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A sharp boundary for SURE-based admissibility for the normal means problem under unknown scale

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ABSTRACT

We consider quasi-admissibility/inadmissibility of Stein-type shrinkage estimators of the mean of a multivariate normal distribution with covariance matrix an unknown multiple of the identity. Quasi-admissibility/inadmissibility is defined in terms of non-existence/existence of a solution to a differential inequality based on Stein's unbiased risk estimate (SURE). We find a sharp boundary between quasi-admissible and quasi-inadmissible estimators related to the optimal James–Stein estimator. We also find a class of priors related to the Strawderman class in the known variance case where the boundary between quasi-admissibility and quasi-inadmissibility corresponds to the boundary between admissibility and inadmissibility in the known variance case. Additionally, we also briefly consider generalization to the case of general spherically symmetric distributions with a residual vector.

1. Introduction

Let

$$X \sim \mathcal{N}_p(\theta, \sigma^2 I_p), \quad S \sim \sigma^2 \chi^2_n, \tag{1}$$

where X and S are independent and θ and σ^2 are both unknown, and where $p \geq 3$ and $n \geq 3$. Consider the problem of estimating the mean vector θ under the loss function $L(\theta, \sigma^2; d) = \|d - \theta\|^2/\sigma^2$. We study the question of admissibility/inadmissibility of shrinkage-type estimators of the form

$$\delta_\phi(X, S) = \{1 - \phi(W)/W\}X, \tag{2}$$

where $W = \|X\|^2/S$ and ϕ is nonnegative. Some additional assumptions on ϕ will be given later in this section. We do so by examining the existence of solutions, g, to a differential inequality which arises from an unbiased estimate of the difference in risk between δ_ϕ and

$$\delta_{\phi+g} = [1 - \{\phi(W) + g(W)\}/W]X,$$

where $\phi(w) + g(w)$ is not necessarily nonnegative. Hence we are more properly studying what may be termed quasi-admissibility and quasi-inadmissibility of such estimators. Quasi-inadmissibility implies inadmissibility under conditions of risk finiteness, while quasi-admissibility is relatively weaker.

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Stein, in his unpublished lecture notes, Brown [3], Bock [1], Rukhin [13] and Brown and Zhao [4] among others, have studied the admissibility question from this point of view (without necessarily using the term quasi-admissibility) under known σ^2. Of course, Brown [2] has largely settled the admissibility/inadmissibility question when σ^2 is known.

Our efforts focus generally on finding a boundary between quasi-admissibility and quasi-inadmissibility for shrinkage estimators of the form (2); see Theorem 1. We also apply the result to a class of generalized Bayes estimators related to the class of Strawderman [16] priors for the known variance problem and establish a boundary on the tail behavior which also separates quasi-admissibility from quasi-inadmissibility.

While minimaxity of shrinkage estimators in the unknown scale case has been extensively studied by many authors, relatively little is known about admissibility in this case. Strawderman [17] and Zinodiny et al. [19] gave a class of proper Bayes minimax and hence admissible estimators under unknown σ^2. Note that proper Bayes estimators by Strawderman [17] and Zinodiny et al. [19] are not of the form given by (2) whereas generalized Bayes estimators by Maruyama [10], Maruyama and Strawderman [11] and Maruyama and Strawderman [12] are of this form.

While our results on quasi-admissibility do not settle the admissibility issue, it seems likely to us that generalized Bayes estimators satisfying our conditions for quasi-admissibility are admissible, perhaps under mild additional conditions. We are decidedly not claiming that such a result would be easily established! Certainly those found to be quasi-inadmissible are in fact inadmissible under conditions of finiteness of risk.

An unbiased estimator of the risk, $R(\theta, \sigma^2; \delta_\phi)$, for an estimator of the form (2) is given by

$$p + (n+2)D_\phi(W) \tag{3}$$

where

$$D_\phi(w) = \frac{\{\phi(w) - 2c_{p,n}\}\phi(w)}{w} - d_n\phi'(w)\{1+\phi(w)\}, \tag{4}$$

with $c_{p,n} = (p-2)/(n+2)$ and $d_n = 4/(n+2)$.

This result follows from Stein's identity [15] and well known identities for chi-square distributions; see, e.g., Efron and Morris [5]. We may refer to (3) as a SURE estimate of risk and to (6) as a SURE estimate of difference in risk. A sufficient condition for its validity is that ϕ be absolutely continuous and that each term of $\mathrm{E}\{D_\phi(W)\}$ be finite. Let Φ be a family of functions ϕ satisfying these sufficient conditions, viz.

$$\Phi = \{\phi : \mathrm{E}\{D_\phi(W)\} < \infty,\ \phi \text{ is continuously differentiable}\},$$

which we mainly consider in this paper. Note that continuous differentiability implies absolute continuity and hence the family

$$\Phi^* = \{\phi : \mathrm{E}\{D_\phi(W)\} < \infty,\ \phi \text{ is absolutely continuous}\} \tag{5}$$

includes the family Φ. See Remark 2 for an extension with Φ^*.

If $\delta_{\phi+g}$ is of the form (2) with $\phi(w)$ replaced by $\phi(w) + g(w)$, an unbiased estimator of the difference in risk between δ_ϕ and $\delta_{\phi+g}$, namely $R(\theta, \sigma^2; \delta_\phi) - R(\theta, \sigma^2; \delta_{\phi+g})$, is given by

$$(n+2)\Delta(w;\phi,g) = (n+2)\{D_\phi(w) - D_{\phi+g}(w)\} = (n+2)g(w)\{\Delta_1(w;\phi) + \Delta_2(w;\phi,g)\} \tag{6}$$

where

$$\Delta_1(w;\phi) = 2\frac{c_{p,n} - \phi(w)}{w} + d_n\phi'(w) \tag{7}$$

and

$$\Delta_2(w;\phi,g) = -\frac{g(w)}{w} + d_n g'(w) + d_n\frac{g'(w)}{g(w)}\{1+\phi(w)\}. \tag{8}$$

One may find an estimator dominating δ_ϕ by finding a non-zero solution $g \in \Phi$ to the differential inequality $\Delta(w;\phi,g) \geq 0$, where $\Delta(w;\phi,g)$ is given by (6), providing the resulting estimator has finite risk. Here is the definition of quasi-admissibility and quasi-inadmissibility used in this paper:

Definition 1.

(i) An estimator δ_ϕ of the form (2) is said to be quasi-admissible if any solution $g \in \Phi$ of the inequality $\Delta(w;\phi,g) \geq 0$ everywhere satisfies $g(w) \equiv 0$.

(ii) δ_ϕ is said to be quasi-inadmissible if there exists a solution, $g \in \Phi$, which is non-vanishing on some open interval, to the differential inequality $\Delta(w;\phi,g) \geq 0$ everywhere.

For technical reasons we will restrict the class of functions ϕ to the subclass Φ_A of Φ, defined as follows,

$$\Phi_A = \{\phi \in \Phi : \phi \text{ satisfies A1, A2, and A3 below}\},$$

A1 $\phi(0) = 0$ and $\phi(w) \geq 0$ for any $w \geq 0$;
A2 $\phi(w)$ has at most finitely many local extrema;
A3 $\liminf_{w\to\infty} w\phi'(w)/\phi(w) \geq 0$ and $\limsup_{w\to\infty} w\phi'(w)/\phi(w) \leq 1$.

Assumption A2 assumes that ϕ does not oscillate excessively and that $\lim_{w\to\infty}\phi(w)$ exists. As far as we know, Assumptions A1–A3 cover all minimax and smooth estimators in the literature including [10–12]. Assumptions A1–A3 are also satisfied by linear estimators of the form $\delta(X) = \alpha X$ for $\alpha \in [0, 1]$ and for which $\phi(w) = (1-\alpha)w$. These estimators are unique proper Bayes and admissible in the normal case for $\alpha \in [0, 1)$. We emphasize that while we address quasi-admissibility and inadmissibility only for δ_ϕ for $\phi \in \Phi_A \subset \Phi$, we allow competitive estimators of the form $\delta_{\phi+g}$ for $g \in \Phi$.

In Section 2 we will show the following result, which establishes

$$\phi(w) = \frac{p-2}{n+2} - \frac{\beta_\star}{\ln w}$$

as the asymptotic boundary between quasi-admissibility and quasi-inadmissibility where

$$\beta_\star = \frac{d_n(1+c_{p,n})}{2} = \frac{2(p+n)}{(n+2)^2}.$$

Quasi-admissibility: If $\phi \in \Phi_A$ and there exists w_* and $b < 1$ such that

$$\forall_{w\geq w_*} \quad \phi(w) \geq \frac{p-2}{n+2} - b\frac{\beta_\star}{\ln w}, \tag{9}$$

then δ_ϕ is quasi-admissible.

Quasi-inadmissibility: If $\phi \in \Phi_A$ and there exists w_* and $b > 1$ such that

$$\forall_{w\geq w_*} \quad \phi(w) \leq \frac{p-2}{n+2} - b\frac{\beta_\star}{\ln w},$$

then δ_ϕ is quasi-inadmissible (and hence inadmissible).

In Section 3, we find a generalized Bayes estimator with asymptotic behavior

$$\lim_{w\to\infty} \ln w \left\{\frac{p-2}{n+2} - \phi(w)\right\} = b\beta_\star,$$

for all $b > 0$. The corresponding generalized prior is given by

$$\frac{1}{\sigma^2} \times \frac{1}{\sigma^p} G(\|\theta\|/\sigma)$$

with

$$G(\|\mu\|) = \int_0^1 \left(\frac{\lambda}{1-\lambda}\right)^{p/2} \exp\left(-\frac{\lambda}{1-\lambda}\frac{\|\mu\|^2}{2}\right)\lambda^{-2}\left(\ln\frac{1}{\lambda}\right)^b \mathrm{d}\lambda.$$

Hence, $b < 1$ and $b > 1$ imply quasi-admissibility and quasi-inadmissibility, respectively, of the associated generalized Bayes estimators. Interestingly, the boundary $b = 1$ also appears in the known σ^2 case when estimating μ with $Z \sim \mathcal{N}_p(\mu, I_p)$. By using Brown's sufficient condition [2], the generalized Bayes estimator with respect to $G(\|\mu\|)$ above is admissible (resp. inadmissible) when $b \leq 1$ (resp. $b > 1$). This nice correspondence leads naturally to the conjecture: a quasi-admissible generalized Bayes estimator satisfying (9) is admissible.

An extension to the general class of spherically symmetric distributions is briefly considered in Section 2.1. We give some concluding remarks in Section 4. Some technical proofs are given in Appendix.

2. Quasi-admissibility

The main result of this paper, Theorem 1, gives sufficient conditions for quasi-admissibility and quasi-inadmissibility for estimators δ_ϕ of the form (2), for $\phi \in \Phi_A$. In preparation, we first give several lemmas. Recall that the unbiased estimator of the difference in risk between δ_ϕ and $\delta_{\phi+g}$ is given by

$$(n+2)\Delta(w; \phi, g) = (n+2)\{D_\phi(w) - D_{\phi+g}(w)\} = (n+2)g(w)\{\Delta_1(w; \phi) + \Delta_2(w; \phi, g)\}$$

where

$$\Delta_1(w; \phi) = 2\frac{c_{p,n} - \phi(w)}{w} + d_n\phi'(w)$$

and

$$\Delta_2(w; \phi, g) = -\frac{g(w)}{w} + d_n g'(w) + d_n\frac{g'(w)}{g(w)}\{1 + \phi(w)\}, \tag{10}$$

and where $c_{p,n} = (p-2)/(n+2)$ and $d_n = 4/(n+2)$. Note that $\Delta_2(w; \phi, g)$ is well-defined for w such that $g(w) \neq 0$, but $\Delta(w; \phi, g)$ is well-defined even when $g(w) = 0$.

The first lemma gives necessary conditions on $g(w)$ for $\Delta(w; \phi, g)$ to be nonnegative for all $w \geq 0$.

Lemma 1. *Suppose $\Delta(w; \phi, g) \geq 0$ for all $w \geq 0$ with $\phi \in \Phi_A$ and $g \in \Phi$. Then*

B1 $g(w) \geq 0$ *for all* $w \geq 0$;
B2 *if* $g(w_0) > 0$, *then, for any* $w \geq w_0$, $g(w) > 0$.

Proof. See Appendix A.1. □

Recall that finiteness of $\mathrm{E}\{\phi(W)^2/W\}$ is a necessary condition for ϕ to be in Φ. Lemma 2 provides a necessary condition for $\mathrm{E}\{\phi(W)^2/W\}$ to be finite and hence for ϕ to be in Φ. It is needed in the proof of Lemma 3.

Lemma 2. *A necessary condition for* $\mathrm{E}\{\phi(W)^2/W\}$ *to be finite for any* (θ, σ^2) *is that* $\liminf_{t\to\infty} |\phi(t)|^{d_n}/t = 0$.

Proof. See Appendix A.2. □

Let $\mathcal{G} \subset \Phi$ be a class of nonnegative functions which satisfy B1 and B2 of Lemmas 1 and 2. The following lemma is key in proving the main result. Recall that Assumption A2 assumes that ϕ does not oscillate excessively and that $\lim_{w\to\infty}\phi(w)$ exists. In the following lemma, let $\phi_* = \lim_{w\to\infty}\phi(w) \in [0, \infty]$ and

$$\beta_\star = \frac{d_n(1 + c_{p,n})}{2} = \frac{2(p+n)}{(n+2)^2}. \tag{11}$$

Lemma 3. *Suppose* $\phi \in \Phi_A$.

1. *Suppose* $\phi_* < \infty$ *and there exists* w_0 *and* $b < 1$ *such that*

$$\forall_{w \geq w_0} \quad \phi(w) \geq \frac{p-2}{n+2} - b\frac{\beta_\star}{\ln w}. \tag{12}$$

(1.a) *For all* $w \geq w_0$,

$$\Delta_1(w; \phi) - d_n\phi'(w) - \frac{2b\beta_\star}{w \ln w} \leq 0.$$

(1.b) *Also, for any* $g \in \mathcal{G}$ *except* $g \equiv 0$, *there exists* $w_* \in (w_0, \infty)$ *such that*

$$\Delta_2(w_*; \phi, g) + d_n\phi'(w_*) + \frac{2b\beta_\star}{w_* \ln w_*} < 0.$$

2. *Suppose* $\phi_* = \infty$.

(2.a) *Then there exists* w_0 *such that* $\Delta_1(w; \phi) + d_n\phi'(w) \leq 0$ *for all* $w \geq w_0$.
(2.b) *Also, for any* $g \in \mathcal{G}$ *except* $g \equiv 0$, *there exists* $w_* \in (w_0, \infty)$ *such that* $\Delta_2(w_*; \phi, g) - d_n\phi'(w_*) < 0$.

3. *Suppose there exists* w_0 *and* $b > 1$ *such that*

$$\forall_{\forall w \geq w_0} \quad \phi(w) \leq \frac{p-2}{n+2} - b\frac{\beta_\star}{\ln w}. \tag{13}$$

(3.a) *There exists* w_1 *such that*

$$\forall_{w \geq w_1} \quad \Delta_1(w; \phi) - \frac{2b\beta_\star}{w \ln w} \geq 0.$$

(3.b) *Fix*

$$\nu = \min\left\{1, \frac{2b\beta_\star - d_n(1+\phi_*)}{2d_n(3+\phi_*)}\right\}.$$

Then, for any non-decreasing continuous function $k(w)$ *with* $k(0) = 0$ *and* $k(\infty) = 1$, *there exists* w_* *such that*

$$\Delta_2[w; \phi, k(w)\{\ln(w+e)\}^{-1-\nu}] + \frac{2b\beta_\star}{w \ln w} \geq 0$$

for all $w > \max[\max(w_*, w_1), \sup\{w : k(w) = 0\}]$ *and* $e = \exp(1)$.

Proof. See Appendix A.3. □

Note, in Part 3, $\Delta_2[w; \phi, k(w)\{\ln(w+e)\}^{-1-\nu}]$ is well-defined for $w > \sup\{w : k(w) = 0\}$ by the definition of Δ_2 given by (10).

The following result is the main result of this section.

Theorem 1. *Suppose $\phi \in \Phi_A$. Let $\beta_\star$ be as given in (11).*

1. *[Quasi-admissibility] If there exists w_* and $b < 1$ such that*

$$\forall_{w \geq w_*} \quad \phi(w) \geq \frac{p-2}{n+2} - b\frac{\beta_\star}{\ln w},$$

then δ_ϕ is quasi-admissible.

2. *[Quasi-inadmissibility] If there exists w_* and $b > 1$ such that*

$$\forall_{w \geq w_*} \quad \phi(w) \leq \frac{p-2}{n+2} - b\frac{\beta_\star}{\ln w}, \tag{14}$$

then δ_ϕ is quasi-inadmissible (and hence inadmissible).

Proof. [Part 1] By Parts 1 ($\phi_* < \infty$) and 2 ($\phi_* = \infty$) of Lemma 3, there exists w_* such that $\Delta_1(w_\star; \phi) + \Delta_2(w_\star; \phi, g) < 0$ for any $g \in \mathcal{G}$ except $g \equiv 0$. Therefore any solution $g \in \mathcal{G}$ of the differential inequality

$$g(w)\{\Delta_1(w; \phi) + \Delta_2(w; \phi, g)\} \geq 0$$

must be identically equal to 0, or equivalently δ_ϕ is quasi-admissible.

[Part 2] By (14), we have $\phi_* \leq (p-2)/(n+2) = c_{p,n}$ and hence $d_n(1+\phi_*) \leq 2\beta_\star < 2b\beta_\star$ since $b > 1$. As in Part 3 of Lemma 3, let

$$\nu = \min\left\{1, \frac{2b\beta_\star - d_n(1+\phi_*)}{2d_n(3+\phi_*)}\right\}.$$

Let $w_\sharp = \max(w_1, w_*)$ where w_1 and w_* are both determined by Part 3 of Lemma 3. Let k be the cdf of $X + w_\sharp$, where X is a Gamma random variable with pdf $x\exp(-x)\mathbf{1}_{(0,\infty)}(x)$. Then k is non-decreasing, $k(w) = 0$ for $w \in [0, w_\sharp]$, $k(w) > 0$ for $w \in (w_\sharp, \infty)$, $k'(w)|_{w=w_\sharp} = 0$ and $k(\infty) = 1$. Let $g(w) = k(w)\{\ln(w+e)\}^{-1-\nu}$. Then g is continuously differentiable and a member of $\mathcal{G} \subset \Phi$. Then we have

$$\Delta(w) = g(w)\{\Delta_1(w; \phi) + \Delta_2(w; \phi, g)\} \begin{cases} = 0 & \text{if } 0 \leq w \leq w_\sharp, \\ \geq 0 & \text{if } w > w_\sharp, \end{cases}$$

where $\Delta(w) = 0$ for $0 \leq w \leq w_\sharp$ since $g(w) = 0$ and $\Delta(w) \geq 0$ for $w > w_\sharp$ since

$$\left\{\Delta_1(w; \phi) - \frac{2b\beta_\star}{w \ln w}\right\} + \left\{\Delta_2(w; \phi, g) + \frac{2b\beta_\star}{w \ln w}\right\} \geq 0$$

by Part 3 of Lemma 3. Hence δ_ϕ is quasi-inadmissible. □

Remark 1. With our current methodology, we cannot determine whether the estimator satisfying

$$\lim_{w \to \infty} \frac{\ln w}{\beta_\star}\left\{\frac{p-2}{n+2} - \phi(w)\right\} = 1$$

is quasi-admissible or not. However, by analogy with the known scale case which is explained in Section 3, we can make a conjecture that it is quasi-admissible.

Remark 2. An alternative approach would be to define quasi-admissibility and quasi-inadmissibility under Φ^* given by (5),

$$\Phi^* = \{\phi : \mathrm{E}\{D_\phi(W)\} < \infty, \ \phi \text{ is absolutely continuous}\},$$

and add an assumption for Φ_A, viz.

A4 $\quad\phi'$ has only finitely many discontinuities on $(0, \infty)$,

which leads a new Φ_A defined by

$$\Phi_A^* = \left\{\phi \in \Phi^* : \phi \text{ satisfies A1, A2, A3, and A4}\right\}.$$

In this case, $\Delta(w; \phi, g)$, $\Delta_1(w; \phi)$ and $\Delta_2(w; \phi, g)$ given by (6)–(8), would be defined only almost everywhere and Definition 1 of quasi-admissibility and quasi-inadmissibility would be more properly defined in terms of $\Delta(w; \phi, g) \geq 0$ almost everywhere. Our theorems for quasi-admissibility and inadmissibility remain valid under both these scenarios and the proofs are essentially identical with the addition of a number of "almost everywhere" qualifiers in various proofs. The reason that this works is that whenever inequalities in terms ϕ' or g' are developed in the lemmas and theorems, they are always expressed in terms of integrals functions of these quantities which are valid everywhere, and not just almost everywhere. We choose to present the "everywhere" version of the development because it is slightly easier to develop and present.

An interesting estimator, to which the "almost everywhere" version applies and gives quasi-admissibility but where the "everywhere" version does not apply, is the James–Stein positive-part estimator

$$\max\left(0,\, 1-\frac{p-2}{n+2}\frac{1}{W}\right)X.$$

Since the corresponding ϕ, given by $\phi_+(w) = \min\{w, (p-2)/(n+2)\}$, is not a member of Φ, while it is a member of Φ_A^*. Also since $\phi_+(w) = \min\{w, (p-2)/(n+2)\}$ satisfies Part 1 of Theorem 1, the James–Stein positive-part estimator is quasi-admissible in the sense of "almost everywhere" version.

Remark 3. Note that it is possible that an estimator which is quasi-admissible according to the above definition may fail to be admissible for several reasons. Here are some of them. First, there may be an estimator that is not of the form (2) that dominates δ_ϕ. Second, there may be an estimator of the form (2) with $g \notin \Phi$ that dominates δ_ϕ. Third there may be an estimator that dominates δ_ϕ but does not satisfy the differential inequality $\Delta(w;\phi,g) \geq 0$. Hence quasi-admissibility is quite weak as an optimality criterion.

Quasi-inadmissibility, on the other hand, is more compelling in the sense that if δ_ϕ is quasi-inadmissible then it is inadmissible and dominated by $\delta_{\phi+g}$. Note that requiring both ϕ and g to be in Φ implies that the risk of $\delta_{\phi+g}$ is finite.

2.1. General spherically symmetric distributions

We may also study the more general canonical spherically symmetric setting where (X, U) has a spherically symmetric density of the form

$$\sigma^{-p-n} f(\{\|x-\theta\|^2 + \|u\|^2\}/\sigma^2). \tag{15}$$

Here the p-dimensional vector X has mean vector θ, the n-dimensional "residual" vector U has mean vector 0 and (X, S) is sufficient, where $S = \|U\|^2$. The scale parameter, σ^2, is assumed unknown. Consider the problem of estimating the mean vector θ under the loss function $L(\theta, \sigma^2; d) = \|d-\theta\|^2/\sigma^2$. The most important such setting is the Gaussian case $X \sim \mathcal{N}_p(\theta, \sigma^2 I_p)$, $S \sim \sigma^2\chi^2_n$, which is studied in Section 2, but there is considerable interest in the case of heavier tailed distributions such as the multivariate Student t distribution.

In the general spherically symmetric case, (3) is not an unbiased estimate of risk but has been used as a substitute for such an estimator. In particular, if (X, S) has density (15) and F is defined, for all $t > 0$, by

$$F(t) = \frac{1}{2}\int_t^\infty f(v)\mathrm{d}v.$$

Then as essentially shown by several authors in various settings (see, e.g., [7,9])

$$R(\theta, \sigma^2; \delta_\phi) = p + (n+2)\int_{\mathbb{R}^{p+n}} D_\phi(w)\frac{F[\{\|x-\theta\|^2 + \|u\|^2\}/\sigma^2]}{\sigma^{p+n}}\,\mathrm{d}x\mathrm{d}u,$$

where $D_\phi(w)$ is given in (4). Hence, in this setting,

$$R(\theta, \sigma^2; \delta_\phi) - R(\theta, \sigma^2; \delta_{\phi+g}) = (n+2)\int_{\mathbb{R}^{p+n}} g(w)\{\Delta_1(w;\phi) + \Delta_2(w;\phi,g)\}\frac{F[\{\|x-\theta\|^2 + \|u\|^2\}/\sigma^2]}{\sigma^{p+n}}\,\mathrm{d}x\mathrm{d}u$$

where $w = \|x\|^2/\|u\|^2$. Thus, study of existence of solutions to $\Delta(w) \geq 0$ is relevant in the general spherically symmetric case as well as in the Gaussian case, and defining quasi-admissibility/inadmissibility as in Definition 1 implies that Theorem 1 remains valid in this more general setting.

3. Generalized Bayes estimators in the normal case

3.1. Known variance case

Let $Z \sim \mathcal{N}_p(\mu, I_p)$. Consider estimation of μ under quadratic loss $\|\hat{\mu} - \mu\|^2$. The MLE, Z itself, is inadmissible for $p \geq 3$ as shown by Stein [14]. Brown [2] showed that admissible estimators should be proper Bayes or generalized Bayes estimators with respect to an improper prior and gave a sufficient condition for generalized Bayes estimators to be admissible or inadmissible.

Let the prior be of the form $\pi(\mu) = G(\|\mu\|; a, L)$, where

$$G(\|\mu\|; a, L) = \int_0^1 \left(\frac{\lambda}{1-\lambda}\right)^{p/2} \exp\left(-\frac{\lambda}{1-\lambda}\frac{\|\mu\|^2}{2}\right)\lambda^a L(1/\lambda)\mathrm{d}\lambda \tag{16}$$

and $p/2 + a + 1 > 0$. We assume the following conditions on $L : [1,\infty) \to [0,\infty)$:

L1 L is slowly varying at infinity, i.e., for all $c > 0$, $\lim_{y\to\infty} L(cy)/L(y) = 1$.
L2 L is ultimately monotone, i.e., L is monotone on (y_0, ∞) for some $y_0 > 0$.
L3 L is differentiable with ultimately monotone derivative L'.

By Proposition 1.7 (11) of Geluk and de Haan [8], Assumptions L1 and L3 implies $yL'(y)/L(y) \to 0$ as $y \to \infty$. Under the prior given by (16), the marginal density is

$$\begin{aligned} m(\|z\|; a, L) &= \int_{\mathbb{R}^p} \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{\|z-\mu\|^2}{2}\right) G(\|\mu\|; a, L)\mathrm{d}\mu = \int_0^1 \exp\left(-\frac{\lambda\|z\|^2}{2}\right) \lambda^{p/2+a} L(1/\lambda)\mathrm{d}\lambda \\ &= \int_0^\infty \exp\left(-\frac{\lambda\|z\|^2}{2}\right) f(\lambda; a, L)\mathrm{d}\lambda, \end{aligned}$$

where $f(\lambda; a, L) = \lambda^{p/2+a} L(1/\lambda) I_{(0,1)}(\lambda)$. Note that $f(\lambda; a, L)$ is ultimately monotone as a function of $1/\lambda$ since

(i) when $p/2 + a = 0$, $L(1/\lambda)$ itself is ultimately monotone;
(ii) when $p/2+a \neq 0$, $\lim_{\lambda\to 0} \lambda f'(\lambda)/f(\lambda) = p/2+a \neq 0$. Since f is positive, this implies f' for $p/2+a > 0$ and $p/2+a < 0$ is ultimately positive and negative, respectively. Hence f is ultimately monotone.

Since $f(\lambda; a, L)$ is ultimately monotone and since $m(\|z\|; a, L)$ is the Laplace transform of f, a Tauberian Theorem (see, e.g., Theorem 13.5.4 in Feller [6]) implies that

$$\lim_{\|z\|\to\infty} \frac{m(\|z\|; a, L)}{\Gamma(p/2+a+1)\left(2/\|z\|^2\right)^{p/2+a+1} L(\|z\|^2)} = 1. \tag{17}$$

As shown in Appendix A.4, $\|z\| \times \|\nabla \ln m(\|z\|; a, L)\|$ is bounded. By Theorem 6.4.2 of Brown [2], divergence (convergence) of the integral

$$\int_1^\infty \frac{\mathrm{d}r}{r^{p-1} m(r; a, L)} \tag{18}$$

corresponds to admissibility (inadmissibility) of a generalized Bayes estimator with bounded $\|z\| \times \|\nabla \ln m(\|z\|; a, L)\|$. Hence, by (17) and (18), divergence (convergence) of the integral

$$\int_1^\infty \frac{r^{2a+3}}{L(r^2)} \mathrm{d}r$$

corresponds to admissibility (inadmissibility). It is clear that $a > -2$ and $a < -2$ imply admissibility and inadmissibility, respectively. When $a = -2$, the fact that

$$\int_1^\infty \frac{\mathrm{d}r}{r(\ln r)^b} \begin{cases} = \infty & \text{if } b \le 1, \\ < \infty & \text{if } b > 1, \end{cases}$$

is helpful to determine the boundary. Since

$$\int_1^\infty \frac{r^{2a+3}}{L(r^2)} \mathrm{d}r = \frac{1}{2^b} \int_1^\infty \frac{1}{r(\ln r)^b} \frac{(\ln r^2)^b}{L(r^2)} \mathrm{d}r,$$

we have a following result on admissibility and inadmissibility of the (generalized) Bayes estimator with respect to $G(\|\mu\|; a, L)$.

Theorem 2 (*Admissibility*)**.**

1. *Suppose $a > -2$. The generalized Bayes estimator is inadmissible.*
2. *Suppose $a = -2$ and $\ln(y)/L(y)$ is ultimately monotone non-decreasing. The generalized Bayes estimator is admissible.*

Theorem 3 (*Inadmissibility*)**.**

1. *Suppose $a = -2$ and $\{\ln(y)\}^b/L(y)$ for $b > 1$ is ultimately monotone non-increasing. The generalized Bayes estimator is inadmissible.*
2. *Suppose $a < -2$. The generalized Bayes estimator is inadmissible.*

Remark 4 (*A Boundary Estimator for the Known Variance Case*)**.** Consider the particular choice $a = -2$ and $L(1/\lambda) = (\ln 1/\lambda)^b$ for $b > 0$. Then the prior is given by

$$\int_0^1 \left(\frac{\lambda}{1-\lambda}\right)^{p/2} \exp\left(-\frac{\lambda}{1-\lambda}\frac{\|\mu\|^2}{2}\right) \lambda^{-2} \left(\ln\frac{1}{\lambda}\right)^b \mathrm{d}\lambda.$$

Following [16], the corresponding generalized Bayes estimator is $\{1 - \psi_{-2,b}(\|Z\|^2)/\|Z\|^2\}Z$, where

$$\psi_{-2,b}(v) = v\frac{\int_0^1 \lambda^{p/2-1}\{\ln(1/\lambda)\}^b \exp(-v\lambda/2)\mathrm{d}\lambda}{\int_0^1 \lambda^{p/2-2}\{\ln(1/\lambda)\}^b \exp(-v\lambda/2)\mathrm{d}\lambda}.$$

As shown in Appendix A.5, we have

$$\lim_{v\to\infty} (\ln v)\left\{p - 2 - \psi_{-2,b}(v)\right\} = 2b. \tag{19}$$

Hence by Part 2 of Theorem 2 and Part 1 of Theorem 3, the generalized Bayes estimator with asymptotic behavior

$$\left\{1 - \left(p - 2 - \frac{b}{\ln\|Z\|}\right)\frac{1}{\|Z\|^2}\right\}Z$$

is admissible and inadmissible for $b \le 1$ and $b > 1$ respectively. Thus the estimator

$$\left\{1 - \left(p - 2 - \frac{1}{\ln\|Z\|}\right)\frac{1}{\|Z\|^2}\right\}Z$$

is a boundary estimator. See also Corollary 6.3.2 of [2] and Theorem 6.1.1 of [18] for related discussions, but where the $b/\ln\|z\|$ term is not included.

3.2. Unknown variance case

Let X and S be given by (1) and let the prior be of the form

$$\pi(\theta, \sigma^2) = \frac{1}{\sigma^2}\,\pi(\theta \mid \sigma^2) = \frac{1}{\sigma^2} \times \frac{1}{\sigma^p}\,G(\|\theta\|/\sigma), \tag{20}$$

where G is given by (16) and $1/\sigma^2$ is a standard non-informative prior for σ^2.

The following two theorems relate quasi-admissibility/inadmissibility in the unknown variance case to admissibility/inadmissibility in the known variance case as given in Theorems 2 and 3.

Theorem 4 (*Quasi-admissibility*)**.**

1. *Suppose $a > -2$. The generalized Bayes estimator is quasi-admissible.*
2. *Suppose $a = -2$ and $\{\ln(y)\}^b/L(y)$ for $b < 1$ is monotone non-decreasing. The generalized Bayes estimator is quasi-admissible.*

Theorem 5 (*Quasi-inadmissibility*)**.**

1. *Suppose $a = -2$ and $\{\ln(y)\}^b/L(y)$ for $b > 1$ is monotone non-increasing. The generalized Bayes estimator is quasi-inadmissible.*
2. *Suppose $a < -2$. The generalized Bayes estimator is quasi-inadmissible.*

Proof of Theorems 4 and 5. By following [11,12], the generalized Bayes estimator under the prior given by (20) is δ_ϕ with

$$\phi_{a,L}(w) = w\frac{\int_0^1 \lambda^{p/2+a+1}L(1/\lambda)(1 + w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}{\int_0^1 \lambda^{p/2+a}L(1/\lambda)(1 + w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}.$$

By a change of variables ($t = w\lambda$), we have

$$\phi_{a,L}(w) = \frac{\int_0^w t^{p/2+a+1}L(w/t)(1 + t)^{-(p+n)/2-1}\mathrm{d}t}{\int_0^w \lambda^{p/2+a}L(w/t)(1 + t)^{-(p+n)/2-1}\mathrm{d}t}.$$

By Assumption L1 and the Lebesgue dominated convergence theorem,

$$\lim_{w\to\infty} \phi_{a,L}(w) = \frac{\int_0^\infty t^{p/2+a+1}(1 + t)^{-(p+n)/2-1}\mathrm{d}t}{\int_0^\infty t^{p/2+a}(1 + t)^{-(p+n)/2-1}\mathrm{d}t} = \frac{p/2 + a + 1}{n/2 - a - 1} \tag{21}$$

which is increasing in a and is equal to $(p - 2)/(n + 2)$ when $a = -2$. Hence, by Theorem 1, $a > -2$ and $a < -2$ implies quasi-admissibility and quasi-inadmissibility, respectively.

When $a = -2$, take $L(1/\lambda) = (\ln 1/\lambda)^b$ for $b > 0$ and consider

$$\phi_{-2,b}(w) = w\frac{\int_0^1 \lambda^{p/2-1}\{\ln(1/\lambda)\}^b(1 + w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}{\int_0^1 \lambda^{p/2-2}\{\ln(1/\lambda)\}^b(1 + w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}. \tag{22}$$

Then we have

$$\lim_{w\to\infty}(\ln w)\left\{\frac{p-2}{n+2}-\phi_{-2,b}(w)\right\}=b\,\frac{2(p+n)}{(n+2)^2}=b\beta_\star \tag{23}$$

where $\beta_\star$ is given by (11) as used in Theorem 1. See Appendix A.5 for the derivation of (23). Further the inequality

$$\begin{aligned}\phi_{-2,b}(w)&=\frac{\int_0^1\lambda^{p/2-1}\{\ln(1/\lambda)\}^b(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}{\int_0^1\lambda^{p/2-2}\{\ln(1/\lambda)\}^b(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}\\&=\frac{\int_0^1\lambda\frac{\{\ln(1/\lambda)\}^b}{L(1/\lambda)}\lambda^{p/2-2}L(1/\lambda)(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}{\int_0^1\frac{\{\ln(1/\lambda)\}^b}{L(1/\lambda)}\lambda^{p/2-2}L(1/\lambda)(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}\\&\le(\ge)\frac{\int_0^1\lambda\lambda^{p/2-2}L(1/\lambda)(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}{\int_0^1\lambda^{p/2-2}L(1/\lambda)(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}\\&=\phi_{-2,L}(w)\end{aligned} \tag{24}$$

follows for $b>0$ when $\{\ln(y)\}^b/L(y)$ is monotone non-increasing (non-decreasing). From (21), (23), (24) and Theorem 1, the two theorems follow. □

Remark 5 (*A Boundary Estimator for the Unknown Variance Case*)**.** For the unknown variance case, Theorem 1 established the boundary between quasi-admissibility and quasi-inadmissibility for estimators of the form $\{1-\phi(W)/W\}X$ as

$$\left\{1-\left(\frac{p-2}{n+2}-\frac{b\beta_\star}{\ln W}\right)\frac{1}{W}\right\}X$$

with $b<1$ corresponding to quasi-admissibility and $b>1$ corresponding to quasi-inadmissibility. The generalized prior

$$\pi(\theta,\sigma^2)=\frac{1}{\sigma^2}\times\frac{1}{\sigma^p}\,G(\|\theta\|/\sigma)$$

with G given by (16) where $a=-2$ and $L(1/\lambda)=(\ln\ 1/\lambda)^b$ for $b>0$ leads to a generalized Bayes estimator with ϕ given in (22). As in (23), the asymptotic behavior of this ϕ is

$$\lim_{w\to\infty}(\ln w)\left\{\frac{p-2}{n+2}-\phi(w)\right\}=b\,\frac{2(p+n)}{(n+2)^2}=b\beta_\star.$$

Thus we see that the behavior of the generalized Bayes shrinkage function in the cases of known and unknown scale for the related classes of priors are in very close correspondence. Additionally admissibility/inadmissibility in the known scale case corresponds exactly with quasi-admissibility/inadmissibility in the unknown scale case. We conjecture, for this class of priors in the unknown scale case, that quasi-admissibility/inadmissibility in fact corresponds to admissibility/inadmissibility.

4. Concluding remarks

We have studied quasi-admissible and quasi-inadmissible Stein-type shrinkage estimators in the problem of estimating the mean vector of a p-variate normal distribution when the covariance matrix is an unknown multiple of the identity. We have established a sharp boundary of the form

$$\phi_\star(w)=\frac{p-2}{n+2}-\frac{\beta_\star}{\ln w},$$

where $\beta_\star=2(p+2)/(n+2)^2$. Roughly, estimators with shrinkage function $\phi(w)$ ultimately less than $\phi_\star(w)$ are quasi-inadmissible, while those which ultimately shrink more are quasi-admissible. We have also found generalized prior distributions of the form $(1/\sigma^2)\times(1/\sigma^p)G(\|\theta\|/\sigma)$ for which the resulting generalized Bayes estimators are asymptotically of the form

$$\left\{1-\left(\frac{p-2}{n+2}-\frac{b\beta_\star}{\ln W}\right)\frac{1}{W}\right\}X$$

for any $b>0$, thus establishing a boundary behavior for this class of priors between quasi-admissibility and quasi-inadmissibility. We conjecture, for this class of priors, that quasi-admissibility/inadmissibility in fact corresponds to admissibility/inadmissibility.

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Appendix. Proofs

A.1. Proof of Lemma 1

Let $\Delta(w)=\Delta(w;\phi,g),\ \Delta_1(w)=\Delta_1(w;\phi),\ \Delta_2(w)=\Delta_2(w;\phi,g)$ for notational simplicity.

A.1.1. Part B1

We show (i) $g(0)\geq 0$ and then (ii) $g(w)\geq 0$ for any $w>0$.

[(i) $g(0)\geq 0$] Suppose $g(0)<0$. From Assumption A1 and the continuity of ϕ, ϕ' and g, for a sufficiently small $\epsilon>0$, there exists $c_g>0$ and $w_0>0$ such that

$$g(w)\leq -c_g,\quad 0\leq \phi(w)<\epsilon,\quad \text{and } \phi'(w)\geq 0 \tag{A.1}$$

for $0<w<w_0$. Clearly, with the choice $\epsilon\in(0,c_{p,n})$, $\Delta_1(w)>0$ for $w\in(0,w_0)$ by (A.1). Further we consider the integral of $\Delta_2(t)/g(t)$ on $t\in(w,w_0)$. By integration by parts, we have

$$\begin{aligned}\int_w^{w_0}\frac{g'(t)}{g^2(t)}\{1+\phi(t)\}\mathrm{d}t&=\left[-\frac{1+\phi(t)}{g(t)}\right]_w^{w_0}+\int_w^{w_0}\frac{\phi'(t)}{g(t)}\mathrm{d}t\leq\left[-\frac{1+\phi(t)}{g(t)}\right]_w^{w_0}\\&=-\frac{1+\phi(w_0)}{g(w_0)}+\frac{1+\phi(w)}{g(w)}\leq-\frac{1+\phi(w_0)}{g(w_0)}\end{aligned}$$

for $w\in(0,w_0)$, since $\phi'(t)/g(t)$ is nonpositive. Hence we have

$$\begin{aligned}\int_w^{w_0}\frac{\Delta_2(t)}{-g(t)}\mathrm{d}t&=\int_w^{w_0}\left[\frac{1}{t}-d_n\frac{g'(t)}{g(t)}-d_n\frac{g'(t)}{g^2(t)}\{1+\phi(t)\}\right]\mathrm{d}t\\&\geq\ln\frac{w_0}{w}-d_n\ln\frac{|g(w_0)|}{|g(w)|}+d_n\frac{1+\phi(w_0)}{g(w_0)}\\&\geq\ln\frac{w_0}{w}-d_n\ln\frac{|g(w_0)|}{c_g}+d_n\frac{1+\phi(w_0)}{g(w_0)}\end{aligned}$$

which goes to infinity as $w\to 0$. Therefore there must be some point $w_*\in(0,w_0)$ such that $\Delta_2(w_*)>0$. Hence $\Delta(w_*)=g(w_*)\{\Delta_1(w_*)+\Delta_2(w_*)\}<0$ since $\Delta_1(w)$ is positive and $g(w)$ is negative over $(0,w_0)$. But this contradicts $\Delta(w)\geq 0$ for any $w\geq 0$.

[(ii) $g(w)\geq 0$ for any $w>0$] Suppose that there exists $w_1>0$ such that $g(w_1)<0$. Since $g(0)\geq 0$ by Part (i) and $g(w)$ is continuous, there exists $w_2\in[0,w_1)$ such that $g(w_2)=0$ and $g(w)<0$ for all $w_2<w\leq w_1$. Further Assumption A2 ensures that there exists $w_3\in(w_2,w_1)$ such that $\phi(w)$ is monotone on (w_2,w_3).

Since $\phi(w)$ is bounded on $w\in(w_2,w_3)$, we have

$$\int_{w_2}^{w_3}\frac{\Delta_1(t)}{1+\phi(t)}\mathrm{d}t=2\int_{w_2}^{w_3}\frac{c_{p,n}-\phi(t)}{t\{1+\phi(t)\}}\mathrm{d}t+d_n\big[\ln\{1+\phi(t)\}\big]_{w_2}^{w_3}, \tag{A.2}$$

which is bounded from above and below when $w_2>0$ and goes to infinity when $w_2=0$. Further since $g(w)<0$ for $w\in(w_2,w_3)$, we have

$$\frac{\Delta_2(w)}{1+\phi(w)}=\frac{-g(w)}{w\{1+\phi(w)\}}+\frac{d_ng'(w)}{1+\phi(w)}+d_n\frac{g'(w)}{g(w)}\geq\frac{d_ng'(w)}{1+\phi(w)}+d_n\frac{g'(w)}{g(w)}.$$

Then, by integration by parts, we have

$$\begin{aligned}\frac{1}{d_n}\int_{w_2}^{w_3}\left\{\frac{\Delta_2(t)}{1+\phi(t)}-d_n\frac{g'(t)}{g(t)}\right\}\mathrm{d}t&\geq\int_{w_2}^{w_3}\frac{g'(t)}{1+\phi(t)}\mathrm{d}t=\left[\frac{g(t)}{1+\phi(t)}\right]_{w_2}^{w_3}+\int_{w_2}^{w_3}\frac{g(t)\phi'(t)}{\{1+\phi(t)\}^2}\mathrm{d}t\\&\geq\left[\frac{g(t)}{1+\phi(t)}\right]_{w_2}^{w_3}-\max_{t\in[w_2,w_3]}|g(t)|\int_{w_2}^{w_3}\frac{|\phi'(t)|}{\{1+\phi(t)\}^2}\mathrm{d}t\\&=\left\{\frac{g(w_3)}{1+\phi(w_3)}-\frac{g(w_2)}{1+\phi(w_2)}\right\}-\max_{t\in[w_2,w_3]}|g(t)|\left|\frac{1}{1+\phi(w_2)}-\frac{1}{1+\phi(w_3)}\right|,\end{aligned} \tag{A.3}$$

which is bounded from below. For $w \in (w_2, w_3)$, we have

$$\int_w^{w_3} \frac{g'(t)}{g(t)} \mathrm{d}t = \ln|g(w_3)| - \ln|g(w)|$$

which goes to ∞ as $w \to w_2$ since $g(w_2) = 0$. Then the integral

$$\int_w^{w_3} \frac{\Delta_1(t) + \Delta_2(t)}{1 + \phi(t)} \mathrm{d}t$$

goes to infinity as $w \to w_2$.

Therefore there must be some point $w_* \in (w_2, w_3)$ such that $\Delta_1(w_*) + \Delta_2(w_*) > 0$. Hence $\Delta(w_*) = g(w_*)\{\Delta_1(w_*) + \Delta_2(w_*)\} < 0$ since $g(w)$ is negative over (w_2, w_3). But this contradicts $\Delta(w) \geq 0$ for any $w \geq 0$. □

A.1.2. Part B2

Suppose that there exists $w_1 > w_0$ such that $g(w_1) = 0$. Assumption A2 ensures that there exists $w_2 \in (w_0, w_1)$ such that ϕ is monotone on (w_2, w_1). As in (A.2) and (A.3) of Part B1, the integral

$$\int_{w_2}^{w_1} \left\{ \frac{\Delta_1(t) + \Delta_2(t)}{1 + \phi(t)} - \frac{d_n g'(t)}{g(t)} \right\} \mathrm{d}t$$

is bounded from above. Further, for $w \in (w_2, w_1)$,

$$\int_{w_2}^{w} \frac{g'(t)}{g(t)} \mathrm{d}t = \ln g(w) - \ln g(w_2)$$

which goes to $-\infty$ as $w \to w_1$ since $g(w_1) = 0$. Hence there must be a $w_* \in (w_2, w_1) \subset (w_0, w_1)$ such that $\Delta_1(w_*) + \Delta_2(w_*) < 0$. But that means that $\Delta(w_*) = g(w_*)\{\Delta_1(w_*) + \Delta_2(w_*)\} < 0$ since g is positive over (w_0, w_1), which contradicts $\Delta(w) \geq 0$ for any $w \geq 0$. □

A.2. Proof of Lemma 2

When $\theta = 0$, the distribution of $W = \|X\|^2/S$ is $(p/n)F_{p,n}$ where $F_{p,n}$ is a central F-distribution with p and n degrees of freedom. Hence the tail behavior of the density of W is given by $f_W(w) \approx w^{-n/2-1}$. Therefore if $\mathrm{E}\{\phi(W)^2/W\} < \infty$, it must be that

$$\int_1^\infty \frac{\phi(t)^2}{t} t^{-n/2-1} \mathrm{d}t = \int_1^\infty \frac{1}{t} \frac{\phi(t)^2}{t^{n/2+1}} \mathrm{d}t < \infty.$$

Since $\int_1^\infty \mathrm{d}t/t = \infty$, ϕ must satisfy

$$\liminf_{t\to\infty} \frac{\phi(t)^2}{t^{n/2+1}} = 0$$

which implies

$$\liminf_{t\to\infty} \left\{ \frac{|\phi(t)|^{4/(n+2)}}{t} \right\}^{(n+2)/2} = \liminf_{t\to\infty} \frac{|\phi(t)|^{d_n}}{t} = 0.$$

This completes the argument. □

A.3. Proof of Lemma 3

A.3.1. Part 1.a

By (12), it is clear that $\phi_* \geq (p-2)/(n+2)$ and hence $d_n(1 + \phi_*) \geq 2\beta_\star > 2b\beta_\star$, since $b < 1$. Further (12) implies that

$$\Delta_1(w; \phi) - d_n\phi'(w) - \frac{2b\beta_\star}{w \ln w} = \frac{2}{w} \left\{ \frac{p-2}{n+2} - \phi(w) - \frac{b\beta_\star}{\ln w} \right\} \leq 0,$$

for all $w \geq w_0$. □

A.3.2. Preliminary results for parts 1.b and 2.b

Let $w_0 > 0$ and, for $w > w_0$, let

$$h_1(w; w_0) = \int_{w_0}^{w} \left\{ -\frac{1}{t} + d_n \frac{g'(t)}{g(t)} \right\} \mathrm{d}t, \quad h_2(w; w_0) = d_n \int_{w_0}^{w} \left[\frac{g'(t)}{g^2(t)} \{1 + \phi(t)\} - \frac{\phi'(t)}{g(t)} \right] \mathrm{d}t. \tag{A.4}$$

Then we have

$$h_1(w;w_0)=\int_{w_0}^{w}\left\{-\frac{1}{t}+d_n\frac{g'(t)}{g(t)}\right\}\mathrm{d}t=-\ln\frac{w}{w_0}+d_n\ln\frac{g(w)}{g(w_0)}=\ln\frac{g(w)^{d_n}}{w}+\ln\frac{w_0}{g(w_0)^{d_n}}. \tag{A.5}$$

Since $g\in\mathcal{G}$, $\liminf_{w\to\infty}g(w)^{d_n}/w=0$ by Lemma 2. Hence we have

$$\liminf_{w\to\infty}h_1(w;w_0)=-\infty. \tag{A.6}$$

By integration by parts, $h_2(w;w_0)$, divided by d_n is

$$\frac{h_2(w;w_0)}{d_n}=\int_{w_0}^{w}\left[\frac{g'(t)}{g^2(t)}\{1+\phi(t)\}-\frac{\phi'(t)}{g(t)}\right]\mathrm{d}t=\left[-\frac{1+\phi(t)}{g(t)}\right]_{w_0}^{w}=-\frac{1+\phi(w)}{g(w)}+\frac{1+\phi(w_0)}{g(w_0)}, \tag{A.7}$$

which concludes this part. □

A.3.3. Part 1.b

Let $\alpha=2b\beta_\star$ and fix

$$\epsilon=\frac{d_n(1+\phi_*)-\alpha}{6d_n}. \tag{A.8}$$

Then, by Assumption A2 and $\lim_{w\to\infty}\phi(w)=\phi_*$, there exists w_1 such that

$$\phi(w)\text{ is monotone and }\int_{w}^{\infty}|\phi'(t)|\mathrm{d}t=|\phi_*-\phi(w)|<\epsilon \tag{A.9}$$

for all $w\geq w_1$. Since $g(w)\not\equiv 0$ and $g(w)$ satisfies B1 and B2 of Lemma 1, there exists $w_2>0$ such that $g(w)>0$ for all $w\geq w_2$. Define $w_3=\max(w_0,w_1,w_2,1)$ and consider the integral

$$\int_{w_3}^{w}\frac{\Delta_2(t;\phi,g)+d_n\phi'(t)+\alpha/(t\ln t)}{g(t)}\mathrm{d}t\leq\sum_{i=1}^{4}h_i(w;w_3)$$

where $h_1(w;w_3)$ and $h_2(w;w_3)$ are given by (A.4) and

$$h_3(w;w_3)=2d_n\int_{w_3}^{w}\frac{|\phi'(t)|\mathrm{d}t}{g(t)},\quad h_4(w;w_3)=\alpha\int_{w_3}^{w}\frac{1}{g(t)t\ln t}\mathrm{d}t.$$

We are going to show

$$\liminf_{w\to\infty}\sum_{i=1}^{4}h_i(w;w_3)=-\infty,$$

which guarantees that there exists $w_*\in(w_3,\infty)$ such that

$$\Delta_2(w_*;\phi,g)+d_n\phi'(w_*)+\frac{\alpha}{w_*\ln w_*}<0.$$

By (A.7) and (A.9), we have

$$\frac{h_2(w;w_0)}{d_n}\leq-\frac{1+\phi_*-\epsilon}{g(w)}+\frac{1+\phi(w_0)}{g(w_0)}. \tag{A.10}$$

Let

$$G(w)=\frac{1}{g(w)\ln w} \tag{A.11}$$

and recall w_3 is greater than 1. Then, with (A.11), $h_3(w;w_3)$ and $h_4(w;w_3)$ for $w>w_3>1$, are bounded as follows:

$$h_3(w;w_3)=2d_n\int_{w_3}^{w}G(t)\ln t|\phi'(t)|\mathrm{d}t\leq 2d_n\ln w\sup_{t\in(w_3,w)}G(t)\int_{w_3}^{w}|\phi'(t)|\mathrm{d}t<2d_n\epsilon\ln w\sup_{t\in(w_3,w)}G(t), \tag{A.12}$$

by (A.9), and

$$h_4(w;w_3)=\alpha\int_{w_3}^{w}\frac{G(t)\mathrm{d}t}{t}\leq\alpha\sup_{t\in(w_3,w)}G(t)\int_{w_3}^{w}\frac{\mathrm{d}t}{t}\leq\alpha\ln w\sup_{t\in(w_3,w)}G(t). \tag{A.13}$$

Thus, by (A.10), (A.12) and (A.13), we have

$$\sum_{i=2}^{4}h_i(w;w_3)-\frac{1+\phi(w_3)}{g(w_3)}\leq\ln w\left\{(\alpha+2d_n\epsilon)\sup_{t\in(w_3,w)}G(t)-d_n(1+\phi_*-\epsilon)G(w)\right\}. \tag{A.14}$$

Case I: $\limsup_{w\to\infty} G(w)=\infty$

Since there exists $w_4>w_3$ such that $\sup_{t\in(w_3,w_4)} G(t)=G(w_4)>1$, we have

$$(\alpha+2d_n\epsilon)\sup_{t\in(w_3,w_4)} G(t)-d_n(1+\phi_*-\epsilon)G(w_4)=-G(w_4)\frac{d_n(1+\phi_*)-\alpha}{2}.$$

Therefore, by (A.14),

$$\sum_{i=2}^{4} h_i(w_4;w_3)-\frac{1+\phi(w_3)}{g(w_3)}\le -G(w_4)\ln w_4\frac{d_n(1+\phi_*)-\alpha}{2}. \tag{A.15}$$

By (A.5) and (A.11), we have

$$\begin{aligned} h_1(w_4;w_3)-\ln\frac{w_3}{g(w_3)^{d_n}}&=\ln\frac{g(w_4)^{d_n}}{w_4}=\ln\frac{1}{w_4\{G(w_4)\}^{d_n}(\ln w_4)^{d_n}}\\ &=-d_n\ln\ln w_4-\ln w_4-d_n\ln G(w_4)\le -d_n\ln\ln w_4-\ln w_4, \end{aligned} \tag{A.16}$$

since $G(w_4)>1$. By (A.15), (A.16) and choosing w_4 to be sufficiently large, we conclude that

$$\liminf_{w\to\infty}\sum_{i=1}^{4} h_i(w;w_3)=-\infty.$$

Case II: $\limsup_{w\to\infty} G(w)=G_*\in(0,\infty)$

Under the choice of ϵ given by (A.8), fix

$$\nu=\frac{G_*\{d_n(1+\phi_*)-\alpha\}}{4\{\alpha+d_n(1+\phi_*+\epsilon)\}}. \tag{A.17}$$

There exists $w_5\ge w_3$ such that $\sup_{t\ge w_5} G(t)<G_*+\nu$ and $w_6\in(w_5,\infty)$ which satisfies $G(w_6)\ge G_*-\nu$ can be taken. Then we have

$$\begin{aligned} (\alpha+2d_n\epsilon)\sup_{t\in(w_5,w_6)} G(t)-d_n(1+\phi_*-\epsilon)G(w_6)&\le(\alpha+2d_n\epsilon)(G_*+\nu)-d_n(1+\phi_*-\epsilon)(G_*-\nu)\\ &=\nu\{(\alpha+2d_n\epsilon)+d_n(1+\phi_*-\epsilon)\}\\ &\quad+G_*\{(\alpha+2d_n\epsilon)-d_n(1+\phi_*-\epsilon)\}\\ &=\nu\{\alpha+d_n(1+\phi_*+\epsilon)\}-G_*(\{d_n(1+\phi_*)-\alpha\}-3d_n\epsilon)\\ &=G_*\frac{d_n(1+\phi_*)-\alpha}{4}-G_*\left\{d_n(1+\phi_*)-\alpha-\frac{d_n(1+\phi_*)-\alpha}{2}\right\}\\ &=-G_*\frac{d_n(1+\phi_*)-\alpha}{4} \end{aligned} \tag{A.18}$$

by (A.9) and (A.17). Hence, by (A.14) and (A.18), we have

$$\sum_{i=2}^{4} h_i(w_6;w_5)-\frac{1+\phi(w_5)}{g(w_5)}\le -G_*\frac{d_n(1+\phi_*)-\alpha}{4}\ln w_6. \tag{A.19}$$

As in (A.16), we have

$$h_1(w_6;w_5)-\ln\frac{w_5}{g(w_5)^{d_n}}=-d_n\ln\ln w_6-\ln w_6-d_n\ln G(w_6)\le -d_n\ln\ln w_6-\ln w_6-d_n\ln(G_*-\nu). \tag{A.20}$$

By choosing w_6 to be sufficiently large on (A.19) and (A.20), we have

$$\liminf_{w\to\infty}\sum_{i=1}^{4} h_i(w;w_5)=-\infty.$$

Case III: $\limsup_{w\to\infty} G(w)=0$ or equivalently $\lim_{w\to\infty} G(w)=0$

Case III-i: $\limsup_{w\to\infty} G(w)w^{1/(4d_n)}<\infty$

Let $\tau=1/(4d_n)>0$. Note

$$\begin{aligned} h_3(w;w_3)=2d_n\int_{w_3}^{w} G(t)\ln t|\phi'(t)|\mathrm{d}t&\le 2d_n\int_{w_3}^{\infty}\{G(t)t^{\tau}\}\frac{\ln t}{t^{\tau}}|\phi'(t)|\mathrm{d}t\\ &\le 2d_n\sup_{t\in(w_3,\infty)} G(t)t^{\tau}\sup_{t\in(w_3,\infty)}\frac{\ln t}{t^{\tau}}\int_{w_3}^{\infty}|\phi'(t)|\mathrm{d}t\le 2d_n\epsilon\sup_{t\in(w_3,\infty)} G(t)t^{\tau}\sup_{t\in(w_3,\infty)}\frac{\ln t}{t^{\tau}}, \end{aligned}$$

which is bounded from above. Also note

$$h_4(w;w_3)=\alpha\int_{w_3}^{w}\frac{G(t)\mathrm{d}t}{t}\le\alpha\int_{w_3}^{\infty}\frac{G(t)t^{\tau}\mathrm{d}t}{t^{1+\tau}}\le\alpha\sup_{t\in(w_3,\infty)}G(t)t^{\tau}\int_{w_3}^{\infty}\frac{\mathrm{d}t}{t^{1+\tau}}$$

which is bounded from above. Further we have $\liminf_{w\to\infty}h_1(w;w_3)=-\infty$ by (A.6) and $h_2(w;w_3)\le\{1+\phi(w_3)\}/g(w_3)$ by (A.10). Therefore we have

$$\liminf_{w\to\infty}\sum_{i=1}^{4}h_i(w;w_3)=-\infty.$$

Case III-ii: $\limsup_{w\to\infty}G(w)w^{1/(4d_n)}=\infty$

Under the choice of ϵ given by (A.8), there exists $w_7\ge w_3$ such that

$$\sup_{t\in(w_7,\infty)}G(t)<\frac{1}{2(\alpha+2d_n\epsilon)}. \tag{A.21}$$

By (A.21), we have

$$\sum_{i=2}^{4}h_i(w;w_7)-\frac{1+\phi(w_7)}{g(w_7)}\le(\alpha+2d_n\epsilon)\sup_{t\in(w_7,w)}G(t)\ln w\le\frac{\ln w}{2},$$

$$-\frac{3}{4}\ln w+\sum_{i=2}^{4}h_i(w;w_7)\le-\frac{\ln w}{4}+\frac{1+\phi(w_7)}{g(w_7)}$$

and hence

$$\lim_{w\to\infty}\left\{-\frac{3}{4}\ln w+\sum_{i=2}^{4}h_i(w;w_7)\right\}=-\infty. \tag{A.22}$$

Recall $G(w)=1/\{g(w)\ln w\}$. Then we have

$$h_1(w;w_7)+\ln\frac{g(w_7)^{d_n}}{w_7}+\frac{3}{4}\ln w=\ln\frac{g(w)^{d_n}}{w}+\frac{3}{4}\ln w=-d_n\ln\{G(w)w^{1/(4d_n)}\}-d_n\ln\ln w.$$

Since $\limsup_{w\to\infty}G(w)w^{1/(4d_n)}=\infty$,

$$\liminf_{w\to\infty}\left\{h_1(w;w_7)+\frac{3}{4}\ln w\right\}=-\infty \tag{A.23}$$

follows. Note

$$\sum_{i=1}^{4}h_i(w;w_7)=\left\{h_1(w;w_7)+\frac{3}{4}\ln w\right\}+\left\{\sum_{i=2}^{4}h_i(w;w_7)-\frac{3}{4}\ln w\right\}. \tag{A.24}$$

By (A.22), (A.23) and (A.24), we have

$$\liminf_{w\to\infty}\sum_{i=1}^{4}h_i(w;w_7)=-\infty,$$

and hence we can conclude. □

A.3.4. Part 2.a

We have

$$\frac{w}{\phi(w)}\{\Delta_1(w;\phi)+d_n\phi'(w)\}=2\left\{\frac{c_{p,n}}{\phi(w)}-1+d_n\frac{w\phi'(w)}{\phi(w)}\right\}.$$

By Assumption A3 and the assumption $n\ge 3$, we have

$$d_n\limsup_{w\to\infty}w\frac{\phi'(w)}{\phi(w)}\le d_n=\frac{4}{n+2}<1.$$

Since $\lim_{w\to\infty}1/\phi(w)=0$, there exists w_1 such that, for all $w\ge w_1$, $\Delta_1(w;\phi)+d_n\phi'(w)\le 0$. □

A.3.5. Part 2.b

Consider the integral

$$\int_{w_1}^{w} \frac{\Delta_2(t;\phi,g)-d_n\phi'(t)}{g(t)}\mathrm{d}t = h_1(w;w_1)+h_2(w;w_1)$$

where $h_1(w;\cdot)$ and $h_2(w;\cdot)$ are given by (A.4). We are going to show

$$\liminf_{w\to\infty}\{h_1(w;w_1)+h_2(w;w_1)\} = -\infty, \tag{A.25}$$

which guarantees that there exists $w_* \in (w_1,\infty)$ such that $\Delta_2(w_*;\phi,g)-d_n\phi'(w_*)<0$. By (A.6), $\liminf_{w\to\infty}h_1(w;w_1) = -\infty$ follows. Also, by (A.10), $h_2(w;w_1)\le \{1+\phi(w_1)\}/g(w_1)$. Therefore (A.25) follows. □

A.3.6. Part 3.a

By (13), we have $\phi_* \le (p-2)/(n+2) = c_{p,n}$ and hence

$$d_n(1+\phi_*) \le 2\beta_\star < 2b\beta_\star \tag{A.26}$$

since $b>1$. When $\phi_* = c_{p,n}$, $\phi(w)$ is ultimately monotone nondecreasing and hence without the loss of generality, $\phi'(w)\ge 0$ for all $w \ge w_0$. Then we have

$$\Delta_1(w;\phi) - \frac{2b\beta_\star}{w\ln w} = 2\frac{c_{p,n}-\phi(w)}{w} + \phi'(w) - \frac{2b\beta_\star}{w\ln w} \ge \frac{2}{w}\left\{c_{p,n} - \frac{b\beta_\star}{\ln w} - \phi(w)\right\} \ge 0, \tag{A.27}$$

for all $w \ge w_0$ by (13).

Consider the case where $c_{p,n}-\phi_* = \delta > 0$. By Assumption A3, there exists w_2 such that $w\phi'(w)/\phi(w) > -\delta/(4\phi_*)$ for all $w\ge w_2$. Further, by $\lim_{w\to\infty}\phi(w)=\phi_*$, there exists w_3 such that

$$|\phi(w)-\phi_*| < \frac{\delta}{4\{1+\delta/(4\phi_*)\}}$$

for all $w\ge w_3$. Then, for all $w \ge \max(w_2, w_3, e^{4b\beta_\star/\delta})$, we have

$$\begin{aligned}\frac{w}{2}\left\{\Delta_1(w;\phi)-\frac{2b\beta_\star}{w\ln w}\right\} &= c_{p,n}-\phi(w)+\phi(w)\frac{w\phi'(w)}{\phi(w)} - \frac{b\beta_\star}{\ln w}\\ &\ge \delta - \frac{\delta}{4\{1+\delta/(4\phi_*)\}} - \left[\phi_* + \frac{\delta}{4\{1+\delta/(4\phi_*)\}}\right]\frac{\delta}{4\phi_*} - \frac{\delta}{4} = \frac{\delta}{4}.\end{aligned} \tag{A.28}$$

Hence, under the condition (A.26), by (A.27) and (A.28), there exists w_1 such that $\Delta_1(w;\phi)-2\beta/(w\ln w) \ge 0$ for all $w\ge w_1$. □

A.3.7. Part 3.b

There exists w_4 such that $\phi_*-\nu < \phi(w) < \phi_*+\nu$ for all $w\ge w_4$. Recall

$$\Delta_2(w;\phi,g) = \frac{-g(w)}{w} + d_n g'(w) + d_n\frac{g'(w)}{g(w)}\{1+\phi(w)\}.$$

Hence, for all $w\ge \max(e, w_4, w_\sharp)$, we have

$$\begin{aligned}\Delta_2[w;\phi,\{\ln(w+e)\}^{-1-\nu}k(w)] &= \frac{-k(w)}{w\{\ln(w+e)\}^{1+\nu}} - \frac{d_n(1+\nu)k(w)}{(w+e)\{\ln(w+e)\}^{2+\nu}} + \frac{d_nk'(w)}{\{\ln(w+e)\}^{1+\nu}}\\ &\quad + d_n\left\{\frac{k'(w)}{k(w)} - \frac{1+\nu}{(w+e)\ln(w+e)}\right\}\{1+\phi(w)\}\\ &\ge -\frac{d_n(1+\nu)(1+\phi_*+\nu)}{w\ln w} - \frac{d_n(1+\nu)+1}{w(\ln w)^{1+\nu}}\\ &\ge -\frac{d_n(1+\phi_*)}{w\ln w} - \frac{\nu d_n(3+\phi_*)}{w\ln w} - \frac{2d_n+1}{w(\ln w)^{1+\nu}}\\ &= -\frac{\alpha}{w\ln w} + \frac{\{\alpha - d_n(1+\phi_*) - \nu d_n(3+\phi_*)\}(\ln w)^\nu - (2d_n+1)}{w(\ln w)^{1+\nu}}\\ &\ge -\frac{\alpha}{w\ln w} + \frac{\{\alpha - d_n(1+\phi_*)\}(\ln w)^\nu - 2(2d_n+1)}{2w(\ln w)^{1+\nu}}.\end{aligned}$$

Let $w_* = \max(e, w_4, w_5)$ where

$$w_5 = \exp\left[\left\{\frac{2(2d_n+1)}{\alpha - d_n(1+\phi_*)}\right\}^{1/\nu}\right].$$

Then we have $\Delta_2[w; \phi, \{\ln(w+e)\}^{-1-\nu}k(w)] \geq -\alpha/(w \ln w)$ for all $w \geq \max(w_*, w_\sharp)$. □

A.4. Boundedness of $\|z\| \times \|\nabla \ln m(\|z\|; a, l)\|$

Note that

$$\nabla \ln m(\|z\|; a, L) = -z\frac{\int_0^1 \lambda^{p/2+a+1}L(1/\lambda)\exp(-\|z\|^2\lambda/2)\mathrm{d}\lambda}{\int_0^1 \lambda^{p/2+a}L(1/\lambda)\exp(-\|z\|^2\lambda/2)\mathrm{d}\lambda}.$$

We have $\|z\| \times \|\nabla \ln m(\|z\|; a, L)\| = 0$ at $\|z\| = 0$. By a Tauberian Theorem which is also applied in (17),

$$\begin{aligned}\lim_{\|z\|\to\infty} \|z\| \times \|\nabla \ln m(\|z\|; a, L)\| &= \lim_{\|z\|\to\infty} \|z\|^2\frac{\int_0^1 \lambda^{p/2+a+1}L(1/\lambda)\exp(-\|z\|^2\lambda/2)\mathrm{d}\lambda}{\int_0^1 \lambda^{p/2+a}L(1/\lambda)\exp(-\|z\|^2\lambda/2)\mathrm{d}\lambda} \\ &= p + 2a + 2,\end{aligned}$$

the boundedness of $\|z\| \times \|\nabla \ln m(\|z\|; a, L)\|$ follows. □

A.5. Derivation of (19) and (23)

A.5.1. Derivation of (19)

Recall

$$\psi_{-2,b}(v) = v\frac{\int_0^1 \lambda^{p/2-1}\{\ln(1/\lambda)\}^b\exp(-v\lambda/2)\mathrm{d}\lambda}{\int_0^1 \lambda^{p/2-2}\{\ln(1/\lambda)\}^b\exp(-v\lambda/2)\mathrm{d}\lambda}.$$

By integration by parts,

$$\begin{aligned}\frac{v}{2}\int_0^1 \lambda^{p/2-1}\{\ln(1/\lambda)\}^b\exp(-v\lambda/2)\mathrm{d}\lambda = &\left[-\lambda^{p/2-1}\{\ln(1/\lambda)\}^b\exp(-v\lambda/2)\right]_0^1 \\ &+ (p/2-1)\int_0^1 \lambda^{p/2-2}\{\ln(1/\lambda)\}^b\exp(-v\lambda/2)\mathrm{d}\lambda \\ &- b\int_0^1 \lambda^{p/2-1}\{\ln(1/\lambda)\}^{b-1}\lambda^{-1}\exp(-v\lambda/2)\mathrm{d}\lambda.\end{aligned}$$

Thus we have

$$\psi_{-2,b}(v) = p - 2 - 2b\frac{\int_0^1 \lambda^{p/2-2}\{\ln(1/\lambda)\}^{b-1}\exp(-v\lambda/2)\mathrm{d}\lambda}{\int_0^1 \lambda^{p/2-2}\{\ln(1/\lambda)\}^b\exp(-v\lambda/2)\mathrm{d}\lambda}.$$

By a Tauberian theorem as in (17), we have

$$\lim_{v\to\infty}(\ln v)\frac{\int_0^1 \lambda^{p/2-2}\{\ln(1/\lambda)\}^{b-1}\exp(-v\lambda/2)\mathrm{d}\lambda}{\int_0^1 \lambda^{p/2-2}\{\ln(1/\lambda)\}^b\exp(-v\lambda/2)\mathrm{d}\lambda} = 1,$$

and hence

$$\lim_{v\to\infty}(\ln v)\left\{p - 2 - \psi_{-2,b}(v)\right\} = 2b.$$

This completes the derivation of (19). □

A.5.2. Derivation of (23)

Recall

$$\phi_{-2,b}(w) = w\frac{\int_0^1 \lambda^{p/2-1}\{\ln(1/\lambda)\}^b(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}{\int_0^1 \lambda^{p/2-2}\{\ln(1/\lambda)\}^b(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}. \tag{A.29}$$

Note

$$(1+w\lambda)^{-(p+n)/2-1} = (1+w\lambda)^{-p/2+1}(1+w\lambda)^{-n/2-2},$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\frac{(1+w\lambda)^{-n/2-1}}{w(-n/2-1)}\right\}=(1+w\lambda)^{-n/2-2},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\lambda}{1+w\lambda}\right)^{p/2-1}=(p/2-1)\left(\frac{\lambda}{1+w\lambda}\right)^{p/2-2}\frac{1}{(1+w\lambda)^2}.$$

Then, by integration by parts, we have

$$\begin{aligned}(n/2+1)w\int_0^1 \lambda^{p/2-1}\left(\ln\frac{1}{\lambda}\right)^b(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda \\ &= \left[(1+w\lambda)^{-n/2-1}\left(\frac{\lambda}{1+w\lambda}\right)^{p/2-1}\left(\ln\frac{1}{\lambda}\right)^b\right]_0^1 \\ &\quad + (p/2-1)\int_0^1\left(\frac{\lambda}{1+w\lambda}\right)^{p/2-2}\frac{(1+w\lambda)^{-n/2-1}}{(1+w\lambda)^2}\left(\ln\frac{1}{\lambda}\right)^b\mathrm{d}\lambda \\ &\quad - b\int_0^1\left(\frac{\lambda}{1+w\lambda}\right)^{p/2-1}\frac{(1+w\lambda)^{-n/2-1}}{(1+w\lambda)^2}\left(\ln\frac{1}{\lambda}\right)^{b-1}\frac{1}{\lambda}\mathrm{d}\lambda,\end{aligned} \tag{A.30}$$

which is equal to

$$(p/2-1)\int_0^1\lambda^{p/2-2}\left(\ln\frac{1}{\lambda}\right)^b(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda - b\int_0^1\lambda^{p/2-2}\left(\ln\frac{1}{\lambda}\right)^{b-1}(1+w\lambda)^{-(p+n)/2}\mathrm{d}\lambda. \tag{A.31}$$

By (A.29), (A.30) and (A.31), we have

$$\phi_{-2,b}=\frac{p-2}{n+2}-\frac{2b}{n+2}\frac{\int_0^1\lambda^{p/2-2}\{\ln(1/\lambda)\}^{b-1}(1+w\lambda)^{-(p+n)/2}\mathrm{d}\lambda}{\int_0^1\lambda^{p/2-2}\{\ln(1/\lambda)\}^{b}(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}. \tag{A.32}$$

As in (21), we have

$$\lim_{w\to\infty}(\ln w)\frac{\int_0^1\lambda^{p/2-2}\{\ln(1/\lambda)\}^{b-1}(1+w\lambda)^{-(p+n)/2}\mathrm{d}\lambda}{\int_0^1\lambda^{p/2-2}\{\ln(1/\lambda)\}^{b}(1+w\lambda)^{-(p+n)/2-1}\mathrm{d}\lambda}=\frac{\int_0^\infty t^{p/2-2}(1+t)^{-(p+n)/2}\mathrm{d}t}{\int_0^\infty t^{p/2-2}(1+t)^{-(p+n)/2-1}\mathrm{d}t}=\frac{p+n}{n+2} \tag{A.33}$$

and hence, by (A.32) and (A.33), (23) follows. □

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