Lesson 11:

I. Inverse Kinematics Example

II. Velocity of a Rigid Body (MLS 2.4)

  A. Rotational Velocity

  B. General Velocity (rotation + translation)

I. IK Example

Ex: Elbow Manipulator (MLS Fig. 3.11)

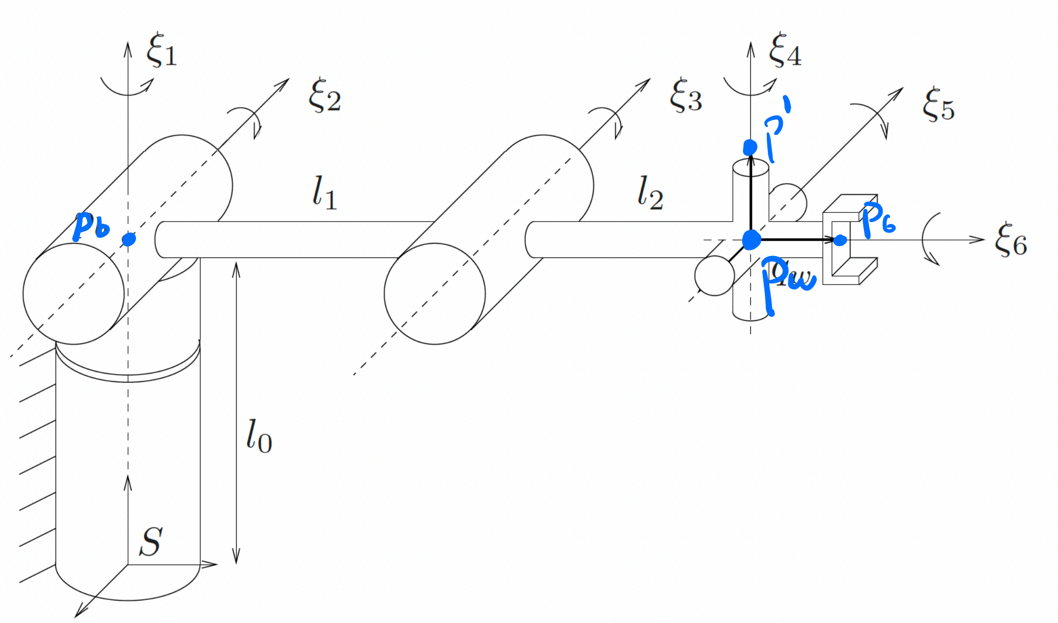


Figure 3.11: Elbow manipulator.

shorthand for $e^{\hat{\xi}_6 \theta_6}$

$$g_{st}(\theta) = \underbrace{e^1 e^2 e^3 e^4 e^5 e^6}_{\text{unknown}} \underbrace{g_{st}(0)}_{\text{known}} = \underbrace{g_d}_{\text{given}}$$

$$\underbrace{e^1 e^2 e^3 e^4 e^5 e^6}_{\text{unknown}} = \underbrace{g_d \, g_{st}^{-1}(0)}_{g_1 \text{ (known)}}$$

<u>Strategy</u>: Solve for the first 3 angles, then solve wrist angles.

Step 1: Solve for elbow angle θ_3

Apply both sides of equation to known point $p_w \in \mathbb{R}^3$, the common point of intersection for all 3 wrist axes. $\Rightarrow$ $e^4 e^5 e^6 \bar{p}_w = \bar{p}_w$

$$\Rightarrow e^1 e^2 e^3 \underbrace{e^4 e^5 e^6 \bar{p}_w}_{} = g_1 \bar{p}_w$$

$$\Rightarrow e^1 e^2 e^3 \bar{p}_w = g_1 \bar{p}_w$$

Subtract a known point p_b at the intersection ξ_1 and ξ_2 $\Rightarrow$ $e^1 e^2 \bar{p}_b = \bar{p}_b$

$$e^1 e^2 e^3 \bar{p}_w - \bar{p}_b = g_1 \bar{p}_w - \bar{p}_b$$

$$e^1 e^2 e^3 \bar{p}_w - e^1 e^2 \bar{p}_b = g_1 \bar{p}_w - \bar{p}_b$$

$$e^1 e^2 (e^3 \bar{p}_w - \bar{p}_b) = \text{''}$$

Distance preserved by RBT

$$\| e^1 e^2 (e^3 \bar{p}_w - \bar{p}_b) \| = \| g_1 \bar{p}_w - \bar{p}_b \|$$

$$\| e^3 \bar{p}_w - \bar{p}_b \| = \underbrace{\| g_1 \bar{p}_w - \bar{p}_b \|}_{:= \delta}$$

$\Rightarrow$ Solve for θ_3 via SP3 ✓

Step 2: Solve for base joint angles θ_1, θ_2:

$$e^1 e^2 (\underbrace{e^3 \bar{p}_w}_{\bar{p}}) = \underbrace{g_1 \bar{p}_w}_{\bar{q}} \implies \text{Solve using SP2 for } \theta_1 \text{ and } \theta_2$$

Step 3: Solve for 2 of 3 wrist angles (I think I should be able to use SP2)

Isolate remaining unknown terms

$$e^4 e^5 e^6 = \underbrace{e^{-3} e^{-2} e^{-1} g_d \, g_{st}^{-1}(c)}_{g_2 \text{ (known)}}$$

Apply both sides to a point p_6 on axis ξ_6 but not on ξ_4, ξ_5 $\implies e^6 \bar{p}_6 = \bar{p}_6$

$$\implies e^4 e^5 (\underbrace{e^6 \bar{p}_6}) = g_2 \bar{p}_6$$

$$\implies e^4 e^5 \bar{p}_6 = \underbrace{g_2 \bar{p}_6}_{\text{known}} \implies \text{SP2 to find } \theta_4, \theta_5$$

Step 4: Solve for θ_6

$$e^6 = \underbrace{e^{-5} e^{-4} e^{-3} e^{-2} e^{-1} g_d \, g_{st}^{-1}(c)}_{g_3}$$

Apply point p' not ξ_6

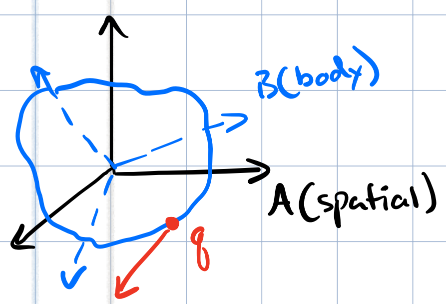
$$e^6 \underset{\text{known}}{\bar{p}'} = \underbrace{g_3 \bar{p}'}_{\text{known}} \implies \text{SP1 to find } \theta_6$$

Note: Obtain up to 8 solutions $(2 \times 2 \times 2 \times 1)$

Note: Sometimes no applicable P-K subproblems, but can still use geometry.
→ See MLS Fig. 3.12

II. Velocity of a Rigid Body

A. Rotational Velocity



$R_{ab}(t) \in SO(3)$
describes rotational path

$q_a(t) = R_{ab}(t) \cdot q_b$
where q_b is constant because attached to frame B.

$$V_{q_a}(t) := \frac{d}{dt} q_a(t) = \dot{R}_{ab}(t)\, q_b \in \mathbb{R}^3$$

so $\dot{R}_{ab}$ maps body coords of a point to spatial velocity

Note: $V_{q_a}(t) = \left(\dot{R}_{ab}(t) \underbrace{R_{ab}^{-1}(t)\right)\left(R_{ab}(t)}_{I} q_b\right) = \underbrace{\left(\dot{R}_{ab}(t) R_{ab}^{-1}(t)\right)}_{\text{skew-symmetric}} q_a$

Lemma 2.12: Given $R(t) \in SO(3)$,
$\dot{R}(t) R^{-1}(t) \in so(3)$ and $R^{-1}(t) \dot{R}(t) \in so(3)$
i.e. they are skew-symmetric matrices.

Proof: $R R^{-1} = R R^T = I$ $\qquad R^T R = I$

$\frac{d}{dt} \Rightarrow \dot{R} R^T + R \dot{R}^T = 0_{3\times 3}$

$\Rightarrow \dot{R} R^T = - R \dot{R}^T = -(\dot{R} R^T)^T$

$\Rightarrow \dot{R} R^T$ is skew-symmetric ✓

same steps ✓

□

* Lemma 2.12 allows us to represent velocity of rotating body using elements of $\mathbb{R}^3$ ("coordinates") rather then $\mathbb{R}^{3\times 3}$.

<u>Def</u>: Angular Velocities

- $\hat{\omega}^s_{ab} := \dot{R}_{ab} R^{-1}_{ab} \in so(3)$ is instantaneous spatial angular velocity (as seen from A)

  $\Rightarrow$ coords $\omega^s_{ab} = (\hat{\omega}^s_{ab})^\vee \in \mathbb{R}^3$ (spatial angular velocity coords)

- $\hat{\omega}^b_{ab} := R^{-1}_{ab} \dot{R}_{ab}$ is inst. body angular velocity (as seen from B)

  $\Rightarrow$ coords $\omega^b_{ab} = (\hat{\omega}^b_{ab})^\vee \in \mathbb{R}^3$ (body ang. velocity coords)

<u>Note</u>: $\hat{\omega}^b_{ab} = R^{-1}_{ab} \hat{\omega}^s_{ab} R_{ab}$ or $\omega^b_{ab} = R^{-1}_{ab} \omega^s_{ab}$

* How to use to compute velocity of a point?

$$V_{q_a}(t) = \underbrace{\dot{R}_{ab} R^{-1}_{ab}}_{\hat{\omega}^s_{ab}} \underbrace{R_{ab} q_b}_{q_a} = \omega^s_{ab}(t) \times q_a(t)$$

$$\begin{aligned} V_{q_b}(t) &= R^{-1}_{ab}(t)\, V_{q_a}(t) = R^{-1}_{ab}(t)\left(\omega^s_{ab}(t) \times q_a(t)\right) \\ &= R^{-1}_{ab}(t)\, \omega^s_{ab}(t) \times R^{-1}_{ab}\, q_a(t) \\ &= \omega^b_{ab}(t) \times q_b \end{aligned}$$

- Can express velocity of <u>any</u> <u>point</u> of rigid body.