Problem 1.

① $\exists a, a \in m \wedge \exists x, x \in S \wedge (Y = f(x)) \wedge (Y_i - Y_a \leq 0)$

② $\exists a, a \in m \wedge \exists x, x \in S \wedge (Y = f(x)) \wedge (Y_a - Y_j \leq 0)$

③ Maybe: $g(x) = (Y_i - Y_a)(Y_a - Y_j)$

Problem 2.

① $z^{(1)} = W^{(1)}x + b^{(1)} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 + 1 \\ 2x_1 - 2x_2 + 1 \end{pmatrix}$

$\because -1 \leq x_1 \leq 1 \quad -1 \leq x_2 \leq 1$

$\therefore \bar{z}^{(1)} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \underline{z}^{(1)} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$ and $\hat{z}^{(1)} = ReLU(z^{(1)})$

then $\bar{\hat{z}}^{(1)} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \underline{\hat{z}}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$z^{(2)} = W^{(2)}\hat{z}^{(1)} + b^{(2)}$

$= \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\hat{z}^{(1)} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\underline{z}^{(2)} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \quad \bar{z}^{(2)} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\hat{z}^{(2)} = ReLU(z^{(2)})$

$\underline{\hat{z}}^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\hat{z}}^{(2)} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\because Y = W^{(3)}(\hat{z}^{(2)} + z^{(1)}) \quad . \quad W^{(3)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\therefore \bar{Y} = 2 \quad \underline{Y} = -2$

$\therefore \quad -2 \leq Y \leq 2$

②

$$Z^{(2)} = W^{(2)}\hat{z}^{(1)} + b^{(2)} = W^{(2)}\,\mathrm{ReLU}(Z^{(1)}) + b^{(2)}$$

$$\because \begin{pmatrix}-1\\-3\end{pmatrix} \le Z^{(1)} \le \begin{pmatrix}3\\5\end{pmatrix} \Rightarrow \begin{pmatrix}\frac{3}{4}Z^{(1)}\\ \frac{5}{8}Z^{(1)}\end{pmatrix} \le \mathrm{ReLU}(Z^{(1)}) \le \begin{pmatrix}\frac{3}{4}Z^{(1)}+\frac{3}{4}\\ \frac{5}{8}Z^{(1)}+\frac{15}{8}\end{pmatrix}$$

$$\therefore W^{(2)}\begin{pmatrix}\frac{3}{4}Z^{(1)}\\ \frac{5}{8}Z^{(1)}\end{pmatrix} + b^{(2)} \le Z^{(2)} \le W^{(2)}\begin{pmatrix}\frac{3}{4}Z^{(1)}+\frac{3}{4}\\ \frac{5}{8}Z^{(1)}+\frac{15}{8}\end{pmatrix} + b^{(2)}$$

$$\begin{pmatrix}\frac{1}{8}Z^{(1)}+2\\ \frac{1}{4}Z^{(1)}+2\end{pmatrix} \le Z^{(2)} \le \begin{pmatrix}\frac{1}{8}Z^{(1)}+\frac{7}{8}\\ \frac{1}{4}Z^{(1)}-\frac{1}{4}\end{pmatrix}$$

$$\because Z^{(1)} = \begin{pmatrix}x_1 - x_2 + 1\\ 2x_1 - 2x_2 + 1\end{pmatrix}$$

$$\therefore \begin{pmatrix}\frac{1}{8}x_1 - \frac{1}{8}x_2 + \frac{17}{8}\\ \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{9}{4}\end{pmatrix} \le Z^{(2)} \le \begin{pmatrix}\frac{1}{8}x_1 - \frac{1}{8}x_2 + 1\\ \frac{1}{2}x_1 - \frac{1}{2}x_2\end{pmatrix}$$

$$\because |u_j^{(i)}| < |l_j^{(i)}| \text{ and } -1 \le x_1 \le 1 \quad -1 \le x_2 \le 1$$

$$\therefore \quad 0 \le \hat{z}^{(2)} \le \begin{pmatrix}\frac{5}{4}\\ 1\end{pmatrix}$$

③ $Y = W^{(3)}(\hat{z}^{(2)} + z^{(1)})$

$\Downarrow$

$$\begin{pmatrix}x_1 - x_2 + 1\\ 2x_1 - 2x_2 + 1\end{pmatrix} \le \hat{z}^{(2)} + z^{(1)} \le \begin{pmatrix}\frac{9}{8}x_1 - \frac{9}{8}x_2 + 2\\ \frac{5}{2}x_1 - \frac{5}{2}x_2 + 1\end{pmatrix}$$

$$x_1 - x_2 \le Y \le \frac{11}{8}x_1 - \frac{11}{8}x_2 - 1$$

$$-2 \le Y \le \frac{14}{8}$$

④ 4α.