Acoustic Wavefield Imaging Course

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Agenda

Finite Differences

Introduction

- The hyperbolic partial differential equation of acoustic waves in a medium with constant density simplifies the model.

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- At each point in the medium a different velocity value (**c**) is used to attribute the respective heterogeneity.

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Finite Differences

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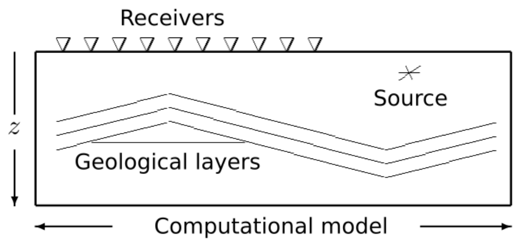
- The figure shows the computational model in surface coordinates x with depth z.



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$$\left(\frac{\partial^2 p}{\partial t^2}\right)_{i,j}^{n} = \frac{p_{i,j}^{n+1} - 2p_{i,j}^{n} + p_{i,j}^{n-1}}{(\Delta t)^2} \tag{2}$$

$$\left(\frac{\partial^2 p}{\partial x^2}\right)_{i,j}^{n} = \frac{p_{i+1,j}^{n} - 2p_{i,j}^{n} + p_{i-1,j}^{n}}{(\Delta x)^2} \tag{3}$$

$$\left(\frac{\partial^2 p}{\partial z^2}\right)_{i,j}^{n} = \frac{p_{i,j+1}^{n} - 2p_{i,j}^{n} + p_{i,j-1}^{n}}{(\Delta z)^2} \tag{4}$$

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$$p_{i,j}^{n+1} = 2(1-2G^2)p_{i,j}^{n} - p_{i,j}^{n-1} + G^2(p_{i+1,j}^{n} + p_{i-1,j}^{n} + p_{i,j+1}^{n} + p_{i,j-1}^{n}) \tag{5}$$

- where $G = \frac{c\Delta t}{\Delta h}$ is called the Courant parameter and will take into account square cells defined as follows: $\Delta h = \Delta x = \Delta z$.

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- where $I = \sqrt{-1}$, ω is the angular frequency, k_x and k_z are the wave vector components.

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- If we clear ω and set $\alpha = \frac{\mathbf{c}\Delta t}{\Delta h}$, $\Delta h = \Delta x = \Delta z$, $k_x = kcos(\theta)$, and $k_z = ksin(\theta)$ we have that

Finite Differences

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$$\omega = \frac{2}{\Delta t} sin^{-1}\left(\alpha\left[sin^2\left(\frac{kcos(\theta)\Delta h}{2}\right) + sin^2\left(\frac{ksin(\theta)\Delta h}{2}\right)\right]^{1/2}\right) \tag{9}$$

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- Stability is achieved when $\alpha \leq \frac{1}{\sqrt{2}}$, because the real-valued field is maintained.

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- If we substitute $k = \frac{2\pi f}{c}$ in the previous equation we have:

$$\frac{v_p}{c} = \frac{1}{\pi f\Delta t} sin^{-1}\left(\alpha\left[sin^2\left(\frac{\pi fcos(\theta)\Delta h}{c}\right) + sin^2\left(\frac{\pi fsin(\theta)\Delta h}{c}\right)\right]^{1/2}\right) \quad (12)$$

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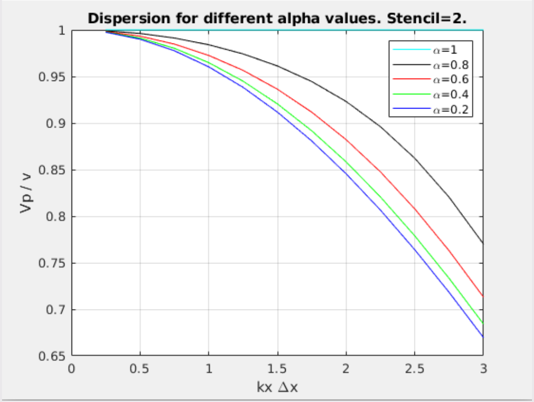
Finite Differences

Numerical dispersion curves



Wave Equation

Forward Operator

$$\frac{1}{\mathbf{c}^2(x,z)}\frac{\partial^2 p(x,z)}{\partial t^2} = \frac{\partial^2 p(x,z)}{\partial x^2} + \frac{\partial^2 p(x,z)}{\partial z^2} + src(x,z,t), \tag{14}$$

$$Gm = d \tag{15}$$

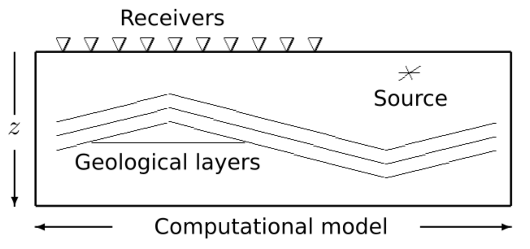
What is?

- G=?
- m=?
- d=?

Seismic Adquisition

Example

- The figure shows the computational model in surface coordinates *x* with depth *z*.



Modeled Data

d; vector or matrix?

- We must simulate synthetic receptors.

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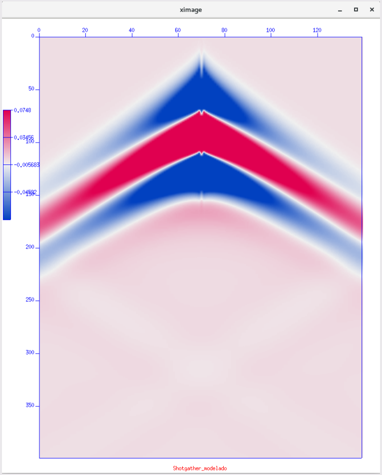
Modeled Data

d; vector or matrix?

- We must simulate synthetic receptors.
- How many receivers should we use?
- How many samples are we going to keep?
- Spatial and Temporal Resolution.
- How to handle the non-natural boundaries?

Seismic Adquisition

Shotgather



Cost Function

Full Waveform Inversion

L_2 squared norm of the differences between the modeled data $s(y_R)$ and the observations $d(y_R)$ for all (N_R) receivers (R).

$$f(s,m) = \frac{1}{2}\sum_{R}^{N_R} \|s(y_R) - d(y_R)\|_2^2, \tag{16}$$

Wave equation

Forward Operator

Wave equation

$$\frac{1}{\mathbf{c}^2(x,z)}\frac{\partial^2 p(x,z)}{\partial t^2} = \frac{\partial^2 p(x,z)}{\partial x^2} + \frac{\partial^2 p(x,z)}{\partial z^2} + src(x,z,t), \tag{17}$$

Written in matrix form

$$T(m)\ddot{s} - C(m)\dot{s} - A(m)s - b(m) = 0 \tag{18}$$

We have left

$$T(m) = \frac{1}{c^2},\ -C(m) = 0,\ -A(m) = \left[-D_x^2 - D_z^2\right] \tag{19}$$

with $b(m) = src(x,z,t)$, $s = [p]$, $\dot{s}(0) = 0$, and $s(0) = 0$.

Adjoint Equation

Adjoint Operator

Written in matrix form

$$T^T(m)\ddot{\lambda} = -C^T(m)\dot{\lambda} + A^T(m)\lambda - \frac{\partial f}{\partial s} \tag{20}$$

with $\dot{\lambda}(T) = 0$ and $\lambda(T) = 0$. This brings us to the equation

$$\frac{1}{\mathbf{c}^2(x,z)}\frac{\partial^2\lambda(x,z)}{\partial t^2} = \frac{\partial^2\lambda(x,z)}{\partial x^2} + \frac{\partial^2\lambda(x,z)}{\partial z^2} - \frac{\partial f}{\partial s}, \tag{21}$$

where

$$\frac{\partial f}{\partial s} = \sum_R^{N_R}(s(y_R) - d(y_R)). \tag{22}$$

Gradient

Definition

$$\frac{df}{dm} = \int_0^T \lambda^T \left(\frac{\partial T}{\partial m}\ddot{s} - \frac{\partial C}{\partial m}\dot{s} - \frac{\partial A}{\partial m}s - \frac{\partial b}{\partial m}\right)dt. \tag{23}$$

Since we are interested in the variation of the cost function in the direction of the speed c, we are left with:

Specific Case

$$\frac{df}{dc} = \int_0^T \lambda\left(\frac{\partial T}{\partial c}\ddot{p} - \cancel{\frac{\partial C}{\partial c}}\dot{p} - \cancel{\frac{\partial A}{\partial c}}p - \cancel{\frac{\partial b}{\partial c}}\right)dt. \tag{24}$$

Therefore

$$\frac{df}{dc} = \int_0^T \lambda\left(\frac{-2}{c^3(x,z)}\frac{\partial^2 p(x,z)}{\partial t^2}\right)dt. \tag{25}$$

Gradient

Specific Case

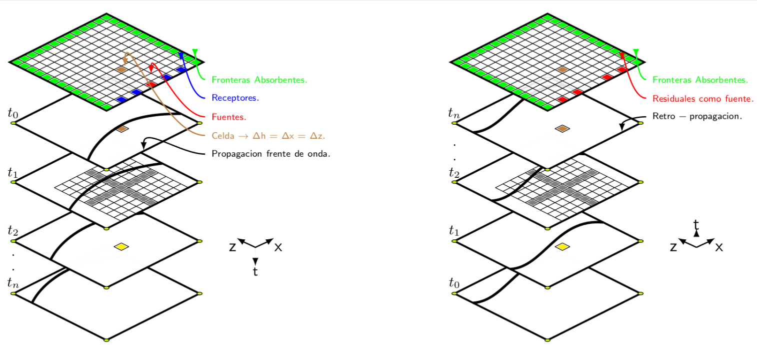


Figure 1: Taken from [Abreo et al., 2015]

Summary

FWI elements

- $\frac{\partial^2 p(x,z)}{\partial t^2}$: Forward-field's derivative.
- λ: Adjoint field.
- $\frac{-2}{c^3(x,z)}$: Scale factor.
- $\frac{df}{dc} = g(c)$: Velocity's gradient.
- $c^{k+1} = c^k - \alpha g(c^k)$: Steepest descent update.
- $c^{k+1} = c^k - \alpha H^{-1}(c^k) g(c^k)$: Newton method.
- Quasi-newton method: When working with an approximation of "$H^{-1}(c^k)g(c^k)$".

Update

Quasi-newton

- L-BFGS [Liu and Nocedal, 1989] allows to obtain this approximation from the history of the last "L" gradients and models where $2 \leq L \leq 20$.
- In the Figure 2 Algorithm, $\mathbf{s}_k = \mathbf{v}_{k+1} - \mathbf{v}_k$, $\mathbf{y}_k = \mathbf{g}_{k+1} - \mathbf{g}_k$ and $\sigma_k = 1/\mathbf{y}_k^T \mathbf{s}_k$.
- The matrix $\mathbf{D}_k^0$ is approximated by $\mathbf{D}_k^0 = \gamma_k \mathbf{I}$, with

$$\gamma_k = \frac{\mathbf{s}_{k-1}^T \mathbf{y}_{k-1}}{\mathbf{y}_{k-1}^T \mathbf{y}_{k-1}}. \tag{26}$$

L-BFGS

L-BFGS' Algorithm

$\mathbf{q} \leftarrow \mathbf{g}_k$

for $i = k-1, k-2, \cdots, k-m$ **do**

$\quad \varepsilon_i \leftarrow \sigma_i \mathbf{s}_i^T \mathbf{q};$

$\quad \mathbf{q} \leftarrow \mathbf{q} - \varepsilon_i \mathbf{y}_i;$

end for

$\mathbf{r} \leftarrow \mathbf{D}_k^0 \mathbf{q};$

for $i = k-m, k-m+1, \cdots, k-1$ **do**

$\quad \beta \leftarrow \sigma_i \mathbf{y}_i^T \mathbf{r};$

$\quad \mathbf{r} \leftarrow \mathbf{r} + \mathbf{s}_i(\varepsilon_i - \beta);$

end for

Figure 2: Taken from [Liu and Nocedal, 1989]

L-BFGS

L-BFGS' attempts

$\alpha = 2$

$\mathbf{h}_k = r$;

Compute $\phi_{(\mathbf{v}_k)}$;

$\phi_{(attempt)} = \phi_{(\mathbf{v}_k)} + 100$;

while $\phi_{(attempt)} > \phi_{(\mathbf{v}_k)}$ **do**

  $\alpha = \alpha/2$;

  $attempt = \mathbf{v}_k - \alpha \mathbf{h}_k$;

  Compute $\phi_{(attempt)}$;

end while

$\mathbf{v}_{k+1} = attempt$

Figure 3: Taken from [Liu and Nocedal, 1989]

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