CS 514/ Math 514 Numerical Analysis
Spring 2018
Homework 7
Given: April 12, 2018; Due: **April 19, 2018** 11:45 A.M. (15 minutes before class).
Note that you have only one week to do this HW.

You are permitted to discuss these problems with **one** other student, and you should indicate the name of your collaborator in your answer to the first problem in this set. If you worked on your own, please indicate this on your answer. You should write up your own solution, and you are not permitted to either share a written copy of your solution or copy some one else's written solution. Similarly with programs, you are not permitted to share your code with another student. If you need help with debugging your code, you should describe what the problem is without sharing or showing your code to another student. If you are stuck on a problem or on a code, you could ask the TAs for help on Piazza. Please do not discuss solutions to the HW problems on Piazza before they are graded.

If you copy someone else's work, or let your work be copied, you will get zero points for the entire HW, a loss of one letter grade in the course, and you will be reported to the Dean of Students. I follow the course policies described by Professor Gene Spafford (of Purdue Computer Science department) at the URL `http://spaf.cerias.purdue.edu/~spaf/cpolicy.html` If you are not familiar with this policy, please read it! Please submit your assignments using Blackboard, preferably using either a Tex-generated or Word document.

0. If you collaborated with a student in answering these questions, please write down your collaborator's name. If you did not collaborate with anyone, please indicate that too.

1. We wish to approximate a function

$$f(x) = \begin{cases} -1 & \text{for} \quad -1 \le x < 0, \\ 0 & \text{for} \quad x = 0, \\ +1 & \text{for} \quad 0 < x \le 1. \end{cases}$$

   (a) **10 points**
   Suppose we wish to approximate $f(x)$ by by a constant a in the continuous least squares norm (L_2 norm) over the interval $[-1\ 1]$. Find the value of the constant a.

   (b) **10 points**
   Now we wish to approximate $f(x)$ by a constant a over an odd number of points x_1, $\ldots$, x_n, again in the least squares sense. Here $x_i = -1 + 2(i-1)/(n-1)$, for $i = 1$, $\ldots$, n. Compute the value of the constant a.

2. **10 points**
   Recall that the Legendre polynomials $P_i(x)$ are given by Rodriguez's formula

$$P_i(x) = \frac{1}{2^i\, i!} \frac{d^i}{dx^i} (x^2 - 1)^i.$$

Show that they satisfy

$$\int_{-1}^{1} P_i(x)^2 dx = \frac{2}{2i+1}.$$

Hint. Follow the integration by parts technique that we used in class to show that these polynomials were orthogonal. Use the fact

$$\frac{d^{2n}}{dx^{2n}}(x^2-1)^n = (2n)!.$$

To integrate $(x^2-1)^n$, substitute $s = (x+1)/2$, write down the transformed integral in s, and then compute the resulting integral by parts.

3. **5 points**
   In class we learned of n Chebyshev points t_i, for $i = 1, \ldots, n$, which are roots of the Chebyshev polynomial of degree n in the interval $[-1\ 1]$. Now write down an expression for the Chebyshev points x_i in the interval $[a\ b]$ in terms of t_i. Then compute the maximum value over all $x \in [a\ b]$ of the product $P = |\prod_1^n (x - x_i)|$, by first expressing it in terms of the product of the terms $(t - t_i)$, where the point $t \in [-1\ 1]$ maps to $x \in [a\ b]$. Hence obtain an upper bound for P.
   Hint. Recall how to map the interval $[-1\ 1]$ to the interval $[a\ b]$.

4. (a) **10 points**
   We interpolate a function $f(x)$ at the three points $x_0 = x_1 - h$, x_1, and $x_2 = x_1 + h$ by a quadratic interpolating polynomial. Write down an expression for the interpolation error using the error term in the Newton interpolation formula. Your answer should involve the third derivative of $f(x)$ and terms involving x_i, $i = 0$, 1 and 2. Let M denote the maximum of the absolute value of the third derivative of f in the interval $[x_0\ x_2]$. Find an upper bound for the error in terms of M and h.
   Hint: You can simplify the algebra by changing the variable to $x = x_1 + th$, where $-1 \leq t \leq 1$.

   (b) **5 points**
   Now consider three Chebyshev points for x_0, x_1, x_2, which are roots of the Chebyshev polynomial of degree 3, and write down a corresponding error term, and an upper bound on the error in terms of M and h.