APPLIED NUMERICAL METHODS
Homework # 1

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Question 1

[Range, Linear Mapping and Matrix] Let $f_1, ..., f_8$ be a set of functions defined on the interval $[1, 8]$ with the property that for any numbers $d_1, ..., d_8$, there exists a set of coefficients $c_1, ..., c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, \quad i = 1, ..., 8$$

(a) Show by appealing to the theorems of this lecture that $d_1, ..., d_8$ determine $c_1, ..., c_8$ uniquely.
(b) Let A be the 8×8 matrix representing the linear mapping from data $d_1, ..., d_8$ to coefficients $c_1, ..., c_8$. What is the i, j entry of A^{-1} ?

Answer

(a). For proof, we will use two theorems from APPLIED NUMERICAL ANALYSIS below:

Theorem 1. *Range(A) is the space spanned by the columns of A.*

Theorem 2. *A matrix $A \in \mathbb{C}^{m \times n}$ has full rank if and only if maps no two distinct vectors to the same vector.*

Proof. Denote $F = \begin{bmatrix} f_1(1) & ... & f_8(1) \\ ... & ... & ... \\ f_1(8) & ... & f_8(8) \end{bmatrix}$. We can re-write the mapping f on $c_1, ..., c_8$ to $d_1, ..., d_8$ in a matrix-vector-product way:

$$F \begin{bmatrix} c_1 \\ ... \\ c_8 \end{bmatrix} = \begin{bmatrix} d_1 \\ ... \\ d_8 \end{bmatrix}.$$

Note that the value of d_i can be any number, and we can view $[d_i]_{i \in [1,8]}$ to be spanned by the columns of F, so $Range(F) = 8$. By applying Theorem 1, we get that F is full-rank.

So, $d_1, ..., d_8$ determine $c_1, ..., c_8$ uniquely. Otherwise, we'll find two different vectors that map the columns of full-rank matrix F to a same vector, which conflicts Theorem 2. Until now We have finished sub-question (a). □

(b). As F is full-rank, () can be written as:

$$\begin{bmatrix} c_1 \\ ... \\ c_8 \end{bmatrix} = F^{-1} \begin{bmatrix} d_1 \\ ... \\ d_8 \end{bmatrix}.$$

So $A = F^{-1} \rightarrow A^{-1} = F \rightarrow A^{-1}_{i,j} = F_{ij} = f_j(i)$. Until now We have finished sub-question (b).

Question 2

[Rank-one perturbation of the identity] 2.6. If u and v are m-vectors, the matrix $A = I + uv*$ is known as a rank-one perturbation of the identity.

- Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + auv*$ for some scalar a, and give an expression for a.
- For what u and v is A singular? If it is singular, what is $null(A)$?

The first part of proof

Suppose that A is non-singular, let $B = I + auv^*$, we will prove that for any A, we can find a corresponding a, and thus a corresponding B, so that $B = A^{-1}$.

If $AB = I$, we get:

$$\begin{aligned} AB = I &\to (I + uv^*)(I + auv^*) = I \\ &\to I + auv^* + uv^* + au(v^*u)v* = I \\ &\to (a + 1 + av^*u)uv^* = 0 \\ &\to a(1 + v^*u) + 1 = 0 \text{ } or \text{ } uv^* = 0 \end{aligned}$$

As been given, $rank(u) = rank(v) = 1 > 0$, so the a we need satisfies:

$$a(1 + v^*u) = -1 \tag{1}$$

Next we show that $v^*u \neq -1$. Assume that $v^*u = -1$, then $Au = (I + uv^*)u = u + u * (-1) = \mathbf{0}$, conflicting that $u \neq \mathbf{0}$ and A is non-singular.

So from (1) we further get:

$$a = -\frac{1}{1 + v^*u} \tag{2}$$

The second part of proof

In this part, we show that $v^*u = -1$ is the necessary and sufficient condition when A is singular.

sufficiency

As been shown in the former part, when $v^*u = -1$, $Au = (I + uv^*)u = u + u * (-1) = \mathbf{0}$, then A is singular.

necessity

Once $v^*u \neq -1$, according to the former part, we can construct a matrix $B = I - \dfrac{1}{1 + v^*u}uv^*$ that $AB = I$, which indicates that A is non-singular. So we have proved the necessity.

The Null Space of A

$$\begin{aligned} &Ax = 0 \\ \to &(I + uv^*)x = 0 \\ &\to x = -(v^*x)u \\ &\to x//u \to Null(A) = span([u]) \end{aligned}$$

Question 3

[Singular Values of Rotated Matrix] Suppose A is an $m \times n$ matrix and B is the $n \times m$ matrix obtained by rotating A ninety degrees clockwise on paper (not exactly a standard mathematical transformation!). Do A and B have the same singular values? Prove that the answer is yes or give a counterexample.

Answer

Yes.

Proof

Notation 1. *We use A^R as a notation for the ninety-degree clock-wisely rotated A, and A^T for the transpose of A.*

For proof, we will use the theorem and corollary below:

Theorem 3. *The nonzero singular values of A are the square roots of the nonzero eigenvalues of $A * A$ or $AA*$.[NLA Theorem 5.4]*

We will also use the immediate corollary of 3 below:

Corollary 1. *For any matrix $M \in \mathbb{R}^{m\times n}$, AA^* and A^*A share the same singular values.*

Lemma 1 (Relation between A^R and A^T)**.** *For any matrix $A \in \mathbb{R}^{m\times n}$, $A^T_{i,j} = A^R_{i,m-j-1}$*

Proof. $A^T_{i,j} = Aj, i = A^R_{i,m-j-1}$. □

Lemma 2 (Singular value of A^R and A^T)**.** *A^R and A^T share the same singular values.*

Proof. Denote A^T to be C, A^R to be D. According to 1, the elements in each row i of C and D are the same, they are just permuted (more precisely, permuted in reversed orders), and the permutation are parallel among each row. So the inner product of any two rows in C and D are the same:

$$\langle C[i,:], C[j,:]\rangle = \langle D[i,:], D[j,:]\rangle$$

So:

$$CC^T = [\langle C[i,:], C[j,:]\rangle]_{ij} = \langle D[i,:], D[j,:]\rangle = DD^T \tag{3}$$

By Theorem 3 and (3) we derive that C and D share the same singular values . Thus, we have finished the proof of Lemma 2. □

Proof. Back to the question. By 2, B shares the same singular values with A^T; By 1, A^T shares the same singular values with A. So A and B have the same singular values. □

Question 4

[Practice with SVD]

Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$$

1. Determine, on paper, a real SVD of A in the form $A = U\Sigma V^T$. The SVD is not unique, so find the one that has the minimal number of minus signs in U and V;
2. List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked;
3. What are the 1-, 2-, ∞-, and Frobenius norms of A?
4. Find A^{-1} not directly, but via the SVD;
5. Find the eigenvalues λ_1, λ_2 of A;
6. Verify that $\det A = \lambda_1\lambda_2$ and $|\det A| = \sigma_1\sigma_2$;
7. What is the area of the ellipsoid onto which A maps the unit ball in $\mathbb{R}^2$?

Answer

1. SVD Calculation

$$
\begin{aligned}
A = U\Sigma V^T &= \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \\
\rightarrow AA^T = U\Sigma^2 U^T &= \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix}, \\
A^T A = V\Sigma^2 V^T &= \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix}
\end{aligned}
$$

Conduct eigen decomposition on AA^T and A^TA, we get:

$$
U = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 10\sqrt{2} \end{bmatrix}, \quad V = \begin{bmatrix} -4/5 & -3/5 \\ -3/5 & 4/5 \end{bmatrix}
$$

We adjust U and V so that we get the minimal number of minus signs, we turn the sign of U's first column and V's first column, and get:

$$
U = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 10\sqrt{2} \end{bmatrix}, \quad V = \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix} \tag{4}
$$

2. Visualization of SVD.

From (4) we directly get:

- Singular values:
$$
\sigma_1 = 5\sqrt{2}, \quad \sigma_2 = 10\sqrt{2}; \tag{5}
$$
- Left singular vectors: $[-\sqrt{2}/2, \sqrt{2}/2]^T, [\sqrt{2}/2, \sqrt{2}/2]^T$;
- Right singular vectors : $[4/5, -3/5]^T, [3/5, 4/5]^T$.

The unit ball's image under A is show as follow:

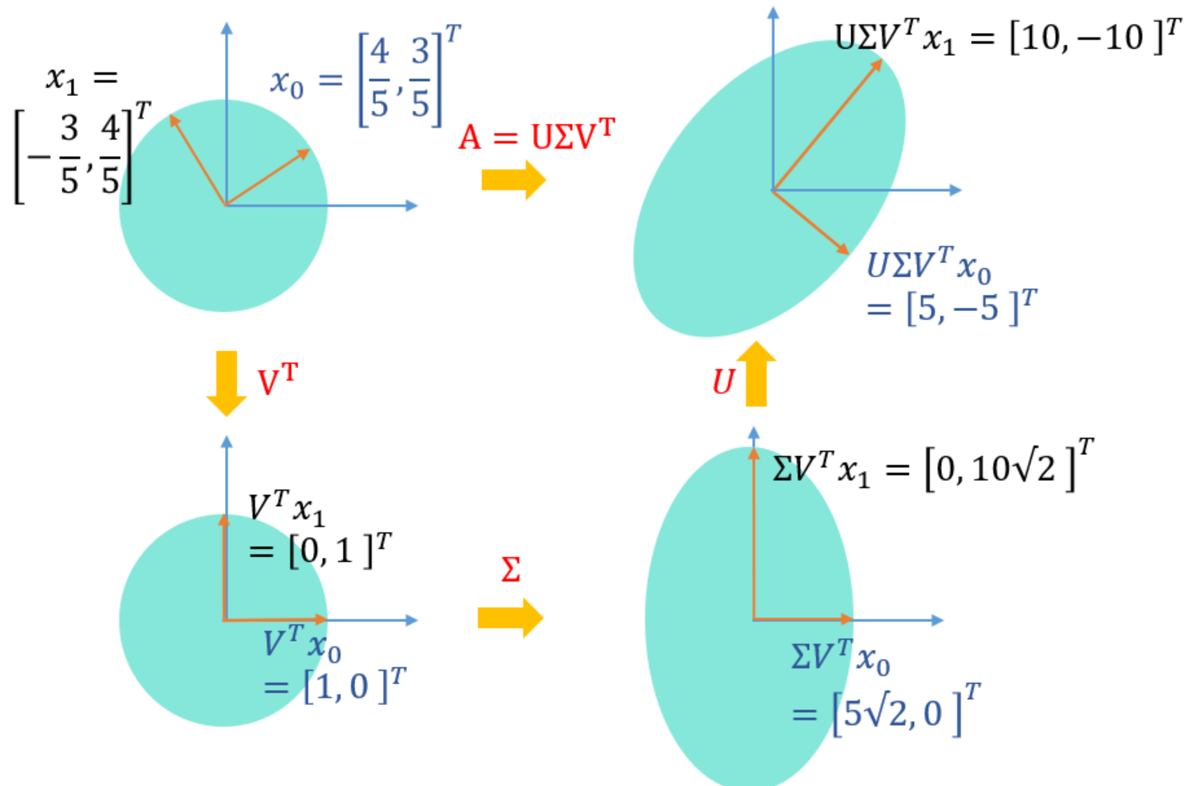

Figure 1: The unit ball's image under A

3. Matrix's Norms

$\|A\|_1 = 28,$
$\|A\|_2 = 5\sqrt{10},$
$\|A\|_\infty = 11,$
$\|A\|_F = 5\sqrt{10}$

4. A^{-1} By SVD [TODO: Check again!]

$$A^{-1} = (U\Sigma V^T)^{-1} = V\Sigma^{-1}U^T = \begin{bmatrix} 0.05 & 0.11 \\ -0.1 & -0.02 \end{bmatrix}$$

5. Eigen Values of A^{-1}

The charateristic polynomial of A is:

$$p_A(\lambda) = \det(A - \lambda I) = \lambda^2 - 3\lambda + 100,$$

so that we obtain:

$$\begin{cases} \lambda_1 = \dfrac{3 + \sqrt{-391}}{2} \\ \lambda_2 = \dfrac{3 - \sqrt{-391}}{2} \end{cases} \tag{6}$$

6. Relationship of Eigen Values and Matrixs' Determinant and Trace

From (6) and (5), we get:

$$\begin{cases} \lambda_1\lambda_2 = 100 = \det A \\ \sigma_1\sigma_2 = 100 = |\det A| \end{cases}$$

7. Ellipsoid Mapped A^{-1}

Figure 1 has been shown in sub-question 2, now we describe the mapped-to ellipsoid analytically:
The unit ball before mapped by A is :

$$\{\ (x, y) \mid [x, y] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq 1\}.$$

After being mapped by A, the set of points comes to:

$$\{\ (x, y) \mid ([x, y]A^T) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (A \begin{bmatrix} x \\ y \end{bmatrix}) \leq 1\},$$

which can be written as:

$$\{\ (x, y) \mid [x, y]A^T A \begin{bmatrix} x \\ y \end{bmatrix} \leq 1\}. \tag{7}$$

Equation (7) is just the analytical solution of obtain ellipsoid.

Question 5

[(Orthogonal) Projector onto Column Range] Consider the matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Answer the following questions by hand calculation.
(a) What is the orthogonal projector P onto $range(A)$, and what is the image under P of vector $[1, 2, 3]^*$?
(b) Same question for B.

Answer

(a). The projector is:

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

The image under P of $[1, 2, 3]^*$ is:

$$P[1, 2, 3]^* = [2, 2, 2]^*.$$

(b). The projector is:

$$P = B(B^T B)^{-1} B^T = \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix}.$$

The image under P of $[1, 2, 3]^*$ is:

$$P[1, 2, 3]^* = [2, 0, 2]^*.$$

Question 6

[QR Factorization] Consider again the matrices A and B of question 6.

(a) Using any method you like, determine (on paper) a reduced QR factorization $A = \hat{Q}\hat{R}$ and a full QR factorization $A = QR$.

(b) Again using any method you like, determine reduced and full QR factorizations $B = \hat{Q}\hat{R}$ and $B = QR$.

Answer

(a1)[Reduced QR factorization of A]. Denote the reduced QR factorization of A as $A = \hat{Q}\hat{R}$, $\hat{Q} \in \mathbb{R}^{3\times 2}$, $\hat{R} \in \mathbb{R}^{2\times 2}$,

$$\begin{aligned} &A = [\mathbf{a_1}, \mathbf{a_2}], \quad \hat{Q} = [\mathbf{q_1}, \mathbf{q_2}], \quad \hat{R} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}, \\ &\mathbf{a_1}, \mathbf{a_2} \in \mathbb{R}^{3\times 1}, \quad \mathbf{q_1}, \mathbf{q_2} \in \mathbb{R}^{3\times 1}, \\ &r_{11}, r_{12}, r_{22} \in \mathbb{R}. \end{aligned}$$

As been given, $\mathbf{a_1} \perp \mathbf{a_2}$, so we directly derive that:

$$\mathbf{q_1} = \mathbf{a_1}/\|\mathbf{a_1}\|, \quad r_{11} = \|\mathbf{a_1}\| \tag{8}$$

$$\mathbf{q_2} = \mathbf{a_2}/\|\mathbf{a_2}\|, \quad r_{22} = \|\mathbf{a_2}\| \tag{9}$$

, and

$$\begin{aligned} \mathbf{a_2} &= r_{12}\mathbf{q_1} + r_{22}\mathbf{q_2} \\ \Rightarrow \langle \mathbf{a_2}, \mathbf{q_1} \rangle &= r_{12}\langle \mathbf{q_1}, \mathbf{q_1} \rangle + r_{22}\langle \mathbf{q_2}, \mathbf{q_1} \rangle \\ \Rightarrow r_{12} &= 0. \end{aligned} \tag{10}$$

Compounding equations (8) (9) and (10), we get we the reduced QR factorization of A:

$$\hat{Q} = \begin{bmatrix} \sqrt{2}/2 & 0 \\ 0 & 1 \\ \sqrt{2}/2 & 0 \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$$

(a2)[Full QR factorization of A]. Denote the full QR factorization of A as $A = QR$, $Q \in \mathbb{R}^{3\times 3}$, $R \in \mathbb{R}^{3\times 2}$, note that in the equations below, all the values except for $\mathbf{q_3}$ keep the same with in reduced QR factorization:

$$A = [\mathbf{a_1}, \mathbf{a_2}], \quad Q = [\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}], \quad R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \\ 0 & 0 \end{bmatrix},$$

As $\mathbf{q_3} \perp \mathbf{q_1}$ and $\mathbf{q_3} \perp \mathbf{q_2}$, $\mathbf{q_3} \in Null(\hat{Q})$,

$$\begin{aligned} &\hat{Q}^T\mathbf{q_3} = 0, \quad \|\mathbf{q_3}\| = 1 \\ &\Rightarrow \mathbf{q_3} = \pm[\sqrt{2}/2, 0, -\sqrt{2}/2]^T \end{aligned}$$

Let $\mathbf{q_3} = [\sqrt{2}/2, 0, -\sqrt{2}/2]^T$, we get the full QR factorization of A :

$$Q = \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(b1)[Reduced QR factorization of B]. Denote the reduced QR factorization of B as $B = \hat{Q}\hat{R}$, $\hat{Q} \in \mathbb{R}^{3\times 2}$, $\hat{R} \in \mathbb{R}^{2\times 2}$,

$$B = [\mathbf{b_1}, \mathbf{b_2}], \quad \hat{Q} = [\mathbf{q_1}, \mathbf{q_2}], \quad \hat{R} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix},$$
$$\mathbf{b_1}, \mathbf{b_2} \in \mathbb{R}^{3\times 1}, \mathbf{q_1}, \mathbf{q_2} \in \mathbb{R}^{3\times 1}$$
$$r_{11}, r_{12}, r_{22} \in \mathbb{R}.$$

By the inherent property of QR factorization,

$$\begin{aligned} &\mathbf{q_1} \perp \mathbf{q_2}, \\ &span[\mathbf{q_1}] = span[\mathbf{b_1}], \\ &span[\mathbf{q_1}, \mathbf{q_2}] = span[\mathbf{b_1}, \mathbf{b_2}] \end{aligned}$$

we derive that:

$$\begin{aligned} \mathbf{q_1} &= \mathbf{b_1}/\|\mathbf{b_1}\|, \quad r_{11} = \|\mathbf{b_1}\| \\ \Rightarrow \mathbf{q_1} &= [\sqrt{2}/2, 0, \sqrt{2}/2]^T, \quad r_{11} = \sqrt{2} \end{aligned} \tag{11}$$

,

and

$$\mathbf{b_2} = r_{12}\mathbf{q_1} + r_{22}\mathbf{q_2} \tag{12}$$
$$\Rightarrow \langle \mathbf{b_2}, \mathbf{q_1} \rangle = r_{12}\langle \mathbf{q_1}, \mathbf{q_1} \rangle + r_{22}\langle \mathbf{q_2}, \mathbf{q_1} \rangle$$
$$\Rightarrow r_{12} = \langle \mathbf{b_2}, \mathbf{q_1} \rangle = \sqrt{2}. \tag{13}$$

Feed (13) into (12), and apply $\|\mathbf{q_2}\| = 1$we get:

$$\begin{aligned} r_{22}\mathbf{q_2} &= \mathbf{b_2} - r_{12}\mathbf{q_1} = [1, 1, -1]^T \\ &\Rightarrow r_{22} = \sqrt{3}, \mathbf{q_2} = [\sqrt{3}/3, \sqrt{3}/3, -\sqrt{3}/3] \end{aligned} \tag{14}$$

Compounding equations (11) to (14), we get we the reduced QR factorization of B:

$$B = \hat{Q}\hat{R}, \quad \hat{Q} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{3}/3 \\ 0 & \sqrt{3}/3 \\ \sqrt{2}/2 & -\sqrt{3}/3 \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

(b2)[Full QR factorization of B]. Denote the full QR factorization of B as $B = QR$, $Q \in \mathbb{R}^{3\times 3}$, $R \in \mathbb{R}^{3\times 2}$, note that in the equations below, all the values except for $\mathbf{q_3}$ keep the same with in reduced QR factorization:

$$B = [\mathbf{b_1}, \mathbf{b_2}], \quad Q = [\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}], \quad R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \\ 0 & 0 \end{bmatrix},$$

As $\mathbf{q_3} \perp \mathbf{q_1}$ and $\mathbf{q_3} \perp \mathbf{q_2}$, $\mathbf{q_3} \in Null(\hat{Q})$,

$$\begin{aligned} &\hat{Q}^T\mathbf{q_3} = 0, \quad \|\mathbf{q_3}\| = 1 \\ &\Rightarrow \mathbf{q_3} = \pm[-\sqrt{6}/6, \sqrt{6}/3, \sqrt{6}/6]^T \end{aligned}$$

Let $\mathbf{q_3} = [-\sqrt{6}/6, \sqrt{6}/3, \sqrt{6}/6]^T$, we get the full QR factorization of B :

$$B = QR, \quad Q = \begin{bmatrix} \sqrt{2}/2 & \sqrt{3}/3 & -\sqrt{6}/6 \\ 0 & \sqrt{3}/3 & \sqrt{6}/3 \\ \sqrt{2}/2 & -\sqrt{3}/3 & \sqrt{6}/6 \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}.$$

Question 7

[QR Factorization with MATLAB Householder]
(a) Write a Matlab function $[W, R] = house(A)$ that computes an implicit representation of a full QR factorization $A = QR$ of an $m \times n$ matrix A with $m > n$ using Householder reflections. The output variables are a lower-triangular matrix $W \in \mathbb{C}^{m\times n}$ whose columns are the vectors v_k defining the successive Householder reflections, and a triangular matrix $R \in \mathbb{C}^{n\times n}$.
(b) Write a Matlab function $Q = formQ(W)$ that takes the matrix W produced by *house* as input and generates a corresponding $m \times m$ orthogonal matrix Q.

Answers / Matlab Codes

[(a). Code for $house(A)$]

```
function [W, R] = house(A)
    [m, n] = size(A)
    W = zeros(m, n)
    e1 = eye(m, 1)
    for k=1:n
        x = A(k:m, k)
        % We use e1_sub together with x to decide reflector,
        % as the dimension of subspace (submatrix) is reduced in each loop,
        % e1_sub comes shorter and shorter.
        e1_sub = e1(1:m-k+1)
        % Choose the better Reflector:
        vk = sign(x(1))*norm(x,2) * e1_sub + x
        vk = vk / norm(vk,2)
        % HouseHolder mapping. To improve efficiency,
        % it's important to calculate (vk'*A(k:m,k:n) first.
        A(k:m,k:n) = A(k:m,k:n) - 2*vk*(vk'*A(k:m,k:n))
        W(k:m,k) = vk
    end
    R=A(1:n,:)
end
```

[(b). Test Code for $house(A)$]

```
m = 8
n = 4
x = (-m/2:m/2-1)'/(m/2)
A = [x.^0  x.^1  x.^2 x.^3]
W, R = house(A)
```