Markowitz

Asset Pricing I

Yvan Lengwiler, University of Basel

1 Exercise. Take a look at the file `Markowitz-unfinished.r`. The file contains `R` code that downloads data from `finance.yahoo.com` for 16 large US stocks as well as the S&P 500 index. You can study this code to understand how it works, or you can just use it.

Add to this program so that it does the following: Compute the mean return and the volatility (standard deviation) or the return of each equity separately. Make a scatterplot of these computations with the mean on the vertical and the volatility (same a standard deviation) on the horizontal axis. This means that each equity will be represented by one point (representing its mean return and volatility) in the chart.

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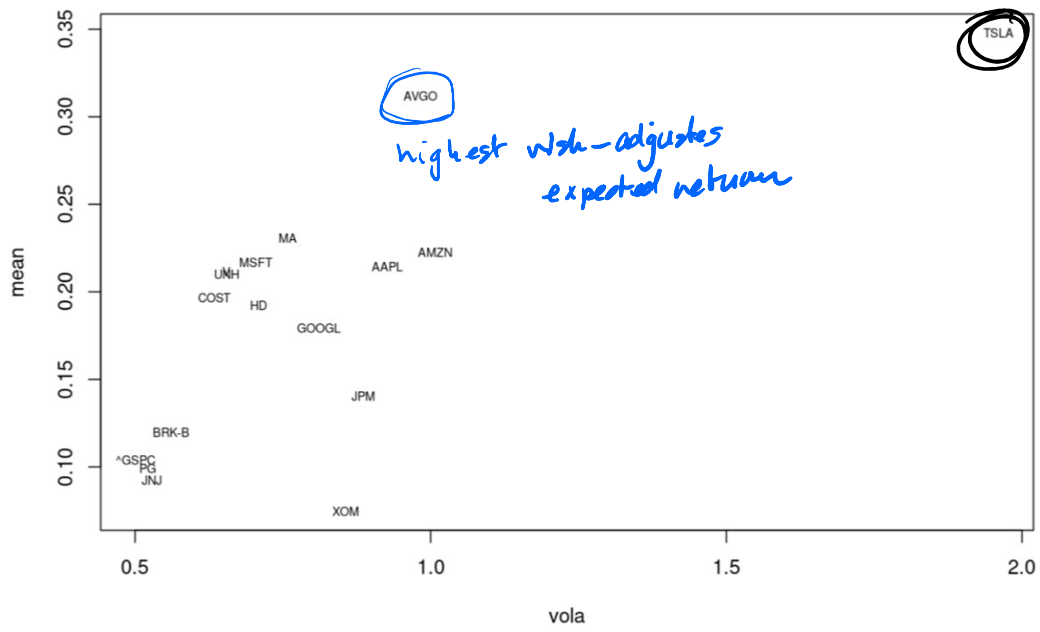
```
# **** compute moments *******************************

mu    <- colMeans(yield)      # mean return of assets
Omega <- cov(yield)           # variance-covariance matrix
sigma_assets <- sqrt(diag(Omega))  # vol. of each asset

# **** mean-volatility plot of assets ******************

plot(sigma_assets, mu, type='n', ylab='mean', xlab='vola')
text(sigma_assets, mu, assets, cex=0.6)
```



2 Question. Read the first paragraph. Markowitz seems to suggest a functional form for a utility function. What form would this function take?

PORTFOLIO SELECTION*

HARRY MARKOWITZ
The Rand Corporation

THE PROCESS OF SELECTING a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio. This paper is concerned with the second stage. We first consider the rule that the investor does (or should) maximize discounted expected, or anticipated, returns. This rule is rejected both as a hypothesis to explain, and as a maximum to guide investment behavior. We next consider the rule that the investor does (or should) consider expected return a desirable thing *and* variance of return an undesirable thing. This rule has many sound points, both as a maxim for, and hypothesis about, investment behavior. We illustrate geometrically relations between beliefs and choice of portfolio according to the "expected returns—variance of returns" rule.

$\max E[r] \longrightarrow$ zero diversification

$\max E[r] - var(r)$, $E[r] - std(r)$, ...

$E[r] - \gamma \cdot std(r)$, ...

3 Question. Read the next paragraph. Since payoffs of investments happen in the future, we need to discount them. Markowitz mentions — as other authors of his time have done — that we could use a different discount rate depending on the risk assiciated with a particular payoff. Question: Would the discount rate be higher or lower if the payoff is more risky, and why?

One type of rule concerning choice of portfolio is that the investor does (or should) maximize the discounted (or capitalized) value of future returns.[1] Since the future is not known with certainty, it must be "expected" or "anticipated" returns which we discount. Variations of this type of rule can be suggested. Following Hicks, we could let "anticipated" returns include an allowance for risk.[2] Or, we could let the rate at which we capitalize the returns from particular securities vary with risk.

more risk → higher discount rate

4 Question. Read the third and fourth paragraph (bottom of page 77 and top of page 78). Explain his argument why the risk-adjusted discounting is unsatisfactory.

The hypothesis (or maxim) that the investor does (or should) maximize discounted return must be rejected. If we ignore market imperfections the foregoing rule never implies that there is a diversified portfolio which is preferable to all non-diversified portfolios. Diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim.

The foregoing rule fails to imply diversification no matter how the anticipated returns are formed; whether the same or different discount rates are used for different securities; no matter how these discount rates are decided upon or how they vary over time.[3] The hypothesis implies that the investor places all his funds in the security with the greatest discounted value. If two or more securities have the same value, then any of these or any combination of these is as good as any other.

$$\max \frac{1}{\rho} E[r]$$

$\rho \uparrow$ the more risky the asset is.

$\rightarrow$ no diversification

5 Task. Read from the paragraph on page 79 that starts with "The portfolio with maximum [...]" until the end of page 81. This is essentially a somewhat tedious statement of the formulas for expected return and variance of a portfolio (which is just a weighted sum of assets).

We first look at an example with just two assets, and then state the situation with n assets more compactly.

6 Task. There are two assets with stochastic returns $\tilde{r}_1$ and $\tilde{r}_2$. The means and volatilities are (μ_1, σ_1) and (μ_2, σ_2), respectively. Let ρ be the correlation coefficient between $\tilde{r}_1$ and $\tilde{r}_2$.

A portfolio is a mixture of the two assets, containing a share w of the first and $(1-w)$ of the second asset. Let x_w denote the stochastic return of the portfolio w, $x_w = wr_1 + (1-w)r_2$, and let μ_w and σ_w^2 denote the expected mean and variance of the porfolio's return.

- Derive a formula for μ_w.
- Derive a formula for σ_w^2.

$w = (w_1, w_2)$ weights

$$\mu_w = w_1 \mu_1 + w_2 \mu_2$$

$$\sigma_w^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2 \rho$$

$$w \Omega w' \qquad w^T \Omega w$$

$$Var(r_w) = E[(r_w - \mu_w)^2]$$

$$r_w = w_1 r_1 + w_2 r_2 \qquad r_w - \mu_w$$

$$\mu_w = w_1 \mu_1 + w_2 \mu_2$$

$$Var(r_w) = E[(r_w - \mu_w) \cdot (r_w - \mu_w)]$$

$$= E[(w_1 r_1 + w_2 r_2 - w_1 \mu_1 - w_2 \mu_2)^2]$$

$$(w_1 r_1 + w_2 r_2 - w_1 \mu_1 - w_2 \mu_2)(w_1 r_1 + w_2 r_2 - w_1 \mu_1 - w_2 \mu_2)$$

$$w_1^2 r_1^2 + w_1 w_2 r_1 r_2 \ldots$$

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- Derive a formula for μ_w.
- Derive a formula for σ_w^2.

Your answer here: We obviously have $\mu_w = w\mu_1 + (1-w)\mu_2$.
What about the variance? To understand the variance of the mixture, note that

$$(wr_1 + (1-w)r_2) - (w\mu_1 + (1-w)\mu_2) = w(r_1 - \mu_1) + (1-w)(r_2 - \mu_2), \quad \text{and}$$
$$[w(r_1-\mu_1) + (1-w)(r_2-\mu_2)]^2 = \\ w^2(r_1-\mu_1)^2 + 2w(1-w)(r_1-\mu_1)(r_2-\mu_2) + (1-w)^2(r_2-\mu_2)^2.$$

The variance of the mixture is the expectation of this, so

$$\sigma_w^2 = w^2\sigma_1^2 + 2w(1-w)\sigma_{12} + (1-w)^2\sigma_2^2.$$

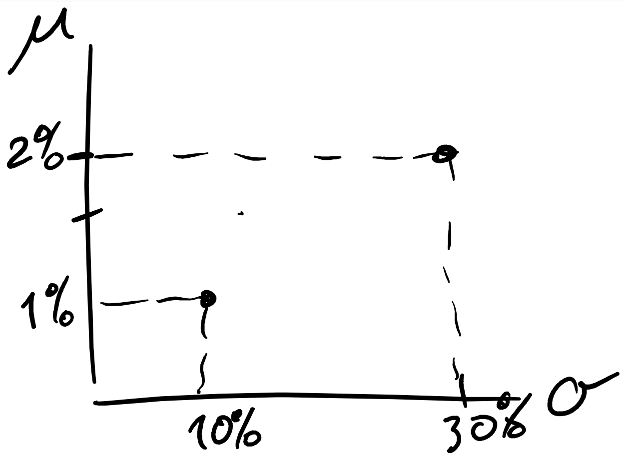
σ_{12} is the covariance of the two returns, and ρ is their correlation coefficient, so $\sigma_{12} = \rho\sigma_1\sigma_2$, and we can rewrite as,

$$\sigma_w^2 = w^2\sigma_1^2 + 2w(1-w)\rho\sigma_1\sigma_2 + (1-w)^2\sigma_2^2.$$

7 Task. Asset 1 and 2 have expected returns of 1% and 2% and standard errors of 10% and 30%, respectively. Using your formulae derived in the previous task, make a chart with the volatility (= standard deviation) of the portfolio (σ) on the horizontal and the expected return (μ) of the portfolio on the vertical axis as w goes from 0 to 1.

1. assume $\rho = 1$,
2. assume $\rho = -1$,
3. assume $\rho = 0$.

1%, 10%
2%, 30%



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1. assume $\rho = 1$,
2. assume $\rho = -1$,
3. assume $\rho = 0$.

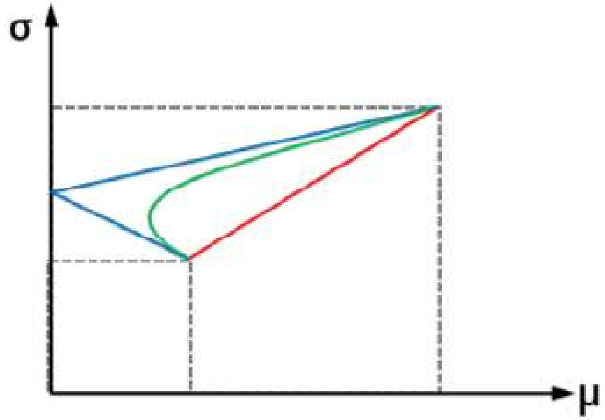
Your answer here: The volatility is the square root of the variance, so

$$\sigma(w) = \sqrt{w^2\sigma_1^2 + 2w(1-w)\rho\sigma_1\sigma_2 + (1-w)^2\sigma_2^2}.$$

For $\rho = 1$ and $\rho = -1$, this boils down to a binomial formula,

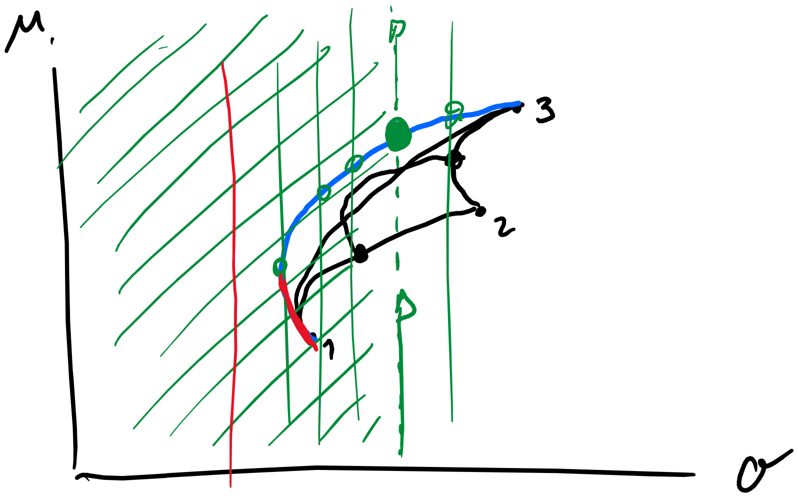
$$\begin{aligned} \rho = +1: &\quad \sigma = w\sigma_1 + (1-w)\sigma_2, \\ \rho = -1: &\quad \sigma = \left|w\sigma_1 - (1-w)\sigma_2\right|. \end{aligned}$$

This is linear in w for $\rho = +1$. Because $\mu(w)$ is also linear, this produces a straight line. For $\rho = -1$, we can make this zero by setting $w = \sigma_2/(\sigma_1+\sigma_2)$. So this produces a piecewise linear graph that hits zero somewhere in the middle.
The cases with interior ρ are somewhere between the two extremes and produce parabolas (quadratic curves).



8 Exercise. Add to your program. Compute the efficient frontier and plot it with σ on the horizontal and μ on the vertical axis (the section called `'unconstrained optimization'` in the R file). Do not disallow short sales to begin with, but make sure that 100% of capital is invested (meaning the sum of the weights of the protfolio must sum exactly to 1).

`A %*% B`	is the matrix product, and
`t(A)`	is the transpose of `A`.



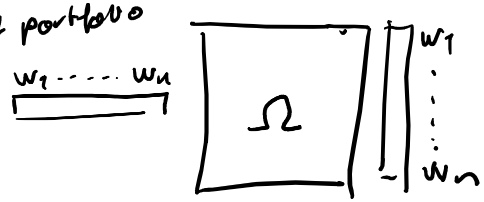
σ_0

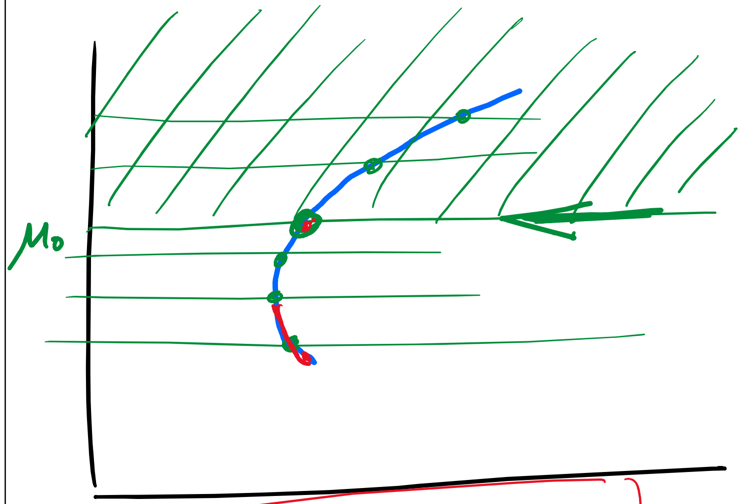
max μ by choosing w

so that $\sigma \le \sigma_0$

$$\min \; \underbrace{w^T \Omega w}_{} = \text{var of portfolio}$$

Sigma





$$\boxed{\text{min } \sigma \text{ by choosing } w \quad \text{so that } \mu \ge \mu_0}$$

8 Exercise. Add to your program. Compute the efficient frontier and plot it with σ on the horizontal and μ on the vertical axis (the section called `'unconstrained optimization'` in the R file). Do not disallow short sales to begin with, but make sure that 100% of capital is invested (meaning the sum of the weights of the protfolio must sum exactly to 1).

Minimize $V = w^{\mathsf{T}}\Omega w.$ subject to wμ >= some value and sum w_i = 1.

Solve a Quadratic Programming Problem

Description

This routine implements the dual method of Goldfarb and Idnani (1982, 1983) for solving quadratic programming problems of the form $\min(-d^T b + 1/2 b^T D b)$ with the constraints $A^T b >= b_0$.

Usage

```
solve.QP(Dmat, dvec, Amat, bvec, meq= , factorized=     )
```

Arguments

`Dmat`	matrix appearing in the quadratic function to be minimized.
`dvec`	vector appearing in the quadratic function to be minimized.
`Amat`	matrix defining the constraints under which we want to minimize the quadratic function.
`bvec`	vector holding the values of b_0 (defaults to zero).
`meq`	the first `meq` constraints are treated as equality constraints, all further as inequality constraints (defaults to 0).
`factorized`	logical flag: if `TRUE`, then we are passing R^{-1} (where $D = R^T R$) instead of the matrix D in the argument `Dmat`.

We have 2 constraints, namely

Sum w_i = 1
wμ = Sum w_i μ_i >= some value

We have to formulate this as a matrix A'w >= b_0. → HOW?

```
sum w = 1          ====> rep(1, nassets) %*% w = 1
mu * w >= mu(k)  ====>              mu %*% w >= b0
```

Amat should contain the left-hand sides of these equations:

```
    Amat <- t(rbind(rep(1, nassets), mu))
```

The first constraint is equality constraint, the 2nd is inequality constraint:

```
    meq  <- 1
```

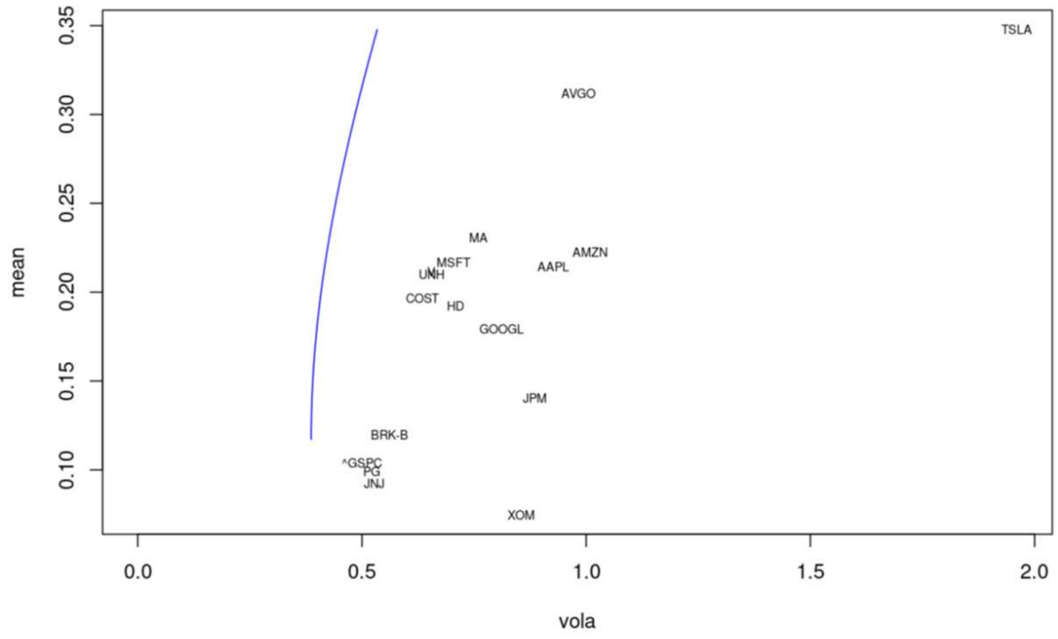
The right-hand sides are 1 and b_0.
b_0 is the minimum expected return we require.

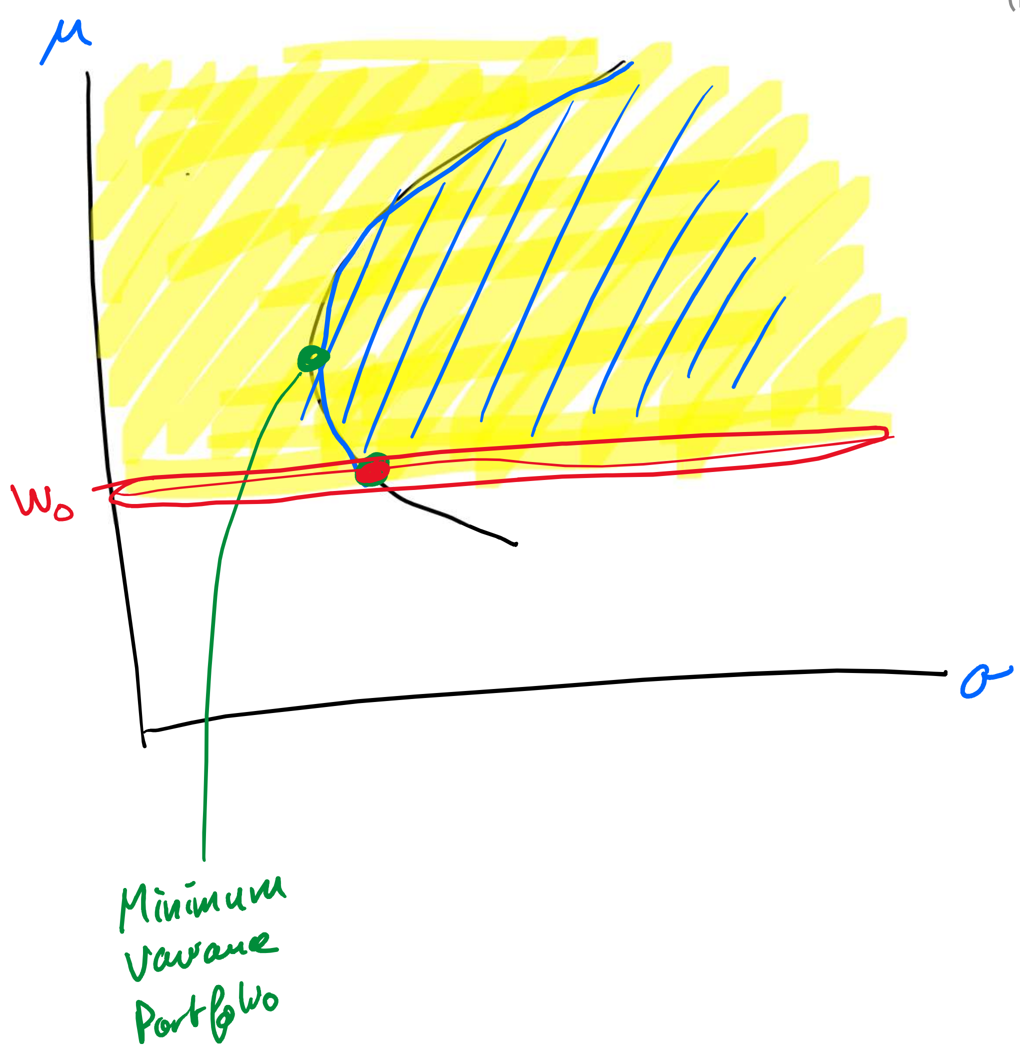
```
    bvec <- c(1, b0)
    out <- solve.QP(
               Omega, w_init, Amat=Amat,
               bvec=bvec, meq=meq
           )
    w_star <- out$solution
    mu_star <- mu %*% w_star
    sigma_star <- sqrt( t(w_star) %*% Omega %*% w_star )
```

Now make a vector b0_vec and loop over it to compute the efficient frontier.

8 Exercise. Add to your program. Compute the efficient frontier and plot it with σ on the horizontal and μ on the vertical axis (the section called `'unconstrained optimization'` in the R file). Do not disallow short sales to begin with, but make sure that 100% of capital is invested (meaning the sum of the weights of the protfolio must sum exactly to 1).

Minimize $V = w^{\mathsf{T}}\Omega w.$ subject to wμ >= some value and w >= 0.





9 Exercise. Repeat the last task but this time exclude short sales, meaning no individual weight can be negative.
Plot the shares of the components along the efficient frontier using an "area plot" or using stacked bars, so that all the weights sum to a bar of constant height (100%) but their composition changes as we move up the efficient frontier.

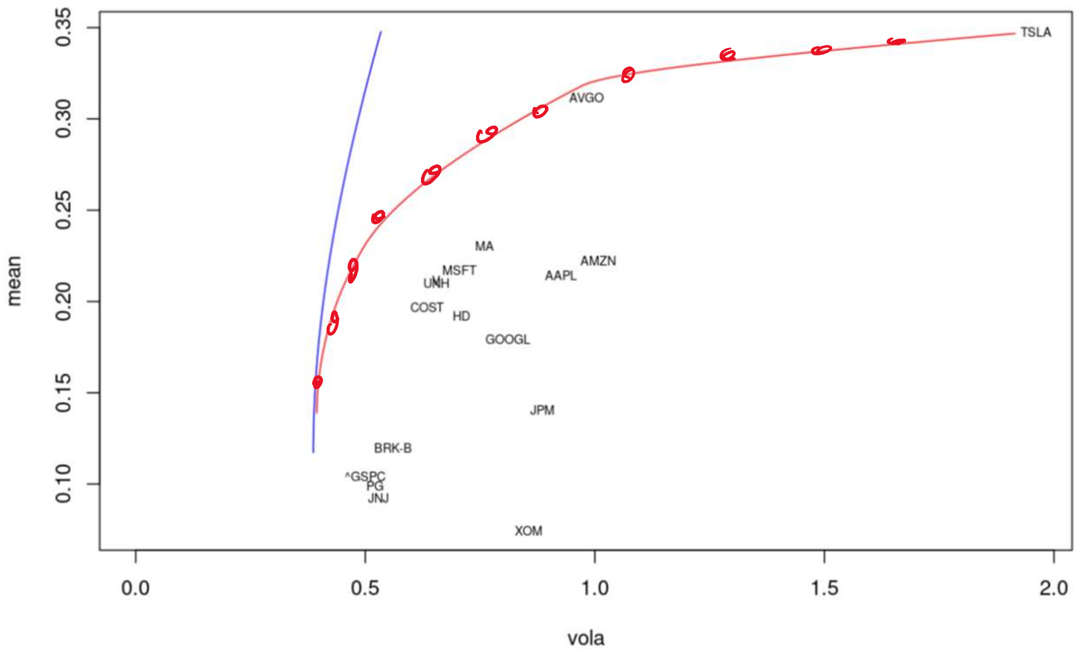
We have 2+n constraints, namely

$\text{Sum } w_i = 1$
$w\mu = \text{sum } w_i \mu_i >= b_0$
$w_1 >= 0$
$w_2 >= 0$
...
$w_n >= 0$

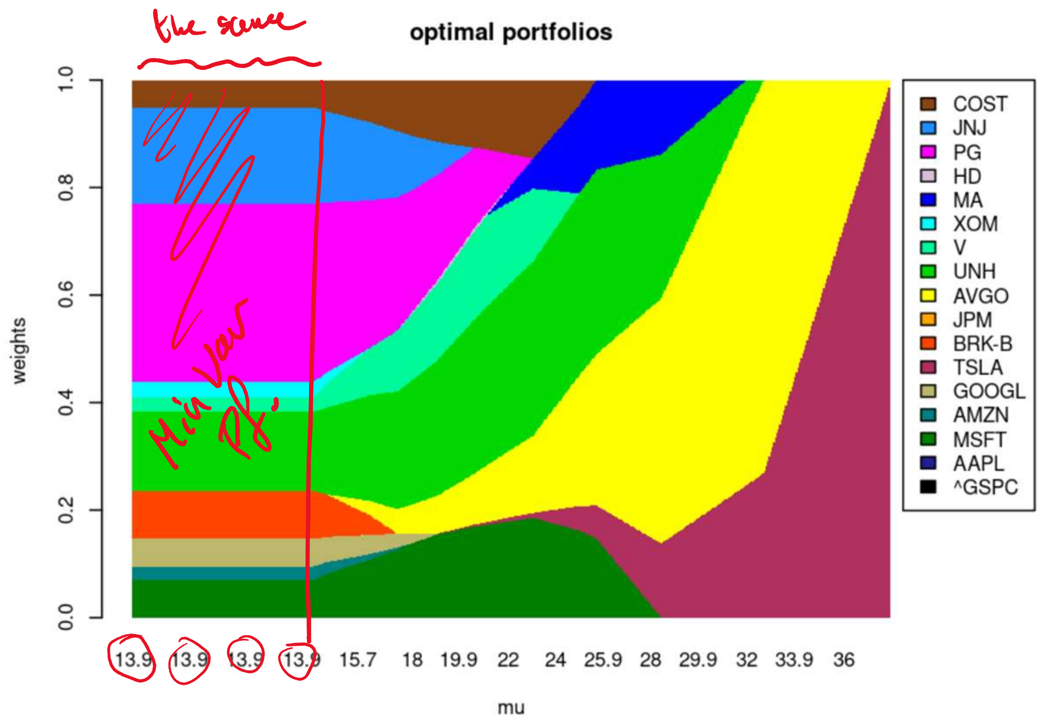
We have to formulate this as a matrix A'w >= 0.

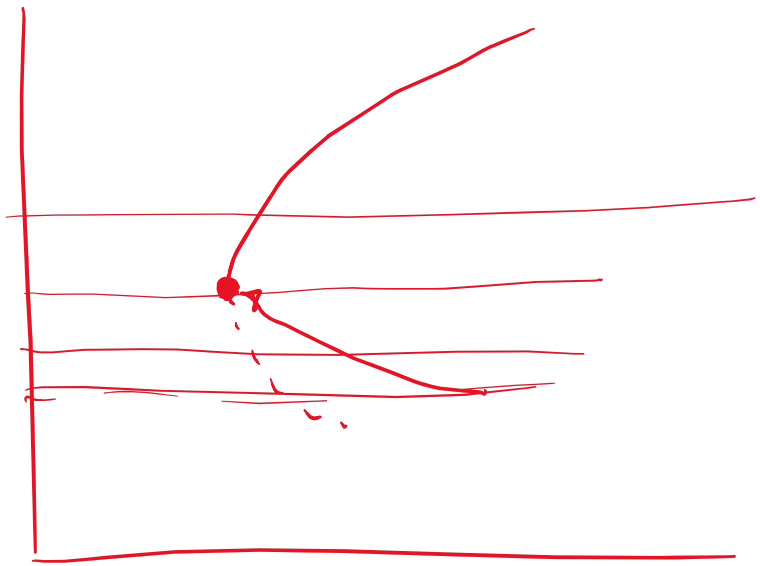
```
each w >= 0 ====> diag(nassets) %*% w >= 0

# LHS of constraint:
    Amat <- t(rbind(rep(1, nassets), mu, diag(nassets)))

# RHS of constraint:
    bvec = c(1, b0, rep(0, nassets))
```

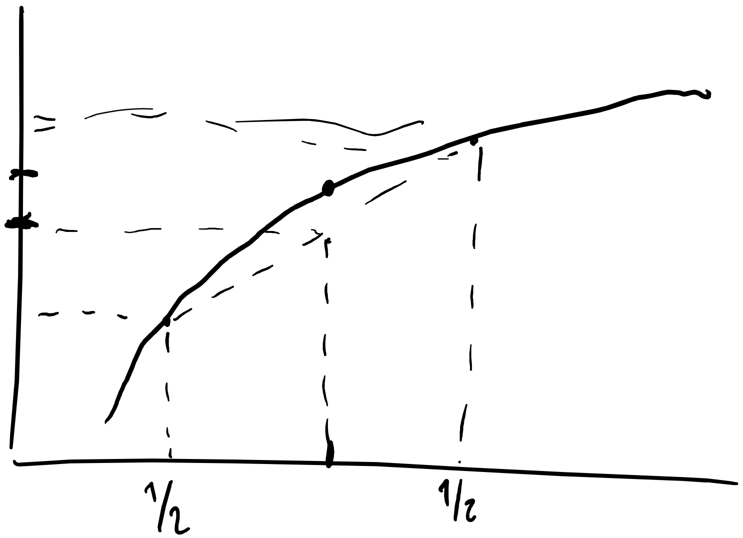
The rest of the code is the same.
Now make a vector b0_vec and loop over it to compute the efficient frontier with short sale constraints.



```
barplot(
    w_con,
    legend.text = assets,
    border = NA,
    space = 0,
)
# w_con is a matrix with optimal weights
along the constrained efficient frontier
filled in the loop.
```







Jensen's inequality