Lesson 15:

I. Wrenches & Joint Torques (MLS 3.4.2)

II. Dynamics (MLS 4)

I. Wrenches & Joint Torques

Recall $J_{st}(\theta) : \dot{\theta} \mapsto V_{st}$

Also describes relationship between wrenches applied at end effector and joint torques.

- fundamental for interacting with the environment (e.g. controlling force)

Recall $\delta W = V_{st}^{b} \cdot F_b$ for body wrench F_b at frame T.

$= \dot{\theta} \cdot \tau$ for joint velocities $\dot{\theta} \in \mathbb{R}^n$ and joint torques $\tau \in \mathbb{R}^n$

If assume no energy loss or storage

$$\delta W_{joints} = \delta W_{end\ eff.}$$

$$\dot{\theta} \cdot \tau = V_{st}^{b} \cdot F_b$$

$$\dot{\theta}^T \tau = (\underbrace{V_{st}^{b}}_{J_{st}^{b}(\theta)\dot{\theta}})^T F_b$$

$$\dot{\theta}^T \tau = \dot{\theta}^T (J_{st}^{b}(\theta))^T F_b$$

$$\boxed{\begin{aligned} \tau &= (J_{st}^{b}(\theta))^T F_b, && \text{for body wrench} \\ &= (J_{st}^{s}(\theta))^T F_s, && \text{for spatial wrench} \end{aligned}}$$

- If J_{st} is square & non-singular, then $F = (J_{st}^T)^{-1} \tau$
  or pseudo-inverse in some cases of non-square.
- At singular configs., get a rank drop in J_{st}, so
  J_{st} has a non-trivial nullspace, i.e. if F is in null space of J_{st}^T, then $\tau = J_{st}^T F = 0$.
  - $\Rightarrow$ Robot structure resists forces at end-effector (not joint actuators)
    e.g. locked knees while standing.

<u>Ex</u>: SCARA Manipulator

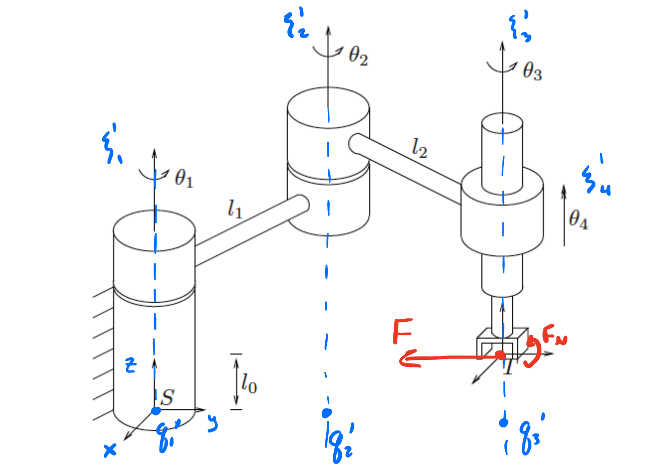

Figure 3.13: SCARA manipulator in non-reference configuration.

Recall: $J_{st}^s(\theta) = \left[\xi_1', \xi_2', \xi_3', \xi_4' \right]$

where $\xi_i' = \begin{bmatrix} -\omega_i \times q_i' \\ \omega_i \end{bmatrix}$ for $i = 1,2,3$

and $\xi_4' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$q_1' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad q_2' = \begin{bmatrix} -l_1 \sin\theta \\ l_1 \cos\theta \\ 0 \end{bmatrix} \qquad q_3' = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$J_{st}^s = \begin{bmatrix} 0 & l_1 c_1 & l_1 c_1 + l_2 c_{12} & 0 \\ 0 & l_1 s_1 & l_1 s_1 + l_2 s_{12} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Q: What joint torques are needed to resist the spatial wrench applied to end effector

$$F = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

for any $\theta \in \mathbb{R}^4$?

$$\tau = (J^s_{st}(\theta))^T F = \begin{bmatrix} 0 \\ -l_1 s_1 \\ -l_1 s_1 - l_2 s_{12} \\ 0 \end{bmatrix}$$

<u>Note</u>: In reference config, $\tau = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

https://join.iclicker.com/MMAW

Q: What joint torques are needed to resist

$$F_N = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

for any $\theta \in \mathbb{R}^4$?

(a.) $\begin{bmatrix} 0 \\ l_1 c_1 \\ l_1 c_1 + l_2 c_{12} \\ 0 \end{bmatrix}$

(b.) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(c.) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

(d) None of the above

II. Dynamics

Def: Generalized coordinates $q \in \mathbb{R}^n$ of a system are the minimal set of variables that determine position of all particles that make up a robot.

→ joint configuration vector (when assuming rigid links & joints)

→ write Equations of Motion (EoM) for systems in terms of gen. coords.

Def: Dynamic system state $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \in \mathbb{R}^{2n}$

Def: The Lagrangian function $\mathcal{L}(q, \dot{q}) \in \mathbb{R}^1$ given by

$$\mathcal{L}(q, \dot{q}) = \underbrace{K(q, \dot{q})}_{\text{kinetic energy}} - \underbrace{P(q)}_{\text{potential energy}}$$

Thm 4.1: Lagrange's Equations or Euler-Lagrange Equations

The EoM of a mechanical system w/ generalized coords. $q \in \mathbb{R}^n$ and Lagrangian $\mathcal{L}(q, \dot{q})$ are given by:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \Upsilon_i \quad , \quad i = 1, \ldots, n$$

for generalized external force Υ_i acting on i^{th} coordinate.

↳ that contains non-conservative/external forces such actuator torques, friction torques, environment interaction forces (e.g. contact constraints), end effector forces ($\tau = J_{st}^T F$), etc.

Set of n 2^{nd}-order ODE's, i.e. vector ODE:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \tau$$

* From the "principal of least action" (variational principle)