Stochastic volatility (GARCH)

Asset Pricing I

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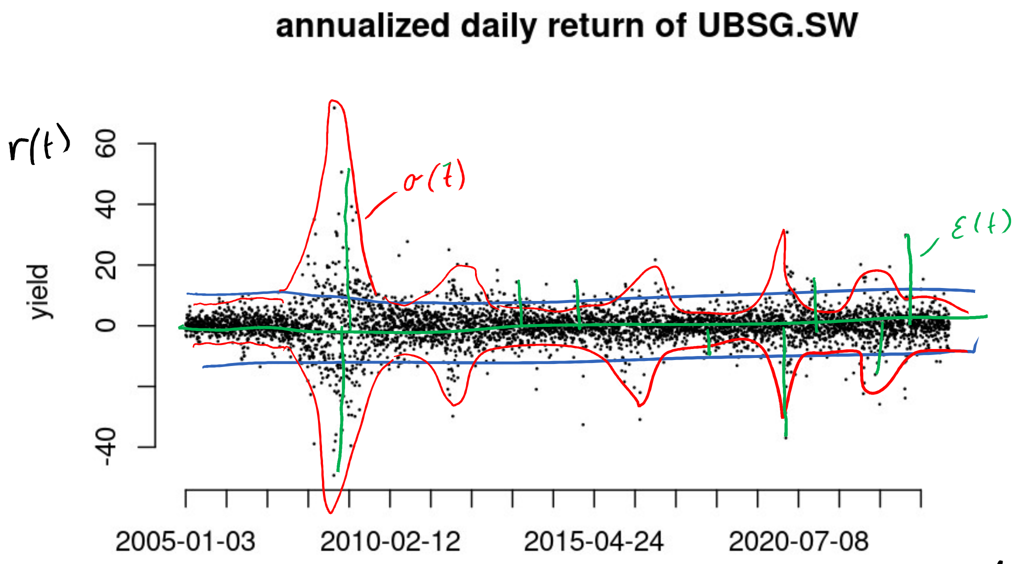
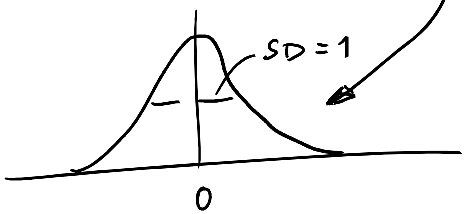


If volatility was constant, we could represent this stochastic process like this:

$$r(t) = \mu + \varepsilon(t)$$
$$\varepsilon(t) = \sigma \cdot z(t)$$

$z(t) \sim N(0,1)$
standard normal



How does the GARCH model deviate from this?

σ is not constant, but changes over time.

$$\underline{\sigma^2(t)} \leftarrow \underline{\varepsilon^2(t-1)} = (r(t)-\mu)^2$$
$$\underline{\sigma^2(t)} \leftarrow \underline{\sigma^2(t-1)}$$

$$r(t) = \mu + \varepsilon(t)$$
$$\varepsilon(t) = \sigma(t)\, z(t)$$
$$\sigma^2(t) = \omega + \alpha\, \varepsilon(t-1)^2 + \beta \sigma^2(t-1)$$

Note: The only observations we have is r.

μ is estimated as $\mu = \frac{\Sigma r(t)}{T}$.

ε is estimated as $\varepsilon = r - \mu$

σ is estimates with the variance equation

The parameters of the variance equation, ω, α, β, are also estimated.

Also note: r is stochastic, and σ is stochastic as well. But they are both driven by z. Thus, there is only one stochastic process here. This is a clever model: we can avoid the difficulties of dealing with mult. stochastic processes.

$\sigma^2(t) = \omega + \alpha\, \varepsilon(t-1)^2 + \beta\, \sigma^2(t-1)$

is GARCH(1,1) because there is only one lagged ε (one α) and one lagged σ^2 (one β). We can have more than one.

Also, note that we could add exogenous explanatory variables to the mix:

$$r(t) = \mu + \varepsilon(t) + \gamma X(t)$$
$$\varepsilon(t) = \sigma(t)\, z(t)$$
$$\sigma^2(t) = \omega + \alpha\, \varepsilon(t-1)^2 + \beta\, \sigma^2(t-1) + \delta X(t)$$

This is an extension known as GARCH X.

Moreover, we could also have a vector-version of GARCH, known as VGARCH, where different r interact with each other, and the variance-equation becomes a co-variance equation.

There are many more variations in the literature...

Unconditional Variance

$$\sigma^2(t) = \omega + \alpha\, \varepsilon(t-1)^2 + \beta\, \sigma^2(t-1)$$

Long-term averages:

$$\bar{\sigma}^2 = \omega + \alpha\, \underbrace{\bar{\varepsilon}^2} + \beta\, \underbrace{\bar{\sigma}^2}$$

$$\bar{\sigma}^2 = \omega + \alpha\bar{\sigma}^2 + \beta\bar{\sigma}^2$$

$$\longrightarrow \bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$$

unconditional variance

Note that this requires $\alpha + \beta < 1$.

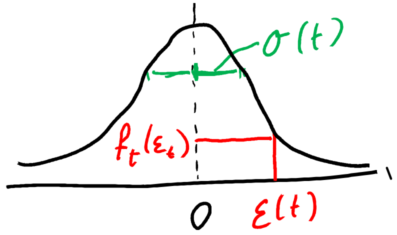
If $\alpha + \beta \geq 1$, the GARCH process is not stable and volatility explodes.

MLE: maximum likelihood estimation

Maximize the "probability of truthfulness" of the estimated parameters.

At time t we observe : $\varepsilon(t) = r(t) - \mu$

" " " " estimate : $\sigma^2(t) = \omega + \alpha\varepsilon(t-1)^2 + \beta\sigma^2(t-1)$



$f_t(\varepsilon_t)$ is the density of the distribution that is valid at time t, evaluated at the point $\varepsilon(t)$.

The likelihood over all days $t = 1 \ldots T$ is:

$$L = f_1(\varepsilon(1)) \cdot f_2(\varepsilon(2)) \cdot f_3(\varepsilon(3)) \cdots f_T(\varepsilon(T))$$

Note that f_t depends on $\sigma(t)$.

Whenever we change any parameter — α, β, ω, $\sigma(1)$ — all f-functions change, and thus L changes.

MLE is the idea to select the parameters of the model in such a way that L is maximized.

Maximizing a product $(f_1 \cdot f_2 \cdot f_3 \ldots)$ is numerically difficult. It is much easier to maximize a sum. For this reason, we always maximize $\log L$:

$$\log L = \log f_1(\varepsilon(1)) + \log f_2(\varepsilon(2)) + \ldots + \log f_T(\varepsilon(T)).$$

$\rightarrow$ This is what we attempt to do with R.

1 Question. Why are we allowed to maximize $\log L$ instead of just plain L? Does this not skew our result? After all, the logarithm is not a linear transformation!

log is not linear, but it is a strictly increasing function. Thus, the maximum of L is attained at the same point as the maximum of $\log L$.
Maximizing $\log L$ yields the same maximizer as maximizing L. ✓

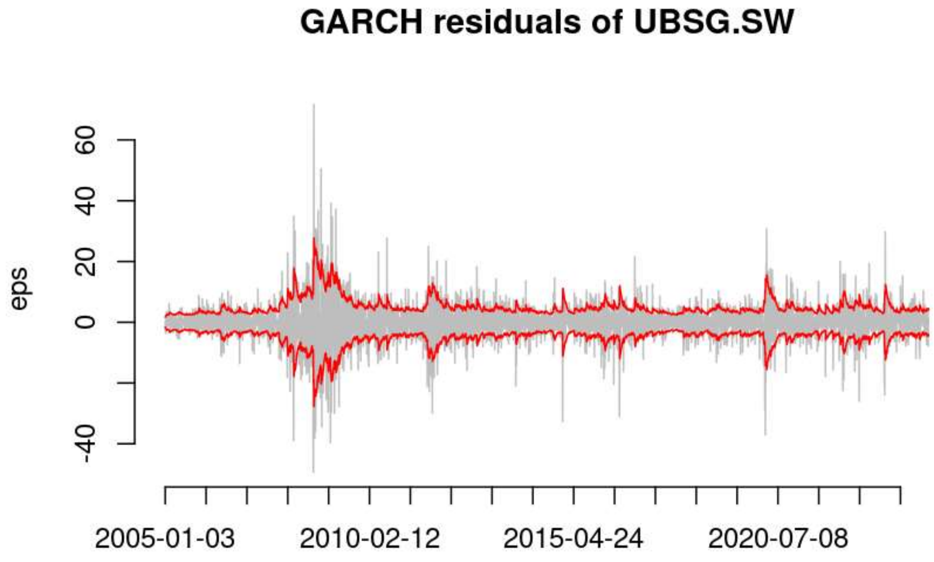
2 Task. Use the file `GARCH-unfinished.r` and complete the script so that it uses the MLE method to fit a GARCH(1,1) model to the daily yields.

In addition, try to think of useful charts that would show what is going on. For instance, the simplest chart would simply plot σ_t. Maybe you can combine that with ε_t somehow?

In GARCH_unfinished.R, there was only one part you had to complete, namely the GARCH function.
This function computes the stochastic sigma and the likelihood (density) of all observations.

Key code for the GARCH function:

```
for (t in 2:nobs) {
    # variance eq of the GARCH model
    sigma2[t] <- omega + alpha * eps[t-1]^2 + beta * sigma2[t-1]
    # likelihood function
    logL[t] <- dnorm(eps[t], mean=0, sd=sqrt(sigma2[t]), log=TRUE)
}
```




```
neg likelihood (minimized) = 14454.87

mu    = -0.00976988
omega = 0.390581
alpha = 0.08692232
beta  = 0.9020931
1-alpha-beta = 0.01098458

initial volatility (t=1) = 1.611454
unconditional volatility = 5.962987
```

3 Task. Start from the file `GARCH-MC-unfinished.r` to generate a Monte Carlo simulation of a fitted GARCH(1,1) model for UBSG.SW for the duration of one month, starting at 2008-09-19 (in the middle of the global financial crisis), 2008-12-19 (three months later), and at 2024-04-02 (recently). Make charts of the simulated volatility and the simulated price of UBSG.SW.

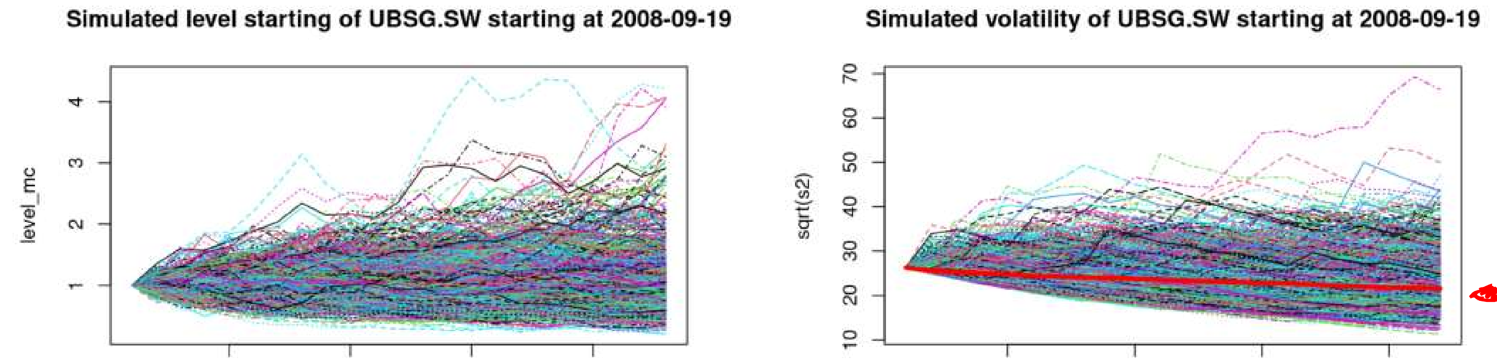
PS: "One month" refers to one calendar month, which is about 22 working days. Since we work with working day data, you should simulate 22 days.

$$r_t = \mu + \varepsilon_t,$$

$$\varepsilon_t = \sigma_t z_t,$$

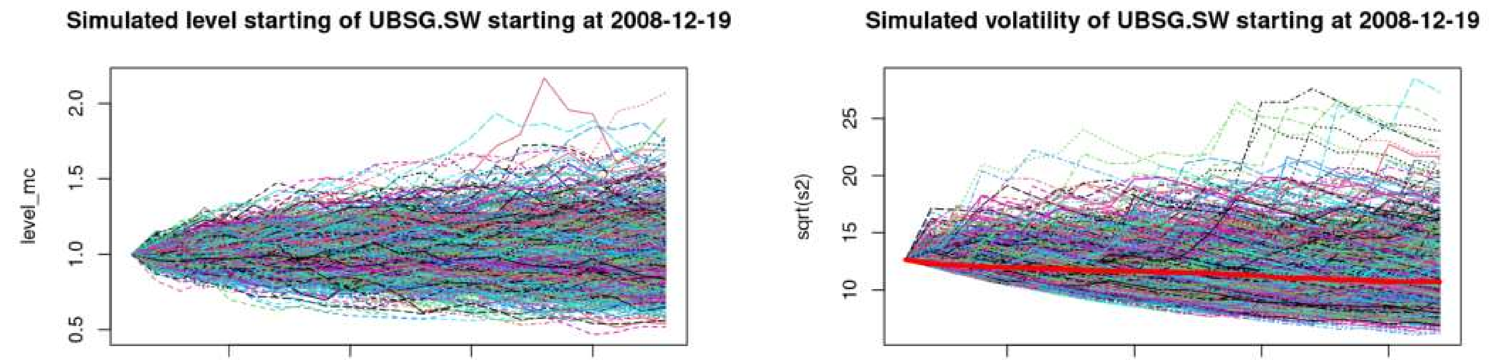
$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2.$$

```
# identify position of date
idx <- which(dates == the_date)

# generate standard normal shocks
z <- matrix(rnorm(sim_length*n, 0, 1), nrow=sim_length)

# first step of simulation (on day after the_date)
s2 <- matrix(omega + alpha*e_init^2 + beta*s2_init, nrow=1, ncol=n)
e  <- matrix(z[1,] * sqrt(s2), nrow=1, ncol=n)
r  <- matrix(mu + e, nrow=1, ncol=n)

# t = 2 ... (keep adding rows to the s2, e, and r matrices)
for (t in seq(2, sim_length)) {
    s2 <- rbind(s2, omega + alpha * e[t-1,]^2 + beta * s2[t-1,])
    e  <- rbind(e, z[t,] * sqrt(s2[t,]))
    r  <- rbind(r, mu + e[t,])
}
```

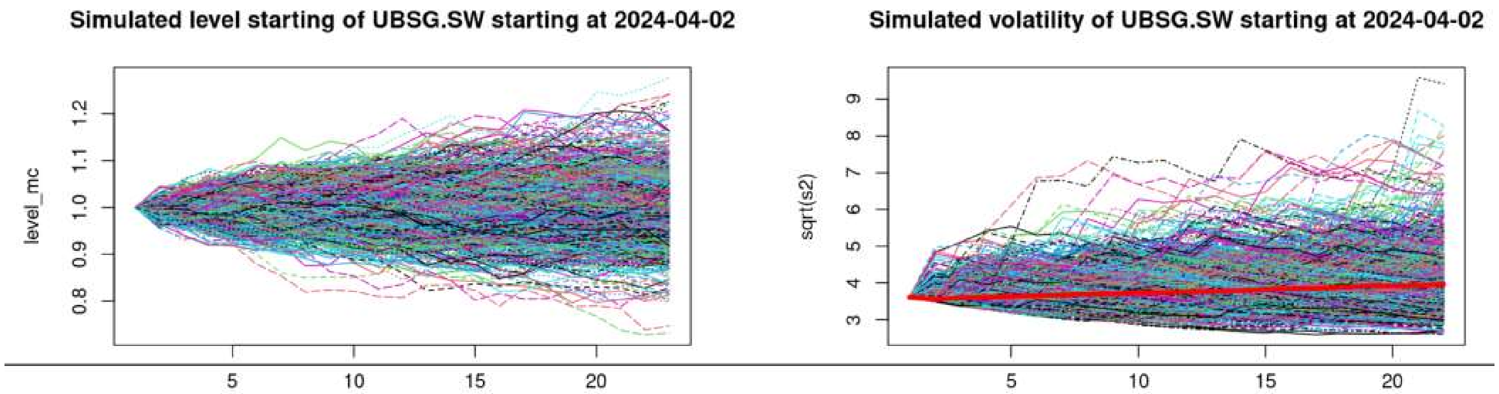


UBS stock at the height of the GFC.

red line is the median: 50% above, 50% below this line



3 months later



~ today

4 Task. Suppose your company has a very non-diversified portfolio: It just holds CHF 1 million in UBS shares (UBSG.SW), nothing else. Compute the $\alpha = 1\%$ and $\alpha = 5\%$ VaR over one month (22 days) at the three dates used in the previous task. Comment on what these numbers and their changes over time mean.

The idea of Value at Risk (VaR) is explained at beginning of section 3.2 of the handout. Can you explain what this is? Do you have an idea how to compute this?

⇨

Just look at the final values that were simulated, and extract the 1% or 5% worst value.
The result is worse than these values with a probability of 1% or 5%, respectively.

```
# sort final levels from low to high
final <- sort(level_mc[sim_length,])

# 1% and 5% worst simulated final levels
level_1 <- as.integer(0.01 * n)
level_5 <- as.integer(0.05 * n)

# VaR levels
K = 1000000       # initial capital
alpha_1 <- -K * (1 - final[level_1])
alpha_5 <- -K * (1 - final[level_5])
```

VaR 1 month	1%	5%
2008-09-19	-662’670	-508’994
2008-12-19	-373’540	-286’856
2024-04-02	-181’201	-108’701