(a.) System is unstable

(b.) " " exponentially stable

(c.) " " asymptotically stable

(d.) " " stable

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$$\begin{aligned} V &= x^T P x \\ \dot{V} &= -x^T Q x \end{aligned}$$

Suppose both P and Q are positive definite. What can you conclude? The conclusion is that the linear system is (exponentially) stable.

The Lyapunov Equation

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positive-definite matrix and consider the equation

$$A^T P + PA + Q = 0$$

where P is the unknown and A and Q are given. Suppose that we solve Equation (1) for P and find that P is positive-definite. What conclusions can you draw?

The Lyapunov Equation

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The equation (1) is called a **Lyapunov Equation**.

$x^T P x > 0$

Stability of Linear Systems

The leads to the following result to determine stability of linear systems

Theorem

(globally) asymptotically stable

The $n \times n$ matrix A is ~~stable~~ (i.e. all eigenvalues of A have negative real part) if and only if for any symmetric positive definite $n \times n$ matrix Q there is a unique symmetric positive definite matrix P satisfying the Lyapunov Equation

$$A^T P + PA + Q = 0$$

Remark

This gives a way to determine stability of linear systems by solving a system of linear algebraic equations; in other words, without computing matrix eigenvalues.

Example

Remark

The Matlab Control Systems Toolbox has the built-in function `lyap`*, which solves the Lyapunov equation.*

Matlab Script

$>>\ \ A = [0\ 1;\ \ -2\ \ -3];$

$>>\ \ Q = [1\ 0;\ \ 0\ 1];$

$>>\ \ P = lyap(A, Q)$

$>>\ \ P =$

$$\begin{array}{cc} 1.0000 & -0.5000 \\ -0.5000 & 0.5000 \end{array}$$

$>>\ \ eig(P)$

ans $=$

0.1910

1.3090

II. Multivariable Control

A. PD control (Proportional - Derivative Control)

$$(1) \sim \quad \underbrace{M(q)}_{\text{could include motor inertias by adding diagonal matrix to the prior definition.}}\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

control torques w/ full actuation

- First assume $G(q) = 0$ (gravity-free)

Define independendent joint PD control law:

constant set point

$$u = -K_p\tilde{q} - K_d\dot{\tilde{q}} \quad \text{where} \quad \tilde{q} = q - q^d, \ \dot{\tilde{q}} = \dot{q}$$

diagonal matrices $K_p, K_d > 0$

Note $x = \begin{bmatrix} \tilde{q} \\ \dot{\tilde{q}} \end{bmatrix}$ and $\bar{x} = 0$ is eq. pt.

Q: Does this controller make $\bar{x} = 0$ a stable eq. pt.?
("is this controller stable"?)

- Try Lyapunov function

$$V(x) = V(q,\dot{q}) = \underbrace{\frac{1}{2}\dot{q}^T M(q)\dot{q}}_{KE} + \underbrace{\frac{1}{2}\tilde{q}^T K_p \tilde{q}}_{\text{PE if actuators replaced by springs (virtual PE from controller)}} > 0 \quad \forall x \neq 0$$

$$\Rightarrow \dot V = \underbrace{\dot q^T M(q)\ddot q}_{} + \frac12 \dot q^T \dot M(q)\dot q + \dot q^T K_p \tilde q$$

$$= \dot q^T\left(u - C(q,\dot q)\dot q\right) + \frac12 \dot q^T \dot M(q)\dot q + \dot q^T K_p \tilde q$$

$$= \dot q^T\left(u + K_p \tilde q\right) + \frac12 \dot q^T \underbrace{\cancel{\left(\dot M - 2C\right)}}_{0}\dot q$$

plug in

$$= \dot q^T\left(-K_d \dot q\right) = \underbrace{-\dot q^T K_d \dot q}_{\text{doesn't depend on } \tilde q} \le 0 \qquad \forall x.$$

$\Rightarrow$ This prove weak form of stability ("in the sense of Lyapunov")

- Try LaSalle's Thm (i.e., invariance principle)

Suppose $\dot V \equiv 0 \Rightarrow \dot q \equiv 0 \Rightarrow \ddot q \equiv 0$

$(1) \Rightarrow u = 0 \iff -K_p \tilde q - \cancel{K_d \dot q}^{\,0} = 0 \Rightarrow \tilde q = 0$

I.e., $\dot V$ vanishes only at eq. pt. $(\tilde q, \dot q) = (0,0)$

$\Rightarrow$ Asymptotically stable (A.S) ✓