# Physics of Waves

## A practical review of the workshop on acoustic wavefield imaging

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# Table of contents

01

# Introduction to Physics of Waves

# Applications Acoustical Imaging

**Table 1.** Applications Acoustical Imaging [1].

| | |
|---|---|
| >50 MHz | Acoustic Microscopy |
| 10 MHz | Laminated materials (e.g. airplanes) → 0.5 cm deep |
| 1 MHz | Inspections of welds → 0.5-5 cm deep<br>Medical ultrasound → 0.2-20 cm deep |
| 100 kHz | |
| 10 kHz | Ship wreck detection → 1-20 m deep |
| 1 kHz | Near surface inspection → 5-500 m deep |
| 100 Hz | Oil & gas exploration → 0.5-5 km deep |
| 10 Hz | |
| 1 Hz | Tectonic imaging → 2-25 km depth |
| 0.1 Hz | Global seismology → full earth |

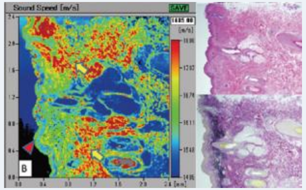



**Figure 1.** SOS map of skin [2].

**Figure 2.** Echography [3].

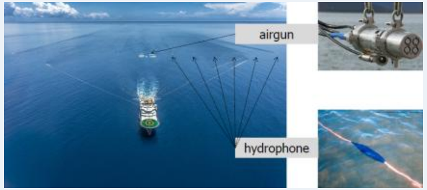



**Figure 3.** Seismology experiment [4].

[1] https://kvandongen.tnw.tudelft.nl/wp-content/uploads/2025/02/Koen-Bucaramanga-day-1.pdf
[2] K. Miura, "Application of Scanning Acoustic Microscopy to Pathological Diagnosis," DOI: 10.5772/63405
[3] http://langeproductions.com/
[4] www.iagc.org

# Basic Concepts

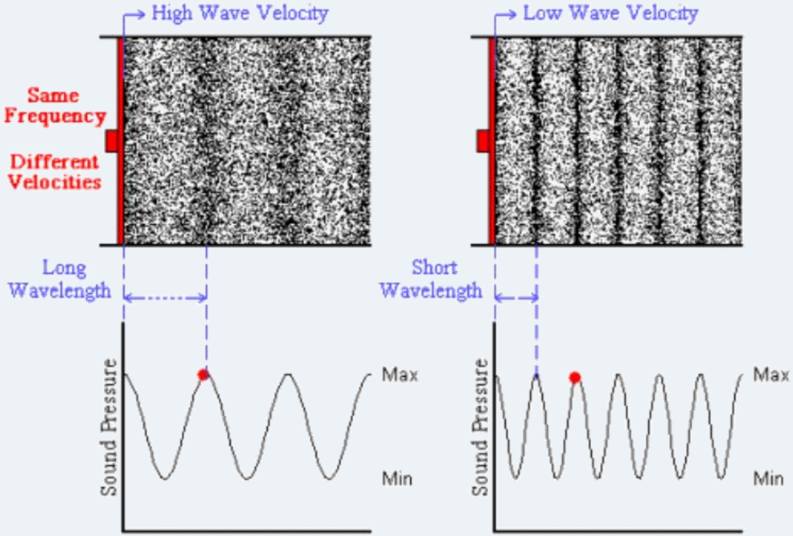



**Figure 4.** Acoustic Wave Propagation [5].

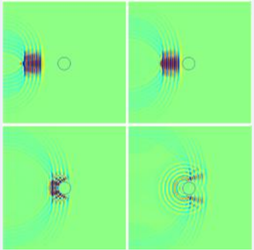


**Figure 5.** Acoustic Wave Propagation [6].

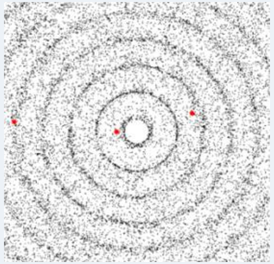


**Figure 6.** Acoustic Monopole [7].

How a single acoustic source produces waves that propagate in all directions and how they interact in the medium

[5] http://resource.isvr.soton.ac.uk/spcg/tutorial/tutorial/Tutorial_files/Web-basics-frequency.htm
[6] http://www.mathworks.com/matlabcentral/fileexchange/screenshots/1347/original.jpg
[7] http://resource.isvr.soton.ac.uk/spcg/tutorial/tutorial/Tutorial_files/Web-basics-pointsources.htm

# Acoustic Field Equations

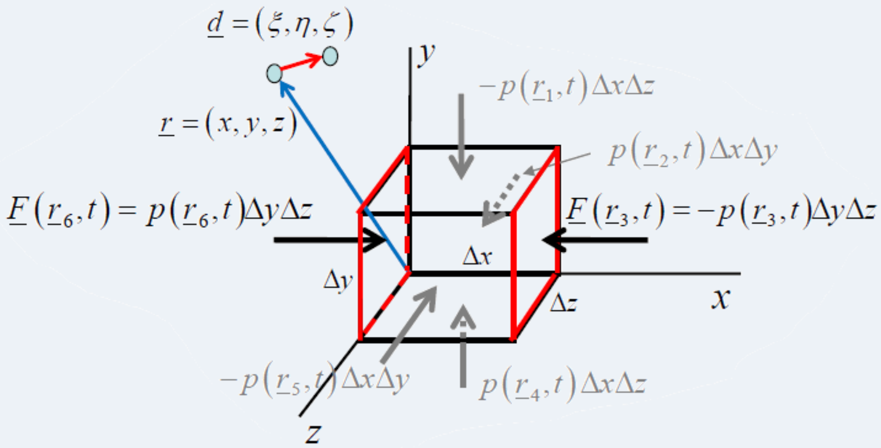
Volumen: $\Delta V = \Delta x \Delta y \Delta z$

Location: $\underline{r} = (x, y, z)$

Displacement field: $\underline{d}(\underline{r},t) = [\xi(\underline{r},t), \eta(\underline{r},t), \zeta(\underline{r},t)]$

Velocity field: $\underline{v}(\underline{r},t) = \frac{d}{dt}\underline{d}(\underline{r},t)$

Pressure field: $p(\underline{r},t) = p_{tot}(\underline{r},t) - p_0$




## Hooke's law

### Equation of deformation

A volume source density of injection rate

$$\boxed{\frac{\partial p(\underline{r},t)}{\partial t} = -\frac{1}{\kappa}\nabla\cdot\underline{v}(\underline{r},t) + \frac{1}{\kappa}q(\underline{r},t)}$$

$$i\omega\hat{p}(\underline{r},\omega) = -\frac{1}{\kappa}\nabla\cdot\hat{\underline{v}}(\underline{r},\omega)$$

## Newton's law

### Equation of motion

A volume source density of volume source

$$\boxed{-\nabla p(\underline{r},t) = \rho\frac{\partial\underline{v}(\underline{r},t)}{\partial t} - \underline{f}(\underline{r},t)}$$

$$-\nabla\hat{p}(\underline{r},\omega) = i\omega\rho\hat{\underline{v}}(\underline{r},\omega)$$

# Acoustic Field Equations

## Wave Equation

In the absence of sources:

- A escalar wave equation for the pressure field

$$\left.\begin{aligned}\rho\kappa\frac{\partial}{\partial t}\left[\frac{\partial p(\underline{r},t)}{\partial t}\right] &= \rho\kappa\frac{\partial}{\partial t}\left[-\frac{1}{\kappa}\nabla\cdot\underline{v}(\underline{r},t)\right]\\ \nabla\cdot\left[-\nabla p(\underline{r},t)\right] &= \nabla\cdot\left[\rho\frac{\partial\underline{v}(\underline{r},t)}{\partial t}\right]\end{aligned}\right\} \Rightarrow \boxed{\nabla^2 p(\underline{r},t)-\rho\kappa\frac{\partial^2 p(\underline{r},t)}{\partial t^2}=0}$$

- A vectorial wave equation for the velocity wavefield $= \frac{1}{c^2}$

$$\left.\begin{aligned}\kappa\nabla\left[\frac{\partial p(\underline{r},t)}{\partial t}\right] &= \kappa\nabla\left[-\frac{1}{\kappa}\nabla\cdot\underline{v}(\underline{r},t)\right]\\ \kappa\frac{\partial}{\partial t}\left[-\nabla p(\underline{r},t)\right] &= \kappa\frac{\partial}{\partial t}\left[\rho\frac{\partial\underline{v}(\underline{r},t)}{\partial t}\right]\end{aligned}\right\} \Rightarrow \boxed{\nabla\left[\nabla\cdot\underline{v}(\underline{r},t)\right]-\rho\kappa\frac{\partial^2 \underline{v}(\underline{r},t)}{\partial t^2}=0}$$

## Helmholtz Equation

$$\boxed{\nabla^2\hat{p}(\underline{r},\omega)+\frac{\omega^2}{c^2}\hat{p}(\underline{r},\omega)=0}$$

$$\nabla^2\hat{p}(\underline{r},\omega)+\frac{\omega^2}{c^2}\hat{p}(\underline{r},\omega)=-\hat{S}(\underline{r},\omega)$$

## Spherical Waves

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\hat{p}(r,\omega)\right)+\frac{\omega^2}{c^2}\hat{p}(r,\omega)=-\delta(\underline{r})\hat{S}(\omega)$$

$$\hat{p}(r,\omega)=\hat{A}(\omega)\frac{e^{-i\omega r/c_0}}{r}+\overline{\hat{B}(\omega)\frac{e^{i\omega r/c_0}}{r}},\quad \text{for } r\neq 0.$$

Green's function or impulse response of the medium

$$\boxed{\hat{G}(\underline{r},\omega)=\frac{e^{-i\omega|\underline{r}|/c_0}}{4\pi|\underline{r}|}}$$

# Acoustic Field Equations

## Green's functions

In the absence of sources:

1D: $\left(\frac{d^2}{dx^2}+k^2\right)\hat{G}(x)=-\delta\left(x-a\right) \quad \rightarrow \quad \hat{G}(x)=\frac{i}{2k}e^{-ikr}$ with $r=|x-a|$

2D: $\left(\nabla^2+k^2\right)\hat{G}(\vec{x})=-\delta\left(\vec{x}-\vec{a}\right) \quad \rightarrow \quad \hat{G}(\vec{x})=-\frac{i}{4}H_0^{(2)}(kr)$ with $r=|\vec{x}-\vec{a}|$

3D: $\left(\nabla^2+k^2\right)\hat{G}(\vec{x})=-\delta\left(\vec{x}-\vec{a}\right) \quad \rightarrow \quad \hat{G}(\vec{x})=\frac{1}{4\pi}\frac{e^{-ikr}}{r}$ with $r=|\vec{x}-\vec{a}|$

$$\begin{cases} \frac{i}{2ka}J_1(ka)H_0^{(1)}(kr) & r\geq a \\ \frac{i}{2ka}\left[H_1^{(1)}(ka)+\frac{2i}{\pi ka}\right] & r=0 \end{cases}$$

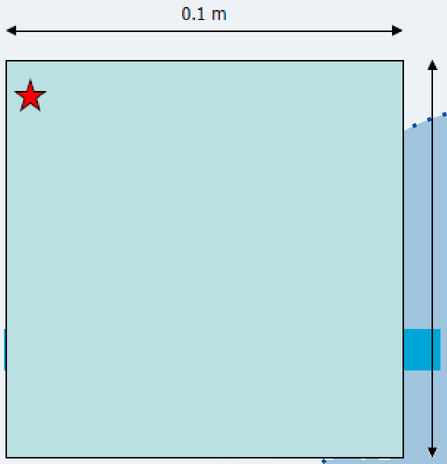
$$\begin{cases} \frac{3e^{-ikr}}{4k^3\pi a^3 r}\left[\sin(ka)-ka\cos(ka)\right] & r\geq a \\ \frac{3}{4k^2\pi a^3}\left[(1+ika)e^{-ika}-1\right] & r=0 \end{cases}$$

# Exercise 1

- Compute the pressure field generated by a point source (volume source density of injection rate)
  - Centre frequency Gaussian pulse $f_0$ = 1 MHz;
  - Medium water ($c$ = 1500 m/s, $\rho$ = 1000 kg/m$^3$);
  - Locate the point source in the centre of a volume (e.g. 0.1 m x 0.1 m x 0.0015 m).

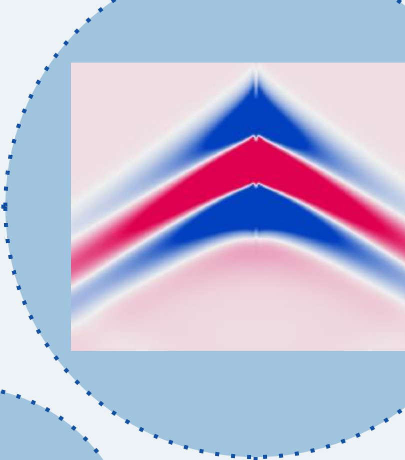
- Compute the pressure field in the frequency domain and transform the resulting field to the time domain using FFT.

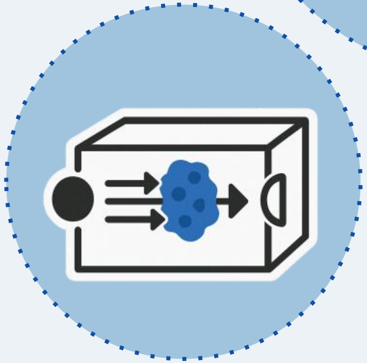
- $\Delta x$ = ?
- $\Delta t$ = ?
- Nt = ?



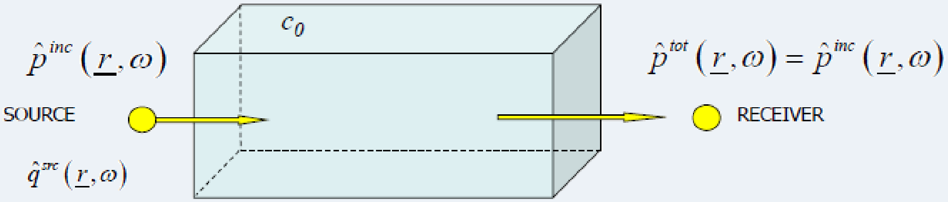
02

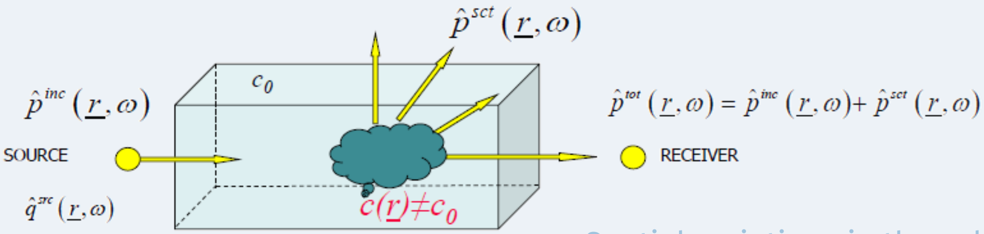
# Heterogeneous Media





# Equations




The acoustic media parameters become spatially varying

## Hooke's law

$$\nabla \cdot \underline{v}(\underline{r},t) + \kappa(\underline{r})\partial_t p(\underline{r},t) = q(\underline{r},t)$$

$$\downarrow$$

$$\nabla \cdot \underline{v}(\underline{r},t) + \kappa_0 \partial_t p(\underline{r},t) = q(\underline{r},t) + \{\kappa_0 - \kappa(\underline{r})\}\partial_t p(\underline{r},t)$$

## Newton's law

$$\nabla p(\underline{r},t) + \rho(\underline{r})\partial_t \underline{v}(\underline{r},t) = \underline{f}(\underline{r},t)$$

$$\downarrow$$

$$\nabla p(\underline{r},t) + \rho_0 \partial_t \underline{v}(\underline{r},t) = \underline{f}(\underline{r},t) + \cancel{\{\rho_0 - \rho(\underline{r})\}\partial_t \underline{v}(\underline{r},t)}$$

Spatial variations in the volume density of mass are neglected

Combining...

$$\nabla^2 p(\underline{r},t) - \frac{1}{c_0^2}\partial_t^2 p(\underline{r},t) = -\underbrace{\{\rho_0 \partial_t q(\underline{r},t) - \nabla \cdot \underline{f}(\underline{r},t)\}}_{S_{pr}(\underline{r},t)} - \underbrace{\{\rho_0(\kappa_0 - \kappa(\underline{r}))\partial_t^2 p(\underline{r},t) - \nabla \cdot \cancel{[\{\rho_0 - \rho(\underline{r})\}\partial_t \underline{v}(\underline{r},t)]}\}}_{S_{cs}(\underline{r},t)}$$

$$\nabla^2 \hat{p}(\underline{r},\omega) + \frac{\omega^2}{c_0^2}\hat{p}(\underline{r},\omega) = -\{\rho_0 i\omega \hat{q}(\underline{r},\omega) - \nabla \cdot \hat{\underline{f}}(\underline{r},\omega)\} - \{-\rho_0(\kappa_0 - \kappa(\underline{r}))\omega^2 \hat{p}(\underline{r},\omega) - \nabla \cdot \cancel{[\{\rho_0 - \rho(\underline{r})\} i\omega \hat{\underline{v}}(\underline{r},\omega)]}\}$$

# Equations

Primary sources

$$S_{pr}\left(\underline{r}',t\right)$$

Green's function

$$\hat{G}(\underline{r},\omega)=\frac{e^{-i\omega|\underline{r}|/c_0}}{4\pi|\underline{r}|}$$

Field generated

$$\hat{p}^{inc}(\underline{r},\omega)=\int_{\underline{r}'\in D}\hat{G}(\underline{r}-\underline{r}',\omega)\,S_{pr}\left(\underline{r}',\omega\right)dV(\underline{r}').$$

Principle of superposition

$$\boxed{\hat{p}(r,\omega)=\hat{p}^{inc}(\underline{r},\omega)+\int_{\underline{r}'\in D}\hat{G}(\underline{r}-\underline{r}',\omega)\chi\left(\underline{r}'\right)\omega^2\;\hat{p}(\underline{r}',\omega)\,dV(\underline{r}')}$$

with $\chi\left(\underline{r}'\right)=\dfrac{1}{c^2\left(\underline{r}'\right)}-\dfrac{1}{c_0^2}$

**Born approximation**

$$\hat{p}(r,\omega)=\hat{p}^{inc}(\underline{r},\omega)+\int_{\underline{r}'\in D}\hat{G}(\underline{r}-\underline{r}',\omega)\,\chi\left(\underline{r}'\right)\omega^2\;\hat{p}^{inc}(\underline{r}',\omega)\,dV(\underline{r}')$$

# Forward and Inverse Problem

## Forward problem

- Source and contrast are known
- Total/actual field is unknown

➔ Linear Problem

## Inverse problem

- Sources are known
- Total/actual field is known at the boundary
- Contrast and field in ROI is unknown

➔ Non-linear problem

# Solutions

## Neumann Series

$\hat{p}^{(1)}$

Born approximation

$$\hat{p}^{(2)}(r,\omega) = \hat{p}^{inc}(\underline{r},\omega) + \int_{\underline{r}'\in D} \hat{G}(\underline{r}-\underline{r}',\omega)\,\chi(\underline{r}')\,\omega^2\,\hat{p}^{(1)}(\underline{r}',\omega)\,dV(\underline{r}')$$

$\hat{p}^{(1)}, \hat{p}^{(2)}, \ldots, \hat{p}^{(n)}$ : Neumann Series

Warning: For strong contrast, the iteration may not converge to the true solution.

## Conjugate gradient method

$$\hat{p}(r,\omega) = \hat{p}^{inc}(\underline{r},\omega) + \int_{r'\in D} \hat{G}(\underline{r}-\underline{r}',\omega)\,\chi(\underline{r}')\,\omega^2\,\hat{p}(\underline{r}',\omega)\,dV(\underline{r}')$$

f=Lu

Known incident field $\quad \mathrm{f} = \hat{p}^{inc}(\underline{r},\omega)$

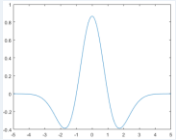
Unknown total field $\quad \mathrm{u} = \hat{p}(\underline{r},\omega)$

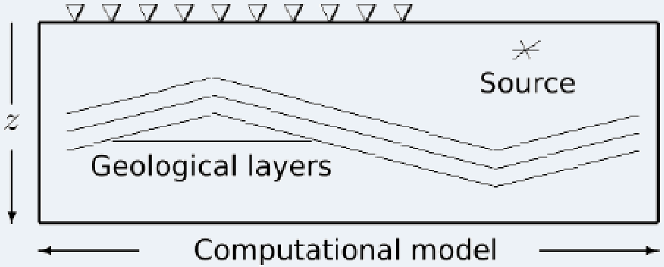
Remaining integral operator $\quad \mathrm{L}$

$$\mathrm{E_n} = \|\mathrm{r_n}\|^2 = \|\mathrm{f} - \mathrm{Lu_n}\|^2$$
$$\mathrm{u_n} = \mathrm{u_{n-1}} + \alpha_n \mathrm{d_n}$$
$$\mathrm{d_n} = \mathrm{L}^{\dagger}\mathrm{r_n}$$

# Finite Difference Time Domain (FDTD)

$$\frac{1}{\mathbf{c}^2(x,z)}\frac{\partial^2 p(x,z)}{\partial t^2} = \frac{\partial^2 p(x,z)}{\partial x^2} + \frac{\partial^2 p(x,z)}{\partial z^2} + src(x,z,t)$$






The finite difference method can be applied to the acoustical scalar wave equation

$$\left(\frac{\partial^2 p}{\partial t^2}\right)^n_{i,j} = \frac{p^{n+1}_{i,j} - 2p^n_{i,j} + p^{n-1}_{i,j}}{(\Delta t)^2}$$

$$\left(\frac{\partial^2 p}{\partial x^2}\right)^n_{i,j} = \frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{(\Delta x)^2}$$

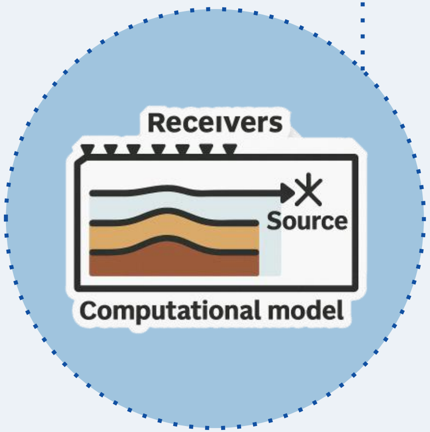
$$\left(\frac{\partial^2 p}{\partial z^2}\right)^n_{i,j} = \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{(\Delta z)^2}$$

$$p^{n+1}_{i,j} = 2(1-2G^2)p^n_{i,j} - p^{n-1}_{i,j} + G^2(p^n_{i+1,j} + p^n_{i-1,j} + p^n_{i,j+1} + p^n_{i,j-1})$$

$G = \frac{c\Delta t}{\Delta h}$ Courant parameter $\quad \Delta h = \Delta x = \Delta z$

Stability is archieved when $\alpha \leq \frac{1}{\sqrt{2}} \quad \alpha = \frac{\mathbf{c}\Delta t}{\Delta h}$

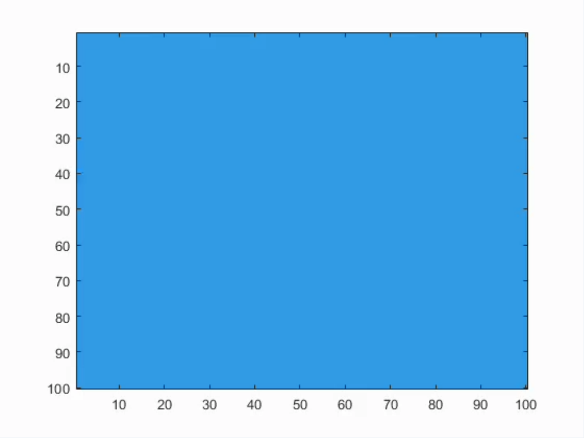
Numerical dispersion is avoided when $\frac{v_p}{c} = 1 \quad v_p = \frac{\omega}{k}$




# Exercise 2

Simulate wave propagation using FDTD using the Ricket function as source.
Define your parameters properly and check the stability of your simulation.

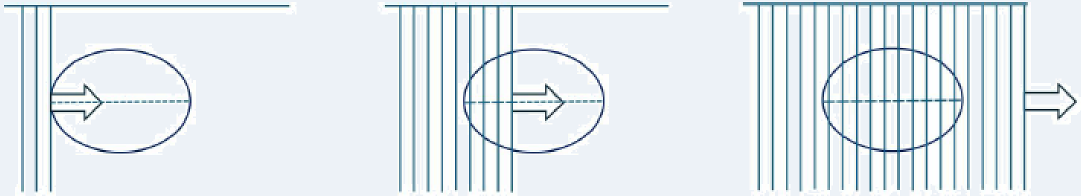
$$\frac{1}{\mathbf{c}^2(x,z)}\frac{\partial^2 p(x,z)}{\partial t^2} = \frac{\partial^2 p(x,z)}{\partial x^2} + \frac{\partial^2 p(x,z)}{\partial z^2} + src(x,z,t),$$



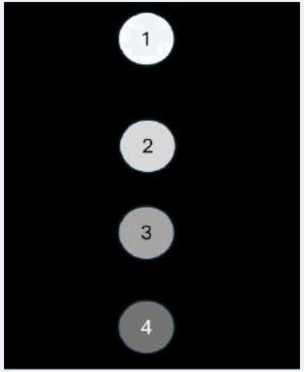

03

# Ultrasound Imaging

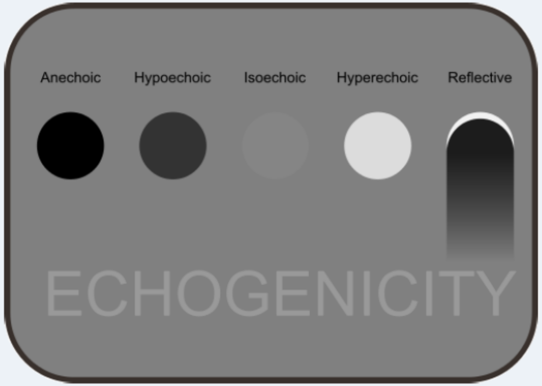
# B-mode ultrasound



An 2D B-mode image is formed line-by-line as the beam moves along the transducer array.

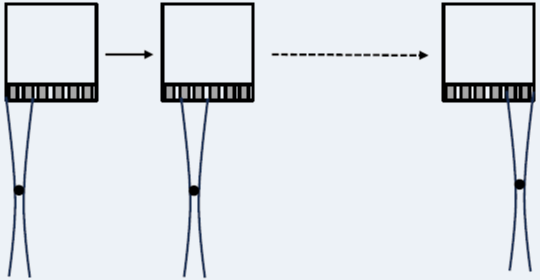


B-mode




## Linear Array Imaging

- In linear array imaging, a few elements are grouped and fired to get an A-line
- Delays can be used to create a focused beam



$$\textit{Number of A lines to make an image} = \textit{number of elements} - \textit{number of grouped elements} + 1$$

# Exercise 3

A series of ultrasound data acquisitions were conducted using the L11-5V linear array transducer, which has a center frequency of 7.6 MHz. The experiment was performed in a controlled water tank environment to ensure consistent acoustic propagation conditions. Obtain the B-mode image from the channel data.

Cross-sectional image | Longitudinal image

Tube

Needle

Finger