1 Short Answers - Graphs

(a) Bob removed a degree 3 node from an n-vertex tree. How many connected components are there in the resulting graph?

(b) Given an n-vertex tree, Bob added 10 edges to it and then Alice removed 5 edges. If the resulting graph has 3 connected components, how many edges must be removed in order to remove all cycles from the resulting graph?

Solution:

(a) **3**.

In this solution, "neighbor" will refer to the neighbors of the removed vertex. The neighbors must be in different connected components. This follows from a tree having a unique path between each neighbor in the tree as it is acyclic (see dis 2B q1). The removed vertex broke that path, so each neighbor must be in a separate component.

Also, there cannot be more than 3 connected components. This is because every other node is connected to one of the neighbors since every other vertex has a path to the removed node which must first go through a neighbor.

(b) **7**

We first note that in any connected graph if we remove an edge belonging to a cycle, then the resulting graph is still connected. Hence for any connected graph, we can repeatedly remove edges belonging to cycles, until no more cycles remain. This process will give rise to a connected acyclic graph, i.e., a tree.

Since the final graph we wish to obtain is acyclic, each of its connected component must be a tree. Thus the components should have $n_1 - 1$, $n_2 - 1$ and $n_3 - 1$ edges each, where n_1, n_2, n_3 are the number of vertices in each of these components. Let n denote the total number of vertices and hence $n = n_1 + n_2 + n_3$. As a result, the total number of edges in the final graph is $n - 3$. The total number of edges after Bob and Alice did their work was $n - 1 + 10 - 5 = n + 4$. Thus one needs to remove 7 edges.

2 Planarity

(a) Prove that $K_{3,3}$ is nonplanar.

(b) Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Use a proof by contradiction to show that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Solution:

(a) Assume toward contradiction that $K_{3,3}$ were planar. In $K_{3,3}$, there are $v = 6$ vertices and $e = 9$ edges. If $K_{3,3}$ were planar, from Euler's formula we would have $v - e + f = 2 \Rightarrow f = 5$. On the other hand, each region is bounded by at least four edges, so $4f \leq 2e$, i.e., $20 \leq 18$, which is a contradiction. Thus, $K_{3,3}$ is not planar.

(b) In this problem, we use proof by contradiction. Assume G is planar. Select any five vertices out of the seven. Consider the subgraph formed by these five vertices. They cannot form K_5, since G is planar. So some pair of vertices amongst these five has no edge between them. Label these vertices v_1 and v_2. The remaining five vertices of G besides v_1 and v_2 cannot form K_5 either, so there is a second pair of vertices amongst these new five that has no edge between them. Label these v_3 and v_4. Label the remaining three vertices v_5, v_6 nd v_7. Since $v_1 v_2$ is not an edge, by property T (which states any three vertices must have at least two edges between them)it must be that $\{v_1, v\}$ and $\{v_2, v\}$ are edges, where $v \in \{v_3, v_4, v_5, v_6, v_7\}$. Similarly for v_3, v_4 we have that $\{v_3, v\}$ and $\{v_4, v\}$ are edges, where $v \in \{v_1, v_2, v_5, v_6, v_7\}$. Now consider the subgraph induced by $\{v_1, v_2, v_3, v_5, v_6, v_7\}$. With the three vertices $\{v_1, v_2, v_3\}$ on one side and $\{v_5, v_6, v_7\}$ on the other, we observe that $K_{3,3}$ is a subgraph of this induced graph. This contradicts the fact that G is planar.

The above shows that any graph with 7 vertices and property T is non-planar. Any graph with greater than 7 vertices and property T will also be non-planar because it will contain a subgraph with 7 vertices and property T.

3   Graph Coloring

Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

Solution:

The natural way to try to prove this theorem is to use induction on the graph's maximum degree, k. Unfortunately, this approach is extremely difficult because covering all possible types of graphs when maximum degree changes requires extreme caution. You might be envisioning a certain graph as you write your proof, but your argument will likely not generalize. In graphs, typical good choices for the induction parameter are n, the number of nodes, or e, the number of edges. We typically shy away from inducting on degree.

We use induction on the number of vertices in the graph, which we denote by n. Let $P(n)$ be the proposition that an n-vertex graph with maximum degree at most k is $(k+1)$-colorable.

Base Case $n = 1$: A 1-vertex graph has maximum degree 0 and is 1-colorable, so $P(1)$ is true.

Inductive Step: Now assume that $P(n)$ is true, and let G be an $(n+1)$-vertex graph with maximum degree at most k. Remove a vertex v (and all edges incident to it), leaving an n-vertex subgraph, H. The maximum degree of H is at most k, and so H is $(k+1)$-colorable by our assumption $P(n)$. Now add back vertex v. We can assign v a color (from the set of $k+1$ colors) that is different from all its adjacent vertices, since there are at most k vertices adjacent to v and so at least one of the $k+1$ colors is still available. Therefore, G is $(k+1)$-colorable. This completes the inductive step, and the theorem follows by induction.

4 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V,E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.

Solution:

(a) The three hypercubes are a line, a square, and a cube, respectively. See also note 6 for pictures.

(b) Consider the vertices with an even number of 0 bits and the vertices with an odd number of 0 bits. Each vertex with an even number of 0 bits is adjacent only to vertices with an odd number of 0 bits, since each edge represents a single bit change (either a 0 bit is added by flipping a 1 bit, or a 0 bit is removed by flipping a 0 bit). Let L be the set of the vertices with an even number of 0 bits and let R be the vertices with an odd number of 0 bits, then no two adjacent vertices will belong to the same set.

Alternate solution: We can also prove that the hyper-cube can be 2-colorable through induction. Base case: When $n = 1$, there are only 2 vertices and it is 2-colorable. Induction step: Assume that the hypercube can be 2-colored in the case of n. We will show that in the case of $n+1$, the hypercube is also 2-colorable: Suppose we have 2 already 2-colored n-dimensional hypercube G_1, G_2 (which we know can be done from our induction hypothesis). We add corresponding edges to the two n-dimensional hypercube to form the $n+1$-dimensional hypercube. Every newly added edge connects a vertex u in G_1 to a vertex v in G_2. For each (u,v) that we add, flip the color of the vertex v in G_2. By doing this, we've successfully found a way that the $n+1$-dimensional hypercube can be 2-colorable. And therefore, it must be a bipartite.