LGT6006:
Game Theoretical Methods

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Last Updated: December 5, 2023

1 Games with Incomplete Information

The *information* of concern can be market demand, private cost, and any efforts exerted by players that can influence payoffs. *Information asymmetry* does not only mean some knowledge is unknown to decision-making players, but the difference in timing for players to know that knowledge.

Two potential conflicts may arise:

- They may want to share that information with the other players, but are afraid that the other players do not simply believe what they reveal.
- They may want to hide that information or pass the wrong information to earn benefits, while the other players can design mechanisms to force real information revelation.

1.1 Introductory example

Example 1.1. Consider an industry with two firms: One incumbent (player 1), and one entrant (player 2). The two players engage in a simultaneous move game: player 1 decides whether to build a new plant; player 2 decides whether to enter. Player 2 is uncertain about Player 1's cost, which can be either 3 or 0, while Player 1 knows its cost.

Player 1 (H)	Enter	Not	Player 1 (L)	Enter	Not
Build	0,-1	2,0	Build	3,-1	5,0
Not	2,1	3,0	Not	2,1	3,0

Table 1: Payoffs of the two players

We introduce the *belief*. Player 2 has the belief that player 1 is of high type with probability p and of low type with probability $1-p$. If high type, the dominant strategy of player 1 is not to build; otherwise, the dominant strategy is to build. Entering is profitable for player 2 if and only if player 1 does not build. In expectation, the utility of player 2 is $u_1(E) = 2p - 1, u_1(N) = 0$. So, player 2 enters when $p > \frac{1}{2}$, stays out when $p < \frac{1}{2}$, and be indifferent when $p = \frac{1}{2}$.

Example 1.2. If the payoffs are changed as follows.

Player 1 (H)	Enter	Not	Player 1 (L)	Enter	Not
Build	0,-1	2,0	Build	1.5,-1	3.5,0
Not	2,1	3,0	Not	2,1	3,0

Table 2: Payoffs of the two players

If high type, the dominant strategy of player 1 is not to build; otherwise, there is no dominant strategy anymore. We suppose player 1 will make a *mixed* strategy that builds with probability x and does not build with probability $1-x$. Also, suppose player 2 will make a *mixed* strategy that enters with probability y and does not enter with probability $1-y$.

We calculate the utility of low-type player 1 building or not building.

$$u_1^L(B) = 1.5y + 3.5(1-y) = 3.5 - 2y, u_1^L(N) = 2y + 3(1-y) = 3 - y$$

Compare the two values, we have:

$$\begin{cases} x = 0 & u_1^L(B) < u_1^L(N) \Rightarrow y > \frac{1}{2} \\ x = 1 & u_1^L(B) > u_1^L(N) \Rightarrow y < \frac{1}{2} \\ x \in [0,1] & u_1^L(B) = u_1^L(N) \Rightarrow y = \frac{1}{2} \end{cases}$$

Also, we calculate the utility of player 2 entering or not entering.

$$u_2(E) = p \cdot 1 + (1-p) \cdot (-1 \cdot x + 1 \cdot (1-x)) = 1 - (1-p)2x, u_2(N) = 0$$

Compare the two values, we have:

$$\begin{cases} y = 0 & u_2(E) < u_2(N) \Rightarrow x > \frac{1}{2(1-p)} \\ y = 1 & u_2(E) > u_2(N) \Rightarrow x < \frac{1}{2(1-p)} \\ y \in [0,1] & u_2(E) = u_2(N) \Rightarrow x = \frac{1}{2(1-p)} \end{cases}$$

Finally, we can derive the best response of player 1(2) to player 2(1), and their intersection is the equilibrium(s), which is also dependent on player 2's belief (i.e., p). The illustrative figures of the three cases are shown as follows.

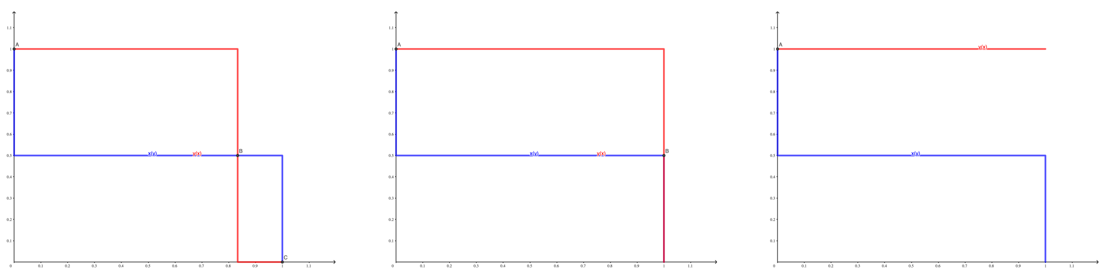

Figure 1: Figures of $y(x)$ and $x(y)$ to derive the equilibrium(s)

1.2 Formal definition

Definition 1. Bayesian Game. A Bayesian Game with n players has the form

$$G = (S_1, \ldots, S_n; \Theta_1, \ldots, \Theta_n; p_1, \ldots, p_n; u_1, \ldots, u_n)$$

- S_i is the set of player i's pure strategies;
- Θ_i is the set of possible types of player i, and $\Theta = \Theta_1 \times \Theta_2 \times \ldots \times \Theta_n$. Type $\theta_i \in \Theta_i$ is a description of i 's private information that is not common knowledge;
- $p_i\left(\theta_{-i} | \theta_i\right)$ is the probability that type θ_i assigns to every configuration of the opponents' types. We usually assume consistent beliefs that all p_i are derived from some common probability measure using the Bayesian rule;
- $u_i(s, \theta)$ where $\theta \in \Theta$ is player i 's payoff that depends on the actions and types of all the players;

- The game structure is common knowledge, but the actual realization of the vector of types is not common knowledge;
- A strategy for player i is a function $d_i : \Theta_i \to \Delta(S_i)$.

Definition 2. Bayesian Nash Equilibrium. A Bayesian Nash Equilibrium (BNE) is a strategy profile d^* such that for all i, θ_i, and s_i

$$\mathbb{E}\left[u_i\left(d^*(\theta), \theta\right) | \theta_i\right] \geq \mathbb{E}\left[u_i\left(\left(s_i, d^*_{-i}(\theta)\right), \theta\right) \middle| \theta_i\right]$$

Based on a realization of player i's type $\theta_i \in \Theta_i$, the player will have a belief of what the opponents' types (i.e., $\theta_{-i} \in \Theta_{-i}$) are. The player will further anticipate the opponents' strategies based on their types (i.e., $d^*_{-i}(\theta_{-i})$), and then obtain his/her best strategy (i.e., $d^*_i(\theta_i)$) to maximize the utility.

However, solving the system of inequalities is very difficult. Therefore, we introduce several classes of games where we can get elegant results.

1.3 Auction

An auction is a market situation with rules governing resource allocation based on bids from participants. The point lies in that, the participants do not know other participants' evaluations of the resource.

Example 1.3. First-Price Auction. In a first-price auction, the bidder who offers the highest price gets the good and pays the bid, and the other bidders pay nothing.

Let's consider a symmetric-uncertainty first-price auction when the valuations belong to an interval:

- There are two bidders, $i = 1, 2$ and one unit of good for sale;
- The seller imposes a reservation price $s_0 > \theta_L$;
- Player i 's valuation θ_i belongs to $[\theta_L, \theta_U]$, where $\theta_L \geq 0$. Each player knows his/her own valuation and has belief F with positive density f on $[\theta_L, \theta_U]$ about the rival's valuation;
- Suppose player i's bid is s_i. The utility of player i is $u_i = \theta_i - s_i$ if $s_i > s_j$ and $s_i \geq s_0$, and $u_i = 0$ if $s_i < s_j$ or $s_i < s_0$. The case where $s_i = s_j$ is omitted.

We study the (pure) bidding strategy, that is, $s_i : \Theta_i \to S_i$. When $\theta_i \leq s_0$, the player has no incentive to participate in the auction since the utility is always non-positive. We assume s_i is strictly increasing and continuous over $[s_0, \theta_U]$, then the inverse exists (denoted as $\theta_i(s_i)$) and the player's utility can be given by:

$$u_i(s_i|\theta_i) = (\theta_i - s_i)\mathbb{P}\left\{s_i > s_j(\theta_j)\right\} = (\theta_i - s_i)\mathbb{P}\left\{\theta_j < \theta_j(s_i)\right\} = (\theta_i - s_i)F(\theta_j(s_i))$$

Then player i is to optimize $L = (\theta_i - s_i)F(\theta_j(s_i))$ over s_i. By the First Order Condition:

$$\frac{dL}{d(s_i)} = -F(\theta_j(s_i)) + (\theta_i - s_i)f(\theta_j(s_i))\frac{d}{d(s_i)}(\theta_j(s_i)) = 0$$

we have

$$\frac{1}{\theta_i - s_i}d(s_i) = \frac{f(\theta_j(s_i))}{F(\theta_j(s_i))}d(\theta_j(s_i))$$

We focus on the symmetric setting, so $\theta_i = \theta_j$ and $s_i = s_j$. By taking the integration of both sides, we have

$$\int \frac{1}{\theta(s) - s} d(s) = \int \frac{f(\theta(s))}{F(\theta(s))} d(\theta(s)) = \ln\left[F(\theta(s))\right]$$

with $\theta(s_0) = s_0$ as the boundary condition, since we can easily find that $s_i = s_0$ is one of the best responses when $\theta_i = s_0$.

Note. Riley and Samuelson (AER, 1981). Under the symmetric setting, also by the First Order Condition,

$$\frac{dL}{ds(x)} = \frac{dL}{dx}\frac{dx}{ds} = 0 \Leftrightarrow \frac{dL}{dx} = \frac{d\left[(\theta - s(x))F(x)\right]}{dx} = \theta f(x) - (s(x)F(x))' = 0, \forall x = \theta$$

Then we have $s(\theta)F(\theta) = \int_{s_0}^{\theta} \theta f(\theta) d\theta + s_0 F(s_0) = \theta F(\theta) - \int_{s_0}^{\theta} F(x)dx$.

Therefore, the best response of type-θ bidder and the average payment are respectively

$$s(\theta) = \theta - \frac{\int_{s_0}^{\theta} F(x)dx}{F(\theta)}, P = \int_{s_0}^{\theta_U} \left[\theta F(\theta) - \int_{s_0}^{\theta} F(x)dx\right] f(\theta)d\theta$$

Example 1.4. Second-Price Auction. The highest bidder wins the object and pays the *second* highest bid. That is, $u_i = v_i - \max_{j \neq i} s_j$ if $s_i > \max_{j \neq i} s_j$, and $u_i = 0$ otherwise, where v_i is player i's evaluation.

We first consider the auction with complete information. For Bidder i, the valuation and strategy are denoted as v_i and s_i respectively, and we suppose the second highest price is r_i. We divide our discussion into the following cases:

- $v_i < s_i$. When $r_i \le v_i < s_i$, the utility is $v_i - r_i$, and the player is indifferent between v_i and s_i; when $v_i < s_i \le r_i$, the utility is 0, and the player is also indifferent between v_i and s_i; however, when $v_i < r_i < s_i$, the utility is $v_i - r_i$ which is negative, and the player has the incentive to decrease s_i to be no more than r_i (and then indifferent to v_i).

- $s_i < v_i$. When $r_i < s_i < v_i$, the utility is $v_i - r_i$, and the player is indifferent between v_i and s_i; when $s_i < v_i \le r_i$, the utility is 0, and the player is also indifferent between v_i and s_i; however, when $s_i \le r_i < v_i$, the player has the incentive to increase s_i to be more than r_i (and then indifferent to v_i).

To conclude, for each player i, the strategy of bidding his/her valuation (i.e., $s_i = v_i$) weakly dominates other strategies. At the equilibrium, the bidder I with the highest valuation wins the auction and has utility $v_I - \max_{j \neq I} v_j$.

It is important to note that, since bidding one's valuation is a *dominant* strategy, it does not matter whether the bidders have information about others' valuations. That is why the Second-Price Auction is so widely used around the world.

A problem arising here is how to compare the payment of the two kinds of actions. For the First-Price Auction, the strategy is determined by the first order condition and the payment is the maximal one of them, while for the Second-Price Auction, the payment at equilibrium is just v_I.

For the Second-Price Auction with the same settings as the above First-Price Auction, the average payment can be calculated by:

$$P = \int_{s_0}^{\theta_U} \left[\int_{s_0}^{\theta} x f(x)dx + s_0 F(s_0)\right] f(\theta)d\theta$$

where we assume that, when the second price $x < s_0$, the winner should pay the reservation price of the seller (i.e., s_0).

1.4 Adverse selection

Adverse selection refers to a situation where individuals have hidden characteristics and a selection process results in a pool of individuals with undesirable characteristics.

Example 1.5. Lemons Model (Akerlof, 1970). Suppose there are two types of cars: lemons and peaches. A peach is worth \$3,000 to a buyer and \$2,500 to a seller. A lemon is worth \$2,000 to a buyer and \$1,000 to a seller. There are twice as many lemons as peaches.

If both buyers and sellers have full information, the price of peaches and lemons will be 3k and 2k respectively. On the other hand, if both with no information, buyers value the car at $(1 \cdot 3 + 2 \cdot 2)/3 = 2.3$k, sellers value the car at $(1 \cdot 2.5 + 2 \cdot 1)/3 = 1.5$k, and the price of both types will be 2.3k in the seller market. However, in the adverse selection where buyers have no information but sellers do, buyers are only willing to pay 2.3k. Sellers do not sell any cars at a price below 1k, sell only lemons at a price between 1k and 2.5k, and sell both lemons and peaches at a price above 2.5k, resulting in the situation where sellers will not supply peach cars. At the equilibrium, only lemon cars are put on the market and they are sold at 2k.

Faced with adverse selection, there are some mechanisms to mitigate the market inefficiency.

Example 1.6. Signaling Game (Spence, 1974). In the labor market, there are two types of workers, one of high quality with productivity θ_H and the other of low quality with productivity θ_L. The proportion of workers of high quality is λ. $\theta_H > \theta_L$ are the values of the workers in the company. Workers can choose an education level $e \geq 0$ (which is the "signal"). Utility of workers is $U_t(w, e) = w - k_t g(e)$, where $t \in (H, L), 0 < k_H < k_L$, and $g(\cdot)$ is continuous and strictly increasing in e with $g(0) = 0$. So the inverse of g exists.

Nature determines each worker's type. Each worker chooses an education level. The firms observe the worker's education choices and make simultaneous wage offers. The worker decides whether to work and for which firm.

In order to maintain a utility level $u_t = u_0$, the wage w should satisfy $w - k_t g(e) = u_0 \Rightarrow w = u_0 + k_t g(e)$, which is also continuous and strictly increasing in e and $w(0) = u_0$. We call the curve $w(e) = u_0 + k_t g(e)$ iso-utility curve(s). Intuitively, we have

Definition 3. Single-crossing Property. If any two iso-utility curves $w(e) = u_0^H + k_H g(e)$ and $w(e) = u_0^L + k_L g(e)$ intersect, they can intersect only once.

If the firm has no information about workers' qualifications, it will give the offer $w = \lambda\theta_H + (1-\lambda)\theta_L$. But now, the firm's strategy is $w(e) = \lambda(e)\theta_H + (1-\lambda(e))\theta_L$, in which process λ is updated to $\lambda(e)$ based on workers' signals e:

$$e \to \lambda(e) \leftrightarrow w(e)$$

where the first arrow connecting the signal to the firm's strategy is very critical.

There may be many equilibria in the game, and we will focus on two credible types of pure strategy equilibrium. One is the separating equilibrium where $e_H^* \neq e_L^*$, the firm will update the belief to $\lambda^*(e_H^*) = 1$ and $\lambda^*(e_L^*) = 0$ (i.e., $w^*(e_t^*) = \theta_t$). The other is the pooling equilibrium where $e_H^* = e_L^*$, which will make the firm not differentiable to the workers' qualifications, thereby the $\lambda^*(e)$ will be the same as λ.

Consider the separating equilibrium. Define e_1, e_2 as $\theta_H = \theta_L + k_L g(e_1)$ and $\theta_H = \theta_L + k_H g(e_2)$. The firms' wage schedules will be

$$w^*(e) = \begin{cases} \theta_L & e < e_H^* \\ \theta_H & e \geq e_H^* \end{cases}$$

where $e_H^* \in [e_1, e_2]$ and $e_L^* = 0$. From the workers' perspective, we are to prove no worker will deviate from the equilibrium. For low-type workers, intuitively, based on the iso-utility curve, the "high" wage does not deserve their effort to be e_H^*. Mathematically, if $0 < e_L < e_H^*$, $u_L(e_L) = \theta_L - k_L g(e_L) < \theta_L$; if $e_L \geq e_H^*$, $u_L(e_L) = \theta_H - k_L g(e_L) \leq \theta_H - k_L g(e_1) = \theta_L$. For high-type workers, $e_H^* \leq e_2$ guarantees that they can earn at least θ_L thus they have no incentive to "lie flat", that is, the utility $u_H(e_H^*) = \theta_H - k_H g(e_H^*) \geq \theta_H - k_H g(e_2) = \theta_L$. For individuals of high-type workers, increasing or decreasing the effort will lower the utility (mathematically, $u_H(e_H) = \theta_H - k_H g(e_H) < \theta_H - k_H g(e_H^*) = u_H(e_H^*), e_H > e_H^*$ and $u_H(e_H) = \theta_L - k_H g(e_H) < \theta_L - k_H g(0) = \theta_L \leq u_H(e_H^*), 0 < e_H < e_H^*$). From the firm's perspective, it correctly schedules the wage according to the qualifications of the workers.

Then, consider the pooling equilibrium. Define e_p as $\mathbb{E}[\theta] = \theta_L + k_L g(e_p)$. The firms' wage schedules will be

$$w^*(e) = \begin{cases} \theta_L & e < e^* \\ \mathbb{E}[\theta] & e \geq e^* \end{cases}$$

where $e^* \leq e_p$. Also, both types of workers have a guaranteed utility of θ_L by signaling as e^*, that is, $u_H(e^*) = \mathbb{E}[\theta] - k_H g(e^*) > \mathbb{E}[\theta] - k_L g(e^*) = u_L(e^*) \geq \mathbb{E}[\theta] - k_L g(e_p) = \theta_L$. So, no one would be willing to decrease (then utility no more than θ_L) or increase (then wasted effort) the effort.

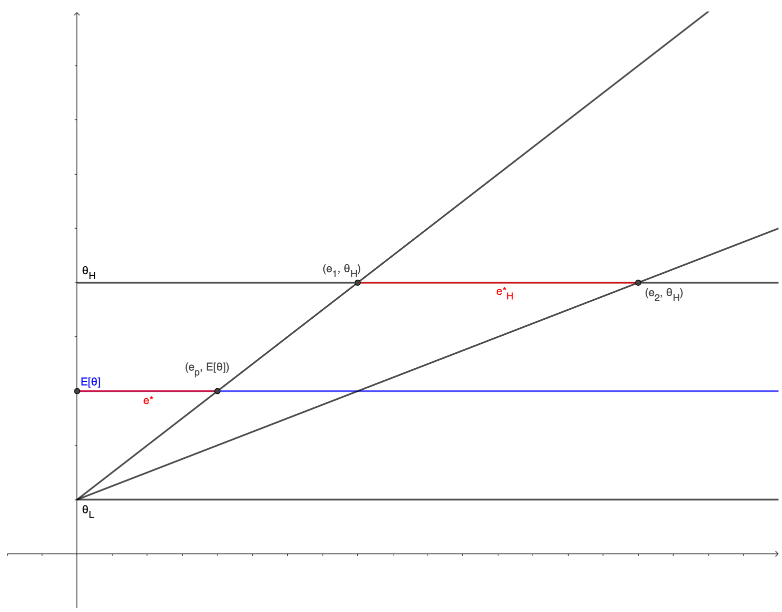

Figure 2: Positions of threshold effort and equilibria

However, as we can see here, there is much flexibility in both cases, thereby existing many (actually infinite) equilibria. Several approaches have been proposed to refine the equilibrium. We introduce the domination-based refinement here.

Suppose that the education levels e and e' are such that $\theta_L - k_t g(e) > \theta_H - k_t g(e')$. Then an equilibrium has reasonable beliefs if it is possible to sustain this equilibrium by beliefs that put zero probability on education level e' being chosen by a type-t worker. Intuitively, the LHS of the inequality is the worst-case utility of a worker choosing e, and the RHS of the inequality is the best-case utility of a worker choosing e'. If RHS is less than LHS, the worker has no incentive to choose e'.

In our story, the low-type workers have no incentive to choose efforts larger than e_1 since

$\theta_H - k_L g(e_1) = \theta_L$, which is the utility of the worker "lies flat". Based on the belief, the firm will schedule a wage of θ_H for signals no less than e_1. Further, for the separating equilibrium, high-type workers always choose education level e_1, and for the pooling equilibrium, situation where the pooling education level is such that $\theta_H - k_H g(e_1) > \mathbb{E}[\theta] - k_H g(e^*)$ will be eliminated by refinement, where the LHS means that the high-type worker directly choose e_1, and the firm knows they are of high type (remember that, the low-type workers have no incentive to choose efforts larger than e_1), and the RHS means that the high-type workers choose e^*.

Instead of the party with private information (i.e., informed party) setting out signals to the uninformed party, *dually*, the uninformed party can also develop some contract terms for the informed party to choose, thus seeking some information.

Example 1.7. Sorting Game (Mussa and Rosen, 1978). A monopolist produces a product that can have different quality levels. The marginal production cost for a product of quality q is $c(q)$, where $c(0) = c'(0) = 0$, and $c'(q) > 0, c''(q) > 0$ for $q > 0$. We have two consumer types, $i = \alpha, \beta$, where $\beta > \alpha > 0$. Utility of type i consumer buying a unit of the good of quality q at price p is $iq - p$, and there are N_i type-i consumers.

If with perfect information, the monopolist can exercise perfect price discrimination by solving the optimization problem:

$$\begin{aligned} \max \quad & p - c(q) \\ \text{s.t.} \quad & iq - p \geq 0 \end{aligned}$$

At the optimum, the inequality must be binding. We can prove by contradiction: if $iq - p > 0$, the monopolist can increase p or decrease q to increase the objective. Given the equality, by substitution, we have $\max L(q) = iq - c(q)$. By the First Order Condition and the strict convexity of $c(q)$, the optimal solution (p_i, q_i) can be obtained by letting $i - c'(q_i) = 0$ and $p_i = iq_i$, for $i = \alpha, \beta$.

However, with incomplete information, the monopolist does not know the specific type of the arriving customer. If offering (p_α, q_α), it can extract all the consumer surplus for type α; but for type β, since $\beta q_\alpha - p_\alpha > \alpha q_\alpha - p_\alpha = 0$, it cannot exercise perfect price discrimination any more. Similarly, if offering (p_β, q_β), it can extract all the consumer surplus for type β; but for type α, since $\alpha q_\beta - p_\beta < \beta q_\beta - p_\beta = 0$, it will lose type α customers.

Faced with this situation, the monopolist can provide two combinations. When a customer arrives, they pick one combination. Then to determine how the combinations can be set, the monopolist solves the following problem:

$$\begin{aligned} \max \quad & \pi\left(q_\alpha, p_\alpha, q_\beta, p_\beta\right) && (1.1) \\ \text{s.t.} \quad & \pi = N_\alpha\left(p_\alpha - c\left(q_\alpha\right)\right) + N_\beta\left(p_\beta - c\left(q_\beta\right)\right) && (1.2) \\ & \beta q_\beta - p_\beta \geq \beta q_\alpha - p_\alpha && (1.3) \\ & \alpha q_\alpha - p_\alpha \geq \alpha q_\beta - p_\beta && (1.4) \\ & \beta q_\beta - p_\beta \geq 0 && (1.5) \\ & \alpha q_\alpha - p_\alpha \geq 0 && (1.6) \end{aligned}$$

The first two constraints, (1.3) and (1.4), guarantee that type i customers will choose combination (p_i, q_i) respectively (incentive compatibility). The second two constraints, (1.5) and (1.6), guarantee that customers will not directly leave (rationality), where the reservation price is set as zero.

We take a further look at the optimization problem.

Proposition 1.1. *Constraint* (1.3) *and* (1.6) *implies* (1.5).

Proof by:

$$\beta q_\beta - p_\beta \overset{(1.3)}{\geq} \beta q_\alpha - p_\alpha \overset{\beta>\alpha}{>} \alpha q_\alpha - p_\alpha \overset{(1.6)}{\geq} 0$$

Proposition 1.2. *At optimum, Constraint (1.3) must be binding.*

Proof by contradiction. If $\beta q_\beta - p_\beta > \beta q_\alpha - p_\alpha$, increasing p_β can make (1.3) binding while increasing the objective but not violating Constraint (1.4).

Proposition 1.3. *$q_\beta \geq q_\alpha$. And with $q_\beta \geq q_\alpha$, Constraint (1.4) is satisfied.*

Proof by:

$$q_\beta - q_\alpha \overset{(1.3)}{=} \frac{1}{\beta}(p_\beta - p_\alpha) \overset{(1.4)}{\geq} \frac{\alpha}{\beta}(q_\beta - q_\alpha) \overset{\beta>\alpha}{\Leftrightarrow} q_\beta - q_\alpha \geq 0$$

Proposition 1.4. *At optimum, Constraint (1.6) must be binding.*

Proof by contradiction. If $\alpha q_\alpha - p_\alpha > 0$, increasing p_α and p_β together can still make (1.3) binding while increasing the objective.

Finally, we simplify the problem as maximizing

$$\begin{aligned}\pi = N_\alpha(\alpha q_\alpha - c(q_\alpha)) + N_\beta(\beta(q_\beta - q_\alpha) + \alpha q_\alpha - c(q_\beta))\\ = (N_\alpha\alpha + N_\beta\alpha - N_\beta\beta)q_\alpha - N_\alpha c(q_\alpha) + N_\beta(\beta q_\beta - c(q_\beta))\end{aligned}$$

and we can obtain the optimal solution

$$c'(q_\alpha^*) = \alpha - \frac{N_\beta}{N_\alpha}(\beta - \alpha), c'(q_\beta^*) = \beta$$

where $q_\beta^* \geq q_\alpha^*$ by $c'' > 0$.

The implication here is that the highest type always has incentive compatibility binding, while the lowest type has rationality binding.

Then we conclude the discussion on finite types of consumers, and we will continue to discuss the case of infinite types of consumers.

Example 1.8. Sorting Game with Continuous Buyer's Type. There is continuous buyer's type on $[0, V]$ with CDF F. F is continuously differentiable with density f. The utility of a buyer with type v after buying quantity q for a payment of t is given by $u(q, v) - t$, where $u(0, v) = 0$. u is twice continuously differentiable with $u_1 \geq 0, u_2 \geq 0, u_{11} \leq 0$, and $u_{12} \geq 0$.

The seller commits to a menu of offers from which buyers choose the one to fits their needs. Let the menu be $\{q(i), t(i)\}$, where $q(i)$ and $t(i)$ are the quantity received and payment made by a consumer who chooses offer i. Let $\{q(v), t(v)\}$ be the offer optimally selected by type v. Denote $U(v) = u(q(v), v) - t(v)$. The seller's profit from a type-v buyer is $t(v) - cq(v)$, and the total profit is $\pi = \int_0^V [t(v) - cq(v)]f(v)dv$.

The seller is going to solve an optimization problem with infinite constraints:

$$\begin{aligned}\max \quad & \pi = \int_0^V [t(v) - cq(v)]f(v)dv && (1.7)\\ \text{s.t.} \quad & u(q(v), v) - t(v) \geq u(q(v'), v) - t(v') && (1.8)\\ & U(v) \geq 0 && (1.9)\end{aligned}$$

The problem is of very large size but with beautiful structure. So we can derive the closed-form solution with some standard procedures.

Proposition 1.5. *Constraints (1.8) and (1.9) is equivalent to Constraint (1.8) with $U(0) \geq 0$.*

Proof by:

$$U(v) \overset{(1.8)}{\geq} u(q(0), v) - t(0) \overset{u_2 \geq 0}{\geq} u(q(0), 0) - t(0) = U(0)$$

Proposition 1.6. *There exists a function $t : [0, V] \to R$ such that $\{q(v), t(v)\}$ satisfy Constraint (1.8) if and only if $q(v)$ is increasing in v.*

We first prove $\Rightarrow$. Suppose there are $v_1, v_2 \in [0, V]$ and $v_1 < v_2$. Then by (1.8) we have:

$$U(v_1) = u(q(v_1), v_1) - t(v_1) \geq u(q(v_2), v_1) - t(v_2)$$
$$U(v_2) = u(q(v_2), v_2) - t(v_2) \geq u(q(v_1), v_2) - t(v_1)$$

So,

$$u(q(v_1), v_2) - u(q(v_1), v_1) \leq U(v_2) - U(v_1) \leq u(q(v_2), v_2) - u(q(v_2), v_1)$$

where

$$u(q(v_1), v_2) - u(q(v_1), v_1) = \int_{v_1}^{v_2} u_2(q(v_1), x) dx$$
$$u(q(v_2), v_2) - u(q(v_2), v_1) = \int_{v_1}^{v_2} u_2(q(v_2), x) dx$$

By contradiction, if $q(v_1) > q(v_2)$, then by $u_{21} > 0$, $u_2(q(v_1), x) > u_2(q(v_2), x)$ and then $U(v_2) - U(v_1) \in \emptyset$.

We then prove $\Leftarrow$ by construction. Let $t(v) = u(q(v), v) - \int_0^v u_2(q(v), v) dv - U(0)$. If $q(v)$ is increasing in v, then

$$U(v_2) - U(v_1) = \int_{v_1}^{v_2} u_2(q(v), v) dv \in \left[\int_{v_1}^{v_2} u_2(q(v_1), v) dv, \int_{v_1}^{v_2} u_2(q(v_2), v) dv \right]$$

which means $\{q(v), t(v)\}$ satisfy Constraint (1.8).

The intuition of the construction is from that if Constraint (1.8) is satisfied, by the Intermediate Value Theorem,

$$u(q(v_2), v_1) - u(q(v_2), v_2) = \int_{v_2}^{v_1} u_2(q(v_2), x) dx = u_2(q(v_2), \zeta_1)(v_1 - v_2)$$
$$u(q(v_1), v_1) - u(q(v_1), v_2) = \int_{v_2}^{v_1} u_2(q(v_1), x) dx = u_2(q(v_1), \zeta_2)(v_1 - v_2)$$

Then, we have

$$u_2(q(v_2), \zeta_1) \leq \frac{U(v_1) - U(v_2)}{v_1 - v_2} \leq u_2(q(v_1), \zeta_2)$$

When $v_2 \to v_1$, ζ_1 and ζ_2 both converge to v_1. By the Sandwich Theorem,

$$U'(v) = u_2(q(v), v)$$

Take integration on both sides, we obtain

$$U(v) = \int_0^v u_2(q(v), v) dv + U(0)$$

Substitute $t(v) = u(q(v), v) - \int_0^v u_2(q(v), v)dv - U(0)$ into π, we obtain

$$\pi = \int_0^V \left[u(q(v), v) - \int_0^v u_2(q(x), x)dx - U(0) - cq(v) \right] f(v)dv$$

where

$$\int_0^V \int_0^v u_2(q(x), x)dxf(v)dv = \int_0^V u_2(q(x), x)dxF(V) - \int_0^V u_2(q(v), v)F(v)dv$$
$$= \int_0^V (1 - F(v))u_2(q(v), v)dv$$

so,

$$\pi = \int_0^V \left[u(q(v), v) - \frac{1 - F(v)}{f(v)} u_2(q(v), v) - cq(v) \right] f(v)dv - U(0)$$

For the seller, it then should choose a non-decreasing function $q(v)$ and $U(0) \geq 0$ to maximize π. It is obvious that $U^*(0) = 0$ (the lowest type has rationality binding), and by the First Order Condition,

$$u_1(q^*, v) - \frac{1 - F(v)}{f(v)} u_{21}(q^*, v) - c = 0$$

We further assume that $u(q, v) = vu(q)$, where u is concave, $Vu'(0) > c$ and $\frac{1-F(v)}{f(v)}$ is decreasing in v. Then, we can conclude that, there exists v^* such that $v^* - \frac{1-F(v^*)}{f(v^*)} = \frac{c}{u'(0)}$ where for $v \leq v^*, q^*(v) = 0$ and for $v > v^*$, $q^*(v)$ satisfies $u'(q^*(v)) = \frac{cf(v)}{vf(c) - (1-F(v))}$.

1.5 Moral hazard

Moral hazard is a topic under the principal-agent model. The principal cannot perfectly observe the agent's effort and then has to depend pay on the performance to increase output at the expense of imposing extra risk on the agent.

Example 1.9. Observable Effort. Let the principal be the owner of a firm and the agent is the manager of the firm. Wages are denoted by w, and effort level is denoted by e. Agent's utility function is $u(w, e) = \sqrt{w} - e$ (risk aversion) and agent's reservation utility is 1. There are two effort levels for agents: $e_L = 0, e_H = \frac{1}{2}$. Let the profit of the firm be denoted by $\pi \in [10, 20]$, but profit depends stochastically on the effort level. Let $f(\pi|e)$ be the probability that the profit will be π when the effort level is e, and

$$f(\pi|e_L) = \frac{1}{10}, f(\pi|e_H) = \begin{cases} \frac{1}{100} & \pi \in [10, 15] \\ \frac{19}{100} & \pi \in [15, 20] \end{cases}$$

The principal offers a contract to the agent based on performance, as effort is not contractible. Let the contract take the form $w(\pi)$.

The firm should solve the optimization problem:

$$\begin{aligned} \max \quad & \int (\pi - w(\pi))f(\pi|e)d\pi \\ \text{s.t.} \quad & \int (\sqrt{w(\pi)} - e)f(\pi|e)d\pi \geq 1 \end{aligned}$$

Let λ be the Lagrange multiplier of the constraint, then the Lagrangian

$$L(\lambda, w(\pi)) = \int \left[(\pi - w(\pi)) + \lambda \left(\sqrt{w(\pi)} - e - 1 \right) \right] f(\pi|e) d\pi$$

According to the First Order Condition

$$-1 + \lambda \frac{1}{2\sqrt{w^*}} = 0 \Rightarrow w^* = \frac{\lambda^2}{4}$$

The wage cannot be 0 all the time. So $\lambda \neq 0$, which means the constraint is binding, that is

$$\int \left(\sqrt{w^*(\pi)} - e - 1 \right) f(\pi|e) d\pi = 0 \Rightarrow w^*(\pi) = (e+1)^2$$

By calculating $\int (\pi - (e_L + 1)^2) f(\pi|e_L) d\pi$ and $\int (\pi - (e_H + 1)^2) f(\pi|e_H) d\pi$, we conclude that the firm should enforce the high-level effort.

Example 1.10. Non-observable Effort with Risk Neutral Utility. Here, the agent's utility function is $u(w, e) = w - e$ (risk-neutral).

Reconsider the optimization problem of the firm:

$$\begin{aligned} \max \quad & \int (\pi - w(\pi)) f(\pi|e) d\pi \\ \text{s.t.} \quad & \int (w(\pi) - e) f(\pi|e) d\pi \geq 1 \end{aligned}$$

where

$$\int (w(\pi) - e) f(\pi|e) d\pi = \int w(\pi) f(\pi|e) d\pi - e$$

The firm has the incentive to decrease $w(\pi)$ to make the constraint binding. Then it is to maximize:

$$\int \pi f(\pi|e) d\pi - (e+1)$$

Then, $\int \pi f(\pi|e^*) d\pi - (e^* + 1)$ is the first-best solution with optimal e^*. We say that the firm can achieve the same value with a compensation contract when effort is non-observable.

Consider a compensation schedule of the form $w(\pi) = \pi - \alpha$, which is effectively like selling the project to the agent for α. Then the agent is to solve

$$\max \int (\pi - \alpha - e) f(\pi|e) d\pi = \int \pi f(\pi|e) d\pi - (e + \alpha)$$

which has the same form as the firm's decision problem. Though without the incentive-compatible constraint, the effort that maximizes the firm's utility will maximize the agent's utility. The first-best solution is obtained when $\alpha = \int \pi f(\pi|e^*) d\pi - (e^* + 1)$.

So, it is not a big issue for risk-neutral agents.

Example 1.11. Non-observable Effort with Risk Aversion Utility. Here, the agent's utility function is $u(w, e) = \sqrt{w} - e$ (risk aversion).

If the firm wants to impose low effort e_L, setting $w(\pi) = 1$ is enough. So, we focus on the situation when the principal wants to impose a high effort level. Then, the firm is to solve:

$$\max \quad \int (\pi - w(\pi)) f(\pi|e = 1/2) d\pi \tag{1.10}$$

$$\text{s.t.} \quad \int (\sqrt{w(\pi)} - 1/2) f(\pi|e = 1/2) d\pi \geq 1 \tag{1.11}$$

$$\int (\sqrt{w(\pi)} - 1/2) f(\pi|e = 1/2) d\pi \geq \int (\sqrt{w(\pi)} - 0) f(\pi|e = 0) d\pi \tag{1.12}$$

Let λ and μ be the Lagrange multiplier of the two constraints, then the Lagrangian

$$\begin{aligned} L(\lambda, \mu, w(\pi)) = \int \left[(\pi - w(\pi)) + (\lambda + \mu) \left(\sqrt{w(\pi)} - 1/2 \right) \right] f(\pi|e = 1/2) d\pi \\ - \lambda - \mu \int \sqrt{w(\pi)} f(\pi|e = 0) d\pi \end{aligned}$$

According to the First Order Condition

$$\begin{aligned} \int \left[-f(\pi|e = 1/2) + \frac{\lambda + \mu}{2\sqrt{w^*}} f(\pi|e = 1/2) - \frac{\mu}{2\sqrt{w^*}} f(\pi|e = 0) \right] d\pi = 0 \\ \Rightarrow 2\sqrt{w^*} = \lambda + \mu \left(1 - \frac{f(\pi|e = 0)}{f(\pi|e = 1/2)} \right) \end{aligned}$$

Proposition 1.7. *The two multipliers λ and μ must be positive.*

Proof by contradiction. If $\mu = 0$, the wage is a constant and the agent will make the lower effort. If $\lambda = 0$, then the RHS will become negative when $\pi \in [10, 15]$.

Therefore, the wage has two values for $\pi \in [10, 15]$ and $\pi \in [15, 20]$ respectively, and let w_1 and w_2 denote them respectively. Then, by the two binding constraints, we can obtain w_1^* and w_2^* by solving:

$$\begin{aligned} \sqrt{w_1} + 19\sqrt{w_2} = 30 \\ -9\sqrt{w_1} + 9\sqrt{w_2} = 10 \end{aligned}$$

By calculation, the firm's utility by imposing low effort is 14, and that by imposing high effort is 14.575, so it still prefers high effort by the agent, but compared with the case with observable effort (14.625), the firm's utility is sacrificed.

We make a few more extensions to the problem. Suppose there are n tasks that the agent performs. The agent chooses a vector of efforts $e = (e_1, \ldots, e_n) \in \mathbb{R}^n$. Cost of effort is $c(e)$, the resulting profit is $\pi = b(e) + \eta$, where η is a random variable with mean zero. Some signals about the agent's effort are $x = \theta(e) + \epsilon$, where $\epsilon \sim N(0, \Sigma)$.

The agent's compensation is based on the observable x. Assume that it has constant absolute risk aversion (CARA) utility $u(w(x) - c(e))$, where $u(z) = -e^{-rz}$. The principal offers only linear contract $w(x) = \alpha^T x + \beta$.

The agent's expected utility is

$$\mathbb{E} \left[-\exp \left\{ -r \left[\alpha^T (\theta(e) + \epsilon) + \beta - c(e) \right] \right\} \right] = -\exp \left\{ -r \left[\alpha^T \theta(e) + \beta - c(e) \right] \right\} \mathbb{E} \left[\exp \left\{ -r\alpha^T \epsilon \right\} \right]$$

where $\mathbb{E} \left[\exp \left\{ -r\alpha^T \epsilon \right\} \right]$ can be calculated by the moment generating function

$$M(t) = \mathbb{E} \left[\exp\{tX\} \right] = \exp \left\{ t^T \mu + \frac{1}{2} t^T \Sigma t \right\}$$

Then

$$\mathbb{E}\left[\exp\{-r\alpha^T\epsilon\}\right] = M(-r\alpha) = \exp\left\{-r\alpha^T\mu + \frac{1}{2}(-r\alpha)^T\Sigma(-r\alpha)\right\} = \exp\left\{\frac{r^2}{2}\alpha^T\Sigma\alpha\right\}$$

The agent's expected utility equals

$$u(\alpha^T\theta(e) + \beta - c(e) - \frac{r}{2}\alpha^T\Sigma\alpha) = -\exp\left\{-r\left[\alpha^T\theta(e) + \beta - c(e) - \frac{r}{2}\alpha^T\Sigma\alpha\right]\right\}$$

So, the certainty equivalent is $\alpha^T\theta(e) + \beta - c(e) - \frac{r}{2}\alpha^T\Sigma\alpha$.

The principal's expected profit under the linear contract is

$$\mathbb{E}\left[b(e) + \eta - (\alpha^T(\theta(e) + \epsilon) + \beta)\right] = b(e) - \alpha^T\theta(e) - \beta$$

The principal chooses (α, β) to solve

$$\begin{aligned} \max \quad & b(e) - \alpha^T\theta(e) - \beta \\ \text{s.t.} \quad & \alpha^T\theta(e) + \beta - c(e) - \frac{r}{2}\alpha^T\Sigma\alpha \geq 1 \\ & \alpha^T\theta(e) - c(e) \geq \alpha^T\theta(e') - c(e') \end{aligned}$$

At optimum, the first constraint must be binding, so the optimization problem is equivalent to

$$\begin{aligned} \max \quad & b(e) - c(e) - \frac{r}{2}\alpha^T\Sigma\alpha \\ \text{s.t.} \quad & \alpha^T\theta(e) - c(e) \geq \alpha^T\theta(e') - c(e') \end{aligned}$$