Markowitz

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Paper to read: Markowitz (1952), read pages 77–82.

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1 Return vs Risk

No risk, no fun, right? In the financial market, it is generally true, that more volatile assets have a higher return on average than very stable ones. This is the risk-return tradeoff. It is far from perfect. We can study data to check if we see this tradeoff.

1 Exercise. Take a look at the file `Markowitz-unfinished.r`. The file contains `R` code that downloads data from `finance.yahoo.com` for 16 large US stocks as well as the S&P 500 index. You can study this code to understand how it works, or you can just use it.

Add to this program so that it does the following: Compute the mean return and the volatility (standard deviation) or the return of each equity separately. Make a scatterplot of these computations with the mean on the vertical and the volatility (same a standard deviation) on the horizontal axis. This means that each equity will be represented by one point (representing its mean return and volatility) in the chart.

Your answer here:

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2 Diversification

Diversification is the simple insight that carrying your eggs in separate baskets is prudent, because it diminishes the risk of losing everything. It is the guiding principle of so-called "passive investing." This is a well-understood idea and has been around for centuries. It was formally put to paper in an investment context by Markowitz (1952), who received a Nobel award for this work in 1990. By today's standards, this is trivial work, but it was revolutionary at the time in formalizing a vague idea. We will read the first half of Markowitz paper and then put his idea to code.

2 Question. Read the first paragraph. Markowitz seems to suggest a functional form for a utility function. What form would this function take?

Your answer here:

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3 Question. Read the next paragraph. Since payoffs of investments happen in the future, we need to discount them. Markowitz mentions — as other authors of his time have done — that we could use a different discount rate depending on the risk assiciated with a particular payoff. Question: Would the discount rate be higher or lower if the payoff is more risky, and why?

Your answer here:

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4 Question. Read the third and fourth paragraph (bottom of page 77 and top of page 78). Explain his argument why the risk-adjusted discounting is unsatisfactory.

Your answer here:

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5 Task. Read from the paragraph on page 79 that starts with "The portfolio with maximum [...]" until the end of page 81. This is essentially a somewhat tedious statement of the formulas for expected return and variance of a portfolio (which is just a weighted sum of assets).

We first look at an example with just two assets, and then state the situation with n assets more compactly.

6 Task. There are two assets with stochastic returns $\tilde{r}_1$ and $\tilde{r}_2$. The means and volatilities are (μ_1, σ_1) and (μ_2, σ_2), respectively. Let ρ be the correlation coefficient between $\tilde{r}_1$ and $\tilde{r}_2$.

A portfolio is a mixture of the two assets, containing a share w of the first and $(1-w)$ of the second asset. Let x_w denote the stochastic return of the portfolio w, $x_w = wr_1 + (1-w)r_2$, and let μ_w and σ_w^2 denote the expected mean and variance of the porfolio's return.

- Derive a formula for μ_w.
- Derive a formula for σ_w^2.

Your answer here:

7 Task. Asset 1 and 2 have expected returns of 1% and 2% and standard errors of 10% and 30%, respectively. Using your formulae derived in the previous task, make a chart with the volatility (= standard deviation) of the portfolio (σ) on the horizontal and the expected return (μ) of the portfolio on the vertical axis as w goes from 0 to 1.

1. assume $\rho = 1$,
2. assume $\rho = -1$,
3. assume $\rho = 0$.

Your answer here:

For the n asset case, I will restate Markowitz derivations more elegantly using matrix algebra. Let $\tilde{r}_i$ denote the random return of asset i, w_i be the share of asset i in the portfolio (Markowitz uses X_i for this).

Let $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ be a column vector of expected returns of n assets, and let Ω denote the variance-covariance matrix. We now derive an expression for the mean and the variance of the portfolio using matrix notation. We do this because we will use this notation directly in our program.

A portfolio is just a list of weights $(w_1, w_2, \dots, w_n)$ (again a column vector) that sum to one. The expected return of the portfolio (Markowitz denotes this with E)) is just the weighted sum of the individual portfolio returns,

$$E = w^\intercal \mu. \tag{1}$$

The variance of the portfolio (V) is given by Markowitz as

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} w_i w_j,$$

where σ_{ij} are the components of Ω. We can express this more succinctly as

$$V = w^\intercal \Omega w. \tag{2}$$

This is what is known as a quadratic form.

Look at the chart on page 82. Note that the expected return is shown on the horizontal axis and the variance of the portfolio on the vertical axis. A point is potentially efficients if there is no other portfolio with the same expected return and smaller variance, so the efficient points are on the lower right edge of the graph. Today, we depict the expected return on the vertical axis and the standard deviation (called volatility), not the variance, on the horizontal axis. Hence, in this representation, the efficient frontier is in the top left fontier of the choice space.

3 Putting Markowitz calculus into code

8 Exercise. Add to your program. Compute the efficient frontier and plot it with σ on the horizontal and μ on the vertical axis (the section called `'unconstrained optimization'` in the R file). Do not disallow short sales to begin with, but make sure that 100% of capital is invested (meaning the sum of the weights of the protfolio must sum exactly to 1).

There are packages for `R` that make this very easy (such as `'PortfolioAnalytics'`, `'quantmod'`, or `'yfinance'`). I want you not to use those for now. If you do it more “manually”, you will better understand what these packages do under the hood. Only if you cannot solve it yourself you should use these specialized packages.

Since this is a quadratic minimization, you can use a specialized solver, such as `'solve.QP'`. However, there are many other optimizers available for R and you are free to use something else. Also, note that matrix multiplication in `R` has a somewhat unappealing syntax: if `A` and `B` are matrices or vectors, then

`A %*% B`	is the matrix product, and
`t(A)`	is the transpose of `A`.

Your answer here:

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9 Exercise. Repeat the last task but this time exclude short sales, meaning no individual weight can be negative.
Plot the shares of the components along the efficient frontier using an "area plot" or using stacked bars, so that all the weights sum to a bar of constant height (100%) but their composition changes as we move up the efficient frontier.

Your answer here:

The efficient frontier reveals the risk-return tradeoff much more clearly than the scatter plot of the individual equities. Mean-variance optimization of this kind is actually performed in professional investment. In practice, one imposes much stronger constraints, though. For instance, it is customary to exclude very lopsided portfolios that contain more than, say, 10% in one asset (in practice this limit is often set much lower if the investment universe is broad). Also, if you optimize a mixed porfolio for an investment fund that contains various asset classes, such as equities, bonds, alternatives, etc., then the strategic allocation often imposes intervals (upper and lower bounds) for these classes that have to be observed. You might want to play around and try to implement such a multi-asset constraint as well.

4 Ready for sharing in Zoom

Have this ready for the Zoom meeting. Have your notes, charts, programs etc. ready for the Zoom lecture, so that you can share them with us and we can discuss them.
For Exercises 6 to 8, have your `R` programs ready so that you can run them during class. Alternatively, prepare the charts of the efficient frontier to show that. If you had trouble completing the tasks, prepare an explanation where exactly you failed.

References

Markowitz, Harry (1952). "Portfolio Selection." *Journal of Finance*, 7, 77–91.