Course: MECH 567: Robot Kinematics & Dynamics
Instructor: Daniel Bruder, PhD

Assigned Date: 01/10/2024
Due: 11:59 PM on Thursday, 02/01/2024
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Homework 1

Problem 1

Use Lemma 2.3 on page 28 of MLS to prove Rodrigues' formula, Equation (2.14).

Hint: We can prove the following two lemmas first using mathematical induction, where
Lemma 0.1: $\theta^{2n-1}\hat{\omega}^{2n-1} = (-1)^{n-1}\theta^{2n-1}\hat{\omega}$ for $n = 1, \dots, \infty$.
Lemma 0.2: $\theta^{2n}\hat{\omega}^{2n} = (-1)^{n+1}\theta^{2n}\hat{\omega}^2$ for $n = 1, \dots, \infty$.
Using the results of the lemmas and Taylor expansion of $\sin\theta$ and $\cos\theta$, we can get the Rodrigues' formula.

Problem 2

Equation (2.18) on page 30 of MLS shows one way to extract ω from a 3×3 rotation matrix R. This mapping breaks down for $\theta = 0$ and $\theta = \pi$, however, which can be found from (2.17). This is to be expected at $\theta = 0$, because this is a singularity (the axis of rotation ω is undefined when $\theta = 0$). When $\theta = \pi$, there is not a singularity; one should be able to find ω. Use Rodrigues' formula to show that when $R = I$, $\theta = 0$ and ω is arbitrary.

Problem 3

Equation (2.18) is just one way of finding ω from the nine terms in R−you have nine equations you can use and only three unknowns. From (2.16), find an alternative to equation (2.18) that will give ω for the particular case that $\theta = \pi$.

Note: when we see methods to solve the inverse kinematics of a multi-link manipulator in a few weeks, our knowledge of the structure of the manipulator will give us ω, and we will use a more robust method of finding θ from the known ω.

Problem 4

Let $R \in SO(3)$ be a rotation matrix generated by rotating about a unit vector ω by θ radians. That is, R satisfies $R = e^{\hat{\omega}\theta}$.

(a) Show that the eigenvalues of $\hat{\omega}$ are 0, i, and $-i$, where $i = \sqrt{-1}$. Show that the eigenvector associated with $\lambda = 0$ is ω.

(b) Verify that the eigenvalues of R are 1, $e^{i\theta}$, and $e^{-i\theta}$ and that the eigenvectors of R are the same as $\hat{\omega}$.

Hint: Note that the eigenvalues of R are the exponentials of the eigenvalues of $\hat{\omega}$ times θ. You don't need to find the eigenvectors to do this problem.

Problem 5

Let $SO(2)$ be the set of all 2×2 orthogonal matrices with determinant equal to $+1$.

(a) Show that $SO(2)$ can be identified with the $\mathbb{S}^1$, the unit circle in $\mathbb{R}^2$.

Hint: Where you are on a circle can be identified by a single parameter θ with periodicity 2π. Show, using the definition of the properties of $SO(n)$ in section 2.1, that any member of $SO(2)$ can also be written as a function of θ.

(b) Let $\omega \in \mathbb{R}$ be a real number and define $\hat{\omega} \in so(2)$ as the skew-symmetric matrix

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}.$$

Show that

$$e^{\hat{\omega}\theta} = \begin{bmatrix} \cos\omega\theta & -\sin\omega\theta \\ \sin\omega\theta & \cos\omega\theta \end{bmatrix}.$$

Is the exponential map $\exp : so(2) \to SO(2)$ surjective? injective?

Hints: An *injective* mapping is one-to-one. Don't use Rodrigues' formula–it is only proven for $so(3)$.

Problem 6

Let $R \in SO(2)$ and $\hat{\omega} \in so(2)$.

(a) Show that $R\hat{\omega}R^T = \hat{\omega}$.

(b) Verify that $Re^{\hat{\omega}\theta}R^T = e^{\hat{\omega}\theta}$ and $\frac{d}{dt}e^{\hat{\omega}\theta} = (\hat{\omega}\dot{\theta})e^{\hat{\omega}\theta} = e^{\hat{\omega}\theta}(\hat{\omega}\dot{\theta})$.

Hints: Matrix exponential of Λ is defined as: $e^{\Lambda} = I + \Lambda + \frac{\Lambda^2}{2!} + \cdots$.