# Stochastic volatility (GARCH)

Asset Pricing I

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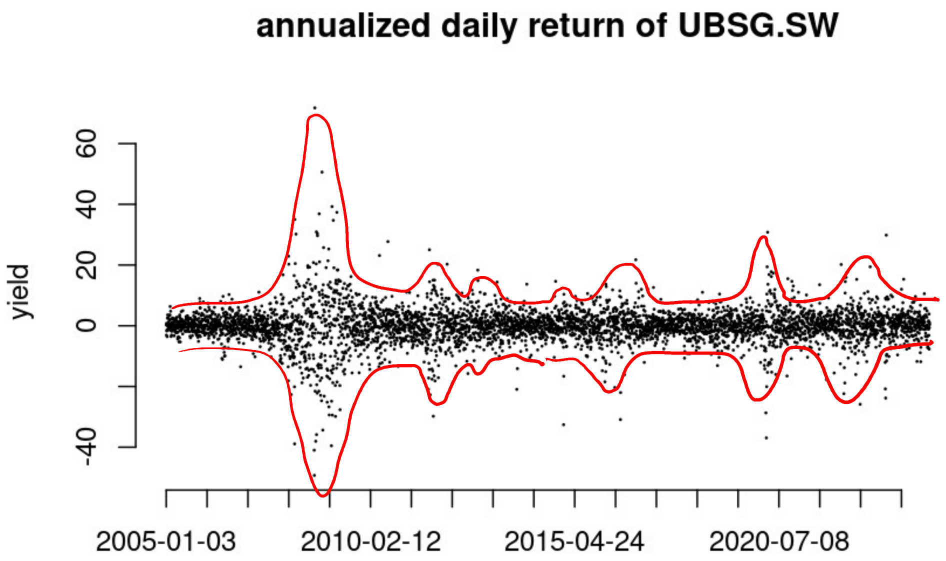



It appears that not only are the returns stochastic, the volatility of the returns is fluctuating as well.

If volatility was constant, we could describe this process as follows:

$$r(t) = \mu + \varepsilon(t)$$
$$\varepsilon(t) = \sigma z(t)$$

where $\mu$ is a constant (the mean return), $\sigma$ is the standard deviation, and $z$ is a standard normally distributed shock

$$z(t) \sim N(0,1).$$

How does the GARCH model deviate from this?

$$\sigma^2(t) = \underbrace{\underbrace{\omega + \alpha\,\varepsilon(t-1)^2}_{ARCH(1)} + \beta\,\sigma^2(t-1)}_{GARCH(1,1)}$$

One process ($z$) drives returns <u>and</u> volatilities

# Estimation with Maximum Likelihood

Maximize the propability of the truthfulness of the estimated coefficients.

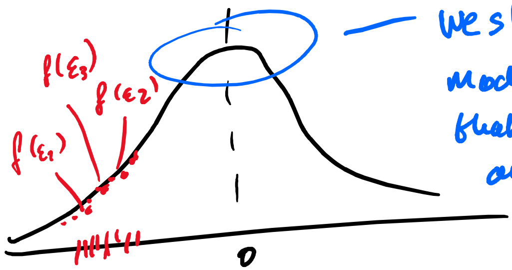
If we observe a residual $\varepsilon(t)$ at time $t$, if we observe a residual $\varepsilon(t)$ at time $t$ is some specific value then,

If we observe a residual $\varepsilon(t)$ at time $t$,
is some specific value then,
and $\sigma(t)$ is some specific value
then we know the density ($\approx$ probability) of $\varepsilon(t)$: Evaluate normal density function with mean $0$ and variance $\sigma(t)^2$:

$$f(\varepsilon_t) = \frac{1}{\sigma_t\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\varepsilon_t}{\sigma_t}\right)^2}.$$

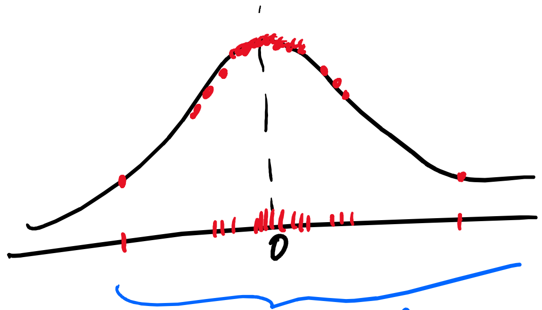
$L = f_1(\varepsilon_1) \cdot f_2(\varepsilon_2) \cdot \ldots \cdot f_T(\varepsilon_T) =$ likelyhood $\to$ max

by choosing parameters $\omega, \alpha, \beta$, and $\sigma^2(1)$ that govern the development of $\sigma(2, \ldots, T)$.




We should try to select model parameters so that the residuals are located here.

This is not a good fit. (Likelihood not maximized.)




Much better fit

**1 Question.** Why are we allowed to maximize $\log L$ instead of just plain $L$? Does this not skew our result? After all, the logarithm is not a linear transformation!

log is monotone function, so $L(\alpha, \beta, \omega, \sigma^2(1))$ and $\log L(\alpha, \beta, \omega, \sigma^2(1))$ attain their respective maximum at the same location.

Numerically, maximizing a sum

$$\log L = \log(f_1(\varepsilon_1)) + \log(f_2(\varepsilon_2)) + \dots$$

is much easier than maximizing a product

$$L = f_1(\varepsilon_1) \cdot f_2(\varepsilon_2) \cdot \dots$$

**2 Task.** Use the file `GARCH.r` and complete the script so that it uses the MLE method to fit a GARCH(1,1) model to the daily yields. Report the estimates $\mu$, $\omega$, $\alpha$, and $\beta$.

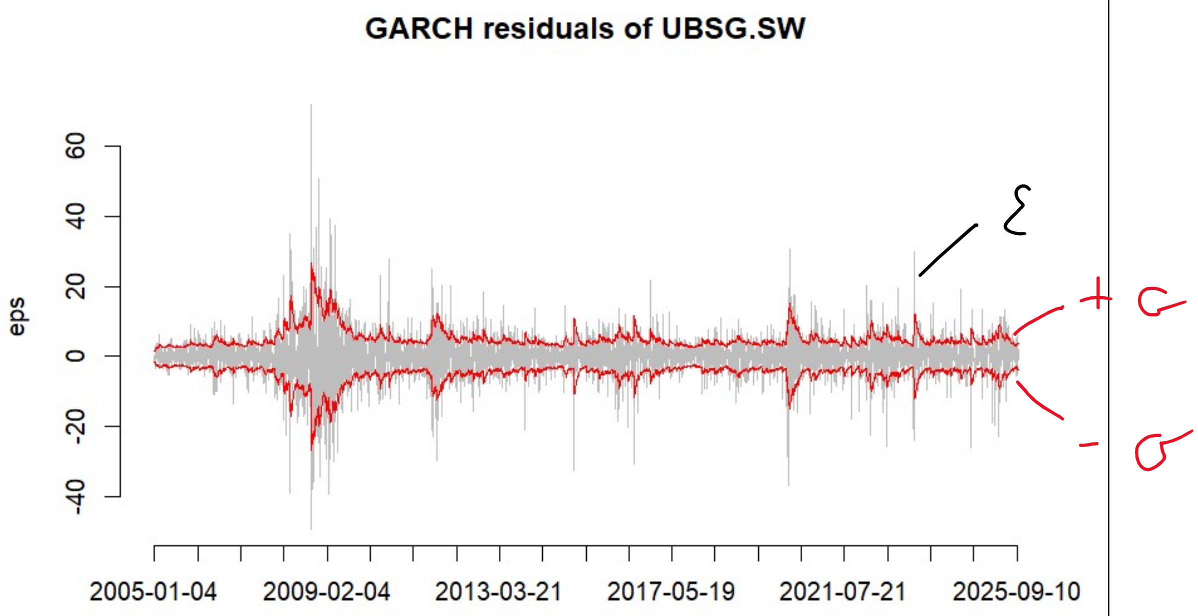
In GARCH.R, there was only one part you had to complete, namely the GARCH function.
This function computes the stochastic sigma and the likelihood (density) of all observations.

```
# number of observations
nobs        <- length(eps)

# declare some empty vectors
logL        <- rep(NA, times=nobs)
sigma2      <- rep(NA, times=nobs)

# initial variance
sigma2[1]   <- init_sigma2
# alternatively, set sigma2[1] equal to unconditional variance:
# sigma2[1] <- omega / (1 - alpha - beta)

# compute likelihood at t=1
logL[1]     <- dnorm(eps[1], mean=0, sd=sqrt(sigma2[1]), log=TRUE)

# compute likelihood sequentially for t=2 and later
for (t in 2:nobs) {
    # variance eq of the GARCH model
    sigma2[t] <- omega + alpha * eps[t-1]^2 + beta * sigma2[t-1]
    # likelihood function
    logL[t] <- dnorm(eps[t], mean=0, sd=sqrt(sigma2[t]), log=TRUE)
}
```

```
param <- c(
    var(yield) * (1 - 0.9),   # omega
    0.1,                      # alpha
    0.8,                      # beta
    var(yield)                # initial sigma2
)

# running the NL opt
mu  <- mean(yield)
eps <- yield - mu
out <- nlminb(param, negLL, eps=eps,
    lower=c(0,0,0,0), upper=c(Inf,1,1,Inf),
    control=list("iter.max"=1000, "eval.max"=2000))
```

```
neg likelihood (minimized) = 15509.26

mu    = 0.00560067
omega = 0.4123963
alpha = 0.07921658
beta  = 0.9081301
1-alpha-beta = 0.01265331

initial volatility (t=1) = 1.578407
unconditional volatility = 5.708937
```

**3 Task.** Make a chart showing the residuals $\epsilon(t)$ and $+\sigma(t)$ and $-\sigma(t)$, all in the same chart.



**4 Task.** Start from the file `GARCH-MC.r` to generate a Monte Carlo simulation of a fitted GARCH(1,1) model for UBSG.SW for the duration of one week, starting at 2008-09-19 (in the middle of the global financial crisis), 2008-12-19 (three months later), and at 2009-09-18 (one year after the height of the GFC). Make charts of the density of the final price at the end of the simulation, together with the densiy of a normal distribution with the sma emean and volatility.

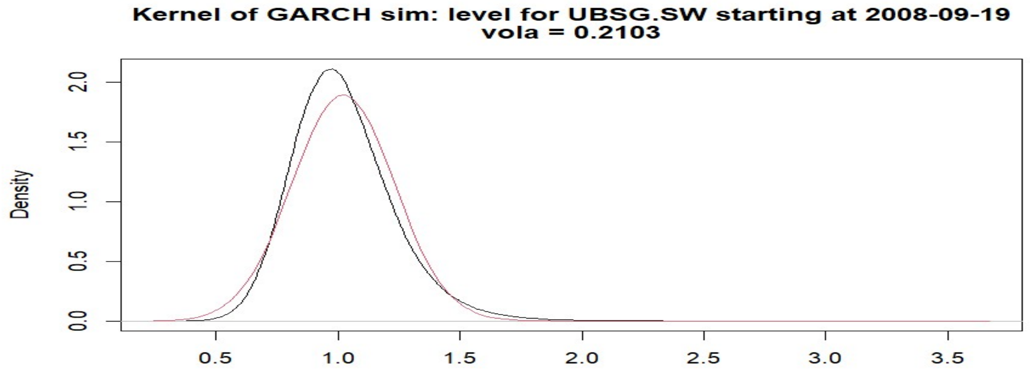
PS: "One week" refers to five working days.

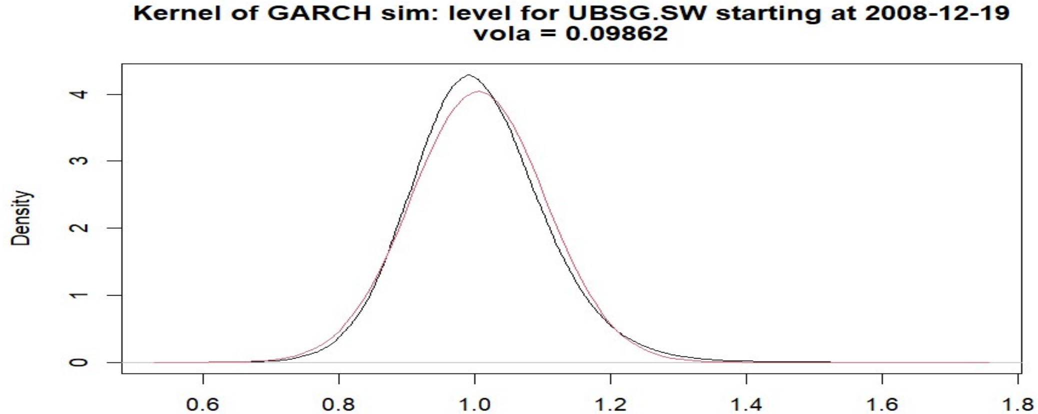
$$r_t = \mu + \varepsilon_t,$$

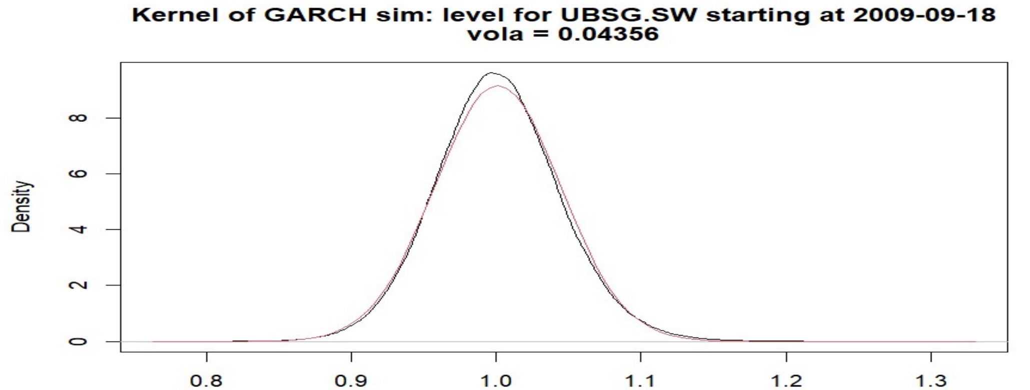
$$\varepsilon_t = \sigma_t z_t,$$

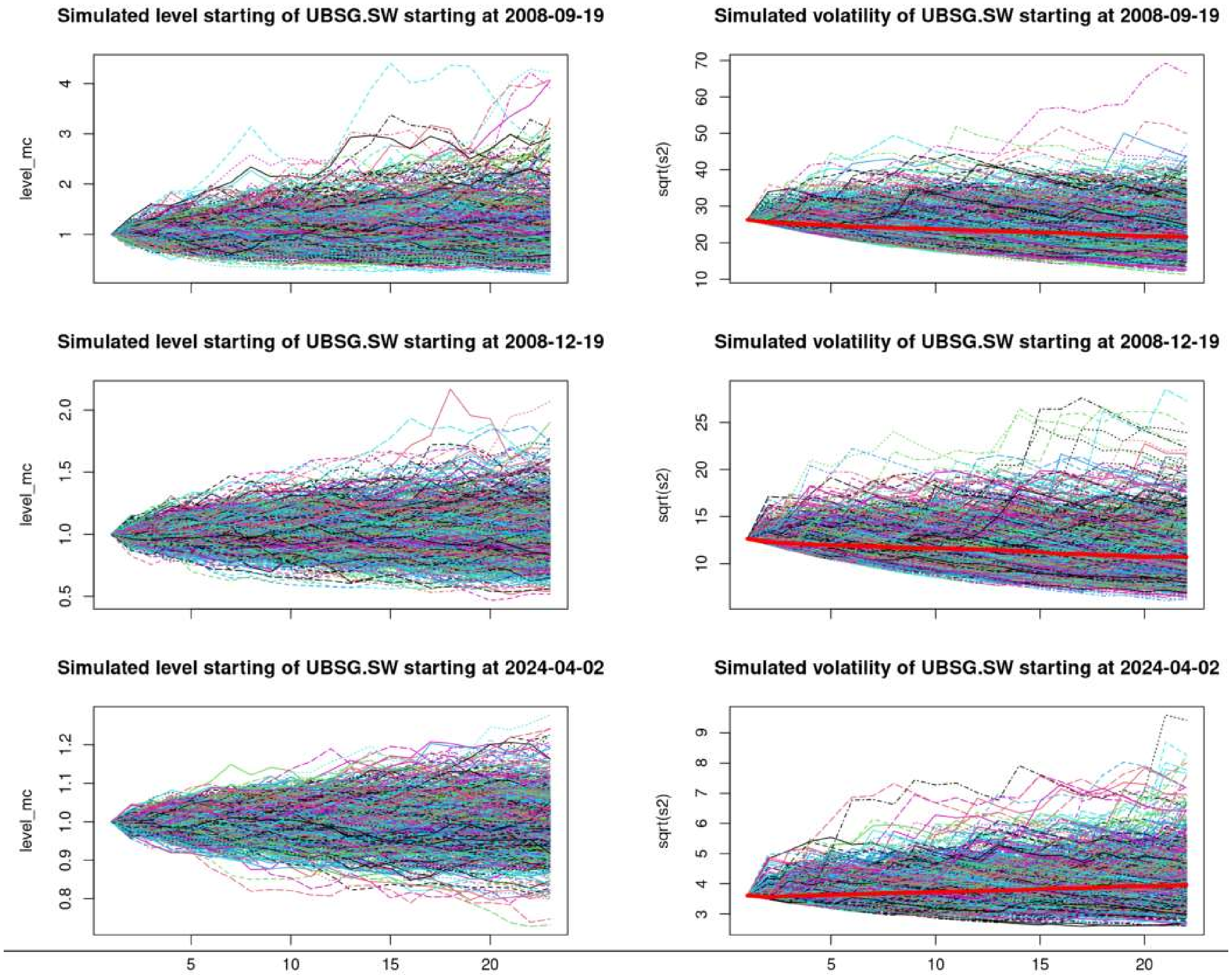
$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2.$$

```r
# identify position of date
idx <- which(dates == the_date)

# generate standard normal shocks
z <- matrix(rnorm(sim_length*n, 0, 1), nrow=sim_length)

# initial sigma2 and eps
s2_init <- matrix(sigma2[idx], nrow=1, ncol=n)
e_init  <- matrix(eps[idx], nrow=1, ncol=n)

# first step of simulation (one day after 'the_date')
s2 <- matrix(omega + alpha*e_init^2 + beta*s2_init, nrow=1, ncol=n)
e  <- matrix(z[1,] * sqrt(s2), nrow=1, ncol=n)
r  <- matrix(mu + e, nrow=1, ncol=n)

# t = 2 ... (keep adding rows to s2, e, and r matrices)
for (t in seq(2, sim_length)) {
    s2 <- rbind(s2, omega + alpha * e[t-1,]^2 + beta * s2[t-1,])
    e  <- rbind(e, z[t,] * sqrt(s2[t,]))
    r  <- rbind(r, mu + e[t,])
}
```




Simulated level starting of UBSG.SW starting at 2008-09-19
Simulated volatility of UBSG.SW starting at 2008-09-19
Simulated level starting of UBSG.SW starting at 2008-12-19
Simulated volatility of UBSG.SW starting at 2008-12-19
Simulated level starting of UBSG.SW starting at 2024-04-02
Simulated volatility of UBSG.SW starting at 2024-04-02
level_mc
sqrt(s2)

**5 Task.** Suppose your company has a very non-diversified portfolio: It just holds CHF 1 million in UBS shares (UBSG.SW), nothing else. Compute the $\alpha = 1\%$ and $\alpha = 5\%$ VaR over one month (22 days) at the three dates used in the previous task. Comment on what these numbers and their changes over time mean.

The idea of Value at Risk (VaR) is explained at beginning of section 3.2 of the handout. Can you explain what this is? Do you have an idea how to compute this?

Just look at the final values that were simulated, and extract the 1% or 5% worst value.
The result is worse than these values with a probability of 1% or 5%, respectively.

```
# value at risk
K = 1000000     # capital
final <- sort(level_mc[sim_length,])
level_1 <- as.integer(0.01 * n)
level_5 <- as.integer(0.05 * n)
alpha_1 <- rbind(alpha_1, -K * (1 - final[level_1]))
alpha_5 <- rbind(alpha_5, -K * (1 - final[level_5]))
```

```
> print(VaR)
              alpha_1    alpha_5
2008-09-19 -385550.64 -282143.06
2008-12-19 -209638.01 -147600.64
2009-09-18  -99144.39  -68712.27
```

**Caution:** I did it for 5 days, not for 22 days!

**6 Task.** Produce a short report for your boss. Your boss is not a technical person, but is responsible for risk management. Your boss needs your estimates or the VaR over one week at 5% and 1% levels for the last available day, 1 week earlier, and 2 weeks earlier.

He needs to know what data you used and some confidence that you did this correctly (maybe just mention the methods without going into details). Provide a very short interpretation what these numbers mean.

Provide additional information if you think your boss appreciates this; do *not* provide irrelevant information (everyone dislikes that). The report must be succinct and brief (certainly not more than one page, can easily be less). You want to be recognized for your work so document your name and date.

Please hand in your report to Tri by email. If you work in groups, hand in just one report together.

**Quant Management Ltd**

Author: Joe Shmo

To: CRO

**VaR report for September 22, 2025**

**Result**

The portfolio consists of USD 1 million worth of UBS stock. The VaR for 5 days at 1% level is currently USD 65'600, at 5% it is USD 45'300.

This means that with 5% probability, the portfolio will lose more than USD 45'300 within one working week; with 1% probability, it will lose more than USD 65'600.

The VaR has been steadily increasing in recent weeks:

| DATE | 5 days, 1% | 5 days; 5% |
|---|---|---|
| 2025-09-08 | 60'500 | 41'600 |
| 2025-09-15 | 63'700 | 44'000 |
| 2025-09-22 | 65'600 | 45'300 |

**Methodology and data**

The data are collected at a daily frequency, stating in 2005. A GARCH model is estimated on these daily returns. The GARCH estimate converges nicely and seems to capture the stochastic volatility well.

Given the GARCH model, a Monte Carlo simulation is run with 1 million draws, starting from the respective dates in the table above, simulating five future days. The returns over these five days are accumulated to compute the final value of the portfolio.

The worst 5% and 1% quantiles, respectively, of the final outcome are the VaR numbers reported above.

Possibly add the location of your program on the company's server to allow reproduction of your work and other workers to use it.