ME C231 Assignment: Optimization 2 and Finite-Time Optimal Control

1. **Linear Programming and Quadratic Programming**

   The following problems include some of the examples from homework 2. This time, use `YALMIP` to solve them. You should submit your code with names LPQP.m to bCourse. The file shall contain the solutions to the 4 optimization problems stated below. For each problem also verify that the KKT conditions hold.

   In your code print out the optimal value, the optimal solution and the value of a boolean variable, 'true' if KTT are satisfied, 'false' otherwise.

   In the hard copy, you only need to write down the optimal objective values, the optimal solutions and if the KKT are satisfied. If the problem is infeasible, please write "infeasible" as the answer.

   (a)

$$\begin{aligned}
\min_{z_1,z_2} \quad & -5z_1 - 7z_2 \\
\text{s.t.} \quad & -3z_1 + 2z_2 \leq 30 \\
& -2z_1 + z_2 \leq 12 \\
& z_1 \geq 0 \\
& z_2 \geq 0
\end{aligned}$$

   (b)

$$\begin{aligned}
\min_{z_1,z_2} \quad & 3z_1 + z_2 \\
\text{s.t.} \quad & z_1 - z_2 \leq 1 \\
& 3z_1 + 2z_2 \leq 12 \\
& 2z_1 + 3z_2 \leq 3 \\
& -2z_1 + 3z_2 \geq 9 \\
& z_1 \geq 0 \\
& z_2 \geq 0
\end{aligned}$$

(c)

$$\begin{aligned} \min \quad & \left\| \begin{bmatrix} z_1 - 2 \\ z_2 \end{bmatrix} \right\|_\infty + \left\| \begin{bmatrix} z_1 \\ z_2 + 5 \end{bmatrix} \right\|_1 \\ \text{subject to} \quad & 3z_1 + 2z_2 \leq -3 \\ & 0 \leq z_1 \leq 2 \\ & -2 \leq z_2 \leq 3 \end{aligned}$$

(d)

$$\begin{aligned} \min_{z_1, z_2} \quad & z_1^2 + z_2^2 \\ \text{s.t.} \quad & z_1 \leq -3 \\ & z_2 \leq 4 \\ & 0 \geq 4z_1 + 3z_2 \end{aligned}$$

2. **1-Norm and ∞-Norm**

Consider the Optimization Problem (c) in the previous exercise. In Homework 2 you computed the LP formulation of the problem. In Yalmip you can directly solve the problem or solve the corresponding LP. Use both approaches with Yalmip and confirm that the solution is the same. In the hard copy, please write "Confirmed" once you are done. You should submit your code with names Norm1.m showing the result of the comparison.

3. **Nonlinear Programming**

(a) Solve the following NLP many times with random initial guesses so you can observe convergence to different local optimizers. Check KKT conditions at each local optimizer. Submit your code with name NLP.m.

In the hard copy, print out as many local optimal solution you can find. For each local optimizer you find, confirm whether the KKT conditions are satisfied or not.

$$\begin{aligned} \min_{z_1, z_2} \quad & 3\sin(-2\pi z_1) + 2z_1 + 4 + \cos(2\pi z_2) + z_2 \\ \text{s.t.} \quad & -1 \leq z_1 \leq 1 \\ & -1 \leq z_2 \leq 1 \end{aligned}$$

(b) In **Matlab/yalmip** print out a plot of the cost function contour, mark out the initial guesses and the optimal solutions. Finally, by using a 3D plot show whether the obtained solutions are global or local optima. Provide printed plots as well as your code.
Hint: You can make use of `meshgrid.m`, `contour.m` and `mesh.m` to do the plotting.

(c) Solve the following NLP many times with random initial guesses so you can observe convergence to different local optimizers. Check KKT conditions at each local optimizer. Submit your code with name NLP2.m. In the hard copy, print out as many local optimal solution you can find. For each local optimizer you find, confirm whether KKT conditions are satisfied or not. Plot the cost function contour and the feasible region in one plot. Mark the optimal solution. Provide the printed plot as well as your code in the file NLP2.m.

$$\begin{aligned}\min_{z_1,z_2}\ & \log(1+z_1^2)-z_2\\ \text{s.t. }\ & -1(1+z_1^2)^2+z_2^2=4\end{aligned}$$

Hint: You can make use of `meshgrid.m`, `contour.m` and `ezplot.m` to do the plotting.

4. **Mixed-integer Programming**
Use `YALMIP` to solve the two following optimization problems. You should submit your code with names MIP.m to bCourse.

(a)

$$\begin{aligned}\min_{z_1,z_2}\ & -6z_1-5z_2\\ \text{s.t. }\ & z_1+4z_2<=16\\ & 6z_1+4z_2\le 28\\ & 2z_1-5z_2\le 6\\ & 0\le z_1\le 10\\ & 0\le z_2\le 10\\ & z_1,z_2\in \mathbf{Z}, \text{(integer)}\end{aligned}$$

(b)

$$\begin{aligned}\min_{z_1,z_2}\ & -z_1-2z_2\\ \text{s.t. }\ & \text{either } 3z_1+4z_2\le 12 \ \text{ or } 4z_1+3z_2\le 12\\ & z_1\ge 0\\ & z_2\ge 0\end{aligned}$$

5. **Finite-Time Optimal Control of a Vehicle**

Submit the solution to the problem below as a m-file named "`parkingVehicle.m`" on bCourse. For each question, please print out 4 subplots (one for each state vs time). Also print a simple plot showing the vehicle motion (x_k vs y_k) and its heading angle.

Consider the same simplified kinematic bicycle model showed in Lab.

$$
\begin{aligned}
\dot{x} &= v\cos(\psi+\beta) \\
\dot{y} &= v\sin(\psi+\beta) \\
\dot{v} &= a \\
\dot{\psi} &= \frac{v}{l_r}\sin(\beta) \\
\beta &= \tan^{-1}\left(\frac{l_r}{l_f+l_r}\tan(\delta_f)\right)
\end{aligned}
$$

where

$x =$ global x CoG coordinate
$y =$ global y CoG coordinate
$v =$ speed of the vehicle
$\psi =$ global heading angle
$\beta =$ angle of the current velocity with respect to the longitudinal axis of the car
$a =$ acceleration of the center of mass into this direction
$l_r =$ distance from the center of mass of the vehicle to the rear axle
$l_f =$ distance from the center of mass of the vehicle to the front axle
$\delta_f =$ steering angle of the front wheels with respect to the longitudinal axis of the car

Collect the states in one vector $z = [x, y, v, \psi]^T$, and the inputs as $u = [a, \beta]^T$. Obtain a discrete-time model by using Forward Euler Discretization with sampling time $\Delta t = 0.2$. Use $l_f = l_r = 1.738$ in the simulation.

You are asked to formulate and solve a parking problem as a finite-time optimal control problem.

(a) Parking Problem Formulation 1.

$$
\begin{aligned}
\min_{z_0,\ldots,z_N,u_0,\ldots,u_{N_1}} \quad & \sum_{k=N-2}^{k=N} \|z_k - z_{ref}\|_2^2 \\
& z_{k+1} = z_k + f(z_k, u_k)\Delta t && \forall k = \{0,\ldots,N-1\} \\
& z_{min} \le z_k \le z_{max} && \forall k = \{0,\ldots,N\} \\
& u_{min} \le u_k \le u_{max} && \forall k = \{0,\ldots,N\} \\
& |\beta_{k+1} - \beta_k| \le \beta_d && \forall k = \{0,\ldots,N-1\} \\
& z_0 = \bar{z}_0 \\
& z_N = \bar{z}_N
\end{aligned}
$$

The vehicle starts from the initial state $\bar{z}_0 = [0,\ 3,\ 0,\ 0]^T$. Our goal is to park the vehicle in the terminal state $\bar{z}_N = [0, 0, 0, -\pi/2]$ Set the horizon $N = 70$ and consider the following constraints:

- The difference of current and previous steering commands are bounded by ± 0.2 rad. (i.e. $\beta_d = 0.2$)
- The accelerations are bounded by $|a(k)| \leq 1.5\Delta t$ m/s^2.
- The steering control inputs are limited to $|\beta(k)| \leq 0.6$ rad.
- The state has the constraints of $[-20, \quad -5, \quad -10, \quad -2\pi]^T \leq z(k) \leq [20, \quad 10, \quad 10, \quad 2\pi]^T$

Submit the plots on the pdf solution file and the code.

(b) Parking Problem Formulation 2
Same as formulation 1, use now the state constraints :$[-20, \quad -0.2, \quad -10, \quad -2\pi]^T \leq z(k) \leq [20, \quad 10, \quad 10, \quad 2\pi]^T$. I have changed only the lower bound on the lateral position (from -5 to -0.2) to see if you can generate a different manoeuver that does not violate the constraints. Recall that z constrains are constraints on the vehicle CoG.

(c) Parking Problem Formulation 3
Use formulation 1 and add the additional constraints on the difference between current and previous acceleration to get a smoother parking maneuver:

$$|a(k+1) - a(k)| \leq a_d \quad \forall k = \{0, \dots, H_p - 1\}$$

with $a_d = 0.06$ m/s^2 . Show by using plots that you get a smoother control action.