The Risk Neutral Density

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1 Financial derivatives

1.1 *Puts and Calls*

A financial asset is a contract that delivers a certain payoff that may or may not be conditional on the "state of the world" (meaning, on random events). For instance, a bond is a promise to repay principal and interest when the bond matures no matter what (so this is not conditional, assuming there is no event in which the borrower *cannot* pay back). An equity is a promise to paz annual dividends, but these dividends are somehow related to the profit of the firm and are ultimately decided every year by the shareholders. An insurance contract is also a financial asset: it pays you damages but only if your phone or car was broken or stolen, or if you got ill (depending on what the object of the insurance is).

A derivative is a financial assets whose payoff is a defined function of the price of some other traded financial asset. It's payoff is "derived" from another financial asset.

Some of the most basic derivatives are puts and calls. A call on an equity is defined by its expiration date T and its strike price K. The payoff of a call depends on the price x of the underlying share on the day the call matures. If x exceeds K, the call pays out the difference, $x-K$. In this case, we say the call is "in the money". If it is out of the money, $x<K$, the call option expired without any payoff. So the payoff function is $\max\{0, x-K\}$.

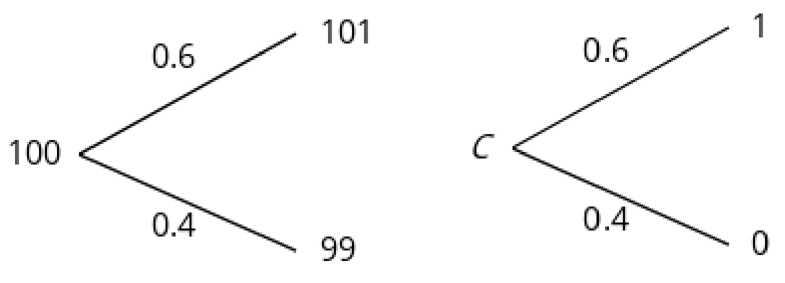
The payoff function of a put is the referse: $\max\{0, K-x\}$. Thus, a put become more valuable the more depressed the equity's price is.

1.2 *Delta hedging*

onsider a very simple world: The is a share that costs 100 today. Tomorrow, with 60% probability, it will be worth either 101 or, with 40% probability, it will be worth 99. So this is binomially distributed (nothing between or beyong 11 and 99 is possible).

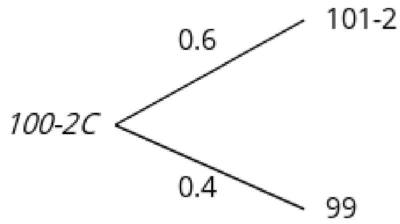
In addition there is also cash, and cash is just worth the same tomorrow as it is today (no inflation or interest).

Moreover, there is a call option with strike $K=100$. Depending on what the equity is doing, the call option will pay out $\max 0, 101-100 = 1$ or $\max 0, 99-100+0$.



How much is the call option C worth? It's expected payoff is $0.6 \cdot 1 + 0.4 \cdot 0 = 0.6$, so is this today's value? — It is not. This asset is risky. You might end up with nothing, so presumably there should be some risk premium?

Consider the following portfolio: You *buy* 1 share and you *sell* 2 call options short. So you spend 100 on the share and you get $2C$ from selling the option. This portfolio costs $100 - 2C$. The payoffs of this portfolio are



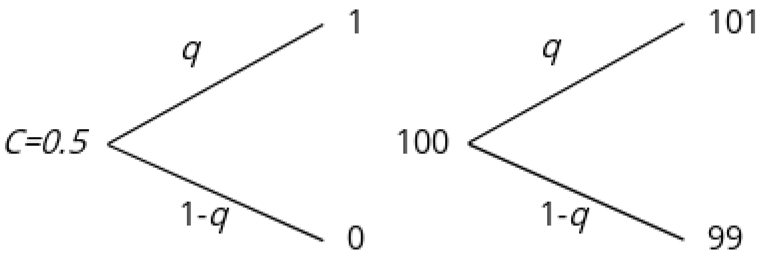
Wow! This is risk free. It's like holding 99 cash. For this reason, the value of the portfolio, $100 - 2C$, must equal 99, or $C = 0.5$. We have just determined the price of an option. The procedure shown here to remove risk from equity with the help of options is called *delta hedging*.[1]

1.3 *Risk-neutral densities*

So wait a minute: The call option has an expected payoff of 0.6, but costs only 0.5. This is a 20% risk premium?

In fact, it is. Note that the equitz also has a risk premium: it's expected payoff is $0.6 \cdot 101 + 0.4 \cdot 99 = 100.2$, but it costs only 100. It carries a resk premium of 0.2%. Options are equities on steroids.

We can simplify these calculations by distorting the probabilities we assign to the two states. Which binomial probability q would allow us to compute the value of the option (left chart below) and the price of equity (right chart below) by simply taking the expected payoff under these distorted probabilities?



So, for the call option, the answer is $q = 0.5$. And for the equity ... it is *also* $q = 0.5$. This is not a conincidence. It is a fundamental result of asset pricing theory that there is a *risk-neutral density* (RND) that allows us to compute asset prices by simply computung expected outcomes (under a non-empirical, but distorted risk-neutral probability).

This is true not only in this simple binomial world. In fact, we can extract the RND of an equity if we have sufficiently many options with different strikes. This is what we do next.

[1]The idea goes back to Bachelier (1900), Samuelson (1965), Black and Scholes (1973), Merton (1973).

2 Option chains

An option chain is a collection of multiple put and call options, all on the same underlying and the same expiration date. They differ with respect to their strike prices.

1 Task. Consider a portfolio of call options: 1 unit at strike $K_0 = 100$, minus 2 units and strike $K_1 = 1 - 5$, and 1 unit at strike $K_2 = 110$. This produces a so-called *butterfly option* at 105. Draw the payoff function of this portfolio.

Your answer here:

Of we know the prices of the three options contained in this portfolio, and given this payoff function, we can estimate the risk-neutral density around 105. (The price of the portolio should be aproximately equal to the density multiplied with the area under the payoff function.)

Given this logic, if we have a whole chain of options, we should be able to estimate the RND at multiple places, ideally, over the whole relevant range.

2 Task. Use the script `implied-rnd.R`. Running the first 30 lines or should should load the option price data contained in `option-prices.exe`. The options are on the S&P 500 index, were measured on 2025-09-29 and expire on 2025-12-19. Make a chart of these data and discuss what you see.

Your answer here:

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3 Mixed normals

We will estimate a RND that can be described with a handful of parameters. We will combine multiple normal distributions into one.

The normal distribution is defined by the mean and variance. The density is given by

$$f_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right).$$

Because this is a density, by definition, $\int_{-\infty}^{+\infty} f_1(x)dx = 1$.

Now consider a weighted average of two normal densities,

$$f_m(x) = w_1 f_1(x) + w_2 f_2(x),$$

where w_1 and w_2 are positive numbers that sum to 1, and f_2 is a normal density with parameters (μ_2, σ_2). This new f_m function is clearly also a density because is it positive everywhere and integrates to 1. The f_m function is completely described by the parameters $(w_2, w_2, \mu_1, \mu_2, \sigma_1, \sigma_2)$, with the restriction that $w_1 + w_2 = 1$. You can see how this can be extended to a mixtures of more than two normals.

3 Task. Define a function f taking as arguments a number x and three vectors w, m, and s (all of the same length). The function should return the density of the mixed normal at point x. The function must be able to accommodate an arbitraty number of components of the mixed normal. Play around with the parameters and try to create a skewed or fat tailed distribution, or maybe even multi-modal one. Make some charts.

Your answer here:

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4 Pricing with an RND

So, the strategy is this: We start with some RDN, described by $(w_1, w_2, \mu_1, \mu_2, \sigma_1, \sigma_2)$. For each put and call option in our sample, we compute the payoff function. We multiply this payoff function with the RND. Then we integrate to compute the expected payoff under the RND.

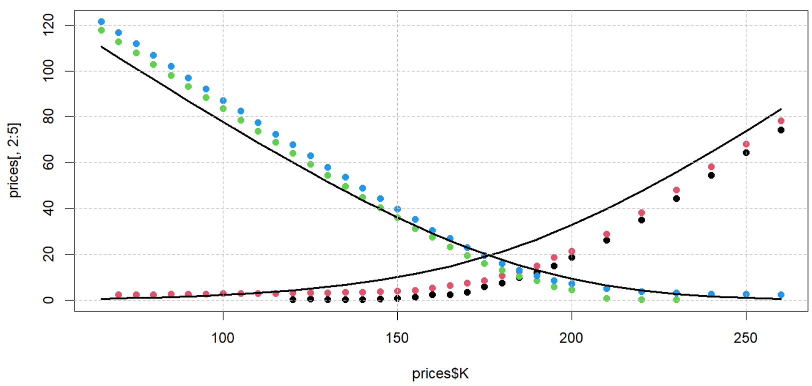
Finally, we discount the result by multiplying with $\exp(-rdT)$, where r is the market interest at the time we observed the option prices, and dT is the time in years until the options mature.[2] Discounting is necessaty because the options' payoffs happen in the future and we want to compare this to today's prices.

So, the price of an option can be computed as

$$\begin{aligned}
\text{put price} &= \exp(-rdT)\int_0^\infty \max\{0, K-x\} f(x)dx \\
&= \exp(-rdT)\int_0^K (K-x) f(x)dx. \\
\text{call price} &= \exp(-rdT)\int_0^\infty \max\{0, x-K\} f(x)dx \\
&= \exp(-rdT)\int_K^\infty (x-K) f(x)dx.
\end{aligned}$$

4 Task. The function under the integration for the put option is provided for you in the R script (function `integrand_put`). Your task (*a*) is to program a function `put_price` that performs the numerical integration, and the disciunting. The function will return the price of a put option given a strike price K, a RND, and an interest rate r. (*b*) Repeat the same for call options.

If you implemented this correctly, the script will produce a chart for you showing the data and the theoretical prices.



[2] r and dT are given in the R script.

You can see here that the theoretical prices (the black lines) do not match the observed prices well. The prices of deep out-of-the-money all options are underestimated, and the prices of put options are generally overestimated. The reason is that our RND is not a good fit. We need to improve this.

5 Numerical estimation

Finally, we come to the estimation or fitting of the RND to the data. We would like to find parameters $(w_1, w_2, \mu_1, \mu_2, \sigma_1, \sigma_2)$, with the restriction that $w_1 + w_2 = 1$, so that the computed prices are as close as possible to the observed empirical option prices. The strategy is to compute the differences bewteen the observed and computed prices at all strike prices where we have observations (these are the *residuals*), compute the sum of the squared residuals (ssr), and then select the parameters to the RND so that the ssr becomes as small as possible. In other words, we are performing a least squares estimation. But it is more challenging than the normal OLS, because our problem is not linear. Thus, we need to optimize numerically (we did that already when we estimated the GARCH process).

5 Task. Create a function `resid` that takes as asuments (r, w, m, s). It computes the put and call prices at all the strike prices we observe and returns the difference (the residual) for each position in our data table containing the observed prices.

6 Task. Create a function ssr that computes the sum of squared residuals, using the function `resid`.

When you have done all that, we now come to the final step.

7 Task. Use R's `nlminb` procedure[a] to minimize ssr by choice of the parameters that define the RND.

[a] `nlminb` is a legacy function. R offers other numerical minimizers which I do not know but you are free to explore them.

If you are successful, the script will print out the final parameters, show you a chart of the observed data and the estimated prices, as well as the shape of the RND. Ponder what you can do with this RND.

6 Ready for sharing in Zoom

Have this ready for the Zoom meeting. Have your notes, charts, programs etc. ready for the Zoom lecture, so that you can share them with us and we can discuss them. If you had trouble completing the tasks, prepare an explanation where exactly you failed.

References

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Black, F. and **M. Scholes** (1973). "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81(3), 637—654.

Merton, Robert C. (1973). "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science*, 4(1), 141–183.

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