CS 514/ Math 514 Numerical Analysis
Spring 2018
Homework 4
Given: March 6, 2018; Due: **March 27, 2018** 11:45 A.M. (15 minutes before class. Note the changed date!)

You are permitted to discuss these problems with **one** other student, and you should indicate the name of your collaborator in your answer to the first problem in this set. If you worked on your own, please indicate this on your answer. You should write up your own solution, and you are not permitted to either share a written copy of your solution or copy some one else's written solution. Similarly with programs, you are not permitted to share your code with another student. If you need help with debugging your code, you should describe what the problem is without sharing or showing your code to another student. If you are stuck on a problem or on a code, you could ask the TAs for help on Piazza. Please do not discuss solutions to the HW problems on Piazza before they are graded.

If you copy someone else's work, or let your work be copied, you will get zero points for the entire HW, a loss of one letter grade in the course, and you will be reported to the Dean of Students. I follow the course policies described by Professor Gene Spafford (of Purdue Computer Science department) at the URL `http://spaf.cerias.purdue.edu/~spaf/cpolicy.html` If you are not familiar with this policy, please read it! Please submit your assignments using Blackboard, preferably using either a Tex-generated or Word document.

0. If you collaborated with a student in answering these questions, please write down your collaborator's name. If you did not collaborate with anyone, please indicate that too.

1. **5 points**
   Polynomial interpolation may be viewed as a linear combination of the basis functions $\{\phi_i(x)\}$ in the form
   $$\sum_{i=1}^{n+1} c_i \phi_i(x_j) = y_j, \quad \text{for } j = 1, \ldots, n+1.$$
   For the power basis, this equation leads to the Van der Monde matrix; for the Lagrange basis we obtain the identity matrix. Write down the system of linear equations for the Newton basis, and identify the form of the matrix involved.

2. **20 points**
   We consider the Lagrange interpolation formula using barycentric weights given in the Algorithm on page 305 of the text book, and the Newton interpolation formula. For these questions, give your answer in terms of the leading power of n, including the constant term that multiplies it. E.g., $(2/3)n^3$. You can ignore lower order terms.

   (a) Given $n+1$ points (x_i, y_i), how many operations does it take to construct the Lagrange basis functions? (We need to compute and store the weights w_j, the values $w_j y_j$, and the differences $x - x_j$, for $j = 1, \ldots, n+1$.)

(b) Given the values you computed and stored in the previous part, how many additional operations does it take to evaluate the polynomial $p(x)$ at a given point x?

(c) Given $n + 1$ points (x_i, y_i), how many operations does it take to compute the divided difference coefficients in the Newton form?

(d) Given the divided difference coefficients, how many additional operations does it take to evaluate the polynomial at some point x?

3. **20 points**

(a) Compute a polynomial $p(x)$ that interpolates the function x^4 at the points -1, 0 and 1 using any method of your choice. Simplify the polynomial you obtain, since it will help you to understand your answers to the other parts of this question.

(b) Write down an expression for the error in the interpolating polynomial using the Newton form. Compute an upper bound on the error by (1) bounding an appropriate derivative of the function x^4 in the interval $[-1\ 1]$, and (2) by bounding the product of the terms involving x and the interpolation points in the interval $[-1\ 1]$. Calculate the value of the upper bound of the error.

(c) Since we know the function, we can write down the true error as $|x^4 - p(x)|$. Calculate the maximum value of the error in the interval $[-1\ 1]$. It should be smaller then the upper bound you obtained in the previous part.

(d) Since we know the true error, we can calculate the point $s(x)$ where the derivative is evaluated in the error term for the interpolation formula $|x^4 - p(x)|$. Find $s(x)$ as a function of x.

4. **15 points**
Let x_1, x_2, …, x_{n+1} be $n + 1$ distinct points in $[a, b]$ and denote $f_i = f(x_i)$ for $i = 1, \ldots,$ $n + 1$. Consider the Lagrange form of the interpolating polynomial

$$p_n(x) = \sum_{i=1}^{n+1} f_i L_i(x),$$

where the $L_i(x)$ are the Lagrange basis functions defined in class and in the textbook.

(a) Define the Lebesgue function as

$$\lambda_n(x) = \sum_{i=1}^{n+1} |L_i(x)|.$$

Show that $\lambda_n(x_j) = 1$, for $j = 1, \ldots, n + 1$.

(b) For quadratic interpolation at three equally spaced points, show that $\lambda_2(x) \leq 1.25$.

(c) Recall that $p_n(x)$ denotes the interpolating polynomial of degree at most n that interpolates the function values f_i at the points x_i, for $i = 1, \ldots, n+1$. Suppose that the function value at each point x_i is changed to $f_i + \epsilon_i$, with $\max_i |\epsilon_i| \leq \epsilon$. Consider the interpolating polynomial $q_n(x)$ that interpolates the perturbed function at the same $n+1$ points. Show that

$$|p_n(x) - q_n(x)| \leq \epsilon\, \lambda_n(x), \quad \text{for} \quad a \leq x \leq b.$$

Note that this last inequality helps us to interpret the Lebesgue constant as an absolute condition number for the polynomial interpolation problem.

5. **15 points**
This problem will explore the difference between interpolation at arbitrary points versus Chebyshev points (See Section 10.6).

(a) We consider interpolating the function $\sin x$ at the five points 0, $\pi/6$, $\pi/4$, $\pi/3$ and $\pi/2$ in the interval $[0\ \pi/2]$ with an interpolating polynomial of degree 4. In order to see the symmetries in the interpolation, change the variable to $y = x - \pi/4$, so that the interval changes to $[-\pi/4\ \pi/4]$. Report the interpolating polynomial you obtain. Plot the error $|\sin y - p_4(y)|$ in the interval $[-\pi/4, \pi/4]$, and report the maximum value you obtain.

(b) Now interpolate the function $\sin y$ at five Chebyshev points in the interval $[-\pi/4\ \pi/4]$. For any n, the $(n+1)$ Chebyshev points in $[-1\ 1]$ are $x_i = \cos(\pi * (2i-1)/(2 * (n+1)))$, for $i = 1, \ldots, n+1$. For any interval $[a\ b]$ one can scale the points to $a + (x_i + 1) * (b-a)/2$ to obtain the Chebyshev points, see Section 10.6 of the textbook. Again plot the error $|\sin y - p(y)|$ in the interval $[-\pi/4, \pi/4]$, where $p(y)$ is the interpolating polynomial here. Compare the result with your result from the previous part, by plotting both errors in one Figure. Submit your plot and comment on what you observe.

(c) Let $\{y_i\}$, where $i = 1:5$, be the points you used in the first part of this problem, and similarly let $\{z_i\}$ denote the Chebyshev points. Plot both functions $\prod_{i=1}^{5}(y - y_i)$ and $\prod_{i=1}^{5}(y - z_i)$, again in the interval $[-\pi/4\ \pi/4]$. Submit your plot, and comment on the similarities and differences between these plots and the error plots.