Efficiency

Announcements

Measuring Efficiency

Recursive Computation of the Fibonacci Sequence

Our first example of tree recursion:

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http://en.wikipedia.org/wiki/File:Fibonacci.jpg

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fib(5)

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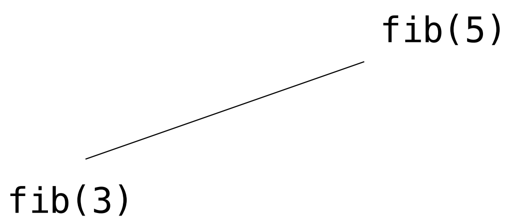

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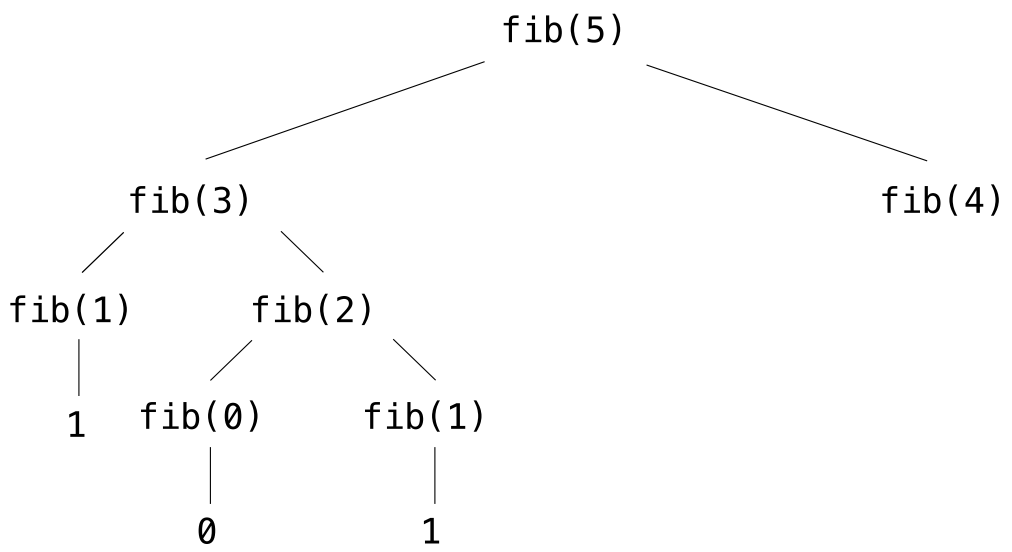


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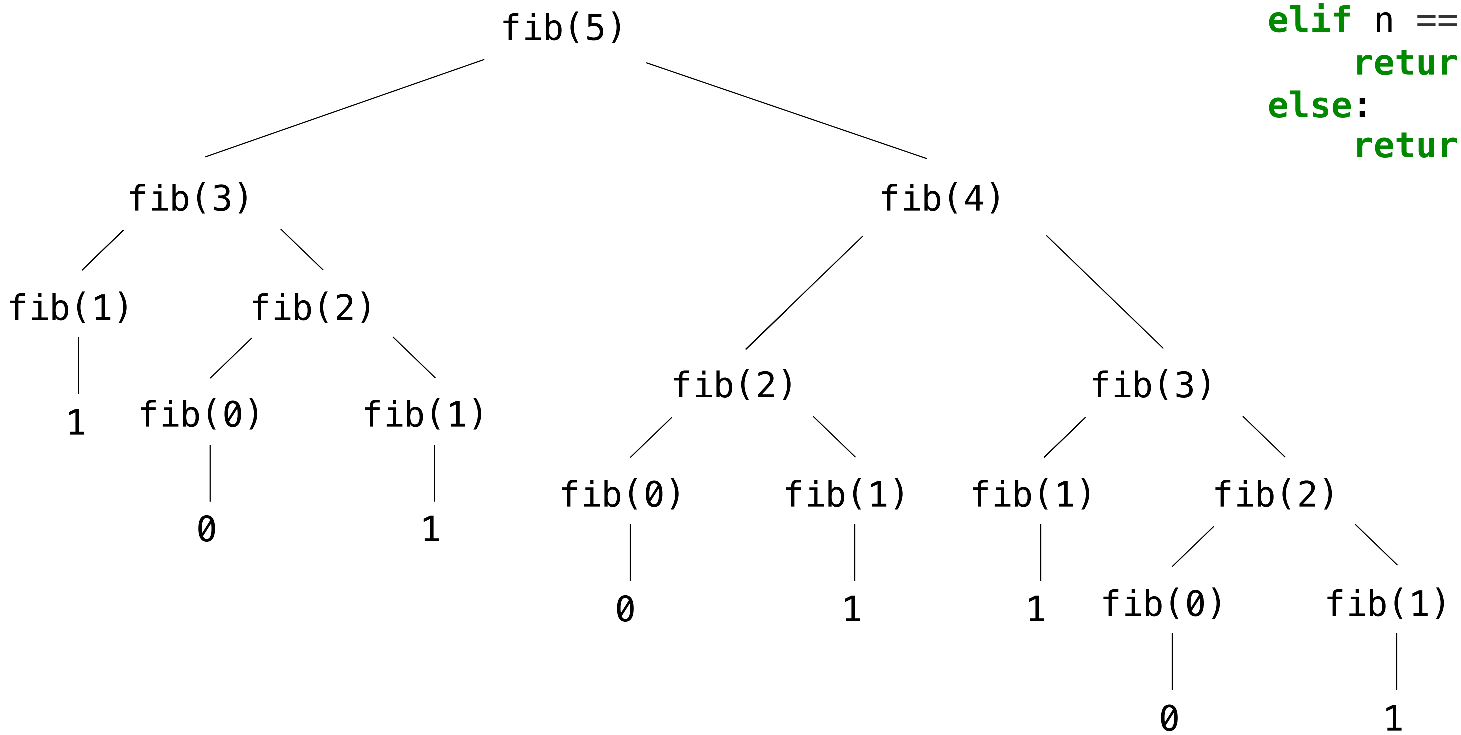

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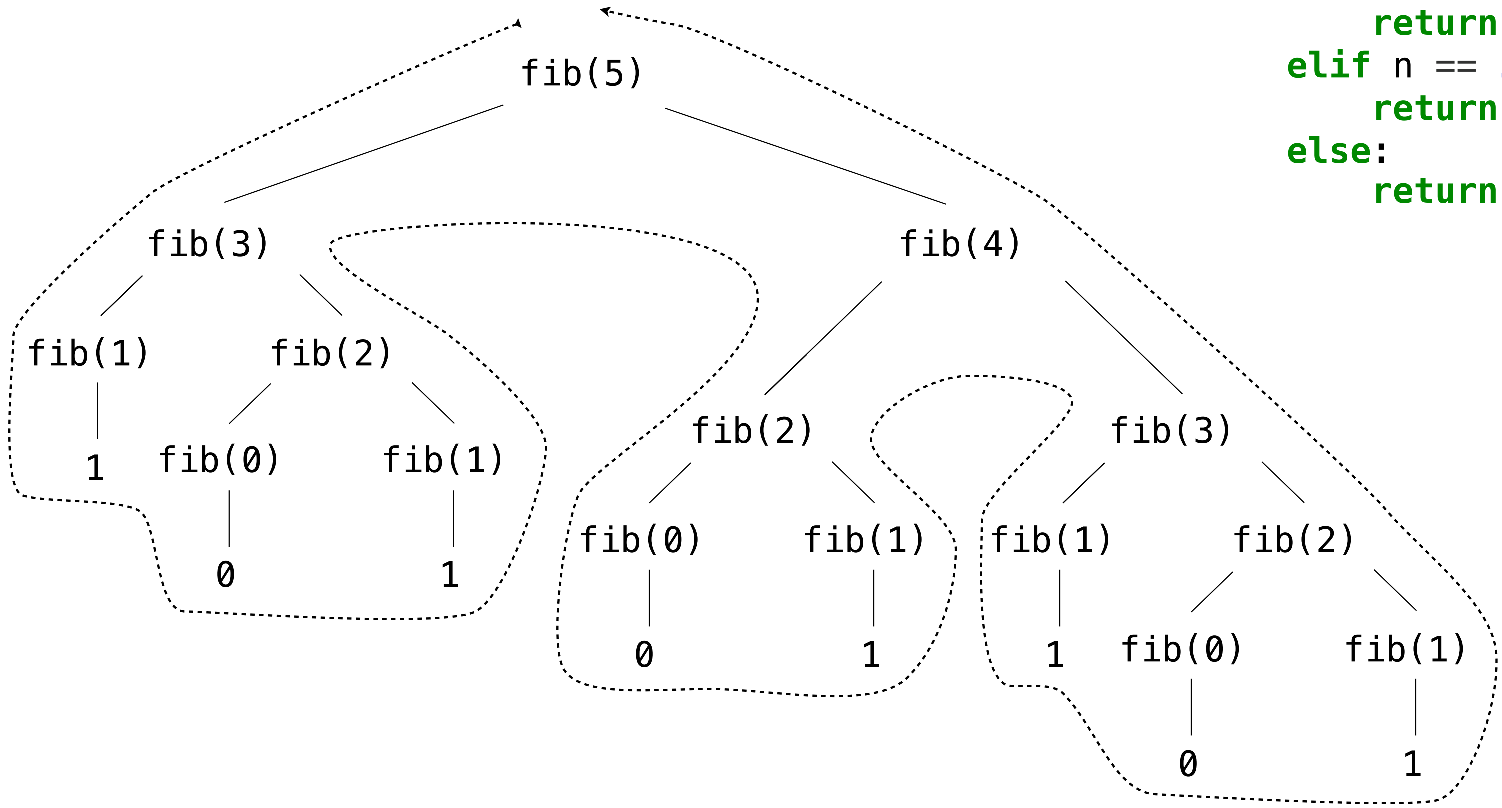


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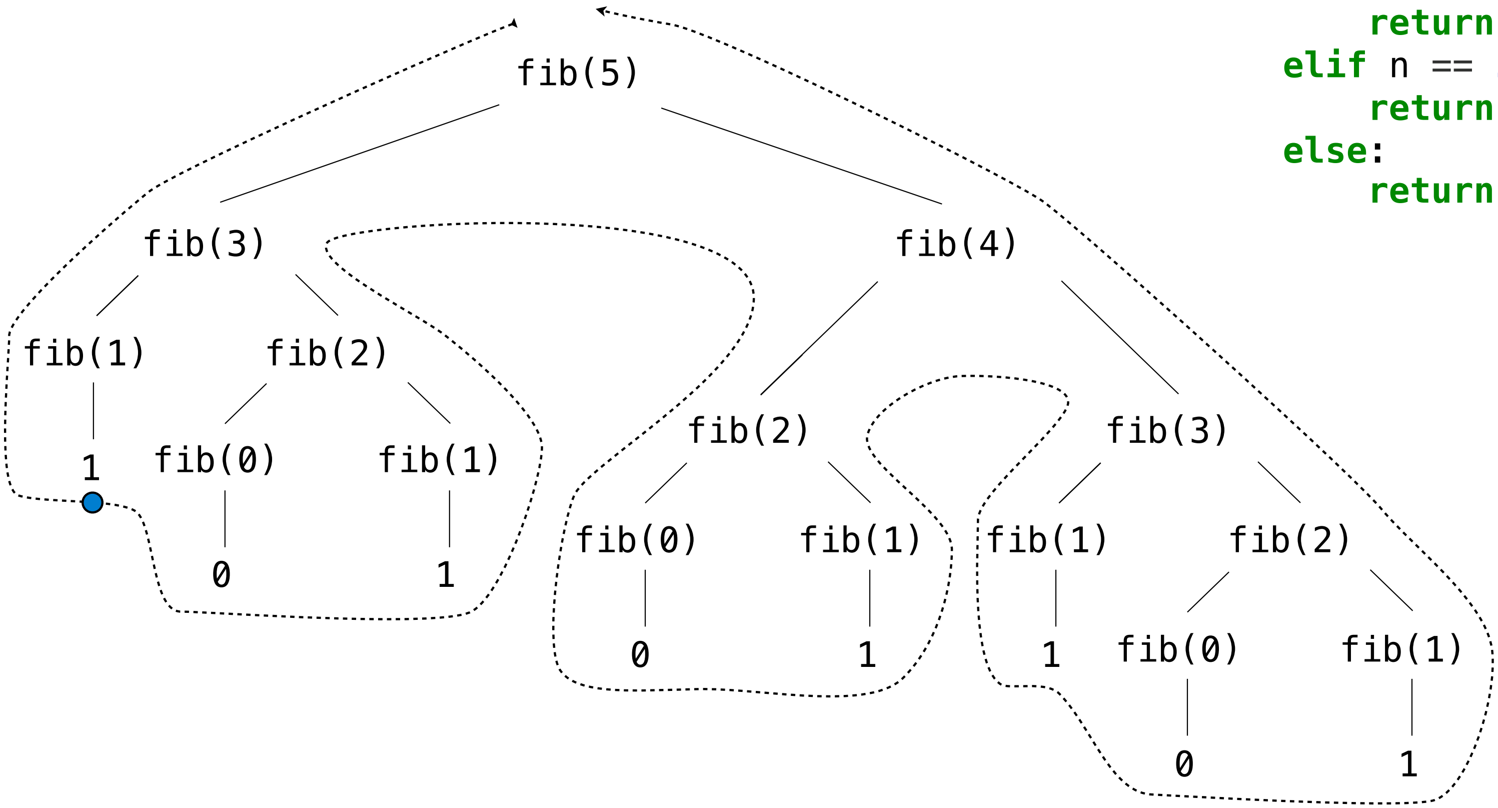

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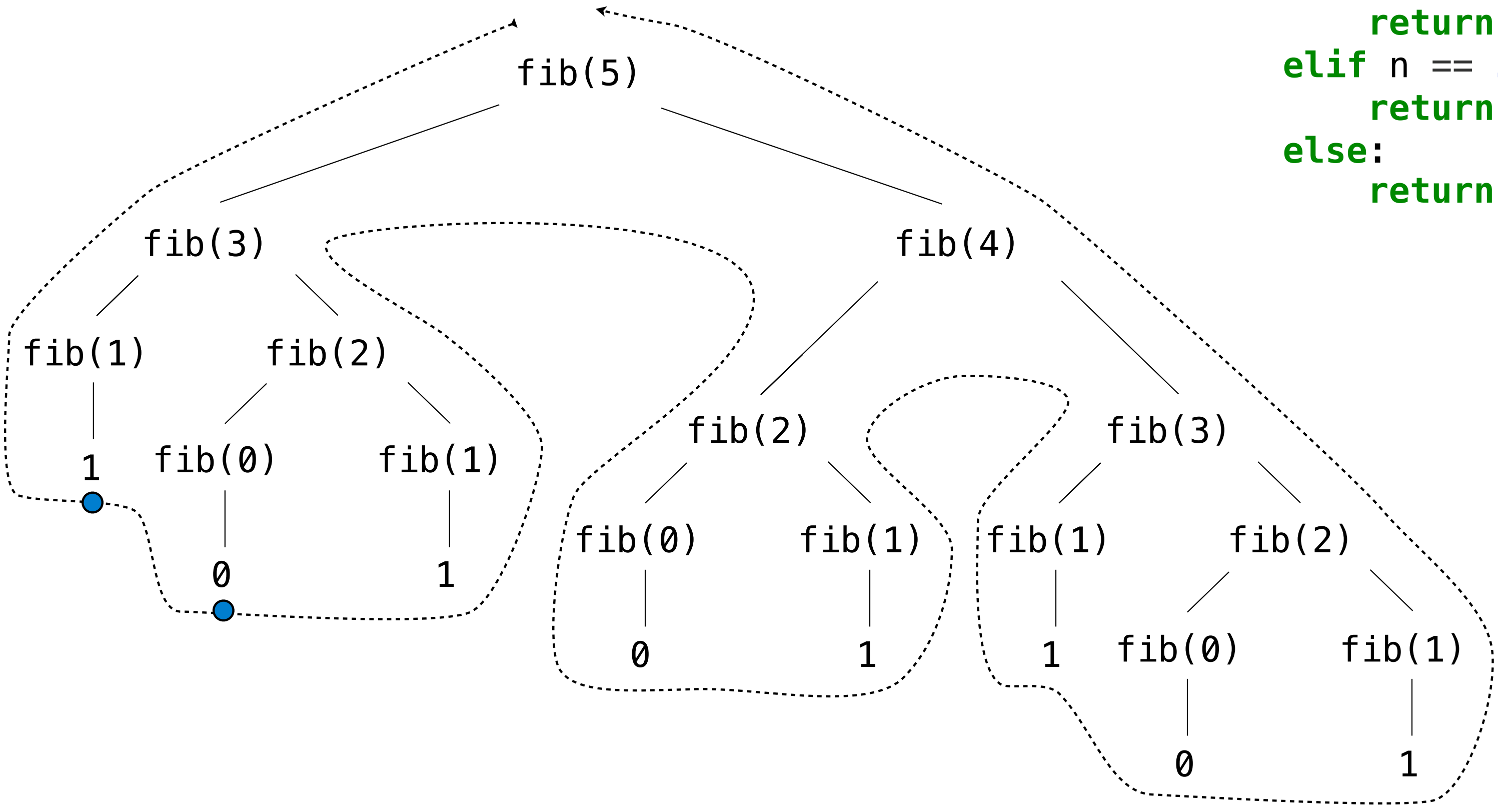

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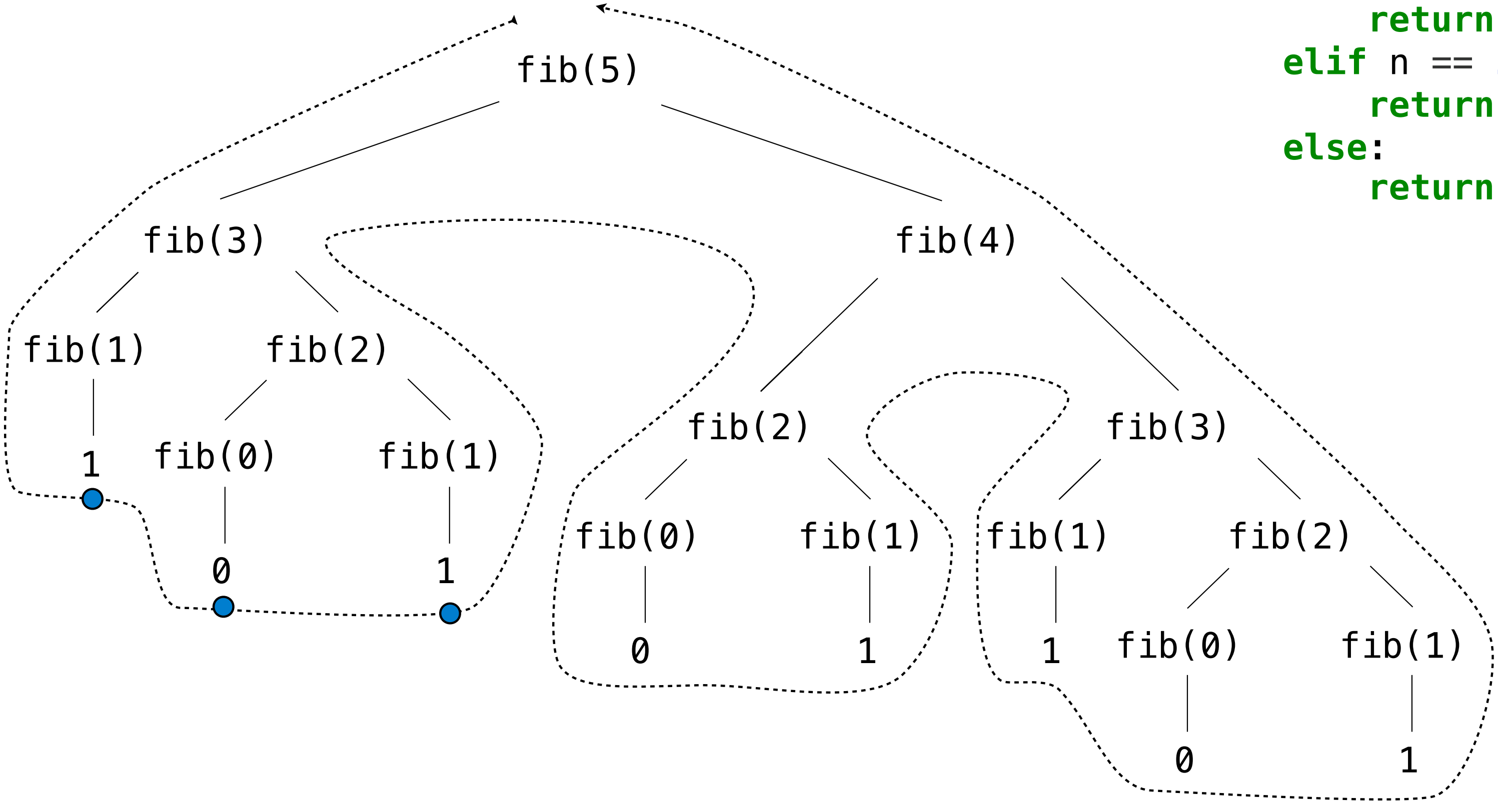

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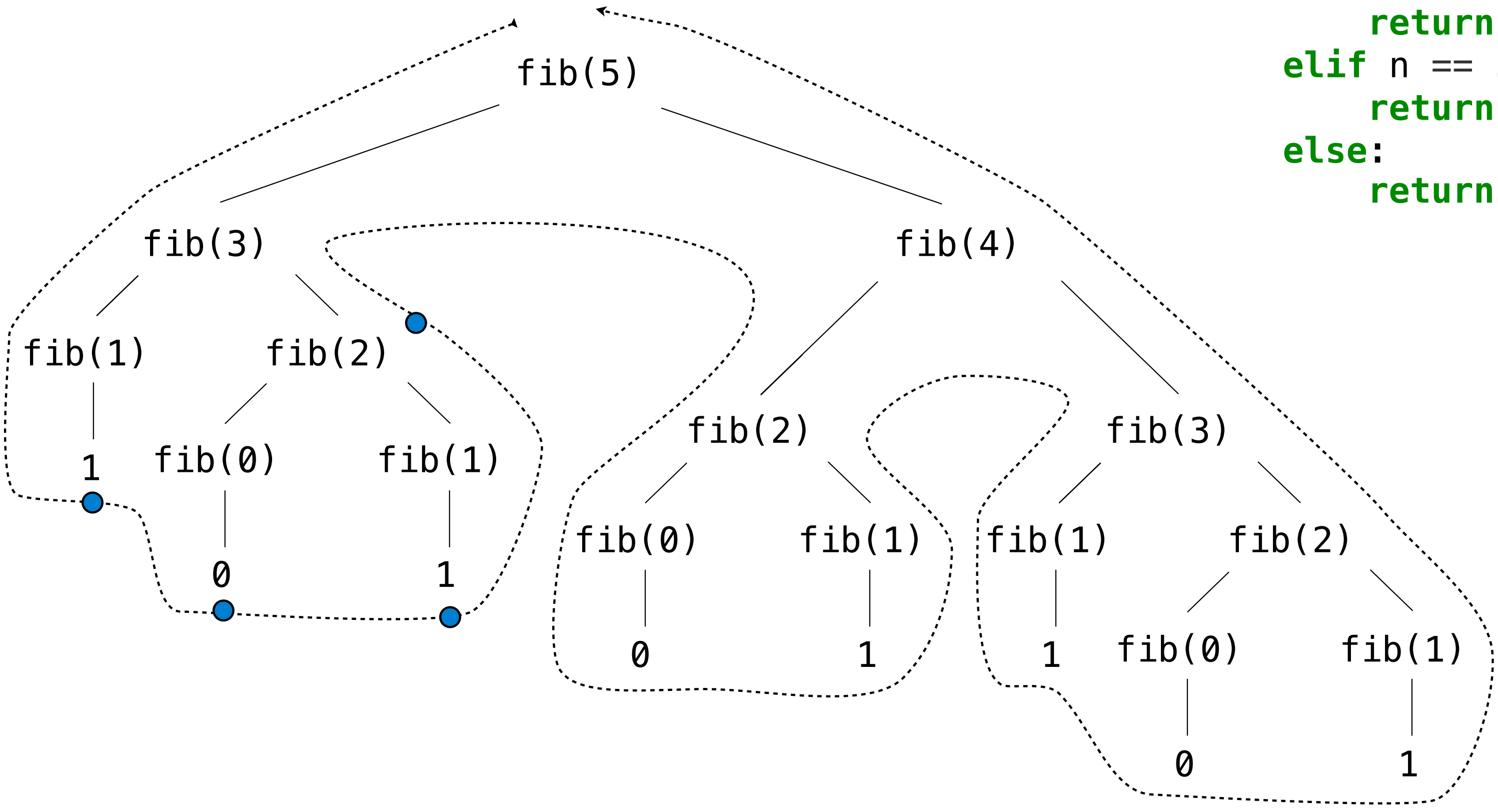

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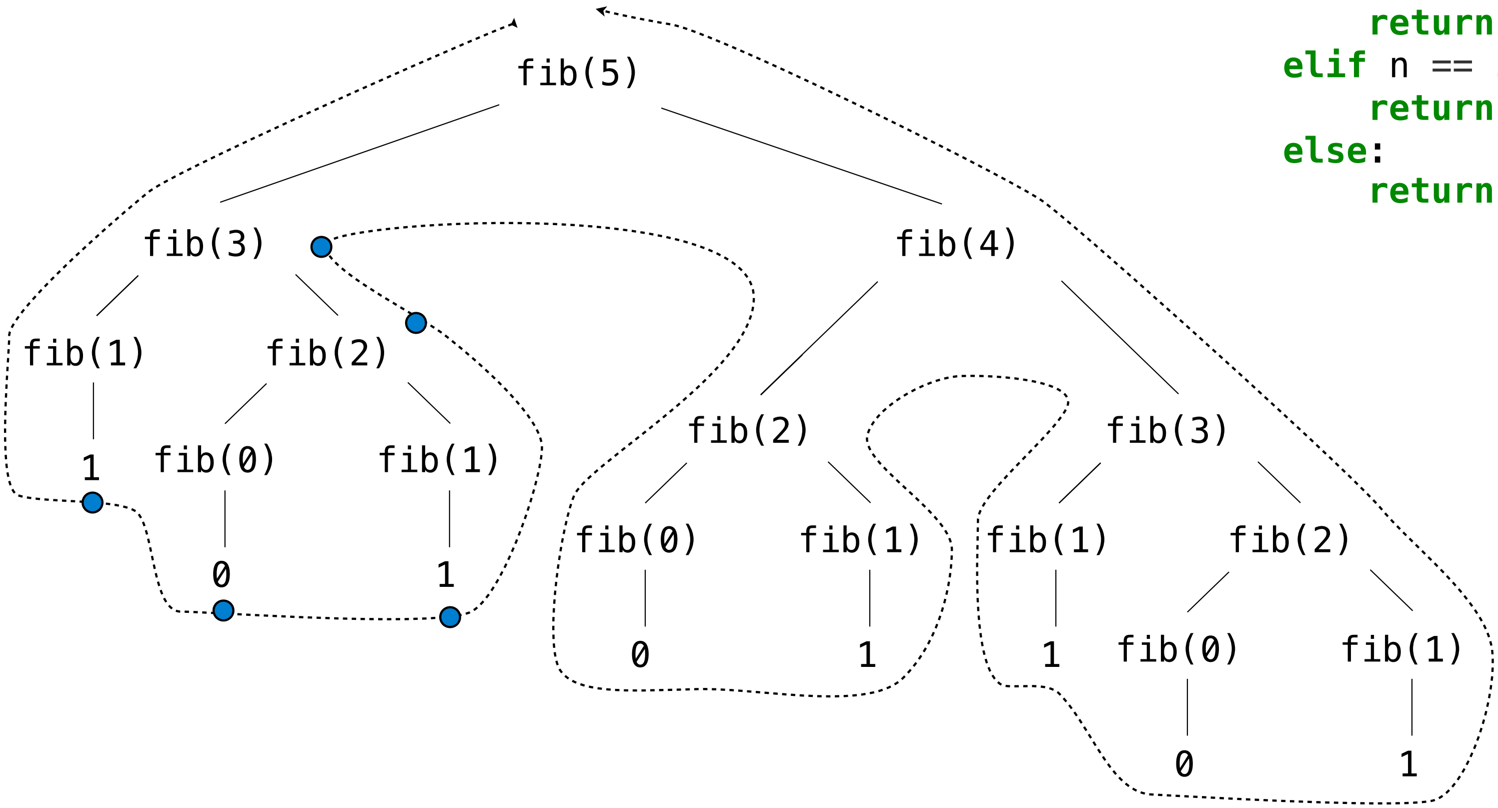

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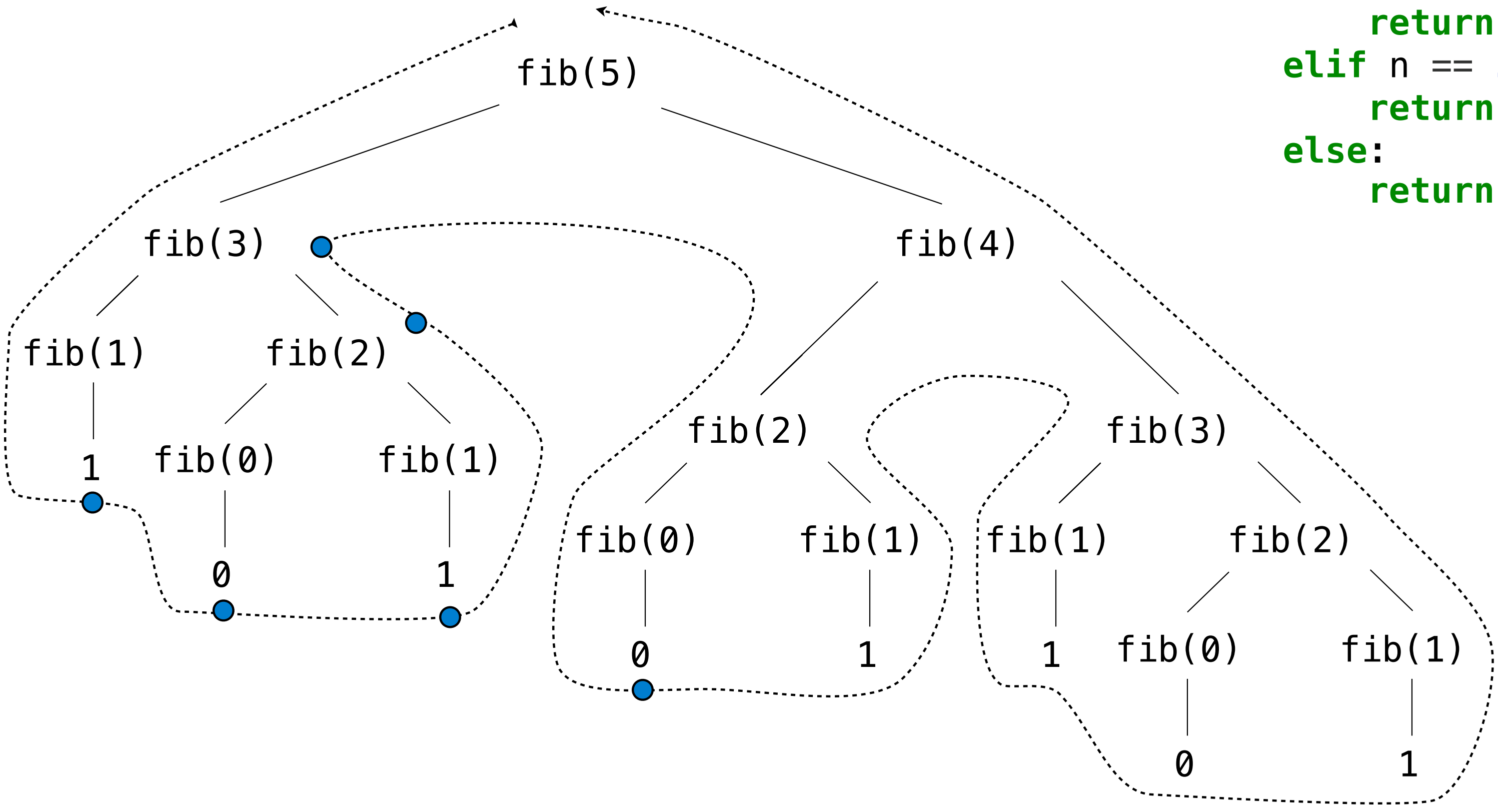

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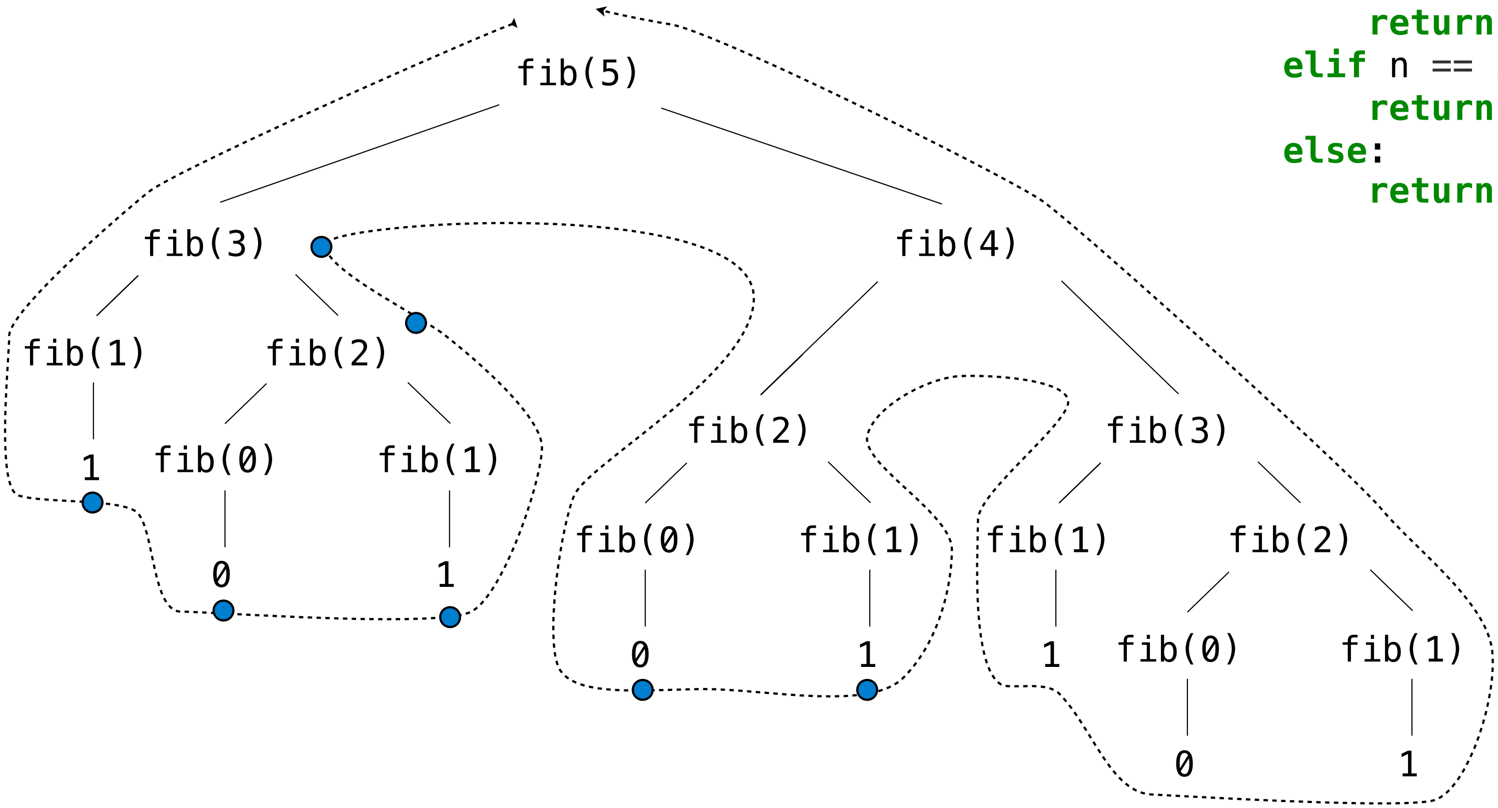

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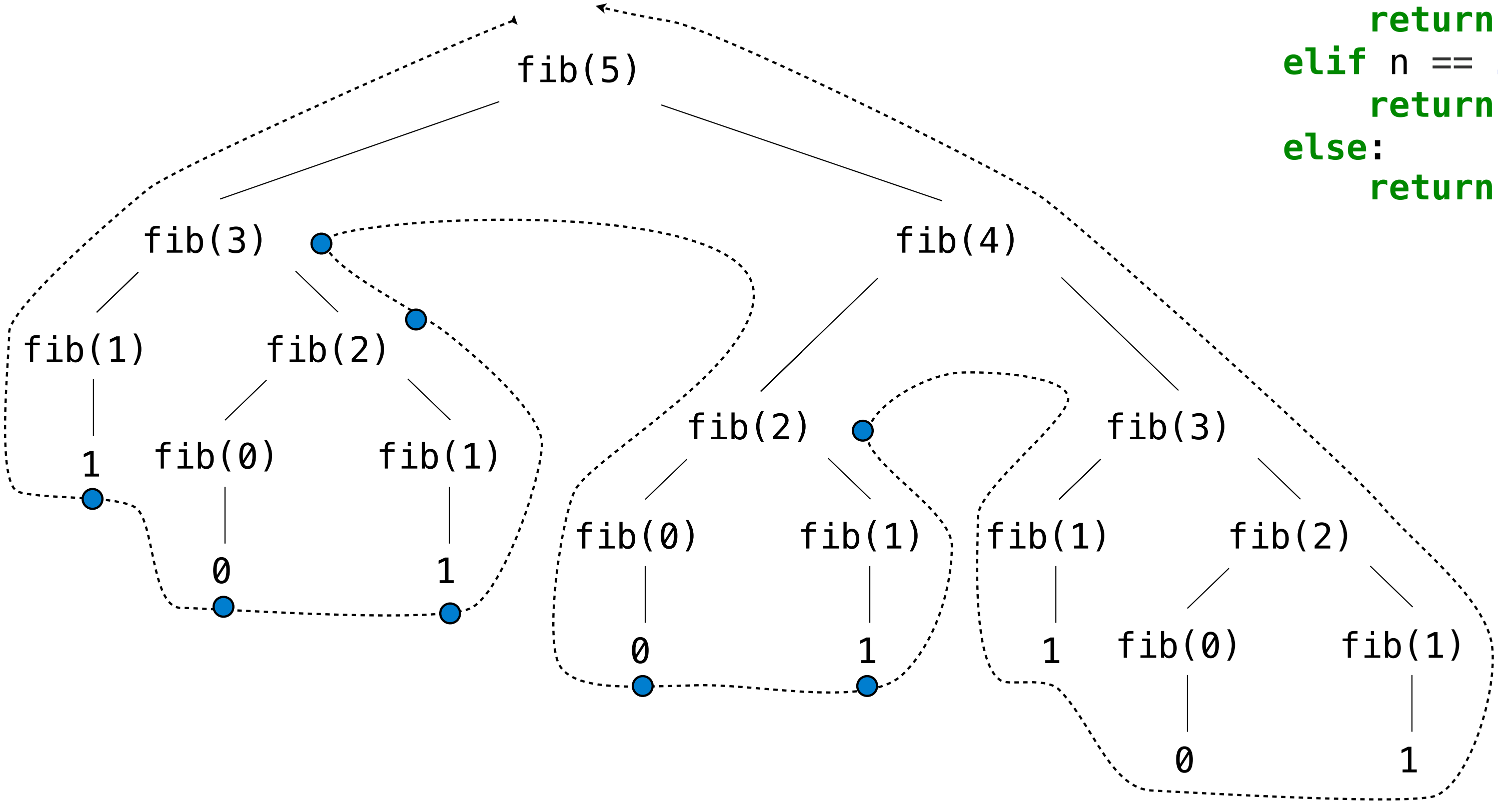

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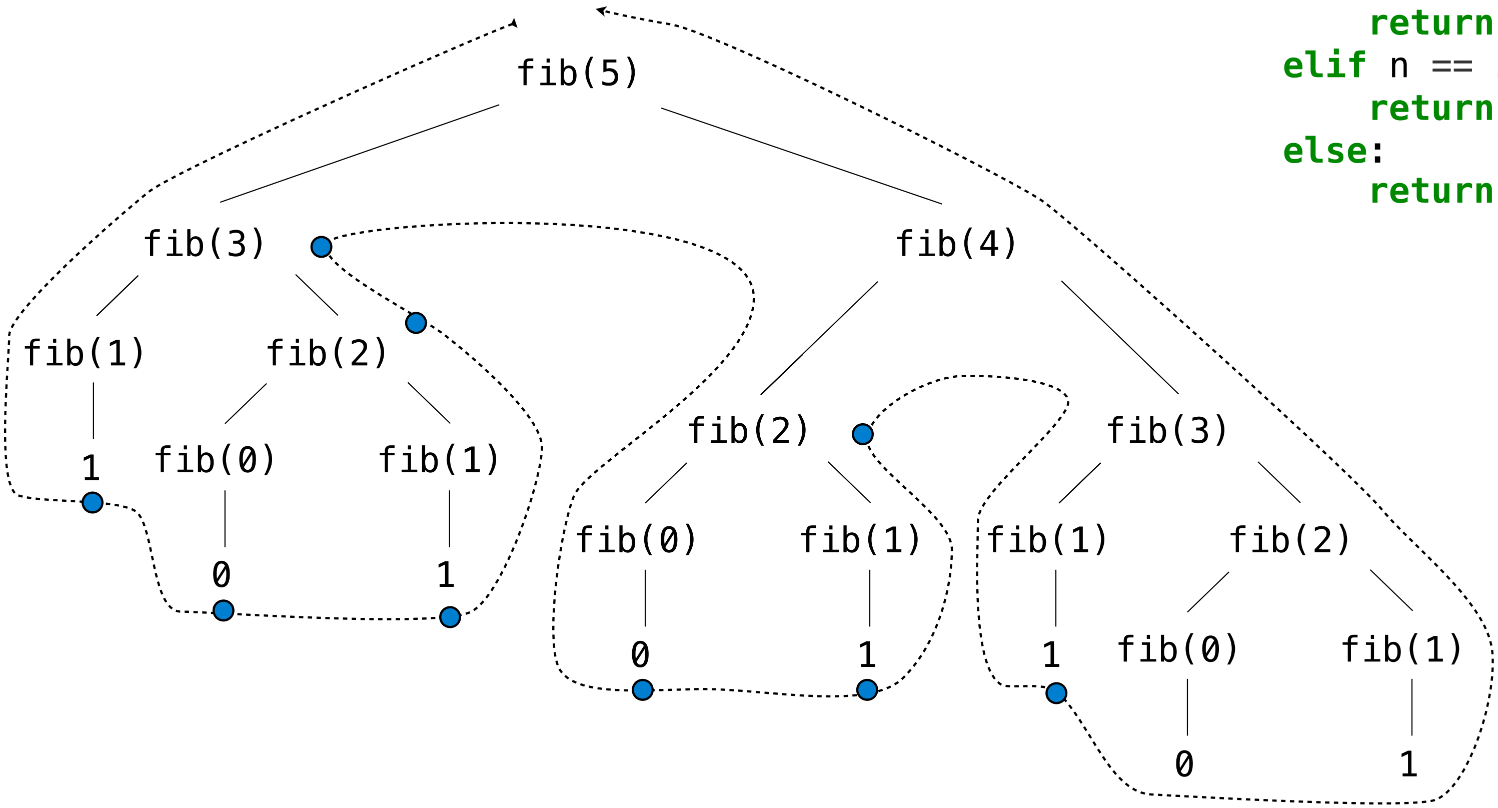

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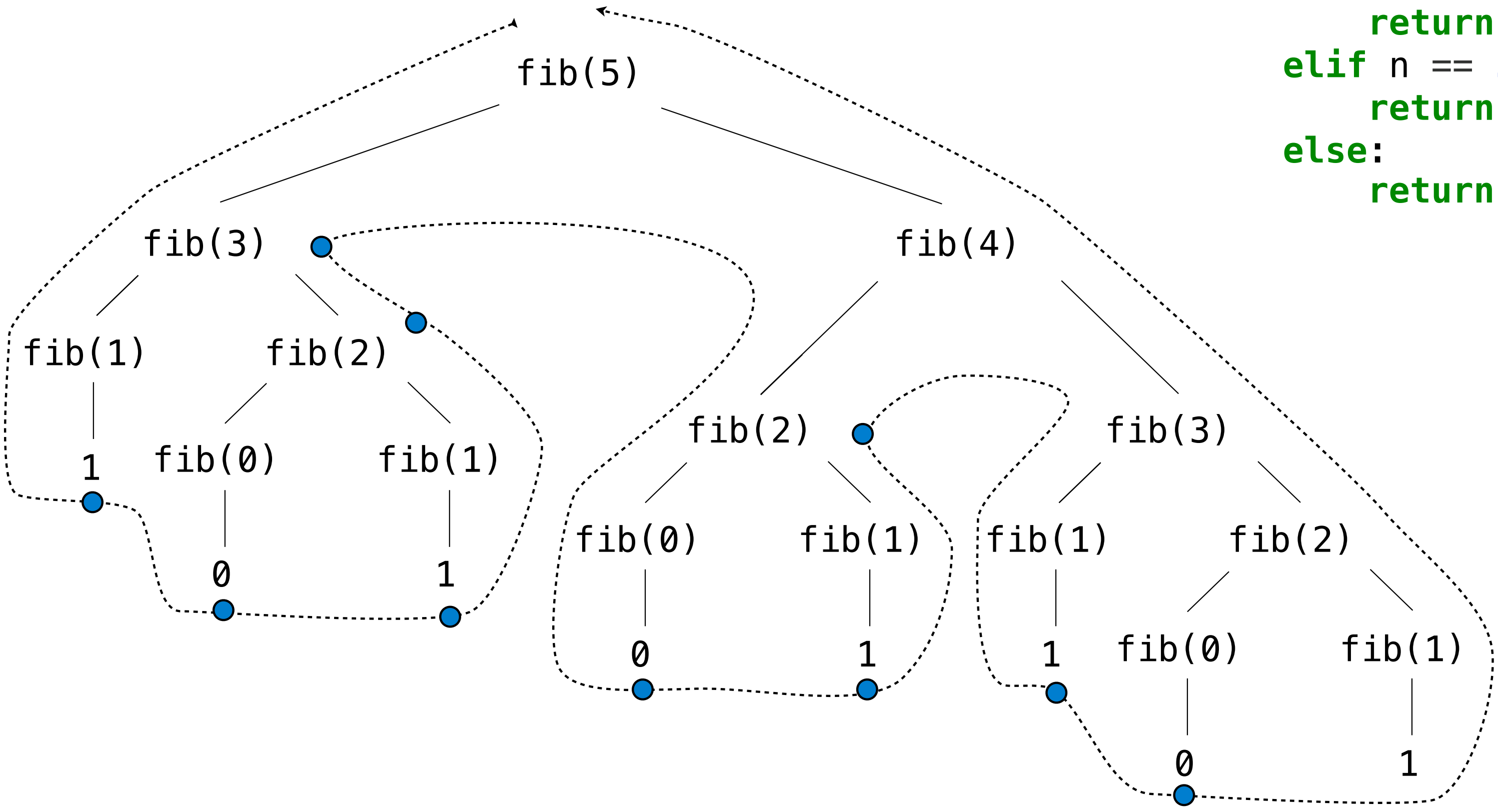

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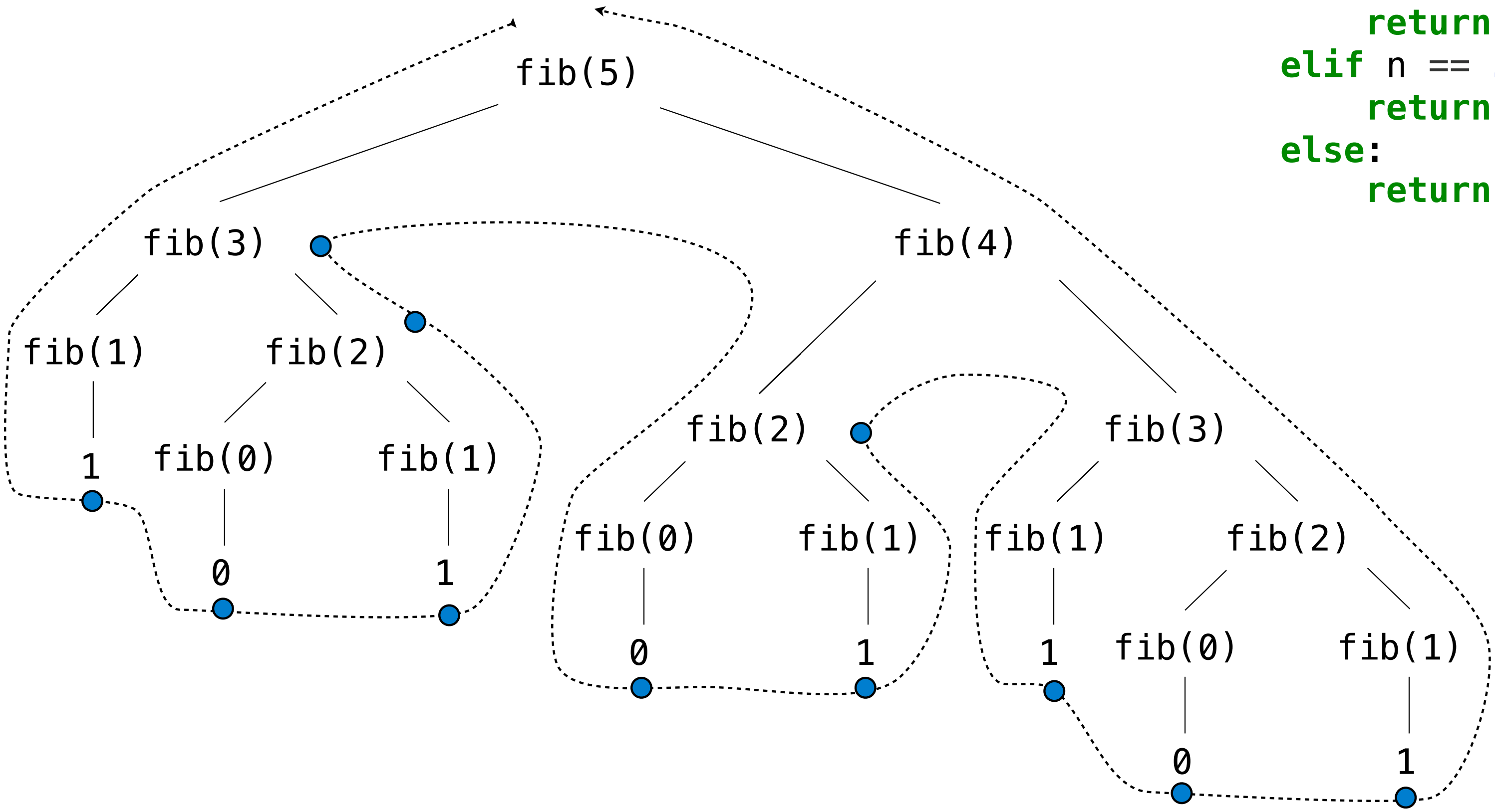

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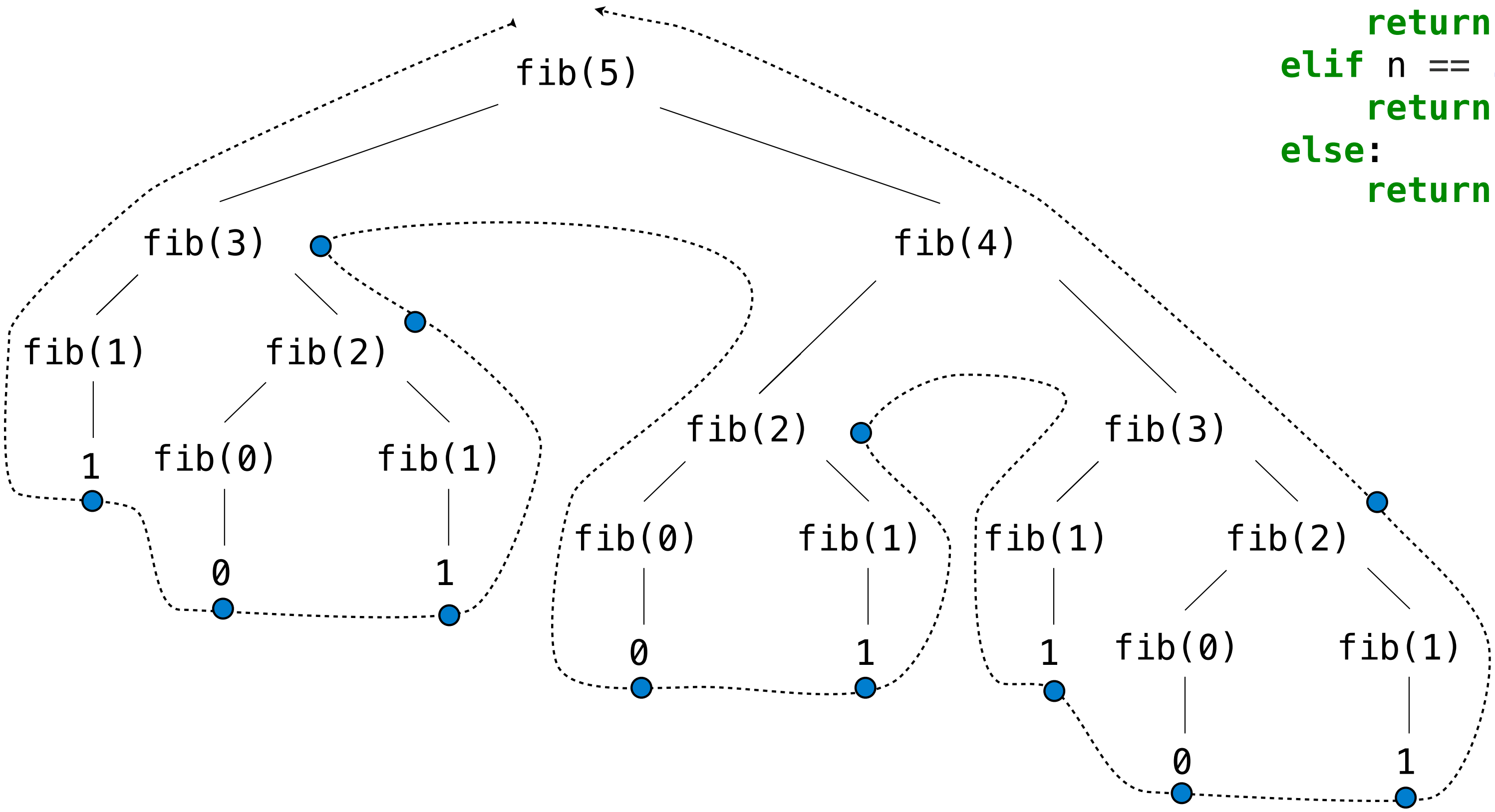

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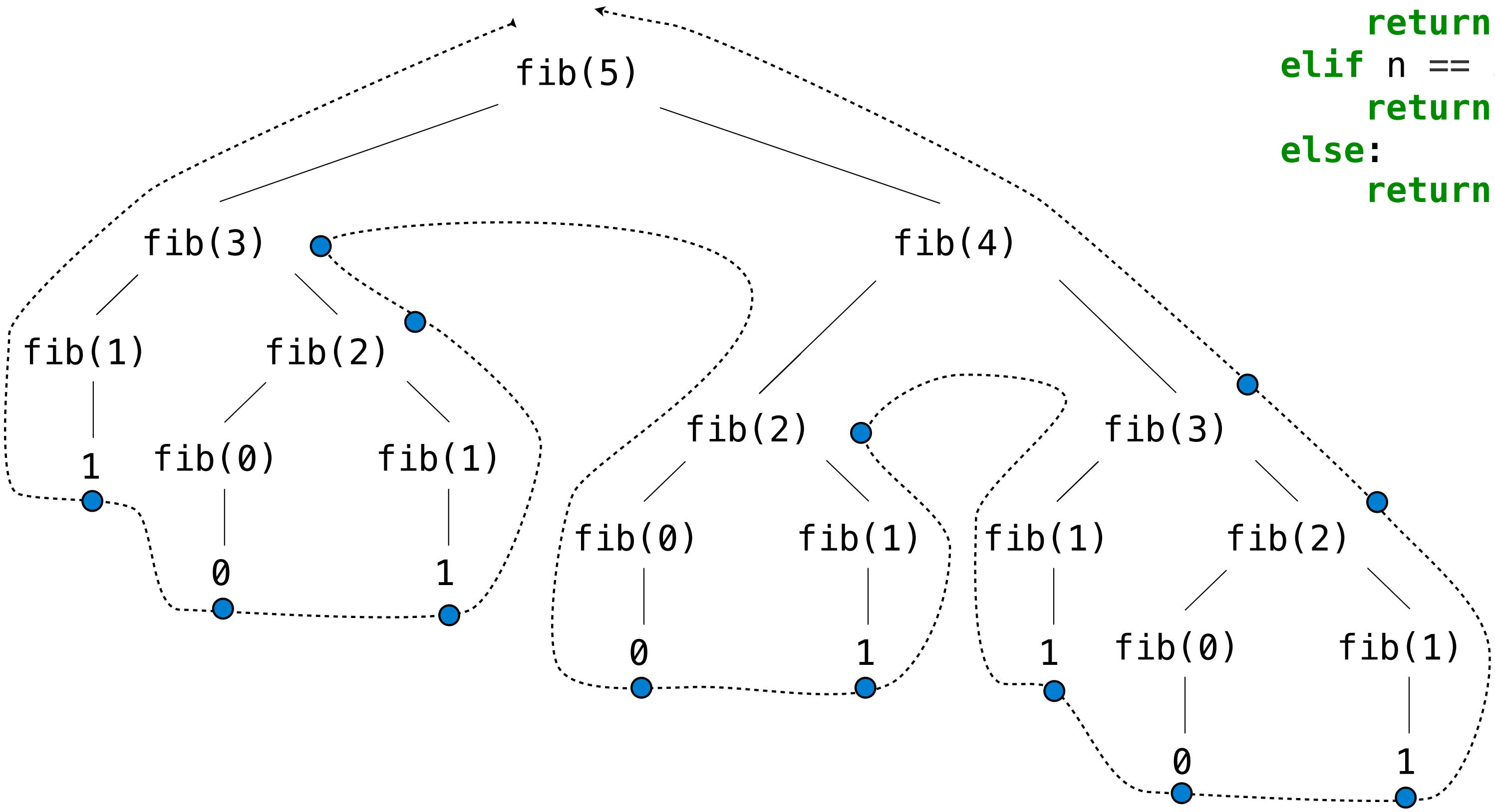

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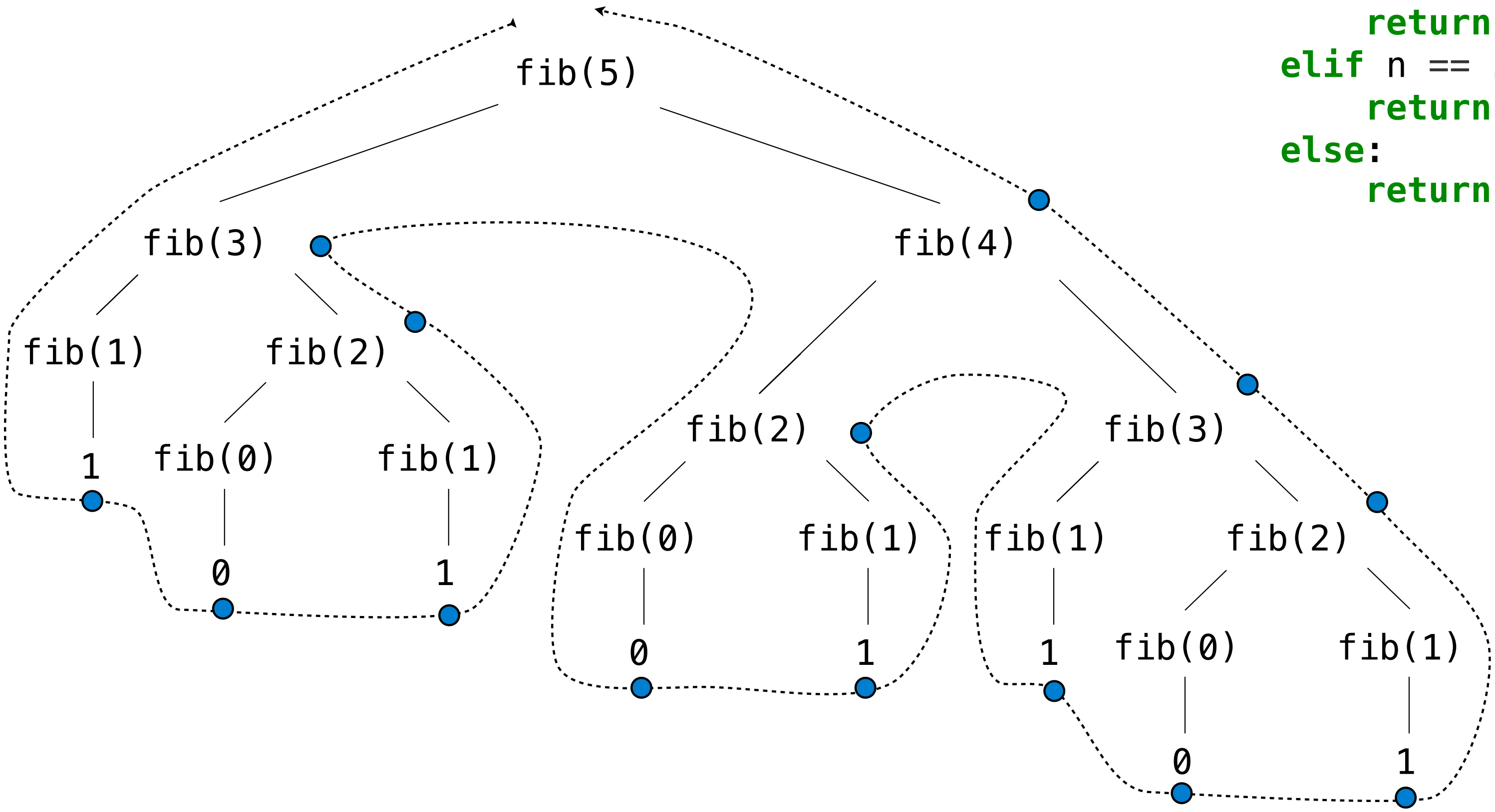

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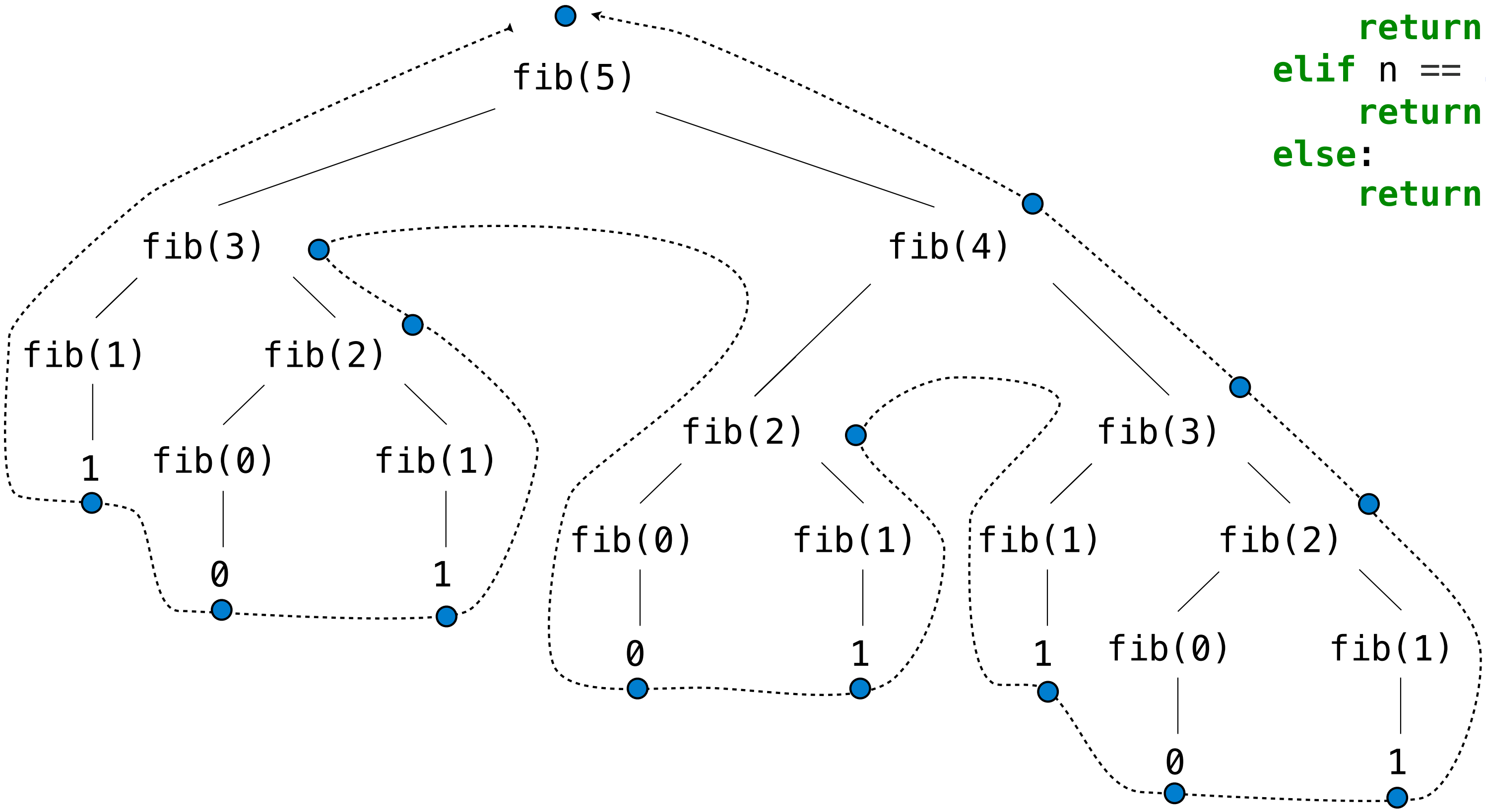

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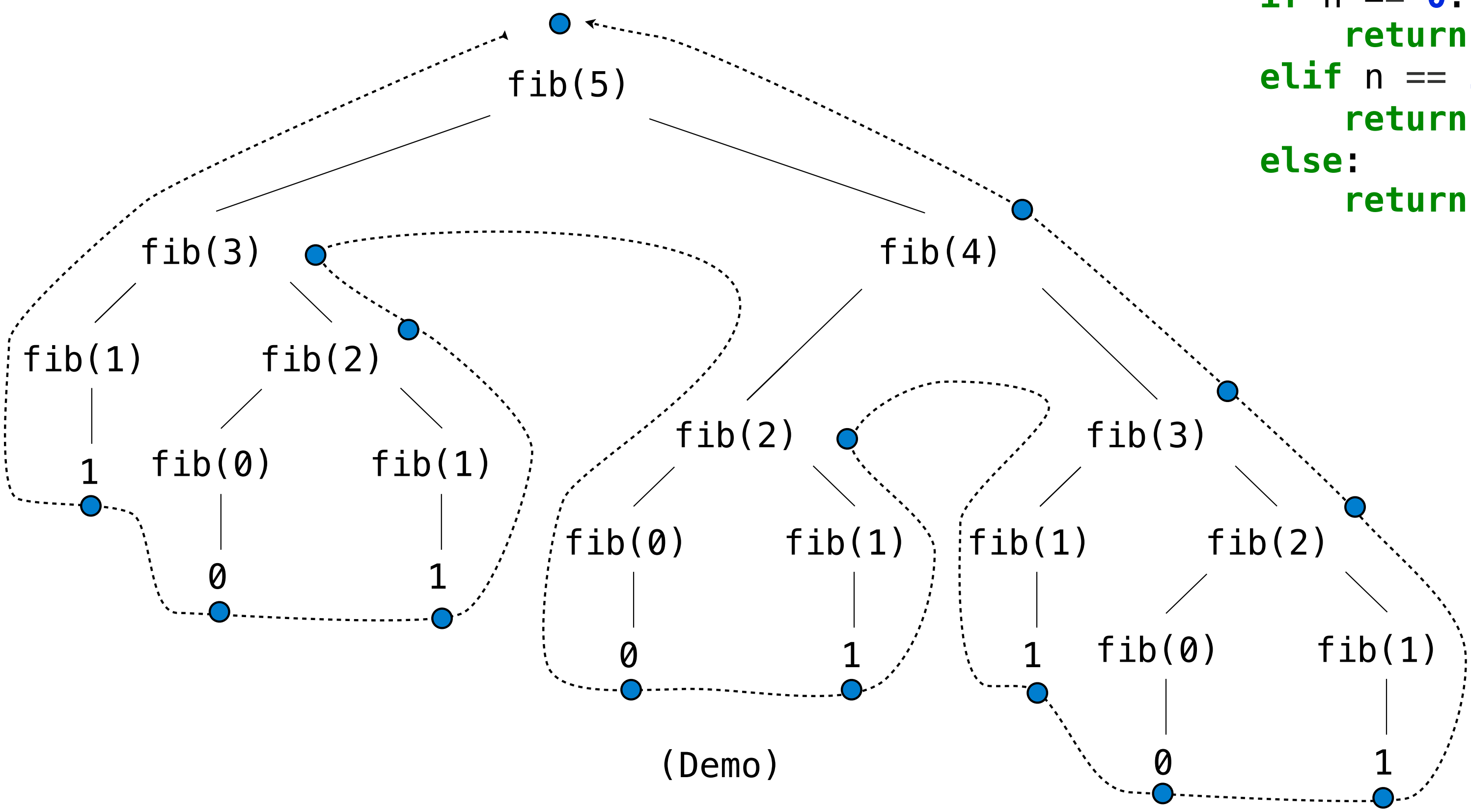

(Demo)

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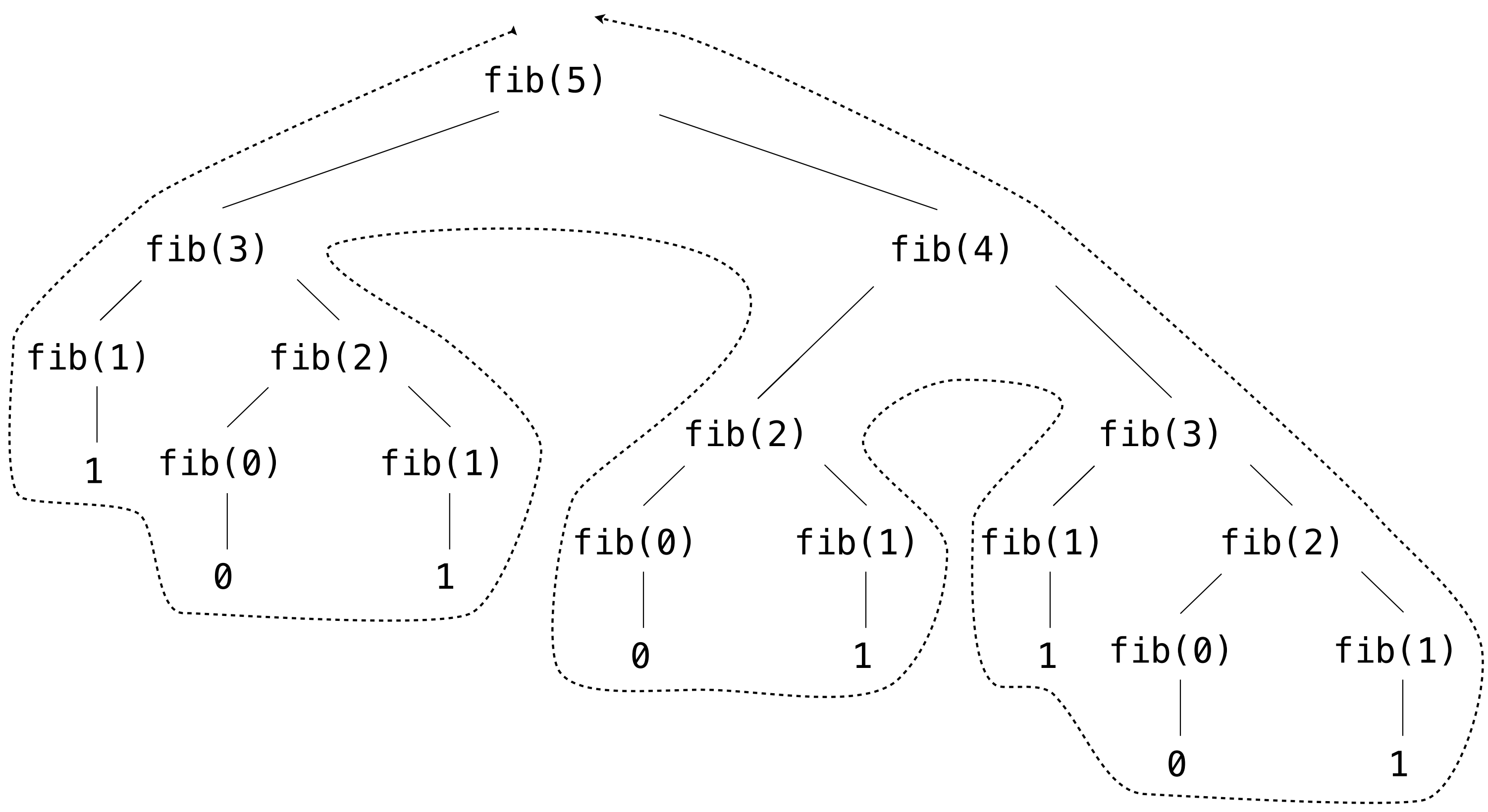

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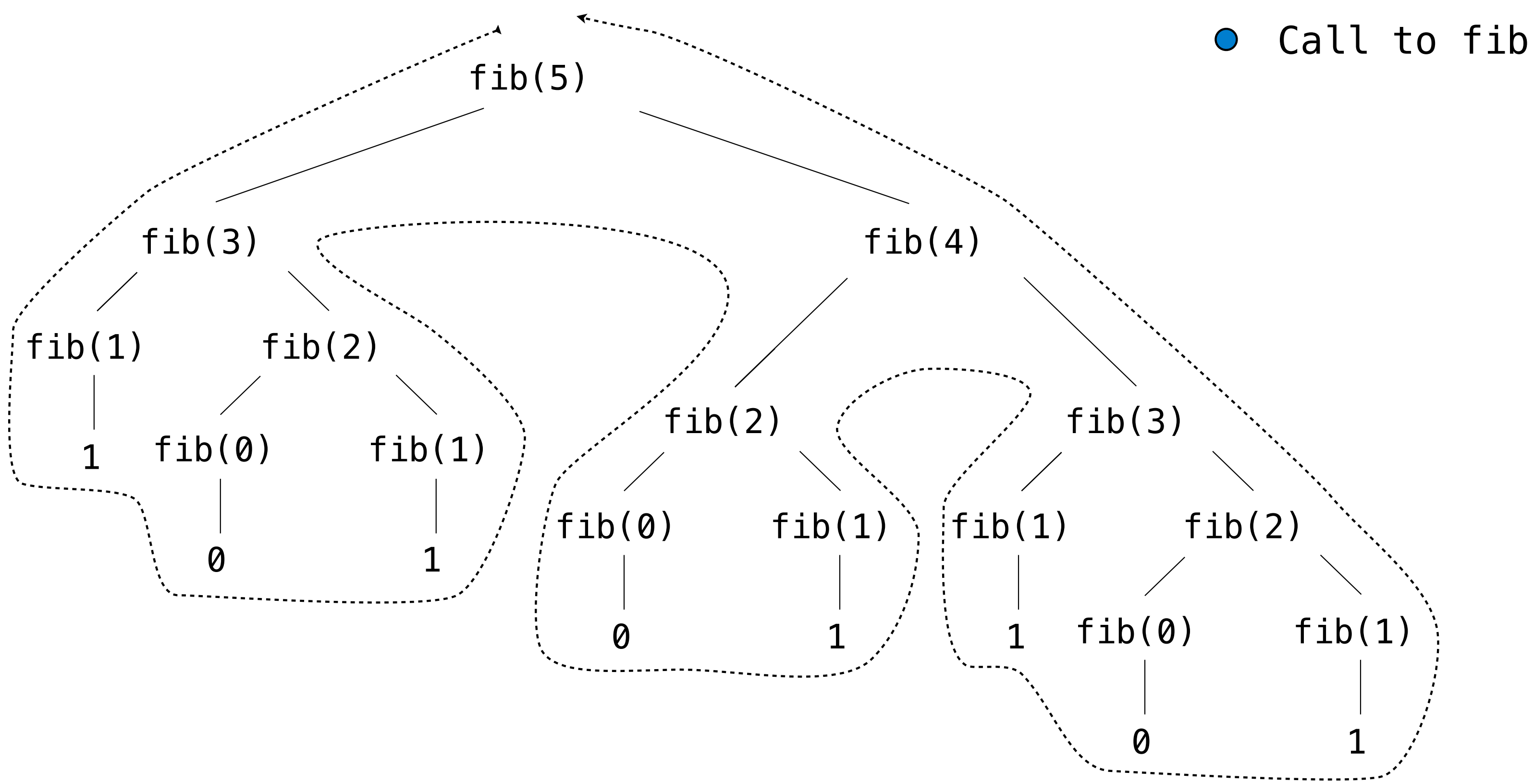

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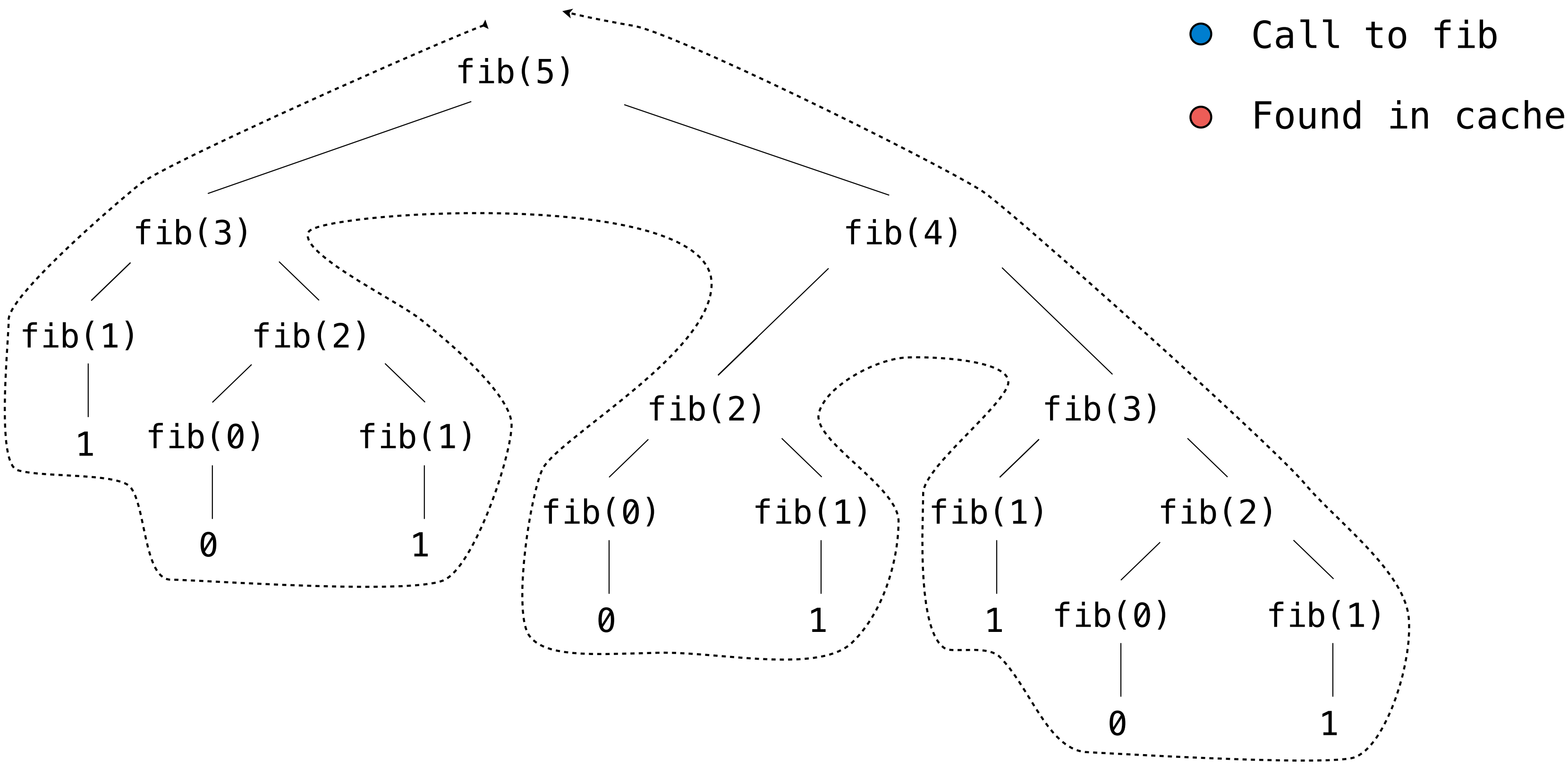

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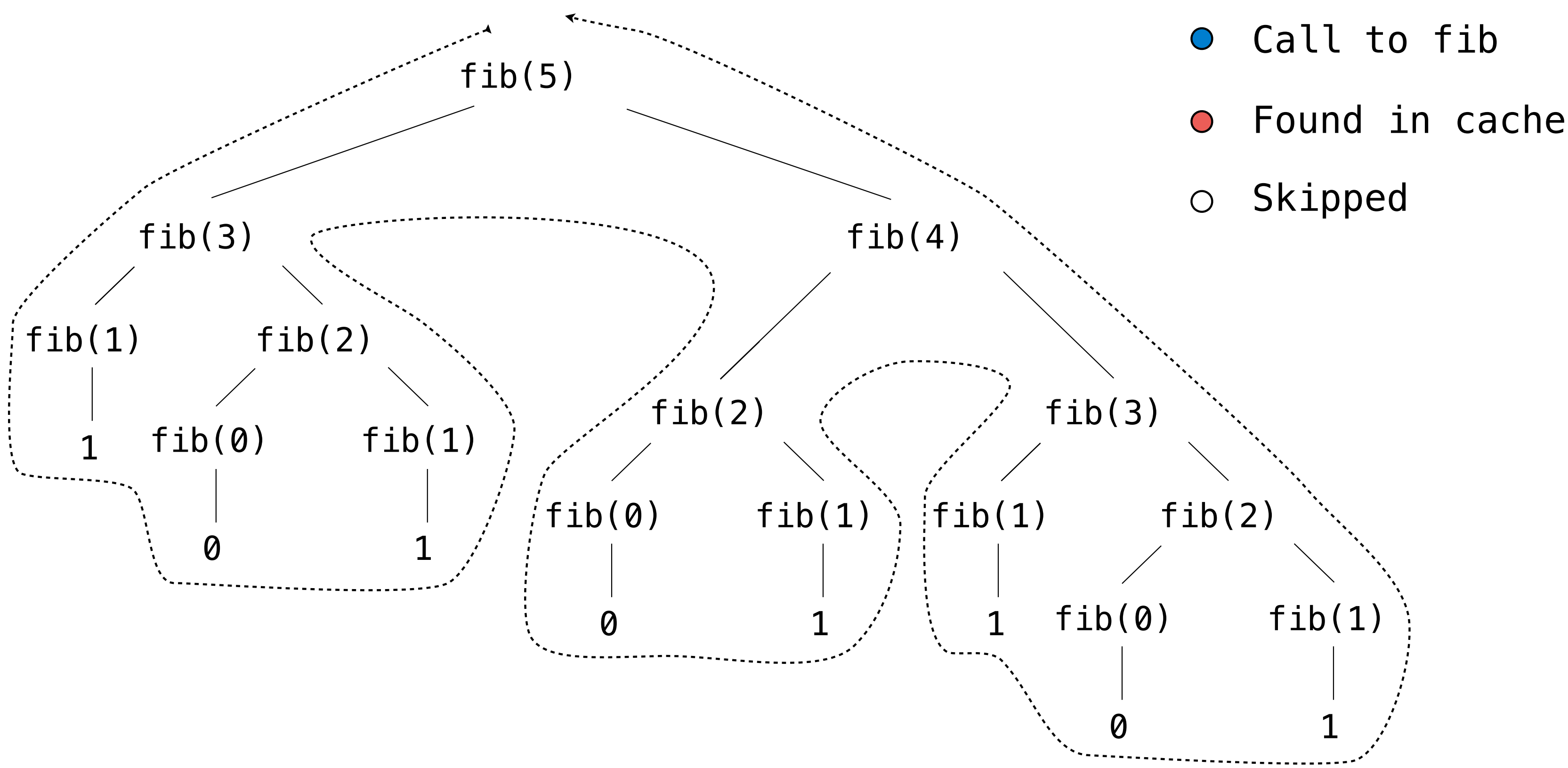

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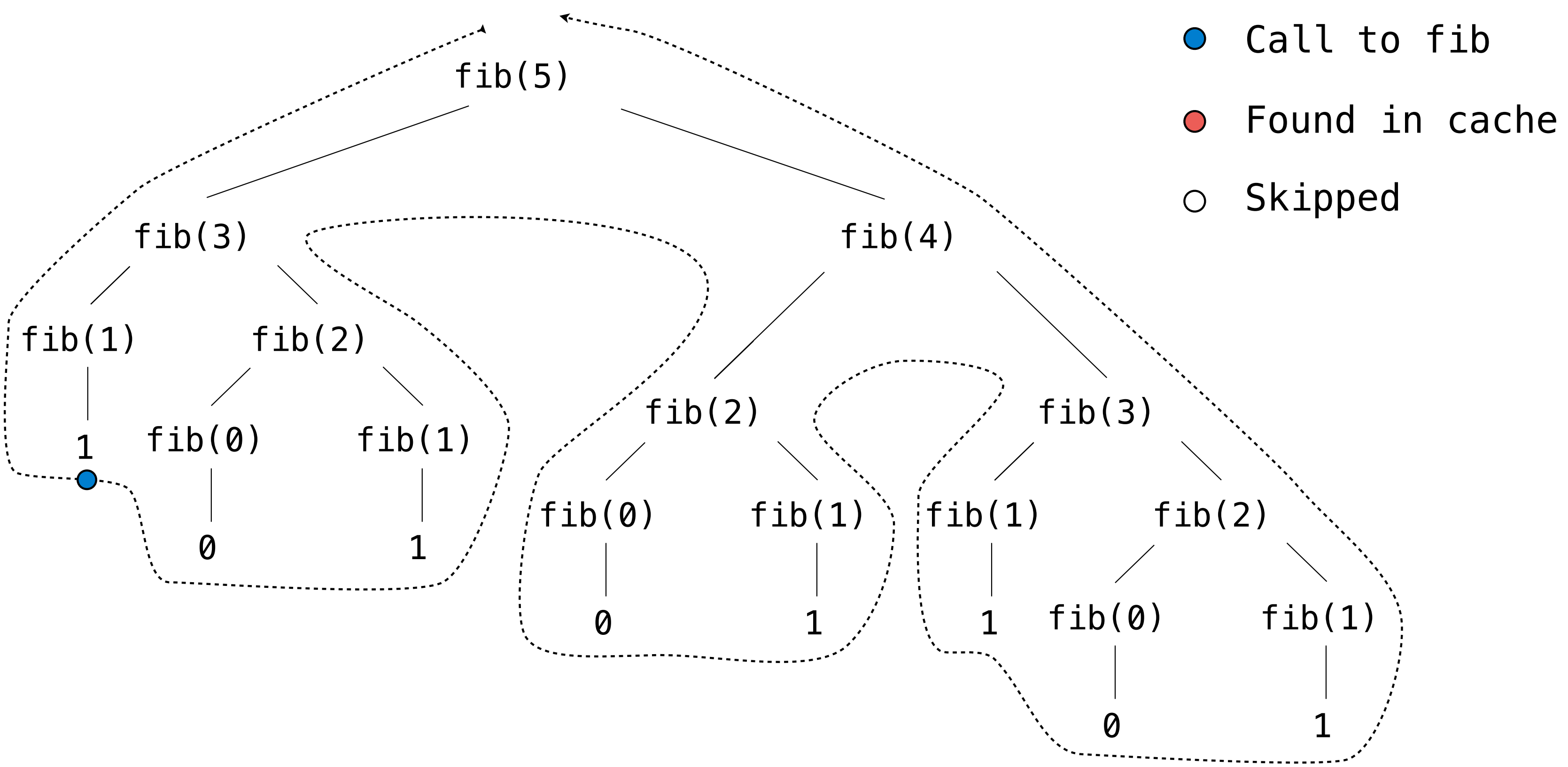

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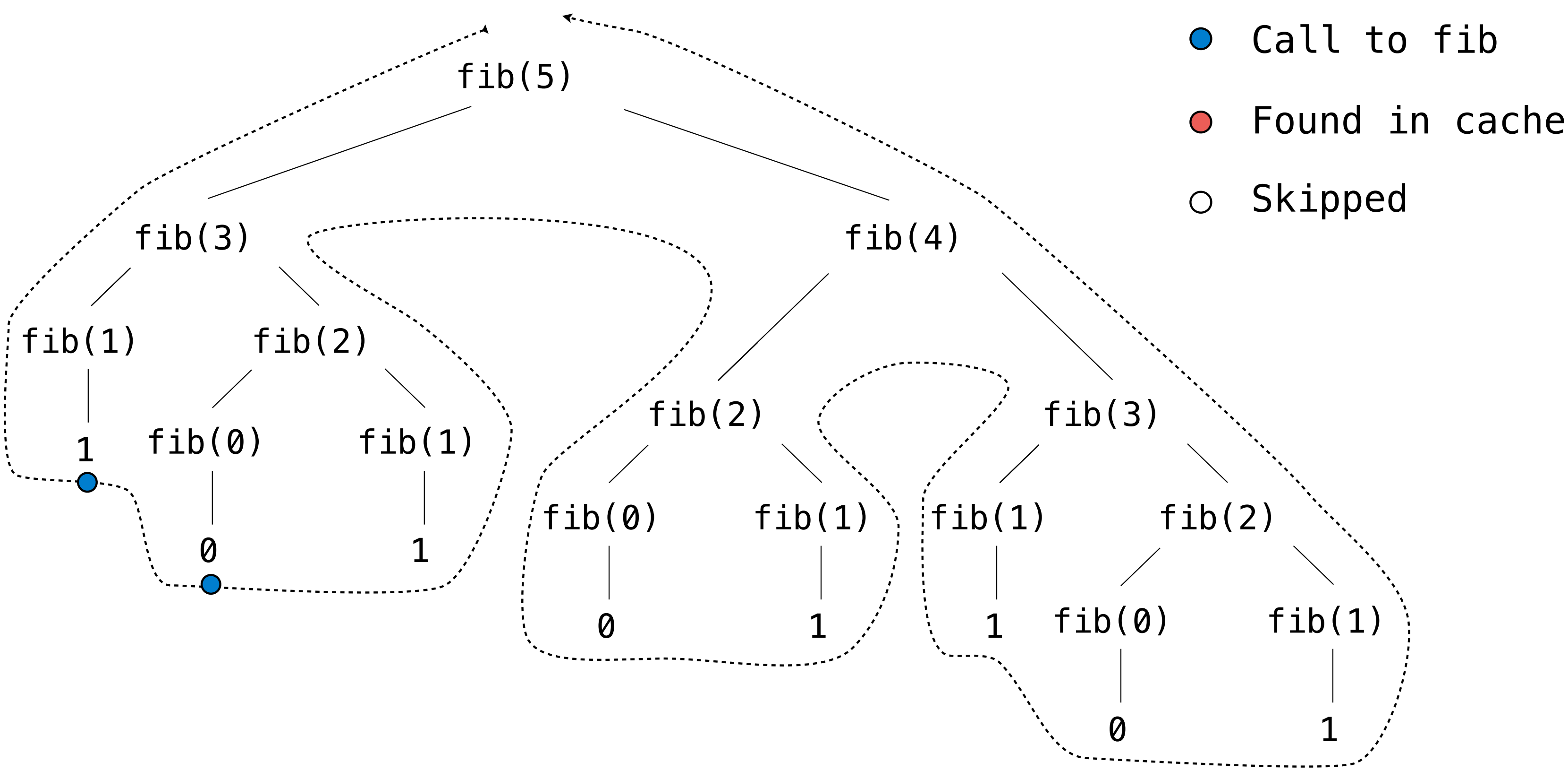

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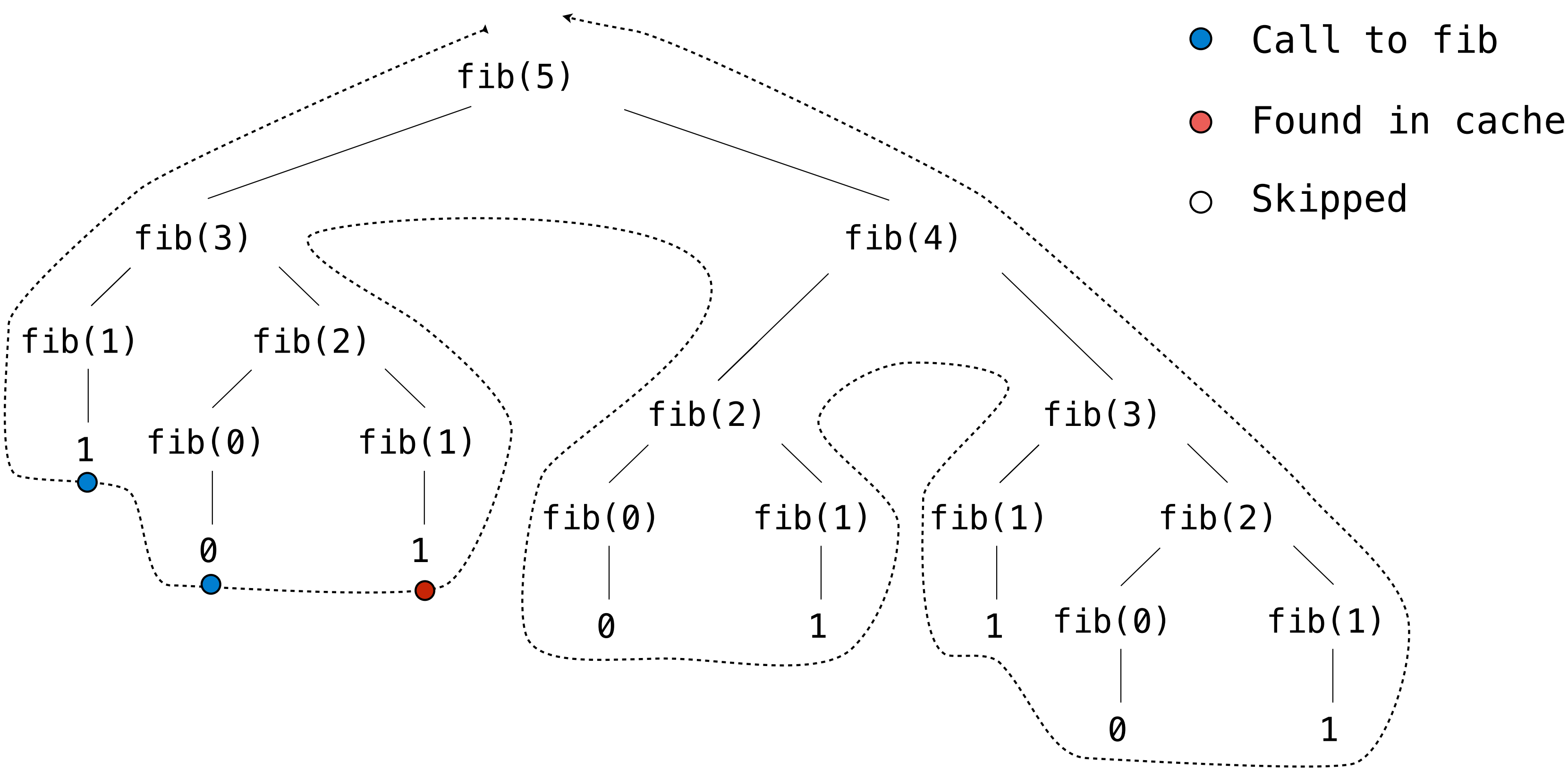

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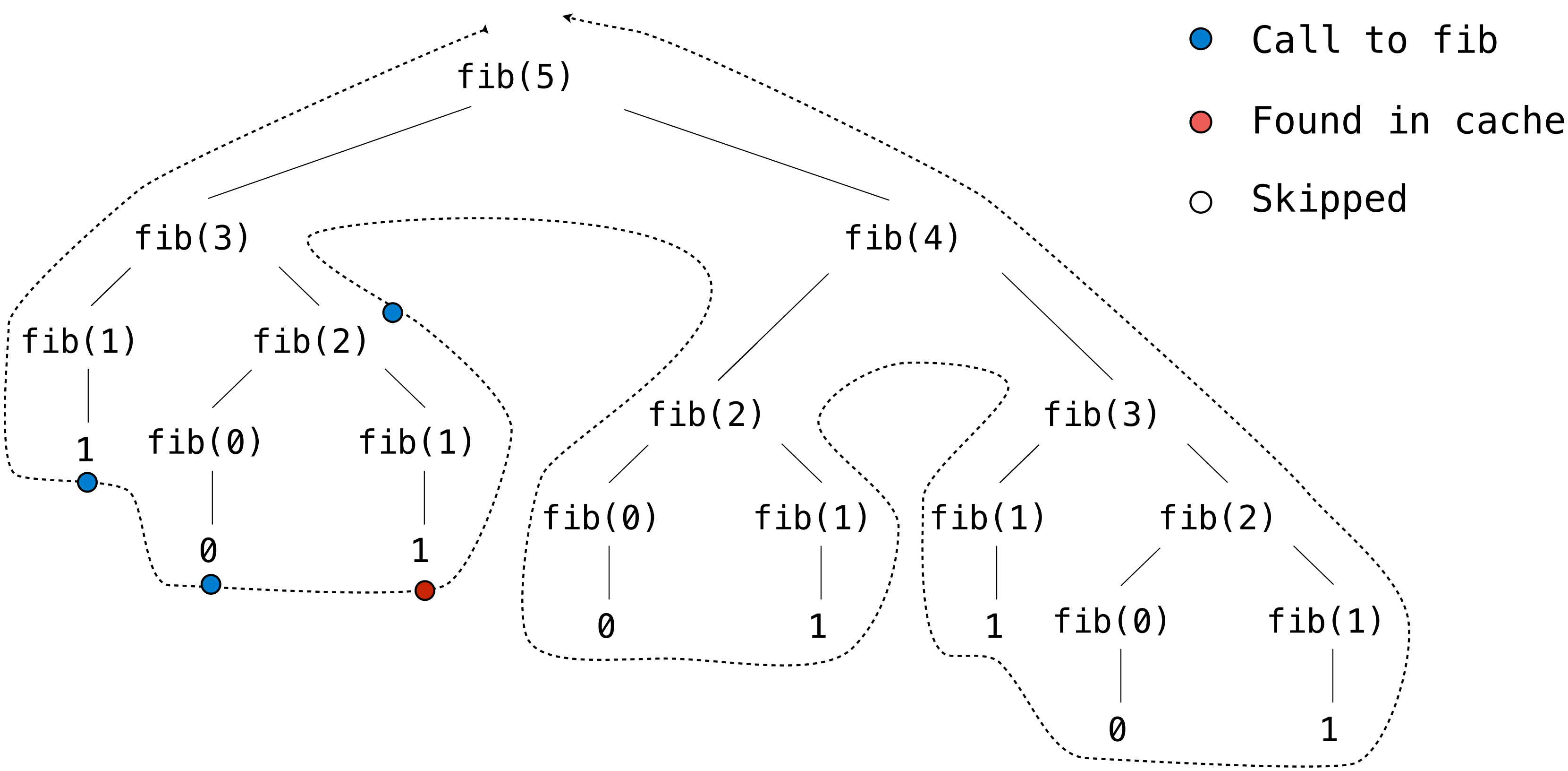

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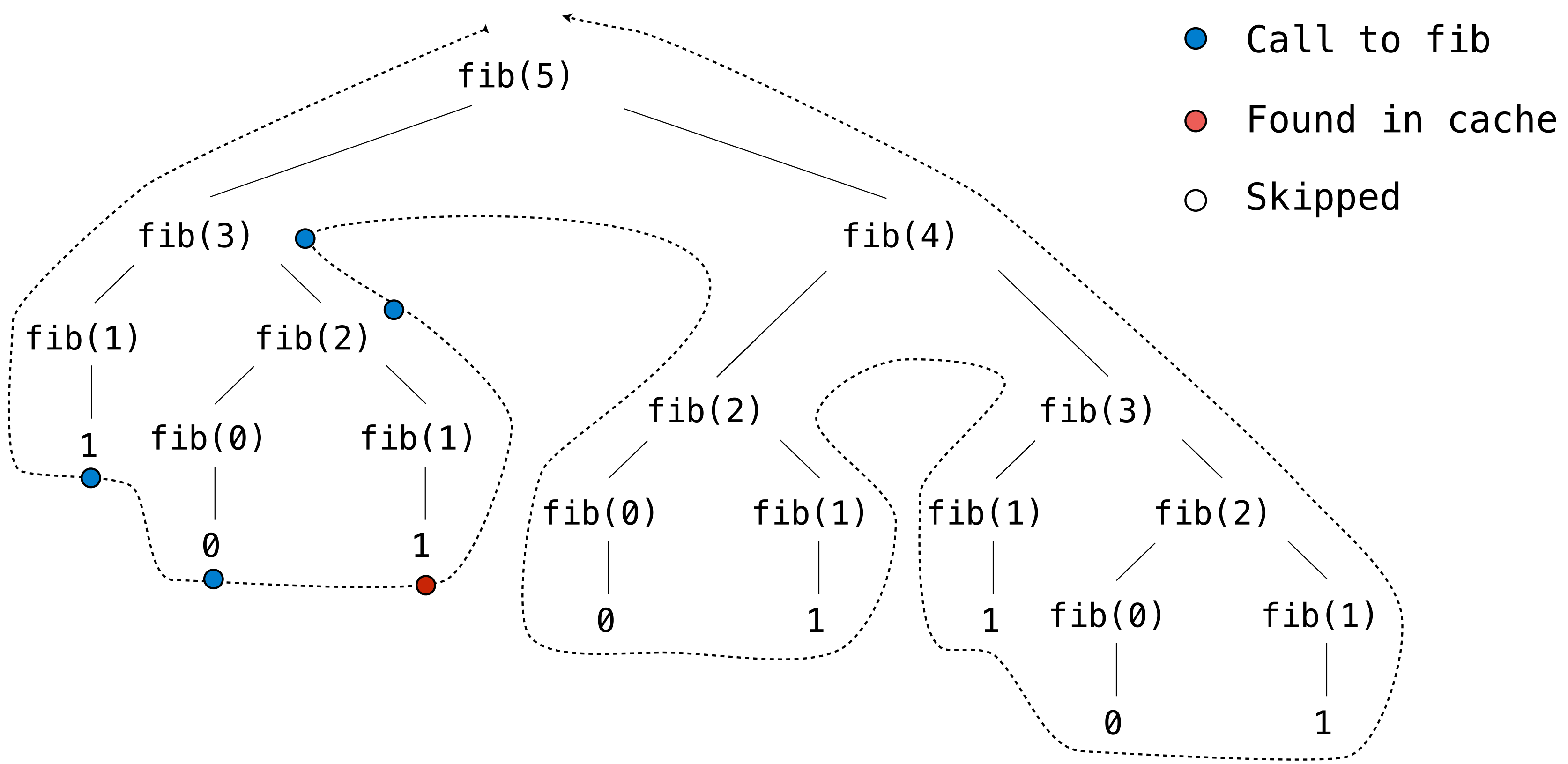

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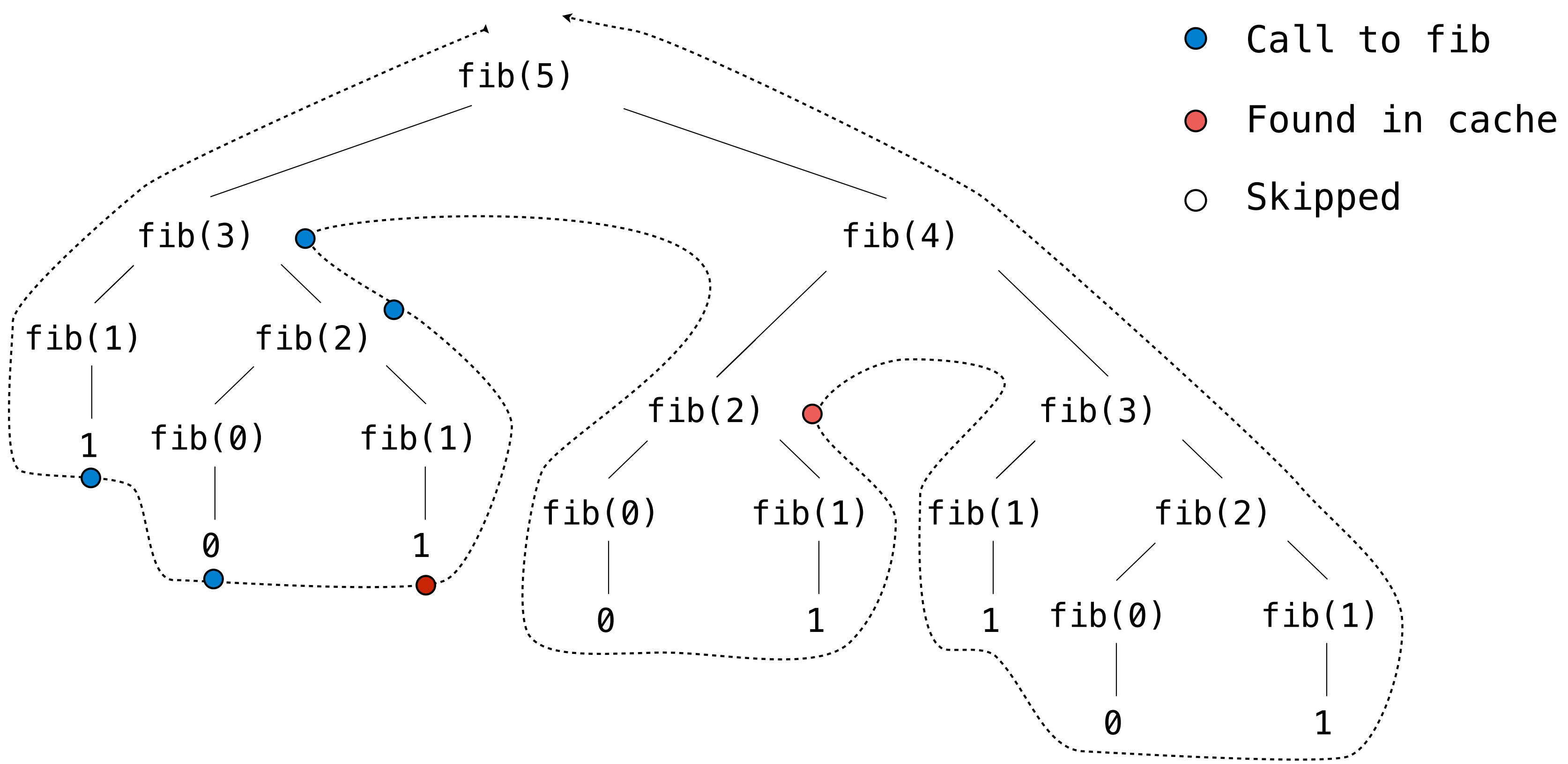

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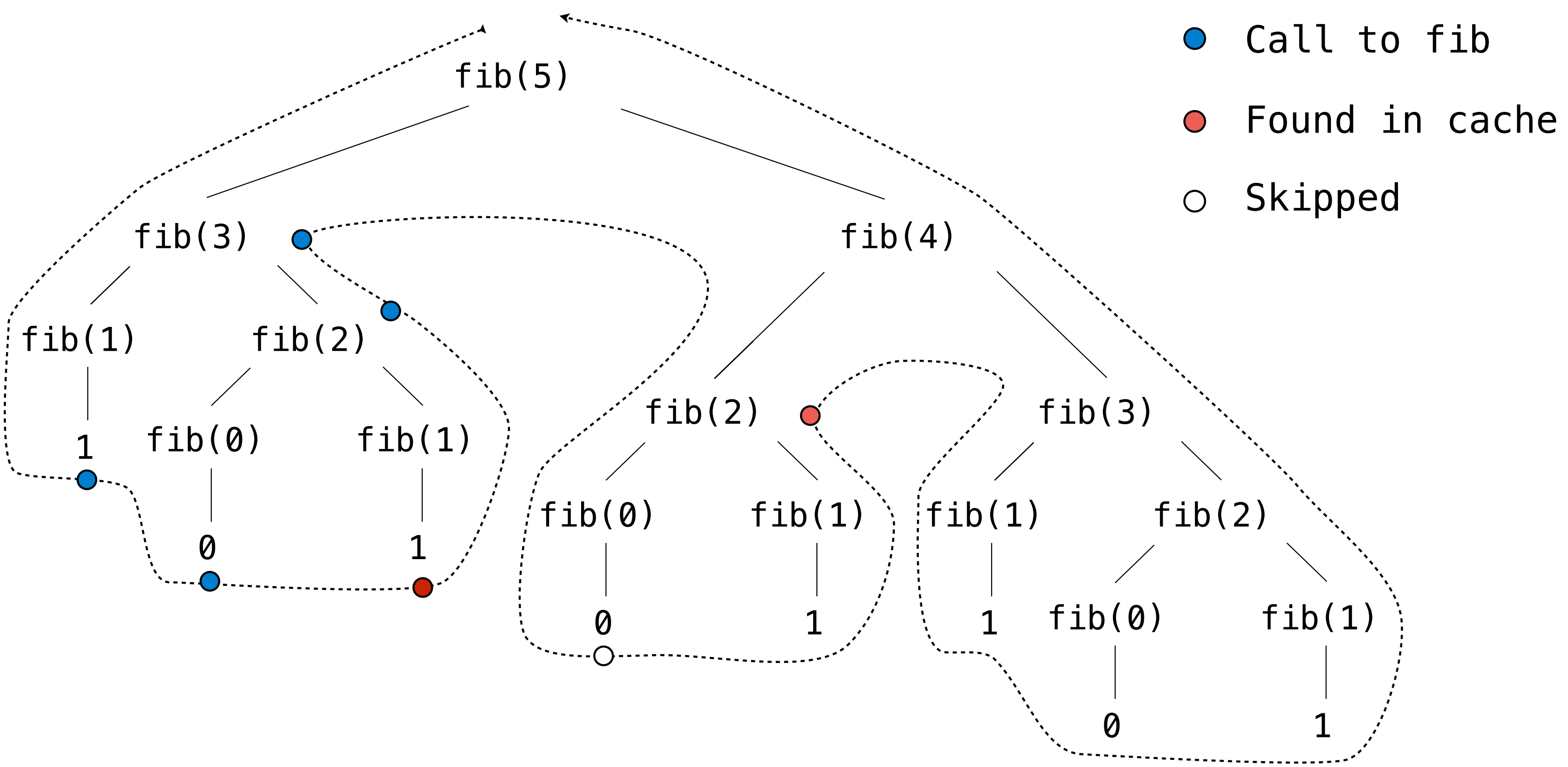

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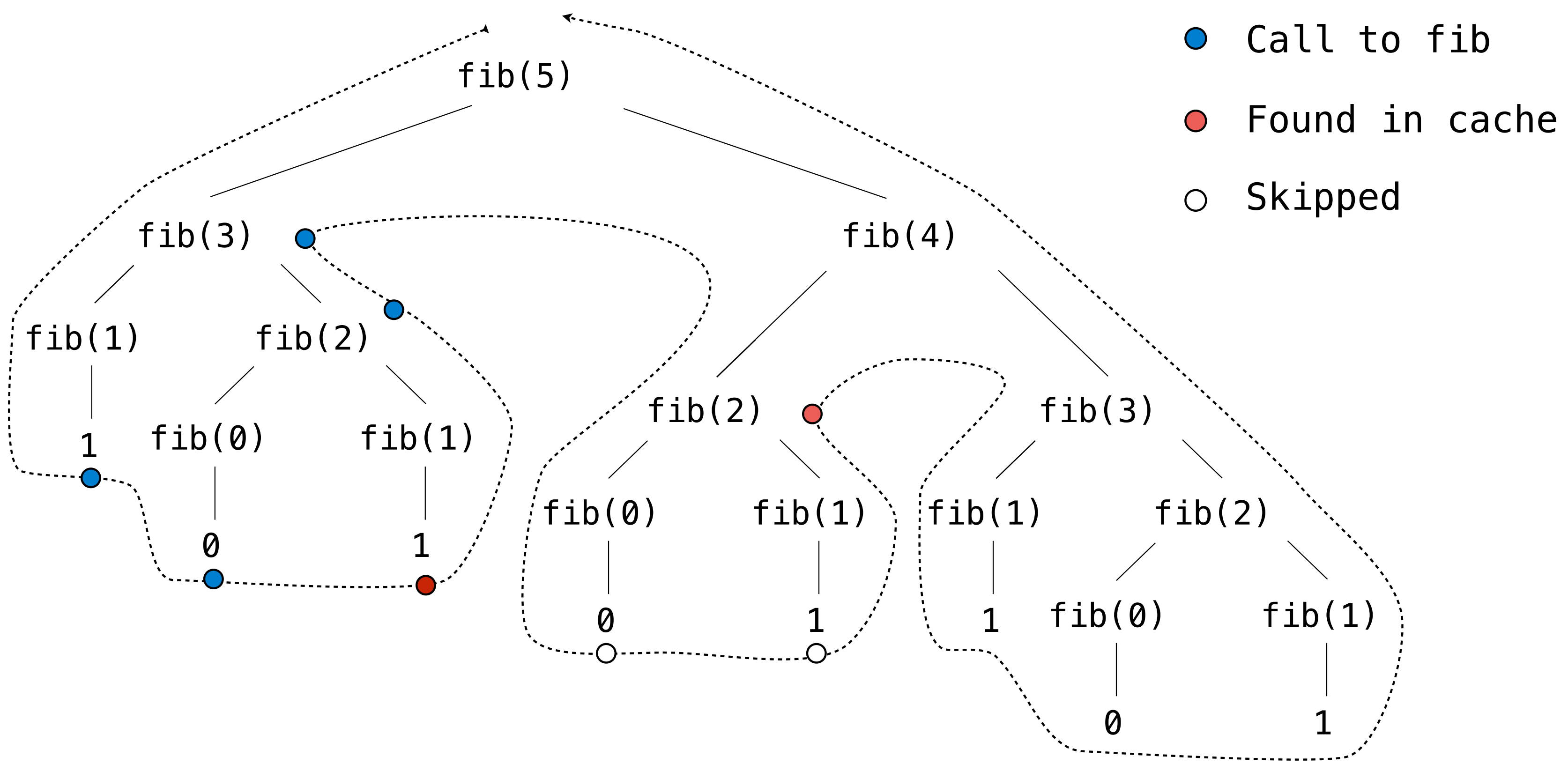

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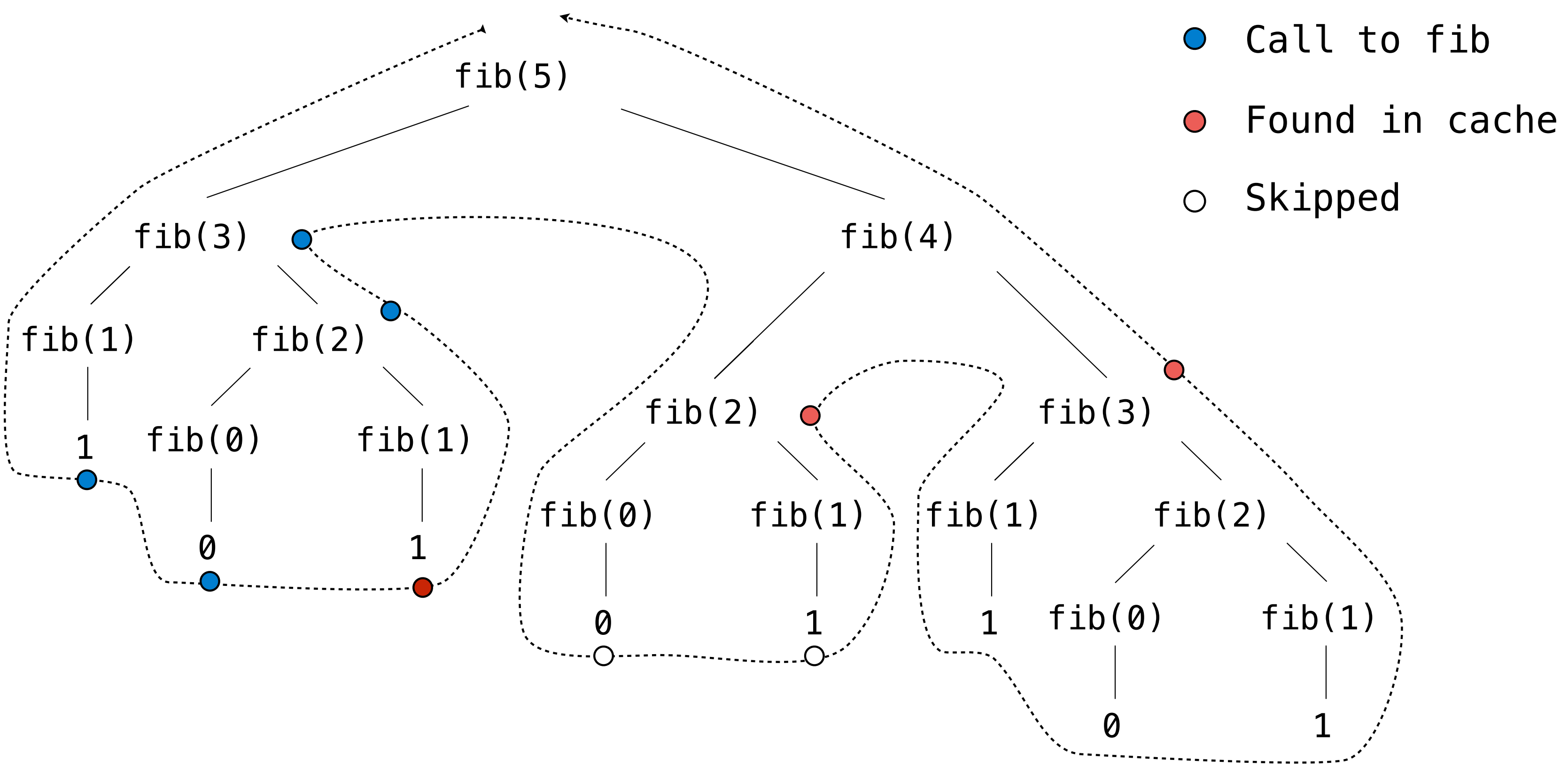

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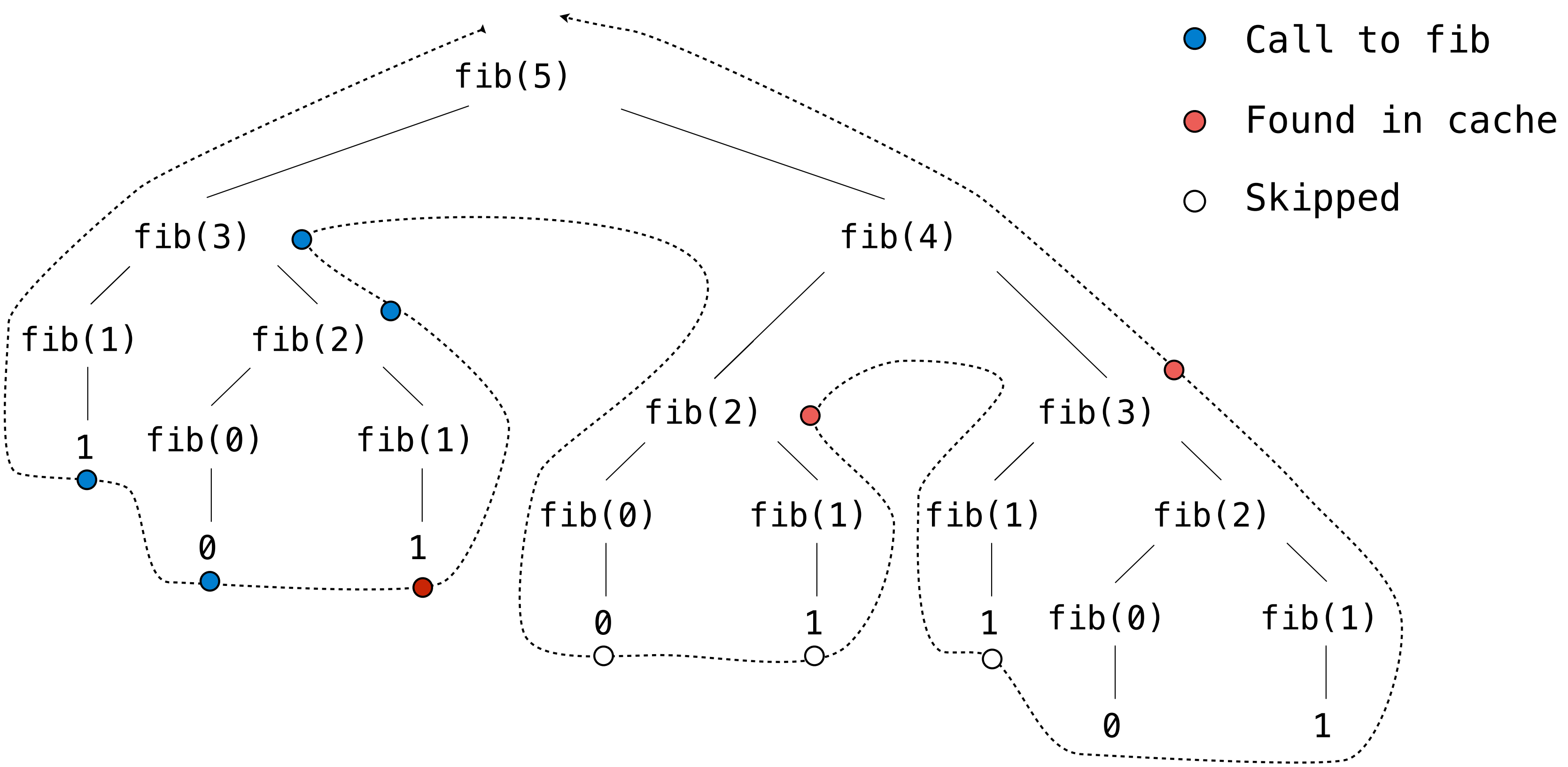

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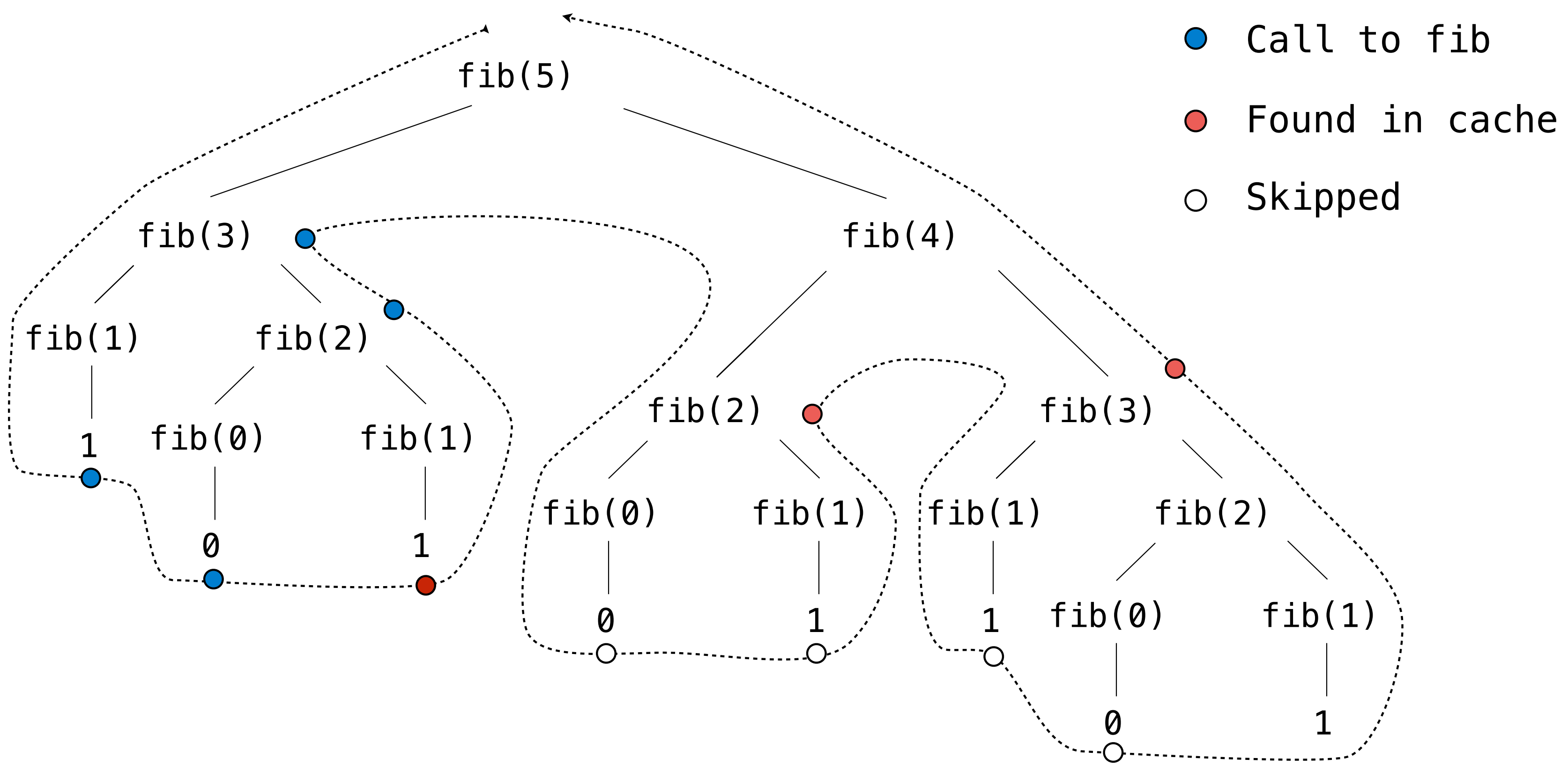

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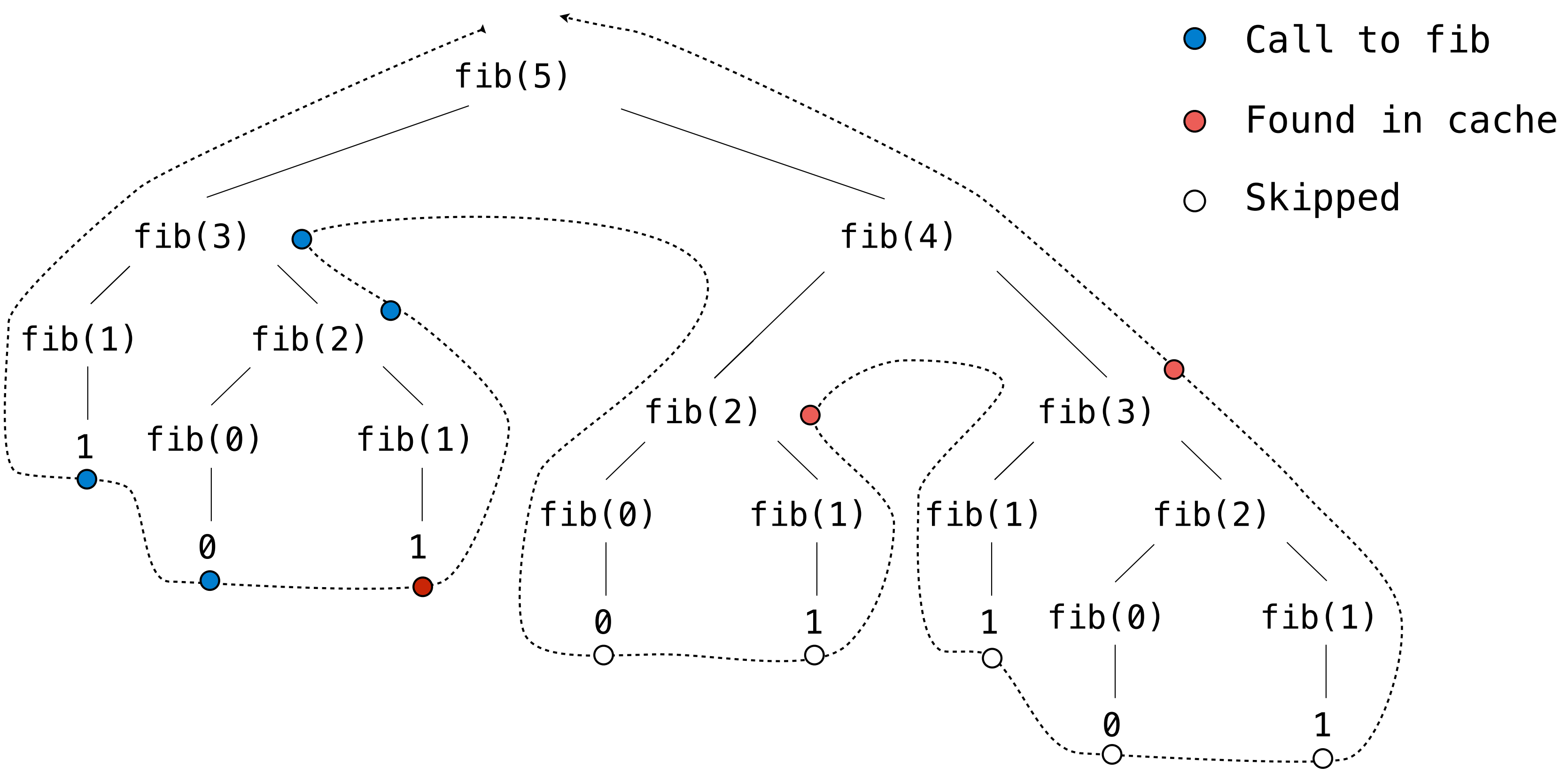

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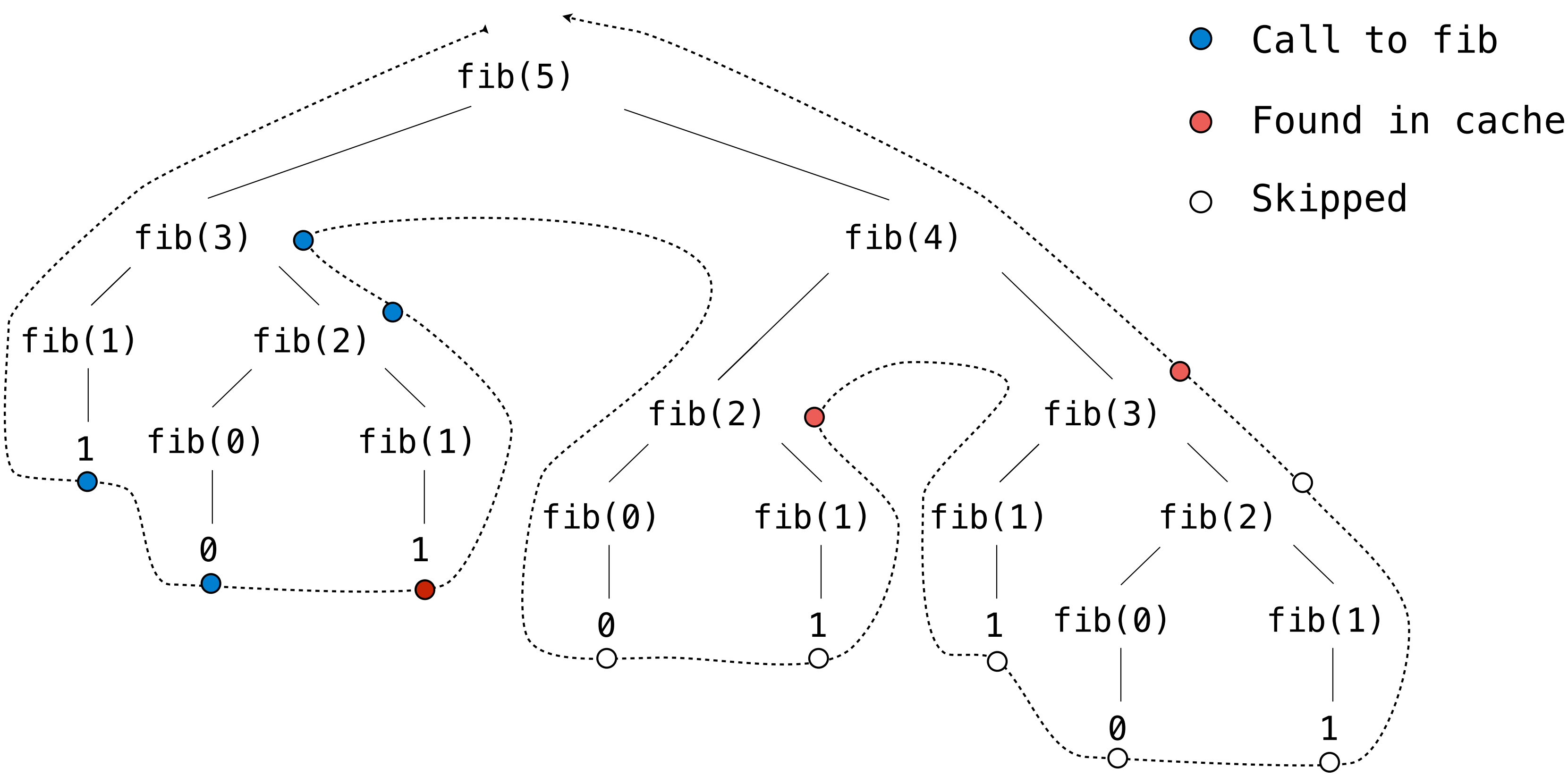

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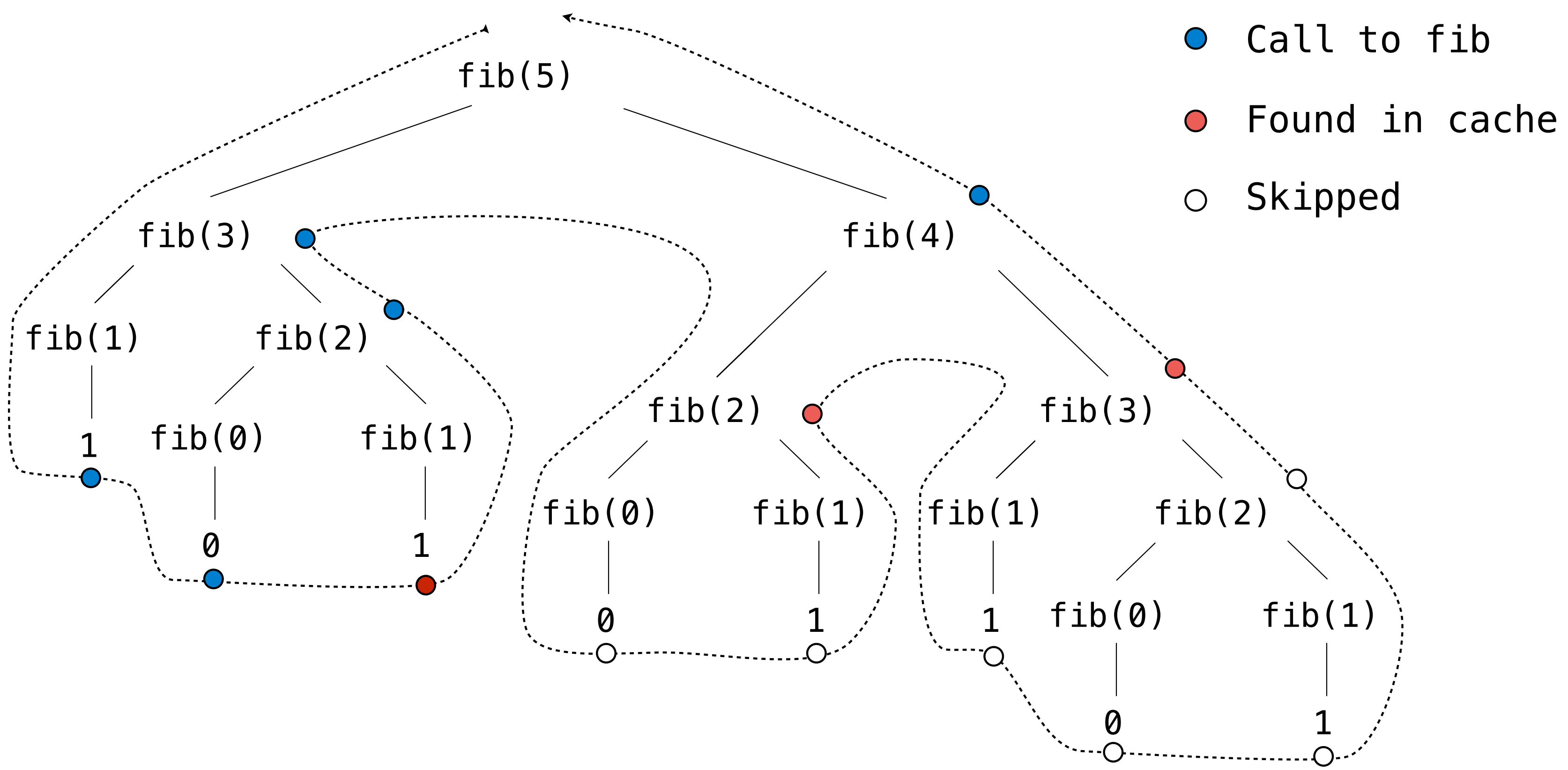

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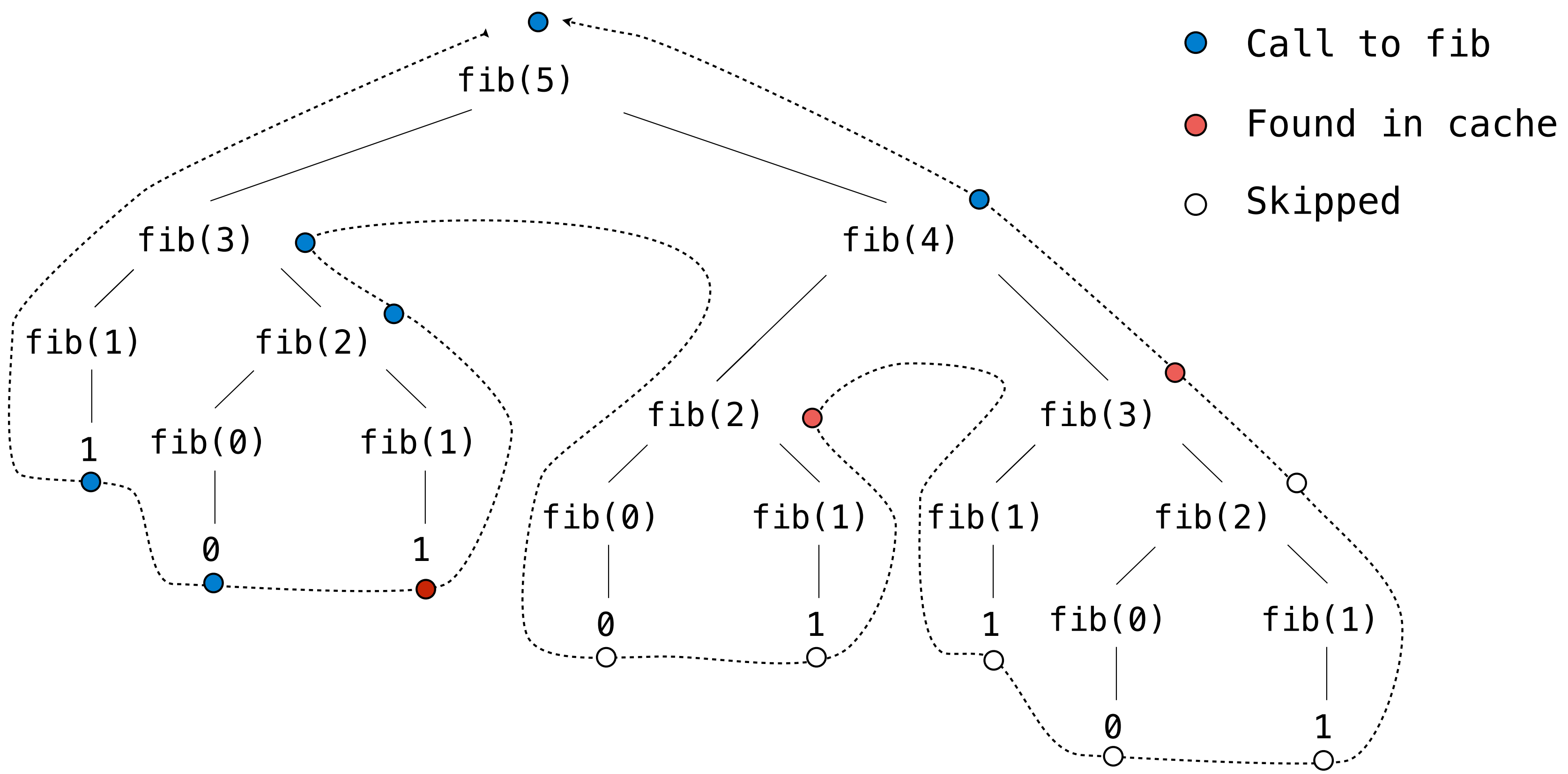

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$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ (b^{\frac{1}{2}n})^2 & \text{if } n \text{ is even} \\ b \cdot b^{n-1} & \text{if } n \text{ is odd} \end{cases}$$

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```
def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)

def square(x):
    return x * x
```

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ (b^{\frac{1}{2}n})^2 & \text{if } n \text{ is even} \\ b \cdot b^{n-1} & \text{if } n \text{ is odd} \end{cases}$$

Exponentiation

Goal: one more multiplication lets us double the problem size

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def exp(b, n):
    if n == 0:
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    else:
        return b * exp(b, n-1)
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$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ b \cdot b^{n-1} & \text{otherwise} \end{cases}$$

```
def exp_fast(b, n):
    if n == 0:
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    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)

def square(x):
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(Demo)

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Linear time:

- Doubling the input **doubles** the time
- 1024x the input takes 1024x as much time

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Logarithmic time:

- Doubling the input **increases** the time by a constant C
- 1024x the input increases the time by only 10 times C

Orders of Growth

Quadratic Time

Functions that process all pairs of values in a sequence of length n take quadratic time

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def overlap(a, b):
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        for other in b:
            if item == other:
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    return count

overlap([3, 5, 7, 6], [4, 5, 6, 5])
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Tree-recursive functions can take exponential time

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def fib(n):
    if n == 0:
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http://en.wikipedia.org/wiki/File:Fibonacci.jpg

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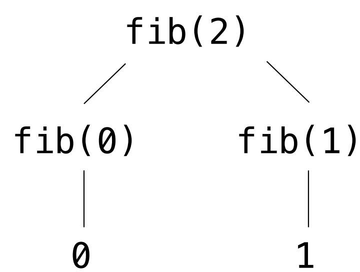

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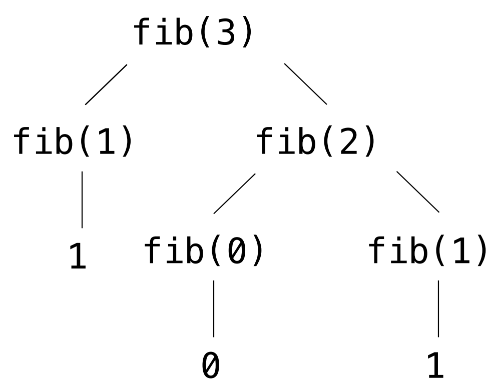

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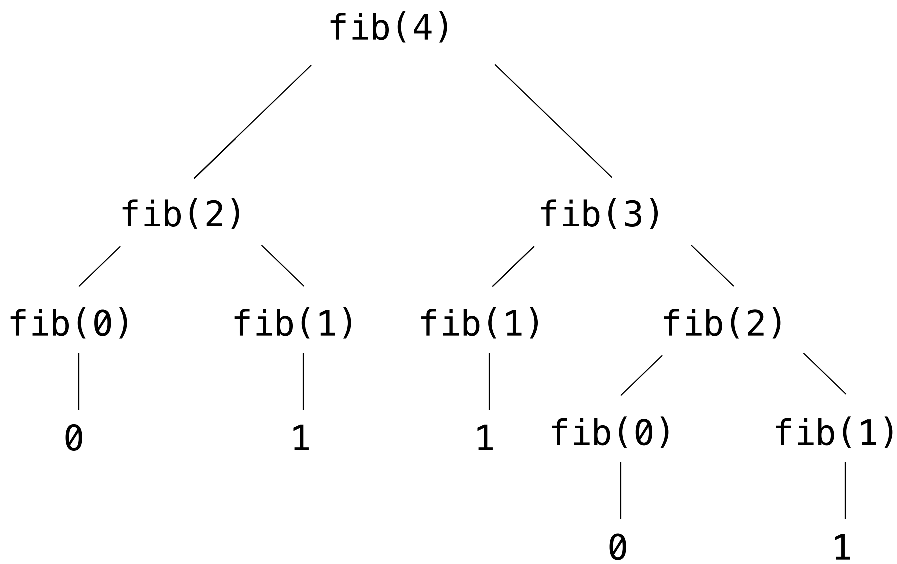

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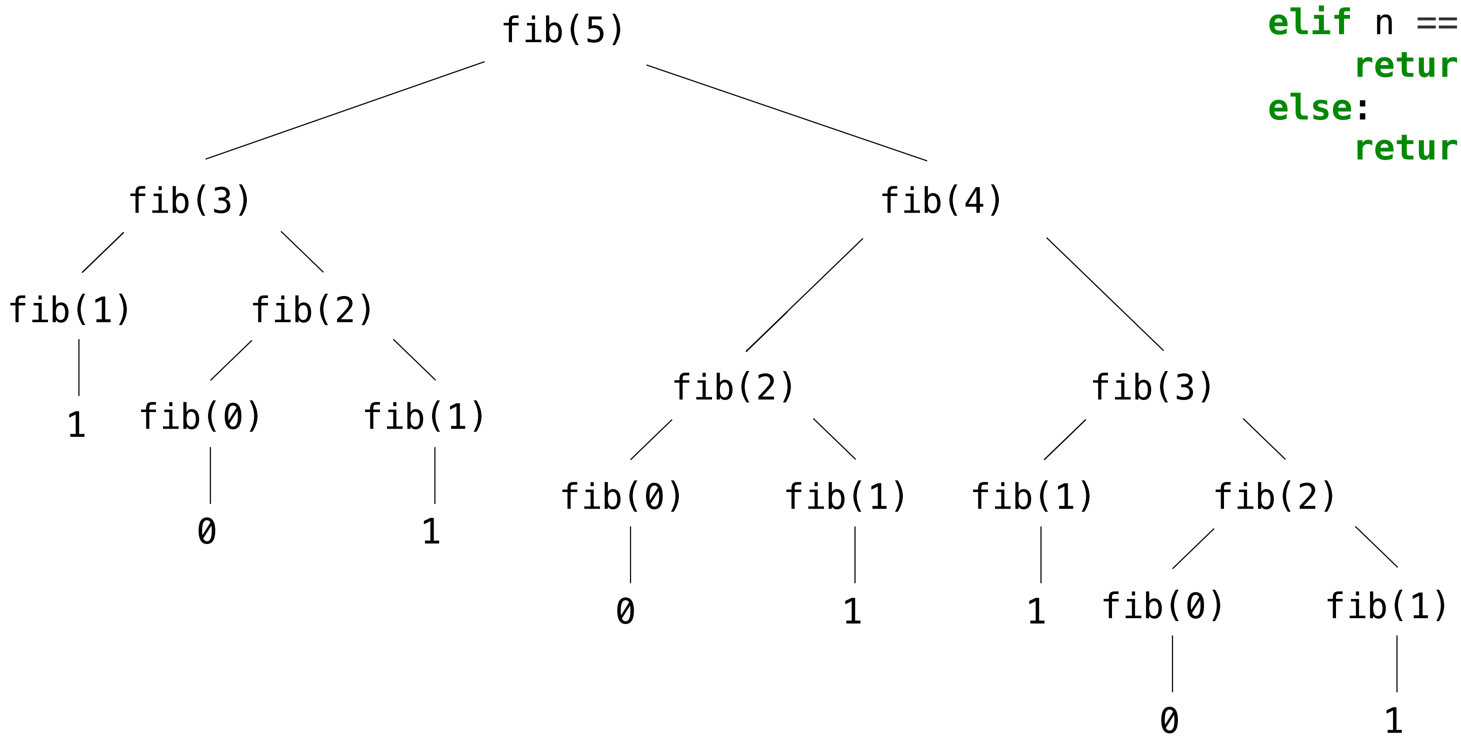

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Common Orders of Growth

Exponential growth. E.g., recursive fib

Quadratic growth. E.g., overlap

Linear growth. E.g., slow exp

Logarithmic growth. E.g., exp_fast

Constant growth. Increasing *n* doesn't affect time

Common Orders of Growth

Exponential growth. E.g., recursive `fib`

$$a \cdot b^{n+1} = (a \cdot b^n) \cdot b$$

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Space

Space and Environments

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Which environment frames do we need to keep during evaluation?

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(Demo)

pythontutor.com/
composingprograms.html#code=def%20fib%28n%29%3A%0A%20%20%20%20if%20n%20%3D%3D%200%20or%20n%20%3D%3D%201%3A%0A%20%20%20%20%20%20%20%20return%20n%0A%20%20%20%20else%3A%0A%20%20%20%20%20%20%20%20return%20fib%28n-2%29%20%2B%20fib%28n-1%29%0A%20%20%20%20%20%20%20%20%0Afib%286%29&mode=display&
origin=composingprograms.js&cumulative=false&py=3&rawInputLstJSON=[]&curInstr=1

Fibonacci Space Consumption

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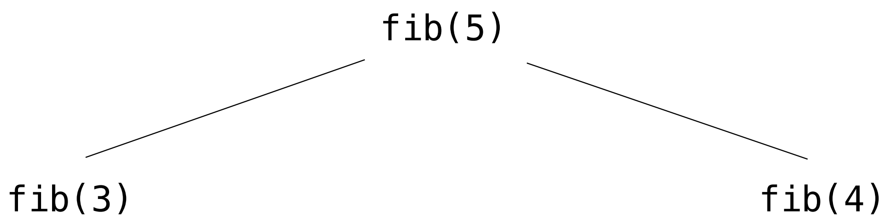
```
fib(5)
```

Fibonacci Space Consumption

```
                fib(5)

        fib(3)
```

Fibonacci Space Consumption



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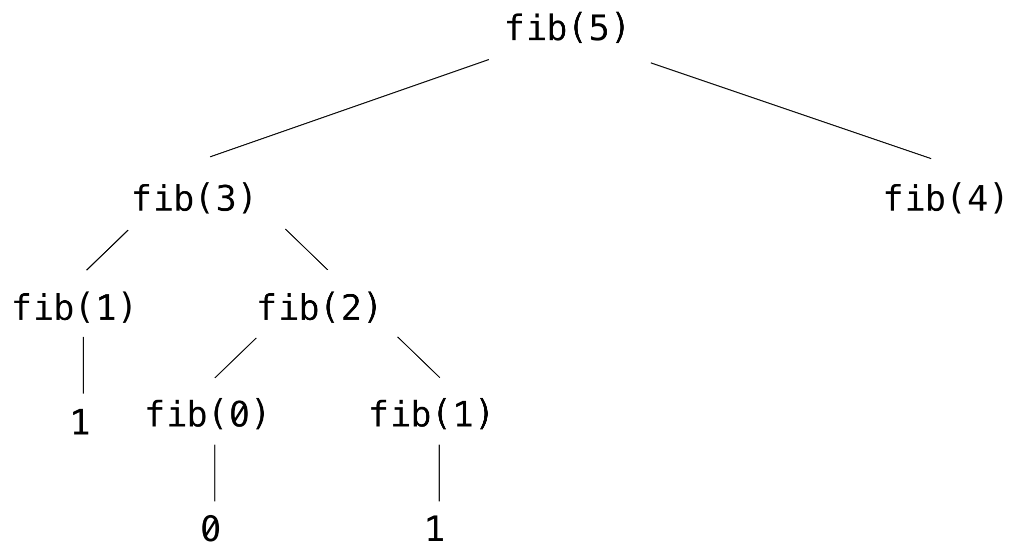

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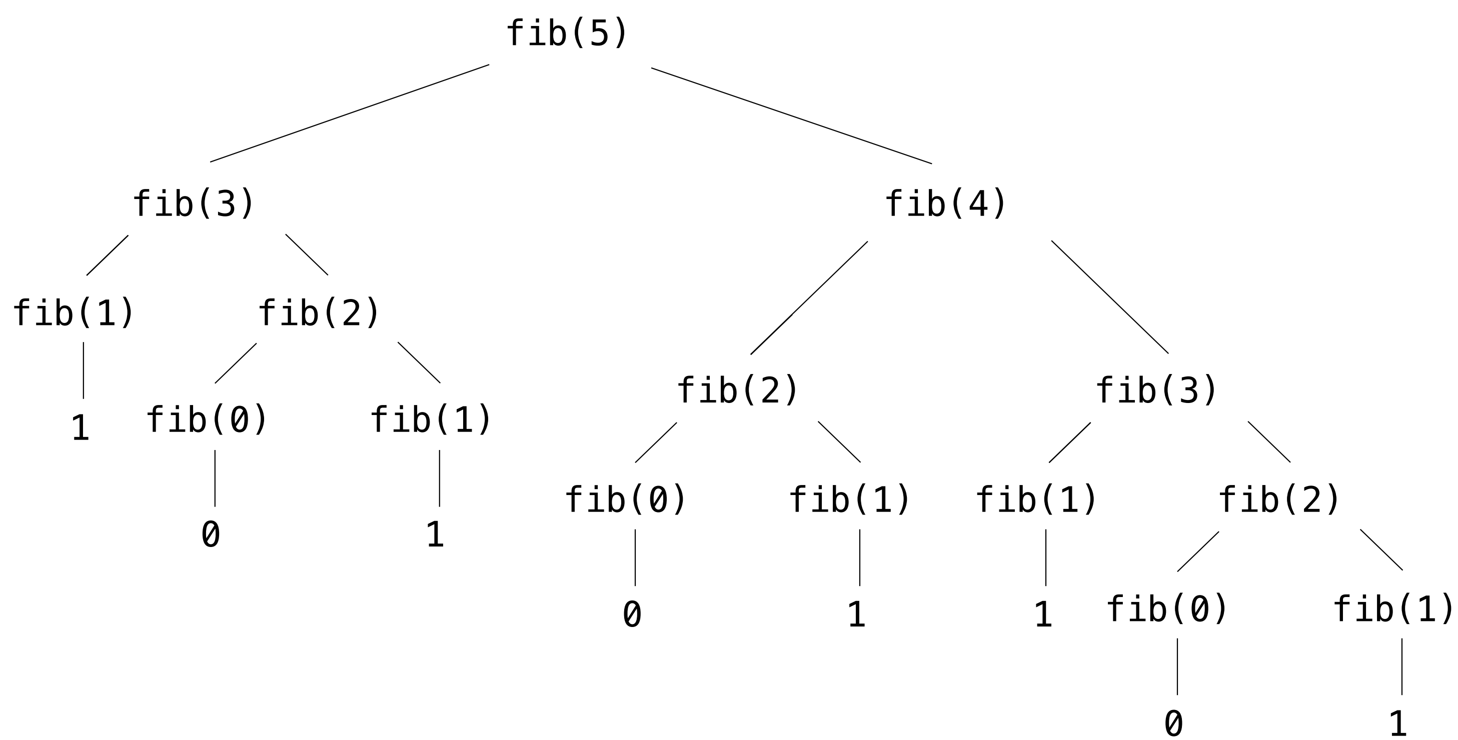

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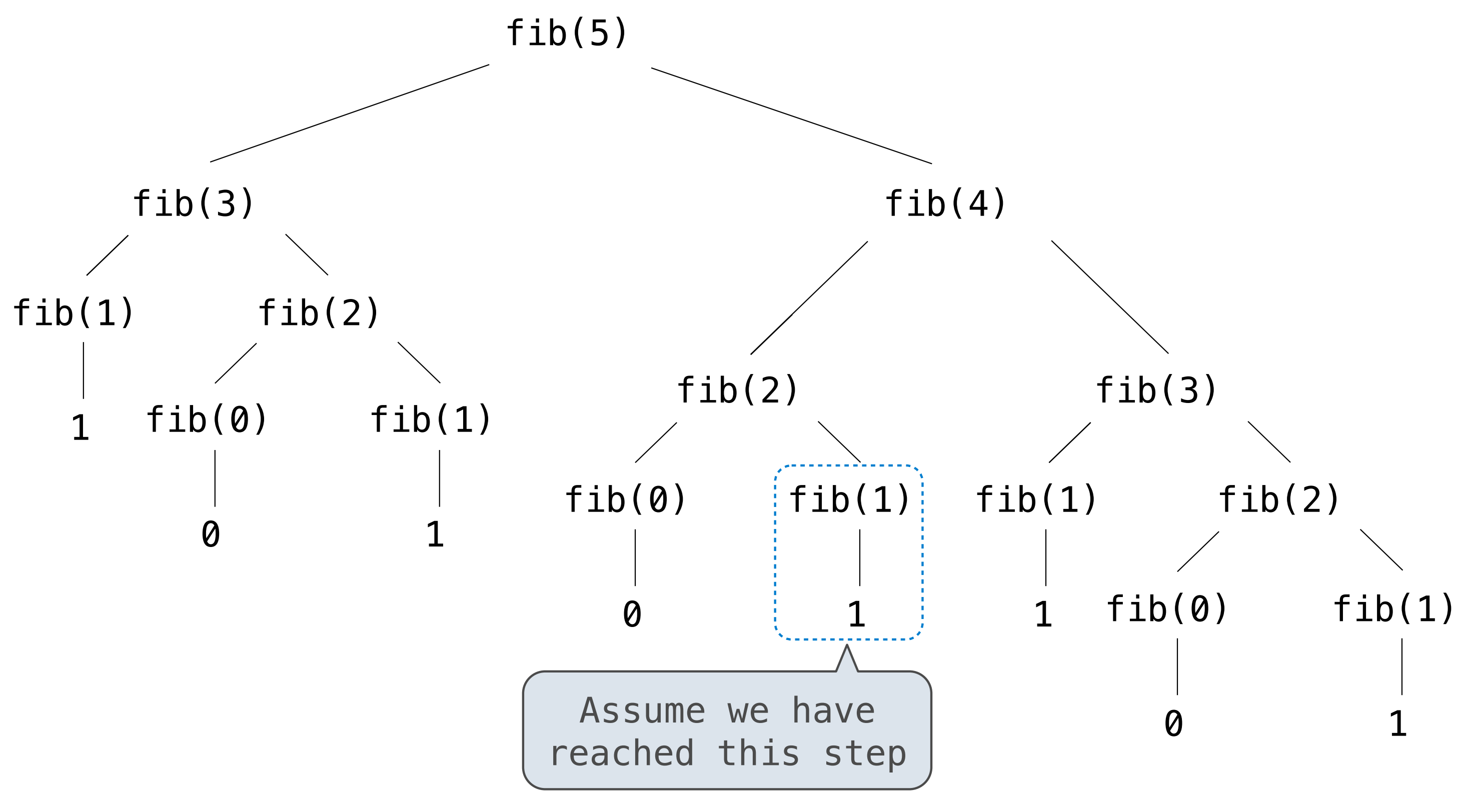

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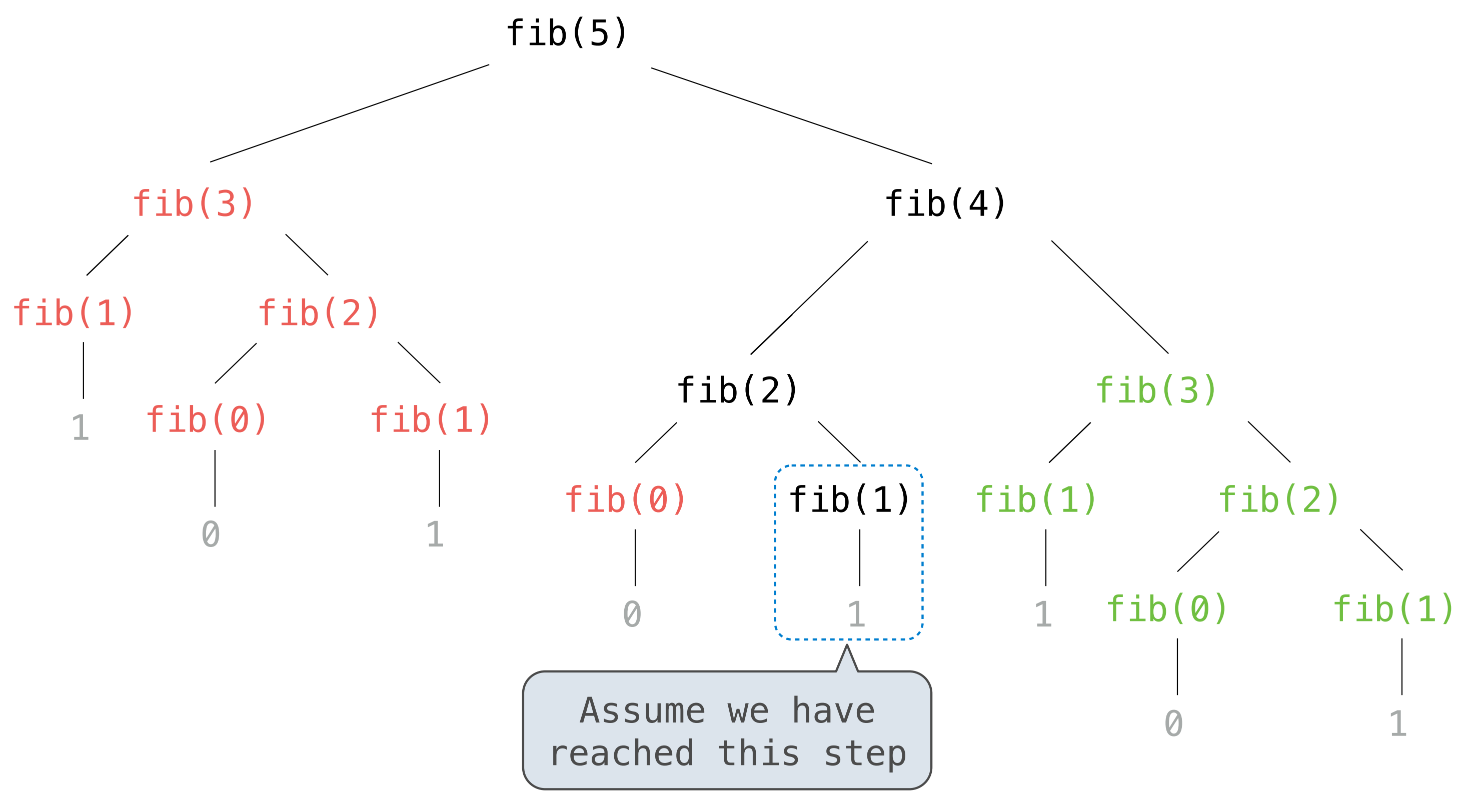

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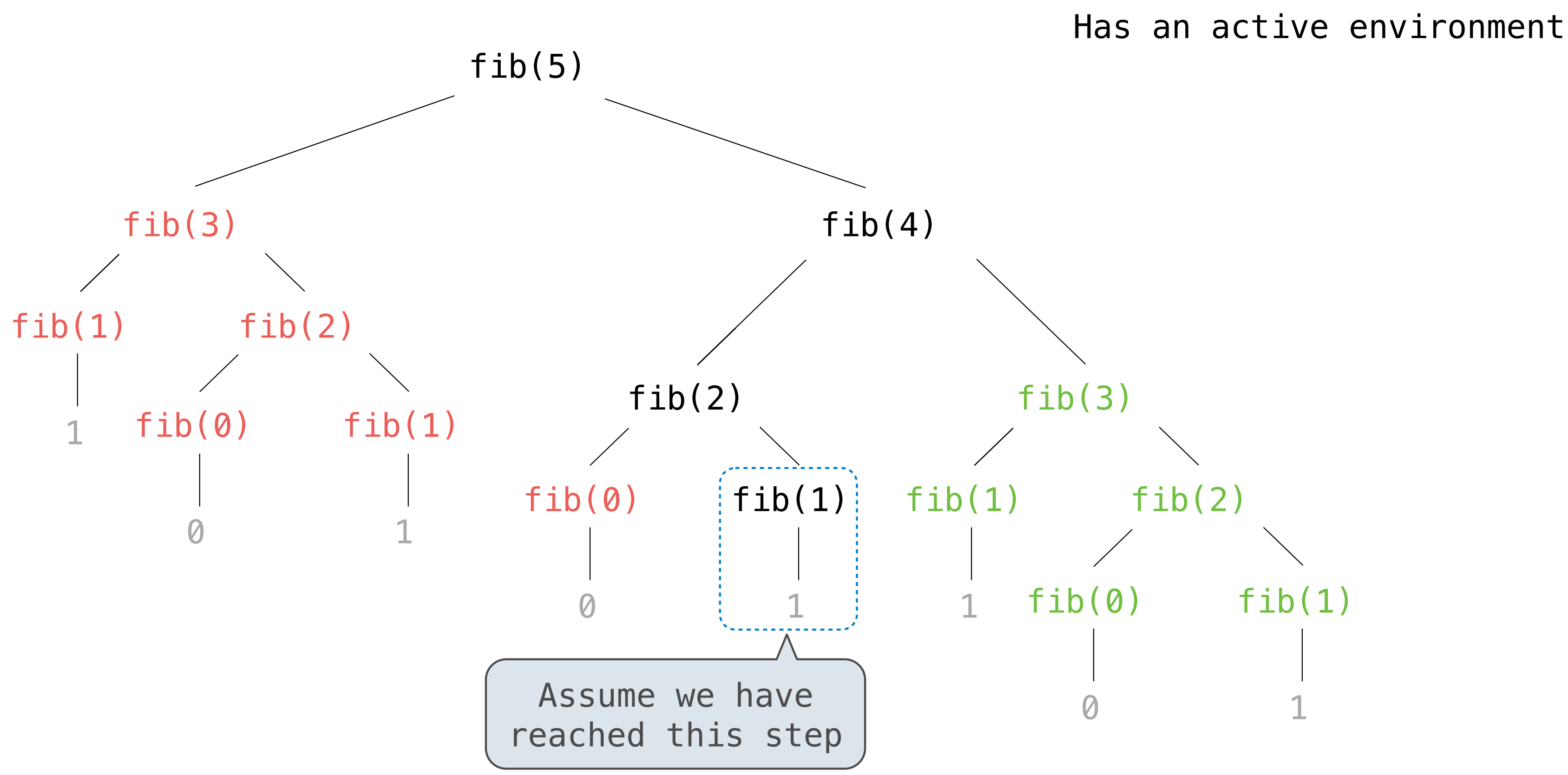
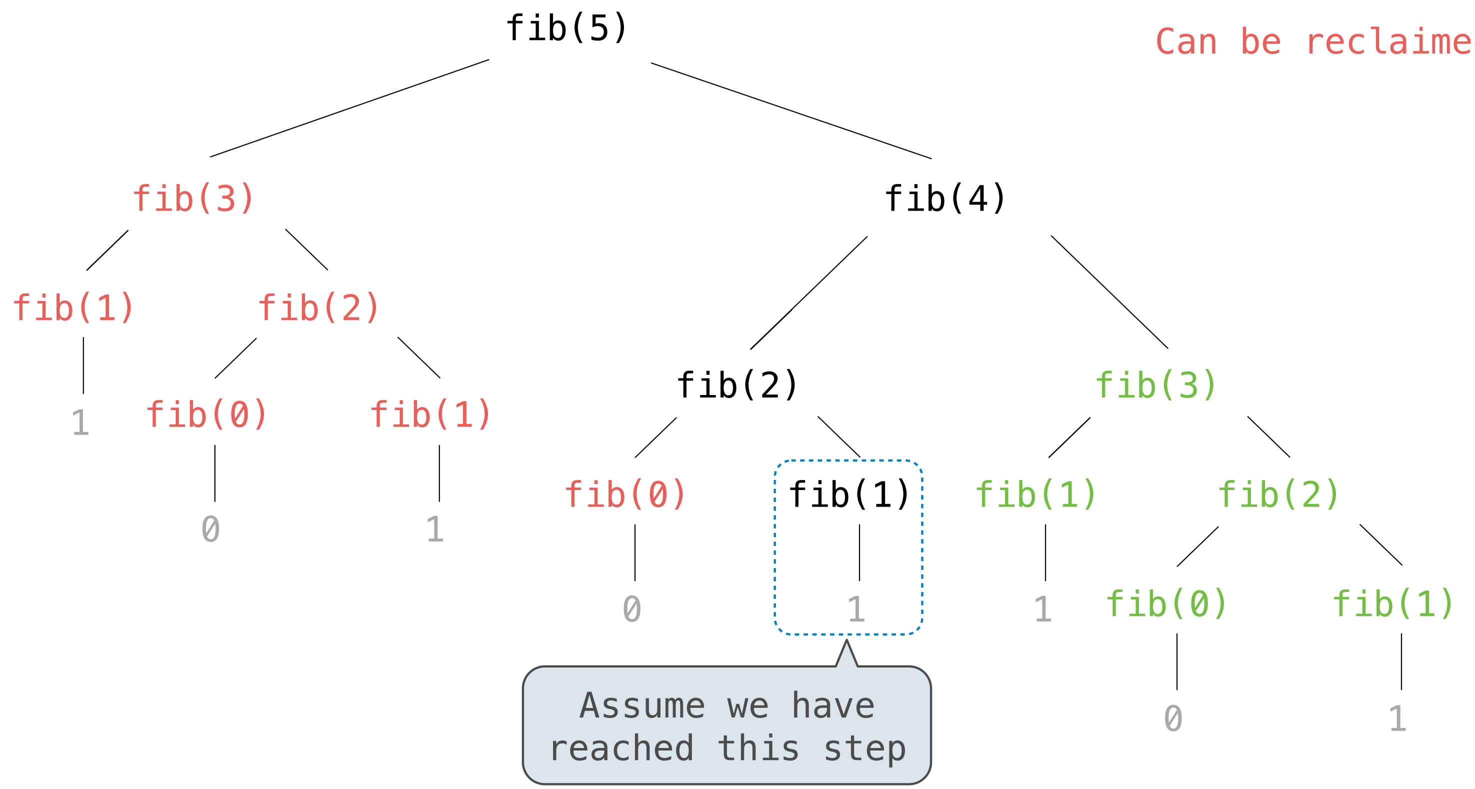


Fibonacci Space Consumption

Has an active environment

Can be reclaimed



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