Course: MECH 567: Robot Kinematics & Dynamics
Instructor: Daniel Bruder, PhD

Homework 2

Problem 1

Q1) $e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\theta) + \frac{\hat{\omega}^2}{\|\omega\|^2}(1-\cos(\|\omega\|\theta))$

Sol:- We know that

$$e^{\hat{\omega}\theta} = I + \theta\hat{\omega} + \frac{\theta^2}{2!}\hat{\omega}^2 + \frac{\theta^3}{3!}\hat{\omega}^3 + \frac{\theta^4}{4!}\hat{\omega}^4 \cdots\cdots$$

$$e^{\hat{\omega}\theta} = I + \left(\theta\hat{\omega} + \frac{\theta^3}{3!}\hat{\omega}^3 + \frac{\theta^5}{5!}\hat{\omega}^5 + \cdots\right) + \left(\frac{\theta^2}{2!}\hat{\omega}^2 + \frac{\theta^4}{4!}\hat{\omega}^4 + \cdots\cdots\right)$$

For Odd powers of $\hat{\omega}$:-

$\hat{\omega} = \hat{\omega}$

$\hat{\omega}^3 = -\|\omega\|^2\hat{\omega}$

$\hat{\omega}^5 = \hat{\omega}^3\hat{\omega}^2 = (-\|\omega\|^2\hat{\omega})(\omega\omega^T - \|\omega\|^2 I) = (-\|\omega\|^2)(\cancel{\hat{\omega}\omega}\omega^T - \|\omega\|^2\hat{\omega}) = \|\omega\|^4\hat{\omega}$

$\therefore$ It can be generalised as :- $\underline{\hat{\omega}^{2n-1} = (-1)^{n-1}\|\omega\|^{2n-2}\hat{\omega}}$

For Even powers of $\hat{\omega}$:-

$$\hat{\omega}^2 = \hat{\omega}^2$$

$$\hat{\omega}^4 = \hat{\omega}^3 \cdot \hat{\omega} = -\|\omega\|^2 \hat{\omega}^2$$

$$\hat{\omega}^6 = \hat{\omega}^5 \cdot \hat{\omega} = \|\omega\|^4 \hat{\omega}^2$$

$\therefore$ It can be generalised as:- $\hat{\omega}^{2n} = (-1)^{n+1} \|\omega\|^{2n-2} \hat{\omega}^2$

$$e^{\hat{\omega}\theta} = I + \left(\theta\hat{\omega} + \frac{\theta^3}{3!}\hat{\omega}^3 + \frac{\theta^5}{5!}\hat{\omega}^5 + \cdots \right) + \left(\frac{\theta^2}{2!}\hat{\omega}^2 + \frac{\theta^4}{4!}\hat{\omega}^4 + \cdots\cdots \right)$$

Using the results found above:-

$$e^{\hat{\omega}\theta} = I + \left(\theta\hat{\omega} - \frac{\theta^3}{3!}\|\omega\|^2\hat{\omega} + \frac{\theta^5}{5!}\|\omega\|^4\hat{\omega} + \cdots\cdots \right) + \left(\frac{\theta^2}{2!}\hat{\omega}^2 - \frac{\theta^4}{4!}\|\omega\|^2\hat{\omega}^2 + \frac{\theta^6}{6!}\|\omega\|^4\hat{\omega}^2 + \cdots \right)$$

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\left(\|\omega\|\theta - \|\omega\|^3\frac{\theta^3}{3!} + \|\omega\|^5\frac{\theta^5}{5!} + \cdots \right) + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(\|\omega\|^2\frac{\theta^2}{2!} - \|\omega\|^4\frac{\theta^4}{4!} + \|\omega\|^6\frac{\theta^6}{6!} + \cdots\cdots \right)$$

$$\boxed{e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\theta) + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(1 - \cos(\|\omega\|\theta)\right)}$$

Problem 2

$$\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad q = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

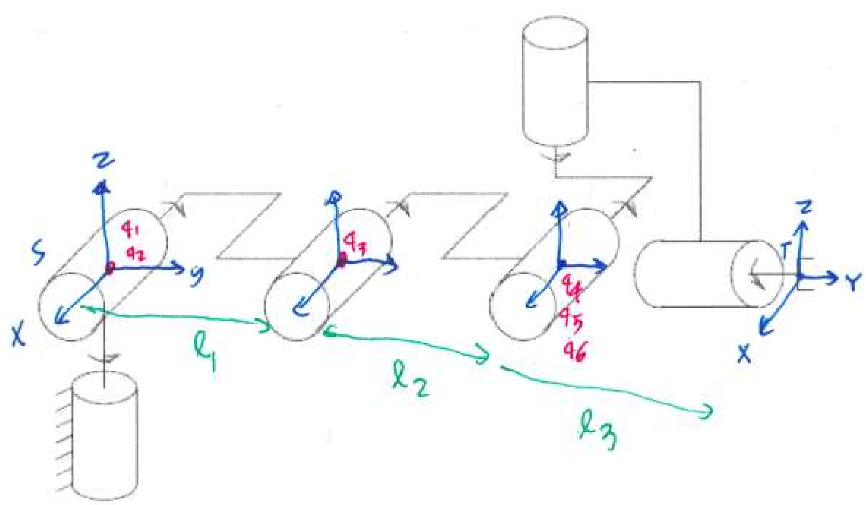
$$h = \frac{1}{10 \cdot 2\pi} = \frac{1}{20\pi} \qquad \text{(translation per revolution)}$$

$$v = -\omega \times q + h w$$

$$\xi = \begin{bmatrix} 1 \\ -1 \\ \frac{1}{20\pi} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Problem 3

(i)



(i) Elbow manipulator

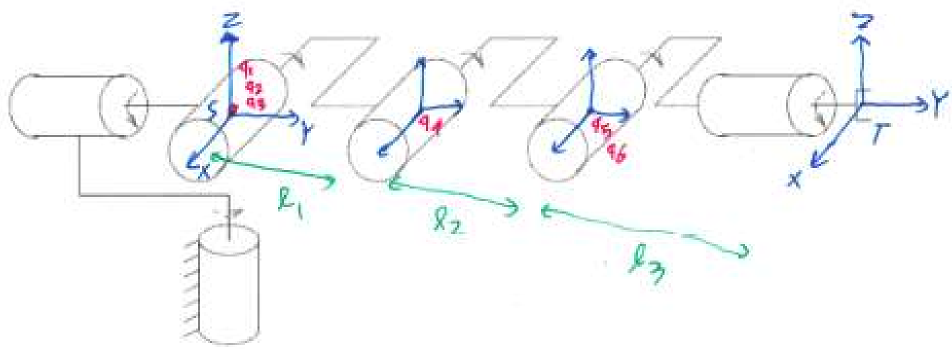
$$\omega_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \omega_2 = \omega_3 = \omega_4 = \begin{bmatrix}-1\\0\\0\end{bmatrix} \qquad \omega_5 = \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \omega_6 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$q_1 = q_2 = \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad q_3 = \begin{bmatrix}0\\L_1\\0\end{bmatrix} \qquad q_4 = q_5 = q_6 = \begin{bmatrix}0\\L_1 + L_2\\0\end{bmatrix}$$

$$g_{st}(0) = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & L_1 + L_2 + L_3\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \qquad \xi = \begin{bmatrix}-\omega \times q\\ \omega\end{bmatrix}$$

$$\xi_1 = \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \quad \xi_2 = \begin{bmatrix}0\\0\\0\\-1\\0\\0\end{bmatrix} \quad \xi_3 = \begin{bmatrix}0\\0\\L_1\\-1\\0\\0\end{bmatrix} \quad \xi_4 = \begin{bmatrix}0\\0\\L_1 + L_2\\-1\\0\\0\end{bmatrix} \quad \xi_5 = \begin{bmatrix}L_1 + L_2\\0\\0\\0\\0\\1\end{bmatrix} \quad \xi_6 = \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}$$

(ii)



(ii) Inverse elbow manipulator

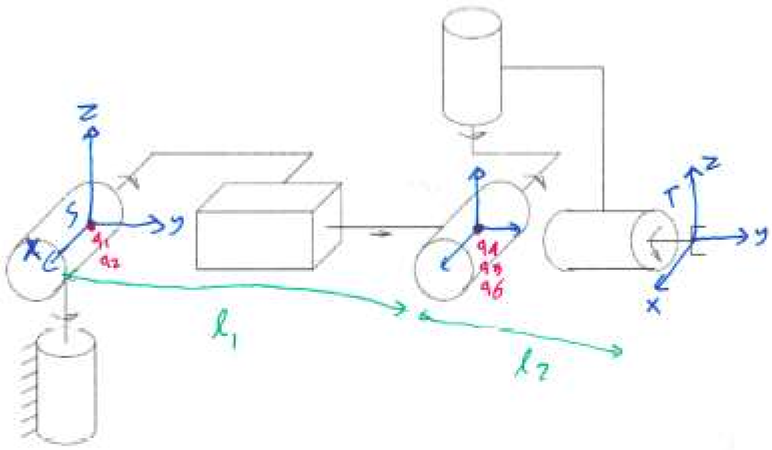
$$\omega_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \omega_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \omega_3 = \omega_4 = \omega_5 = \begin{bmatrix}-1\\0\\0\end{bmatrix} \qquad \omega_6 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$q_1 = q_2 = q_3 = \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad q_4 = \begin{bmatrix}0\\L_1\\0\end{bmatrix} \qquad q_5 = q_6 = \begin{bmatrix}0\\L_1 + L_2\\0\end{bmatrix}$$

$$g_{st}(0) = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & L_1 + L_2 + L_3\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \qquad \xi = \begin{bmatrix}-\omega \times q\\ \omega\end{bmatrix}$$

$$\xi_1 = \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \quad \xi_2 = \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix} \quad \xi_3 = \begin{bmatrix}0\\0\\0\\-1\\0\\0\end{bmatrix} \quad \xi_4 = \begin{bmatrix}0\\0\\L_1\\-1\\0\\0\end{bmatrix} \quad \xi_5 = \begin{bmatrix}0\\0\\L_1+L_2\\-1\\0\\0\end{bmatrix} \quad \xi_6 = \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}$$

(iii)



(iii) Stanford manipulator

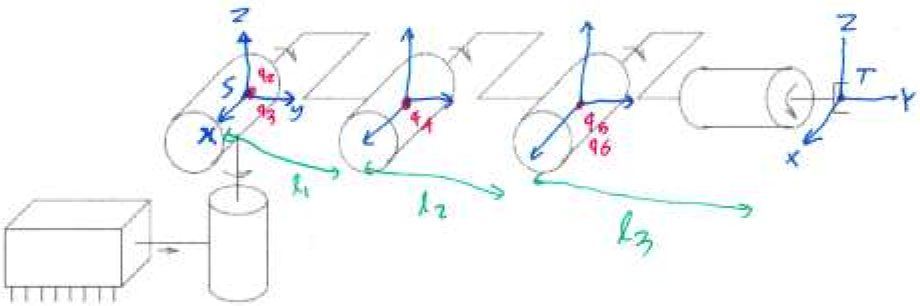
$$\omega_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \omega_2 = \begin{bmatrix}-1\\0\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \omega_4 = \begin{bmatrix}-1\\0\\0\end{bmatrix} \qquad \omega_5 = \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \omega_6 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$q_1 = q_2 = \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad q_4 = q_5 = q_6 = \begin{bmatrix}0\\L_1\\0\end{bmatrix}$$

$$g_{st}(0) = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & L_1+L_2\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \qquad \xi = \begin{bmatrix}-\omega \times q\\ \omega\end{bmatrix}$$

$$\xi_1 = \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \quad \xi_2 = \begin{bmatrix}0\\0\\0\\-1\\0\\0\end{bmatrix} \quad \xi_3 = \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix} \quad \xi_4 = \begin{bmatrix}0\\0\\L_1\\-1\\0\\0\end{bmatrix} \quad \xi_5 = \begin{bmatrix}L_1\\0\\0\\0\\0\\1\end{bmatrix} \quad \xi_6 = \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}$$

(iv)



(iv) Rhino robot

$$v_1 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \omega_2 = \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \omega_3 = \omega_4 = \omega_5 = \begin{bmatrix}-1\\0\\0\end{bmatrix} \qquad \omega_6 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$q_2 = q_3 = \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad q_4 = \begin{bmatrix}0\\L_1\\0\end{bmatrix} \qquad q_5 = q_6 = \begin{bmatrix}0\\L_1+L_2\\0\end{bmatrix}$$

$$g_{st}(0) = \begin{bmatrix}1&0&0&0\\0&1&0&L_1+L_2+L_3\\0&0&1&0\\0&0&0&1\end{bmatrix} \qquad \xi = \begin{bmatrix}-\omega\times q\\ \omega\end{bmatrix}$$

$$\xi_1 = \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix} \quad \xi_2 = \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \quad \xi_3 = \begin{bmatrix}0\\0\\0\\-1\\0\\0\end{bmatrix} \quad \xi_4 = \begin{bmatrix}0\\0\\L_1\\-1\\0\\0\end{bmatrix} \quad \xi_5 = \begin{bmatrix}0\\0\\L_1+L_2\\-1\\0\\0\end{bmatrix} \quad \xi_6 = \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}$$

Problem 4

$$e^{\hat{\xi}_1\theta_1} = \begin{bmatrix}\cos\theta_1 & -\sin\theta_1 & 0 & 0\\ \sin\theta_1 & \cos\theta_1 & 0 & 0\\ 0&0&1&0\\0&0&0&1\end{bmatrix} \qquad e^{\hat{\xi}_2\theta_2} = \begin{bmatrix}1&0&0&0\\0&\cos\theta_2&\sin\theta_2&0\\0&-\sin\theta_2&\cos\theta_2&0\\0&0&0&1\end{bmatrix}$$

$$e^{\hat{\xi}_3\theta_3} = \begin{bmatrix}1&0&0&0\\0&1&0&\theta_3\\0&0&1&0\\0&0&0&1\end{bmatrix} \qquad e^{\hat{\xi}_4\theta_4} = \begin{bmatrix}1&0&0&0\\0&\cos\theta_4&\sin\theta_4&L_1(1-\cos\theta_4)\\0&-\sin\theta_4&\cos\theta_4&L_1\sin\theta_4\\0&0&0&1\end{bmatrix}$$

$$e^{\hat{\xi}_5 \theta_5} = \begin{bmatrix} \cos\theta_5 & -\sin\theta_5 & 0 & L_1 \sin\theta_5 \\ \sin\theta_5 & \cos\theta_5 & 0 & L_1 (1-\cos\theta_5) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad e^{\hat{\xi}_6 \theta_6} = \begin{bmatrix} \cos\theta_6 & 0 & \sin\theta_6 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_6 & 0 & \cos\theta_6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 5

(i)

```
ClearAll["Global`*"]
Needs["Screws`" , "C://Mathematica//Screws.m"]
xi1 = { 0 , 0 , 0 , 0 , 0 , 1 };
xi2 = { 0 , 0 , 0 , -1 , 0 , 0 };
xi3 = { 0 , 0 , L1 , -1 , 0 , 0 };
xi4 = { 0 , 0 , L1 + L2 , -1 , 0 , 0 };
xi5 = { L1 + L2 , 0 , 0 , 0 , 0 , 1 };
xi6 = { 0 , 0 , 0 , 0 , 1 , 0 };
MatrixForm[ e1 = TwistExp[ xi1 , q1[t] ] ];
MatrixForm[ e2 = TwistExp[ xi2 , q2[t] ] ];
MatrixForm[ e3 = TwistExp[ xi3 , q3[t] ] ];
MatrixForm[ e4 = TwistExp[ xi4 , q4[t] ] ];
MatrixForm[ e5 = TwistExp[ xi5 , q5[t] ] ];
MatrixForm[ e6 = TwistExp[ xi6 , q6[t] ] ];
MatrixForm[ gst0 = { { 1 , 0 , 0 , 0 } , { 0 , 1 , 0 , L1 + L2 + L3 } ,
{ 0 , 0 , 1 , 0 } , { 0 , 0 , 0 , 1 } } ];
MatrixForm[ gst = Simplify[e1.e2.e3.e4.e5.e6.gst0] ]
```

(ii)

```
ClearAll["Global`*"]
Needs["Screws`" , "C://Mathematica//Screws.m"]
xi1 = { 0 , 0 , 0 , 0 , 0 , 1 };
xi2 = { 0 , 0 , 0 , 0 , 1 , 0 };
xi3 = { 0 , 0 , 0 , -1 , 0 , 0 };
xi4 = { 0 ,0 , L1 , -1 , 0 , 0 };
xi5 = { 0 , 0 , L1 + L2 , -1 , 0 , 0 };
xi6 = { 0 , 0 , 0 , 0 , 1 , 0 };
MatrixForm[ e1 = TwistExp[ xi1 , q1[t] ] ];
MatrixForm[ e2 = TwistExp[ xi2 , q2[t] ] ];
MatrixForm[ e3 = TwistExp[ xi3 , q3[t] ] ];
MatrixForm[ e4 = TwistExp[ xi4 , q4[t] ] ];
MatrixForm[ e5 = TwistExp[ xi5 , q5[t] ] ];
MatrixForm[ e6 = TwistExp[ xi6 , q6[t] ] ];
MatrixForm[ gst0 = { { 1 , 0 , 0 , 0 } , { 0 , 1 , 0 , L1 + L2 + L3 } ,
{ 0 , 0 , 1 , 0 } , { 0 , 0 , 0 , 1 } } ];
MatrixForm[ gst = Simplify[e1.e2.e3.e4.e5.e6.gst0] ]
```

(iii)

```
ClearAll["Global`*"]
Needs["Screws`" , "C://Mathematica//Screws.m"]
xi1 = { 0 , 0 , 0 , 0 , 0 , 1 };
xi2 = { 0 , 0 , 0 , -1 , 0 , 0 };
xi3 = { 0 , 1 , 0 , 0 , 0 , 0 };
xi4 = { 0 , 0 , L1 , -1 , 0 , 0 };
xi5 = { L1 , 0 , 0 , 0 , 0 , 1 };
xi6 = { 0 , 0 , 0 , 0 , 1 , 0 };
MatrixForm[ e1 = TwistExp[ xi1 , q1[t] ] ];
MatrixForm[ e2 = TwistExp[ xi2 , q2[t] ] ];
MatrixForm[ e3 = TwistExp[ xi3 , q3[t] ] ];
MatrixForm[ e4 = TwistExp[ xi4 , q4[t] ] ];
MatrixForm[ e5 = TwistExp[ xi5 , q5[t] ] ];
MatrixForm[ e6 = TwistExp[ xi6 , q6[t] ] ];
MatrixForm[ gst0 = { { 1 , 0 , 0 , 0 } , { 0 , 1 , 0 , L1 + L2 } , { 0 , 0 ,
{ 0 , 0 , 0 , 1 } } ];
MatrixForm[ gst = Simplify[e1.e2.e3.e4.e5.e6.gst0] ]
```

(iv)

```
ClearAll[" Global `*"]
Needs[" Screws `"  , "C://Mathematica//Screws.m"]
xi1 = { 0 , 1 , 0 , 0 , 0 , 0 };
xi2 = { 0 , 0 , 0 , 0 , 0 , 1 };
xi3 = { 0 , 0 , 0 , -1 , 0 , 0 };
xi4 = { 0 , 0 , L1 , -1 , 0 , 0 };
xi5 = { 0 , 0 , L1 + L2 , -1 , 0 , 0 };
xi6 = { 0 , 0 , 0 , 0 , 1 , 0 };
MatrixForm[ e1 = TwistExp[ xi1 , q1[t] ] ];
MatrixForm[ e2 = TwistExp[ xi2 , q2[t] ] ];
MatrixForm[ e3 = TwistExp[ xi3 , q3[t] ] ];
MatrixForm[ e4 = TwistExp[ xi4 , q4[t] ] ];
MatrixForm[ e5 = TwistExp[ xi5 , q5[t] ] ];
MatrixForm[ e6 = TwistExp[ xi6 , q6[t] ] ];
MatrixForm[ gst0 = { { 1 , 0 , 0 , 0 } , { 0 , 1 , 0 , L1 + L2 + L3 } ,
{ 0 , 0 , 1 , 0 } , { 0 , 0 , 0 , 1 } } ];
MatrixForm[ gst = Simplify[e1.e2.e3.e4.e5.e6.gst0] ]
```