Warm-up Problem: https://join.iclicker.com/MMAW

The manipulator below is in its reference config

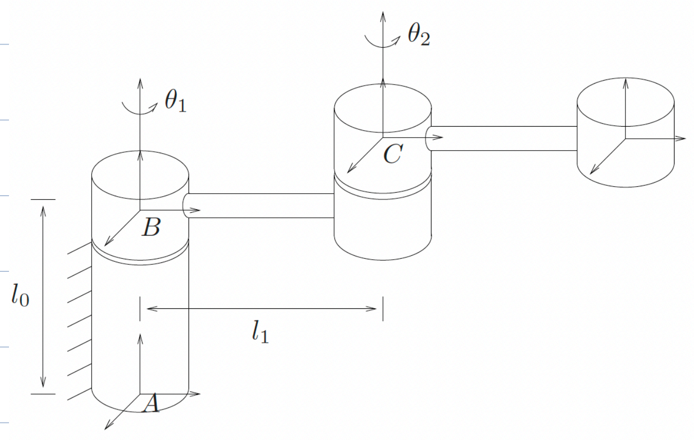


Figure 2.12: Two degree of freedom manipulator.

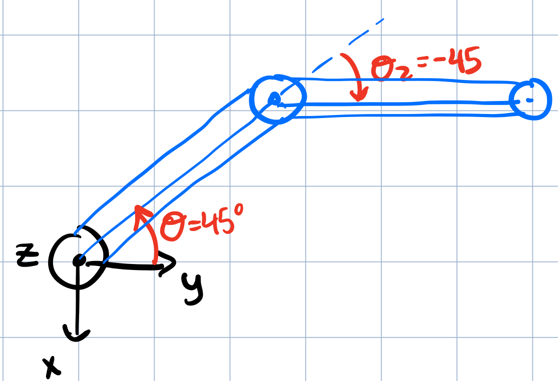
Consider the following 2 cases:

① Joint 1 rotates by $\theta_1 = 45°$, then joint 2 rotates by $\theta_2 = -45°$

② Joint 2 rotates by $\theta_2 = -45°$, then joint 1 rotates by $\theta_1 = 45°$

Q: Is the final configuration of the robot the same in both cases?

(a.) Yes, (b.) No, (c.) Neither



<u>Lesson 9</u>: Forward Kinematics (cont'd)

I. Order of the Product of Exponentials (MLS 3.2.2)

II. D-H / POE Equivalence (MLS 3.2.3)

I. Order of the POE

Recall, POE: $$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0) \qquad (1)$$

Note: You should get the same $g_{st}(\theta)$ no matter the order in which you deform the joints.

Q: What happens if we change the order of the exp. transformations?

- If move θ_1 first, then 2^{nd} axis and beyond would change:

$$\xi_2' = Ad_{(e^{\hat{\xi}_1\theta_1})} \cdot \xi_2$$ to get coords. of 2^{nd} axis after moving θ_1.

Def: Adjoint Transformation associated with $g = (R, p) \in SE(3)$, $Ad_g : \mathbb{R}^6 \to \mathbb{R}^6$, transforming twist coordinates from one frame to another:

$$Ad_g = \begin{bmatrix} R & \hat{p} \cdot R \\ 0 & R \end{bmatrix} \in \mathbb{R}^{6\times6}, \quad \text{where } \hat{p} \in so(3)$$

Note: $Ad_g^{-1} = Ad_{g^{-1}}$

Lemma 2.13: If $\hat{\xi} \in se(3)$ is a twist with coords $\xi = (\hat{\xi})^\vee \in \mathbb{R}^6$, then for any $g \in SE(3)$, $\hat{\xi}' = g\hat{\xi}g^{-1}$ is a twist with coords. $\xi' = (\hat{\xi}')^\vee = Ad_g \cdot \xi \in \mathbb{R}^6$.

I.e., $(Ad_g \cdot \xi)^\wedge = g\hat{\xi} \cdot g^{-1}$ and $(g\hat{\xi}g^{-1})^\vee = Ad_g \xi$

* Adjoint transforms twist coords (vector in $\mathbb{R}^6$)
* Similarity transform transforms twists (matrix, $se(3)$)

- Hence, $\exp(\hat{\xi}_2' \theta_2) = \exp(g \hat{\xi}_2 g^{-1} \theta_2) = g \exp(\hat{\xi}_2 \theta_2) g^{-1}$

Describing the motion about the new joint 2 axis (after moving θ_1)

$g = \exp(\hat{\xi}_1 \theta_1)$ which describes the motion due to joint 1 moving θ_1.

<u>Ex</u>: 2-DOF

$$g_{st}(\theta) = e^{\hat{\xi}_2' \theta_2} e^{\hat{\xi}_1 \theta_1} g_{st}(0)$$

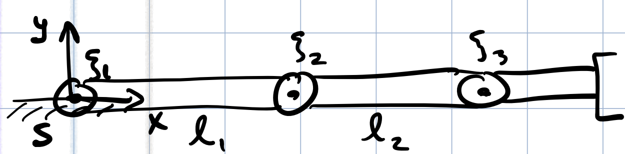
$$= \left(e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{-\hat{\xi}_1 \theta_1}\right) e^{\hat{\xi}_1 \theta_1} g_{st}(0)$$

$$= e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g_{st}(0)$$

* Order of transformations doesn't matter physically, just need to account for it mathematically.

<u>Ex</u>: 3-DOF

Ref. config ($\theta = 0$)

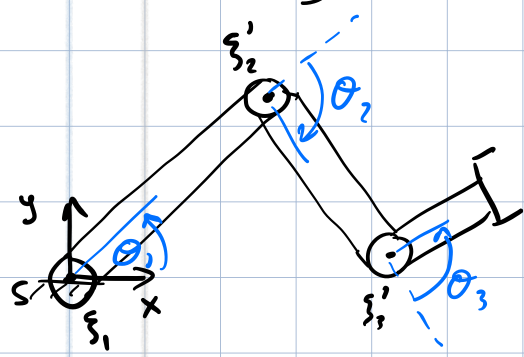


$$\xi_i = \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix}$$

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \omega_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \end{bmatrix}$$

Current config ($\theta \neq 0$)



$$\omega_1' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \omega_2' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \omega_3' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$q_1' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad q_2' = \begin{bmatrix} l_1 \cos\theta_1 \\ l_1 \sin\theta_1 \\ 0 \end{bmatrix}, \quad q_3' = \begin{bmatrix} q_2'(1) + l_2\cos\theta_2 \\ q_2'(2) + l_2 \sin\theta_2 \\ 0 \end{bmatrix}$$

$$\xi_2' = \begin{bmatrix} -\omega_2' \times q_2' \\ \omega_2' \end{bmatrix} = Ad_{e^{\hat{\xi}_1\theta_1}} \xi_2$$

$$\xi_3' = \begin{bmatrix} -\omega_3' \times q_3' \\ \omega_3' \end{bmatrix} = Ad_{(e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2})} \xi_3$$

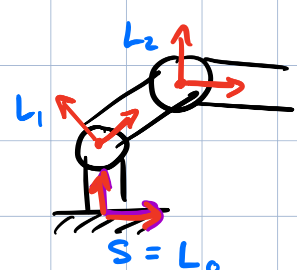
$$\vdots$$

$$\xi_i' = Ad_{(e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}})} \xi_i$$

II. D-H / POE Equivalence

Recall: D-H $\quad g_{st}(\theta) = g_{s,l_1}(\theta_1) \cdots g_{l_{n-1},l_n}(\theta_n)\, g_{l_n,t} \quad (2)$

where each g_{l_{i-1},l_i} transforms between adjacent link frames.



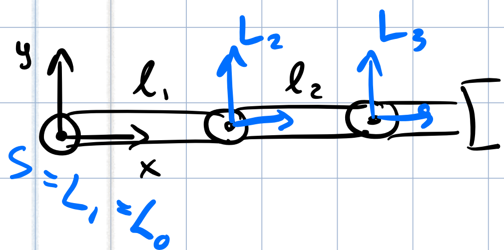
Adjacent transformations can be expressed in exp. coords:

$$g_{l_{i-1}, l_i}(\theta_i) = e^{\hat{\xi}_{i-1,i}\theta_i} g_{l_{i-1}, l_i}(0)$$

where $\hat{\xi}_{i-1,i}$ is the twist for i^{th} joint in coords of frame L_{i-1}.

<u>note</u>: Not the same as POE formula where each twist is defined in coords. of single <u>spatial</u> <u>frame</u>.

Reference Config ($\theta = 0$)



$$\xi_{2,3} = \begin{bmatrix} -\omega_{2,3} \times q_{2,3} \\ \omega_{2,3} \end{bmatrix}$$

$$\omega_{2,3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad q_{2,3} = \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}$$

$$g_{l_2, l_3}(0) = \begin{bmatrix} I & \begin{matrix} l_2 \\ 0 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix}$$

<u>Note</u>:

$$\underset{\text{spatial twist}}{\xi_i} = Ad_{g_{l_0, l_{i-1}}(0)} \underset{\text{relative twist}}{\xi_{i-1,i}}$$

To show that D-H and POE are equivalent, rewrite D-H (2) in exp. coords:

$$(2):\quad g_{st}(\theta) = g_{s,\ell_1}(\theta)\, g_{\ell_1,\ell_2}(\theta_2) \cdots g_{\ell_{n-1},\ell_n}(\theta_n)\, g_{\ell_n,t}$$

$$= \left(e^{\hat{\xi}_{0,1}\theta_1} g_{\ell_0,\ell_1}(0)\right)\left(e^{\hat{\xi}_{1,2}\theta_2} \underbrace{g_{\ell_1,\ell_2}(0)}_{g_{\ell_0,\ell_1}^{-1}(0)\cdot g_{\ell_0,\ell_2}(0)}\right)\cdots\left(e^{\hat{\xi}_{n-1,n}\theta_n} \underbrace{g_{\ell_{n-1},\ell_n}(0)}_{g_{\ell_0,\ell_{n-1}}^{-1}(0)\cdot g_{\ell_0,\ell_n}(0)}\right) g_{\ell_n,t}$$

Aside:

$$g_{\ell_0,\ell_2} = g_{\ell_0,\ell_1} g_{\ell_1,\ell_2}$$

$$\Downarrow$$

$$g_{\ell_1,\ell_2} = g_{\ell_0,\ell_1}^{-1} g_{\ell_0,\ell_2}$$

Plugging the purple expressions in:

Can group terms to get similarity transforms to change coords. of exponentials.

$$= e^{\hat{\xi}_{0,1}\theta_1} \underbrace{\left(g_{\ell_0,\ell_1}(0)\cdot e^{\hat{\xi}_{1,2}\theta_2}\, g_{\ell_0,\ell_1}^{-1}(0)\right)}_{e^{\hat{\xi}_{0,2}\theta_2}} g_{\ell_0,\ell_2}(0) \cdots$$

$$\cdots e^{\hat{\xi}_{n-1,n}\theta_n}\, g_{\ell_0,\ell_{n-1}}^{-1}(0)\Big) \underbrace{g_{\ell_0,\ell_n}(0)\, g_{\ell_n,t}}_{\substack{g_{\ell_0,t}(0) \\ = \\ g_{st}(0)}}$$

$$= e^{\hat{\xi}_1\theta_1}\, e^{\hat{\xi}_2\theta_2} \cdots e^{\hat{\xi}_n\theta_n}\, g_{st}(0) \qquad (1) \quad ✓$$

* This shows that D-H and POE are equivalent.