New Optimization Criteria for Fitting LTI Filters for Audio Signals

Yonatan Vaizman

I. EXCITATION-FILTER MODEL

We model the generation of an observed time signal $x(n)$ as excitation signal $e(n)$ passed through an all-pole filter $A(z)$. The underlying model's excitation signal $e(n)$ can also be viewed as a prediction error signal of linear forward prediction on the observed signal $x(n)$, when the prediction coefficients are $-a_1, -a_2, \ldots, -a_M$: the prediction of the signal at time n is

$$\widehat{x}(n) = \sum_{k=1}^{M} -a_k x(n-k)$$

and the prediction error is $e(n) = x(n) - \widehat{x}(n)$. $e(n)$ can also be described as the output of a FIR filter, with coefficients $1, a_1, a_2, \ldots, a_M$, applied to the observed signal:

$$
\begin{aligned}
e(n) =& (1, \underline{a}^T)\underline{x}_{M+1}(n) \\
=& x(n) + \underline{a}^T \underline{x}_M(n-1) \\
=& x(n) + \sum_{k=1}^{M} a_k x(n-k)
\end{aligned}
$$

Linear predictive coding (LPC) framework aims to optimize some objective over the choice of the all-pole filter $A(z)$, meaning over the choice of the filter coefficient vector $\underline{a} \in \mathbb{R}^M$.

A. Useful definitions

Lets define as $r(m)$ the autocorrelation of the observed signal $(x(n))$ at lag m:

$$r(m) = E[x(n)x(n-m)].$$

Lets also define as R_M (or R for short) the autocorrelation matrix of the observed signal, which is a symmetric, PSD, Toeplitz matrix of dimension $M \times M$:

$$
R_M =
\begin{bmatrix}
r(0) & r(1) & r(2) & \ldots & r(M-1) \\
r(-1) & r(0) & r(1) & \ldots & r(M-2) \\
\vdots & & & & \vdots \\
r(1-M) & r(2-M) & r(3-M) & \ldots & r(0)
\end{bmatrix}
=
\begin{bmatrix}
r(0) & r(1) & r(2) & \ldots & r(M-1) \\
r(1) & r(0) & r(1) & \ldots & r(M-2) \\
\vdots & & & & \vdots \\
r(M-1) & r(M-2) & r(M-3) & \ldots & r(0)
\end{bmatrix}.
$$

Lets also define as $\underline{d}_M$ (or $\underline{d}$ for short) the vector of cross correlation between the predicted variable and the variables used for the linear prediction. In our case (forward prediction in a time series) the values of this cross-correlation vector are taken from the autocorrelation of the observed signal:

$$
d_M =
\begin{bmatrix}
r(1) \\
r(2) \\
\vdots \\
r(M)
\end{bmatrix}.
$$

II. MEAN SQUARED ERROR CRITERION

The traditional approach is to minimize the mean squared error signal:

$$\min_{\underline{a}} E[|e(n)|^2].$$

This approach was studied a lot and used a lot. The problem is convex — a quadratic programming problem — and it has a unique solution that should satisfy the Yule-Walker equations:

$$R\underline{a} = \underline{d}.$$

If R has no zero eigenvalues (if there is still some error with best order-M linear prediction) then it is also invertible and the solution can be calculated by inverting it:

$$\underline{a} = R^{-1}\underline{d}.$$

The complexity of the matrix-inversion approach is $O(M^3)$.

The Levinson-Durbin algorithm (LDA) is an order-recursive algorithm to find the optimal order-1 linear filter, then the optimal order-2 linear filter, ...up to the optimal order-M linear filter. It is an efficient algorithm, which is also based on the autocorrelation $r(m)$ but doesn't involve inverting the correlation matrix. It's complexity is $O(M^2)$.

The mean squared error approach has many advantages:

- Not only is this problem convex, it also has a closed form solution — the Yule-Walker normal equations (no need for any iterative algorithm to converge).
- Not only it has a closed form solution, it also has an even more efficient method to solve — Levinson-Durbin algorithm.
- The optimal filter is guaranteed to be minimum phase, so both the IIR all-pole filter and its inverse — the FIR all-zero filter — are stable filters.
- The scheme lends itself nicely to lattice filters and allows using a modular filter structure, and representation of the filter using reflection coefficients.
- The scheme lends itself nicely to adaptive filtering, with the LMS algorithm for filtering and tracking slow changes.

With all these strengths, it is possible that the criterion of minimizing mean squared error (or $L2$-norm of the excitation signal) is not fitting for modeling generation of musical sounds. MSE fits well when assuming the excitation signal is a Gaussian white noise. However, to model pitched sounds we would like to assume the excitation to be a "colorless" periodic signal. The gross shape of the spectrum should be colorless, but there should be very specific loci of energy, in a fundamental frequency and its integer multiples. We would like the excitation signal to resemble a pulse train, to have a comb-shape spectrum and autocorrelation function.

III. Mean absolute error

IV. Absolute error autocorrelation

If we want the excitation signal to resemble a pulse train, we may want to require it to have no correlation between close time points (up until lags that are reasonable for pitch periods). To require low-magnitude error-correlation in lag l we'll minimize the following cost function:

$$
\begin{aligned}
J^l(\underline{a}) &= |E\left[e(n)e(n-l)\right]| \\
&= \left|E\left[(x(n) + \underline{a}^T\underline{x}_M(n-1))(x(n-l) + \underline{a}^T\underline{x}_M(n-l-1))\right]\right| \\
&= \left|E\left[x(n)x(n-l)\right] + E\left[\underline{a}^T\underline{x}_M(n-1)x(n-l)\right] + E\left[\underline{a}^T\underline{x}_M(n-l-1)x(n)\right] + E\left[\underline{a}^T\underline{x}_M(n-1)\underline{x}_M^T(n-l-1)\underline{a}\right]\right| \\
&= \left| r(l) + \underline{a}^T \begin{bmatrix} r(l-1) \\ r(l-2) \\ \vdots \\ r(l-M) \end{bmatrix} + \underline{a}^T \begin{bmatrix} r(l+1) \\ r(l+2) \\ \vdots \\ r(l+M) \end{bmatrix} + \underline{a}^T \begin{bmatrix} r(l) & r(l+1) & \dots & r(l+M-1) \\ r(l-1) & r(l) & \dots & r(l+M-2) \\ \vdots & & & \\ r(l-M+1) & r(l-M+2) & \dots & r(l) \end{bmatrix} \underline{a} \right|.
\end{aligned}
$$

To simplify the notation, lets define some more variables:

$$
\underline{d}_M^{l-} = \begin{bmatrix} r(l-1) \\ r(l-2) \\ \vdots \\ r(l-M) \end{bmatrix}
$$

$$
\underline{d}_M^{l+} = \begin{bmatrix} r(l+1) \\ r(l+2) \\ \vdots \\ r(l+M) \end{bmatrix}
$$

$$
R_M^l = \begin{bmatrix} r(l) & r(l+1) & \dots & r(l+M-1) \\ r(l-1) & r(l) & \dots & r(l+M-2) \\ \vdots & & & \\ r(l-M+1) & r(l-M+2) & \dots & r(l) \end{bmatrix}
$$

Lets notice that the matrix R_M^l is still in Toeplitz structure, but it is no longer symmetric or positive semi-definite. Using these notations the cost function we present can be expressed as:

$$J^l(\underline{a}) = \left| r(l) + \underline{a}^T \left(\underline{d}_M^{l-} + \underline{d}_M^{l+} \right) + \underline{a}^T R_M^l \underline{a} \right|.$$

The optimization problem is now:

$$\min_{\underline{a}} J^l(\underline{a}).$$

If we didn't require the absolute value in the cost function, we would be able to get a closed form solution, similar to the MSE minimization solution: $\underline{a} = \left(R_M^l + R_M^{lT} \right)^{-1} \left(\underline{d}_M^{l-} + \underline{d}_M^{l+} \right)$. However, such a criterion might find a solution that creates an excitation signal with strong negative correlation at lag l, since we wish to minimize it. This is not what we want; we want the correlation to be close to zero. With the absolute value we can no longer search for a point with zero gradient, but we can still use the gradient for iterative improvements.

$$\begin{aligned}
\frac{\partial J^l}{\partial \underline{a}} &= sign(\cdot) \frac{\partial \cdot}{\partial \underline{a}} \\
&= sign \left(r(l) + \underline{a}^T \left(\underline{d}_M^{l-} + \underline{d}_M^{l+} \right) + \underline{a}^T R_M^l \underline{a} \right) \left[\underline{d}_M^{l-} + \underline{d}_M^{l+} + \left(R_M^l + R_M^{lT} \right) \underline{a} \right].
\end{aligned}$$

We can use this derivative in a gradient descent way in order to minimize the cost. We shouldn't expect to reach a fix point in the filter parameters, where the derivative is zero, but we can attempt to get convergence of the cost value to a locally minimal cost.

Using this cost function we can define a composite cost function to require close-to-zero error correlation in multiple lags:

$$J^L(\underline{a}) = \sum_{l \in L} J^l(\underline{a})$$

and the derivative of the composite cost function will be a sum of the individual components' derivatives:

$$\frac{\partial J^L}{\underline{a}} = \sum_{l \in L} \frac{\partial J^l}{\underline{a}}.$$

A reasonable set of lags for the criterion can be of the form $L = \{0, 1, \ldots, l_{max}\}$.

A. How to solve

1) Stochastic gradient descent: With a stochastic gradient descent (SGD) approach we can optimize the filter for a composite cost function more quickly by selecting a single lag to optimize J^l for in every iteration. As traditional SGD samples a subset (minibatch) of examples from the pool of examples and updates the parameters based on the subset, we can select a component of the sum-cost function and update the parameters over all the examples (the entire signal segment).

2) Convex relaxation: We can observe the cost function:

$$J^L(\underline{a}) = \sum_{l \in L} |r(l) + \underline{a}^T \left(\underline{d}_M^{l-} + \underline{d}_M^{l+} \right) + \underline{a}^T R_M^l \underline{a}|.$$

And reformulate the problem as a quadratically constrained quadratic programming (QCQP), by defining additional parameters (or actually quadratically constrained linear programming):

$$\min_{\underline{a} \in \mathbb{R}^M, \widetilde{\underline{J}} \in \mathbb{R}^{|L|}} \mathbf{1}^T \widetilde{\underline{J}}$$

$$s.t$$

$$\widetilde{\underline{J}} \succeq \underline{0},$$

$$\forall l \in L :$$

$$r(l) + \underline{a}^T \left(\underline{d}_M^{l-} + \underline{d}_M^{l+} \right) + \underline{a}^T R_M^l \underline{a} \leq \widetilde{J}_l,$$

$$r(l) + \underline{a}^T \left(\underline{d}_M^{l-} + \underline{d}_M^{l+} \right) + \underline{a}^T R_M^l \underline{a} \geq -\widetilde{J}_l.$$

We must notice, however, that this QCQP is **not convex** since the matrices R_M^l are not positive semidefinite (PSD). First lets notice that since $\underline{a}^T R_M^l \underline{a}$ is a scalar the following holds:

$$\underline{a}^T R_M^l \underline{a} =$$
$$\frac{1}{2}\left(\underline{a}^T R_M^l \underline{a} + \underline{a}^T R_M^l \underline{a}\right) =$$
$$\frac{1}{2}\left(\underline{a}^T R_M^l \underline{a} + \left(\underline{a}^T R_M^l \underline{a}\right)^T\right) =$$
$$\frac{1}{2}\left(\underline{a}^T R_M^l \underline{a} + \underline{a}^T R_M^{lT} \underline{a}\right) =$$
$$\underline{a}^T \left(\frac{R_M^l + R_M^{lT}}{2}\right)\underline{a}.$$

So the fact that the matrices R_M^l are not symmetric is not an issue — we can calculate their symmetric versions $R_{M,sym}^l = \frac{R_M^l + R_M^{lT}}{2}$ and the problem will stay equivalent to the original non-convex QCQP. The natural convex relaxation for this problem is achieved by positivizing these matrices — projecting the symmetric versions $R_{M,sym}^l$ to the PSD cone (by eigenvalue decomposition, zeroing all the negative eigenvalues and re-composing the matrices) — to the convex versions $R_{M,cvx}^l$. The new problem:

$$\min_{\underline{a}\in\mathbb{R}^M, \widetilde{\underline{J}}\in\mathbb{R}^{|L|}} \mathbf{1}^T \widetilde{\underline{J}}$$
$$s.t$$
$$\widetilde{\underline{J}} \succeq \underline{0},$$
$$\forall l \in L:$$
$$r(l) + \underline{a}^T \left(\underline{d}_M^{l-} + \underline{d}_M^{l+}\right) + \underline{a}^T R_{M,cvx}^l \underline{a} \leq \widetilde{J}_l,$$
$$r(l) + \underline{a}^T \left(\underline{d}_M^{l-} + \underline{d}_M^{l+}\right) + \underline{a}^T R_{M,cvx}^l \underline{a} \geq -\widetilde{J}_l$$

is now a convex QCQP and is guaranteed to have a single global minimal value. It still might have a whole set of optimal solutions, and still there is the issue of efficient computation of the optimum. Of-the-shelf QCQP solvers might take too much time.

3) Order-recursive algorithm: To ease the computation, there is the possibility to define a Burg-like algorithm by solving for a lattice filter structure, looking for the reflection coefficients, one by one, recursively, by using the latest order's residual prediction errors (forward and backward) to find the next reflection coefficient. For every stage the optimization is over a single scalar value in the segment $[-1, +1]$, so a simple line search can be performed to converge to a local minimum (for the general problem, or global minimum for the convex relaxation). I'm not sure if this order-greedy procedure guarantees any optimality for the final resulted filter.