The Term Structure of Interest Rates

Yvan Lengwiler
Faculty of Business and Economics
University of Basel

yvan.lengwiler@unibas.ch

File: prep-05.tex

Papers to read: parts of Gürkaynak, Sack, and Wright (2006) and Nelson and Siegel (1987).

Contents

1 What is the yield curve?

1.1 *Fixed income?*

"Fixed income securities" are called "fixed income" because their cash flow (coupons) returns are defined from the beginning and do not change over time. This is in contrast to, for instance, equities, whose payoff (dividends) are determined each year by the general shareholders' assembly. A fixed income security is defined by principal value, the percentage coupon that it pays, the interval with witch it pays this coupon, and the time to maturity. At maturity, the final coupon payment occurs and the principal capital is returned as well. Depending on the time to maturity, we call such assets bills (up to one year time to maturity), notes (two to ten years), and bonds (longer maturity).

Note: If the emitter of a bond — say a large company — is not certain to still be solvent when the bond matures, then the cash flow of such a bond is not strictly fixed. The coupn payments and the payment of the face value of the bond might not materialize if the company vanishes before that. We still call such corporate bonds "fixed income", even if it is not strictly accurate. Such bonds carry a default risk premium. Their pricing is different from (supposedly) risk free government bonds. When describing a bond, it is therefore not only the coupon rate and the maturity that matter, but also the issuer.

Now, the market value of a fixed income security is anything but fixed. It is determined by the present value of the future cash flow that the asset generates. When the market interest rate shifts, so does the market value of fixed income assets. In fact, if we observe the market prices of sufficiently many fixed income securities, we can extract the interest rate for different times to maturity. The relationship between the time to maturity and the interest rate over that amount of time is called the *term structure of interest rate*, or *yield curve*. It is difficult to underestimate the practical importance of of the yield curve. The prices that make up this curve belong to the most important prices on the financial market, and in the economy as a whole. These yields determine investment decisions and capital allocation like little else.

1.2 *Who cares?*

The yield curve is, in fact, one of the most important prices in the financial market and the economy as a whole. It is difficult to overestimate its importance.

The yield curve can take different shapes. Normally, it is upward sloping (longer maturities have a higher yield), but sometimes it is inverted, and this correlates with impending recessions. Sometimes there is a bump somewhere at medium maturities, related to monetary policy announcements of shifting expectations.

Maybe more fundamentally, interest rates at different maturities gauge investment decisions, and these are not only affected but the business cycle but feed back into it.

The changes of the yield curve are also noteworthy. It can shift parallel, it can steepen or flatten. All of these movements are key ingredients for developing macroeconomic scenarios that are important in investing.

2 Zero bonds, spot rates, and forward rates

2.1 *Zero bonds*

A *zero bond* is a very simple asset. It has a face value, say 100, m periods later (we call this the *time to maturity*, hence the m). The zero bond with these parameters is simply a promise to pay 100 to the bearer of this bond m periods into the future. There is just one payment, at the very end, nothing else. There is no interest, there are no coupon payments.

Of course, that does not mean that the yield of such an asset is zero. If the asset can be purchased for less than 100 today, and you will receive 100 later (you have to wait for a duration of m), this implies that you have earned a return on this investment.

1 Question. Consider a zero bond with principal of 100 payable in 1 year and 6 months from today. Suppose you can purchase this asset for 98.4 today. So the present value of 100 one and a half years from now is 98.4. What is the return on this asset? What is the annualized return rate? What is the return rate using continuous compounding?

Your answer here:

2.2 *The spot rate*

The *spot rate* is the yield (or interest rate) that an investment of a given duration produces, i.e. the yield of a zero bond. We denote this with $y(t,m)$. t denotes the time at which we observe the yield (i.e. today), m is the time until the investment matures.

There is a mechanical relationship between an investment's present value PV, its future value FV, and the yield $y(t,m)$,

$$PV \cdot (1 + y(t,m))^m = FV.$$

This equation simply computes the compounded interest of today's investment (PV) with the rate $y(t,m)$ until the investment matures (i.e. for m periods, assuming one interest payment each period. For simplicity, we normally work with continuous compounding,

$$PV \exp(m\, y(t,m)) = FV.$$

This frees us from the intricacies of the compounding interval and makes the math considerably simpler.

Gürkaynak *et al.* (2006), equation 2 on page 2 define a special function for that which they call the *discount function*,[1]

$$d(t,m) = \exp(-m\,y(t,m)).$$

This is the PV of a zero bond with a face value of 1 if $y(t,m)$ is the spot rate of that time to maturity,

$$PV = d(t,m)\cdot FV = \exp(-y(t,m)\cdot m)\cdot FV.$$

2 Question. It is important that you fully grasp this. Make a few examples where you compute the value of the $d(\cdot)$ function for different maturities and yields. Can the value be greater than 1? Can it be less than 0?

Your answer here:

2.3 *The forward rate*

The *forward rate* $F(t,m_1,m_2)$ is the yield that a forward investment that will start at m_1 and end at m_2 will yield, if we commit to such an investment today (at t) using forward contracts. The topic is covered in Gürkaynak *et al.* (2006, Section 2.3) as well. It seems maybe far-fetched to study such a frivolous investment. The concept is very useful in theoretical asset pricing. It is also used in practice particularly in an international context when hedging exchange rate risk, and generally in option pricing.

We can take this a bit further by studying the *instantaneous forward rate*,

$$f(t,m) := \lim_{\delta\to 0} F(t,m,m+\delta).$$

This is the forward rate of an investment that that we commit to today (t), that starts at time (m) and lasts only for an infinitesimal amount of time, so to speak. In theoretical asset pricing, this is called simply the forward rate (i.e. we drop the "instantaneous"). There is a mechanical relationship between spot rates and forward rates, see appendix.

[1] Gürkaynak *et al.* use a slightly different notation, $d_t(n)$ and $y_t(n)$, where their n is equivalent to our remaining time to maturity m.

3 Types of yield curves

3.1 *Types of issuers*

It is important that the rates used to produce a yield curve are all from the same company — or at least from companies with similar default risks. A common form of yield curve is the *risk-free yield curve* or *government yield curve*, assuming that governments do not go bust.[2] Likewise, one can produce yield curves for corporate bonds of high quality (say ratings AA and better) or of lesser quality (BBB to A), for instance. Mixing yields from different default risk entities does not make sense. The yield curve is the collection of similarly defined rates at a given point in time.

3.2 *Types of rates*

For instance, $(y(t,0.5), y(t,1), y(t,2), y(t,5), y(t,10))$, as observed at time t, is a *spot yield curve*. More precisely, the full spot yield curve at time t is the function $y(t,m)$ where we keep t fixed and interpret this as a function of m only.

Likewise, $(f(t,0.5), f(t,1), f(t,2), f(t,5), f(t,10))$, as observed at time t, is a *forward yield curve*. Figure 1 shows an example of such curves.

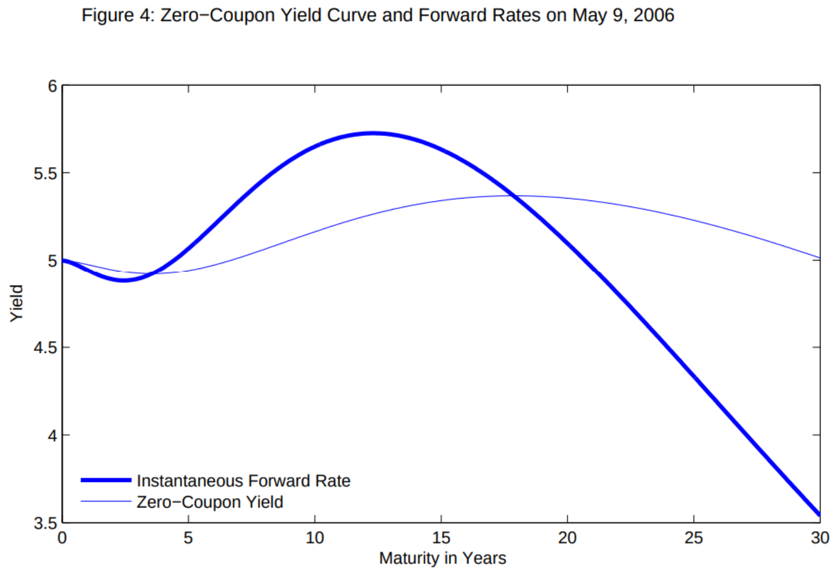

Figure 1: Example of a spot and a forward curve. Source: Gürkaynak *et al.* (2006).

Forward rates can take on dramatic shapes in practice. Spot rates are much more stable because longer maturity rates are related to an average of forward rates of shorter maturities, and being similar to averages, they fluctuate much less.

NOTE: Gürkaynak *et al.* (2006) also discuss the *par-rates*. These are important in practice but cumbersome to work with in theoretical or empirical work. Here, the yields are expressed a coupons of a bonds with a specific maturity congruent with market process. We

[2] . . . which is not always true, `https://en.wikipedia.org/wiki/Sovereign_default`.

cover coupon bonds a little later. These are more complicated assets than zero bonds, and working with par-yields is cumbersome for this reason. We do not cover this here.

4 Taking a first look at the data

For our empirical work that will follow a little later, I provide you with a list of currently circulating bonds from the Swiss confederation, together with their market prices for a specific day (June 25, 2024). Open the file `2024-06-25-CH.csv`.[3]

The market prices are quoted in percent of the face value of the bond, which means that we can treat all face values as 100, see Table 1 for a sample. The relevant fields for us are `coupon`, `maturity`, `bid`, `ask`. Use the script `'FitNelsonSiegel-unfinished.r'` to load the relevant data into R. *Note*: In addition to the outstanding bonds, I have added the SARON (Swiss average overnight) contract to the datafile. The SARON is a very short term money market rate. This helps to stabilize the estimation, because otherwise we have no sufficient data on the short end of the yield curve.

Table 1: One entry of the data on outstanding bonds of Swiss confederation.
(Data are avaliable in a horizontal format in the file `2024-06-25-CH.csv`.)

issuer	Swiss Confederation Government Bond
amount	5612455000
coupon	4
maturity	08.04.2028
bid	111.87
mid	112.07
ask	112.27
par yield (bid)	0.799306818
par yield (mid)	0.749281505
par yield (ask)	0.699372555
ASW Spread (Mid)	N/A
ISIN	CH0008680370
ticker	SWISS
series	N/A
BBG Composite	NR
maturity type	AT MATURITY
currency	CHF
format	conventional

We see that some bonds have 0.0 in the coupon column. These are the zero bonds. Some entries have a positive value in the coupon column. These bonds pay a yearly coupon to the holder.

[3]You might be able to just open this file in Excel. If that does not work right, you have to import the file into Excel using the 'text import wizard'. Specify 'delimited' and 'comma', because it is a comma-delimited table.

5 Bootstrapping and interpolation

If all bonds were zero bonds, our job of computing the yield curve would be trivial: just compute the spot rates for all available matutities and be done. Unfortunately, this is not so. Most bonds are not zero bonds, but coupon bonds, and that makes our job not trivial.

5.1 *Coupon bonds*

A coupon bond is more complicated than a zero bond. It is an asset that pays a coupon (a percentage of the face value of the bond) in regular intervals. In addition, it pays the final coupon and the face value (also called principal) back when it matures. Because such a bond delivers cash at different points in time, it is much more difficult to value.

Consider the example in Table 1. This bond matures in a bit less than four years (8.4.2028) after the observation date (25.6.2024). It has an annual coupon of 4.0%. This means that it will pay 4.0 on 8.4.2025, 8.4.2026, 8.4.2027, and 104.0 when it matues on 8.4.2028. What is the present value of this payment stream? That is not so obvious, because we have cash flows at *multiple* points on time, and if the interest rate is not the same for these various points on time, we have to apply different discounting factors for each embedded payment.

To attack this problem, it is instructive to understand a coupon bond as portfolios of zero bonds. The above example consist of four zero bonds with face value 4.0 and maturity of roughly 10 month, 1 year and 10 month, 2 years and 10 months, as well as one zero bond with face value 104.0 and maturity 3 years and 10 months. The present value of the coupon bond is the sum of the present values of these individual cash flows (i.e. the included zero bonds).

3 Task. Use the file `'FitNelsonSiegel-unfinished.r'` and create a function `'unbundle'` that takes the information of just one bond (one row of the df dataframe) and returns a dataframe with two columns, `t` and `cf`. The first column (`t`) is a time (in years) until a cash flow occurs, the second column (`cf`) is the amount of cash flow that will happen at that time. For each coupon payment and for the final payment, there will be one line in this table. *Tip*: Coupons are once a year and we have to count backwards from the maturity date. *Tip*: The package `lubridate` that contains the function `time_length(..., "years")` is rather helpful here.

You can verify that your program works correctly by doing this:

```
> datafile <- '2024-06-25-CH.csv'
> price_type <- 'mid'
> df <- import_data(datafile, price_type)
> bond <- df[4,]
> unbundle(bond, obs_date)
          t  cf
```

```
1 3.7864476 104
2 2.7864476   4
3 1.7864476   4
4 0.7864476   4
```

5.2 *Duration*

The *duration* of a bond is the average time of all the cash flows the bond delivers, weighted by the relative sizes of these cash flows. The duration of a zero bond is trivially equal to its time to maturity, because there is only one cash flow happening here. For a coupon bond, the duration is always a little less than the time to maturity, because the coupon payments occur before the maturity of the bond.

4 Task. Make a function `'duration'` that computes the average time to maturity of all the cash flows of a bond, weighted by the respective cash flows. Simply use the output of `'unbundle'` and return a weighted average of the `t` column, with the weights being determined by the relative size of the `cf` column.

You can again verify that your program works correctly:

```
> bond <- df[4,]
> duration(bond,obs_date)
[1] 3.579551
```

We see that the duration is a bit shorter than the time to maturity of this bond.

Why is the duration interesting? Consider a zero bond with m time to maturity and market price p. We have $p = \exp(-m\,y)$, thus the yield can be computed as[4]

$$y = -\frac{\log(p/100)}{m},$$

and therefore

$$y\,m = -\log(p/100) \approx -p/100.$$

Thus, when the yield changes by one basis point, the price of the zero bond changes, to a first-order approximation, by m basis points. The longer a zero bond is, the more strongly it reacts to yield changes.

Again, to a first order approximation, the same is true for prices of coupon bonds, if we replace 'time to maturity' with 'duration'.

> *"In estimating the yield curve, we choose the parameters to minimize the weighted sum of the squared deviations between the actual prices of Treasury securities and*

[4]We use $p/100$ in the following derivation because the nominal value of our zerobonds is 100 and not 1.

the predicted prices. The weights chosen are the inverse of the duration of each individual security. To a rough approximation, the deviation between the actual and predicted prices of an individual security will equal its duration multiplied by the deviation between the actual and predicted yields. Thus, this procedure is approximately equal to minimizing the (unweighted) sum of the squared deviations between the actual and predicted yields on all of the securities." (Gürkaynak *et al.*, 2006, page 15)

5.3 *Bootstrapping*

If we know all the interest rates for maturities of 10 month, 1 year and 10 month, 2 years and 10 months, as well as 3 years and 10 months, we can compute the theoretical value of our bond in Table 1. If we know all rates except the last one (for maturity 3 years and 10 month), we can use the observed market price of this coupon bond to infer this interest rate. We just compute the PV of all the embedded coupon payments using the known shorter interest rates and then compute the yield at maturity 3 years and 10 months that makes the sum of these PVs compatible with the market price.

This means that in order to construct longer maturity rates, we have to first compute shorter maturity rates because we need those to value the coupon-payments included in the longer-maturity bonds. This is called *bootstrapping*.

5 Exercise. Let us make a simple toy example of this. Suppose we observe the following market prices:

- A zero bond that matures in 6 months, face value 100, current market price 99.0.
- A coupon bond that matures in 18 months, annual coupon of 3% (every 12 months), face value 1000, current market price is 1021.8.

Your task is to compute the 6-months and the 18-months spot rates. Use continuous compounding.

Your answer here:

5.4 *The need for interpolation*

Challenge #1: Holes in the bootstrapping chain. In practice, we are normally not so lucky to have a bonds that allow us to build such a neatly fitting chain. Instead, we might have zero bonds with 10 months and 2 years and 2 month to maturity, and coupon bonds with maturities at 1 year and 4 months (1y4m), 2y1m, 2y10m, 3y3m, 3y7m, all paying coupons every six months. It is not possible to bootstrap a yield curve from this without interpolation.

Challenge #2: Constant maturity yield curves. Fixed income assets are not issued every day. Instead, a government (or a company) issues such instruments either a few times a year, or in irregular intervals. Also note that a bond's time to maturity keeps getting shorter as we progress though time. A bond at period t with maturity m will become, one period later at time $t+1$, a bond with time to maturity $m-1$. Each day, the bond progresses one day closer to its demise and becomes shorter and shorter. That implies that we cannot observe the interest rate over a fixed time to maturity directly.

For instance, if we want to know what the market interest rate over a two year investment is, we might not be lucky enough to have a traded instrument with exactly that maturity. This means that we have to interpolate from neighboring observed rates.

Interpolation is very useful because it allows us to create constant-maturity time series of interest rates, and because it allows us to overcome the 'holes in the chain' problem discussed before. When you read in a newspaper that the 10-year interest rate has declined over the last two days, then bootstrapping and interpolation is involved.

Typically, central banks perform such computations for the public and publish constant-maturity interest rates that they have bootstrapped and interpolated for you.[5] Here, we will learn how they do that.

5.5 *The Nelson-Siegel model*

A very popular interpolation method for this application was formulated by Nelson and Siegel (1987), and later expanded by Svensson (1994). The Svensson-extension is not fundamentally different from the Nelson-Siegel specification, so we would not learn more by studying Svensson's formulation. We therefore concentrate on Nelson-Siegel. It should be noted, however, that alteriatives are also in use. The Federal Reserve, for instance, uses the Nelson-Siegel-Svensson model as well as cubic splines. (They also estimate a more theory-based arbitrage free model.)

From Nelson and Siegel (1987), equation (2), we copy the interpolation equation,

$$
\begin{aligned}
y(t,m) &= \beta_0(t)w_0(m) + \beta_1(t)w_1(m) + \beta_2(t)w_2(m), \qquad \text{where} \\
w_0(m) &= 1, \\
w_1(m) &= \frac{1-\exp(-m/\tau)}{m/\tau}, \\
w_2(m) &= \frac{1-\exp(-m/\tau)}{m/\tau} - \exp(-m/\tau).
\end{aligned}
$$

This is a model with three linear factors that receive weights β_0, β_1, and β_2. The factors themselves are not linear in the maturity m, and the shape of the factors is determined by a parameter τ.

6 Task. Make a function `NSfactors` that takes a vector of maturities m and a parameter τ and returns the three Neslon-Siegel factors. Use this fuction to create a chart of the three factors for m from 0 to 10, and for $\tau \in \{0.2, 0.6, 1.2\}$.

[5] See here for various rates, `https://fred.stlouisfed.org/searchresults?st=constant+maturity+yields&404-search-button=Search`. For the Federal Reserve, see here, `https://www.federalreserve.gov/releases/h15/`; for the ECB here, `https://data.ecb.europa.eu/methodology/yield-curves`; Bank of England, `https://www.bankofengland.co.uk/statistics/yield-curves`; SNB, `https://data.snb.ch/en/topics/ziredev/doc/explanations_ziredev`.

Your answer here:

Because of the shape of these factors, it is customay to call the first factor *level*, the second factor *slope*, and the third factor *curvature* or *hump*.

7 Question. If you know the three coefficients $(\beta_0(t), \beta_1(t), \beta_2(t))$ that describe a yield curve at a particular day t, how can you quickly know what the ultra short term rate $y(t,0)$ is? Likewise, how can you very quickly determine the very long term rate $y(t,\infty)$?

Your answer here:

---

6 Fitting the Nelson-Siegel model to the data

6.1 *How to do it, and what not to do*

8 Task. Read Section 4, pages 16 to first paragraph of page 19 of Gürkaynak *et al.* (2006). The authors describe in some detail their choices about inclusion or exclusion of data. Why do they make the choices that they make?

Note: This should not shock you. “Cleaning the data” is a standard activity in empirical work.

Your answer here:

6.2 *Doing it yourself*

We are now going to fit the Nelson-Siegel specification to some real data. The Swiss confederation does not issue callable bonds (they all have `maturity type = AT MATURITY`, so we do not have to worry about this aspect.) In fact, we will use all available data points. The strategy for fitting this data to the Nelson-Siegel model proceeds in ten steps (but you have done four of those already):

1. Import the data into R, filtering out just the relevant fields. This part is already done for you in the R script `FitNelsonSiegel-unfinished.r`.
2. Make a function `NSfactors`. We have done this already for task 6.
3. Make a function `NSyield` that takes a maturity m in years (or possibly a vector of maturities) as well as a coefficient/parameter combination $(\beta_0, \beta_1, \beta_2, \tau)$ as inputs and returns the yield according to the Nelson-Siegel factors at the given maturities. (Just compute the weighted sum of the three factors from `NSfactors` and use the βs as weights.)
4. Make a function `PVzero`. It takes a maturity m (in years) and a coefficient/parameter combination $(\beta_0, \beta_1, \beta_2, \tau)$ of the model as inputs and returns the value of a zero bond with face value 100 and maturity m that is implied by the Nelson-Siegel model. Just use `NSyield` and compute the present value using this yield and maturity.
5. Make a function `unbundle`. We have done this already.
6. Make a function `duration`. We have done this already as well.
7. Create a function `PVbond`. This function uses the output of `unbundle` and `PVzero` to compute the present value of a coupon bond, given the parameters $(\beta_0, \beta_1, \beta_2, \tau)$ of the Nelson-Siegel model. It computes the sum of the present values of the bundled zero bonds.
8. Create a function `res` that takes the observed market price and subtracts the theoretical present value according to `PVbond`. It does that for each bond. These differences are the residuals. It returns all residuals as a vector.
9. Create a function `SSR` (for sum of squared residuals) that takes the computed residuals and the weights (one divided by the duration of a bond) and computes the weighted sum of squared residuals.
10. Use a non-linear optimization algorithm to minimize the sum of these weighted squared residuals by choosing the best $(\beta_0, \beta_1, \beta_2, \tau)$. Note that τ is not allowed to be negative.

9 Task. Implement this in R. As output, you should provide your estimation of $(\beta_0, \beta_1, \beta_2, \tau)$, a graph of the spot yield curve, as well as a list of observed vs theoretical prices and the difference (residuals) of the coupon bonds. Use the mid prices. If the script works, you can also estimate using the bid and ask prices.

Warning: This obviously requires a substantial effort. Try to get as far as you can, following the ten steps laid out above. Good luck :-)

Your answer here:

10 Question. Can you discuss this yield curve? Is it normal, or is it unusual?

Your answer here:

---

7 Ready for sharing in Zoom

Have this ready for the Zoom meeting. Have your notes, charts, programs etc. ready for the Zoom lecture, so that you can share them with us and we can discuss them. If you had trouble completing the tasks, prepare an explanation where exactly you failed.

References

Gürkaynak, Refet S., **Brian Sack**, and **Jonathan H. Wright** (2006). *The U.S. Treasury Yield Curve: 1961 to the Present*. *Technical Report*. `https://www.federalreserve.gov/pubs/feds/2006/200628/index.html`.

Nelson, Charles R. and **Andrew D. Siegel** (1987). "Parsimonious Modeling of Yield Curves." *The Journal of Business*, 60, 473–489. `https://www.jstor.org/stable/2352957`.

Svensson, Lars E.O. (1994). *Estimating and Interpreting Forward Interest Rates: Sweden 1992–1994*. *Working Paper 4871*, National Bureau of Economic Research. doi:10.3386/w4871. `http://www.nber.org/papers/w4871`.

A From spot rates to the (instantaneous) forward rate

We denote the PV of a cash flow of 1 unit at m as $d(t,m)$. Using continuous compounding, this PV is

$$d(t,m) = \exp(-m\,y(t,m)).$$

Discounting from m to $m+\delta$ defines the forward rate from m to $m+\delta$,

$$d(t,m)\exp(-\delta\,F(t,m,m+\delta)) = d(t,m+\delta).$$

Solving for $F(t,m,m+\delta)$ yields,

$$F(t,m,m+\delta) = -\frac{\log d(t,m+\delta) - \log d(t,m)}{\delta}.$$

Taking the limit as $\delta \to 0$ reveals that this is the definition of the partial derivative,

$$\begin{aligned} f(t,m) &= -\frac{\partial}{\partial m}\log d(t,m) \\ &= \frac{\partial}{\partial m} m\,y(t,m) \\ &= y(t,m) + m\frac{\partial y(t,m)}{\partial m}. \end{aligned}$$

We see here that the forward rate is related to the spot rate and the local slope of the spot curve. You can see this in Figure 1: The forward curve crosses the spot curve whenever the spot curve is locally flat (that is $\partial y(t,m)/\partial m = 0$), and at $m = 0$ (meaning time to maturity is zero). In these two cases, the second term of the last equation above vanishes and therefore $f(t,m) = y(t,m)$.