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$$\det(A - \lambda I) = 0$$

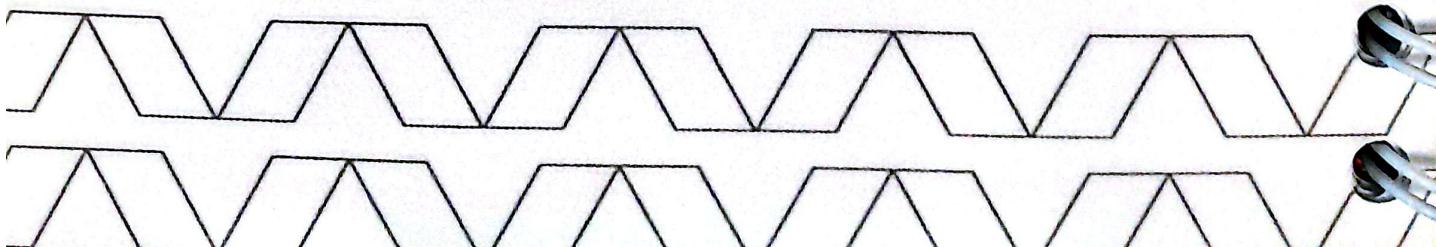
$$\left| \begin{array}{cccc} 4-x & x & -1 & -2 \\ -2 & -9-\lambda & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & -13 & -14 & -13-\lambda \end{array} \right| = 0$$

$$\begin{array}{c|ccc} (4-x) & -9-\lambda & -2 & -4 \\ \hline & 10 & 5-\lambda & -10 \\ & -13 & -14 & -13-\lambda \end{array} \quad -$$

$$\begin{array}{c|ccc} 8 & -2 & -2 & -4 \\ \hline & 0 & 5-\lambda & -10 \\ & -1 & -14 & -13-\lambda \end{array} \quad +$$

$$\begin{array}{c|ccc} -1 & -2 & -9-\lambda & -2 \\ \hline & 0 & 10 & 5-\lambda \\ & -1 & -13 & -13-\lambda \end{array} \quad -$$

$$\begin{array}{c|ccc} -2 & -2 & -9-\lambda & -2 \\ \hline & 0 & 10 & 5-\lambda \\ & -1 & -13 & -14 \end{array}$$



Suppose

$$\begin{vmatrix} -9-x & -2 & -4 \\ 10 & 5-x & -10 \\ -13 & -14 & -13-x \end{vmatrix} = M_1$$

Suppose $x = x$

formula \det of (M_1)

$$= aei + bfg + cdh - ceg - bdi - afh$$

where

$$a = -9-x, b = -2, c = -4, d = 10$$

$$e = 5-x, f = -10, g = -13, h = -14$$

$$i = -13-x$$

$$aei = (-9-x)(5-x)(-13-x)$$

$$(5-x)(-13-x) = -65 - 5x + 13x + x^2 = -65 + 8x + x^2$$

$$ace = -8(-9-x)(-65+8x+x^2)$$

$$= 585 - 72x - 9x^2 + 65x - 8x^2 - x^3$$

$$= \underline{\underline{585 - 7x - 17x^2 - x^3}}$$

$$bfg = (-2)(-10)(-13) = \underline{\underline{-260}}$$

$$cdh = (-4)(10)(-14) = \underline{\underline{560}}$$

$$ceg = (-4)(5-x)(-13) = \underline{\underline{260 - 52x}}$$

$$\text{bedi} = (-2)(10)(-13-x)$$

$$= \underline{\underline{260 + 20x}}$$

$$\text{aph} = (-9-x)(-10)(-14)$$

$$= \underline{\underline{-1260 - 140x}}$$

using the determinant formula we get

$$\det(M_1) = (585 - 7x - 17x^2 - x^3) + (-260)$$

$$+ 560 - (260 - 52x) \cdot$$

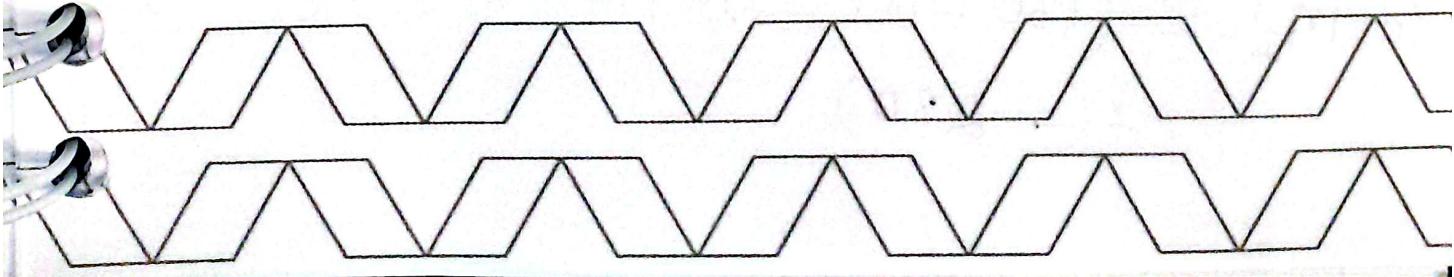
$$- (260 + 20x) - (-1260 - 140x)$$

$$= \underline{\underline{1625 + 165x - 17x^2 - x^3}}$$

Suppose

$$M_2 = \begin{vmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ -1 & -14 & -13-\lambda \end{vmatrix} \text{ and } \lambda = x$$

using the formula we have used above
let's identify our labels (a, b, c, ...)



$$a = -2, b = -2, c = -4, d = 0$$

$$e = 5-x, f = -10, g = -1, h = -14$$

$$i = -13-10$$

now let's compute determinant's terms

$$\begin{aligned}aei &= (-2)(5-x)(-13+x) \\&= (-2)(-65-5x+13x+x^2) \\&= \underline{\underline{130-16x-2x^2}}\end{aligned}$$

$$bef = (-2)(-10)(-1) = \underline{\underline{-20}}$$

$$cbh = (-4)(0)(-14) = \underline{\underline{0}}$$

$$\begin{aligned}ceg &= (-4)(5-x)(-1) \\&= \underline{\underline{20-4x}}\end{aligned}$$

$$bdj = (-2)(0)(-13-x) = \underline{\underline{0}}$$

$$ehf \cancel{= (-2)(-10)(-14)} = \underline{\underline{-280}}$$

using the determinant formula we get

$$\det(M_2) = (130-16x-2x^2) + (-20) - (20-4x)$$

$$+ (-280)$$

$$\det(M_3) = \underline{370 - 12x - 2x^2}$$

Suppose

$$M_3 = \begin{vmatrix} -2 & -9-x & -4 \\ 0 & 10 & 5-x \\ -1 & -13 & -14 \end{vmatrix} \text{ and } x = 2$$

Then the table for the formulae
will be

$$a = -2, b = -9-x, c = -4, d = 0$$

$$e = 10, f = 5-x, g = -1$$

$$h = -13, i = -14$$

Then the terms will be

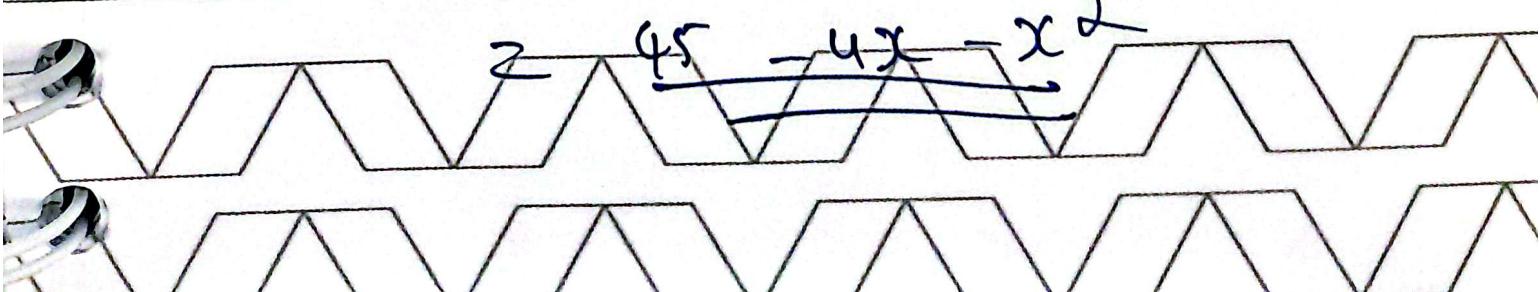
$$aci = (-2)(10)(-14) = \underline{\underline{280}}$$

$$bfg = (-9-x)(5-x)(-1)$$

$$= \underline{\underline{-45 + 14x - 5x^2}} - 1$$

g

$$2 \cancel{45} - \cancel{4x} - \cancel{x^2}$$



$$\text{col}_h = (-4)(-15)(0) = \underline{\underline{0}}$$

$$\text{co}_g = (-4)(10)(-1) = \underline{\underline{40}}$$

$$\text{bot}_i = (-9-x)(0)(-14) = \underline{\underline{0}}$$

$$\alpha_{fh} = (-2)(5-x)(-13)$$

$$= \underline{\underline{130 - 26x}}$$

now let's combine all terms

$$\begin{aligned} \text{det}(M_3) &= (280) + (4r - 4x - x^2) - (40) \\ &\quad - (130 - 25x) \\ &= 155 + 22x - x^2 \end{aligned}$$

Suppose

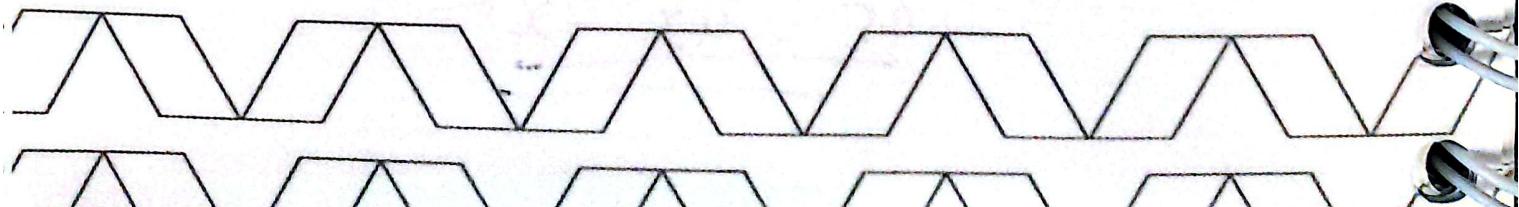
$$M_4 = \left| \begin{array}{ccc} -2 & -9-x & -2 \\ 0 & 10 & 5-x \\ -1 & -13 & -14 \end{array} \right| \text{ and } \lambda = x$$

then labels will be

$$a = (-2), b = -9-x, c = -2, d = \underline{\underline{0}}$$

$$e = 10, f = 5-x, g = -1, h = -13$$

$$i = -14$$



and terms will be

$$\text{axi} = (-2)(10)(-14) = 2 \underline{\underline{280}}$$

$$\text{bfg} = (-9-x)(5-x)(-1)$$

$$= 2(-45 + \underbrace{9x - 5x + x^2}_{4x})(-1)$$

$$= \underline{\underline{45 - 4x - x^2}}$$

$$\text{cet} = (-2)(0)(-13) = \underline{\underline{0}}$$

$$\text{ceg} = (-2)(10)(-1) = 20$$

$$\text{bdi} = (-9-x)(0)(-14) = 0$$

$$\text{gfa} = (-2)(5-x)(-13)$$

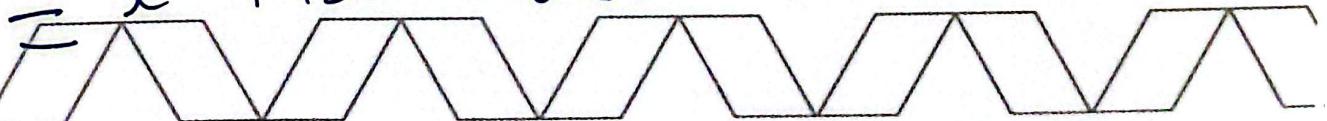
$$= \underline{\underline{130 - 25x}}$$

$$\text{now } \det(M_4) = (280) + (45 - 4x - x^2) - (20) \\ - (130 - 25x)$$

$$= 175 + 22x - x^2$$

$$\text{finally } \det(A - \lambda I) = (4-x)(1625 + 165x - x^2) \\ - 8(370 - 12x - 2x^2) - 1(155 + 22x - x^2) \\ + 2(x^2 + 22x - x^2)$$

$$= x^4 + 13x^3 - 219x^2 - 835x + 3500$$



so first now we have this equation

$$\lambda^4 + 13\lambda^3 - 215\lambda^2 - 835\lambda + 3500 = 0$$

because the leading coefficient is 1
 the possible rational roots
 are the factors of 3500

Since the factors 3500 are many
 we decided to use a library to
 compute this equation and we
 got

$$\lambda_1 \approx 2.6$$

$$\lambda_2 \approx -5.6$$

$$\lambda_3 \approx 11.05$$

$$\lambda_4 \approx -21.12$$

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Finding eigenvector for eigen value

$$\lambda \approx -21.12$$

formular $\vec{v}^T (A - \lambda I) = 0$

$$\vec{v}^T (A - 21.12 I) = 0$$

$$A - 21.12 I \approx \begin{pmatrix} 25.12 & 8 & -1 & -2 \\ -2 & 12.12 & -2 & -4 \\ 0 & 10 & 26.12 & -10 \\ -1 & -13 & -14 & 8.12 \end{pmatrix}$$

then

$$\begin{pmatrix} 25.12 & 8 & -1 & -2 \\ -2 & 12.12 & -2 & -4 \\ 0 & 10 & 26.12 & -10 \\ -1 & -13 & -14 & 8.12 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Suppose } v_4 = 1$$

from row (3)

$$10v_2 + 26.12v_3 + 10v_4 = 0$$

$$10v_2 = -26.12v_3 + 10$$

$$v_2 = -2.612v_3 + 1$$

from Row 1

$$25.12x_1 + 8(-2.612x_3 + 1) - x_3 - 2 = 0$$

$$25.12x_1 - 20.896x_3 + 8 - x_3 - 2 = 0$$

$$25.12x_1 - 21.896x_3 + 6 = 0$$

$$x_1 = \frac{21.896x_3 - 6}{25.12}$$

$$= 0.87x_3 - 0.2388$$

From Row 3

$$-(0.87x_3 - 0.2388) - 13(-2.612x_3 + 1) - 14x_3 + 8.12 = 0$$

$$\Rightarrow -0.87x_3 + 0.2388 - 33.956x_3 - 13 - 14x_3 + 8.12 = 0$$

$$+ 8.12 = 0$$

$$\Rightarrow \cancel{-48.826} \cancel{x_3} = -20.7612$$

$$\therefore x_3 = \frac{-20.7612}{-48.826}$$

$$x_3 \approx 0.421$$

then

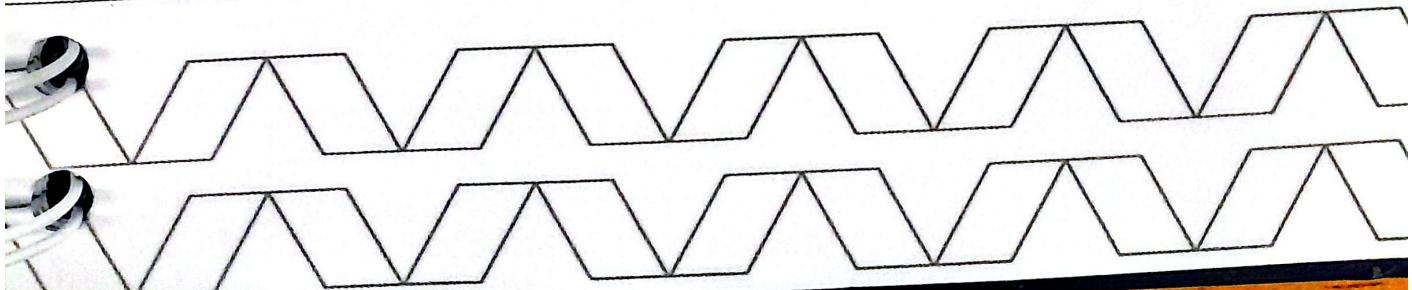
$$\begin{aligned} \mathbf{v}_1 &= 0.87 \mathbf{v}_3 - 0.2388 \\ &= 0.87 (0.242) - 0.2388 \\ &= -0.028 \end{aligned}$$

$$\begin{aligned} \mathbf{v}_2 &= -2.612 \mathbf{v}_3 + 1 \\ &= -2.612 (0.242) + 1 \\ &= 0.3678 \end{aligned}$$

So

eigen vector for eigenvalue $\lambda = -21.12$

is $\vec{v}_2 = \begin{pmatrix} -0.028 \\ 0.3678 \\ 0.242 \\ 1 \end{pmatrix}$



$$\lambda_1 \approx 2.6, \lambda_2 = -5.6, \lambda_3 \approx 11, \lambda_4 \approx -21.6$$

$$\vec{v}(A - \lambda I) = 0 \Rightarrow A - (-5.6)I = A + 5.6I$$

$$A + 5.6I = \begin{pmatrix} 4 + 5.6 & 8 & -1 & -2 \\ -2 & -9 + 5.6 & -2 & -4 \\ 0 & 10 & 5 + 5.6 & -10 \\ -1 & -13 & -14 & -13 + 5.6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 9.6 & 8 & -1 & -2 \\ -2 & -3.4 & -2 & -4 \\ 0 & 10 & 10.6 & -10 \\ -1 & -13 & -14 & -7.4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0.5634 \\ -0.6161 \\ 0.5495 \\ -0.0334 \end{pmatrix}$$

PART 1

$$A = \begin{bmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{bmatrix}$$

Finding the eigen values and eigen vector of one matrix A

1) Eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -9-\lambda & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & -13 & -14 & -13-\lambda \end{vmatrix} = 0$$

Using cofactor expansion along the first row

* First term: $(4-\lambda) \begin{vmatrix} -9-\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{vmatrix}$

* Second term:

$$+ 8 \begin{vmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ -1 & -14 & -13-\lambda \end{vmatrix}$$

* Third term: $-1(-1) \begin{vmatrix} -2 & -9-\lambda & -4 \\ 0 & 10 & -10 \\ -1 & -13 & -13-\lambda \end{vmatrix}$

* fourth term

$$\begin{array}{c|cccc} & -2 & -9-2 & -2 \\ \begin{array}{c} -2 \\ 0 \\ -1 \end{array} & \left| \begin{array}{ccc} -2 & -9-2 & -2 \\ 0 & 10 & 5-2 \\ -1 & -13 & -14 \end{array} \right| \end{array}$$

\Rightarrow substituting and simplifying the minors

$$\begin{aligned} & \Rightarrow (4-2)(-2^3 - 17\lambda^2 + 165\lambda + 1625) - 8(-2\lambda^2 - 22\lambda + 370) - 1(2^3 + 390) + 2(2^2 + 22\lambda + 275) \\ & \Rightarrow \lambda^4 + 13\lambda^3 - 214\lambda^2 - 835\lambda + 3500 = 0 \end{aligned}$$

Finding the eigen values yields

$$\begin{aligned} \lambda_1 & \approx -21.125, \quad \lambda_2 \approx -5.604, \quad \lambda_3 \approx 2.675 \\ \lambda_4 & \approx 11.054 \end{aligned}$$

Part 2) finding the eigen vector the value

$$\lambda_3 = 2.675$$

$$\Rightarrow (A - \lambda I) \vec{v} = 0$$

$$\text{Substituting } \lambda = 2.675$$

\Rightarrow

$$A - 2.675 I = \begin{bmatrix} 1.325 & 8 & -1 & -2 \\ -2 & -11.675 & -2 & -4 \\ 0 & 10 & 2.325 & -10 \\ -1 & -13 & -14 & -15.675 \end{bmatrix}$$

$$\text{Let } \vec{v} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Setting up the Augmented Matrix

$$\left[\begin{array}{cccc|c} 1.325 & 8 & -1 & -2 & 0 \\ -2 & -11.675 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array} \right]$$

performing the Gaussian elimination

1) Divide Row 1 (R_1) by 1.325

$$R_1 \div 1.325 \Rightarrow \boxed{1 \quad 6.0377 \quad -0.7547 \dots}$$

$$R_1 \div 1.325 \Rightarrow \boxed{\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5094 & 0 \\ -2 & -11.675 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array}}$$

2) Eliminating below Row 1 (R_1)

$$\times R_2 + 2 \cdot R_1$$

$$\times R_4 + R_1$$

New matrix:

$$\boxed{\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5094 & 0 \\ 0 & 0.4004 & -3.5094 & -7.0189 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ 0 & -6.9623 & -14.7547 & -17.1844 & 0 \end{array}}$$

3) Divide R_2 by 0.4004:

$$R_2 \div 0.4004 \Rightarrow \boxed{\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5094 & 0 \\ 0 & 1 & -8.766 & -17.533 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ 0 & -6.9623 & -14.7547 & -17.1844 & 0 \end{array}}$$

- Eliminating using new R_2
- a) $R_3 - 10 \cdot R_2$
 - b) $R_4 + 6 \cdot R_2$

$$\Rightarrow \begin{array}{|cccc|c|} \hline & 1 & 6.0377 & -0.7547 & -1.5044 & 0 \\ & 0 & 1 & -8.768 & -17.533 & 0 \\ & 0 & 0 & 90.005 & 165.33 & 0 \\ & 0 & 0 & 46.257 & 105.3 & 0 \\ \hline \end{array}$$

Back Substitution
from the now reduced system,

$$\text{Row 3: } 90.005z + 165.33w = 0 \Rightarrow z = \frac{-165.33w}{90.005}$$

$$\Rightarrow z \approx -1.837w$$

$$\text{Row 2: } y - 8.768z - 17.533w = 0$$

$$\Rightarrow y = 8.768z + 17.533w$$

Substituting $z = -1.837w$

$$\Rightarrow y = 8.768(-1.837w) + 17.533w = 16.107w$$

$$\text{Row 1: } x + 6.0377y - 0.7547z - 1.5044w = 0$$

$$\Rightarrow x = -6.0377y + 0.7547z + 1.5044w$$

$$\Rightarrow x = -6.0377(-16.107w) + 0.7547(-1.837w)$$

$$+ 1.5044w$$

$$x \approx 97.23w - 1.387w + 1.504w \cancel{- 97.352w}$$

$$x \approx 97.352w$$

General solution
let $w = 1$, then

$$\vec{v}_3 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 97.352 \\ -16.107 \\ -1.837 \\ 1 \end{pmatrix} //$$

Normalizing eigen vector \vec{N}_3

$$\|\vec{v}_3\| = \sqrt{(97.352)^2 + (-16.107)^2 + (-1.837)^2 + 1^2} = \sqrt{9741.1}$$

$$\boxed{97.491.1} \approx 98.7$$

Normalized eigenvector \vec{v}_3^{norm}

$$\vec{v}_3^{\text{norm}} \approx \frac{1}{98.7} \begin{pmatrix} 97.352 \\ -16.107 \\ -1.837 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 0.987 \\ -0.1683 \\ -0.019 \\ 0.010 \end{pmatrix}$$

In conclusion, the eigenvector for the eigen value $\lambda_3 = 2.675$, base is

$$\vec{v}_3 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 97.352 \\ -16.107 \\ -1.837 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 97.352 \\ -16.107 \\ -1.837 \\ 1 \end{pmatrix}$$

and the Normalized eigenvector is

$$\vec{v}_3^{\text{norm}} = \begin{pmatrix} 0.987 \\ -0.163 \\ -0.019 \\ 0.010 \end{pmatrix}$$

$$\lambda_3 = 2.675 \quad \vec{v}_3 = \begin{pmatrix} 97.352 \\ -16.107 \\ -1.837 \\ 1 \end{pmatrix} \quad \vec{v}_3^{\text{norm}} = \begin{pmatrix} 0.987 \\ -0.163 \\ -0.019 \\ 0.010 \end{pmatrix}$$

For $\lambda_2 \approx 11.05$

$$\text{formula} = (A - \lambda I)$$

$$A - 11.05I \Rightarrow \begin{bmatrix} 4 - 11.05 & 8 & -2 & -2 \\ -2 & -9 - 11.05 & -2 & -4 \\ 0 & 10 & 5 - 11.05 & -10 \\ 1 & 13 & -14 & -13 - 11.05 \end{bmatrix}$$

$$= \begin{bmatrix} -7.05 & 8 & -1 & -2 \\ -2 & -20.05 & -2 & -4 \\ 0 & 10 & -6.05 & -10 \\ 1 & -13 & -14 & -24.05 \end{bmatrix}$$

$$\text{Vector} : (A - \lambda I)x = 0$$

$$\begin{bmatrix} -7.05 & 8 & -1 & -2 \\ -2 & -20.05 & -2 & -4 \\ 0 & 10 & -6.05 & -10 \\ 1 & -13 & -14 & -24.05 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Let's use equation 3.

$$10x_2 - 6.05x_3 - 10x_4 = 0 \Rightarrow x_2 = 0.605x_3 + x_4$$

Let's replace x_2 in equation 1

$$-7.05x_1 + 8(0.605x_3 + x_4) - x_3 - 2x_4 = 0$$

$$-7.05x_1 + 4.84x_3 + 8x_4 - x_2 = 9.84 = 0$$

$$-7.05x_1 + 3.84x_3 + 6x_4 = 0$$

$$x_1 = \frac{3.84x_3 + 6x_4}{7.05}$$

pick $x_3 = 0.5, x_4 = -0.8$

$$x_2 =$$

$$x_1 = \frac{3.84(0.5) + 6(-0.8)}{7.05}$$

$$x_1 = 0.408$$

$$x_2 = 0.6x_3 + x_4$$

$$x_2 = 0.6(0.5) + -0.8$$

$$x_2 = -0.49$$

$$\vec{y}_1 = \begin{pmatrix} -0.408 \\ -0.49 \\ -0.8 \\ -0.5 \end{pmatrix}$$