forward lateral vertical

$$\xi = (P, V^b, \Theta, \omega) = (x, y, z, V_x^b, V_y^b, V_z^b, \phi, \theta, \psi, p, q, r) \in \mathbb{R}^{12}$$

$$u = (F_{mag}, \alpha, \beta, \tau_\psi) \in \mathbb{R}^4 \qquad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

rot matrix from body to inertial frame

$$\dot\xi = f(\xi, u) = \begin{pmatrix} \dot P \\ \dot V^b \\ \dot\Theta \\ \dot\omega \end{pmatrix} = \begin{pmatrix} R(\Theta)V^b \\ \frac{1}{m}[F_{tot}^b + R^T(\Theta)F_g^i] - \omega \times V^b \\ W(\Theta)\omega \\ I^{-1}[\tau_{tot}^b - \omega \times I\omega] \end{pmatrix}$$

$$\dot P = RV^b$$
$$m\dot V^b = F_{tot}^b + R^T F_g^i - \omega \times m V^b$$
$$F_g^i = (0, 0, -mg)^T$$
$$F_{tot}^b = F_{rotor}^b + F_{drag}^b = T(\alpha,\beta)F_{mag} - A_{trans}V^b$$
$$F_{\ell/u}^b = T(\alpha,\beta)|F_{m\ell/u}|$$

determined by blade inclination
=> by cyclic pitch

$$T(\alpha,\beta) = \begin{pmatrix} -\tan\alpha\cos\gamma \\ \tan\beta\cos\gamma \\ -\cos\gamma \end{pmatrix}, \quad \gamma = \operatorname{atan}\left(\sqrt{\tan^2\alpha + \tan^2\beta}\right)$$

assumption: the only external force is gravity

$$F_e^i = F_g^i = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

$$\begin{pmatrix} \dot V_x^b \\ \dot V_y^b \\ \dot V_z^b \end{pmatrix} = \frac{1}{m}[F_{tot}^b + R^T F_g^i] - \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} V_x^b \\ V_y^b \\ V_z^b \end{pmatrix}$$

$$\begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix} = W(\phi, \theta) \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot p \\ \dot q \\ \dot r \end{pmatrix} = I^{-1}\left[\tau_{tot}^b - \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times I \begin{pmatrix} p \\ q \\ r \end{pmatrix}\right]$$

rate of change of Euler angles

$$\dot\Theta = W(\Theta)\,\omega$$

body angular velocity

$$W(\Theta) = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{pmatrix} \Rightarrow \dot\psi = \frac{\sin\phi}{\cos\theta} q + \frac{\cos\phi}{\cos\theta} r$$

$$I\dot\omega = -\omega \times I\omega + \tau_{tot}^b + R^T\tau_e^i$$

yaw
$$\dot\psi = \overbrace{\frac{\sin\phi}{\cos\theta} q + \frac{\cos\phi}{\cos\theta} r}^{(s_\phi\sec\theta)q + (c_\phi\sec\theta)r} = f(\phi, \theta, q, r) \text{ function of the state}$$

$$\ddot\psi = \underbrace{\frac{d}{dt}(s_\phi\sec\theta\, q)}_{1} + \underbrace{\frac{d}{dt}(c_\phi\sec\theta\, r)}_{2}$$

$$1 = (\dot\phi c_\phi\sec\theta + s_\phi\dot\theta\sec\theta\tan\theta)q + (s_\phi\sec\theta)\dot q$$

input hidden here

$$2 = (-\dot\phi s_\phi\sec\theta + c_\phi\dot\theta\sec\theta\tan\theta)r + (c_\phi\sec\theta)\dot r \quad ⊛$$

Output to be tracked: $y_{out} = \begin{pmatrix} x \\ y \\ z \\ \psi \end{pmatrix} = \begin{pmatrix} P \\ \psi \end{pmatrix}$

$$⊛\ \ddot\psi = \underbrace{(s_\phi\sec\theta)\dot q + (c_\phi\sec\theta)\dot r}_{\text{input-dependent part}} + \underbrace{(\dot\phi c_\phi\sec\theta + s_\phi\dot\theta\sec\theta\tan\theta)q + (-\dot\phi s_\phi\sec\theta + c_\phi\dot\theta\sec\theta\tan\theta)r}_{\text{state-dependent part } L_\psi(\xi)}$$

$$\hookrightarrow (0 \;\; s_\phi\sec\theta \;\; c_\phi\sec\theta)\underbrace{\begin{pmatrix} \dot p \\ \dot q \\ \dot r \end{pmatrix}}_{\dot\omega = I^{-1}[\tau_{tot}^b - \omega\times I\omega]}$$

$$\Rightarrow \ddot\psi = L_\psi(\xi) + (0 \;\; s_\phi\sec\theta \;\; c_\phi\sec\theta) I^{-1}[\tau_{tot}^b - \omega \times I\omega] = L_\psi(\xi) + (0 \;\; s_\phi\sec\theta \;\; c_\phi\sec\theta) I^{-1}[\tau_{thrust}^b + \tau_r^b - A_{rot}\omega - \omega\times I\omega] \quad ⊗$$

$$\hookrightarrow \underbrace{\tau_{thrust}^b}_{d_{cm}\times F_{rotor}^b} + \underbrace{\tau_{reaction}^b}_{\begin{pmatrix} 0 \\ 0 \\ \tau_\psi \end{pmatrix}} \underbrace{- A_{rot}\,\omega}_{\tau_{drag}}$$

$$d_{cm} \times F_{rotor}^b = d_{cm} \times T(\alpha,\beta)F_{mag} \hookrightarrow d_{cm} \times T(u_2, u_3)u_1$$

$$\begin{pmatrix} 0 \\ 0 \\ \tau_\psi \end{pmatrix} \hookrightarrow (0, 0, u_4)^T$$

$$⊗\ \ddot\psi = L_\psi(\xi) + (0 \;\; s_\phi\sec\theta \;\; c_\phi\sec\theta) I^{-1}\left[d_{cm} \times T(\underset{u_2}{\alpha}, \underset{u_3}{\beta})\underset{u_1}{F_{mag}} + \begin{pmatrix} 0 \\ 0 \\ \underset{u_4}{\tau_\psi} \end{pmatrix} - A_{rot}\omega - \omega\times I\omega\right]$$

$\dot{P} = R(\Theta)V^b = f(V^b, \Theta) = f(\phi, \vartheta, \psi, V_x^b, V_y^b, V_z^b)$ function of the state

$\ddot{P} = \underbrace{\dot{R}}_{\hookrightarrow RS(\omega)}V^b + R\dot{V}^b = RS(\omega)V^b + R\left[\frac{1}{m}\left(\underbrace{F_{tot}^b}_{\hookrightarrow F_{rotor}^b + F_{drag}^b = T(\alpha,\beta)F_{mag} - A_{trans}V^b} + R^T F_g^i\right) - \omega \times V^b\right] = \cancel{R(\omega \times V^b)} + R\left[\frac{1}{m}\left(T(\alpha,\beta)F_{mag} - A_{trans}V^b + R^T F_g^i\right) \cancel{- \omega \times V^b}\right]$ ⊛

⊛ $\ddot{P} = \frac{1}{m}RT(\underset{u_2}{\alpha},\underset{u_3}{\beta})\underset{u_1}{F_{mag}} - \frac{1}{m}RA_{trans}V^b + \frac{1}{m}F_g^i$

$\Rightarrow$ Decoupling matrix $J(\xi) = \left(\begin{array}{ccc|c} J_{11} & J_{12} & J_{13} & 0 \\ J_{21} & J_{22} & J_{23} & 0 \\ J_{31} & J_{32} & J_{33} & 0 \\ \hline J_{41} & J_{42} & J_{43} & J_{44} \end{array}\right)\begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \ddot{P} \\ \} \ddot{\psi} \end{array}$ nonsingular $\Rightarrow$ invertible $\Rightarrow u = J^{-1}(\xi)[r - \ell(\xi)]$

r: desired accel vector