Supplementary Material for "Admissible Bayes equivariant estimation of location vectors for spherically symmetric distributions with unknown scale"

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Contents

Appendix A: Proof of Theorem 2.1

In Theorem 2.1, we show that if an estimator of the form

$$\delta_\psi(X,U) = \left\{1 - \psi(\|X\|^2/\|U\|^2)\right\} X, \text{ where } \psi : \mathbb{R}_+ \to \mathbb{R}, \tag{A.1}$$

is admissible within the class of all such estimators, then it is also admissible within the larger class of estimators of the form

$$\delta_\xi(X,U) = \{1 - \xi(X/\|U\|)\} X, \text{ where } \xi : \mathbb{R}^p \to \mathbb{R}. \tag{A.2}$$

Suppose the estimator $\delta_\xi(X,S) \in \mathcal{D}_\xi$ is strictly better than the estimator $\delta_\psi \in \mathcal{D}_\psi$, that is,

$$E\left[\eta \|\delta_\xi(X,S) - \theta\|^2\right] \le E\left[\eta \|\delta_\psi(X,S) - \theta\|^2\right] \tag{A.3}$$

for all $\eta^{1/2}\theta \in \mathbb{R}^p$ with strict inequality for some value. Because of the continuity of $\delta_\xi(X,S)$ and $\delta_\psi(X,S)$, strict inequality will hold for $\eta^{1/2}\theta \in \mathbb{R}^p$ in some nonempty open set $U \subset \mathbb{R}^p$. The inequality (A.3) will remain true if $\delta_\xi(X,S)$ is replaced by $\Gamma\delta_\xi(\Gamma^{-1}X,S)$ with Γ orthogonal, since

$$E\left[\eta \left\|\Gamma\delta_\xi(\Gamma^{-1}X,S) - \theta\right\|^2\right] = E\left[\eta \left\|\delta_\xi(\Gamma^{-1}X,S) - \Gamma^{-1}\theta\right\|^2\right].$$

Thus, for fixed $\eta^{1/2}\theta \in U \subset \mathbb{R}^p$, the set of Γ for which

$$E\left[\eta \left\|\Gamma\delta_\xi(\Gamma^{-1}X,S) - \theta\right\|^2\right] < E\left[\eta \|\delta_\psi(X,S) - \theta\|^2\right]$$

will be a nonempty open set. Let μ be the invariant probability measure on $\mathcal{O}(p)$ which assigns strictly positive measure to any nonempty open set (for the existence of such a measure, see Chapter 2 of Weil (1940)). Then the weighted estimator

$$\delta_{\xi\star} = \int_{\mathcal{O}(p)} \Gamma\delta_\xi(\Gamma^{-1}X,S)\mathrm{d}\mu(\Gamma)$$

is a member of the class $\mathcal{D}_\psi$, and because of the convexity of the loss function in δ, we have

$$\begin{aligned} E\left[\eta \|\delta_{\xi\star}(X,S) - \theta\|^2\right] &\le \int E\left[\eta \left\|\Gamma\delta_\xi(\Gamma^{-1}X,S) - \theta\right\|^2\right] \mathrm{d}\mu(\Gamma) \\ &\le E\left[\eta \|\delta_\psi(X,S) - \theta\|^2\right] \end{aligned}$$

with strict inequality for $\eta^{1/2}\theta \in U$. This implies that $\delta_\psi(X,S)$ is not admissible among $\mathcal{D}_\psi$ as assumed and hence completes the proof.

Appendix B: Proof of Theorem 3.1

Parts 3 and 4 are important. [Part 3] Given the proper prior $\pi(\lambda)$ on $\lambda = \eta\|\theta\|^2$ with $\int_0^\infty \pi(\lambda)\mathrm{d}\lambda = 1$, the Bayes equivariant estimator is equivalent to the generalized Bayes estimator of θ with respect to the joint prior density

$$\eta^{-1}\eta^{p/2}\bar{\pi}(\eta\|\theta\|^2)$$

where $\bar{\pi}(\lambda) = c_p^{-1}\lambda^{1-p/2}\pi(\lambda)$. [Part 4] The Bayes equivariant estimator is admissible among the class $\mathcal{D}_\psi$.

[Parts 1 and 2] The Bayes equivariant risk given by (3.4) is rewritten as

$$\begin{aligned} B(\delta_\psi, \pi) &= \int_{\mathbb{R}^p} \tilde{R}(\|\mu\|^2, \delta_\psi)\bar{\pi}(\|\mu\|^2)\mathrm{d}\mu \\ &= \int_{\mathbb{R}^p} \tilde{R}(\eta\|\theta\|^2, \delta_\psi)\eta^{p/2}\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta \\ &= \int_{\mathbb{R}^p} R(\theta, \eta, \delta_\psi)\eta^{p/2}\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta, \end{aligned} \tag{B.1}$$

where the third equality follows from (3.2). Further $B(\delta_\psi, \pi)$ given by (B.1) is expanded as

$$\begin{aligned} B(\delta_\psi, \pi) = &\int_{\mathbb{R}^p} E\left[\eta\|X\|^2\psi^2(\|X\|^2/S)\right] \eta^{p/2}\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta \\ &- 2\int_{\mathbb{R}^p} E\left[\eta\|X\|^2\psi(\|X\|^2/S)\right] \eta^{p/2}\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta \\ &+ 2\int_{\mathbb{R}^p} E\left[\eta\psi(\|X\|^2/S)X^{\mathrm{T}}\theta\right] \eta^{p/2}\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta \\ &+ \int_{\mathbb{R}^p} E\left[\eta\|X-\theta\|^2\right] \eta^{p/2}\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta. \end{aligned} \tag{B.2}$$

Note that, by (1.7) and the propriety of the prior given by (3.7), the third term is equal to p, that is,

$$\int_{\mathbb{R}^p} E\left[\eta\|X-\theta\|^2\right] \eta^{p/2}\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta = \int_{\mathbb{R}^p} p\bar{\pi}(\|\mu\|^2)\mathrm{d}\mu = p. \tag{B.3}$$

The first and second terms of (B.2) with $\psi^j(\|X\|^2/S)$ for $j = 2, 1$ respectively, are rewritten as

$$\int_{\mathbb{R}^p} E\left[\eta\|X\|^2\psi^j(\|X\|^2/S)\right] \eta^{p/2}\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta \tag{B.4}$$

$$
\begin{aligned}
&= c_n \iiint \eta\|x\|^2\psi^j(\|x\|^2/s)\eta^{(2p+n)/2}s^{n/2-1}f(\eta\{\|x-\theta\|^2+s\}) \\
&\qquad \times \bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}x\mathrm{d}s \\
&= c_n \iiint \eta s\|z\|^2\psi^j(\|z\|^2)\eta^{(2p+n)/2}s^{(p+n)/2-1}f(\eta\{\|\sqrt{s}z-\theta\|^2+s\}) \\
&\qquad \times \bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}z\mathrm{d}s \quad (z = x/\sqrt{s},\ J = s^{p/2}) \\
&= c_n \iiint \eta s\|z\|^2\psi^j(\|z\|^2)\eta^{(2p+n)/2}s^{(2p+n)/2-1}f(s\eta\{\|z-\theta_*\|^2+1\}) \\
&\qquad \times \bar{\pi}(\eta s\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}z\mathrm{d}s \quad (\theta_* = \theta/\sqrt{s},\ J = s^{p/2}) \\
&= c_n \iiint \|z\|^2\psi^j(\|z\|^2)\eta_*^{(2p+n)/2}f(\eta_*\{\|z-\theta_*\|^2+1\}) \\
&\qquad \times \bar{\pi}(\eta_*\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}z\mathrm{d}\eta_* \quad (\eta_* = \eta s,\ J = 1/\eta) \\
&= c_n \int_{\mathbb{R}^p} \|z\|^2\psi^j(\|z\|^2)M_1(z,\pi)\mathrm{d}z,
\end{aligned}
$$

where c_n is given by (3.5), $z = x/\sqrt{s}$, J is the Jacobian, and

$$
M_1(z,\pi) = \iint \eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta. \tag{B.5}
$$

Similarly, the third term of (B.2) is rewritten as

$$
\begin{aligned}
&\int_{\mathbb{R}^p} E\left[\eta\psi(\|X\|^2/S)X^{\mathrm{T}}\theta\right]\eta^{p/2}\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta \qquad\qquad \text{(B.6)} \\
&= c_n \iiint \eta\psi^2(\|x\|^2/s)x^{\mathrm{T}}\theta\eta^{(2p+n)/2}s^{n/2-1}f(\eta\{\|x-\theta\|^2+s\}) \\
&\qquad \times \bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}x\mathrm{d}s \\
&= c_n \iiint \eta\psi(\|z\|^2)\sqrt{s}z^{\mathrm{T}}\theta\eta^{(2p+n)/2}s^{(p+n)/2-1}f(\eta\{\|\sqrt{s}z-\theta\|^2+s\}) \\
&\qquad \times \bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}z\mathrm{d}s \quad (z = x/\sqrt{s},\ J = s^{p/2}) \\
&= c_n \iiint \eta s\psi(\|z\|^2)z^{\mathrm{T}}\theta_*\eta^{(2p+n)/2}s^{(2p+n)/2-1}f(s\eta\{\|z-\theta_*\|^2+1\}) \\
&\qquad \times \bar{\pi}(\eta s\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}z\mathrm{d}s \quad (\theta_* = \theta/\sqrt{s},\ J = s^{p/2}) \\
&= c_n \iiint \psi(\|z\|^2)z^{\mathrm{T}}\theta_*\eta_*^{(2p+n)/2}f(\eta_*\{\|z-\theta_*\|^2+1\}) \\
&\qquad \times \bar{\pi}(\eta_*\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}z\mathrm{d}\eta_* \quad (\eta_* = \eta s,\ J = 1/\eta) \\
&= c_n \int_{\mathbb{R}^p} \psi(\|z\|^2)z^{\mathrm{T}}M_2(z,\pi)\mathrm{d}z,
\end{aligned}
$$

where

$$M_2(z,\pi) = \iint \theta \eta^{(2p+n)/2} f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta. \tag{B.7}$$

Hence, by (B.3), (B.4) and (B.6), we have

$$\begin{aligned} B(\delta_\psi,\pi) = c_n \int_{\mathbb{R}^p} & \{\psi^2(\|z\|^2)\|z\|^2 M_1(z,\pi) \\ & -2\psi(\|z\|^2)\{\|z\|^2 M_1(z,\pi) - z^{\mathrm{T}} M_2(z,\pi)\}\}\,\mathrm{d}z + p. \end{aligned} \tag{B.8}$$

Then the Bayes equivariant solution or minimizer of $B(\delta_\psi,\pi)$ is

$$\psi_\pi(\|z\|^2) = \arg\min_{\psi}(B(\delta_\psi,\pi)) = 1 - \frac{z^{\mathrm{T}} M_2(z,\pi)}{\|z\|^2 M_1(z,\pi)} \tag{B.9}$$

and hence the corresponding Bayes equivariant estimator is

$$\delta_\pi = \frac{Z^{\mathrm{T}} M_2(Z,\pi)}{\|Z\|^2 M_1(Z,\pi)} X, \tag{B.10}$$

where $Z = X/\sqrt{S}$.

[Part 3] The generalized Bayes estimator of θ with respect to the density on (θ,η),

$$\eta^\nu \eta^{p/2} g(\eta\|\theta\|^2)$$

is given by

$$\begin{aligned} \delta_{g,\nu} &= \frac{E[\eta\theta \mid x,s]}{E[\eta \mid x,s]} \\ &= \frac{\iint \eta\theta c_n \eta^{(p+n)/2} s^{n/2-1} f(\eta\{\|x-\theta\|^2+s\})\eta^\nu \eta^{p/2} g(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta c_n \eta^{(p+n)/2} s^{n/2-1} f(\eta\{\|x-\theta\|^2+s\})\eta^\nu \eta^{p/2} g(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta} \\ &= \frac{\iint \theta \eta^{(2p+n)/2+\nu+1} f(\eta\{\|x-\theta\|^2+s\}) g(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2+\nu+1} f(\eta\{\|x-\theta\|^2+s\}) g(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}. \end{aligned} \tag{B.11}$$

By the change of variables $\theta_* = \theta/\sqrt{s}$ and $\eta_* = s\eta$, we have

$$\delta_{g,\nu} = \sqrt{s}\frac{\iint \theta_* \eta_*^{(2p+n)/2+\nu+1} f(\eta_*\{\|x/\sqrt{s}-\theta_*\|^2+1\}) g(\eta_*\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}\eta_*}{\iint \eta_*^{(2p+n)/2+\nu+1} f(\eta_*\{\|x/\sqrt{s}-\theta_*\|^2+1\}) g(\eta_*\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}\eta_*}.$$

Comparing $\delta_{g,\nu}$ with δ_π given by (B.10), we see that $\delta_{g,\nu}$ with $\nu = -1$ is

$$\delta_{g,-1} = \sqrt{s}\frac{M_2(z,g)}{M_1(z,g)} = \sqrt{s}\frac{zz^{\mathrm{T}} M_2(z,g)}{\|z\|^2 M_1(z,g)} = \frac{z^{\mathrm{T}} M_2(z,g)}{\|z\|^2 M_1(z,g)} x.$$

The second equality follows since $M_2(z,g)$ is proportional to z and the length of $M_2(z,g)$ is $z^{\mathrm{T}} M_2(z,g)/\|z\|$.

[Part 4] Since the quadratic loss function is strictly convex, the Bayes solution is unique, and hence admissibility within $\mathcal{D}_\psi$ follows.

Appendix C: Proof that $\delta_{g,\nu} \in \mathcal{D}_\psi$ needed in Remark 3.1

This is for Remark 3.1. The generalized Bayes estimator of θ with respect to the joint prior density $\eta^\nu \eta^{p/2}\bar{\pi}(\eta\|\theta\|^2)$ for any ν is a member of the class $\mathcal{D}_\psi$. Part 3 of Theorem 3.1 applies only to the special case of $\nu = -1$. Additionally the admissibility results of this section and of Section 4 apply only to this special case of $\nu = -1$ and imply neither admissibility or inadmissibility of generalized Bayes estimators if $\nu \neq -1$.

As in (B.11), the generalized Bayes estimator of θ with respect to $\eta^\nu \eta^{p/2} g(\eta\|\theta\|^2)$ is given by

$$\delta_{g,\nu}(x,s) = \frac{\iint \theta\eta^{(2p+n)/2+\nu+1} f(\eta\{\|x-\theta\|^2+s\})g(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2+\nu+1} f(\eta\{\|x-\theta\|^2+s\})g(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}.$$

The estimator $\delta_{g,\nu}(x,s)$ with $x = \gamma\Gamma x$ and $s = \gamma^2 s$ is

$$\delta_{g,\nu}(\gamma\Gamma x, \gamma^2 s) = \frac{\iint \theta\eta^{(2p+n)/2+\nu+1} f(\eta\{\|\gamma\Gamma x-\theta\|^2+\gamma^2 s\})g(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2+\nu+1} f(\eta\{\|\gamma\Gamma x-\theta\|^2+\gamma^2 s\})g(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}$$

and, by the change of variables $\theta = \gamma\Gamma\theta_*$ and $\eta_* = \gamma^2\eta$, is rewritten as

$$\begin{aligned}\delta_{g,\nu}(\gamma\Gamma x, \gamma^2 s) &= \gamma\Gamma\frac{\iint \theta_*\eta_*^{(p+n)/2+\nu+1} f(\eta_*\{\|x-\theta_*\|^2+s\})g(\eta_*\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}\eta_*}{\iint \eta_*^{(p+n)/2+\nu+1} f(\eta_*\{\|x-\theta_*\|^2+s\})g(\eta_*\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}\eta_*} \\ &= \gamma\Gamma\delta_{g,\nu}(x,s).\end{aligned}$$

Hence $\delta_{g,\nu} \in \mathcal{D}_\psi$.

Appendix D: Proof of Theorem 4.1

When $\pi(\lambda)$ on $\mathbb{R}_+$ (and hence $\bar{\pi}(\|\mu\|^2)$ on $\mathbb{R}^p$) is improper, that is

$$\int_{\mathbb{R}^p} \bar{\pi}(\|\mu\|^2)\mathrm{d}\mu = \int_0^\infty \pi(\lambda)\mathrm{d}\lambda = \infty,$$

the admissibility of δ_π within the class of equivariant estimators can be investigated through Blyth's (1951) method.

There are several versions of the Blyth method. For our purpose, Theorem 4.1, a version from Brown (1971) and Brown and Hwang (1982), is useful.

Suppose that $\delta_\pi \in \mathcal{D}_\psi$ is inadmissible within the class $\mathcal{D}_\psi$ and hence $\delta' \in \mathcal{D}_\psi$ satisfies $\tilde{R}(\lambda, \delta') \leq \tilde{R}(\lambda, \delta_\pi)$ for all λ with strict inequality for some λ. Let $\delta'' = (\delta_\pi + \delta')/2$. Clearly δ'' is also a member of $\mathcal{D}_\psi$. Then, using Jensen's inequality, we have

$$\begin{aligned}
\tilde{R}(\lambda, \delta'') &= E\left[\eta\|\delta'' - \theta\|^2\right] \\
&< (1/2)E\left[\eta\|\delta' - \theta\|^2\right] + (1/2)E\left[\eta\|\delta_\pi - \theta\|^2\right] \\
&= \frac{1}{2}\left\{\tilde{R}(\lambda, \delta') + \tilde{R}(\lambda, \delta_\pi)\right\} \\
&\leq \tilde{R}(\lambda, \delta_\pi),
\end{aligned}$$

for any λ. Since $\tilde{R}(\lambda, \delta'')$ and $\tilde{R}(\lambda, \delta_\pi)$ are both continuous functions of λ, there exists an $\epsilon > 0$ such that $\tilde{R}(\lambda, \delta'') < \tilde{R}(\lambda, \delta_\pi) - \epsilon$ for $0 \leq \lambda \leq 1$. Then

$$\begin{aligned}
\operatorname{diff} B(\delta_\pi, \delta_{\pi i}; \pi_i) &= \int_0^\infty \left\{\tilde{R}(\lambda, \delta_\pi) - \tilde{R}(\lambda, \delta_{\pi i})\right\} \pi_i(\lambda) \mathrm{d}\lambda \\
&\geq \int_0^\infty \left\{\tilde{R}(\lambda, \delta_\pi) - \tilde{R}(\lambda, \delta'')\right\} \pi_i(\lambda) \mathrm{d}\lambda \\
&\geq \int_0^1 \left\{\tilde{R}(\lambda, \delta_\pi) - \tilde{R}(\lambda, \delta'')\right\} \pi_1(\lambda) \mathrm{d}\lambda \\
&\geq \epsilon\gamma > 0,
\end{aligned}$$

which contradicts $\operatorname{diff} B(\delta_\pi, \delta_{\pi i}; \pi_i) \to 0$ as $i \to \infty$.

Appendix E: Preliminary results on π, π_i and f

E.1. *Preliminary results on π*

Here are some preliminary results on π needed in Section 4.

Lemma E.1. 1. *Under Assumptions A.1–A.3,*

$$\sup_{\lambda \in \mathbb{R}_+} \lambda \frac{|\pi'(\lambda)|}{\pi(\lambda)}$$

is bounded.

2. *Under Assumption A.2,* $\displaystyle\int_0^1 \frac{\pi(\lambda)}{\lambda^{1/2}} \mathrm{d}\lambda < \infty$.
3. *Under Assumption A.2 with* $\alpha > 0$, $\displaystyle\int_0^1 \frac{\pi(\lambda)}{\lambda} \mathrm{d}\lambda < \infty$.

4. *Under Assumption A.3.1,* $\displaystyle\int_1^\infty \frac{\pi(\lambda)}{\lambda}\mathrm{d}\lambda < \infty$.
5. *Under Assumption A.3,* $\displaystyle\int_1^\infty \frac{\pi(\lambda)}{\lambda^2}\mathrm{d}\lambda < \infty$.
6. *If* $\lim_{\lambda\to\infty} \lambda\pi'(\lambda)/\pi(\lambda) < -1$, $\displaystyle\int_1^\infty \pi(\lambda)\mathrm{d}\lambda < \infty$.
7. *Under Assumption A.3, there exist* $\epsilon \in (0,1)$ *and* $\lambda_* > \exp(1)$ *such that* $\pi(\lambda)/\{\log\lambda\}^{1-\epsilon}$ *for* $\lambda \geq \lambda_*$ *is bounded from above.*
8. *Under Assumption A.3.2,* $\displaystyle\int_1^\infty \frac{\pi(\lambda)}{\lambda}\kappa^2(\lambda)\mathrm{d}\lambda < \infty$.

Proof. [Part 1] This follows from Assumptions A.1–A.3 in a straightforward way.

[Part 2] We have

$$\int_0^1 \frac{\pi(\lambda)}{\lambda^{1/2}}\mathrm{d}\lambda \leq \sup_{\lambda\in[0,1]} \nu(\lambda)\int_0^1 \lambda^{1/2+\alpha-1}\mathrm{d}\lambda = \sup_{\lambda\in[0,1]} \nu(\lambda)\frac{1}{1/2+\alpha} < \infty. \tag{E.1}$$

[Part 3] As in (E.1) of Part 2, we have

$$\int_0^1 \frac{\pi(\lambda)}{\lambda}\mathrm{d}\lambda \leq \sup_{\lambda\in[0,1]} \nu(\lambda)\frac{1}{\alpha} < \infty.$$

[Part 4] By Assumption A.3.1, there exist $\epsilon > 0$ and $\lambda_1 > 0$ such that

$$\lambda\frac{\pi'(\lambda)}{\pi(\lambda)} \leq -\epsilon,$$

for all $\lambda \geq \lambda_1$ and hence we have

$$\int_{\lambda_1}^\lambda \frac{\pi'(s)}{\pi(s)}\mathrm{d}s \leq -\epsilon\int_{\lambda_1}^\lambda \frac{1}{s}\mathrm{d}s \Leftrightarrow \log\frac{\pi(\lambda)}{\pi(\lambda_1)} \leq -\epsilon\log\frac{\lambda}{\lambda_1} \tag{E.2}$$

for $\lambda \geq \lambda_1$, which implies that

$$\pi(\lambda) \leq \frac{\pi(\lambda_1)}{\lambda_1^{-\epsilon}}\lambda^{-\epsilon} \text{ for all } \lambda \geq \lambda_1. \tag{E.3}$$

Hence we have

$$\int_{\lambda_1}^\infty \frac{\pi(\lambda)}{\lambda}\mathrm{d}\lambda \leq \frac{\pi(\lambda_1)}{\lambda_1^{-\epsilon}}\int_{\lambda_1}^\infty \frac{\mathrm{d}\lambda}{\lambda^{1+\epsilon}} = \frac{\pi(\lambda_1)}{\epsilon} < \infty. \tag{E.4}$$

[Parts 5 and 6] The proof is omitted since it is similar to that of Part 4.

[Part 7] Under Assumptions A.3.1, by (E.3), $\pi(\lambda) \to 0$ as $\lambda \to \infty$. Under Assumption A.3.2.1, it is clear that $\pi(\lambda)$ is bounded. Under Assumption A.3.2.2, there exist $\epsilon \in (0,1)$ and $\lambda_2 > \exp(1)$ such that

$$\lambda \frac{\pi'(\lambda)}{\pi(\lambda)} \leq \frac{1-\epsilon}{\log \lambda} \tag{E.5}$$

for all $\lambda \geq \lambda_2$. As in (E.2) and (E.3), we have

$$\begin{aligned} &\int_{\lambda_2}^{\lambda} \frac{\pi'(s)}{\pi(s)} \mathrm{d}s \leq (1-\epsilon) \int_{\lambda_2}^{\lambda} \frac{\mathrm{d}s}{s \log s} \\ \Leftrightarrow \ &\log \frac{\pi(\lambda)}{\pi(\lambda_2)} \leq (1-\epsilon) \left\{ \log\log\lambda - \log\log\lambda_2 \right\} \end{aligned}$$

and hence

$$\pi(\lambda) \leq \pi(\lambda_2) \{\log \lambda\}^{1-\epsilon} \text{ for all } \lambda \geq \lambda_2, \tag{E.6}$$

which completes the proof.

[Part 8] Under Assumption A.3.2.1, there exists $\lambda_3 > 0$ such that $|\kappa(\lambda)|$ for $\lambda \geq \lambda_3$ is monotone decreasing. Then $\pi(\lambda)$ for $\lambda \geq \lambda_3$ is expressed by

$$\pi(\lambda) = \pi(\lambda_3) \exp\left(- \int_{\lambda_3}^{\lambda} \frac{|\kappa(s)|}{s} \mathrm{d}s \right)$$

and

$$\begin{aligned} \int_{\lambda_3}^{\infty} \frac{\pi(\lambda)}{\lambda} \kappa^2(\lambda) \mathrm{d}\lambda &= \pi(\lambda_3) \int_{\lambda_3}^{\infty} \frac{\{\kappa(\lambda)\}^2}{\lambda} \exp\left(- \int_{\lambda_3}^{\lambda} \frac{|\kappa(s)|}{s} \mathrm{d}s \right) \mathrm{d}\lambda \\ &\leq \pi(\lambda_3) |\kappa(\lambda_3)| \int_{\lambda_3}^{\infty} \frac{|\kappa(\lambda)|}{\lambda} \exp\left(- \int_{\lambda_3}^{\lambda} \frac{|\kappa(s)|}{s} \mathrm{d}s \right) \mathrm{d}\lambda \\ &= \pi(\lambda_3) |\kappa(\lambda_3)| \left[- \exp\left(- \int_{\lambda_3}^{\lambda} \frac{|\kappa(s)|}{s} \mathrm{d}s \right) \right]_{\lambda_3}^{\infty} \\ &\leq \pi(\lambda_3) |\kappa(\lambda_3)| < \infty. \end{aligned}$$

Under Assumption A.3.2.2, by (E.6), we have

$$\int_{\lambda_2}^{\infty} \frac{\pi(\lambda)}{\lambda} \kappa^2(\lambda) \mathrm{d}\lambda \leq \int_{\lambda_2}^{\infty} \frac{\pi(\lambda_2)}{\lambda \{\log \lambda\}^{1+\epsilon}} \mathrm{d}\lambda = \frac{\pi(\lambda_2)}{\epsilon \{\log \lambda_2\}^{\epsilon}} < \infty,$$

which completes the proof. □

Remark E.1. By Parts 2 and 6 of Lemma E.1, if $\lim_{\lambda \to \infty} \lambda \pi'(\lambda)/\pi(\lambda) < -1$, the prior $\pi(\lambda)$ with Assumption A.2 is proper and hence Part 4 of Theorem 3.1 can be applied. This is why we assume $\lim_{\lambda \to \infty} \lambda \pi'(\lambda)/\pi(\lambda) \geq -1$ as the asymptotic behavior in Assumption A.3.

E.2. Preliminary results on the sequence π_i

Here are some preliminary results on the sequence π_i. In particular, together with π satisfying Assumptions A.1, A.2 and A.3, $\pi_i = \pi h_i^2$ where $h_i(\lambda)$ is given by (4.5)

$$h_i(\lambda) = 1 - \frac{\log\log(\lambda + e)}{\log\log(\lambda + e + i)}, \quad e = \exp(1),$$

satisfies BL.1, BL.2 and BL.3 in of Theorem 4.1.

Lemma E.2. *1. $h_i(\lambda)$ is increasing in i for fixed λ, and decreasing in λ for fixed i. Further $\lim_{i\to\infty} h_i(\lambda) = 1$ for fixed $\lambda \geq 0$.*

2. For fixed i,

$$\lim_{\lambda\to\infty} \{(\lambda + e + i)\log(\lambda + e + i)\log\log(\lambda + e + i)\}h_i(\lambda) = i.$$

3. For $\lambda \geq 0$,

$$\sup_i |h_i'(\lambda)| \leq \frac{2}{(\lambda + e)\log(\lambda + e)\log\log(\lambda + e + 1)}.$$

4. $h_1(1) > 1/8$.

5. $\sup_{i,\lambda} |h_i'(\lambda)| < 5$.

6. Under Assumption A.2 on π,

$$\int_0^1 \pi_1(\lambda)\mathrm{d}\lambda > 0.$$

7. Under Assumptions A.1, A.2 and A.3 on π,

$$\int_0^\infty \lambda\pi(\lambda)\sup_i\{h_i'(\lambda)\}^2\mathrm{d}\lambda < \infty.$$

8. Under Assumptions A.1, A.2 and A.3 on π,

$$\int_0^\infty \pi_i(\lambda)\mathrm{d}\lambda < \infty, \text{ for fixed } i.$$

Proof. [Part 1] This part is straightforward given the form of $h_i(\lambda)$.

[Part 2] This follows from the expression,

$$h_i(\lambda) = \frac{1}{\log\log(\lambda + e + i)}\log\frac{\log(\lambda + e + i)}{\log(\lambda + e)}$$

$$= \frac{\log(\{\lambda+e+i\}/\{\lambda+e\})}{\log(\lambda+e+i)\log\log(\lambda+e+i)}\zeta\left(\frac{\log(\{\lambda+e+i\}/\{\lambda+e\})}{\log(\lambda+e+i)}\right)$$
$$= \frac{i}{(\lambda+e+i)\log(\lambda+e+i)\log\log(\lambda+e+i)}$$
$$\times\zeta\left(\frac{i}{\lambda+e+i}\right)\zeta\left(\frac{\log(\{\lambda+e+i\}/\{\lambda+e\})}{\log(\lambda+e+i)}\right),$$

where $\zeta(x) = -\log(1-x)/x$ which satisfies $\lim_{x\to 0+}\zeta(x) = 1$.

[Part 3] The derivative is

$$h_i'(\lambda) = -\frac{1}{(\lambda+e)\log(\lambda+e)\log\log(\lambda+e+i)} + \frac{\log\log(\lambda+e)}{(\lambda+e+i)\log(\lambda+e+i)\{\log\log(\lambda+e+i)\}^2}.$$

Hence we have

$$|h_i'(\lambda)| \le \left|\frac{1}{(\lambda+e)\log(\lambda+e)\log\log(\lambda+e+i)}\right| + \left|\frac{\log\log(\lambda+e)}{(\lambda+e+i)\log(\lambda+e+i)\{\log\log(\lambda+e+i)\}^2}\right|$$
$$\le \frac{2}{(\lambda+e)\log(\lambda+e)\log\log(\lambda+e+1)}$$

which does not depend on i.

[Part 4] At $\lambda = 1$, $h_1(\lambda)$ is

$$h_1(1) = 1 - \frac{\log\log(1+e)}{\log\log(2+e)} = 1 - \frac{\int_e^{1+e} 1/(\lambda\log\lambda)\mathrm{d}\lambda}{\int_e^{2+e} 1/(\lambda\log\lambda)\mathrm{d}\lambda} = \frac{\int_{1+e}^{2+e} 1/(\lambda\log\lambda)\mathrm{d}\lambda}{\int_e^{2+e} 1/(\lambda\log\lambda)\mathrm{d}\lambda},$$

which is greater than

$$\frac{1/\{(2+e)\log(2+e)\}}{2(1/e)} > \frac{1}{2}\frac{e}{2+e}\frac{1}{\log e^2} > \frac{1}{8}.$$

[Part 5] The upper bound of $\sup_i |h_i'(\lambda)|$, derived in Part 3, is decreasing in λ and hence

$$\sup_i |h_i'(\lambda)| \le \sup_i |h_i'(\lambda)|\big|_{\lambda=0} = \frac{2}{e\log\log(e+1)} \le \frac{1}{\log\log(e+1)}.$$

Further we have

$$\log\log(e+1) = \int_e^{e+1}\frac{\mathrm{d}s}{s\log s} > \frac{\log(e+1)-\log(e)}{\log(e+1)} = 1 - \frac{1}{\log(e+1)},$$

$$\log(e+1) = \log(e) + \log\frac{e+1}{e} = 1 - \log\left(1 - \frac{1}{e+1}\right) > 1 + \frac{1}{e+1},$$

and hence $\sup_{\lambda,i}|h_i'(\lambda)| \leq e+2 < 5$.

[Part 6] By Parts 1 and 4, $h_1^2(\lambda) \geq 1/64$ for $\lambda \in [0,1]$. By Assumption A.2 on π, there exists $\lambda_1 \in (0,1)$ such that $\pi(\lambda) \geq \lambda^\alpha\{\nu(0)/2\}$ for $\lambda \in [0,\lambda_1]$. Then

$$\int_0^1 \pi(\lambda)h_1^2(\lambda)\mathrm{d}\lambda \geq \frac{\nu(0)}{2}\frac{1}{64}\int_0^{\lambda_1}\lambda^\alpha \mathrm{d}\lambda = \frac{\nu(0)\lambda_1^{\alpha+1}}{128(\alpha+1)} > 0.$$

[Part 7] As in Part 7 of Lemma E.1, there exist $\epsilon \in (0,1)$ and $\lambda_2 > \exp(1)$ such that

$$\pi(\lambda) \leq \pi(\lambda_2)\{\log\lambda\}^{1-\epsilon} \text{ for all } \lambda \geq \lambda_2. \tag{E.7}$$

Then, by Part 5 and (E.7), we have

$$\begin{aligned}
&\int_0^\infty \lambda\pi(\lambda)\sup_i\{h_i'(\lambda)\}^2\mathrm{d}\lambda \\
&\leq 25\int_0^{\lambda_2}\lambda\pi(\lambda)\mathrm{d}\lambda + \int_{\lambda_2}^\infty \lambda\pi(\lambda)\sup_i\{h_i'(\lambda)\}^2\mathrm{d}\lambda \\
&\leq 25\int_0^{\lambda_2}\lambda\pi(\lambda)\mathrm{d}\lambda + \pi(\lambda_2)\int_{\lambda_2}^\infty \frac{4(\lambda+e)\log(\lambda+e)\mathrm{d}\lambda}{\{(\lambda+e)\log(\lambda+e)\log\log(\lambda+e)\}^2} \\
&= 25\int_0^{\lambda_2}\lambda\pi(\lambda)\mathrm{d}\lambda + \frac{4\pi(\lambda_2)}{\log\log(\lambda_2+e)},
\end{aligned}$$

where $\int_0^{\lambda_2}\lambda\pi(\lambda)\mathrm{d}\lambda$ in the first term of the right-hand side is bounded by Part 2 of Lemma E.1.

[Part 8] The proof is omitted since it is similar to the proof of Part 7. □

E.3. Preliminary results on the underlying density f

Recall BL.1–BL.3 are in terms of π_i. For BL.4, note that $\mathrm{diff}\,B(\delta_\pi, \delta_{\pi i}; \pi_i)$ given by (4.2) is a functional of f as well as π and π_i. Some additional assumptions on f, Assumptions F.1–F.3, as well as (1.7), are required. Here are some preliminary results on the underlying density f.

Lemma E.3. *Let Assumptions F.1–F.3 hold.*

1. Also assume

$$\limsup_{t\to\infty} t\frac{f'(t)}{f(t)} < -\frac{p+n}{2} - 2 - j \tag{E.8}$$

for $j \geq 0$ (hence $j = 0$ for Assumption F.3.1 and $j = 1$ for Assumption F.3.2).

1.A Then there exist $\epsilon \in (0,1)$ and $t_* > 1$ such that

$$\begin{aligned} f(t) &\leq \frac{f(t_*)}{t_*^{-(p+n)/2-2-j-\epsilon}} t^{-(p+n)/2-2-j-\epsilon}, \\ F(t) &\leq \frac{tf(t)}{(p+n)+2+2j+2\epsilon}, \end{aligned} \tag{E.9}$$

for all $t \geq t_*$, where

$$F(t) = \frac{1}{2}\int_t^\infty f(s)\mathrm{d}s.$$

1.B

$$\int_0^\infty t^{(p+n)/2-1+j}\left\{\frac{F(t)}{f(t)}\right\}^2 f(t)\mathrm{d}t < \infty.$$

2. Assume Assumption F.3.2. Also assume $p \geq 3$. Let

$$\begin{aligned} \tilde{\mathcal{F}}(t) &= t^{1/2}F(t)/f(t), \\ f_\star(t) &= \int_0^\infty \eta^{n/2-1}\tilde{\mathcal{F}}^2(t+\eta)f(t+\eta)\mathrm{d}\eta. \end{aligned}$$

Then there exists $\mathcal{Q}_f > 0$ such that

$$\int_{\mathbb{R}^p} \frac{1}{\|y\|^2} f_\star(\|y-\mu\|^2)\mathrm{d}y \leq \mathcal{Q}_f \min(1, 1/\|\mu\|^2). \tag{E.10}$$

Proof. [Part 1.A] By (E.8), there exist $t_* > 1$ and $\epsilon \in (0,1)$ such that

$$t\frac{f'(t)}{f(t)} \leq -\frac{p+n}{2} - 2 - \epsilon - j \tag{E.11}$$

for all $t \geq t_*$. Then, by (E.11), we have

$$\int_{t_*}^t \frac{f'(s)}{f(s)}\mathrm{d}s \leq \left(-\frac{p+n}{2} - 2 - j - \epsilon\right)\int_{t_*}^t \frac{\mathrm{d}s}{s} \quad \text{for } t \geq t_*,$$

$$
\begin{aligned}
&\Leftrightarrow \log \frac{f(t)}{f(t_*)} \leq \left(-\frac{p+n}{2} - 2 - j - \epsilon\right) \log \frac{t}{t_*} \quad \text{for } t \geq t_*, \\
&\Leftrightarrow f(t) \leq \frac{f(t_*)}{t_*^{-(p+n)/2-2-j-\epsilon}} t^{-(p+n)/2-2-j-\epsilon} \quad \text{for } t \geq t_*.
\end{aligned} \tag{E.12}
$$

Further, by (E.11), we have

$$
tf'(t) \leq -\left(\frac{p+n}{2} + 2 + j + \epsilon\right) f(t),
$$

for all $t \geq t_*$, and hence

$$
\int_t^\infty sf'(s)\mathrm{d}s \leq -\left(\frac{p+n}{2} + 2 + j + \epsilon\right) \int_t^\infty f(s)\mathrm{d}s. \tag{E.13}
$$

By an integration by parts, the left-hand side is rewritten as

$$
\int_t^\infty sf'(s)\mathrm{d}s = [sf(s)]_t^\infty - \int_t^\infty f(s)\mathrm{d}s = -tf(t) - 2F(t),
$$

where the second equality follows from $[sf(s)]_t^\infty = -tf(t)$ by (E.12). Then the inequality (E.13) is equivalent to

$$
\begin{aligned}
-tf(t) - 2F(t) &\leq -2\left(\frac{p+n}{2} + 2 + j + \epsilon\right) F(t), \\
\Leftrightarrow \quad \frac{F(t)}{f(t)} &\leq \frac{t}{(p+n) + 2 + 2j + 2\epsilon},
\end{aligned} \tag{E.14}
$$

for all $t \geq t_*$. Hence Part 1.A follows from (E.12) and (E.14).

[Part 1.B] By Assumption F.1, we have

$$
\int_0^1 f(s)\mathrm{d}s < \infty. \tag{E.15}
$$

Also the integrability given by (1.7),

$$
\int_1^\infty s^{(p+n)/2-1} f(s)\mathrm{d}s < \infty,
$$

implies

$$
\int_1^\infty f(s)\mathrm{d}s < \infty. \tag{E.16}
$$

By (E.15) and (E.16), we have

$$F(0) = \frac{1}{2}\int_0^\infty f(s)\mathrm{d}s < \infty. \tag{E.17}$$

Note $0 < f(0) < \infty$ by Assumption F.1. Also by (E.14) and (E.17), it follows that there exists $C_f > 0$ such that

$$\frac{F(t)}{f(t)} \leq C_f \max(t, t_*), \quad \forall t \geq 0. \tag{E.18}$$

By (E.18), for $t \in [0, 1]$, we have

$$t^j \left\{\frac{F(t)}{f(t)}\right\}^2 f(t) \leq C_f^2 t_*^2 \max_{t\in[0,1]} f(t) \tag{E.19}$$

and hence

$$\int_0^1 t^{(p+n)/2-1+j} \left\{\frac{F(t)}{f(t)}\right\}^2 f(t)\mathrm{d}t \leq \frac{2C_f^2 t_*^2}{p+n} \max_{t\in[0,1]} f(t) < \infty. \tag{E.20}$$

By (E.12) and (E.18), we have

$$t^j \left\{\frac{F(t)}{f(t)}\right\}^2 f(t) \leq \frac{f(t_*)C_f^2}{t_*^{-(p+n)/2-2-j-\epsilon}} t^{-(p+n)/2-\epsilon} \tag{E.21}$$

for $t \geq t_*$ and hence

$$\begin{aligned}\int_{t_*}^\infty t^{(p+n)/2-1+j} \left\{\frac{F(t)}{f(t)}\right\}^2 f(t)\mathrm{d}t &\leq \frac{f(t_*)C_f^2}{t_*^{-(p+n)/2-2-j-\epsilon}} \int_{t_*}^\infty t^{-1-\epsilon}\mathrm{d}t \\ &= \frac{f(t_*)C_f^2}{\epsilon t_*^{-(p+n)/2-2-j}} < \infty.\end{aligned} \tag{E.22}$$

Combining (E.20) and (E.22), completes the proof of Part 1.B.

[Part 2] Note, by Part 1 of this lemma with $j = 1$,

$$\begin{aligned}\int_0^\infty t^{(p+n)/2-1+1} \left\{\frac{F(t)}{f(t)}\right\}^2 f(t)\mathrm{d}t &= \int_0^\infty t^{(p+n)/2-1} \tilde{\mathcal{F}}^2(t) f(t)\mathrm{d}t \\ &< \infty.\end{aligned} \tag{E.23}$$

To prove Part 2, it suffices to show that, for $\|\mu\| = 0$,

$$\int_{\mathbb{R}^p} \frac{1}{\|y\|^2} f_\star(\|y\|^2)\mathrm{d}y < \infty \tag{E.24}$$

and also that there exist $a > 0$ and $b > 0$ such that

$$\|\mu\|^2 \int_{\mathbb{R}^p} \frac{1}{\|y\|^2} f_\star(\|y-\mu\|^2)\mathrm{d}y < b \tag{E.25}$$

for all $\|\mu\|^2 \geq a$.

[Bound in (E.24)] Note $f_\star(0)$ is decomposed as

$$\begin{aligned} f_\star(0) &= \int_0^\infty \eta^{n/2-1}\tilde{\mathcal{F}}^2(\eta)f(\eta)\mathrm{d}\eta \\ &= \int_0^1 \eta^{n/2-1}\tilde{\mathcal{F}}^2(\eta)f(\eta)\mathrm{d}\eta + \int_1^{t_*} \eta^{n/2-1}\tilde{\mathcal{F}}^2(\eta)f(\eta)\mathrm{d}\eta \\ &\quad + \int_{t_*}^\infty \eta^{n/2-1}\tilde{\mathcal{F}}^2(\eta)f(\eta)\mathrm{d}\eta, \end{aligned} \tag{E.26}$$

where t_* is from (E.11). The first and third terms both are integrable since, by (E.19),

$$\begin{aligned} \int_0^1 \eta^{n/2-1}\tilde{\mathcal{F}}^2(\eta)f(\eta)\mathrm{d}\eta &\leq C_f^2 t_*^2 \max_{\eta\in[0,1]} f(\eta) \int_0^1 \eta^{n/2-1}\mathrm{d}\eta \\ &= C_f^2 t_*^2 \frac{2}{n} \max_{\eta\in[0,1]} f(\eta), \end{aligned} \tag{E.27}$$

and by (E.21),

$$\begin{aligned} \int_{t_*}^\infty \eta^{n/2-1}\tilde{\mathcal{F}}^2(\eta)f(\eta)\mathrm{d}\eta &\leq \frac{f(t_*)C_f^2}{t_*^{-(p+n)/2-3-\epsilon}} \int_{t_*}^\infty \eta^{n/2-1-(p+n)/2-\epsilon}\mathrm{d}\eta \\ &= \frac{f(t_*)C_f^2}{t_*^{-(p+n)/2-3-\epsilon}} \frac{t_*^{-p/2-\epsilon}}{p/2+\epsilon}. \end{aligned} \tag{E.28}$$

By (E.26), (E.27) and (E.28), we have $f_\star(0) < \infty$. Then, by continuity of $f_\star$, it follows that

$$\sup_{t\in[0,1]} f_\star(t) < \infty. \tag{E.29}$$

Further the integrability of $\int_{\mathbb{R}^p} f_\star(\|y\|^2)\mathrm{d}y$ follows since

$$\begin{aligned} \int_{\mathbb{R}^p} f_\star(\|y\|^2)\mathrm{d}y &= \int_{\mathbb{R}^p}\int_0^\infty \eta^{n/2-1}\tilde{\mathcal{F}}^2(\|y\|^2+\eta)f(\|y\|^2+\eta)\mathrm{d}\eta\mathrm{d}y \\ &= \frac{1}{c_n}\int_{\mathbb{R}^{p+n}} \tilde{\mathcal{F}}^2(\|q\|^2)f(\|q\|^2)\mathrm{d}q \\ &= \frac{c_{p+n}}{c_n}\int_0^\infty t^{(p+n)/2-1}\tilde{\mathcal{F}}^2(t)f(t)\mathrm{d}t \\ &< \infty \text{ (by (E.23))}. \end{aligned} \tag{E.30}$$

Then, by (E.29) and (E.30), we have

$$\begin{aligned}\int_{\mathbb{R}^p} \frac{f_\star(\|y\|^2)}{\|y\|^2}\mathrm{d}y &\leq \sup_{\|y\|\leq 1} f_\star(\|y\|^2) \int_{\|y\|\leq 1} \frac{\mathrm{d}y}{\|y\|^2} + \int_{\|y\|\geq 1} f_\star(\|y\|^2)\mathrm{d}y \\ &\leq \frac{2c_p}{p-2} \sup_{\|y\|\leq 1} f_\star(\|y\|^2) + \int_{\mathbb{R}^p} f_\star(\|y\|^2)\mathrm{d}y \\ &< \infty.\end{aligned}$$

Hence the bound in (E.24) is established.

[Bound in (E.25)] Let $\|\mu\|^2 > 2t_*$ where t_* is from (E.11). Under the decomposition of the integral region,

$$\begin{aligned}\mathbb{R}^p &= \left\{y : \|y-\mu\|^2 \leq \|\mu\|^2/2\right\} \\ &\quad \cup \left\{y : \|y-\mu\|^2 \geq \|\mu\|^2/2 \text{ and } 0 \leq \|y\|^2 \leq \|\mu\|^2\right\} \\ &\quad \cup \left\{y : \|y-\mu\|^2 \geq \|\mu\|^2/2 \text{ and } \|y\|^2 \geq \|\mu\|^2\right\} \\ &= \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3,\end{aligned}$$

we have

$$\begin{aligned}&\|\mu\|^2 \int_{\mathbb{R}^p} \frac{f_\star(\|y-\mu\|^2)}{\|y\|^2}\mathrm{d}y \\ &= \|\mu\|^2 \left(\int_{\mathcal{R}_1} + \int_{\mathcal{R}_2} + \int_{\mathcal{R}_3}\right) \frac{f_\star(\|y-\mu\|^2)}{\|y\|^2}\mathrm{d}y.\end{aligned}$$

For the region $\mathcal{R}_1$, $\|y-\mu\|^2 \leq \|\mu\|^2/2$ implies $\|y\|^2 \geq \|\mu\|^2/2$ and hence

$$\begin{aligned}\|\mu\|^2 \int_{\mathcal{R}_1} \frac{f_\star(\|y-\mu\|^2)}{\|y\|^2}\mathrm{d}y &\leq 2\int_{\mathcal{R}_1} f_\star(\|y-\mu\|^2)\mathrm{d}y \\ &\leq 2\int_{\mathbb{R}^p} f_\star(\|y-\mu\|^2)\mathrm{d}y,\end{aligned}$$

which is bounded by (E.30). Similarly, for $\mathcal{R}_1$, since $\|y\|^2 \geq \|\mu\|^2$, we have

$$\|\mu\|^2 \int_{\mathcal{R}_3} \frac{f_\star(\|y-\mu\|^2)}{\|y\|^2}\mathrm{d}y \leq \int_{\mathbb{R}^p} f_\star(\|y-\mu\|^2)\mathrm{d}y < \infty.$$

For the region

$$\mathcal{R}_2 = \left\{y : \|y-\mu\|^2 \geq \|\mu\|^2/2 \text{ and } 0 \leq \|y\|^2 \leq \|\mu\|^2\right\},$$

we have

$$\mathcal{R}_2 \subset \left\{ y : \|y-\mu\|^2 \geq \|\mu\|^2/2 \right\}, \quad \mathcal{R}_2 \subset \left\{ y : 0 \leq \|y\|^2 \leq \|\mu\|^2 \right\}.$$

Hence

$$\begin{aligned} &\|\mu\|^2 \int_{\mathcal{R}_2} \frac{f_\star(\|y-\mu\|^2)}{\|y\|^2} \mathrm{d}y \\ &\leq \sup_{y:\|y-\mu\|^2 \geq \|\mu\|^2/2} f_\star(\|y-\mu\|^2) \int_{y:0\leq\|y\|^2\leq\|\mu\|^2} \frac{\|\mu\|^2}{\|y\|^2} \mathrm{d}y, \end{aligned} \tag{E.31}$$

where

$$\int_{\{y:0\leq\|y\|^2\leq\|\mu\|^2\}} \frac{\|\mu\|^2}{\|y\|^2} \mathrm{d}y = \|\mu\|^2 c_p \int_0^{\|\mu\|^2} r^{p/2-2} \mathrm{d}r = \frac{2c_p}{p-2} \|\mu\|^p. \tag{E.32}$$

Recall the assumption $\|\mu\|^2 > 2t_*$ and hence note

$$\|y-\mu\|^2 \geq \|\mu\|^2/2 > t_*. \tag{E.33}$$

By (E.21), the integrand of $f_\star$, for $t \geq t_*$, is bounded as

$$\tilde{\mathcal{F}}^2(t) f(t) \leq \frac{f(t_*) C_f^2}{t_*^{-(p+n)/2-3-\epsilon}} t^{-(p+n)/2-\epsilon}$$

and hence $f_\star(t)$ for $t \geq t_*$ is bounded as

$$\begin{aligned} f_\star(t) &= \int_0^\infty \eta^{n/2-1} \tilde{\mathcal{F}}^2(t+\eta) f(t+\eta) \mathrm{d}\eta \\ &\leq \frac{f(t_*) C_f^2}{t_*^{-(p+n)/2-3-\epsilon}} \int_0^\infty \eta^{n/2-1} (t+\eta)^{-(p+n)/2-\epsilon} \mathrm{d}\eta \\ &= \tilde{C}_f t^{-p/2-\epsilon}, \end{aligned} \tag{E.34}$$

where

$$\tilde{C}_f = f(t_*) C_f^2 t_*^{(p+n)/2+3+\epsilon} B(p/2+\epsilon, n/2).$$

Then, for any $y \in \{y : \|y-\mu\|^2 \geq \|\mu\|^2/2\}$ with $\|\mu\|^2 > 2t_*$,

$$f_\star(\|y-\mu\|^2) \leq \tilde{C}_f \{\|y-\mu\|^2\}^{-p/2-\epsilon} \leq \frac{\tilde{C}_f}{2^{p/2+\epsilon}} \|\mu\|^{-p-2\epsilon}, \tag{E.35}$$

where the first and second inequalities follow from (E.34) and (E.33), respectively. By (E.31), (E.32), and (E.35),

$$\|\mu\|^2 \int_{\mathcal{R}_2} \frac{f_\star(\|y-\mu\|^2)}{\|y\|^2} \mathrm{d}y \leq \frac{1}{\|\mu\|^{2\epsilon}} \left\{ \tilde{C}_f 2^{p/2+\epsilon} \frac{2c_p}{p-2} \right\}$$

which is bounded under the assumption $\|\mu\|^2 > 2t_*$. □

Appendix F: Proof of satisfaction of A.1–A.3 by (4.4)

A typical prior $\pi(\lambda)$ satisfying Assumptions A.1–A.3, corresponding to a generalized Strawderman's (1971) prior, is given by (4.4)

$$\pi(\lambda;\alpha,\beta,b)=c_p\lambda^{p/2-1}\int_b^\infty\frac{1}{(2\pi\xi)^{p/2}}\exp\left(-\frac{\lambda}{2\xi}\right)(\xi-b)^\alpha(1+\xi)^\beta\mathrm{d}\xi.$$

Lemma F.1. *Let* $\pi(\lambda)=c_p\lambda^{p/2-1}\bar{\pi}(\lambda)$ *where*

$$\bar{\pi}(\lambda)=\int_b^\infty\frac{1}{(2\pi\xi)^{p/2}}\exp\left(-\frac{\lambda}{2\xi}\right)(\xi-b)^\alpha(1+\xi)^\beta\mathrm{d}\xi. \tag{F.1}$$

Then Assumptions A.1–A.3 are satisfied when $\{b>0,\ -1\le\alpha+\beta\le 0,\ \text{and } \alpha>-1\}$ *or* $\{b=0,\ -1\le\alpha+\beta\le 0,\ \text{and } \alpha>-1/2\}$.

Note the integrability of (F.1) under $(b,b+\epsilon)$ follows for $b=0$ or $b>0$ as well as $\alpha>-1$. Also the integrability of (F.1) under $(b+\epsilon,\infty)$ follows when $\alpha+\beta-p/2<-1$. Further note if $\alpha+\beta<-1$ as well as $\alpha>-1$, $\int_b^\infty(\xi-b)^\alpha(1+\xi)^\beta\mathrm{d}\xi<\infty$ and hence the prior on λ is proper.

Proof. Clearly $\pi(\lambda)$ is differentiable as

$$\bar{\pi}'(\lambda)=-\frac{1}{2}\int_b^\infty\frac{(2\pi)^{-p/2}}{\xi^{p/2+1}}\exp\left(-\frac{\lambda}{2\xi}\right)(\xi-b)^\alpha(1+\xi)^\beta\mathrm{d}\xi. \tag{F.2}$$

and hence Assumption A.1 is satisfied.

[Assumption A.2 with $\alpha>-1/2$ and $b=0$] When $b=0$, by a Tauberian theorem (see, e.g., Theorem 4 of Section 5 of Chapter 13 in Feller (1971)), we have, in (F.1) and (F.2),

$$\begin{aligned}\lim_{\lambda\to 0}\frac{(2\pi)^{p/2}\bar{\pi}(\lambda)}{(\lambda/2)^{-(p/2-\alpha-1)}\Gamma(p/2-\alpha-1)}&=1\\ \lim_{\lambda\to 0}\frac{-2(2\pi)^{p/2}\bar{\pi}'(\lambda)}{(\lambda/2)^{-(p/2-\alpha)}\Gamma(p/2-\alpha)}&=1,\end{aligned} \tag{F.3}$$

which implies that

$$\lim_{\lambda\to 0}\lambda\frac{\bar{\pi}'(\lambda)}{\bar{\pi}(\lambda)}=-\frac{p}{2}+\alpha+1. \tag{F.4}$$

Recall $\pi(\lambda)=c_p\lambda^{p/2-1}\bar{\pi}(\lambda)$ and let $\nu(\lambda)=\lambda^{-\alpha}\pi(\lambda)=c_p\lambda^{p/2-1-\alpha}\bar{\pi}(\lambda)$. Then we have

$$\nu(0)=c_p2^{p/2-\alpha-1}\Gamma(p/2-\alpha-1)(2\pi)^{-p/2}$$

by (F.3) and

$$\lim_{\lambda\to 0}\lambda\frac{\nu'(\lambda)}{\nu(\lambda)}=\lim_{\lambda\to 0}\lambda\nu'(\lambda)=0$$

by (F.4).

[Assumption A.2 with $\alpha>-1$ and $b>0$] When $\alpha>-1$ and $b>0$, it follows that $0<\bar{\pi}(0)<\infty$ and $0<|\bar{\pi}'(0)|<\infty$. For this case take $\nu(\lambda)=c_p\bar{\pi}(\lambda)$. Then $\pi(\lambda)=\lambda^{p/2-1}\nu(\lambda)$, where $p/2-1>-1/2$ and $\nu(\lambda)$ satisfies

$$0<\nu(0)<\infty \text{ and } \lim_{\lambda\to 0}\lambda\frac{\nu'(\lambda)}{\nu(\lambda)}=0.$$

[Assumption A.3] Again, by a Tauberian theorem, we have, in (F.1) and (F.2),

$$\begin{aligned}\lim_{\lambda\to\infty}\frac{(2\pi)^{p/2}\bar{\pi}(\lambda)}{(\lambda/2)^{-(p/2-\alpha-\beta-1)}\Gamma(p/2-\alpha-\beta-1)}&=1\\ \lim_{\lambda\to\infty}\frac{-2(2\pi)^{p/2}\bar{\pi}'(\lambda)}{(\lambda/2)^{-(p/2-\alpha-\beta)}\Gamma(p/2-\alpha-\beta)}&=1,\end{aligned}\tag{F.5}$$

which implies that

$$\lim_{\lambda\to\infty}\lambda\frac{\bar{\pi}'(\lambda)}{\bar{\pi}(\lambda)}=-\frac{p}{2}+\alpha+\beta+1 \text{ and } \lim_{\lambda\to\infty}\lambda\frac{\pi'(\lambda)}{\pi(\lambda)}=\alpha+\beta.\tag{F.6}$$

Hence when $-1\le\alpha+\beta<0$, Assumption A.3.1 is satisfied.

When $\alpha+\beta=0$, note

$$(\xi-b)^{\alpha}(1+\xi)^{\beta}=\left(1-\frac{1+b}{1+\xi}\right)^{\alpha}$$

and

$$\lim_{\xi\to\infty}\xi\left\{\left(1-\frac{1+b}{1+\xi}\right)^{\alpha}-1\right\}=-\alpha(1+b).$$

Then

$$\lim_{\lambda\to\infty}\frac{(2\pi)^{p/2}\bar{\pi}(\lambda)-(\lambda/2)^{-(p/2-1)}\Gamma(p/2-1)}{(\lambda/2)^{-p/2}\Gamma(p/2)}=-\alpha(1+b)$$

and

$$\lim_{\lambda\to\infty}\frac{-2(2\pi)^{p/2}\bar{\pi}'(\lambda)-(\lambda/2)^{-p/2}\Gamma(p/2)}{(\lambda/2)^{-p/2-1}\Gamma(p/2+1)}=-\alpha(1+b).$$

Hence we get

$$\lim_{\lambda\to\infty}\lambda\left(\lambda\frac{\bar\pi'(\lambda)}{\bar\pi(\lambda)}+\frac{p}{2}-1\right)=\lim_{\lambda\to\infty}\lambda^2\frac{\pi'(\lambda)}{\pi(\lambda)}=2(p-2)\alpha(b+1),$$

which satisfies Assumption A.3.2.2. Hence Assumption A.3 is satisfied by $-1\le\alpha+\beta\le 0$. $\square$

Appendix G: Proof of Theorem 4.2 for Case I

In Case I, we assume Assumptions A.1, A.2 and A.3.1 on π and Assumptions F.1, F.2 and F.3.1 on f.

As in (4.10), $\overline{\mathrm{diff}\,B}(z;\delta_\pi,\delta_{\pi i};\pi_i)$ is re-expressed as

$$\begin{aligned}&\overline{\mathrm{diff}\,B}(z;\delta_\pi,\delta_{\pi i};\pi_i)\\&=\left\|\frac{\iint\eta^{(2p+n)/2-1}F(\eta\{\|z-\theta\|^2+1\})\nabla_\theta\bar\pi(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}{\iint\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar\pi(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}\right.\\&\qquad\left.-\frac{\iint\eta^{(2p+n)/2-1}F(\eta\{\|z-\theta\|^2+1\})\nabla_\theta\bar\pi_i(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}{\iint\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar\pi_i(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}\right\|^2\\&\qquad\times\iint\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar\pi_i(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta.\end{aligned}\tag{G.1}$$

By the decomposition

$$\begin{aligned}\nabla_\theta\bar\pi_i(\eta\|\theta\|^2)&=\nabla_\theta\left\{\bar\pi(\eta\|\theta\|^2)h_i^2(\eta\|\theta\|^2)\right\}\\&=\{\nabla_\theta\bar\pi(\eta\|\theta\|^2)\}h_i^2(\eta\|\theta\|^2)+\bar\pi(\eta\|\theta\|^2)\{\nabla_\theta h_i^2(\eta\|\theta\|^2)\},\end{aligned}$$

we have

$$\begin{aligned}&\overline{\mathrm{diff}\,B}(z;\delta_\pi,\delta_{\pi i};\pi_i)\\&=c_n\left\|\frac{\iint\eta^{(2p+n)/2-1}F(\circ)\nabla_\theta\bar\pi(\bullet)\mathrm{d}\theta\mathrm{d}\eta}{\iint\eta^{(2p+n)/2}f(\circ)\bar\pi(\bullet)\mathrm{d}\theta\mathrm{d}\eta}-\frac{\iint\eta^{(2p+n)/2-1}F(\circ)\nabla_\theta\bar\pi(\bullet)h_i^2(\bullet)\mathrm{d}\theta\mathrm{d}\eta}{\iint\eta^{(2p+n)/2}f(\circ)\bar\pi_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta}\right.\\&\qquad\left.-\frac{\iint\eta^{(2p+n)/2-1}F(\circ)\bar\pi(\bullet)\nabla_\theta h_i^2(\bullet)\mathrm{d}\theta\mathrm{d}\eta}{\iint\eta^{(2p+n)/2}f(\circ)\bar\pi_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta}\right\|^2\iint\eta^{(2p+n)/2}f(\circ)\bar\pi_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta,\end{aligned}$$

where, for notational convenience and to control the size of expressions,

$$\bullet=\eta\|\theta\|^2,\quad\circ=\eta(\|z-\theta\|^2+1).$$

Further, by the triangle inequality and the fact $h_i^2 \leq 1$, we have

$$\overline{\mathrm{diff}B}(z;\delta_\pi,\delta_{\pi i};\pi_i) \leq 2c_n(\Delta_{1i}+\Delta_{2i}),$$

where

$$\Delta_{1i} = \frac{\left\|\iint \eta^{(2p+n)/2-1}F(\circ)\bar{\pi}(\bullet)\nabla_\theta h_i^2(\bullet)\mathrm{d}\theta\mathrm{d}\eta\right\|^2}{\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta}, \tag{G.2}$$

$$\begin{aligned}\Delta_{2i} &= \frac{\left\|\iint \eta^{(2p+n)/2-1}F(\circ)\nabla_\theta\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta\right\|^2}{\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta} \\ &\quad + \frac{\left\|\iint \eta^{(2p+n)/2-1}F(\circ)\nabla_\theta\bar{\pi}(\bullet)h_i^2(\bullet)\mathrm{d}\theta\mathrm{d}\eta\right\|^2}{\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta}.\end{aligned} \tag{G.3}$$

G.1. *The integrability of* $\sup_i \Delta_{1i}$

Note

$$\begin{aligned}\nabla_\theta h_i^2(\eta\|\theta\|^2) &= 2h_i(\eta\|\theta\|^2)\nabla_\theta h_i(\eta\|\theta\|^2), \\ \|\nabla_\theta h_i(\eta\|\theta\|^2)\|^2 &= 4\eta^2\|\theta\|^2\{h_i'(\eta\|\theta\|^2)\}^2.\end{aligned}$$

Then, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\Delta_{1i} &= \frac{\| \iint \eta^{(2p+n)/2-1}F(\circ)\bar{\pi}(\bullet)\{2h_i(\bullet)\nabla_\theta h_i(\bullet)\}\mathrm{d}\theta\mathrm{d}\eta\|^2}{\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta} \\ &\leq 4\iint \eta^{(2p+n)/2-2}\mathcal{F}^2(\circ)f(\circ)\bar{\pi}(\bullet)\|\nabla_\theta h_i(\bullet)\|^2\mathrm{d}\theta\mathrm{d}\eta \\ &= 16\iint \eta^{(2p+n)/2-2}\mathcal{F}^2(\circ)f(\circ)\bar{\pi}(\bullet)\eta^2\|\theta\|^2\{h_i'(\bullet)\}^2\mathrm{d}\theta\mathrm{d}\eta,\end{aligned}$$

where $\mathcal{F}(t) = F(t)/f(t)$. Then

$$\begin{aligned}\frac{\int_{\mathbb{R}^p}\sup_i \Delta_{1i}\mathrm{d}z}{16} &\leq \iiint \eta^{(2p+n)/2-1}\mathcal{F}^2(\eta\{\|z-\theta\|^2+1\})f(\eta\{\|z-\theta\|^2+1\}) \\ &\qquad \times \eta\|\theta\|^2\bar{\pi}(\eta\|\theta\|^2)\sup_i\{h_i'(\eta\|\theta\|^2)\}^2\mathrm{d}\theta\mathrm{d}\eta\mathrm{d}z \\ &= \iiint \eta^{(p+n)/2-1}\mathcal{F}^2(\eta\{\|z\|^2+1\})f(\eta\{\|z\|^2+1\}) \\ &\qquad \times \|\mu\|^2\bar{\pi}(\|\mu\|^2)\sup_i\{h_i'(\|\mu\|^2)\}^2\mathrm{d}\mu\mathrm{d}\eta\mathrm{d}z \\ &\leq A_1c_p\iint w^{p/2-1}\eta^{(p+n)/2-1}\mathcal{F}^2(\eta\{w+1\})f(\eta\{w+1\})\mathrm{d}w\mathrm{d}\eta\end{aligned}$$

$$
\begin{aligned}
&= A_1 c_p \int_0^\infty \frac{w^{p/2-1}\mathrm{d}w}{(1+w)^{(p+n)/2}} \int_0^\infty t^{(p+n)/2-1}\mathcal{F}^2(t)f(t)\mathrm{d}t \\
&\leq A_1 A_2 c_p B(p/2, n/2),
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \int_{\mathbb{R}^p} \|\mu\|^2 \bar{\pi}(\|\mu\|^2) \sup_i \{h_i'(\|\mu\|^2)\}^2 \mathrm{d}\mu \\
&= \int_0^\infty \lambda \pi(\lambda) \sup_i \{h_i'(\lambda)\}^2 \mathrm{d}\lambda && \text{(G.4)} \\
A_2 &= \int_0^\infty t^{(p+n)/2-1}\mathcal{F}^2(t)f(t)\mathrm{d}t && \text{(G.5)}
\end{aligned}
$$

are both bounded as shown in Part 7 of Lemma E.2 and in Part 1 of Lemma E.3, respectively.

G.2. The integrability of $\sup_i \boldsymbol{\Delta}_{2i}$

We consider $\alpha > 0$ and $-1/2 < \alpha \leq 0$ separately in G.2.1 and G.2.2, respectively.

G.2.1. Δ_{2i} *under Assumption A.2 with* $\alpha > 0$

By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\left\| \iint \eta^{(2p+n)/2-1} F(\circ)\nabla_\theta \bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta \right\|^2 \\
&\leq \iint \eta^{(2p+n)/2-2}\mathcal{F}^2(\circ)f(\circ)\|\nabla_\theta\bar{\pi}(\bullet)/\bar{\pi}(\bullet)\|^2\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta && \text{(G.6)} \\
&\quad \times \iint \eta^{(2p+n)/2} f(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta.
\end{aligned}
$$

Similarly, by the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\left\| \iint \eta^{(2p+n)/2-1} F(\circ) \nabla_\theta \bar{\pi}(\bullet) h_i^2(\bullet) \mathrm{d}\theta \mathrm{d}\eta \right\|^2 \\
&\leq \iint \eta^{(2p+n)/2-2} \mathcal{F}^2(\circ) f(\circ) \| \nabla_\theta \bar{\pi}(\bullet)/\bar{\pi}(\bullet)\|^2 \bar{\pi}(\bullet) h_i^2(\bullet) \mathrm{d}\theta \mathrm{d}\eta \\
&\qquad \times \iint \eta^{(2p+n)/2} f(\circ) \bar{\pi}(\bullet) h_i^2(\bullet) \mathrm{d}\theta \mathrm{d}\eta \qquad \text{(G.7)} \\
&\leq \iint \eta^{(2p+n)/2-2} \mathcal{F}^2(\circ) f(\circ) \| \nabla_\theta \bar{\pi}(\bullet)/\bar{\pi}(\bullet)\|^2 \bar{\pi}(\bullet) \mathrm{d}\theta \mathrm{d}\eta \\
&\qquad \times \iint \eta^{(2p+n)/2} f(\circ) \bar{\pi}(\bullet) h_i^2(\bullet) \mathrm{d}\theta \mathrm{d}\eta,
\end{aligned}
$$

where the second inequality follows from the fact $h_i^2 \leq 1$. Hence, by (G.6) and (G.7) with (G.3),

$$
\sup_i \Delta_{2i} \leq 2 \iint \eta^{(2p+n)/2-2} \mathcal{F}^2(\circ) f(\circ) \left\| \frac{\nabla_\theta \bar{\pi}(\bullet)}{\bar{\pi}(\bullet)} \right\|^2 \bar{\pi}(\bullet) \mathrm{d}\theta \mathrm{d}\eta.
$$

By the relationship

$$
\| \nabla_\theta \bar{\pi}(\eta \|\theta\|^2)\|^2 = 4\eta^2 \|\theta\|^2 \{\bar{\pi}'(\eta\|\theta\|^2)\}^2, \tag{G.8}
$$

we have

$$
\begin{aligned}
\frac{\sup_i \Delta_{2i}}{2} &= \iint \eta^{(2p+n)/2-2} \mathcal{F}^2(\circ) f(\circ) \left\| \frac{\nabla_\theta \bar{\pi}(\bullet)}{\bar{\pi}(\bullet)} \right\|^2 \bar{\pi}(\bullet) \mathrm{d}\theta \mathrm{d}\eta \\
&= 4 \iint \eta^{(2p+n)/2-1} \mathcal{F}^2(\eta\{\|z-\theta\|^2+1\}) f(\eta\{\|z-\theta\|^2+1\}) \\
&\qquad \times \eta \|\theta\|^2 \left\{ \frac{\bar{\pi}'(\eta\|\theta\|^2)}{\bar{\pi}(\eta\|\theta\|^2)} \right\}^2 \bar{\pi}(\eta\|\theta\|^2) \mathrm{d}\theta \mathrm{d}\eta \\
&\leq 4\Pi^2 \iint \eta^{(2p+n)/2-1} \mathcal{F}^2(\eta\{\|z-\theta\|^2+1\}) \\
&\qquad \times f(\eta\{\|z-\theta\|^2+1\}) \frac{\bar{\pi}(\eta\|\theta\|^2)}{\eta\|\theta\|^2} \mathrm{d}\theta \mathrm{d}\eta, \qquad \text{(G.9)}
\end{aligned}
$$

where

$$
\Pi = \max_{\lambda \in \mathbb{R}_+} \frac{\lambda |\bar{\pi}'(\lambda)|}{\bar{\pi}(\lambda)} = \max_{\lambda \in \mathbb{R}_+} \left| 1 - \frac{p}{2} + \frac{\lambda \pi'(\lambda)}{\pi(\lambda)} \right|, \tag{G.10}
$$

is bounded by Part 1 of Lemma E.1. Further, by (G.9), we have

$$
\begin{aligned}
&\frac{1}{8\Pi^2}\int \sup_i \Delta_{2i}\mathrm{d}z \\
&\leq \iiint \eta^{(p+n)/2-1}\mathcal{F}^2(\eta\{\|z\|^2+1\})f(\eta\{\|z\|^2+1\})\frac{\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2}\mathrm{d}\mu\mathrm{d}\eta\mathrm{d}z \\
&= c_pA_3\iint w^{p/2-1}\eta^{(p+n)/2-1}\mathcal{F}^2(\eta\{w+1\})f(\eta\{w+1\})\mathrm{d}\eta\mathrm{d}w \qquad \text{(G.11)} \\
&= c_pA_3\int_0^\infty \frac{w^{p/2-1}\mathrm{d}w}{(1+w)^{(p+n)/2}}\int_0^\infty t^{(p+n)/2-1}\mathcal{F}^2(t)f(t)\mathrm{d}t \\
&= c_pA_2A_3B(p/2,n/2),
\end{aligned}
$$

where A_2 given by (G.5) is bounded and

$$
A_3=\int_{\mathbb{R}^p}\frac{\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2}\mathrm{d}\mu=\int_0^\infty\frac{\pi(\lambda)}{\lambda}\mathrm{d}\lambda \qquad \text{(G.12)}
$$

is also bounded as shown in Parts 3 and 4 of Lemma E.1.

G.2.2. Δ_{2i} under Assumption A.2 with $-1/2<\alpha\leq 0$

We start with Assumptions A.2 on π, but more generally with $\alpha>-1$ not with $\alpha>-1/2$. Let

$$
k(\lambda;\beta)=\lambda^\beta I_{[0,1]}(\lambda)+I_{(1,\infty)}(\lambda) \text{ for } \beta>0
$$

where, Eventually, we set $\beta=1/2$. Note that $0\leq k(\lambda;\beta)\leq 1$ and $1/k(\lambda;\beta)\geq 1$ for any $\lambda\geq 0$. Then, by the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\left\|\iint \eta^{(2p+n)/2-1}F(\circ)\nabla_\theta\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta\right\|^2 \\
&\leq \iint \eta^{(2p+n)/2-2}\mathcal{F}^2(\circ)f(\circ)k(\bullet;\beta)\left\|\frac{\nabla_\theta\bar{\pi}(\bullet)}{\bar{\pi}(\bullet)}\right\|^2\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\qquad\times\iint \eta^{(2p+n)/2}f(\circ)\frac{\bar{\pi}(\bullet)}{k(\bullet;\beta)}\mathrm{d}\theta\mathrm{d}\eta \qquad \text{(G.13)} \\
&\leq \mathcal{J}(f,\pi,\beta)\iint \eta^{(2p+n)/2-2}\mathcal{F}^2(\circ)f(\circ)k(\bullet;\beta)\left\|\frac{\nabla_\theta\bar{\pi}(\bullet)}{\bar{\pi}(\bullet)}\right\|^2\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\qquad\times\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta,
\end{aligned}
$$

where the second inequality with the constant $\mathcal{J}(f,\pi,\beta)$ follows from Lemma I.2, provided below in Appendix I. Note, as in Lemma I.2,

$$\mathcal{J}(f,\pi,\beta)<\infty$$

when

$$\alpha-\beta+1>0. \tag{G.14}$$

Similarly, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left\|\iint \eta^{(2p+n)/2-1}F(\circ)\nabla_\theta\bar{\pi}(\bullet)h_i^2(\bullet)\mathrm{d}\theta\mathrm{d}\eta\right\|^2 \\
&\leq \iint \eta^{(2p+n)/2-2}\mathcal{F}^2(\circ)f(\circ)k(\bullet;\beta)\left\|\frac{\nabla_\theta\bar{\pi}(\bullet)}{\bar{\pi}(\bullet)}\right\|^2\bar{\pi}(\bullet)h_i^2(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\qquad\times\iint \eta^{(2p+n)/2}f(\circ)\frac{\bar{\pi}(\bullet)}{k(\bullet;\beta)}h_i^2(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\leq 64\mathcal{J}(f,\pi,\beta)\iint \eta^{(2p+n)/2-2}\mathcal{F}^2(\circ)f(\circ)k(\bullet;\beta)\left\|\frac{\nabla_\theta\bar{\pi}(\bullet)}{\bar{\pi}(\bullet)}\right\|^2\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\qquad\times\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta,
\end{aligned} \tag{G.15}$$

where the second inequality with $\mathcal{J}(f,\pi,\beta)$ follows from Lemma I.2, provided below in Appendix I, and from the fact $h_i^2\leq 1$. Hence, by (G.8), (G.13), (G.15) and with Π given by (G.10), we have

$$\begin{aligned}
\frac{\sup_i\Delta_{2i}}{65\mathcal{J}(f,\pi,\beta)} &\leq \iint \eta^{(2p+n)/2-2}\mathcal{F}^2(\circ)f(\circ)k(\bullet;\beta)\left\|\frac{\nabla_\theta\bar{\pi}(\bullet)}{\bar{\pi}(\bullet)}\right\|^2\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\leq 4\Pi^2\iint \eta^{(2p+n)/2-1}\mathcal{F}^2(\circ)f(\circ)k(\bullet;\beta)\frac{\bar{\pi}(\bullet)}{\eta\|\theta\|^2}\mathrm{d}\theta\mathrm{d}\eta.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\frac{1}{260\Pi^2\mathcal{J}(f,\pi,\beta)}\int\sup_i\Delta_{2i}\mathrm{d}z \\
&\leq \iiint \eta^{(p+n)/2-1}\mathcal{F}^2(\eta\{\|z\|^2+1\})f(\eta\{\|z\|^2+1\})k(\|\mu\|^2)\frac{\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2}\mathrm{d}\mu\mathrm{d}\eta\mathrm{d}z \\
&= A_4c_p\iint w^{p/2-1}\eta^{(p+n)/2-1}\mathcal{F}^2(\eta\{w+1\})f(\eta\{w+1\})\mathrm{d}\eta\mathrm{d}w \qquad\text{(G.16)} \\
&= A_4c_p\int\frac{w^{p/2-1}\mathrm{d}w}{(1+w)^{(p+n)/2}}\int_0^\infty t^{(p+n)/2-1}\mathcal{F}^2(t)f(t)\mathrm{d}t
\end{aligned}$$

$$= A_2 A_4 c_p B(p/2, n/2),$$

where A_2 given by (G.5) is bounded. Also A_4 is

$$\begin{aligned}
A_4 &= \int_{\mathbb{R}^p} k(\|\mu\|^2) \frac{\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2} \mathrm{d}\mu \\
&= \int_{\|\mu\| \le 1} \|\mu\|^{2(\beta-1)} \bar{\pi}(\|\mu\|^2) \mathrm{d}\mu + \int_{\|\mu\|>1} \frac{\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2} \mathrm{d}\mu \\
&= \int_0^1 \lambda^{\beta-1} \pi(\lambda) \mathrm{d}\lambda + \int_1^\infty \frac{\pi(\lambda)}{\lambda} \mathrm{d}\lambda \\
&\le \max_{\nu \in (0,1)} \nu(\lambda) \int_0^1 \lambda^{\alpha+\beta-1} \mathrm{d}\lambda + \int_1^\infty \frac{\pi(\lambda)}{\lambda} \mathrm{d}\lambda \\
&= \frac{\max_{\nu \in (0,1)} \nu(\lambda)}{\alpha+\beta} + \int_1^\infty \frac{\pi(\lambda)}{\lambda} \mathrm{d}\lambda,
\end{aligned}$$

the first term of which is bounded when

$$\alpha + \beta > 0 \tag{G.17}$$

and the second term of which is bounded as shown in 4 of Lemma E.1. Hence, by (G.14) and (G.17),

$$\int \sup_i \Delta_{2i} \mathrm{d}z \le 260 \Pi^2 \mathcal{J}(f, \pi, \beta) A_2 A_4 c_p B(p/2, n/2)$$

which is bounded when $\beta > 0$ and

$$\alpha > \max(-\beta, \beta - 1). \tag{G.18}$$

By

$$\min_{\beta>0} \{\max(-\beta, \beta-1)\} = -\frac{1}{2}, \quad \operatorname*{arg\,min}_{\beta>0} \{\max(-\beta, \beta-1)\} = \frac{1}{2},$$

the best choice is $\beta = 1/2$ and the corresponding lower bound of α is $-1/2$. (In Appendix H below, these are assumed from the beginning.) The proof of Theorem 4.2 Case I is thus completed by applying the dominated convergence theorem to diff $B(\delta_\pi, \delta_{\pi i}; \pi_i)$ as noted above.

Appendix H: Proof of Theorem 4.2 for case II

In Case II, we assume Assumptions A.1, A.2 and A.3.2 on π and Assumptions F.1, F.2 and F.3.2 on f.

As in (4.10), $\overline{\mathrm{diff}\,B}(z;\delta_\pi,\delta_{\pi i};\pi_i)$ is re-expressed as

$$\begin{aligned}
&\overline{\mathrm{diff}\,B}(z;\delta_\pi,\delta_{\pi i};\pi_i)\\
&=\left\|\frac{\iint \eta^{(2p+n)/2-1}F(\eta\{\|z-\theta\|^2+1\})\nabla_\theta\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}\right.\\
&\qquad\left.-\frac{\iint \eta^{(2p+n)/2-1}F(\eta\{\|z-\theta\|^2+1\})\nabla_\theta\bar{\pi}_i(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}_i(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}\right\|^2\\
&\qquad\times\iint \eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}_i(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta.
\end{aligned}\tag{H.1}$$

Recall, under Assumption A.3.2, π and $\bar{\pi}$ satisfy

$$\lambda\frac{\pi'(\lambda)}{\pi(\lambda)}=\kappa(\lambda),\ \text{ and }\lambda\frac{\bar{\pi}'(\lambda)}{\bar{\pi}(\lambda)}=1-\frac{p}{2}+\kappa(\lambda),$$

where $\kappa(\lambda)\to 0$ as $\lambda\to\infty$.

With $\kappa(\lambda)$, we have

$$\begin{aligned}
\nabla_\theta\bar{\pi}(\eta\|\theta\|^2)&=2\eta\theta\bar{\pi}'(\eta\|\theta\|^2)\\
&=2\eta\theta\left\{(1-p/2)\frac{\bar{\pi}(\eta\|\theta\|^2)}{\eta\|\theta\|^2}+\frac{\bar{\pi}(\eta\|\theta\|^2)}{\eta\|\theta\|^2}\kappa(\eta\|\theta\|^2)\right\}.
\end{aligned}\tag{H.2}$$

By Lemma I.3 and the relationship (H.2), the integral included in (H.1) is rewritten as

$$\begin{aligned}
&-z^{\mathrm{T}}\iint \eta^{(2p+n)/2-1}F(\circ)\nabla_\theta\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta\\
&=\left(\frac{p-2}{n+2}-\frac{n+p}{n+2}\right)z^{\mathrm{T}}\iint \eta^{(2p+n)/2-1}F(\circ)\nabla_\theta\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta\\
&=\frac{p-2}{n+2}\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta\\
&\quad-\frac{(n+p)(p-2)}{n+2}\iint \eta^{(2p+n)/2-1}F(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta\\
&\quad-\frac{n+p}{n+2}z^{\mathrm{T}}\iint \eta^{(2p+n)/2-1}F(\circ)\nabla_\theta\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{p-2}{n+2} \iint \eta^{(2p+n)/2} f(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\quad + \frac{(n+p)(p-2)}{n+2} \iint \frac{z^{\mathrm{T}}\theta - \|\theta\|^2}{\|\theta\|^2} \eta^{(2p+n)/2-1} F(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\quad - 2\frac{n+p}{n+2} z^{\mathrm{T}} \iint \theta\eta^{(2p+n)/2-1} F(\circ)\frac{\kappa(\bullet)\bar{\pi}(\bullet)}{\|\theta\|^2}\mathrm{d}\theta\mathrm{d}\eta
\end{aligned}
$$

where, again, we use the notation

$$
\bullet = \eta\|\theta\|^2, \quad \circ = \eta(\|z-\theta\|^2+1).
$$

Similarly, by Lemma I.3 and the relationship (H.2), the integral included in (H.1) is rewritten as

$$
\begin{aligned}
& - z^{\mathrm{T}} \iint \eta^{(2p+n)/2-1} F(\circ)\nabla_\theta \bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&= \frac{p-2}{n+2} \iint \eta^{(2p+n)/2} f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\quad - \frac{(n+p)(p-2)}{n+2} \iint \eta^{(2p+n)/2-1} F(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\quad - \frac{n+p}{n+2} z^{\mathrm{T}} \iint \eta^{(2p+n)/2-1} F(\circ)\nabla_\theta \bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&= \frac{p-2}{n+2} \iint \eta^{(2p+n)/2} f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\quad + \frac{(n+p)(p-2)}{n+2} \iint \frac{z^{\mathrm{T}}\theta - \|\theta\|^2}{\|\theta\|^2} \eta^{(2p+n)/2-1} F(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta \\
&\quad - 2\frac{n+p}{n+2} z^{\mathrm{T}} \iint \theta\eta^{(2p+n)/2-1} F(\circ)\frac{\kappa(\bullet)\bar{\pi}_i(\bullet)}{\|\theta\|^2}\mathrm{d}\theta\mathrm{d}\eta \\
&\quad - \frac{n+p}{n+2} z^{\mathrm{T}} \iint \eta^{(2p+n)/2-1} F(\circ)\bar{\pi}(\bullet)\nabla_\theta h_i^2(\bullet)\mathrm{d}\theta\mathrm{d}\eta.
\end{aligned}
$$

Then $\overline{\mathrm{diff}\,B}(z;\delta_\pi,\delta_{\pi i};\pi_i)$ given by (H.1) is rewritten as

$$
\begin{aligned}
& \overline{\mathrm{diff}\,B}(z;\delta_\pi,\delta_{\pi i};\pi_i) \\
&= \frac{c_n}{\|z\|^2}\frac{(n+p)^2}{(n+2)^2} \left\{ z^{\mathrm{T}} \frac{\iint \eta^{(2p+n)/2-1} F(\circ)\bar{\pi}(\bullet)\nabla_\theta h_i^2(\bullet)\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2} f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta} \right. \\
&\quad +(p-2)\frac{\iint (z^{\mathrm{T}}\theta/\|\theta\|^2 - 1)\eta^{(2p+n)/2-1} F(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2} f(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta} \\
&\quad -(p-2)\frac{\iint (z^{\mathrm{T}}\theta/\|\theta\|^2 - 1)\eta^{(2p+n)/2-1} F(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2} f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta}
\end{aligned}
$$

$$
\begin{aligned}
&-2\frac{z^{\mathrm{T}}\iint \theta\eta^{(2p+n)/2-1}F(\circ)\{\kappa(\bullet)\bar{\pi}(\bullet)/\|\theta\|^2\}\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta} \\
&\left.+2\frac{z^{\mathrm{T}}\iint \theta\eta^{(2p+n)/2-1}F(\circ)\{\kappa(\bullet)\bar{\pi}_i(\bullet)/\|\theta\|^2\}\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta}\right\}^2 \\
&\times\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta.
\end{aligned}
$$

By the triangle inequality and the fact $h_i^2 \leq 1$,

$$
\overline{\mathrm{diff}B}(z;\delta_\pi,\delta_{\pi i};\pi_i) \leq 2\frac{c_n(n+p)^2}{(n+2)^2}\left\{\Delta_{1i}+(p-2)^2\Delta_{3i}+4\Delta_{4i}\right\},
$$

where

$$
\Delta_{1i} = \frac{\left\|\iint \eta^{(2p+n)/2-1}F(\circ)\bar{\pi}(\bullet)\nabla_\theta h_i^2(\bullet)\mathrm{d}\theta\mathrm{d}\eta\right\|^2}{\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta},
$$

$$
\begin{aligned}
\Delta_{3i} = &\frac{1}{\|z\|^2}\frac{\left\{\iint (z^{\mathrm{T}}\theta/\|\theta\|^2-1)\eta^{(2p+n)/2-1}F(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta\right\}^2}{\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta} \\
&+\frac{1}{\|z\|^2}\frac{\left\{\iint (z^{\mathrm{T}}\theta/\|\theta\|^2-1)\eta^{(2p+n)/2-1}F(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta\right\}^2}{\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta},
\end{aligned} \tag{H.3}
$$

$$
\begin{aligned}
\Delta_{4i} = &\frac{\left\|\iint \theta\eta^{(2p+n)/2-1}F(\circ)\kappa(\bullet)\bar{\pi}(\bullet)\|\theta\|^{-2}\mathrm{d}\theta\mathrm{d}\eta\right\|^2}{\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}(\bullet)\mathrm{d}\theta\mathrm{d}\eta} \\
&+\frac{\left\|\iint \theta\eta^{(2p+n)/2-1}F(\circ)\kappa(\bullet)\bar{\pi}_i(\bullet)\|\theta\|^{-2}\mathrm{d}\theta\mathrm{d}\eta\right\|^2}{\iint \eta^{(2p+n)/2}f(\circ)\bar{\pi}_i(\bullet)\mathrm{d}\theta\mathrm{d}\eta}.
\end{aligned} \tag{H.4}
$$

For Δ_{1i}, as seen in Appendix G.1, we have $\int \sup_i \Delta_{1i}\mathrm{d}z < \infty$. We will show integrability $\int \sup_i \Delta_{3i}\mathrm{d}z < \infty$ and integrability $\int \sup_i \Delta_{4i}\mathrm{d}z < \infty$ in Appendices H.1 and H.2, respectively.

H.1. The integrability of $\sup_i \Delta_{3i}$

Note the inequality

$$
\left|\frac{z^{\mathrm{T}}\theta}{\|\theta\|^2}-1\right| = \left|\frac{(z-\theta)^{\mathrm{T}}\theta}{\|\theta\|^2}\right| \leq \frac{\|z-\theta\|}{\|\theta\|} \leq \frac{\sqrt{\|z-\theta\|^2+1}}{\|\theta\|}.
$$

Then, in the first and second terms of (H.3), we have

$$
\iint \left|\frac{z^{\mathrm{T}}\theta-\|\theta\|^2}{\|\theta\|^2}\right|\eta^{(2p+n)/2-1}F(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta
$$

$$
\begin{aligned}
&\le \iint \frac{\eta^{(2p+n)/2-1}}{\eta^{1/2}\|\theta\|}\{\eta(\|z-\theta\|^2+1)\}^{1/2}F(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta \\
&= \iint \frac{\eta^{(2p+n)/2-1}}{\eta^{1/2}\|\theta\|}\tilde{\mathcal{F}}(\eta\{\|z-\theta\|^2+1\})f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta
\end{aligned}
$$

where

$$
\tilde{\mathcal{F}}(t) = t^{1/2}\mathcal{F}(t) = t^{1/2}\frac{F(t)}{f(t)}
$$

and hence

$$
\begin{aligned}
&\iint \left|\frac{z^{\mathrm{T}}\theta-\|\theta\|^2}{\|\theta\|^2}\right|\eta^{(2p+n)/2-1}F(\eta\{\|z-\theta\|^2+1\})\bar{\pi}_i(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta \\
&\le \iint \frac{\eta^{(2p+n)/2-1}}{\eta^{1/2}\|\theta\|}\tilde{\mathcal{F}}(\eta\{\|z-\theta\|^2+1\})f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}_i(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta.
\end{aligned}
$$

Under Assumption A.2 on π with $\alpha > 0$, applying the same technique used in Appendix G.2.1, the integrability of

$$
\begin{aligned}
B_1 = \iiint &\frac{\eta^{(2p+n)/2-2}}{\|z\|^2}\tilde{\mathcal{F}}^2(\eta\{\|z-\theta\|^2+1\})f(\eta\{\|z-\theta\|^2+1\}) \\
&\times \frac{\bar{\pi}(\eta\|\theta\|^2)}{\eta\|\theta\|^2}\mathrm{d}\theta\mathrm{d}\eta\mathrm{d}z
\end{aligned}
$$

implies the integrability of $\int \sup_i \Delta_{3i}\mathrm{d}z$. The integrability of B_1 is shown as follows;

$$
\begin{aligned}
B_1 &= \iiint \frac{\eta^{n/2-1}}{\|y\|^2}\tilde{\mathcal{F}}^2(\|y-\mu\|^2+\eta)f(\|y-\mu\|^2+\eta)\frac{\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2}\mathrm{d}\mu\mathrm{d}\eta\mathrm{d}y \\
&= \int_{\mathbb{R}^p}\left(\int_{\mathbb{R}^p}\frac{f_\star(\|y-\mu\|^2)}{\|y\|^2}\mathrm{d}y\right)\frac{\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2}\mathrm{d}\mu \\
&\le \mathcal{Q}_f\int_{\mathbb{R}^p}\min(1,\|\mu\|^{-2})\frac{\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2}\mathrm{d}\mu \qquad \text{(H.5)} \\
&= \mathcal{Q}_f\left\{\int_{\|\mu\|\le 1}\frac{\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2}\mathrm{d}\mu + \int_{\|\mu\|>1}\frac{\bar{\pi}(\|\mu\|^2)}{\|\mu\|^4}\mathrm{d}\mu\right\} \\
&= \mathcal{Q}_f\left\{\int_0^1\frac{\pi(\lambda)}{\lambda}\mathrm{d}\lambda + \int_1^\infty\frac{\pi(\lambda)}{\lambda^2}\mathrm{d}\lambda\right\} < \infty
\end{aligned}
$$

where

$$
f_\star(t) = \int_0^\infty \eta^{n/2-1}\tilde{\mathcal{F}}^2(t+\eta)f(t+\eta)\mathrm{d}\eta,
$$

the inequality with $\mathcal{Q}_f$ follows from Part 2 of Lemma E.3 and the integrability of the right-hand side follows from Parts 3 and 5 of Lemma E.1.

Under Assumption A.2 on π with $-1/2 < \alpha \leq 0$, applying the same technique used in Appendix G.2.2, the integrability of

$$
\begin{aligned}
B_2 = \iiint & \frac{\eta^{(2p+n)/2-2}}{\|z\|^2} \tilde{\mathcal{F}}^2(\eta\{\|z-\theta\|^2+1\}) f(\eta\{\|z-\theta\|^2+1\}) \\
& \times \frac{k(\eta\|\theta\|^2)\bar{\pi}(\eta\|\theta\|^2)}{\eta\|\theta\|^2} \mathrm{d}\theta \mathrm{d}\eta \mathrm{d}z
\end{aligned}
$$

where $k(\lambda) = \lambda^{1/2} I_{[0,1]}(\lambda) + I_{(1,\infty)}(\lambda)$, implies the integrability of $\int \sup_i \Delta_{3i} \mathrm{d}z$. As in (H.5), B_2 is given by

$$
\begin{aligned}
B_2 &\leq \mathcal{Q}_f \int_{\mathbb{R}^p} \min(1, \|\mu\|^{-2}) \frac{k(\|\mu\|^2)\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2} \mathrm{d}\mu \\
&= \mathcal{Q}_f \left\{ \int_0^1 \frac{\pi(\lambda)}{\lambda^{1/2}} \mathrm{d}\lambda + \int_1^\infty \frac{\pi(\lambda)}{\lambda^2} \mathrm{d}\lambda \right\} < \infty
\end{aligned}
$$

which is bounded by Parts 2 and 5 of Lemma E.1.

H.2. The integrability of $\sup_i \Delta_{4i}$

Under Assumption A.2 on π with $\alpha > 0$, applying the same technique used in Appendix G.2.1, the integrability of

$$
\begin{aligned}
B_3 = \iiint & \eta^{(2p+n)/2-1} \mathcal{F}^2(\eta\{\|z-\theta\|^2+1\}) f(\eta\{\|z-\theta\|^2+1\}) \\
& \times \frac{\kappa^2(\eta\|\theta\|^2)\bar{\pi}(\eta\|\theta\|^2)}{\eta\|\theta\|^2} \mathrm{d}\theta \mathrm{d}\eta \mathrm{d}z
\end{aligned}
$$

implies the integrability of $\int \sup_i \Delta_{4i} \mathrm{d}z$. The integrability of B_3 is shown as follows;

$$
\begin{aligned}
B_3 &= c_p A_2 B(p/2, n/2) \int \frac{\kappa^2(\|\mu\|^2)\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2} \mathrm{d}\mu \\
&= c_p A_2 B(p/2, n/2) \left\{ \sup_{\lambda \in (0,1)} \kappa^2(\lambda) \int_0^1 \frac{\pi(\lambda)}{\lambda} \mathrm{d}\mu + \int_1^\infty \frac{\pi(\lambda)\kappa^2(\lambda)}{\lambda} \mathrm{d}\mu \right\},
\end{aligned}
$$

where A_2 is given by (G.5), the first term is bounded by Parts 1 and 3 of Lemma E.1 and the second term is bounded by Part 8 of Lemma E.1.

Under Assumption A.2 on π with $-1/2 < \alpha \leq 0$, applying the same technique used in Appendix G.2.2 the integrability of

$$B_4 = \iiint \eta^{(2p+n)/2-1}\mathcal{F}^2(\eta\{\|z-\theta\|^2+1\})f(\eta\{\|z-\theta\|^2+1\}) \times \frac{k(\eta\|\theta\|^2)\kappa^2(\eta\|\theta\|^2)\bar{\pi}(\eta\|\theta\|^2)}{\eta\|\theta\|^2}\mathrm{d}\theta\mathrm{d}\eta\mathrm{d}z$$

where $k(\lambda) = \lambda^{1/2}I_{[0,1]}(\lambda)+I_{(1,\infty)}(\lambda)$ implies the integrability of $\int \sup_i \Delta_{4i}\mathrm{d}z$. The integrability of B_3 is shown as follows;

$$\begin{aligned} B_4 &= c_p A_2 B(p/2,n/2)\int \frac{k(\|\mu\|^2)\kappa^2(\|\mu\|^2)\bar{\pi}(\|\mu\|^2)}{\|\mu\|^2}\mathrm{d}\mu \\ &= c_p A_2 B(p/2,n/2)\left\{\sup_{\lambda\in(0,1)}\kappa^2(\lambda)\int_0^1 \frac{\pi(\lambda)}{\lambda^{1/2}}\mathrm{d}\mu + \int_1^\infty \frac{\pi(\lambda)\kappa^2(\lambda)}{\lambda}\mathrm{d}\mu\right\}, \end{aligned}$$

where A_2 is given by (G.5), the first term is bounded by Parts 1 and 2 of Lemma E.1 and the second term is bounded by Part 8 of Lemma E.1.

Appendix I: Additional Lemmas used in Sections G and H

We prove three lemmas used in Sections G and H. In first two lemmas, Lemmas I.1 and I.2, we assume Assumptions A.2 on π, but more generally with $\alpha > -1$ not with $\alpha > -1/2$.

Let

$$J(f,\pi,z,\beta) = \frac{\iint_{\eta\|\theta\|^2\leq 1}\eta^{(2p+n)/2}\{\eta\|\theta\|^2\}^{-\beta}\bar{\pi}(\eta\|\theta\|^2)f(\eta\{\|z-\theta\|^2+1\})\mathrm{d}\theta\mathrm{d}\eta}{\iint_{\eta\|\theta\|^2\leq 1}\eta^{(2p+n)/2}\bar{\pi}(\eta\|\theta\|^2)f(\eta\{\|z-\theta\|^2+1\})\mathrm{d}\theta\mathrm{d}\eta}. \tag{I.1}$$

Then we have a following result.

Lemma I.1. *Suppose Assumptions F.1–F.3 on f hold. Let*

$$\alpha - \beta + 1 > 0 \tag{I.2}$$

Then

$$J(f,\pi,z,\beta) \leq \mathcal{J}(f,\pi,\beta) < \infty \tag{I.3}$$

for any $z \in \mathbb{R}^p$*, where*

$$\mathcal{J}(f,\pi,\beta) = 2\frac{\alpha+1}{\alpha-\beta+1}\frac{\max_{a\in\mathbb{R}_+}\varphi(a;\alpha-\beta+1,f)}{\min_{a\in\mathbb{R}_+}\varphi(a;\alpha+1,f)}\frac{\max_{\lambda\in[0,1]}\nu(\lambda)}{\min_{\lambda\in[0,1]}\nu(\lambda)},$$
$$\varphi(a;\gamma,f) = \frac{\int_0^a t^{(p+n)/2+\gamma}f(t)\mathrm{d}t}{\int_0^a t^{(p+n)/2+\gamma}f_G(t)\mathrm{d}t}, \tag{I.4}$$
$$f_G(t) = (2\pi)^{-(p+n)/2}\exp(-t/2).$$

Proof. By Assumptions A.2 on π,

$$J(f,\pi,z,\beta) \le \frac{\max_{\lambda\in[0,1]}\nu(\lambda)}{\min_{\lambda\in[0,1]}\nu(\lambda)}J_1(f,\pi,z,\beta) \tag{I.5}$$

where

$$\begin{aligned} &J_1(f,\pi,z,\beta) \\ &= \frac{\iint_{\eta\|\theta\|^2\le 1}\eta^{(2p+n)/2}\{\eta\|\theta\|^2\}^{\alpha+(2-p)/2-\beta}f(\eta\{\|z-\theta\|^2+1\})\mathrm{d}\theta\mathrm{d}\eta}{\iint_{\eta\|\theta\|^2\le 1}\eta^{(2p+n)/2}\{\eta\|\theta\|^2\}^{\alpha+(2-p)/2}f(\eta\{\|z-\theta\|^2+1\})\mathrm{d}\theta\mathrm{d}\eta} \\ &= \frac{\int_{\mathbb{R}^p}\|\theta\|^{2\alpha+2-p-2\beta}\left\{\int_0^{1/\|\theta\|^2}\eta^{(p+n+2)/2+\alpha-\beta}f(\eta\{\|z-\theta\|^2+1\})\mathrm{d}\eta\right\}\mathrm{d}\theta}{\int_{\mathbb{R}^p}\|\theta\|^{2\alpha+2-p}\left\{\int_0^{1/\|\theta\|^2}\eta^{(p+n+2)/2+\alpha}f(\eta\{\|z-\theta\|^2+1\})\mathrm{d}\eta\right\}\mathrm{d}\theta}. \end{aligned} \tag{I.6}$$

Let $\gamma = \alpha+1$ for the denominator and $\alpha+1-\beta$ for the numerator of $J_1(f,\pi,z,\beta)$. By change of variables, the integral with respect to η is rewritten as

$$\begin{aligned} &\int_0^{1/\|\theta\|^2}\eta^{(p+n)/2+\gamma}f(\eta\{\|z-\theta\|^2+1\})\mathrm{d}\eta \\ &= \{\|z-\theta\|^2+1\}^{-(p+n)/2-1-\gamma}\int_0^a t^{(p+n)/2+\gamma}f(t)\mathrm{d}t \\ &= \varphi(a;\gamma,f)\int_0^{1/\|\theta\|^2}\eta^{(p+n)/2+\gamma}f_G(\eta\{\|z-\theta\|^2+1\})\mathrm{d}\eta \end{aligned}$$

where $\varphi(a;\gamma,f)$ is defined by (I.4) and $a = \{\|z-\theta\|^2+1\}/\|\theta\|^2$. Note

$$\lim_{a\to 0}\varphi(a;\gamma,f) = \frac{f(0)}{f_G(0)} \text{ and } \lim_{a\to\infty}\varphi(a;\gamma,f) = \frac{\int_0^\infty t^{(p+n)/2+\gamma}f(t)\mathrm{d}t}{\int_0^\infty t^{(p+n)/2+\gamma}f_G(t)\mathrm{d}t},$$

which are both positive and bounded from the above under $0<\gamma\le 1$ and under Assumptions F.1–F.3 on f and hence

$$\min_{a\in\mathbb{R}_+}\varphi(a;\gamma,f) > 0 \text{ and } \max_{a\in\mathbb{R}_+}\varphi(a;\gamma,f) < \infty,$$

under $0 < \gamma \le 1$. Therefore we have

$$J_1(f,\pi,z,\beta) \le \frac{\max_{a\in\mathbb{R}_+}\varphi(a;\alpha+1-\beta,f)}{\min_{a\in\mathbb{R}_+}\varphi(a;\alpha+1,f)} J_2(\pi,z,\beta) \tag{I.7}$$

where

$$\begin{aligned} &J_2(\pi,z,\beta) \\ &= \frac{\iint_{\eta\|\theta\|^2\le 1}\eta^{(2p+n)/2}\{\eta\|\theta\|^2\}^{\alpha+(2-p)/2-\beta} f_G(\eta\{\|z-\theta\|^2+1\})\mathrm{d}\theta\mathrm{d}\eta}{\iint_{\eta\|\theta\|^2\le 1}\eta^{(2p+n)/2}\{\eta\|\theta\|^2\}^{\alpha+(2-p)/2} f_G(\eta\{\|z-\theta\|^2+1\})\mathrm{d}\theta\mathrm{d}\eta} \\ &= \frac{\iint_{\|\mu\|^2\le 1}\eta^{(p+n)/2}\{\|\mu\|^2\}^{\alpha+(2-p)/2-\beta}\exp(-\|\eta^{1/2}z-\mu\|^2/2-\eta/2)\mathrm{d}\mu\mathrm{d}\eta}{\iint_{\|\mu\|^2\le 1}\eta^{(p+n)/2}\{\|\mu\|^2\}^{\alpha+(2-p)/2}\exp(-\|\eta^{1/2}z-\mu\|^2/2-\eta/2)\mathrm{d}\mu\mathrm{d}\eta}. \end{aligned} \tag{I.8}$$

Note $\|\mu\|^2$ may be regarded as a non-central chi-square random variable with p degrees of freedom and $\eta\|z\|^2$ non-centrality parameter. For

$$a_j(\eta\|z\|^2) = \frac{1}{\Gamma(p/2+j)2^{p/2+j}}\frac{(\eta\|z\|^2/2)^j}{j!}\exp(-\eta\|z\|^2/2),$$

we have

$$\begin{aligned} &J_2(\pi,z,\beta) \\ &= \frac{\sum_{j=0}^{\infty}\int_0^{\infty}\eta^{(p+n)/2}a_j(\eta\|z\|^2)\exp(-\eta/2)\mathrm{d}\eta\int_0^1 r^{\alpha-\beta+j}\exp(-r/2)\mathrm{d}r}{\sum_{j=0}^{\infty}\int_0^{\infty}\eta^{(p+n)/2}a_j(\eta\|z\|^2)\exp(-\eta/2)\mathrm{d}\eta\int_0^1 r^{\alpha+j}\exp(-r/2)\mathrm{d}r} \\ &= \frac{\sum_{j=0}^{\infty}\tilde{a}_j(\|z\|^2)E[R^{j-\beta}]}{\sum_{j=0}^{\infty}\tilde{a}_j(\|z\|^2)E[R^j]}, \end{aligned}$$

where the expected value is taken under the probability density given by

$$\frac{r^{\alpha}\exp(-r/2)I_{[0,1]}(r)}{\int_0^1 r^{\alpha}\exp(-r/2)\mathrm{d}r}$$

and

$$\begin{aligned} \tilde{a}_j(\|z\|^2) &= \int_0^{\infty}\eta^{(p+n)/2}a_j(\eta\|z\|^2)\exp(-\eta/2)\mathrm{d}\eta \\ &= \frac{\Gamma((p+n)/2+j+1)2^{(p+n)/2+j+1}}{\Gamma(p/2+j)2^{p/2+j}}\frac{(\|z\|^2/2)^j}{j!(\|z\|^2+1)^{(p+n)/2+j+1}}. \end{aligned}$$

Since the correlation inequality gives

$$E[R^{-\beta}] \geq \frac{E[R^{1-\beta}]}{E[R]} \geq \frac{E[R^{2-\beta}]}{E[R^2]} \geq \dots,$$

we have

$$\frac{\sum_{j=0}^{\infty} \tilde{a}_j(\|z\|^2) E[R^{j-\beta}]}{\sum_{j=0}^{\infty} \tilde{a}_j(\|z\|^2) E[R^j]} \leq E[R^{-\beta}] = \frac{\int_0^1 r^{\alpha-\beta} \exp(-r/2) \mathrm{d}r}{\int_0^1 r^{\alpha} \exp(-r/2) \mathrm{d}r}.$$

For $0 \leq r \leq 1$, we have

$$\begin{aligned} 1/2 < \exp(-1/2) \leq \exp(-r/2) \leq 1, \\ E[R^{-\beta}] \leq 2\frac{\alpha+1}{\alpha-\beta+1}, \end{aligned}$$

and hence

$$J_2(\pi, z, \beta) \leq 2\frac{\alpha+1}{\alpha-\beta+1}, \text{ for any } z \in \mathbb{R}^p. \tag{I.9}$$

Finally, by (I.5), (I.6), (I.7), (I.8) and (I.9), we have

$$J(f, \pi, z, \beta) \leq 2\frac{\alpha+1}{\alpha-\beta+1} \frac{\max_{a\in\mathbb{R}_+} \varphi(a; \alpha-\beta+1, f)}{\min_{a\in\mathbb{R}_+} \varphi(a; \alpha+1, f)} \frac{\max_{\lambda\in[0,1]} \nu(\lambda)}{\min_{\lambda\in[0,1]} \nu(\lambda)}.$$

□

Recall, in this section, we assume Assumptions A.2 on π, but more generally with $\alpha > -1$ not with $\alpha > -1/2$. Using Lemma I.1, we have the following result.

Lemma I.2. *Suppose Assumptions F.1–F.3 on f hold. Assume Assumptions A.2 on π with $\alpha > -1$. Let*

$$k(\lambda; \beta) = \lambda^{\beta} I_{[0,1]}(\lambda) + I_{(1,\infty)}(\lambda),$$

where

$$\alpha - \beta + 1 > 0.$$

1. *Then*

$$\begin{aligned} &\frac{\iint \eta^{(2p+n)/2} f(\eta\{\|z-\theta\|^2+1\})\{\bar{\pi}(\eta\|\theta\|^2)/k(\eta\|\theta\|^2;\beta)\}\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2} f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta} \\ &\leq \mathcal{J}(f, \pi, \beta), \end{aligned} \tag{I.10}$$

where $\mathcal{J}(f, \pi, \beta)$ is given by (I.4) of Lemma I.1.

2. We have

$$\begin{aligned}&\frac{\iint \eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\{\bar{\pi}_i(\eta\|\theta\|^2)/k(\eta\|\theta\|^2;\beta)\}\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}_i(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta} \\ &\le 64\mathcal{J}(f,\pi,\beta).\end{aligned} \tag{I.11}$$

Proof. Let $\mathcal{R} = \{(\theta,\eta) : \eta\|\theta\|^2 \le 1\}$. The parameter space for (θ,η) is decomposed as

$$\mathbb{R}^p \times \mathbb{R}_+ = \mathcal{R}\cup\mathcal{R}^C \text{ and } \mathcal{R}\cap\mathcal{R}^C = \emptyset.$$

Then

$$\begin{aligned}&\iint \eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\frac{\bar{\pi}(\eta\|\theta\|^2)}{k(\eta\|\theta\|^2;\beta)}\mathrm{d}\theta\mathrm{d}\eta \\ &= \left(\iint_{\mathcal{R}} + \iint_{\mathcal{R}^C}\right)\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\frac{\bar{\pi}(\eta\|\theta\|^2)}{k(\eta\|\theta\|^2;\beta)}\mathrm{d}\theta\mathrm{d}\eta \\ &\le \mathcal{J}(f,\pi,\beta)\iint_{\mathcal{R}}\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta \\ &\qquad + \iint_{\mathcal{R}^C}\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta \\ &\le \mathcal{J}(f,\pi,\beta)\iint \eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta,\end{aligned} \tag{I.12}$$

where the first inequality follows from Lemma I.1, the second inequality follows from the fact that $\mathcal{J}(f,\pi,\beta)\ge 1$. This completes the proof of Part 1.

For Part 2, note the following relationship;

$$\begin{aligned}&\iint_{\mathcal{R}}\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\frac{\bar{\pi}_i(\eta\|\theta\|^2)}{k(\eta\|\theta\|^2;\beta)}\mathrm{d}\theta\mathrm{d}\eta \\ &\le \iint_{\mathcal{R}}\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\frac{\bar{\pi}_i(\eta\|\theta\|^2)}{k(\eta\|\theta\|^2;\beta)}\mathrm{d}\theta\mathrm{d}\eta \\ &\le \mathcal{J}(f,\pi,\beta)\iint_{\mathcal{R}}\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta \\ &= \frac{\mathcal{J}(f,\pi,\beta)}{h_1^2(1)}\iint_{\mathcal{R}}\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)h_1^2(1)\mathrm{d}\theta\mathrm{d}\eta \\ &\le \frac{\mathcal{J}(f,\pi,\beta)}{h_1^2(1)}\iint_{\mathcal{R}}\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)h_i^2(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta \\ &\le 64\mathcal{J}(f,\pi,\beta)\iint_{\mathcal{R}}\eta^{(2p+n)/2}f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)h_i^2(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta.\end{aligned}$$

where the first inequality follows from the fact $h_i^2 \leq 1$, the second inequality follows from Lemma I.1, the third inequality follows from Part 1 of Lemma E.2. The last inequality follows from Part 4 of Lemma E.2. Then, as in (I.12), the inequality (I.11) can be established. $\square$

Lemma I.3. *Under Assumptions F.1–F.3 on f and Assumptions A.1, A.2 A.3 on π,*

$$\begin{aligned}
&z^{\mathrm{T}} \iint \eta^{(2p+n)/2-1} F(\eta\{\|z-\theta\|^2+1\}) \nabla_\theta \bar{\pi}(\eta\|\theta\|^2) \mathrm{d}\theta \mathrm{d}\eta \\
&= \iint \eta^{(2p+n)/2} f(\eta\{\|z-\theta\|^2+1\}) \bar{\pi}(\eta\|\theta\|^2) \mathrm{d}\theta \mathrm{d}\eta \\
&\qquad - (p+n) \iint \eta^{(2p+n)/2-1} F(\eta\{\|z-\theta\|^2+1\}) \bar{\pi}(\eta\|\theta\|^2) \mathrm{d}\theta \mathrm{d}\eta.
\end{aligned}$$

Proof. First, note the following relationship;

$$\begin{aligned}
&z^{\mathrm{T}} \iint \eta^{(2p+n)/2-1} F(\eta\{\|z-\theta\|^2+1\}) \nabla_\theta \bar{\pi}(\eta\|\theta\|^2) \mathrm{d}\theta \mathrm{d}\eta \\
&= z^{\mathrm{T}} \iint \eta^{(2p+n)/2-1} F(\eta\{\|z-\theta\|^2+1\}) 2\theta\eta \bar{\pi}'(\eta\|\theta\|^2) \mathrm{d}\theta \mathrm{d}\eta \\
&= 2z^{\mathrm{T}} \iint \eta^{(p+n)/2} F(\|\eta^{1/2}z-\mu\|^2+\eta) \eta^{-1/2} \mu \bar{\pi}'(\|\mu\|^2) \mathrm{d}\mu \mathrm{d}\eta \\
&= 2 \iint \eta^{(p+n)/2} F(\|\mu\|^2+\eta) \eta^{-1/2} z^{\mathrm{T}}(\mu+\eta^{1/2}z) \bar{\pi}'(\|\mu+\eta^{1/2}z\|^2) \mathrm{d}\mu \mathrm{d}\eta \\
&= 2 \iint \eta^{(p+n)/2} F(\|\mu\|^2+\eta) \frac{\partial}{\partial\eta} \bar{\pi}(\|\mu+\eta^{1/2}z\|^2) \mathrm{d}\mu \mathrm{d}\eta.
\end{aligned}$$

By an integration by parts, the integral with respect to η in the above is

$$\begin{aligned}
&\int_0^\infty \eta^{(p+n)/2} F(\|\mu\|^2+\eta) \frac{\partial}{\partial\eta} \bar{\pi}(\|\mu+\eta^{1/2}z\|^2) \mathrm{d}\eta \\
&= \Big[\eta^{(p+n)/2} F(\|\mu\|^2+\eta) \bar{\pi}(\|\mu+\eta^{1/2}z\|^2)\Big]_0^\infty \\
&\quad + \frac{1}{2} \int_0^\infty \eta^{(p+n)/2} f(\|\mu\|^2+\eta) \bar{\pi}(\|\mu+\eta^{1/2}z\|^2) \mathrm{d}\eta \\
&\quad - \frac{p+n}{2} \int_0^\infty \eta^{(p+n)/2-1} F(\|\mu\|^2+\eta) \bar{\pi}(\|\mu+\eta^{1/2}z\|^2) \mathrm{d}\mu \mathrm{d}\eta,
\end{aligned}$$

where the first term becomes zero for any fixed μ under Assumptions. Then

$$z^{\mathrm{T}} \iint \eta^{(2p+n)/2-1} F(\eta\{\|z-\theta\|^2+1\}) \nabla_\theta \bar{\pi}(\eta\|\theta\|^2) \mathrm{d}\theta \mathrm{d}\eta$$

$$
\begin{aligned}
&= \iint \eta^{(p+n)/2} f(\|\mu\|^2+\eta)\bar{\pi}(\|\mu+\eta^{1/2}z\|^2)\mathrm{d}\eta \\
&\quad - (p+n)\iint \eta^{(p+n)/2-1} F(\|\mu\|^2+\eta)\bar{\pi}(\|\mu+\eta^{1/2}z\|^2)\mathrm{d}\mu\mathrm{d}\eta \\
&= \iint \eta^{(2p+n)/2} f(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta \\
&\quad - (p+n)\iint \eta^{(2p+n)/2-1} F(\eta\{\|z-\theta\|^2+1\})\bar{\pi}(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta,
\end{aligned}
$$

which completes the proof. □

Appendix J: Proof of Corollary 4.2, Part 3

Let $\phi_\alpha(w) = w\psi_\alpha(w)$ where, as in (4.19),

$$
\psi_\alpha(w) = \frac{\int_0^\infty \xi^\alpha(1+\xi)^{n/2}(1+w+\xi)^{-(p+n)/2-1}\mathrm{d}\xi}{\int_0^\infty \xi^\alpha(1+\xi)^{n/2+1}(1+w+\xi)^{-(p+n)/2-1}\mathrm{d}\xi},
$$

for $\alpha \in (-1,0)$. Maruyama and Strawderman (2009) showed that

1. $\phi_\alpha(w)$ is not monotonic,
2. $0 \le \phi_\alpha(w) \le \phi_\star(\alpha)$
$$
\phi_\star(\alpha) = \frac{p/2-\alpha-1}{n/2+\alpha+1+\alpha(p/2+n/2)},
$$
3. $w\phi'_\alpha(w)/\phi_\alpha(w) \ge -c(\alpha)$
$$
c(\alpha) = -\frac{(p/2-\alpha)\alpha}{\alpha+1}.
$$

in Part (iii) of Corollary 3.1, Part (iv) of Corollary 3.1 and Lemma 3.4, respectively. Kubokawa (2009) proposed a sufficient condition of $\delta_\phi = \{1 - \phi(W)/W\}X$ to be minimax as follows;

$$
w\phi'(w)/\phi(w) \ge -c, \text{ and } 0 \le \phi \le 2\frac{p-2-2c}{n+2+2c}, \text{ for some } c > 0,
$$

where the upper bound is larger than the upper bound which Maruyama and Strawderman (2009) and Wells and Zhou (2008) applied. Note that $\phi_\star(\alpha)$ is increasing in $\alpha \in (-1/2,0)$ and that $c(\alpha)$ is decreasing in $\alpha \in (-1/2,0)$. Then the inequalities

$$
\begin{aligned}
\frac{p/2-\alpha-1}{n/2+\alpha+1+\alpha(p/2+n/2)} &\le 2\frac{p-2-2c(\alpha)}{n+2+2c(\alpha)} \\
&= 2\frac{(p-2)(\alpha+1)+2(p/2-\alpha)\alpha}{(n+2)(\alpha+1)-2(p/2-\alpha)\alpha}
\end{aligned}
$$

as well as $-1 < \alpha < 0$ are a sufficient condition for minimaxity of δ_{ψ_α}. Let

$$\begin{aligned} f(\alpha) &= 2\left\{(p-2)(\alpha+1)+2(p/2-\alpha)\alpha\right\}\left\{n/2+\alpha+1+\alpha(p/2+n/2)\right\} \\ &\quad -\left\{(n+2)(\alpha+1)-2(p/2-\alpha)\alpha\right\}(p/2-\alpha-1) \\ &= \frac{(p-2)(n+2)}{2}+\frac{(n+2)(5p-8)+3p(p-2)}{2}\alpha \\ &\quad +\left\{2(p-1)^2+(2p-3)(n+2)\right\}\alpha^2-2(p+n+1)\alpha^3. \end{aligned}$$

For $\alpha \in (-1/2, 0)$,

$$f(\alpha) \geq \frac{(p-2)(n+2)}{2}+\frac{5(n+2)(p-2)+2(n+2)+3p(p-2)}{2}\alpha$$

which is nonnegative when

$$-\left(5+\frac{2}{p-2}+\frac{3p}{n+2}\right)^{-1} \leq \alpha < 0.$$

Hence Part 3 follows.

Appendix K: Proof of Corollary 4.3

Let

$$\bar{\pi}(\eta\|\theta\|^2)=\int_b^\infty \frac{1}{(2\pi)^{p/2}\xi^{p/2}}\exp\left(-\frac{\eta\|\theta\|^2}{2\xi}\right)g_\pi(\xi)\mathrm{d}\xi$$

where

$$g_\pi(\xi)=(\xi-b)^\alpha(\xi+1)^\beta.$$

Eventually set $-1 < \alpha \leq n/2$, $\beta = -n/2$ and $b \geq 0$.

Note the underlying density is Gaussian and let

$$f_G(t)=\frac{1}{(2\pi)^{(p+n)/2}}\exp(-t/2).$$

Note also that

$$\|z-\theta\|^2+\frac{\|\theta\|^2}{\xi}=\frac{\xi+1}{\xi}\left\|\theta-\frac{\xi}{\xi+1}z\right\|^2+\frac{\|z\|^2}{\xi+1}.$$

Then we have

$$M_1(z;\pi)$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{(p+n)/2}} \int_b^\infty \int_0^\infty \eta^{(p+n)/2} \exp\left(-\frac{\eta}{2}\right) g_\pi(\xi) \\
&\quad \times \left\{ \int_{\mathbb{R}^p} \frac{\eta^{p/2}}{(2\pi)^{p/2}\xi^{p/2}} \exp\left(-\eta \frac{\|z-\theta\|^2}{2} - \frac{\eta\|\theta\|^2}{2\xi}\right) \mathrm{d}\theta \right\} \mathrm{d}\eta \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{(p+n)/2}} \int_b^\infty \int_0^\infty \eta^{(p+n)/2} \exp\left(-\frac{\eta}{2}\right) g_\pi(\xi) \\
&\quad \times \frac{1}{(\xi+1)^{p/2}} \exp\left(-\frac{\eta\|z\|^2}{2(\xi+1)}\right) \mathrm{d}\eta \mathrm{d}\xi \\
&= c \int_b^\infty \left(1 + \frac{\|z\|^2}{\xi+1}\right)^{-(p+n)/2-1} \frac{g_\pi(\xi)}{(\xi+1)^{p/2}} \mathrm{d}\xi \\
&= c \int_b^\infty \frac{(\xi-b)^\alpha (1+\xi)^{\beta+n/2+1}}{(1+\xi+\|z\|^2)^{(p+n)/2+1}} \mathrm{d}\xi \\
&= c \int_0^\infty \frac{\xi^\alpha (1+b+\xi)^{\beta+n/2+1}}{(1+b+\|z\|^2+\xi)^{(p+n)/2+1}} \mathrm{d}\xi
\end{aligned}
$$

where

$$
c = \frac{\Gamma((p+n)/2+1)2^{(p+n)/2+1}}{(2\pi)^{(p+n)/2}}.
$$

Also we have

$$
\begin{aligned}
&z^{\mathrm{T}} M_2(z;\pi) \\
&= \frac{1}{(2\pi)^{(p+n)/2}} \int_b^\infty \int_0^\infty \eta^{(p+n)/2} \exp\left(-\frac{\eta}{2}\right) g_\pi(\xi) \\
&\quad \times \left\{ \int_{\mathbb{R}^p} \frac{z^{\mathrm{T}}\theta\eta^{p/2}}{(2\pi)^{p/2}\xi^{p/2}} \exp\left(-\eta \frac{\|z-\theta\|^2}{2} - \frac{\eta\|\theta\|^2}{2\xi}\right) \mathrm{d}\theta \right\} \mathrm{d}\eta \mathrm{d}\xi \\
&= \frac{\|z\|^2}{(2\pi)^{(p+n)/2}} \int_b^\infty \int_0^\infty \eta^{(p+n)/2} \exp\left(-\frac{\eta}{2}\right) g_\pi(\xi) \\
&\quad \times \frac{1}{(\xi+1)^{p/2}} \frac{\xi}{\xi+1} \exp\left(-\frac{\eta\|z\|^2}{2(\xi+1)}\right) \mathrm{d}\eta \mathrm{d}\xi \\
&= \|z\|^2 M_1(z;\pi) - \|z\|^2 c \int_0^\infty \frac{\xi^\alpha (1+b+\xi)^{\beta+n/2}}{(1+b+\|z\|^2+\xi)^{(p+n)/2+1}} \mathrm{d}\xi.
\end{aligned}
$$

Recall

$$
\psi_\pi(z) = \frac{z^{\mathrm{T}} z M_1(z,\pi) - z^{\mathrm{T}} M_2(z,\pi)}{\|z\|^2 M_1(z,\pi)}.
$$

Under the choice $\beta = -n/2$, we have

$$
\begin{aligned}
\psi_\pi(z) &= \frac{\int_0^\infty \xi^\alpha (1+b+\xi)^{\beta+n/2}(1+b+\|z\|^2+\xi)^{-(p+n)/2-1}\mathrm{d}\xi}{\int_0^\infty \xi^\alpha (1+b+\xi)^{\beta+n/2+1}(1+b+\|z\|^2+\xi)^{-(p+n)/2-1}\mathrm{d}\xi} \\
&= \frac{\int_0^\infty \xi^\alpha (1+b+\|z\|^2+\xi)^{-(p+n)/2-1}\mathrm{d}\xi}{\int_0^\infty \xi^\alpha (1+b+\xi)(1+b+\|z\|^2+\xi)^{-(p+n)/2-1}\mathrm{d}\xi} \\
&= \left(1+c+\frac{\int_0^\infty \xi^{\alpha+1}(1+b+\|z\|^2+\xi)^{-(p+n)/2-1}\mathrm{d}\xi}{\int_0^\infty \xi^\alpha (1+b+\|z\|^2+\xi)^{-(p+n)/2-1}\mathrm{d}\xi}\right)^{-1} \qquad (K.1) \\
&= \left(1+b+(1+b+\|z\|^2)\frac{B(\alpha+2,(p+n)/2-\alpha-1)}{B(\alpha+1,(p+n)/2-\alpha)}\right)^{-1} \\
&= \left(1+b+(1+b+\|z\|^2)\frac{\alpha+1}{(p+n)/2-\alpha-1}\right)^{-1}.
\end{aligned}
$$

Let $a = \{(p+n)/2-\alpha-1\}/(\alpha+1)$. Then the Bayes equivariant estimator is

$$
\left(1-\frac{a}{\|X\|^2/S+(a+1)(b+1)}\right)X. \qquad (K.2)
$$

When $\alpha+\beta < -1$ or equivalently $\alpha < n/2-1$ as well as $\alpha > -1$, this is a proper Bayes equivariant estimator. When $-1 \le \alpha+\beta \le 0$ or equivalently $n/2-1 \le \alpha \le n/2$ as well as $\alpha > -1/2$, this is an admissible generalized Bayes equivariant estimator. Hence when $a \ge (p-2)/(n+2)$ and $b \ge 0$, the estimator (K.2) is admissible within the class of equivariant estimators.

Appendix L: Lemma for the risk comparison in Section 4.2

Lemma L.1. *1.* $\psi^+_{\mathrm{JS}}(w) > \psi_0(w) > \psi_{\mathrm{SB}}(w)$ *for all* $w > 0$.

2. $\liminf_{\lambda\to\infty} \lambda^2 \{R(\theta,\eta,\delta_0) - R(\theta,\eta,\delta_{\mathrm{SB}})\} \ge 4c_{p,n}(c_{p,n}+1)^2(n-2)^2$.

3. $\liminf_{\lambda\to\infty} \lambda^{n/2+2} \left\{R(\theta,\eta,\delta^+_{\mathrm{JS}}) - R(\theta,\eta,\delta_0)\right\} \ge \dfrac{(n-2)^{n/2+2}}{B(p/2-1,n/2+2)}$.

Proof. [Part 1] For the part $\psi^+_{\mathrm{JS}}(w) > \psi_0(w)$, it is clear that

$$
\psi_0(w) = \frac{\int_0^\infty (1+\xi)^{n/2}(1+w+\xi)^{-(p+n)/2-1}\mathrm{d}\xi}{\int_0^\infty (1+\xi)^{n/2+1}(1+w+\xi)^{-(p+n)/2-1}\mathrm{d}\xi} < 1.
$$

Further $w\psi_0(w)|_{w=0} = 0$, $w\psi_0(w)$ is increasing in w and $\lim_{w\to\infty} w\psi_0(w) = (p-2)/(n+2)$ as shown in Kubokawa (1991) and Kubokawa (1994). Hence $\psi_0(w) < \min(1,(p-2)/\{(n+2)w\}) = \psi^+_{\mathrm{JS}}(w)$.

For the part $\psi_0(w) > \psi_{\mathrm{SB}}(w)$, recall $\psi_{\mathrm{SB}}(w)$ corresponds to ψ_π given by (K.1) with $\alpha = n/2$, $\beta = -n/2$ and $b = 0$, which is

$$\psi_{\mathrm{SB}}(w) = \frac{\int_0^\infty \{\xi/(1+\xi)\}^{n/2}(1+\xi)^{n/2}(1+w+\xi)^{-(p+n)/2-1}\mathrm{d}\xi}{\int_0^\infty \{\xi/(1+\xi)\}^{n/2}(1+\xi)^{n/2+1}(1+w+\xi)^{-(p+n)/2-1}\mathrm{d}\xi}.$$

Since $\{\xi/(1+\xi)\}^{n/2}$ is increasing in ξ, the correlation inequality gives

$$\psi_{\mathrm{SB}}(w) < \frac{\int_0^\infty (1+\xi)^{n/2}(1+w+\xi)^{-(p+n)/2-1}\mathrm{d}\xi}{\int_0^\infty (1+\xi)^{n/2+1}(1+w+\xi)^{-(p+n)/2-1}\mathrm{d}\xi} = \psi_0(w).$$

[Parts 2 and 3] As in Efron and Morris (1976) as well as Maruyama and Strawderman (2017), an unbiased estimator of the risk, $R(\theta, \eta; \delta_\phi)$, for an estimator of the form $\delta_\phi = \{1 - (S/\|X\|^2)\phi(\|X\|^2/S)\}X$ is given by

$$p + (n+2)D(W;\phi) \tag{L.1}$$

where

$$D(w;\phi) = \frac{\{\phi(w) - 2c_{p,n}\}\phi(w)}{w} - d_n\phi'(w)\{1+\phi(w)\}, \tag{L.2}$$

with $c_{p,n} = (p-2)/(n+2)$ and $d_n = 4/(n+2)$. For the James-Stein positive-part estimator with $\phi^+_{\mathrm{JS}}(w) = w\psi^+_{\mathrm{JS}}(w)$, we have

$$D(0;\phi^+_{\mathrm{JS}}) = -(2c_{p,n} + d_n) \text{ and } D(w;\phi^+_{\mathrm{JS}}) = -\frac{c^2_{p,n}}{w}.$$

For δ_{SB} with $\phi_{\mathrm{SB}}(w) = w\psi_{\mathrm{SB}}(w)$ and δ_0 with $\phi_0(w) = w\psi_0(w)$, we have

$$\begin{aligned} &D(0;\phi_{\mathrm{SB}}) = -\frac{c_{p,n}}{c_{p,n}+1}(2c_{p,n}+d_n) \\ &\text{and } \lim_{w\to\infty} w^2\left(D(w;\phi_{\mathrm{SB}}) + \frac{c^2_{p,n}}{w}\right) = -d_n c_{p,n}(c_{p,n}+1)^2, \end{aligned}$$

and

$$\begin{aligned} &D(0;\phi_0) = -\frac{p-2}{p}(2c_{p,n}+d_n) \\ &\text{and } \lim_{w\to\infty} w^{n/2+2}\left(D(w;\phi_0) + \frac{c^2_{p,n}}{w}\right) = -\frac{1}{B(p/2-1, n/2+2)}. \end{aligned}$$

Hence there exists $q > 0$ such that

$$D(w;\phi_0) - D(w;\phi_{\mathrm{SB}}) \geq \frac{d_n c_{p,n}(c_{p,n}+1)^2}{(w+q)^2} - qI_{[0,1]}(w)$$

for all $w > 0$. Then we have

$$
\begin{aligned}
&R(\theta, \eta, \delta_0) - R(\theta, \eta, \delta_{\mathrm{SB}}) \\
&\geq 4c_{p,n}(c_{p,n}+1)^2 E\left[(W+q)^{-2}\right] - q(n+2)\Pr\left(0 < \frac{W}{W+1} < \frac{1}{2}\right)
\end{aligned}
$$

where $W/(W+1)$ has the type I noncentral beta distribution with shape parameters $p/2$ and $n/2$ and noncentrality parameter $\lambda = \eta\|\theta\|^2$.

Let $\nu = \min\{x \in \mathbb{Z} \mid n/2 \leq x\}$ and let V be the type I noncentral beta random variable with shape parameters $p/2$ and ν and noncentrality parameter λ. Then Nicholson (1954) and Hodges (1955) show that

$$
\Pr\left(0 < V < \frac{1}{2}\right) = \exp(-\lambda/2)\sum_{k=0}^{\nu-1}\frac{(\lambda/2)^k}{k!}\frac{\int_0^{1/2} t^{p/2+k-1}(1-t)^{\nu-k-1}\mathrm{d}t}{B(p/2+k, \nu-k)},
$$

which is smaller than $\nu\exp(-\lambda/2)\lambda^{\nu-1}$ for $\lambda > 1$. Note

$$
\Pr\left(0 < W/(1+W) < 1/2\right) \leq \Pr\left(0 < V < 1/2\right).
$$

Then, by Jensen's inequality,

$$
\begin{aligned}
&R(\theta, \eta, \delta_0) - R(\theta, \eta, \delta_{\mathrm{SB}}) \\
&\geq 4c_{p,n}(c_{p,n}+1)^2 (E\left[W\right]+q)^{-2} - q(n+2)\nu\exp(-\lambda/2)\lambda^{\nu-1} \\
&= 4c_{p,n}(c_{p,n}+1)^2 \left(\frac{p+\lambda}{n-2}+q\right)^{-2} - q(n+2)\nu\exp(-\lambda/2)\lambda^{\nu-1}
\end{aligned}
$$

for $\lambda \geq 1$, which implies that

$$
\liminf_{\lambda\to\infty} \lambda^2\left\{R(\theta, \eta, \delta_0) - R(\theta, \eta, \delta_{\mathrm{SB}})\right\} \geq 4c_{p,n}(c_{p,n}+1)^2(n-2)^2.
$$

The proof of Part 3 is omitted since it is similar to the proof of Part 2. The key is the existence of $r > 0$ such that

$$
D(w;\phi_{\mathrm{JS}}^{+}) - D(w;\phi_0) \geq \frac{1}{(w+r)^{n/2+2}B(p/2-1, n/2+2)} - rI_{[0,1]}(w)
$$

for all $w > 0$. □

Appendix M: Numerical study of the risk in different cases

In Section 4.2 of the main paper, the risk functions of several of the estimators in Corollaries 4.1 and 4.3 are presented in Figure 1 of the main paper, for $p = 10$ and $n = 10$. Graphs in the cases ($p = 15$ and $n = 5$) and ($p = 5$ and $n = 15$) are provided in Figures 2 and 3, respectively.

In each figure, the first graph presents the risks in the Gaussian case, for four estimators (James-Stein estimator, James-Stein positive part estimator, "simple Bayes" estimator given by Corollary 4.3 with $a = (p-2)/(n+2)$ and $b = 0$, and "Harmonic Bayes" estimator given by Corollary 4.1). The second graph gives the corresponding risks for the case of a scaled multivariate-t distribution with 10 degrees of freedom.

As we pointed out in Section 4.2 of the main paper, the risk of the Harmonic Bayes estimator is uniformly smaller than that of James-Stein estimator. Also when $\eta^{1/2}\|\theta\| = 0$, the two risks are equal as shown in Kubokawa (1994). For larger $\eta^{1/2}\|\theta\|$, the risk of the simple Bayes estimator is smallest among four estimators. This is somewhat natural since the simple Bayes estimator is admissible among the class $\mathcal{D}_\psi$ and its risk for smaller $\eta^{1/2}\|\theta\|$ is relatively large. The relative risk behaviors in all cases given in Figures 1, 2 and 3 are largely similar.

Appendix N: Proof of Theorem 5.1

Let δ_{gi} be the proper Bayes estimator with respect to $g_i(\theta, \eta)$. Then the Bayes risk difference of x and δ_{gi} with respect to $g_i(\theta, \eta)$ is

$$\begin{aligned}
\Delta_i &= \int_{\mathbb{R}^p} \int_0^\infty \{R(\theta, \eta, X) - R(\theta, \eta, \delta_{gi})\}\, g_i(\theta, \eta) \mathrm{d}\theta \mathrm{d}\eta \\
&= \int_{\mathbb{R}^p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^p} \int_0^\infty \eta \left\{\|x - \theta\|^2 - \|\delta_{gi} - \theta\|^2\right\} \\
&\quad \times \eta^{(p+n)/2} f(\eta\{\|x - \theta\|^2 + \|u\|^2\}) g_i(\theta, \eta) \mathrm{d}x \mathrm{d}u \mathrm{d}\theta \mathrm{d}\eta \\
&= \int_{\mathbb{R}^p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^p} \int_0^\infty \|\delta_{gi} - x\|^2 \\
&\quad \times \eta^{(p+n)/2+1} f(\eta\{\|x - \theta\|^2 + \|u\|^2\}) g_i(\theta, \eta) \mathrm{d}x \mathrm{d}u \mathrm{d}\theta \mathrm{d}\eta \\
&= \int_{\mathbb{R}^p} \int_{\mathbb{R}^n} \frac{\left\|\int_{\mathbb{R}^p} \int_0^\infty \eta^{(p+n)/2} F(\eta\{\|x - \theta\|^2 + \|u\|^2\}) \nabla_\theta g_i(\theta, \eta) \mathrm{d}\theta \mathrm{d}\eta\right\|^2}{\int_{\mathbb{R}^p} \int_0^\infty \eta^{(p+n)/2+1} f(\eta\{\|x - \theta\|^2 + \|u\|^2\}) g_i(\theta, \eta) \mathrm{d}\theta \mathrm{d}\eta} \mathrm{d}x \mathrm{d}u.
\end{aligned}$$

Note

$$\nabla_\theta g_i(\theta, \eta) = 4\eta\theta h_i(\eta\|\theta\|^2) h_i'(\eta\|\theta\|^2).$$

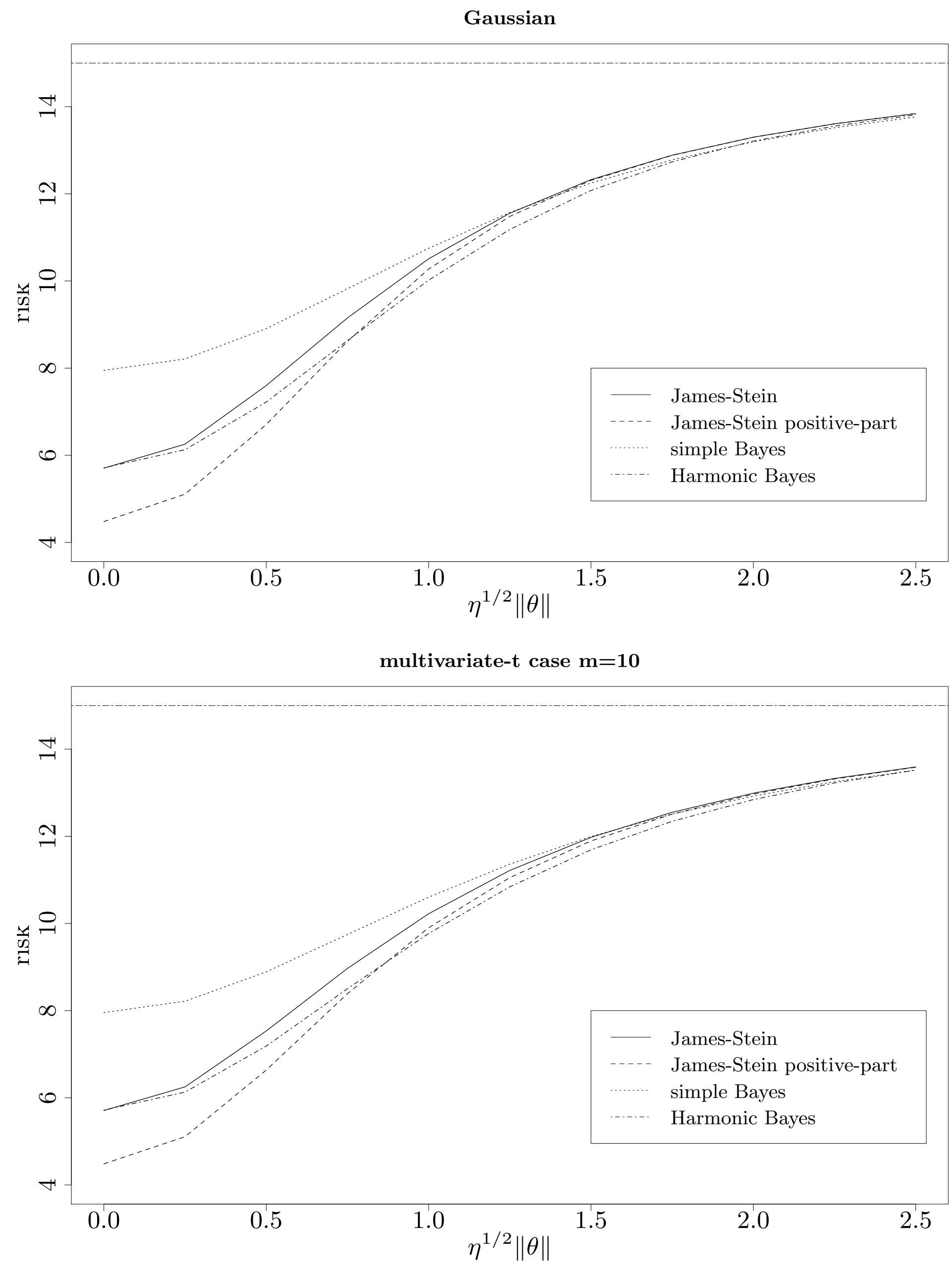

Fig 2. *Risk (Gaussian and multivariate-t$(m = 10)$) with $p = 15$ and $n = 5$*

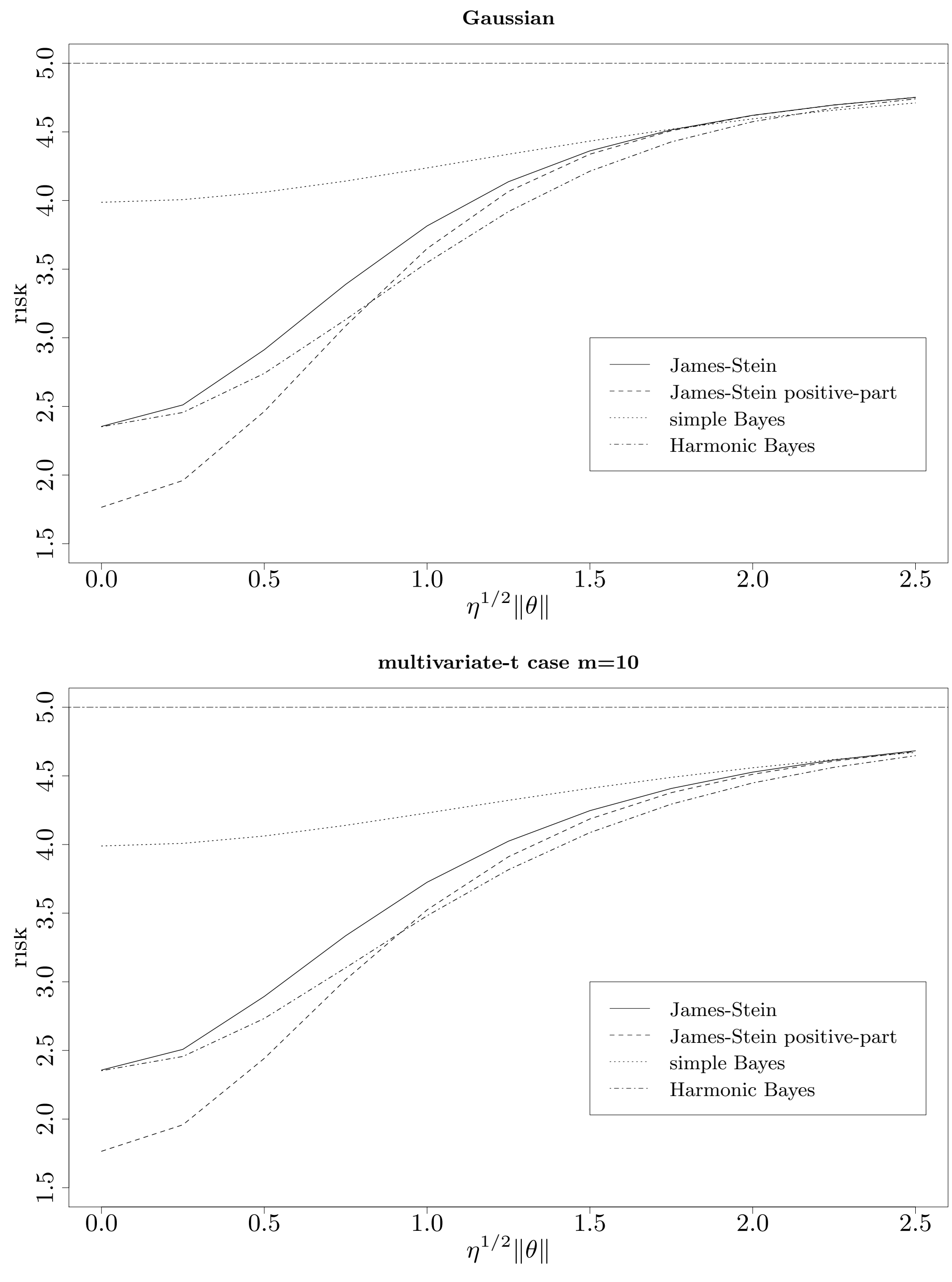

FIG 3. *Risk (Gaussian and multivariate-t($m = 10$)) with $p = 5$ and $n = 15$*

Then, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\Delta_i &\leq \int_{\mathbb{R}^p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^p}\int_0^\infty \eta^{(p+n)/2-1}\frac{F(\eta\{\|x-\theta\|^2+\|u\|^2\})^2}{f(\eta\{\|x-\theta\|^2+\|u\|^2\})} \\
&\quad\times \frac{\|\nabla_\theta g_i(\theta,\eta)\|^2}{g_i(\theta,\eta)}\mathrm{d}x\mathrm{d}u\mathrm{d}\theta\mathrm{d}\eta \\
&= c_{p+n}\int_0^\infty t^{(p+n)/2-1}\frac{F^2(t)}{f(t)}\mathrm{d}t\int_{\mathbb{R}^p}\int_0^\infty \frac{\|\nabla_\theta g_i(\theta,\eta)\|^2}{\eta g_i(\theta,\eta)}\mathrm{d}\theta\mathrm{d}\eta \qquad \text{(N.1)} \\
&= 16c_{p+n}A_2\int_{\mathbb{R}^p}\int_0^\infty \eta^{p/2}\eta\|\theta\|^2\left\{h_i'(\eta\|\theta\|^2)\right\}^2\eta^{-p/2}\pi(\eta)\mathrm{d}\theta\mathrm{d}\eta \\
&= 16c_pc_{p+n}A_2\int_0^\infty \frac{\pi(\eta)}{\eta^{p/2}}\mathrm{d}\eta\int_0^\infty \lambda^{p/2}\sup_i\left\{h_i'(\lambda)\right\}^2\mathrm{d}\lambda
\end{aligned}$$

where $A_2 = \int_0^\infty t^{(p+n)/2-1}\{F^2(t)/f(t)\}\mathrm{d}t$ and it is bounded under Assumptions F.1, F.2 and F.3.1 on f, as in Part 1.B of Lemma E.3. Further, by Lemma E.2 of Appendix E, $\int_0^\infty \lambda^{p/2}\sup_i\left\{h_i'(\lambda)\right\}^2\mathrm{d}\lambda < \infty$ for $p = 1,2$. Hence, by the dominated convergence theorem, we have $\Delta_i \to 0$ as $i \to \infty$. By the Blyth sufficient condition, the admissibility of X for $p = 1,2$ follows.

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