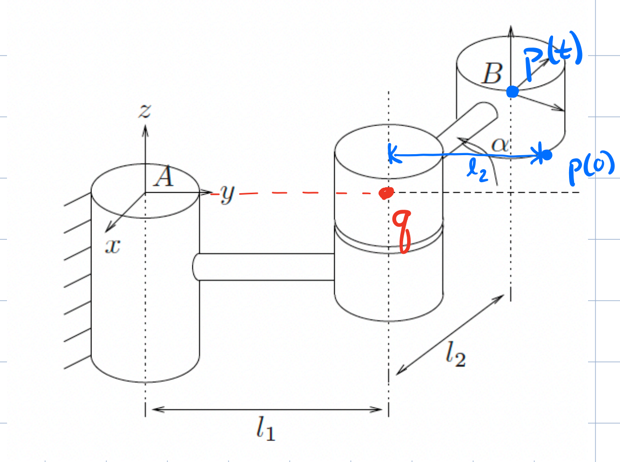
Warm-up Problem : https://join.iclicker.com/MMAW



What twist coordinates $\xi = (v, \omega)$, θ generate the RBT that transforms $p(0)$ to $p(t)$? (Expressed in frame A)

$$\boxed{(a)\quad v = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \theta = \alpha}$$

(b.) $v = \begin{bmatrix} -l_2 \sin(\alpha) \\ l_2 \cos(\alpha) \\ 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \theta = \alpha$

(c.) $v = \begin{bmatrix} 0 \\ l_2 \\ 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \theta = -\alpha$

(d.) None of the above

Recall:

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Axis of rotation is:

$$\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{\omega} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since this is pure rotation

$$v = -\omega \times q$$

some pt. on axis

$$= -\hat{\omega} q$$

$$= -\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding twist

$$\hat{\xi} = \begin{bmatrix} 0 & -1 & 0 & l_1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \theta = \alpha$$

This yields the transformation:

$$\bar{p}(t) = e^{\hat{\xi}\theta} \bar{p}(0)$$

Lesson 6:

I. Exponential Coordinates of Rigid Motion (cont'd)

A. Exponential Map from se(3) to SE(3)

B. Surjectivity of exponential map onto SE(3)

II. Chasles' Theorem (if time permits)

I. Exponential Coordinates of Rigid Motion (cont'd)

- Last time we showed how the exponential of a twist can represent pure rotation/translation about some axis.

- Now want to show that any RBT $\in SE(3)$ can be represented as the exponential of some twist $\hat{\xi} \in se(3)$.

  i.e. - Exponentials of arbitrary twists are in SE(3)

  - Every $g \in SE(3)$ can be "generated" by some $\hat{\xi} \in se(3)$.

A. Exponential Map frome se(3) to SE(3)

<u>Prop 2.8</u>: Given twist $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the matrix exp. of $\hat{\xi}\theta$ is an element of SE(3)

<u>Proof:</u> (brute force)

- <u>Case 1</u> ($\omega = 0$, translation)

$$\hat{\xi} = \begin{bmatrix} 0_{3\times3} & v \\ 0_{1\times3} & 0 \end{bmatrix} \Rightarrow \hat{\xi}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \hat{\xi}^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ldots$$

$$\exp(\hat{\xi}\theta) = I + \hat{\xi}\theta + \cancel{\frac{1}{2!}\hat{\xi}^2\theta^2} + \cdots$$

0

$$= \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \in SE(3) \checkmark$$

- <u>Case 2</u> ($\omega \neq 0$)

Recall $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$, assume $\|\omega\| = 1$

Define RBT $g = \begin{bmatrix} I & \omega \times v \\ 0 & 1 \end{bmatrix}$ by "intuition"

Similarity transformation of the twist (change of coordinates)

$$\hat{\xi}' = g^{-1}\hat{\xi}g = \begin{bmatrix} I & -\omega\times v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & \omega\times v \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\omega} & \omega\,\omega^T v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & h\omega \\ 0 & 0 \end{bmatrix}$$

where scalar $h = \omega^T v$

use identity $\exp(\hat{\xi}\theta) = \exp(g\,\hat{\xi}'g^{-1}\theta)$

can be shown using Taylor series $= g\,\exp(\hat{\xi}'\theta)\,g^{-1}$

$\Rightarrow$ It suffices to calculate $\exp(\hat{\xi}'\theta)$ then transform back.

- Step 1: Calculate $\exp(\hat{\xi}'\theta)$

$$\hat{\xi}'^2 = \begin{bmatrix} \hat{\omega}^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\xi}'^3 = \begin{bmatrix} \hat{\omega}^3 & 0 \\ 0 & 0 \end{bmatrix}, \dots$$

because $h \cdot \hat{\omega}\,\omega = h(\omega \times \omega) = 0$

$$\Rightarrow \exp(\hat{\xi}'\theta) = I + \hat{\xi}'\theta + \frac{1}{2!}\hat{\xi}'^2\theta^2 + \dots$$

$$= \begin{bmatrix} e^{\hat{\omega}\theta} & h\omega\theta \\ 0 & 1 \end{bmatrix}$$

- Step 2: Transform back

$$\exp(\hat{\xi}\theta) = g\,\exp(\hat{\xi}'\theta)\,g^{-1}$$

$$= \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} \in SE(3)$$

✓

$\square$

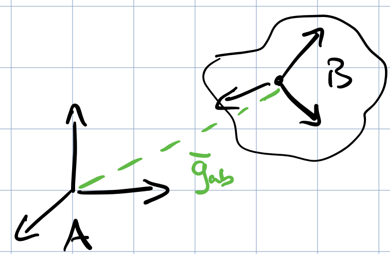
How to use this?

- **Points:** Interpret $\exp(\hat{\xi}\theta)$ as mapping a point from initial coords $\bar{p}(0)$ to coords. after a rigid motion of angle/displacement θ:

$$\bar{p}(\theta) = \exp(\hat{\xi}\theta) \cdot \bar{p}(0)$$

- **Configurations:** Let $\bar{g}_{ab}(0) \in SE(3)$ represents initial configuration of a rigid body (frame B) wrt frame A. Then the final config. of the rigid body w.r.t. A is

$$\underbrace{\bar{g}_{ab}(\theta)}_{\substack{\text{maps B to A} \\ \text{after motion}}} = \exp(\hat{\xi}\theta)\, \underbrace{\bar{g}_{ab}(0)}_{\substack{\text{maps B to A} \\ \text{before motion}}}$$



* Exp. map for a twist gives relative motion of a rigid body.

B. Surjectivity of the exp. map onto $SE(3)$

Prop 2.9: Given $g \in SE(3)$, $\exists\, \hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$ s.t. $\bar{g} = \exp(\hat{\xi}\theta)$.

Proof: (By construction)

- Let $g = (R, p) \in SE(3)$ where $R \in SO(3)$, $p \in \mathbb{R}^3$
- Ignore singular case $(I, 0)$, solved by $\theta = 0$ and arbitrary $\hat{\xi}$
- Case 1: $R = I$ w/translation $(p \neq 0)$

Set $\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$, where $v = \frac{p}{\|p\|}$ and $\theta = \|p\|$

$\Rightarrow p = v\theta$

Then, $\exp(\hat{\xi}\theta) = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix} = \bar{g}$ ✓

- Case 2: $R \neq I$ (pure rotation or combination)

Find $\hat{\xi} = (v, \omega)$ by equating $e^{\hat{\xi}\theta} = \bar{g}$

$$\begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}}_{\text{given}}$$

First solve $e^{\hat{\omega}\theta} = R \Rightarrow$ obtain ω, θ by prop 2.5 (surjectivity of exp map onto $SO(3)$)

Finally solve for v:

Let $A = (I - e^{\hat{\omega}\theta})\hat{\omega} + \omega\omega^T\theta$, then $Av = p$

$$\Rightarrow v = A^{-1} p$$

Can show A non-singular $\forall \theta \neq 0$ because its two matrix parts have mutually orthogonal nullspaces (when $R \neq I$) $\square$

- Recall $\omega\theta \in \mathbb{R}^3$ are exp. coords of $R \in SO(3)$
- Now $\xi\theta \in \mathbb{R}^6$ are exp. coords of $g \in SE(3)$

- <u>Note</u>: $\exp: se(3) \to SE(3)$ is many-to-one (not injective) because $\exp: so(3) \to SO(3)$ was many-to-one.