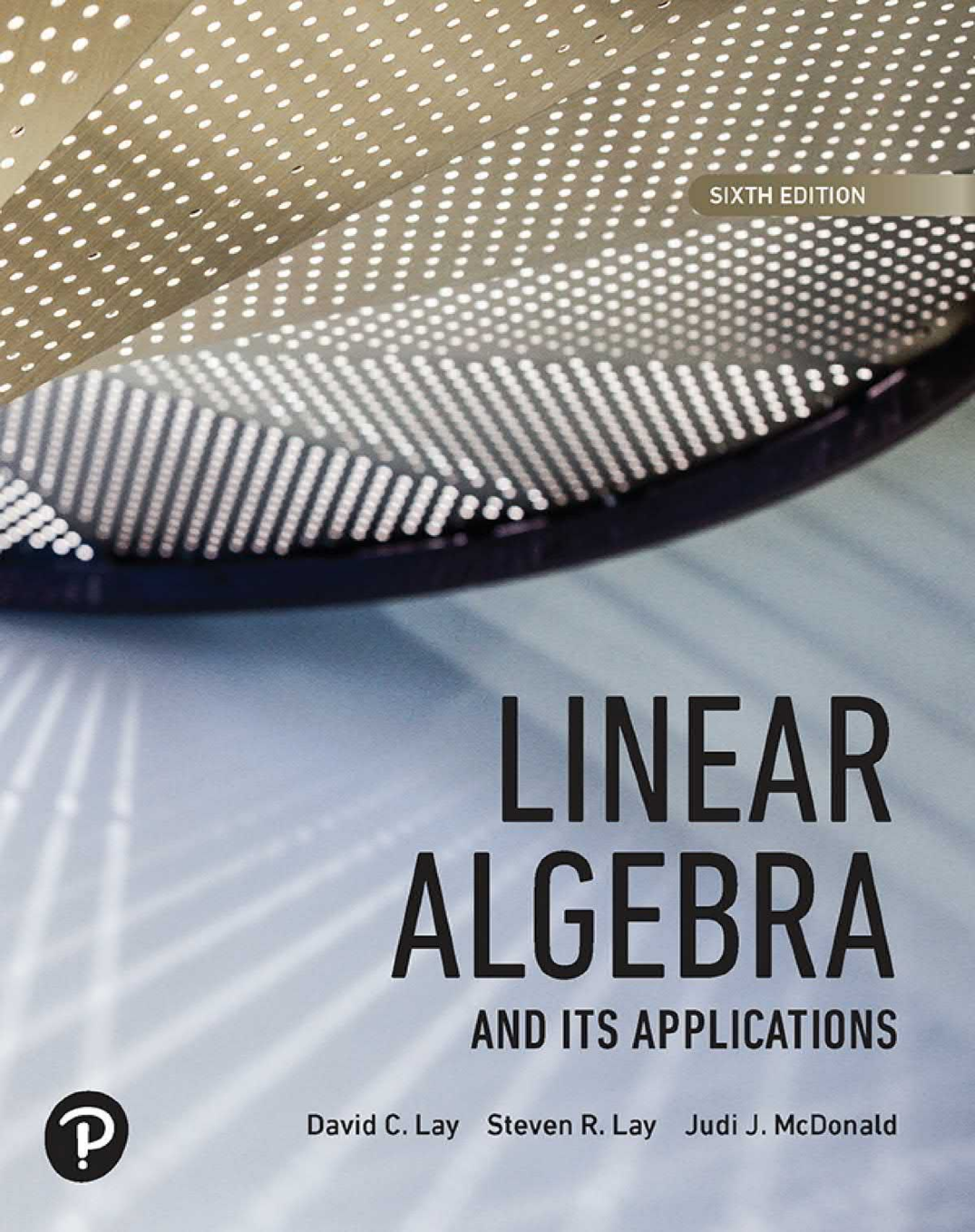

SIXTH EDITION

LINEAR ALGEBRA

AND ITS APPLICATIONS

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1 Linear Equations in Linear Algebra

Introductory Example

LINEAR MODELS IN ECONOMICS AND ENGINEERING

It was late summer in 1949. Harvard Professor Wassily Leontief was carefully feeding the last of his punched cards into the university's Mark II computer. The cards contained information about the U.S. economy and represented a summary of more than 250,000 pieces of information produced by the U.S. Bureau of Labor Statistics after two years of intensive work. Leontief had divided the U.S. economy into 500 "sectors," such as the coal industry, the automotive industry, communications, and so on. For each sector, he had written a linear equation that described how the sector distributed its output to the other sectors of the economy. Because the Mark II, one of the largest computers of its day, could not handle the resulting system of 500 equations in 500 unknowns, Leontief had distilled the problem into a system of 42 equations in 42 unknowns.

Programming the Mark II computer for Leontief's 42 equations had required several months of effort, and he was anxious to see how long the computer would take to solve the problem. The Mark II hummed and blinked for 56 hours before finally producing a solution. We will discuss the nature of this solution in Sections 1.6 and 2.6.

Leontief, who was awarded the 1973 Nobel Prize in Economic Science, opened the door to a new era in mathematical modeling in economics. His efforts at Harvard in 1949 marked one of the first significant uses of computers to analyze what was then a large-scale mathematical model. Since that time, researchers in many other fields have employed computers to analyze mathematical models. Because of the massive amounts of data involved, the models are usually *linear*; that is, they are described by *systems of linear equations*.

The importance of linear algebra for applications has risen in direct proportion to the increase in computing power, with each new generation of hardware and software triggering a demand for even greater capabilities. Computer science is thus intricately linked with linear algebra through the explosive growth of parallel processing and large-scale computations.

Scientists and engineers now work on problems far more complex than even dreamed possible a few decades ago. Today, linear algebra has more potential value for students in many scientific and business fields than any other undergraduate mathematics subject! The material in this text provides the foundation for further work in many interesting areas. Here are a few possibilities; others will be described later.

- *Oil exploration.* When a ship searches for offshore oil deposits, its computers solve thousands of separate systems of linear equations *every day*. The seismic data for the equations are obtained from underwater shock waves created by explosions from air guns. The waves bounce off subsurface rocks and are measured by geophones attached to mile-long cables behind the ship.

- *Linear programming.* Many important management decisions today are made on the basis of linear programming models that use hundreds of variables. The airline industry, for instance, employs linear programs that schedule flight crews, monitor the locations of aircraft, or plan the varied schedules of support services such as maintenance and terminal operations.
- *Electrical networks.* Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software relies on linear algebra techniques and systems of linear equations.
- *Artificial intelligence.* Linear algebra plays a key role in everything from scrubbing data to facial recognition.
- *Signals and signal processing.* From a digital photograph to the daily price of a stock, important information is recorded as a signal and processed using linear transformations.
- *Machine learning.* Machines (specifically computers) use linear algebra to learn about anything from online shopping preferences to speech recognition.

Systems of linear equations lie at the heart of linear algebra, and this chapter uses them to introduce some of the central concepts of linear algebra in a simple and concrete setting. Sections 1.1 and 1.2 present a systematic method for solving systems of linear equations. This algorithm will be used for computations throughout the text. Sections 1.3 and 1.4 show how a system of linear equations is equivalent to a *vector equation* and to a *matrix equation*. This equivalence will reduce problems involving linear combinations of vectors to questions about systems of linear equations. The fundamental concepts of spanning, linear independence, and linear transformations, studied in the second half of the chapter, will play an essential role throughout the text as we explore the beauty and power of linear algebra.

1.1 Systems of Linear Equations

A **linear equation** in the variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where b and the **coefficients** $a_1, \dots, a_n$ are real or complex numbers, usually known in advance. The subscript n may be any positive integer. In textbook examples and exercises, n is normally between 2 and 5. In real-life problems, n might be 50 or 5000, or even larger.

The equations

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2\left(\sqrt{6} - x_1\right) + x_3$$

are both linear because they can be rearranged algebraically as in equation (1):

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

The equations

$$4x_1 - 5x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6$$

are not linear because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second.

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables—say, $x_1, \dots, x_n$. An example is

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad - 4x_3 &= -7 \end{aligned} \tag{2}$$

A **solution** of the system is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \dots, s_n$ are substituted for $x_1, \dots, x_n$, respectively. For instance, $(5, 6.5, 3)$ is a solution of system (2) because, when these values are substituted in (2) for x_1, x_2, x_3, respectively, the equations simplify to $8 = 8$ and $-7 = -7$.

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. A typical problem is

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

The graphs of these equations are lines, which we denote by ℓ_1 and ℓ_2. A pair of numbers (x_1, x_2) satisfies *both* equations in the system if and only if the point (x_1, x_2) lies on both ℓ_1 and ℓ_2. In the system above, the solution is the single point $(3, 2)$, as you can easily verify. See Figure 1.

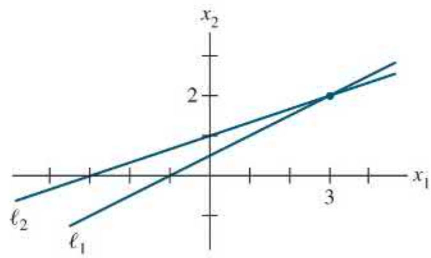
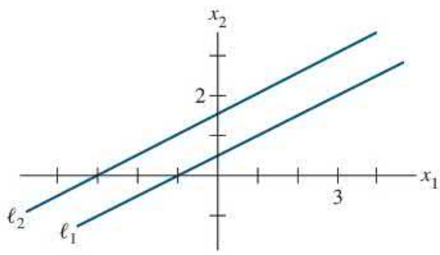


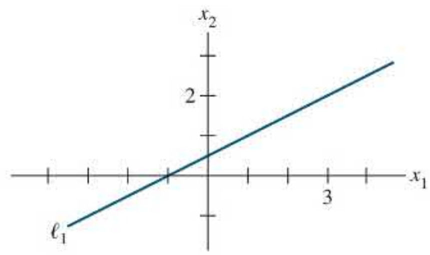
FIGURE 1 Exactly one solution.

Of course, two lines need not intersect in a single point—they could be parallel, or they could coincide and hence "intersect" at every point on the line. Figure 2 shows the graphs that correspond to the following systems:

(a) $\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$ (b) $\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$

Figures 1 and 2 illustrate the following general fact about linear systems, to be verified in Section 1.2.





(a) (b)

FIGURE 2 (a) No solution. (b) Infinitely many solutions.

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Matrix Notation

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned} \tag{3}$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

is called the **coefficient matrix** (or **matrix of coefficients**) of the system (3), and the matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \tag{4}$$

is called the **augmented matrix** of the system. (The second row here contains a zero because the second equation could be written as $0 \cdot x_1 + 2x_2 - 8x_3 = 8$.) An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the respective right sides of the equations.

The **size** of a matrix tells how many rows and columns it has. The augmented matrix (4) above has 3 rows and 4 columns and is called a 3×4 (read "3 by 4") matrix. If m and n are positive integers, an $\boldsymbol{m \times n}$ **matrix** is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.) Matrix notation will simplify the calculations in the examples that follow.

Solving a Linear System

This section and the next describe an algorithm, or a systematic procedure, for solving linear systems. The basic strategy is *to replace one system with an equivalent system (that is one with the same solution set) that is easier to solve.*

Roughly speaking, use the x_1 term in the first equation of a system to eliminate the x_1 terms in the other equations. Then use the x_2 term in the second equation to eliminate the x_2 terms in the other equations, and so on, until you finally obtain a very simple equivalent system of equations.

Three basic operations are used to simplify a linear system: Replace one equation by the sum of itself and a multiple of another equation, interchange two equations, and multiply all the terms in an equation by a nonzero constant. After the first example, you will see why these three operations do not change the solution set of the system.

EXAMPLE 1 Solve system (3).

SOLUTION The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Keep x_1 in the first equation and eliminate it from the other equations. To do so, add -5 times equation 1 to equation 3. After some practice, this type of calculation is usually performed mentally:

$$\begin{array}{r r} -5 \cdot [\text{equation 1}] & -5x_1 + 10x_2 - 5x_3 = 0 \\ + [\text{equation 3}] & 5x_1 \qquad\quad - 5x_3 = 10 \\ \hline [\text{new equation 3}] & 10x_2 - 10x_3 = 10 \end{array}$$

The result of this calculation is written in place of the original third equation:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Now, multiply equation 2 by $\frac{1}{2}$ in order to obtain 1 as the coefficient for x_2. (This calculation will simplify the arithmetic in the next step.)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Use the x_2 in equation 2 to eliminate the $10x_2$ in equation 3. The "mental" computation is

$$\begin{array}{r r} -10 \cdot [\text{equation 2}] & -10x_2 + 40x_3 = -40 \\ + [\text{equation 3}] & 10x_2 - 10x_3 = 10 \\ \hline [\text{new equation 3}] & 30x_3 = -30 \end{array}$$

The result of this calculation is written in place of the previous third equation (row):

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 30x_3 &= -30 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Now, multiply equation 3 by $\frac{1}{30}$ in order to obtain 1 as the coefficient for x_3. (This calculation will simplify the arithmetic in the next step.)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The new system has a *triangular* form (the intuitive term *triangular* will be replaced by a precise term in the next section):

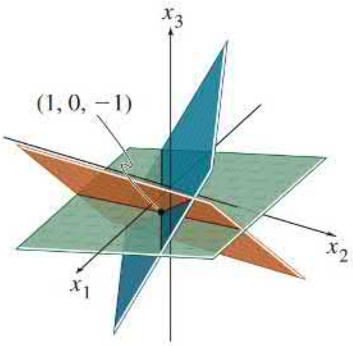
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Eventually, you want to eliminate the $-2x_2$ term from equation 1, but it is more efficient to use the x_3 in equation 3 first, to eliminate the $-4x_3$ and $+x_3$ terms in equations 2 and 1. The two "mental" calculations are

$$\begin{array}{rr} 4 \cdot [\text{equation } 3] & 4x_3 = -4 \\ + [\text{equation } 2] & x_2 - 4x_3 = 4 \\ \hline [\text{new equation } 2] & x_2 = 0 \end{array} \qquad \begin{array}{rr} -1 \cdot [\text{equation } 3] & -x_3 = 1 \\ + [\text{equation } 1] & x_1 - 2x_2 + x_3 = 0 \\ \hline [\text{new equation } 1] & x_1 - 2x_2 = 1 \end{array}$$

It is convenient to combine the results of these two operations:

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$



Each of the original equations determines a plane in three-dimensional space. The point $(1, 0, -1)$ lies in all three planes.

Now, having cleaned out the column above the x_3 in equation 3, move back to the x_2 in equation 2 and use it to eliminate the $-2x_2$ above it. Because of the previous work with x_3, there is now no arithmetic involving x_3 terms. Add 2 times equation 2 to equation 1 and obtain the system:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The work is essentially done. It shows that the only solution of the original system is $(1, 0, -1)$. However, since there are so many calculations involved, it is a good practice to check the work. To verify that $(1, 0, -1)$ *is* a solution, substitute these values into the left side of the original system, and compute:

$$\begin{aligned} 1(1) - 2(0) + 1(-1) &= 1 - 0 - 1 = 0 \\ 2(0) - 8(-1) &= \phantom{1 -{}} 0 + 8 = 8 \\ 5(1) - 5(-1) &= 5 + 5 = 10 \end{aligned}$$

The results agree with the right side of the original system, so $(1, 0, -1)$ is a solution of the system. ■

Example 1 illustrates how operations on equations in a linear system correspond to operations on the appropriate rows of the augmented matrix. The three basic operations listed earlier correspond to the following operations on the augmented matrix.

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[1]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

[1] A common paraphrase of row replacement is "Add to one row a multiple of another row."

Row operations can be applied to any matrix, not merely to one that arises as the augmented matrix of a linear system. Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

It is important to note that row operations are *reversible*. If two rows are interchanged, they can be returned to their original positions by another interchange. If a row is scaled by a nonzero constant c, then multiplying the new row by $1/c$ produces the original row. Finally, consider a replacement operation involving two rows—say, rows 1 and 2—and suppose that c times row 1 is added to row 2 to produce a new row 2. To "reverse" this operation, add $-c$ times row 1 to (new) row 2 and obtain the original row 2. See Exercises 39–42 at the end of this section.

At the moment, we are interested in row operations on the augmented matrix of a system of linear equations. Suppose a system is changed to a new one via row operations. By considering each type of row operation, you can see that any solution of the original system remains a solution of the new system. Conversely, since the original system can be produced via row operations on the new system, each solution of the new system is also a solution of the original system. This discussion justifies the following statement.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Though Example 1 is lengthy, you will find that after some practice, the calculations go quickly. Row operations in the text and exercises will usually be extremely easy to perform, allowing you to focus on the underlying concepts. Still, you must learn to perform row operations accurately because they will be used throughout the text.

The rest of this section shows how to use row operations to determine the size of a solution set, without completely solving the linear system.

Existence and Uniqueness Questions

Section 1.2 will show why a solution set for a linear system contains either no solutions, one solution, or infinitely many solutions. Answers to the following two questions will determine the nature of the solution set for a linear system.

To determine which possibility is true for a particular system, we ask two questions.

TWO FUNDAMENTAL QUESTIONS ABOUT A LINEAR SYSTEM

1. Is the system consistent; that is, does at least one solution *exist*?
2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

These two questions will appear throughout the text, in many different guises. This section and the next will show how to answer these questions via row operations on the augmented matrix.

EXAMPLE 2 Determine if the following system is consistent:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

SOLUTION This is the system from Example 1. Suppose that we have performed the row operations necessary to obtain the triangular form

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

At this point, we know x_3. Were we to substitute the value of x_3 into equation 2, we could compute x_2 and hence could determine x_1 from equation 1. So a solution exists; the system is consistent. (In fact, x_2 is uniquely determined by equation 2 since x_3 has only one possible value, and x_1 is therefore uniquely determined by equation 1. So the solution is unique.) ■

EXAMPLE 3 Determine if the following system is consistent:

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned} \tag{5}$$

SOLUTION The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

To obtain an x_1 in the first equation, interchange rows 1 and 2:

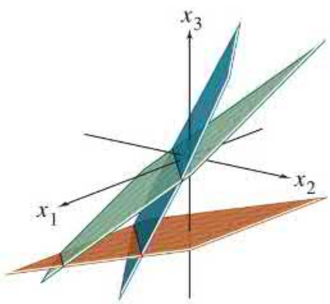
$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

To eliminate the $4x_1$ term in the third equation, add -2 times row 1 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix} \tag{6}$$

Next, use the x_2 term in the second equation to eliminate the $-2x_2$ term from the third equation. Add 2 times row 2 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix} \tag{7}$$



The system is inconsistent because there is no point that lies on all three planes.

The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation:

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 15 \end{aligned} \tag{8}$$

The equation $0 = 15$ is a short form of $0x_1 + 0x_2 + 0x_3 = 15$. This system in triangular form obviously has a built-in contradiction. There are no values of x_1, x_2, x_3 that satisfy (8) because the equation $0 = 15$ is never true. Since (8) and (5) have the same solution set, the original system is inconsistent (it has no solution). ■

Pay close attention to the augmented matrix in (7). Its last row is typical of an inconsistent system in triangular form.

Reasonable Answers

Once you have one or more solutions to a system of equations, remember to check your answer by substituting the solution you found back into the original equation. For example, if you found $(2, 1, -1)$ was a solution to the system of equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ x_1 - 2x_3 &= -2 \\ x_2 + x_3 &= 3 \end{aligned}$$

you could substitute your solution into the original equations to get

$$\begin{array}{rcrcrcrcr} 2 & - & 2(1) & + & (-1) & = & -1 & \neq & 2 \\ 2 & & & - & 2(-1) & = & 4 & \neq & -2 \\ & & 1 & + & (-1) & = & 0 & \neq & 3 \end{array}$$

It is now clear that there must have been an error in your original calculations. If upon rechecking your arithmetic, you find the answer $(2, 1, 2)$, you can see that

$$\begin{array}{rcrcrcrcr} 2 & - & 2(1) & + & (2) & = & 2 & = & 2 \\ 2 & & & - & 2(2) & = & -2 & = & -2 \\ & & 1 & + & 2 & = & 3 & = & 3 \end{array}$$

and you can now be confident you have a correct solution to the given system of equations.

Numerical Note

In real-world problems, systems of linear equations are solved by a computer. For a square coefficient matrix, computer programs nearly always use the elimination algorithm given here and in Section 1.2, modified slightly for improved accuracy.

The vast majority of linear algebra problems in business and industry are solved with programs that use *floating point arithmetic*. Numbers are represented as decimals $\pm .d_1 \cdots d_p \times 10^r$, where r is an integer and the number p of digits to the right of the decimal point is usually between 8 and 16. Arithmetic with such numbers typically is inexact, because the result must be rounded (or truncated) to the number of digits stored. "Roundoff error" is also introduced when a number such as $1/3$ is entered into the computer, since its decimal representation must be approximated by a finite number of digits. Fortunately, inaccuracies in floating point arithmetic seldom cause problems. The numerical notes in this book will occasionally warn of issues that you may need to consider later in your career.

Practice Problems

Throughout the text, practice problems should be attempted before working the exercises. Solutions appear after each exercise set.

1. State in words the next elementary row operation that should be performed on the system in order to solve it. [More than one answer is possible in (a).]

Practice Problems (Continued)

a. $$\begin{aligned} x_1 + 4x_2 - 2x_3 + 8x_4 &= 12 \\ x_2 - 7x_3 + 2x_4 &= -4 \\ 5x_3 - x_4 &= 7 \\ x_3 + 3x_4 &= -5 \end{aligned}$$

b. $$\begin{aligned} x_1 - 3x_2 + 5x_3 - 2x_4 &= 0 \\ x_2 + 8x_3 \qquad &= -4 \\ 2x_3 \qquad &= 3 \\ x_4 &= 1 \end{aligned}$$

2. The augmented matrix of a linear system has been transformed by row operations into the form below. Determine if the system is consistent.

$$\begin{bmatrix} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

3. Is $(3, 4, -2)$ a solution of the following system?

$$\begin{aligned} 5x_1 - x_2 + 2x_3 &= 7 \\ -2x_1 + 6x_2 + 9x_3 &= 0 \\ -7x_1 + 5x_2 - 3x_3 &= -7 \end{aligned}$$

4. For what values of h and k is the following system consistent?

$$\begin{aligned} 2x_1 - x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$$

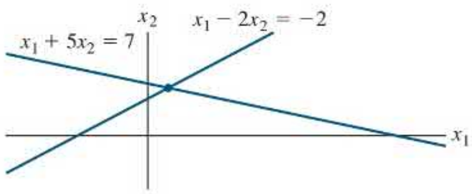
1.1 Exercises

Solve each system in Exercises 1–4 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

1. $$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$$

2. $$\begin{aligned} 2x_1 + 4x_2 &= -4 \\ 5x_1 + 7x_2 &= 11 \end{aligned}$$

3. Find the point (x_1, x_2) that lies on the line $x_1 + 5x_2 = 7$ and on the line $x_1 - 2x_2 = -2$. See the figure.



4. Find the point of intersection of the lines $x_1 - 5x_2 = 1$ and $3x_1 - 7x_2 = 5$.

Consider each matrix in Exercises 5 and 6 as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

5. $$\begin{bmatrix} 1 & -4 & 5 & 0 & 7 \\ 0 & 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$$

6. $$\begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & 1 & 6 \end{bmatrix}$$

In Exercises 7–10, the augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

7. $$\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

8. $$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

9. $$\begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

10. $\begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Solve the systems in Exercises 11–14.

11. $$\begin{aligned} x_2 + 4x_3 &= -4 \\ x_1 + 3x_2 + 3x_3 &= -2 \\ 3x_1 + 7x_2 + 5x_3 &= 6 \end{aligned}$$

12. $$\begin{aligned} x_1 - 3x_2 + 4x_3 &= -4 \\ 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 + 2x_3 &= 4 \end{aligned}$$

13. $$\begin{aligned} x_1 \qquad - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

14. $$\begin{aligned} x_1 - 3x_2 \qquad &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

15. Verify that the solution you found to Exercise 11 is correct by substituting the values you obtained back into the original equations.

16. Verify that the solution you found to Exercise 12 is correct by substituting the values you obtained back into the original equations.

17. Verify that the solution you found to Exercise 13 is correct by substituting the values you obtained back into the original equations.

18. Verify that the solution you found to Exercise 14 is correct by substituting the values you obtained back into the original equations.

Determine if the systems in Exercises 19 and 20 are consistent. Do not completely solve the systems.

19. $$\begin{aligned} x_1 \qquad + 3x_3 \qquad &= 2 \\ x_2 \qquad - 3x_4 &= 3 \\ -2x_2 + 3x_3 + 2x_4 &= 1 \\ 3x_1 \qquad\qquad + 7x_4 &= -5 \end{aligned}$$

20. $$\begin{aligned} x_1 \qquad\qquad - 2x_4 &= -3 \\ 2x_2 + 2x_3 \qquad &= 0 \\ x_3 + 3x_4 &= 1 \\ -2x_1 + 3x_2 + 2x_3 + x_4 &= 5 \end{aligned}$$

21. Do the three lines $x_1 - 4x_2 = 1$, $2x_1 - x_2 = -3$, and $-x_1 - 3x_2 = 4$ have a common point of intersection? Explain.

22. Do the three planes $x_1 + 2x_2 + x_3 = 4$, $x_2 - x_3 = 1$, and $x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

In Exercises 23–26, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

23. $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$

24. $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 3 & -2 \\ -4 & h & 8 \end{bmatrix}$

26. $\begin{bmatrix} 2 & -3 & h \\ -6 & 9 & 5 \end{bmatrix}$

In Exercises 27–34, key statements from this section are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and *justify* your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.) Similar true/false questions will appear in many sections of the text and will be flagged with a **(T/F)** at the beginning of the question.

27. **(T/F)** Every elementary row operation is reversible.

28. **(T/F)** Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

29. **(T/F)** A 5×6 matrix has six rows.

30. **(T/F)** Two matrices are row equivalent if they have the same number of rows.

31. **(T/F)** The solution set of a linear system involving variables $x_1, \dots, x_n$ is a list of numbers $(s_1, \dots, s_n)$ that makes each equation in the system a true statement when the values $s_1, \dots, s_n$ are substituted for $x_1, \dots, x_n$, respectively.

32. **(T/F)** An inconsistent system has more than one solution.

33. **(T/F)** Two fundamental questions about a linear system involve existence and uniqueness.

34. **(T/F)** Two linear systems are equivalent if they have the same solution set.

35. Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

36. Construct three different augmented matrices for linear systems whose solution set is $x_1 = -2$, $x_2 = 1$, $x_3 = 0$.

37. Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients c and d? Justify your answer.

$$\begin{aligned} x_1 + 3x_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

38. Suppose a, b, c, and d are constants such that a is not zero and the system below is consistent for all possible values of f and g. What can you say about the numbers a, b, c, and d? Justify your answer.

$$\begin{aligned} ax_1 + bx_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

In Exercises 39–42, find the elementary row operation that transforms the first matrix into the second, and then find the reverse row operation that transforms the second matrix into the first.

39. $\begin{bmatrix} 0 & -2 & 5 \\ 1 & 4 & -7 \\ 3 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 4 & -7 \\ 0 & -2 & 5 \\ 3 & -1 & 6 \end{bmatrix}$

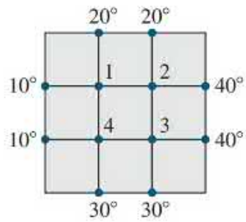
40. $\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -5 & 9 \end{bmatrix}$

41. $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 4 & -1 & 3 & -6 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 0 & 7 & -1 & -6 \end{bmatrix}$

42. $\begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & -3 & 9 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let $T_1, \dots, T_4$ denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.[2] For instance,

$$T_1 = (10 + 20 + T_2 + T_4)/4, \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



43. Write a system of four equations whose solution gives estimates for the temperatures $T_1, \dots, T_4$.

44. Solve the system of equations from Exercise 43. [*Hint:* To speed up the calculations, interchange rows 1 and 4 before starting "replace" operations.]

[2] See Frank M. White, *Heat and Mass Transfer* (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.

Solutions to Practice Problems

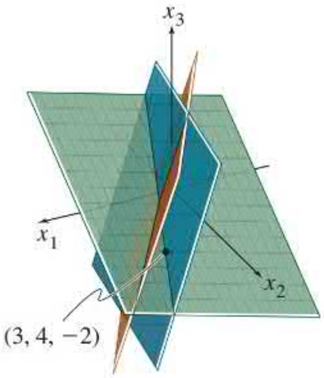
1. a. For "hand computation," the best choice is to interchange equations 3 and 4. Another possibility is to multiply equation 3 by $1/5$. Or, replace equation 4 by its sum with $-1/5$ times row 3. (In any case, do not use the x_2 in equation 2 to eliminate the $4x_2$ in equation 1. Wait until a triangular form has been reached and the x_3 terms and x_4 terms have been eliminated from the first two equations.)

b. The system is in triangular form. Further simplification begins with the x_4 in the fourth equation. Use the x_4 to eliminate all x_4 terms above it. The appropriate step now is to add 2 times equation 4 to equation 1. (After that, move to equation 3, multiply it by $1/2$, and then use the equation to eliminate the x_3 terms above it.)

2. The system corresponding to the augmented matrix is

$$\begin{aligned} x_1 + 5x_2 + 2x_3 &= -6 \\ 4x_2 - 7x_3 &= 2 \\ 5x_3 &= 0 \end{aligned}$$

The third equation makes $x_3 = 0$, which is certainly an allowable value for x_3. After eliminating the x_3 terms in equations 1 and 2, you could go on to solve for unique values for x_2 and x_1. Hence a solution exists, and it is unique. Contrast this situation with that in Example 3.



Since $(3, 4, -2)$ satisfies the first two equations, it is on the line of the intersection of the first two planes. Since $(3, 4, -2)$ does not satisfy all three equations, it does not lie on all three planes.

3. It is easy to check if a specific list of numbers is a solution. Set $x_1 = 3$, $x_2 = 4$, and $x_3 = -2$, and find that

$$\begin{aligned} 5(3) - (4) + 2(-2) &= 15 - 4 - 4 = 7 \\ -2(3) + 6(4) + 9(-2) &= -6 + 24 - 18 = 0 \\ -7(3) + 5(4) - 3(-2) &= -21 + 20 + 6 = 5 \end{aligned}$$

Although the first two equations are satisfied, the third is not, so $(3, 4, -2)$ is not a solution of the system. Notice the use of parentheses when making the substitutions. They are strongly recommended as a guard against arithmetic errors.

4. When the second equation is replaced by its sum with 3 times the first equation, the system becomes

$$\begin{aligned} 2x_1 - x_2 &= h \\ 0 &= k + 3h \end{aligned}$$

If $k + 3h$ is nonzero, the system has no solution. The system is consistent for any values of h and k that make $k + 3h = 0$.

1.2 Row Reduction and Echelon Forms

This section refines the method of Section 1.1 into a row reduction algorithm that will enable us to analyze any system of linear equations.[1] By using only the first part of the algorithm, we will be able to answer the fundamental existence and uniqueness questions posed in Section 1.1.

The algorithm applies to any matrix, whether or not the matrix is viewed as an augmented matrix for a linear system. So the first part of this section concerns an arbitrary rectangular matrix and begins by introducing two important classes of matrices that include the "triangular" matrices of Section 1.1. In the definitions that follow, a *nonzero* row or column in a matrix means a row or column that contains at least one nonzero entry; a **leading entry** of a row refers to the leftmost nonzero entry (in a nonzero row).

DEFINITION

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

[1] The algorithm here is a variant of what is commonly called *Gaussian elimination*. A similar elimination method for linear systems was used by Chinese mathematicians in about 250 B.C. The process was unknown in Western culture until the nineteenth century, when a famous German mathematician, Carl Friedrich Gauss, discovered it. A German engineer, Wilhelm Jordan, popularized the algorithm in an 1888 text on geodesy.

An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form). Property 2 says that the leading entries form an *echelon* ("steplike") pattern that moves down and to the right through the matrix. Property 3 is a simple consequence of property 2, but we include it for emphasis.

The "triangular" matrices of Section 1.1, such as

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

are in echelon form. In fact, the second matrix is in reduced echelon form. Here are additional examples.

EXAMPLE 1 The following matrices are in echelon form. The leading entries (■) may have any nonzero value; the starred entries (∗) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

■

Any nonzero matrix may be **row reduced** (that is, transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique. The following theorem is proved in Appendix A at the end of the text.

THEOREM 1

Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

If a matrix A is row equivalent to an echelon matrix U, we call U **an echelon form** (or row echelon form) **of** A; if U is in reduced echelon form, we call U **the reduced echelon form of** A. [Most matrix programs and calculators with matrix capabilities use the abbreviation RREF for reduced (row) echelon form. Some use REF for (row) echelon form.]

Pivot Positions

When row operations on a matrix produce an echelon form, further row operations to obtain the reduced echelon form do not change the positions of the leading entries. Since the reduced echelon form is unique, *the leading entries are always in the same positions* *in any echelon form obtained from a given matrix.* These leading entries correspond to leading 1's in the reduced echelon form.

DEFINITION

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

In Example 1, the squares (■) identify the pivot positions. Many fundamental concepts in the first four chapters will be connected in one way or another with pivot positions in a matrix.

EXAMPLE 2 Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

SOLUTION Use the same basic strategy as in Section 1.1. The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position. A good choice is to interchange rows 1 and 4 (because the mental computations in the next step will not involve fractions).

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot column

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain matrix (1) below. The pivot position in the second row must be as far left as possible—namely in the second column. Choose the 2 in this position as the next pivot.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \tag{1}$$

Next pivot column

Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \tag{2}$$

The matrix in (2) is different from any encountered in Section 1.1. There is no way to create a leading entry in column 3! (We can't use row 1 or 2 because doing so would destroy the echelon arrangement of the leading entries already produced.) However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{General form: } \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns.

Pivot positions

$$A = \begin{bmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \tag{3}$$

Pivot columns ■

A **pivot**, as illustrated in Example 2, is a nonzero number in a pivot position that is used as needed to create zeros via row operations. The pivots in Example 2 were 1, 2, and −5. Notice that these numbers are not the same as the actual elements of A in the highlighted pivot positions shown in (3).

With Example 2 as a guide, we are ready to describe an efficient procedure for transforming a matrix into an echelon or reduced echelon matrix. Careful study and mastery of this procedure now will pay rich dividends later in the course.

The Row Reduction Algorithm

The algorithm that follows consists of four steps, and it produces a matrix in echelon form. A fifth step produces a matrix in reduced echelon form. We illustrate the algorithm by an example.

EXAMPLE 3 Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

SOLUTION

Step 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Pivot column

> **Step 2**
>
> Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\begin{array}{l} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

> **Step 3**
>
> Use row replacement operations to create zeros in all positions below the pivot.

As a preliminary step, we could divide the top row by the pivot, 3. But with two 3's in column 1, it is just as easy to add -1 times row 1 to row 2.

$$\begin{array}{l} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

> **Step 4**
>
> Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the "top" entry in that column.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \\ \text{New pivot column} \end{array}$$

For step 3, we could insert an optional step of dividing the "top" row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the "top" row to the row below. This produces

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix having only one row:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Pivot

Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. If we want the reduced echelon form, we perform one more step.

Step 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 1} + (-6)\cdot\text{row 3} \\ \leftarrow \text{Row 2} + (-2)\cdot\text{row 3} \\ \end{array}$$

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{Row scaled by } \frac{1}{2} \\ \end{array}$$

Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 1} + (9)\cdot\text{row 2} \\ \\ \end{array}$$

Finally, scale row 1, dividing by the pivot, 3.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row scaled by } \frac{1}{3} \\ \\ \end{array}$$

This is the reduced echelon form of the original matrix. ■

The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

Numerical Note

In step 2 on page 17, a computer program usually selects as a pivot the entry in a column having the largest absolute value. This strategy, called **partial pivoting**, is used because it reduces roundoff errors in the calculations.

Solutions of Linear Systems

The row reduction algorithm leads directly to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent *reduced* echelon form

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{aligned} x_1 \qquad - 5x_3 &= 1 \\ x_2 + \ \ x_3 &= 4 \\ 0 &= 0 \end{aligned} \tag{4}$$

The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**.[2] The other variable, x_3, is called a **free variable**.

Whenever a system is consistent, as in (4), the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables. This operation is possible because the reduced echelon form places each basic variable in one and only one equation. In (4), solve the first equation for x_1 and the second for x_2. (Ignore the third equation; it offers no restriction on the variables.)

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases} \tag{5}$$

The statement "x_3 is free" means that you are free to choose any value for x_3. Once that is done, the formulas in (5) determine the values for x_1 and x_2. For instance, when $x_3 = 0$, the solution is $(1, 4, 0)$; when $x_3 = 1$, the solution is $(6, 3, 1)$. *Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3.*

EXAMPLE 4 Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

SOLUTION The matrix is in echelon form, but we want the reduced echelon form before solving for the basic variables. The row reduction is completed next. The symbol $\sim$ before a matrix indicates that the matrix is row equivalent to the preceding matrix.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

[2] Some texts use the term *leading variables* because they correspond to the columns containing leading entries.

There are five variables because the augmented matrix has six columns. The associated system now is

$$\begin{aligned} x_1 + 6x_2 \quad\;\; + 3x_4 \quad\;\; &= 0 \\ x_3 - 4x_4 \quad\;\; &= 5 \\ x_5 &= 7 \end{aligned} \tag{6}$$

The pivot columns of the matrix are 1, 3, and 5, so the basic variables are x_1, x_3, and x_5. The remaining variables, x_2 and x_4, must be free. Solve for the basic variables to obtain the general solution:

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases} \tag{7}$$

Note that the value of x_5 is already fixed by the third equation in system (6). ■

Parametric Descriptions of Solution Sets

The descriptions in (5) and (7) are *parametric descriptions* of solution sets in which the free variables act as parameters. *Solving a system* amounts to finding a parametric description of the solution set or determining that the solution set is empty.

Whenever a system is consistent and has free variables, the solution set has many parametric descriptions. For instance, in system (4), we may add 5 times equation 2 to equation 1 and obtain the equivalent system

$$\begin{aligned} x_1 + 5x_2 \qquad &= 21 \\ x_2 + x_3 &= \;\;4 \end{aligned}$$

We could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2, and we would have an accurate description of the solution set. However, to be consistent, we make the (arbitrary) convention of always using the free variables as the parameters for describing a solution set. (The answer section at the end of the text also reflects this convention.)

Whenever a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has *no* parametric representation.

Back-Substitution

Consider the following system, whose augmented matrix is in echelon form but is *not* in reduced echelon form:

$$\begin{aligned} x_1 - 7x_2 + 2x_3 - 5x_4 + 8x_5 &= 10 \\ x_2 - 3x_3 + 3x_4 + \;\; x_5 &= -5 \\ x_4 - \;\; x_5 &= \;\;4 \end{aligned}$$

A computer program would solve this system by back-substitution, rather than by computing the reduced echelon form. That is, the program would solve equation 3 for x_4 in terms of x_5 and substitute the expression for x_4 into equation 2, solve equation 2 for x_2, and then substitute the expressions for x_2 and x_4 into equation 1 and solve for x_1.

Our matrix format for the backward phase of row reduction, which produces the reduced echelon form, has the same number of arithmetic operations as back-substitution. But the discipline of the matrix format substantially reduces the likelihood of errors during hand computations. The best strategy is to use only the *reduced* echelon form to solve a system! The *Study Guide* that accompanies this text offers several helpful suggestions for performing row operations accurately and rapidly.

Numerical Note

In general, the forward phase of row reduction takes much longer than the backward phase. An algorithm for solving a system is usually measured in flops (or floating point operations). A **flop** is one arithmetic operation $(+, -, *, /)$ on two real floating point numbers.[3] For an $n \times (n+1)$ matrix, the reduction to echelon form can take $2n^3/3 + n^2/2 - 7n/6$ flops (which is approximately $2n^3/3$ flops when n is moderately large—say, $n \geq 30$). In contrast, further reduction to reduced echelon form needs at most n^2 flops.

Existence and Uniqueness Questions

Although a nonreduced echelon form is a poor tool for solving a system, this form is just the right device for answering two fundamental questions posed in Section 1.1.

EXAMPLE 5 Determine the existence and uniqueness of the solutions to the system

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

SOLUTION The augmented matrix of this system was row reduced in Example 3 to

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \tag{8}$$

The basic variables are x_1, x_2, and x_5; the free variables are x_3 and x_4. There is no equation such as $0 = 1$ that would indicate an inconsistent system, so we could use back-substitution to find a solution. But the *existence* of a solution is already clear in (8). Also, the solution is *not unique* because there are free variables. Each different choice of x_3 and x_4 determines a different solution. Thus the system has infinitely many solutions. ■

When a system is in echelon form and contains no equation of the form $0 = b$, with b nonzero, every nonzero equation contains a basic variable with a nonzero coefficient. Either the basic variables are completely determined (with no free variables) or at least one of the basic variables may be expressed in terms of one or more free variables. In the former case, there is a unique solution; in the latter case, there are infinitely many solutions (one for each choice of values for the free variables).

These remarks justify the following theorem.

[3] Traditionally, a *flop* was only a multiplication or division because addition and subtraction took much less time and could be ignored. The definition of *flop* given here is preferred now, as a result of advances in computer architecture. See Golub and Van Loan, *Matrix Computations*, 2nd ed. (Baltimore: The Johns Hopkins Press, 1989), pp. 19–20.

THEOREM 2

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[\,0 \quad \cdots \quad 0 \quad b\,] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

The following procedure outlines how to find and describe all solutions of a linear system.

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Reasonable Answers

Remember that each augmented matrix corresponds to a system of equations. If you row reduce the augmented matrix $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & -1 & 2 & 5 \\ 0 & 1 & 1 & 3 \end{bmatrix}$ to get the matrix $\begin{bmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the solution set is

$$\begin{cases} x_1 = 8 - 3x_3 \\ x_2 = 3 - x_3 \\ x_3 \text{ is free} \end{cases}$$

The system of equations corresponding to the original augmented matrix is

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ x_1 - x_2 + 2x_3 &= 5 \\ x_2 + x_3 &= 3 \end{aligned}$$

You can now check whether your solution is correct by substituting it into the original equations. Notice that you can just leave the free variables in the solution.

$$\begin{aligned}
(8-3x_3) \;-\; 2(3-x_3) \;+\; (x_3) &= 8-3x_3-6+2x_3+\ x_3 = 2\\
(8-3x_3) \;-\; (3-x_3) \;+\; 2(x_3) &= 8-3x_3-3+\ x_3+2x_3 = 5\\
(3-x_3) \;+\; (x_3) &= 3-\ x_3+\ x_3 = 3
\end{aligned}$$

You can now be confident you have a correct solution to the system of equations represented by the augmented matrix.

Practice Problems

1. Find the general solution of the linear system whose augmented matrix is

$$\begin{bmatrix} 1 & -3 & -5 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

2. Find the general solution of the system

$$\begin{aligned}
x_1 - 2x_2 - \ x_3 + 3x_4 &= 0\\
-2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3\\
3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2
\end{aligned}$$

3. Suppose a 4×7 coefficient matrix for a system of equations has 4 pivots. Is the system consistent? If the system is consistent, how many solutions are there?

1.2 Exercises

In Exercises 1 and 2, determine which matrices are in reduced echelon form and which others are only in echelon form.

1. a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

2. a. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row reduce the matrices in Exercises 3 and 4 to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

3. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$

5. Describe the possible echelon forms of a nonzero 2×2 matrix. Use the symbols ■, $*$, and 0, as in the first part of Example 1.

6. Repeat Exercise 5 for a nonzero 3×2 matrix.

Find the general solutions of the systems whose augmented matrices are given in Exercises 7–14.

7. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 11 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -4 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$

13. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & -5 & -4 & 0 & -5 \\ 0 & 1 & -6 & -4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

You may find it helpful to review the information in the Reasonable Answers box from this section before answering Exercises 15–18.

15. Write down the equations corresponding to the augmented matrix in Exercise 9 and verify your answer to Exercise 9 is correct by substituting the solutions you obtained back into the original equations.

16. Write down the equations corresponding to the augmented matrix in Exercise 10 and verify your answer to Exercise 10 is correct by substituting the solutions you obtained back into the original equations.

17. Write down the equations corresponding to the augmented matrix in Exercise 11 and verify your answer to Exercise 11 is correct by substituting the solutions you obtained back into the original equations.

18. Write down the equations corresponding to the augmented matrix in Exercise 12 and verify your answer to Exercise 12 is correct by substituting the solutions you obtained back into the original equations.

Exercises 19 and 20 use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

19. a. $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$

20. a. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$

In Exercises 21 and 22, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

21. $\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix}$

22. $\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix}$

In Exercises 23 and 24, choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

23. $\begin{aligned} x_1 + hx_2 &= 2 \\ 4x_1 + 8x_2 &= k \end{aligned}$

24. $\begin{aligned} x_1 + 3x_2 &= 2 \\ 3x_1 + hx_2 &= k \end{aligned}$

In Exercises 25–34, mark each statement True or False **(T/F)**. Justify each answer.[4]

25. **(T/F)** In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.

26. **(T/F)** The echelon form of a matrix is unique.

27. **(T/F)** The row reduction algorithm applies only to augmented matrices for a linear system.

28. **(T/F)** The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.

29. **(T/F)** A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

30. **(T/F)** Reducing a matrix to echelon form is called the *forward phase* of the row reduction process.

31. **(T/F)** Finding a parametric description of the solution set of a linear system is the same as *solving* the system.

32. **(T/F)** Whenever a system has free variables, the solution set contains a unique solution.

33. **(T/F)** If one row in an echelon form of an augmented matrix is $[0 \quad 0 \quad 0 \quad 0 \quad 5]$, then the associated linear system is inconsistent.

34. **(T/F)** A general solution of a system is an explicit description of all solutions of the system.

35. Suppose a 3×5 *coefficient* matrix for a system has three pivot columns. Is the system consistent? Why or why not?

36. Suppose a system of linear equations has a 3×5 *augmented* matrix whose fifth column is a pivot column. Is the system consistent? Why (or why not)?

37. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.

38. Suppose the coefficient matrix of a linear system of three equations in three variables has a pivot in each column. Explain why the system has a unique solution.

39. Restate the last sentence in Theorem 2 using the concept of pivot columns: "If a linear system is consistent, then the solution is unique if and only if ______."

40. What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?

41. A system of linear equations with fewer equations than unknowns is sometimes called an *underdetermined system.*

[4] True/false questions of this type will appear in many sections. Methods for justifying your answers were described before the True or False exercises in Section 1.1.

Suppose that such a system happens to be consistent. Explain why there must be an infinite number of solutions.

42. Give an example of an inconsistent underdetermined system of two equations in three unknowns.

43. A system of linear equations with more equations than unknowns is sometimes called an *overdetermined system*. Can such a system be consistent? Illustrate your answer with a specific system of three equations in two unknowns.

44. Suppose an $n \times (n+1)$ matrix is row reduced to reduced echelon form. Approximately what fraction of the total number of operations (flops) is involved in the backward phase of the reduction when $n = 30$? when $n = 300$?

Suppose experimental data are represented by a set of points in the plane. An **interpolating polynomial** for the data is a polynomial whose graph passes through every point. In scientific work, such a polynomial can be used, for example, to estimate values between the known data points. Another use is to create curves for graphical images on a computer screen. One method for finding an interpolating polynomial is to solve a system of linear equations.

45. Find the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ for the data (1, 12), (2, 15), (3, 16). That is, find a_0, a_1, and a_2 such that

$$\begin{aligned} a_0 + a_1(1) + a_2(1)^2 &= 12 \\ a_0 + a_1(2) + a_2(2)^2 &= 15 \\ a_0 + a_1(3) + a_2(3)^2 &= 16 \end{aligned}$$

T 46. In a wind tunnel experiment, the force on a projectile due to air resistance was measured at different velocities:

Velocity (100 ft/sec)	0	2	4	6	8	10
Force (100 lb)	0	2.90	14.8	39.6	74.3	119

Find an interpolating polynomial for these data and estimate the force on the projectile when the projectile is traveling at 750 ft/sec. Use $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$. What happens if you try to use a polynomial of degree less than 5? (Try a cubic polynomial, for instance.)[5]

[5] Exercises marked with the symbol T are designed to be worked with the aid of a "Matrix program" (a computer program, such as MATLAB, Maple, Mathematica, MathCad, Octave, or Derive, or a programmable calculator with matrix capabilities, such as those manufactured by Texas Instruments or Hewlett-Packard).

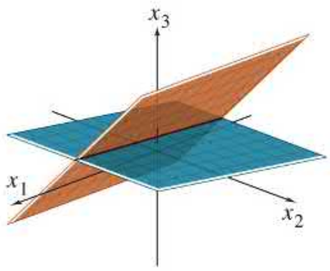
Solutions to Practice Problems

1. The reduced echelon form of the augmented matrix and the corresponding system are

$$\begin{bmatrix} 1 & 0 & -8 & -3 \\ 0 & 1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{aligned} x_1 \quad\;\; - 8x_3 &= -3 \\ x_2 - \;\; x_3 &= -1 \end{aligned}$$

The basic variables are x_1 and x_2, and the general solution is

$$\begin{cases} x_1 = -3 + 8x_3 \\ x_2 = -1 + x_3 \\ x_3 \text{ is free} \end{cases}$$



The general solution of the system of equations is the line of intersection of the two planes.

Note: It is essential that the general solution describe each variable, with any parameters clearly identified. The following statement does *not* describe the solution:

$$\begin{cases} x_1 = -3 + 8x_3 \\ x_2 = -1 + x_3 \\ x_3 = 1 + x_2 \quad \text{Incorrect solution} \end{cases}$$

This description implies that x_2 and x_3 are *both* free, which certainly is not the case.

2. Row reduce the system's augmented matrix:

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Solutions to Practice Problems (Continued)

This echelon matrix shows that the system is *inconsistent*, because its rightmost column is a pivot column; the third row corresponds to the equation 0 = 5. There is no need to perform any more row operations. Note that the presence of the free variables in this problem is irrelevant because the system is inconsistent.

3. Since the coefficient matrix has four pivots, there is a pivot in every row of the coefficient matrix. This means that when the coefficient matrix is row reduced, it will *not* have a row of zeros, thus the corresponding row reduced augmented matrix can never have a row of the form $[0\ \ 0\ \ \cdots\ \ 0\ \ b]$, where b is a nonzero number. By Theorem 2, the system is consistent. Moreover, since there are seven columns in the coefficient matrix and only four pivot columns, there will be three free variables resulting in infinitely many solutions.

1.3 Vector Equations

Important properties of linear systems can be described with the concept and notation of vectors. This section connects equations involving vectors to ordinary systems of equations. The term *vector* appears in a variety of mathematical and physical contexts, which we will discuss in Chapter 4, "Vector Spaces." Until then, *vector* will mean an *ordered list of numbers*. This simple idea enables us to get to interesting and important applications as quickly as possible.

Vectors in $\mathbb{R}^2$

A matrix with only one column is called a **column vector** or simply a **vector**. Examples of vectors with two entries are

$$\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} .2 \\ .3 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where w_1 and w_2 are any real numbers. The set of all vectors with two entries is denoted by $\mathbb{R}^2$ (read "r-two"). The $\mathbb{R}$ stands for the real numbers that appear as entries in the vectors, and the exponent 2 indicates that each vector contains two entries.[1]

Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal. Thus $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ are *not* equal, because vectors in $\mathbb{R}^2$ are *ordered pairs* of real numbers.

Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, their **sum** is the vector $\mathbf{u} + \mathbf{v}$ obtained by adding corresponding entries of $\mathbf{u}$ and $\mathbf{v}$. For example,

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1+2 \\ -2+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Given a vector $\mathbf{u}$ and a real number c, the **scalar multiple** of $\mathbf{u}$ by c is the vector $c\mathbf{u}$ obtained by multiplying each entry in $\mathbf{u}$ by c. For instance,

$$\text{if} \quad \mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad c = 5, \qquad \text{then} \quad c\mathbf{u} = 5\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$$

[1] Most of the text concerns vectors and matrices that have only real entries. However, all definitions and theorems in Chapters 1–5, and in most of the rest of the text, remain valid if the entries are complex numbers. Complex vectors and matrices arise naturally, for example, in electrical engineering and physics.

The number c in $c\mathbf{u}$ is called a **scalar**; it is written in lightface type to distinguish it from the boldface vector $\mathbf{u}$.

The operations of scalar multiplication and vector addition can be combined, as in the following example.

EXAMPLE 1 Given $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4\mathbf{u}$, $(-3)\mathbf{v}$, and $4\mathbf{u} + (-3)\mathbf{v}$.

SOLUTION

$$4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}, \qquad (-3)\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$$

and

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$ ■

Sometimes, for convenience (and also to save space), this text may write a column vector such as $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ in the form $(3, -1)$. In this case, the parentheses and the comma distinguish the vector $(3, -1)$ from the 1×2 row matrix $\begin{bmatrix} 3 & -1 \end{bmatrix}$, written with brackets and no comma. Thus

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 3 & -1 \end{bmatrix}$$

because the matrices have different shapes, even though they have the same entries.

Geometric Descriptions of $\mathbb{R}^2$

Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, *we can identify a geometric point* (a, b) *with the column vector* $\begin{bmatrix} a \\ b \end{bmatrix}$. So we may regard $\mathbb{R}^2$ as the set of all points in the plane. See Figure 1.

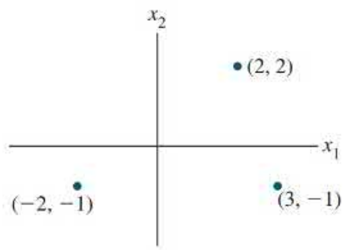


FIGURE 1 Vectors as points.

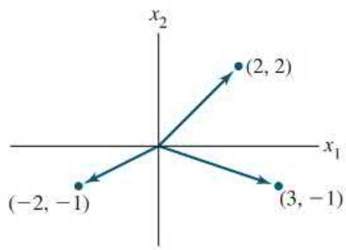


FIGURE 2 Vectors with arrows.

The geometric visualization of a vector such as $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is often aided by including an arrow (directed line segment) from the origin $(0, 0)$ to the point $(3, -1)$, as in Figure 2. In this case, the individual points along the arrow itself have no special significance.[2]

The sum of two vectors has a useful geometric representation. The following rule can be verified by analytic geometry.

[2] In physics, arrows can represent forces and usually are free to move about in space. This interpretation of vectors will be discussed in Section 4.1.

Parallelogram Rule for Addition

If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{u}$, $\mathbf{0}$, and $\mathbf{v}$. See Figure 3.

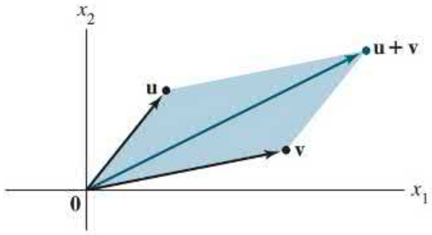


FIGURE 3 The parallelogram rule.

EXAMPLE 2 The vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$, and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ are displayed in Figure 4. ■

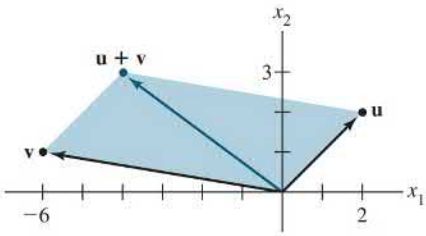
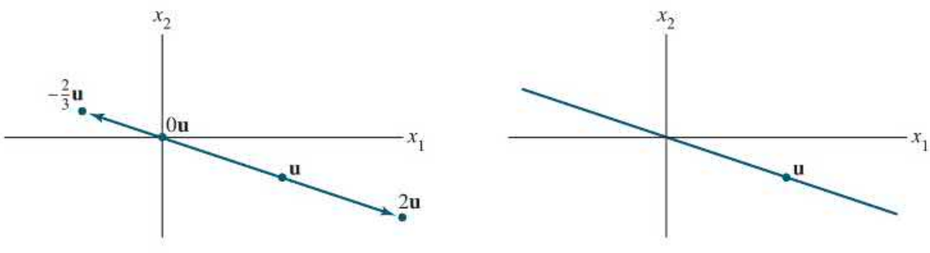


FIGURE 4

The next example illustrates the fact that the set of all scalar multiples of one fixed nonzero vector is a line through the origin, (0, 0).

EXAMPLE 3 Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Display the vectors $\mathbf{u}$, $2\mathbf{u}$, and $-\frac{2}{3}\mathbf{u}$ on a graph.

SOLUTION See Figure 5, where $\mathbf{u}$, $2\mathbf{u} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $-\frac{2}{3}\mathbf{u} = \begin{bmatrix} -2 \\ 2/3 \end{bmatrix}$ are displayed. The arrow for $2\mathbf{u}$ is twice as long as the arrow for $\mathbf{u}$, and the arrows point in the same direction. The arrow for $-\frac{2}{3}\mathbf{u}$ is two-thirds the length of the arrow for $\mathbf{u}$, and the arrows point in opposite directions. In general, the length of the arrow for $c\mathbf{u}$ is $|c|$ times the length of the arrow for $\mathbf{u}$. [Recall that the length of the line segment from $(0, 0)$ to (a, b) is $\sqrt{a^2 + b^2}$.



Typical multiples of $\mathbf{u}$ The set of all multiples of $\mathbf{u}$ ■

FIGURE 5

Vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries. They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin sometimes included for visual clarity. The vectors $\mathbf{a} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $2\mathbf{a}$ are displayed in Figure 6.

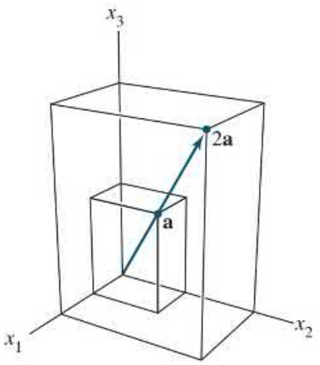


FIGURE 6
Scalar multiples.

Vectors in $\mathbb{R}^n$

If n is a positive integer, $\mathbb{R}^n$ (read "r-n") denotes the collection of all lists (or *ordered n-tuples*) of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The vector whose entries are all zero is called the **zero vector** and is denoted by $\mathbf{0}$. (The number of entries in $\mathbf{0}$ will be clear from the context.)

Equality of vectors in $\mathbb{R}^n$ and the operations of scalar multiplication and vector addition in $\mathbb{R}^n$ are defined entry by entry just as in $\mathbb{R}^2$. These operations on vectors have the following properties, which can be verified directly from the corresponding properties for real numbers. See Practice Problem 1 and Exercises 41 and 42 at the end of this section.

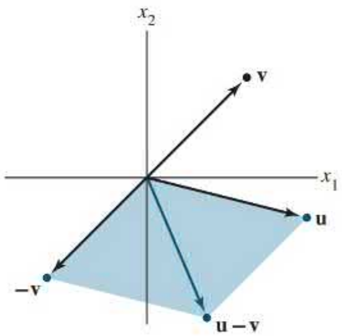


FIGURE 7
Vector subtraction.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

For simplicity of notation, a vector such as $\mathbf{u} + (-1)\mathbf{v}$ is often written as $\mathbf{u} - \mathbf{v}$. Figure 7 shows $\mathbf{u} - \mathbf{v}$ as the sum of $\mathbf{u}$ and $-\mathbf{v}$.

Linear Combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \dots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \dots, \mathbf{v}_p$ with **weights** $c_1, \dots, c_p$. Algebraic Property (ii) above permits us to omit parentheses when forming such a linear combination. The weights in a linear combination can be any real numbers, including zero. For example, some linear combinations of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are

$$\sqrt{3}\,\mathbf{v}_1 + \mathbf{v}_2, \quad \tfrac{1}{2}\mathbf{v}_1 \; (= \tfrac{1}{2}\mathbf{v}_1 + 0\mathbf{v}_2), \quad \text{and} \quad \mathbf{0} \; (= 0\mathbf{v}_1 + 0\mathbf{v}_2)$$

EXAMPLE 4 Figure 8 identifies selected linear combinations of $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. (Note that sets of parallel grid lines are drawn through integer multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$.) Estimate the linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ that generate the vectors $\mathbf{u}$ and $\mathbf{w}$.

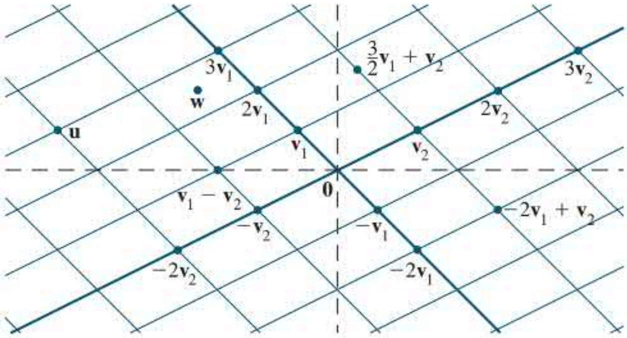


FIGURE 8 Linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.

SOLUTION The parallelogram rule shows that $\mathbf{u}$ is the sum of $3\mathbf{v}_1$ and $-2\mathbf{v}_2$; that is,

$$\mathbf{u} = 3\mathbf{v}_1 - 2\mathbf{v}_2$$

This expression for $\mathbf{u}$ can be interpreted as instructions for traveling from the origin to $\mathbf{u}$ along two straight paths. First, travel 3 units in the $\mathbf{v}_1$ direction to $3\mathbf{v}_1$, and then travel -2 units in the $\mathbf{v}_2$ direction (parallel to the line through $\mathbf{v}_2$ and $\mathbf{0}$). Next, although the vector $\mathbf{w}$ is not on a grid line, $\mathbf{w}$ appears to be about halfway between two pairs of grid lines, at the vertex of a parallelogram determined by $(5/2)\mathbf{v}_1$ and $(-1/2)\mathbf{v}_2$. (See Figure 9.) Thus a reasonable estimate for $\mathbf{w}$ is

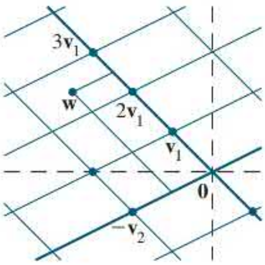


FIGURE 9

$$\mathbf{w} = \tfrac{5}{2}\mathbf{v}_1 - \tfrac{1}{2}\mathbf{v}_2$$ ■

The next example connects a problem about linear combinations to the fundamental existence question studied in Sections 1.1 and 1.2.

EXAMPLE 5 Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether $\mathbf{b}$ can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \tag{1}$$

If vector equation (1) has a solution, find it.

SOLUTION Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \underset{\substack{\uparrow \\ \mathbf{a}_1}}{\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}} + x_2 \underset{\substack{\uparrow \\ \mathbf{a}_2}}{\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}} = \underset{\substack{\uparrow \\ \mathbf{b}}}{\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}}$$

which is the same as

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \tag{2}$$

The vectors on the left and right sides of (2) are equal if and only if their corresponding entries are both equal. That is, x_1 and x_2 make the vector equation (1) true if and only if x_1 and x_2 satisfy the system

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned} \tag{3}$$

To solve this system, row reduce the augmented matrix of the system as follows:[3]

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with weights $x_1 = 3$ and $x_2 = 2$. That is,

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

■

Observe in Example 5 that the original vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$ are the columns of the augmented matrix that we row reduced:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{array} \end{array}$$

For brevity, write this matrix in a way that identifies its columns—namely

$$[\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}\,] \tag{4}$$

It is clear how to write this augmented matrix immediately from vector equation (1), without going through the intermediate steps of Example 5. Take the vectors in the order in which they appear in (1) and put them into the columns of a matrix as in (4).

The discussion above is easily modified to establish the following fundamental fact.

---

[3] The symbol $\sim$ between matrices denotes row equivalence (Section 1.2).

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{5}$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

One of the key ideas in linear algebra is to study the set of all vectors that can be generated or written as a linear combination of a fixed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of vectors.

DEFINITION

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by $\mathbf{v}_1, \dots, \mathbf{v}_p$**. That is, Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \dots, c_p$ scalars.

Asking whether a vector $\mathbf{b}$ is in Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ amounts to asking whether the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix $[\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \ \mathbf{b}\,]$ has a solution.

Note that Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$ (for example), since $c\mathbf{v}_1 = c\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$. In particular, the zero vector must be in Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

A Geometric Description of Span {v} and Span {u, v}

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Then Span $\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$, which is the set of points on the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$. See Figure 10.

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\mathbf{v}$ not a multiple of $\mathbf{u}$, then Span $\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$. In particular, Span $\{\mathbf{u}, \mathbf{v}\}$ contains the line in $\mathbb{R}^3$ through $\mathbf{u}$ and $\mathbf{0}$ and the line through $\mathbf{v}$ and $\mathbf{0}$. See Figure 11.

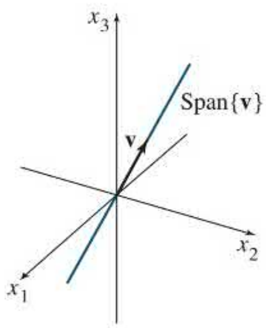


FIGURE 10 Span $\{\mathbf{v}\}$ as a line through the origin.

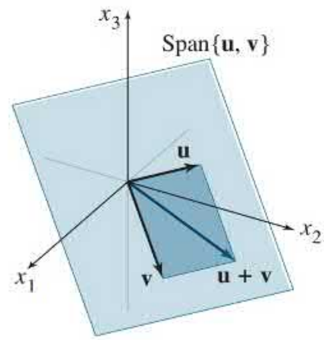


FIGURE 11 Span $\{\mathbf{u}, \mathbf{v}\}$ as a plane through the origin.

EXAMPLE 6 Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. Then Span $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a plane through the origin in $\mathbb{R}^3$. Is $\mathbf{b}$ in that plane?

SOLUTION Does the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ have a solution? To answer this, row reduce the augmented matrix $[\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}\,]$:

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

The third equation is $0 = -2$, which shows that the system has no solution. The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ has no solution, and so $\mathbf{b}$ is *not* in Span $\{\mathbf{a}_1, \mathbf{a}_2\}$. ■

Linear Combinations in Applications

The final example shows how scalar multiples and linear combinations can arise when a quantity such as "cost" is broken down into several categories. The basic principle for the example concerns the cost of producing several units of an item when the cost per unit is known:

$$\left\{\begin{matrix} \text{number} \\ \text{of units} \end{matrix}\right\} \cdot \left\{\begin{matrix} \text{cost} \\ \text{per unit} \end{matrix}\right\} = \left\{\begin{matrix} \text{total} \\ \text{cost} \end{matrix}\right\}$$

EXAMPLE 7 A company manufactures two products. For \$1.00 worth of product B, the company spends \$.45 on materials, \$.25 on labor, and \$.15 on overhead. For \$1.00 worth of product C, the company spends \$.40 on materials, \$.30 on labor, and \$.15 on overhead. Let

$$\mathbf{b} = \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} .40 \\ .30 \\ .15 \end{bmatrix}$$

Then $\mathbf{b}$ and $\mathbf{c}$ represent the "costs per dollar of income" for the two products.

a. What economic interpretation can be given to the vector $100\mathbf{b}$?

b. Suppose the company wishes to manufacture x_1 dollars worth of product B and x_2 dollars worth of product C. Give a vector that describes the various costs the company will have (for materials, labor, and overhead).

SOLUTION

a. Compute

$$100\mathbf{b} = 100\begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} = \begin{bmatrix} 45 \\ 25 \\ 15 \end{bmatrix}$$

The vector $100\mathbf{b}$ lists the various costs for producing \$100 worth of product B—namely \$45 for materials, \$25 for labor, and \$15 for overhead.

b. The costs of manufacturing x_1 dollars worth of B are given by the vector $x_1\mathbf{b}$, and the costs of manufacturing x_2 dollars worth of C are given by $x_2\mathbf{c}$. Hence the total costs for both products are given by the vector $x_1\mathbf{b} + x_2\mathbf{c}$. ■

Practice Problems

1. Prove that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.
2. For what value(s) of h will $\mathbf{y}$ be in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

3. Let $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{u}$, and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Suppose the vectors $\mathbf{u}$ and $\mathbf{v}$ are in Span $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v}$ is also in Span $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. [*Hint:* The solution requires the use of the definition of the span of a set of vectors. It is useful to review this definition before starting this exercise.]

1.3 Exercises

In Exercises 1 and 2, compute $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$.

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

In Exercises 3 and 4, display the following vectors using arrows on an xy-graph: $\mathbf{u}$, $\mathbf{v}$, $-\mathbf{v}$, $-2\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, and $\mathbf{u} - 2\mathbf{v}$. Notice that $\mathbf{u} - \mathbf{v}$ is the vertex of a parallelogram whose other vertices are $\mathbf{u}$, $\mathbf{0}$, and $-\mathbf{v}$.

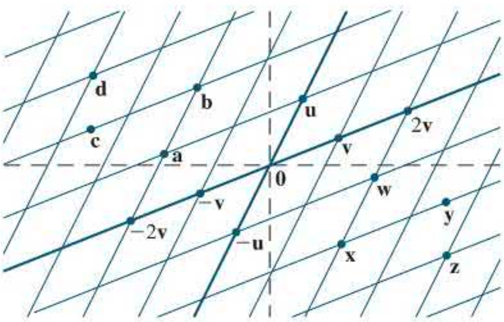
3. $\mathbf{u}$ and $\mathbf{v}$ as in Exercise 1
4. $\mathbf{u}$ and $\mathbf{v}$ as in Exercise 2

In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.

5. $x_1 \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$

6. $x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Use the accompanying figure to write each vector listed in Exercises 7 and 8 as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Is every vector in $\mathbb{R}^2$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$?



7. Vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$
8. Vectors $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$

In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.

9. $$\begin{aligned} x_2 + 5x_3 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0 \\ -x_1 + 3x_2 - 8x_3 &= 0 \end{aligned}$$

10. $$\begin{aligned} 4x_1 + x_2 + 3x_3 &= 9 \\ x_1 - 7x_2 - 2x_3 &= 2 \\ 8x_1 + 6x_2 - 5x_3 &= 15 \end{aligned}$$

In Exercises 11 and 12, determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

11. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

12. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$

In Exercises 13 and 14, determine if $\mathbf{b}$ is a linear combination of the vectors formed from the columns of the matrix A.

13. $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$

In Exercises 15 and 16, list five vectors in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$. For each vector, show the weights on $\mathbf{v}_1$ and $\mathbf{v}_2$ used to generate the vector and list the three entries of the vector. Do not make a sketch.

15. $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$

16. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

17. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what value(s) of h is $\mathbf{b}$ in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$?

18. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$. For what value(s) of h is $\mathbf{y}$ in the plane generated by $\mathbf{v}_1$ and $\mathbf{v}_2$?

19. Give a geometric description of Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors $\mathbf{v}_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$.

20. Give a geometric description of Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors in Exercise 16.

21. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$ for all h and k.

22. Construct a 3×3 matrix A, with nonzero entries, and a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that $\mathbf{b}$ is *not* in the set spanned by the columns of A.

In Exercises 23–32, mark each statement True or False **(T/F)**. Justify each answer.

23. **(T/F)** Another notation for the vector $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is $[-4 \;\; 3]$.

24. **(T/F)** Any list of five real numbers is a vector in $\mathbb{R}^5$.

25. **(T/F)** The points in the plane corresponding to $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ lie on a line through the origin.

26. **(T/F)** The vector $\mathbf{u}$ results when a vector $\mathbf{u} - \mathbf{v}$ is added to the vector $\mathbf{v}$.

27. **(T/F)** An example of a linear combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is the vector $\frac{1}{2}\mathbf{v}_1$.

28. **(T/F)** The weights $c_1, \dots, c_p$ in a linear combination $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ cannot all be zero.

29. **(T/F)** The solution set of the linear system whose augmented matrix is $[\, \mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \;\; \mathbf{b} \,]$ is the same as the solution set of the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$.

30. **(T/F)** When $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, Span $\{\mathbf{u}, \mathbf{v}\}$ contains the line through $\mathbf{u}$ and the origin.

31. **(T/F)** The set Span $\{\mathbf{u}, \mathbf{v}\}$ is always visualized as a plane through the origin.

32. **(T/F)** Asking whether the linear system corresponding to an augmented matrix $[\, \mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \;\; \mathbf{b} \,]$ has a solution amounts to asking whether $\mathbf{b}$ is in Span $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

33. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = \text{Span}\,\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

a. Is $\mathbf{b}$ in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? How many vectors are in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?

b. Is $\mathbf{b}$ in W? How many vectors are in W?

c. Show that $\mathbf{a}_1$ is in W. [*Hint:* Row operations are unnecessary.]

34. Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$, let $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$, and let W be the set of all linear combinations of the columns of A.

a. Is $\mathbf{b}$ in W?

b. Show that the third column of A is in W.

35. A mining company has two mines. One day's operation at mine 1 produces ore that contains 20 metric tons of copper and 550 kilograms of silver, while one day's operation at mine 2 produces ore that contains 30 metric tons of copper and 500 kilograms of silver. Let $\mathbf{v}_1 = \begin{bmatrix} 20 \\ 550 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 30 \\ 500 \end{bmatrix}$.

Then $\mathbf{v}_1$ and $\mathbf{v}_2$ represent the "output per day" of mine 1 and mine 2, respectively.

a. What physical interpretation can be given to the vector $5\mathbf{v}_1$?

b. Suppose the company operates mine 1 for x_1 days and mine 2 for x_2 days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 150 tons of copper and 2825 kilograms of silver. Do not solve the equation.

T c. Solve the equation in (b).

36. A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.

a. How much heat does the steam plant produce when it burns x_1 tons of A and x_2 tons of B?

b. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns x_1 tons of A and x_2 tons of B.

T c. Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.

37. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be points in $\mathbb{R}^3$ and suppose that for $j = 1, \dots, k$ an object with mass m_j is located at point $\mathbf{v}_j$. Physicists call such objects *point masses*. The total mass of the system of point masses is

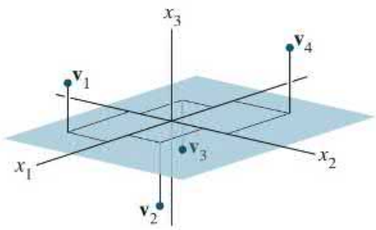
$$m = m_1 + \cdots + m_k$$

The *center of mass* (or *center of gravity*) of the system is

$$\overline{\mathbf{v}} = \frac{1}{m}[m_1\mathbf{v}_1 + \cdots + m_k\mathbf{v}_k]$$

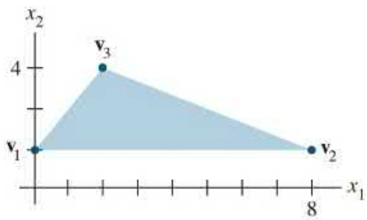
Compute the center of gravity of the system consisting of the following point masses (see the figure):

Point	Mass
$\mathbf{v}_1 = (5, -4, 3)$	2 g
$\mathbf{v}_2 = (4, 3, -2)$	5 g
$\mathbf{v}_3 = (-4, -3, -1)$	2 g
$\mathbf{v}_4 = (-9, 8, 6)$	1 g



38. Let $\mathbf{v}$ be the center of mass of a system of point masses located at $\mathbf{v}_1, \dots, \mathbf{v}_k$ as in Exercise 37. Is $\mathbf{v}$ in Span $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$? Explain.

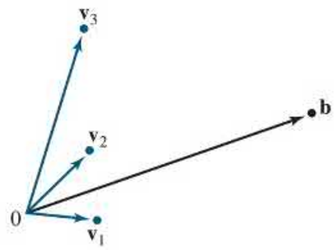
39. A thin triangular plate of uniform density and thickness has vertices at $\mathbf{v}_1 = (0, 1)$, $\mathbf{v}_2 = (8, 1)$, and $\mathbf{v}_3 = (2, 4)$, as in the figure below, and the mass of the plate is 3 g.



a. Find the (x, y)-coordinates of the center of mass of the plate. This "balance point" of the plate coincides with the center of mass of a system consisting of three 1-gram point masses located at the vertices of the plate.

b. Determine how to distribute an additional mass of 6 g at the three vertices of the plate to move the balance point of the plate to $(2, 2)$. [*Hint:* Let w_1, w_2, and w_3 denote the masses added at the three vertices, so that $w_1 + w_2 + w_3 = 6$.]

40. Consider the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{b}$ in $\mathbb{R}^2$, shown in the figure. Does the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ have a solution? Is the solution unique? Use the figure to explain your answers.



41. Use the vectors $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$, and $\mathbf{w} = (w_1, \dots, w_n)$ to verify the following algebraic properties of $\mathbb{R}^n$.

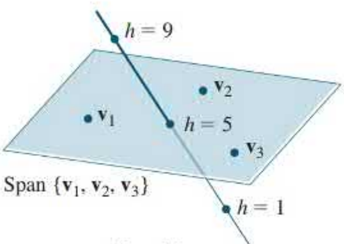
a. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

b. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ for each scalar c

42. Use the vector $\mathbf{u} = (u_1, \dots, u_n)$ to verify the following algebraic properties of $\mathbb{R}^n$.

a. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$

b. $c(d\mathbf{u}) = (cd)\mathbf{u}$ for all scalars c and d



The points $\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$ lie on a line that intersects the plane when $h = 5$.

Solutions to Practice Problems

1. Take arbitrary vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$, and compute

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_1 + v_1, \dots, u_n + v_n) && \text{Definition of vector addition} \\
&= (v_1 + u_1, \dots, v_n + u_n) && \text{Commutativity of addition in } \mathbb{R} \\
&= \mathbf{v} + \mathbf{u} && \text{Definition of vector addition}
\end{aligned}$$

2. The vector $\mathbf{y}$ belongs to Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if there exist scalars x_1, x_2, x_3 such that

$$x_1\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

This vector equation is equivalent to a system of three linear equations in three unknowns. If you row reduce the augmented matrix for this system, you find that

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

The system is consistent if and only if there is no pivot in the fourth column. That is, $h - 5$ must be 0. So $\mathbf{y}$ is in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $h = 5$.

Remember: The presence of a free variable in a system does not guarantee that the system is consistent.

3. Since the vectors $\mathbf{u}$ and $\mathbf{v}$ are in Span $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, there exist scalars c_1, c_2, c_3 and d_1, d_2, d_3 such that

$$\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \quad \text{and} \quad \mathbf{v} = d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3.$$

Notice
$$\begin{aligned} \mathbf{u} + \mathbf{v} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3 \\ &= (c_1 + d_1)\mathbf{w}_1 + (c_2 + d_2)\mathbf{w}_2 + (c_3 + d_3)\mathbf{w}_3 \end{aligned}$$

Since $c_1 + d_1, c_2 + d_2$, and $c_3 + d_3$ are also scalars, the vector $\mathbf{u} + \mathbf{v}$ is in Span $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector. The following definition permits us to rephrase some of the concepts of Section 1.3 in new ways.

DEFINITION

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

EXAMPLE 1

a. $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

b. $$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$$ ■

EXAMPLE 2 For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.

SOLUTION Place $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the columns of a matrix A and place the weights 3, -5, and 7 into a vector $\mathbf{x}$. That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x}$$ ■

Section 1.3 showed how to write a system of linear equations as a vector equation involving a linear combination of vectors. For example, the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \tag{1}$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \tag{2}$$

As in Example 2, the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \tag{3}$$

Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a **matrix equation**, to distinguish it from a vector equation such as is shown in (2).

Notice how the matrix in (3) is just the matrix of coefficients of the system (1). Similar calculations show that any system of linear equations, or any vector equation such as (2), can be written as an equivalent matrix equation in the form $A\mathbf{x} = \mathbf{b}$. This simple observation will be used repeatedly throughout the text.

Here is the formal result.

THEOREM 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{6}$$

Theorem 3 provides a powerful tool for gaining insight into problems in linear algebra, because a system of linear equations may now be viewed in three different but equivalent ways: as a matrix equation, as a vector equation, or as a system of linear equations. Whenever you construct a mathematical model of a problem in real life, you are free to choose whichever viewpoint is most natural. Then you may switch from one formulation of a problem to another whenever it is convenient. In any case, the matrix equation (4), the vector equation (5), and the system of equations are all solved in the same way—by row reducing the augmented matrix (6). Other methods of solution will be discussed later.

Existence of Solutions

The definition of $A\mathbf{x}$ leads directly to the following useful fact.

> The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Section 1.3 considered the existence question, "Is $\mathbf{b}$ in Span $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$?" Equivalently, "Is $A\mathbf{x} = \mathbf{b}$ consistent?" A harder existence problem is to determine whether the equation $A\mathbf{x} = \mathbf{b}$ is consistent *for all* possible $\mathbf{b}$.

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

SOLUTION Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

The third entry in column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\mathbf{x} = \mathbf{b}$ is *not* consistent for every $\mathbf{b}$ because some choices of $\mathbf{b}$ can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero. ■

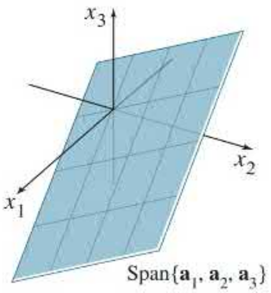


FIGURE 1
The columns of $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3\,]$ span a plane through $\mathbf{0}$.

The reduced matrix in Example 3 provides a description of all $\mathbf{b}$ for which the equation $A\mathbf{x} = \mathbf{b}$ *is* consistent: The entries in $\mathbf{b}$ must satisfy

$$b_1 - \tfrac{1}{2}b_2 + b_3 = 0$$

This is the equation of a plane through the origin in $\mathbb{R}^3$. The plane is the set of all linear combinations of the three columns of A. See Figure 1.

The equation $A\mathbf{x} = \mathbf{b}$ in Example 3 fails to be consistent for all $\mathbf{b}$ because the echelon form of A has a row of zeros. If A had a pivot in all three rows, we would not care about the calculations in the augmented column because in this case an echelon form of the augmented matrix could not have a row such as $[\,0 \quad 0 \quad 0 \quad 1\,]$.

In the next theorem, the sentence "The columns of A span $\mathbb{R}^m$" means that *every* $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A. In general, a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ **spans** (or **generates**) $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$—that is, if Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = \mathbb{R}^m$.

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

Theorem 4 is one of the most useful theorems in this chapter. Statements (a), (b), and (c) are equivalent because of the definition of $A\mathbf{x}$ and what it means for a set of vectors to span $\mathbb{R}^m$. The discussion after Example 3 suggests why (a) and (d) are equivalent; a proof is given at the end of the section. The exercises will provide examples of how Theorem 4 is used.

Warning: Theorem 4 is about a *coefficient matrix*, not an augmented matrix. If an augmented matrix $[\,A \quad \mathbf{b}\,]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ may or may not be consistent.

Computation of $A\mathbf{x}$

The calculations in Example 1 were based on the definition of the product of a matrix A and a vector $\mathbf{x}$. The following simple example will lead to a more efficient method for calculating the entries in $A\mathbf{x}$ when working problems by hand.

EXAMPLE 4 Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

SOLUTION From the definition,

$$\begin{aligned}
\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= x_1\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \qquad (7) \\
&= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}
\end{aligned}$$

The first entry in the product $A\mathbf{x}$ is a sum of products (sometimes called a *dot product*), using the first row of A and the entries in $\mathbf{x}$. That is,

$$\begin{bmatrix} 2 & 3 & 4 \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ \\ \\ \end{bmatrix}$$

This matrix shows how to compute the first entry in $A\mathbf{x}$ directly, without writing down all the calculations shown in (7). Similarly, the second entry in $A\mathbf{x}$ can be calculated at once by multiplying the entries in the second row of A by the corresponding entries in $\mathbf{x}$ and then summing the resulting products:

$$\begin{bmatrix} & & \\ -1 & 5 & -3 \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ -x_1 + 5x_2 - 3x_3 \\ \\ \end{bmatrix}$$

Likewise, the third entry in $A\mathbf{x}$ can be calculated from the third row of A and the entries in $\mathbf{x}$. ■

Row–Vector Rule for Computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

EXAMPLE 5

a. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1\cdot 4 + 2\cdot 3 + (-1)\cdot 7 \\ 0\cdot 4 + (-5)\cdot 3 + 3\cdot 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2\cdot 4 + (-3)\cdot 7 \\ 8\cdot 4 + 0\cdot 7 \\ (-5)\cdot 4 + 2\cdot 7 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1\cdot r + 0\cdot s + 0\cdot t \\ 0\cdot r + 1\cdot s + 0\cdot t \\ 0\cdot r + 0\cdot s + 1\cdot t \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$ ■

By definition, the matrix in Example 5(c) with 1's on the diagonal and 0's elsewhere is called an **identity matrix** and is denoted by I. The calculation in part (c) shows that $I\mathbf{x} = \mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^3$. There is an analogous $n \times n$ identity matrix, sometimes written as I_n. As in part (c), $I_n\mathbf{x} = \mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^n$.

Properties of the Matrix–Vector Product $A\mathbf{x}$

The facts in the next theorem are important and will be used throughout the text. The proof relies on the definition of $A\mathbf{x}$ and the algebraic properties of $\mathbb{R}^n$.

THEOREM 5

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;
b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

PROOF For simplicity, take $n = 3$, $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3\,]$, and $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^3$. (The proof of the general case is similar.) For $i = 1, 2, 3$, let u_i and v_i be the ith entries in $\mathbf{u}$ and $\mathbf{v}$, respectively. To prove statement (a), compute $A(\mathbf{u} + \mathbf{v})$ as a linear combination of the columns of A using the entries in $\mathbf{u} + \mathbf{v}$ as weights.

$$\begin{aligned} A(\mathbf{u}+\mathbf{v}) &= [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3\,]\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{Entries in } \mathbf{u} + \mathbf{v} \\ &= (u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3 \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{Columns of } A \\ &= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3) \\ &= A\mathbf{u} + A\mathbf{v} \end{aligned}$$

To prove statement (b), compute $A(c\mathbf{u})$ as a linear combination of the columns of A using the entries in $c\mathbf{u}$ as weights.

$$\begin{aligned} A(c\mathbf{u}) = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3\,]\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} &= (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + (cu_3)\mathbf{a}_3 \\ &= c(u_1\mathbf{a}_1) + c(u_2\mathbf{a}_2) + c(u_3\mathbf{a}_3) \\ &= c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) \\ &= c(A\mathbf{u}) \end{aligned}$$

■

Numerical Note

To optimize a computer algorithm to compute $A\mathbf{x}$, the sequence of calculations should involve data stored in contiguous memory locations. The most widely used professional algorithms for matrix computations are written in Fortran, a language that stores a matrix as a set of columns. Such algorithms compute $A\mathbf{x}$ as a linear combination of the columns of A. In contrast, if a program is written in the popular language C, which stores matrices by rows, $A\mathbf{x}$ should be computed via the alternative rule that uses the rows of A.

PROOF OF THEOREM 4 As was pointed out after Theorem 4, statements (a), (b), and (c) are logically equivalent. So, it suffices to show (for an arbitrary matrix A) that (a) and (d) are either both true or both false. This will tie all four statements together.

Let U be an echelon form of A. Given $\mathbf{b}$ in $\mathbb{R}^m$, we can row reduce the augmented matrix $[\,A \quad \mathbf{b}\,]$ to an augmented matrix $[\,U \quad \mathbf{d}\,]$ for some $\mathbf{d}$ in $\mathbb{R}^m$:

$$[\,A \quad \mathbf{b}\,] \sim \cdots \sim [\,U \quad \mathbf{d}\,]$$

If statement (d) is true, then each row of U contains a pivot position and there can be no pivot in the augmented column. So $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$, and (a) is true. If (d) is false, the last row of U is all zeros. Let $\mathbf{d}$ be any vector with a 1 in its last entry. Then $[\,U \quad \mathbf{d}\,]$ represents an *inconsistent* system. Since row operations are reversible, $[\,U \quad \mathbf{d}\,]$ can be transformed into the form $[\,A \quad \mathbf{b}\,]$. The new system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, and (a) is false. ■

Practice Problems

1. Let $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$. It can be shown that $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Use this fact to exhibit $\mathbf{b}$ as a specific linear combination of the columns of A.
2. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. Verify Theorem 5(a) in this case by computing $A(\mathbf{u} + \mathbf{v})$ and $A\mathbf{u} + A\mathbf{v}$.
3. Construct a 3×3 matrix A and vectors $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$ so that $A\mathbf{x} = \mathbf{b}$ has a solution, but $A\mathbf{x} = \mathbf{c}$ does not.

1.4 Exercises

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\mathbf{x}$. If a product is undefined, explain why.

1. $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$

2. $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 3 & 1 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 5–8, use the definition of $A\mathbf{x}$ to write the matrix equation as a vector equation, or vice versa.

5. $\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

6. $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$

7. $x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$

8. $z_1 \begin{bmatrix} 4 \\ -2 \end{bmatrix} + z_2 \begin{bmatrix} -4 \\ 5 \end{bmatrix} + z_3 \begin{bmatrix} -5 \\ 4 \end{bmatrix} + z_4 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$

In Exercises 9 and 10, write the system first as a vector equation and then as a matrix equation.

9. $\begin{aligned} 3x_1 + x_2 - 5x_3 &= 9 \\ x_2 + 4x_3 &= 0 \end{aligned}$

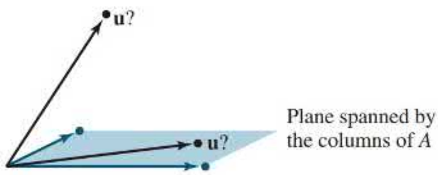
10. $\begin{aligned} 8x_1 - x_2 &= 4 \\ 5x_1 + 4x_2 &= 1 \\ x_1 - 3x_2 &= 2 \end{aligned}$

Given A and $\mathbf{b}$ in Exercises 11 and 12, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

11. $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

13. Let $\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\mathbf{u}$ in the plane in $\mathbb{R}^3$ spanned by the columns of A? (See the figure.) Why or why not?



Where are $\mathbf{u}$?

14. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$. Is $\mathbf{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of A? Why or why not?

15. Let $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$, and describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ *does* have a solution.

16. Repeat Exercise 15: $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Exercises 17–20 refer to the matrices A and B below. Make appropriate calculations that justify your answers and mention an appropriate theorem.

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$$

17. How many rows of A contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each $\mathbf{b}$ in $\mathbb{R}^4$?

18. Do the columns of B span $\mathbb{R}^4$? Does the equation $B\mathbf{x} = \mathbf{y}$ have a solution for each $\mathbf{y}$ in $\mathbb{R}^4$?

19. Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A above? Do the columns of A span $\mathbb{R}^4$?

20. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B above? Do the columns of B span $\mathbb{R}^3$?

21. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^4$? Why or why not?

22. Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}$.

Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$? Why or why not?

In Exercises 23–34, mark each statement True or False **(T/F)**. Justify each answer.

23. **(T/F)** The equation $A\mathbf{x} = \mathbf{b}$ is referred to as a *vector equation*.

24. **(T/F)** Every matrix equation $A\mathbf{x} = \mathbf{b}$ corresponds to a vector equation with the same solution set.

25. **(T/F)** If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ is not in the set spanned by the columns of A.

26. **(T/F)** A vector $\mathbf{b}$ is a linear combination of the columns of a matrix A if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

27. **(T/F)** The equation $A\mathbf{x} = \mathbf{b}$ is consistent if the augmented matrix $[\, A \;\; \mathbf{b} \,]$ has a pivot position in every row.

28. **(T/F)** If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^m$.

29. **(T/F)** The first entry in the product $A\mathbf{x}$ is a sum of products.

30. **(T/F)** Any linear combination vectors can always be written in the form $A\mathbf{x}$ for a suitable matrix A and vector $\mathbf{x}$.

31. **(T/F)** If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^m$.

32. **(T/F)** The solution set of a linear system whose augmented matrix is $[\, \mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \;\; \mathbf{b} \,]$ is the same as the solution set of $A\mathbf{x} = \mathbf{b}$, if $A = [\, \mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \,]$.

33. **(T/F)** If A is an $m \times n$ matrix and if the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^m$, then A cannot have a pivot position in every row.

34. **(T/F)** If the augmented matrix $[\ A \quad \mathbf{b}\ ]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.

35. Note that $\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$. Use this fact (and no row operations) to find scalars c_1, c_2, c_3 such that $\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$.

36. Let $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$.

It can be shown that $3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$. Use this fact (and no row operations) to find x_1 and x_2 that satisfy the equation $\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$.

37. Let $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, and $\mathbf{v}$ represent vectors in $\mathbb{R}^5$, and let x_1, x_2, and x_3 denote scalars. Write the following vector equation as a matrix equation. Identify any symbols you choose to use.

$$x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + x_3\mathbf{q}_3 = \mathbf{v}$$

38. Rewrite the (numerical) matrix equation below in symbolic form as a vector equation, using symbols $\mathbf{v}_1, \mathbf{v}_2, \ldots$ for the vectors and $c_1, c_2, \ldots$ for scalars. Define what each symbol represents, using the data given in the matrix equation.

$$\begin{bmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \\ 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

39. Construct a 3×3 matrix, not in echelon form, whose columns span $\mathbb{R}^3$. Show that the matrix you construct has the desired property.

40. Construct a 3×3 matrix, not in echelon form, whose columns do *not* span $\mathbb{R}^3$. Show that the matrix you construct has the desired property.

STUDY GUIDE offers additional resources for mastering the concept of span.

41. Let A be a 3×2 matrix. Explain why the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. Generalize your argument to the case of an arbitrary A with more rows than columns.

42. Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$ when n is less than m?

43. Suppose A is a 4×3 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^4$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. What can you say about the reduced echelon form of A? Justify your answer.

44. Suppose A is a 3×3 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^3$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^3$.

45. Let A be a 3×4 matrix, let $\mathbf{y}_1$ and $\mathbf{y}_2$ be vectors in $\mathbb{R}^3$, and let $\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2$. Suppose $\mathbf{y}_1 = A\mathbf{x}_1$ and $\mathbf{y}_2 = A\mathbf{x}_2$ for some vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^4$. What fact allows you to conclude that the system $A\mathbf{x} = \mathbf{w}$ is consistent? (*Note:* $\mathbf{x}_1$ and $\mathbf{x}_2$ denote vectors, not scalar entries in vectors.)

46. Let A be a 5×3 matrix, let $\mathbf{y}$ be a vector in $\mathbb{R}^3$, and let $\mathbf{z}$ be a vector in $\mathbb{R}^5$. Suppose $A\mathbf{y} = \mathbf{z}$. What fact allows you to conclude that the system $A\mathbf{x} = 4\mathbf{z}$ is consistent?

T In Exercises 47–50, determine if the columns of the matrix span $\mathbb{R}^4$.

47. $\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix}$

48. $\begin{bmatrix} 5 & -7 & -4 & 9 \\ 6 & -8 & -7 & 5 \\ 4 & -4 & -9 & -9 \\ -9 & 11 & 16 & 7 \end{bmatrix}$

49. $\begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \\ -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{bmatrix}$

50. $\begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix}$

T **51.** Find a column of the matrix in Exercise 49 that can be deleted and yet have the remaining matrix columns still span $\mathbb{R}^4$.

T **52.** Find a column of the matrix in Exercise 50 that can be deleted and yet have the remaining matrix columns still span $\mathbb{R}^4$. Can you delete more than one column?

Solutions to Practice Problems

1. The matrix equation

$$\begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$

is equivalent to the vector equation

$$3\begin{bmatrix}1\\-3\\4\end{bmatrix}-2\begin{bmatrix}5\\1\\-8\end{bmatrix}+0\begin{bmatrix}-2\\9\\-1\end{bmatrix}-4\begin{bmatrix}0\\-5\\7\end{bmatrix}=\begin{bmatrix}-7\\9\\0\end{bmatrix},$$

which expresses $\mathbf{b}$ as a linear combination of the columns of A.

2. $\mathbf{u}+\mathbf{v}=\begin{bmatrix}4\\-1\end{bmatrix}+\begin{bmatrix}-3\\5\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}$

$$A(\mathbf{u}+\mathbf{v})=\begin{bmatrix}2&5\\3&1\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}2+20\\3+4\end{bmatrix}=\begin{bmatrix}22\\7\end{bmatrix}$$

$$\begin{aligned}A\mathbf{u}+A\mathbf{v}&=\begin{bmatrix}2&5\\3&1\end{bmatrix}\begin{bmatrix}4\\-1\end{bmatrix}+\begin{bmatrix}2&5\\3&1\end{bmatrix}\begin{bmatrix}-3\\5\end{bmatrix}\\&=\begin{bmatrix}3\\11\end{bmatrix}+\begin{bmatrix}19\\-4\end{bmatrix}=\begin{bmatrix}22\\7\end{bmatrix}\end{aligned}$$

Remark: There are, in fact, infinitely many correct solutions to Practice Problem 3. When creating matrices to satisfy specified criteria, it is often useful to create matrices that are straightforward, such as those already in reduced echelon form. Here is one possible solution:

3. Let

$$A=\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}3\\2\\0\end{bmatrix},\ \text{and } \mathbf{c}=\begin{bmatrix}3\\2\\1\end{bmatrix}.$$

Notice the reduced echelon form of the augmented matrix corresponding to $A\mathbf{x}=\mathbf{b}$ is

$$\begin{bmatrix}1&0&1&3\\0&1&1&2\\0&0&0&0\end{bmatrix},$$

which corresponds to a consistent system, and hence $A\mathbf{x}=\mathbf{b}$ has solutions. The reduced echelon form of the augmented matrix corresponding to $A\mathbf{x}=\mathbf{c}$ is

$$\begin{bmatrix}1&0&1&3\\0&1&1&2\\0&0&0&1\end{bmatrix},$$

which corresponds to an inconsistent system, and hence $A\mathbf{x}=\mathbf{c}$ does not have any solutions.

1.5 Solution Sets of Linear Systems

Solution sets of linear systems are important objects of study in linear algebra. They will appear later in several different contexts. This section uses vector notation to give explicit and geometric descriptions of such solution sets.

Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). This zero solution is usually called the **trivial solution**. For a given equation $A\mathbf{x} = \mathbf{0}$, the important question is whether there exists a **nontrivial solution**, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$. The Existence and Uniqueness Theorem in Section 1.2 (Theorem 2) leads immediately to the following fact.

> The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

EXAMPLE 1 Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

SOLUTION Let A be the matrix of coefficients of the system and row reduce the augmented matrix $[\,A \quad \mathbf{0}\,]$ to echelon form:

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is a free variable, $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions (one for each nonzero choice of x_3). To describe the solution set, continue the row reduction of $[\,A \quad \mathbf{0}\,]$ to *reduced* echelon form:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \qquad - \tfrac{4}{3}x_3 &= 0 \\ x_2 \qquad\quad &= 0 \\ 0 &= 0 \end{aligned}$$

Solve for the basic variables x_1 and x_2 and obtain $x_1 = \frac{4}{3}x_3$, $x_2 = 0$, with x_3 free. As a vector, the general solution of $A\mathbf{x} = \mathbf{0}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathbf{v}, \quad \text{where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

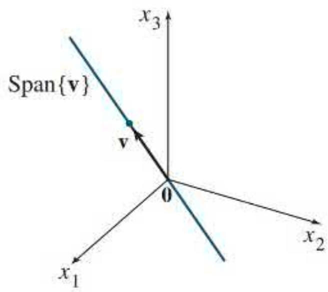


FIGURE 1

Here x_3 is factored out of the expression for the general solution vector. This shows that every solution of $A\mathbf{x} = \mathbf{0}$ in this case is a scalar multiple of $\mathbf{v}$. The trivial solution is obtained by choosing $x_3 = 0$. Geometrically, the solution set is a line through $\mathbf{0}$ in $\mathbb{R}^3$. See Figure 1. ■

Notice that a nontrivial solution $\mathbf{x}$ can have some zero entries so long as not all of its entries are zero.

EXAMPLE 2 A single linear equation can be treated as a very simple system of equations. Describe all solutions of the homogeneous "system"

$$10x_1 - 3x_2 - 2x_3 = 0 \tag{1}$$

SOLUTION There is no need for matrix notation. Solve for the basic variable x_1 in terms of the free variables. The general solution is $x_1 = .3x_2 + .2x_3$, with x_2 and x_3 free. As a vector, the general solution is

$$\begin{aligned}\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix} \\ &= x_2 \underset{\substack{\uparrow \\ \mathbf{u}}}{\begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix}} + x_3 \underset{\substack{\uparrow \\ \mathbf{v}}}{\begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix}} \quad \text{(with } x_2, x_3 \text{ free)} \end{aligned} \tag{2}$$

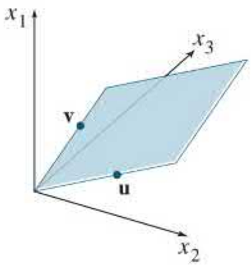


FIGURE 2

This calculation shows that every solution of (1) is a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$, shown in (2). That is, the solution set is Span $\{\mathbf{u}, \mathbf{v}\}$. Since neither $\mathbf{u}$ nor $\mathbf{v}$ is a scalar multiple of the other, the solution set is a plane through the origin. See Figure 2. ■

Examples 1 and 2, along with the exercises, illustrate the fact that the solution set of a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can always be expressed explicitly as Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ for suitable vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$. If the only solution is the zero vector, then the solution set is Span $\{\mathbf{0}\}$. If the equation $A\mathbf{x} = \mathbf{0}$ has only one free variable, the solution set is a line through the origin, as in Figure 1. A plane through the origin, as in Figure 2, provides a good mental image for the solution set of $A\mathbf{x} = \mathbf{0}$ when there are two or more free variables. Note, however, that a similar figure can be used to visualize Span $\{\mathbf{u}, \mathbf{v}\}$ even when $\mathbf{u}$ and $\mathbf{v}$ do not arise as solutions of $A\mathbf{x} = \mathbf{0}$. See Figure 11 in Section 1.3.

Parametric Vector Form

The original equation (1) for the plane in Example 2 is an *implicit* description of the plane. Solving this equation amounts to finding an *explicit* description of the plane as the set spanned by $\mathbf{u}$ and $\mathbf{v}$. Equation (2) is called a **parametric vector equation** of the plane. Sometimes such an equation is written as

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} \quad (s, t \text{ in } \mathbb{R})$$

to emphasize that the parameters vary over all real numbers. In Example 1, the equation $\mathbf{x} = x_3\mathbf{v}$ (with x_3 free), or $\mathbf{x} = t\mathbf{v}$ (with t in $\mathbb{R}$), is a parametric vector equation of a line. Whenever a solution set is described explicitly with vectors as in Examples 1 and 2, we say that the solution is in **parametric vector form**.

Solutions of Nonhomogeneous Systems

When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

EXAMPLE 3 Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

SOLUTION Here A is the matrix of coefficients from Example 1. Row operations on $[\,A \quad \mathbf{b}\,]$ produce

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 \qquad - \tfrac{4}{3}x_3 &= -1 \\ x_2 \qquad\qquad &= 2 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free. As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underset{\mathbf{p}}{\underset{\uparrow}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}} + x_3 \underset{\mathbf{v}}{\underset{\uparrow}{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}}$$

The equation $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter,

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \tag{3}$$

describes the solution set of $A\mathbf{x} = \mathbf{b}$ in parametric vector form. Recall from Example 1 that the solution set of $A\mathbf{x} = \mathbf{0}$ has the parametric vector equation

$$\mathbf{x} = t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \tag{4}$$

[with the same $\mathbf{v}$ that appears in (3)]. Thus the solutions of $A\mathbf{x} = \mathbf{b}$ are obtained by adding the vector $\mathbf{p}$ to the solutions of $A\mathbf{x} = \mathbf{0}$. The vector $\mathbf{p}$ itself is just one particular solution of $A\mathbf{x} = \mathbf{b}$ [corresponding to $t = 0$ in (3)]. ■

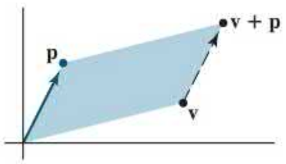


FIGURE 3
Adding $\mathbf{p}$ to $\mathbf{v}$ translates $\mathbf{v}$ to $\mathbf{v} + \mathbf{p}$.

To describe the solution set of $A\mathbf{x} = \mathbf{b}$ geometrically, we can think of vector addition as a *translation*. Given $\mathbf{v}$ and $\mathbf{p}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the effect of adding $\mathbf{p}$ to $\mathbf{v}$ is to *move* $\mathbf{v}$ in a direction parallel to the line through $\mathbf{p}$ and $\mathbf{0}$. We say that $\mathbf{v}$ is **translated by $\mathbf{p}$** to $\mathbf{v} + \mathbf{p}$. See Figure 3. If each point on a line L in $\mathbb{R}^2$ or $\mathbb{R}^3$ is translated by a vector $\mathbf{p}$, the result is a line parallel to L. See Figure 4.

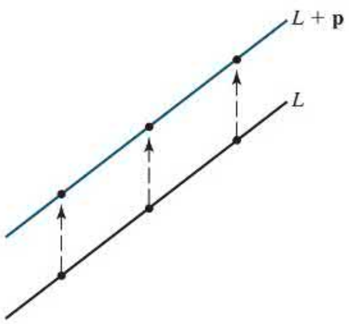


FIGURE 4
Translated line.

Suppose L is the line through $\mathbf{0}$ and $\mathbf{v}$, described by equation (4). Adding $\mathbf{p}$ to each point on L produces the translated line described by equation (3). Note that $\mathbf{p}$ is on the line in equation (3). We call (3) **the equation of the line through $\mathbf{p}$ parallel to $\mathbf{v}$**. Thus *the solution set of* $A\mathbf{x} = \mathbf{b}$ *is a line through* $\mathbf{p}$ *parallel to the solution set of* $A\mathbf{x} = \mathbf{0}$. Figure 5 illustrates this case.

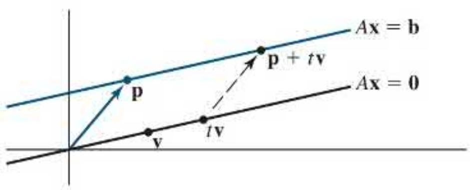


FIGURE 5 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

The relation between the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ shown in Figure 5 generalizes to any *consistent* equation $A\mathbf{x} = \mathbf{b}$, although the solution set will be larger than a line when there are several free variables. The following theorem gives the precise statement. See Exercise 37 at the end of this section for a proof.

THEOREM 6

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Theorem 6 says that if $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, using any particular solution $\mathbf{p}$ of $A\mathbf{x} = \mathbf{b}$ for the translation. Figure 6 illustrates the case in which there are two free variables. Even when $n > 3$, our mental image of the solution set of a consistent system $A\mathbf{x} = \mathbf{b}$ (with $\mathbf{b} \neq \mathbf{0}$) is either a single nonzero point or a line or plane not passing through the origin.

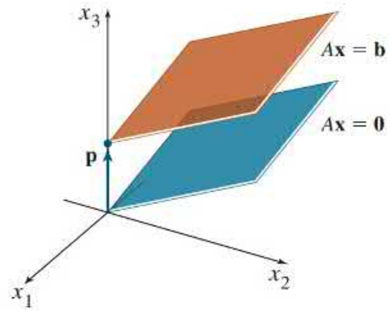


FIGURE 6 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

Warning: Theorem 6 and Figure 6 apply only to an equation $A\mathbf{x} = \mathbf{b}$ that has at least one nonzero solution $\mathbf{p}$. When $A\mathbf{x} = \mathbf{b}$ has no solution, the solution set is empty.

The following algorithm outlines the calculations shown in Examples 1, 2, and 3.

WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Reasonable Answers

To verify that the solutions you found are indeed solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$, simply multiply the matrix by each vector in your solution and check that the result is the zero vector. For example, if $A = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & -1 & 2 & 5 \\ 0 & 1 & 1 & 3 \end{bmatrix}$, and you found the homogeneous solutions to

Reasonable Answers (Continued)

be $x_3\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-8\\-3\\0\\1\end{bmatrix}$, check $\begin{bmatrix}1&-2&1&2\\1&-1&2&5\\0&1&1&3\end{bmatrix}\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1&-2&1&2\\1&-1&2&5\\0&1&1&3\end{bmatrix}\begin{bmatrix}-8\\-3\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. Then $A\left(x_3\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-8\\-3\\0\\1\end{bmatrix}\right)$ $=x_3A\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}+x_4A\begin{bmatrix}-8\\-3\\0\\1\end{bmatrix}$, which is equal to $x_3\begin{bmatrix}0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, as desired.

If you are solving $A\mathbf{x}=\mathbf{b}$, then you can again verify that you have correct solutions by multiplying the matrix by each vector in your solutions. The product of A with the first vector (the one that is *not* part of the solution to the homogeneous equation) should be $\mathbf{b}$. The product of A with the remaining vectors (the ones that are part of the solution to the homogeneous equation) should of course be $\mathbf{0}$.

For example, to verify that $\begin{bmatrix}2\\1\\1\\2\end{bmatrix}+x_3\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-8\\-3\\0\\1\end{bmatrix}$ are solutions to $A\mathbf{x}=\begin{bmatrix}5\\13\\8\end{bmatrix}$, check $\begin{bmatrix}1&-2&1&2\\1&-1&2&5\\0&1&1&3\end{bmatrix}\begin{bmatrix}2\\1\\1\\2\end{bmatrix}=\begin{bmatrix}5\\13\\8\end{bmatrix}$, and use the calculations from above. Notice $A\left(\begin{bmatrix}2\\1\\1\\2\end{bmatrix}+x_3\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-8\\-3\\0\\1\end{bmatrix}\right)$ $=A\begin{bmatrix}2\\1\\1\\2\end{bmatrix}+x_3A\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}+x_4A\begin{bmatrix}-8\\-3\\0\\1\end{bmatrix}$, which is equal to $\begin{bmatrix}5\\13\\8\end{bmatrix}+x_3\begin{bmatrix}0\\0\\0\end{bmatrix}$ $+x_4\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}5\\13\\8\end{bmatrix}$, as desired.

Practice Problems

1. Each of the following equations determines a plane in $\mathbb{R}^3$. Do the two planes intersect? If so, describe their intersection.

$$\begin{aligned} x_1 + 4x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 + 8x_3 &= 9 \end{aligned}$$

2. Write the general solution of $10x_1 - 3x_2 - 2x_3 = 7$ in parametric vector form, and relate the solution set to the one found in Example 2.
3. Prove the first part of Theorem 6: Suppose that $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$, so that $A\mathbf{p} = \mathbf{b}$. Let $\mathbf{v}_h$ be any solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$, and let $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$. Show that $\mathbf{w}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

1.5 Exercises

In Exercises 1–4, determine if the system has a nontrivial solution. Try to use as few row operations as possible.

1. $$\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$$

2. $$\begin{aligned} x_1 - 3x_2 + 7x_3 &= 0 \\ -2x_1 + x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 + 9x_3 &= 0 \end{aligned}$$

3. $$\begin{aligned} -3x_1 + 5x_2 - 7x_3 &= 0 \\ -6x_1 + 7x_2 + x_3 &= 0 \end{aligned}$$

4. $$\begin{aligned} -5x_1 + 7x_2 + 9x_3 &= 0 \\ x_1 - 2x_2 + 6x_3 &= 0 \end{aligned}$$

In Exercises 5 and 6, follow the method of Examples 1 and 2 to write the solution set of the given homogeneous system in parametric vector form.

5. $$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ -4x_1 - 9x_2 + 2x_3 &= 0 \\ -3x_2 - 6x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 0 \\ x_1 + 4x_2 - 8x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

In Exercises 7–12, describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where A is row equivalent to the given matrix.

7. $\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

You may find it helpful to review the information in the Reasonable Answers box from this section before answering Exercises 13–16.

13. Verify that the solutions you found to Exercise 9 are indeed homogeneous solutions.

14. Verify that the solutions you found to Exercise 10 are indeed homogeneous solutions.

15. Verify that the solutions you found to Exercise 11 are indeed homogeneous solutions.

16. Verify that the solutions you found to Exercise 12 are indeed homogeneous solutions.

17. Suppose the solution set of a certain system of linear equations can be described as $x_1 = 5 + 4x_3$, $x_2 = -2 - 7x_3$, with x_3 free. Use vectors to describe this set as a line in $\mathbb{R}^3$.

18. Suppose the solution set of a certain system of linear equations can be described as $x_1 = 3x_4$, $x_2 = 8 + x_4$, $x_3 = 2 - 5x_4$, with x_4 free. Use vectors to describe this set as a line in $\mathbb{R}^4$.

19. Follow the method of Example 3 to describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to that in Exercise 5.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

20. As in Exercise 19, describe the solutions of the following system in parametric vector form, and provide a geometric comparison with the solution set in Exercise 6.

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6 \end{aligned}$$

21. Describe and compare the solution sets of $x_1 + 9x_2 - 4x_3 = 0$ and $x_1 + 9x_2 - 4x_3 = -2$.

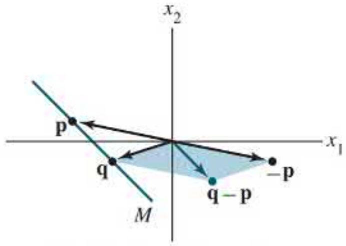
22. Describe and compare the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ and $x_1 - 3x_2 + 5x_3 = 4$.

In Exercises 23 and 24, find the parametric equation of the line through $\mathbf{a}$ parallel to $\mathbf{b}$.

23. $\mathbf{a} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ 24. $\mathbf{a} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -7 \\ 8 \end{bmatrix}$

In Exercises 25 and 26, find a parametric equation of the line M through $\mathbf{p}$ and $\mathbf{q}$. [*Hint:* M is parallel to the vector $\mathbf{q} - \mathbf{p}$. See the figure below.]

25. $\mathbf{p} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ 26. $\mathbf{p} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$



The line through $\mathbf{p}$ and $\mathbf{q}$.

In Exercises 27–36, mark each statement True or False **(T/F)**. Justify each answer.

27. **(T/F)** A homogeneous equation is always consistent.

28. **(T/F)** If $\mathbf{x}$ is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$, then every entry in $\mathbf{x}$ is nonzero.

29. **(T/F)** The equation $A\mathbf{x} = \mathbf{0}$ gives an explicit description of its solution set.

30. **(T/F)** The equation $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$, with x_2 and x_3 free (and neither $\mathbf{u}$ nor $\mathbf{v}$ a multiple of the other), describes a plane through the origin.

31. **(T/F)** The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if the equation has at least one free variable.

32. **(T/F)** The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution.

33. **(T/F)** The equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ describes a line through $\mathbf{v}$ parallel to $\mathbf{p}$.

34. **(T/F)** The effect of adding $\mathbf{p}$ to a vector is to move the vector in a direction parallel to $\mathbf{p}$.

35. **(T/F)** The solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the equation $A\mathbf{x} = \mathbf{0}$.

36. **(T/F)** The solution set of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$.

37. Prove the second part of Theorem 6: Let $\mathbf{w}$ be any solution of $A\mathbf{x} = \mathbf{b}$, and define $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. Show that $\mathbf{v}_h$ is a solution of $A\mathbf{x} = \mathbf{0}$. This shows that every solution of $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, with $\mathbf{p}$ a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}_h$ a solution of $A\mathbf{x} = \mathbf{0}$.

38. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

39. Suppose A is the 3×3 *zero* matrix (with all zero entries). Describe the solution set of the equation $A\mathbf{x} = \mathbf{0}$.

40. If $\mathbf{b} \neq \mathbf{0}$, can the solution set of $A\mathbf{x} = \mathbf{b}$ be a plane through the origin? Explain.

In Exercises 41–44, (a) does the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution and (b) does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$?

41. A is a 3×3 matrix with three pivot positions.

42. A is a 3×3 matrix with two pivot positions.

43. A is a 3×2 matrix with two pivot positions.

44. A is a 2×4 matrix with two pivot positions.

45. Given $A = \begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}$, find one nontrivial solution of $A\mathbf{x} = \mathbf{0}$ by inspection. [*Hint:* Think of the equation $A\mathbf{x} = \mathbf{0}$ written as a vector equation.]

46. Given $A = \begin{bmatrix} 4 & -6 \\ -8 & 12 \\ 6 & -9 \end{bmatrix}$, find one nontrivial solution of $A\mathbf{x} = \mathbf{0}$ by inspection.

47. Construct a 3×3 nonzero matrix A such that the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

48. Construct a 3×3 nonzero matrix A such that the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

49. Construct a 2×2 matrix A such that the solution set of the equation $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b}$ in $\mathbb{R}^2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$. Why does this *not* contradict Theorem 6?

50. Suppose A is a 3×3 matrix and $\mathbf{y}$ is a vector in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ does *not* have a solution. Does there exist a vector $\mathbf{z}$ in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{z}$ has a unique solution? Discuss.

51. Let A be an $m \times n$ matrix and let $\mathbf{u}$ be a vector in $\mathbb{R}^n$ that satisfies the equation $A\mathbf{x} = \mathbf{0}$. Show that for any scalar c, the vector $c\mathbf{u}$ also satisfies $A\mathbf{x} = \mathbf{0}$. [That is, show that $A(c\mathbf{u}) = \mathbf{0}$.]

52. Let A be an $m \times n$ matrix, and let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$ with the property that $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Explain why $A(\mathbf{u} + \mathbf{v})$ must be the zero vector. Then explain why $A(c\mathbf{u} + d\mathbf{v}) = \mathbf{0}$ for each pair of scalars c and d.

Solutions to Practice Problems

1. Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad + 3x_3 &= 4 \\ x_2 - 2x_3 &= -1 \end{aligned}$$

Thus $x_1 = 4 - 3x_3$, $x_2 = -1 + 2x_3$, with x_3 free. The general solution in parametric vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 3x_3 \\ -1 + 2x_3 \\ x_3 \end{bmatrix} = \underset{\mathbf{p}}{\underset{\uparrow}{\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}}} + x_3 \underset{\mathbf{v}}{\underset{\uparrow}{\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}}}$$

The intersection of the two planes is the line through $\mathbf{p}$ in the direction of $\mathbf{v}$.

2. The augmented matrix $\begin{bmatrix} 10 & -3 & -2 & 7 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & -.3 & -.2 & .7 \end{bmatrix}$, and the general solution is $x_1 = .7 + .3x_2 + .2x_3$, with x_2 and x_3 free. That is,

$$\begin{aligned} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 + .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} .7 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix} \\ &= \quad \mathbf{p} \quad + \quad x_2\mathbf{u} \quad + \quad x_3\mathbf{v} \end{aligned}$$

The solution set of the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is the translated plane $\mathbf{p} + \text{Span}\,\{\mathbf{u}, \mathbf{v}\}$, which passes through $\mathbf{p}$ and is parallel to the solution set of the homogeneous equation in Example 2.

3. Using Theorem 5 from Section 1.4, notice

$$A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v}_h = \mathbf{b} + \mathbf{0} = \mathbf{b},$$

hence $\mathbf{p} + \mathbf{v}_h$ is a solution to $A\mathbf{x} = \mathbf{b}$.

1.6 Applications of Linear Systems

You might expect that a real-life problem involving linear algebra would have only one solution, or perhaps no solution. The purpose of this section is to show how linear systems with many solutions can arise naturally. The applications here come from economics, chemistry, and network flow.

A Homogeneous System in Economics

The system of 500 equations in 500 variables, mentioned in this chapter's introduction, is now known as a Leontief "input–output" (or "production") model.[1] Section 2.6 will examine this model in more detail, when more theory and better notation are available. For now, we look at a simpler "exchange model," also due to Leontief.

[1] See Wassily W. Leontief, "Input–Output Economics," *Scientific American*, October 1951, pp. 15–21.

Suppose a nation's economy is divided into many sectors, such as various manufacturing, communication, entertainment, and service industries. Suppose that for each sector we know its total output for one year and we know exactly how this output is divided or "exchanged" among the other sectors of the economy. Let the total dollar value of a sector's output be called the **price** of that output. Leontief proved the following result.

> There exist *equilibrium prices* that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses.

The following example shows how to find the equilibrium prices.

EXAMPLE 1 Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as shown in Table 1, where the entries in a column represent the fractional parts of a sector's total output.

The second column of Table 1, for instance, says that the total output of the Electric sector is divided as follows: 40% to Coal, 50% to Steel, and the remaining 10% to Electric. (Electric treats this 10% as an expense it incurs in order to operate its business.) Since all output must be taken into account, the decimal fractions in each column must sum to 1.

Denote the prices (in dollar values) of the total annual outputs of the Coal, Electric, and Steel sectors by p_C, p_E, and p_S, respectively. If possible, find equilibrium prices that make each sector's income match its expenditures.

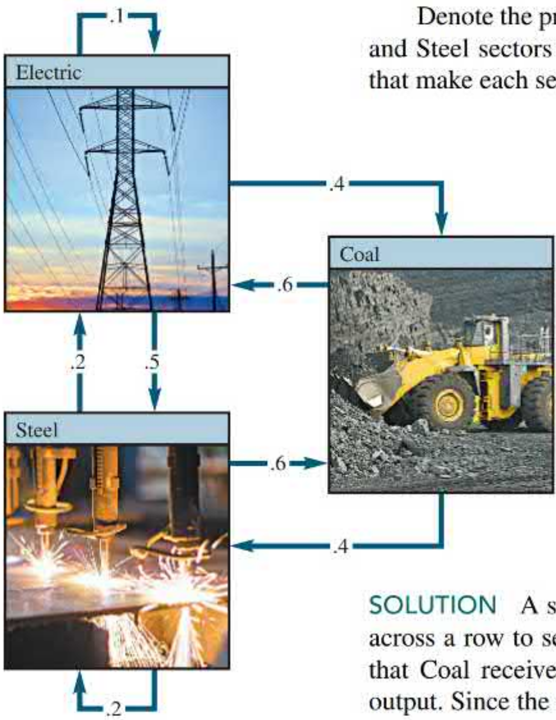


TABLE 1 A Simple Economy

Distribution of Output from			
Coal	**Electric**	**Steel**	**Purchased by**
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

SOLUTION A sector looks down a column to see where its output goes, and it looks across a row to see what it needs as inputs. For instance, the first row of Table 1 says that Coal receives (and pays for) 40% of the Electric output and 60% of the Steel output. Since the respective values of the total outputs are p_E and p_S, Coal must spend $.4p_E$ dollars for its share of Electric's output and $.6p_S$ for its share of Steel's output. Thus Coal's total expenses are $.4p_E + .6p_S$. To make Coal's income, p_C, equal to its expenses, we want

$$p_C = .4p_E + .6p_S \tag{1}$$

The second row of the exchange table shows that the Electric sector spends $.6p_C$ for coal, $.1p_E$ for electricity, and $.2p_S$ for steel. Hence the income/expense requirement for Electric is

$$p_E = .6p_C + .1p_E + .2p_S \tag{2}$$

Finally, the third row of the exchange table leads to the final requirement:

$$p_S = .4p_C + .5p_E + .2p_S \tag{3}$$

To solve the system of equations (1), (2), and (3), move all the unknowns to the left sides of the equations and combine like terms. [For instance, on the left side of (2), write $p_E - .1p_E$ as $.9p_E$.]

$$\begin{aligned} p_C - .4p_E - .6p_S &= 0 \\ -.6p_C + .9p_E - .2p_S &= 0 \\ -.4p_C - .5p_E + .8p_S &= 0 \end{aligned}$$

Row reduction is next. For simplicity here, decimals are rounded to two places.

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ -.4 & -.5 & .8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & -.66 & .56 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $p_C = .94p_S$, $p_E = .85p_S$, and p_S is free. The equilibrium price vector for the economy has the form

$$\mathbf{p} = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

Any (nonnegative) choice for p_S results in a choice of equilibrium prices. For instance, if we take p_S to be 100 (or \$100 million), then $p_C = 94$ and $p_E = 85$. The incomes and expenditures of each sector will be equal if the output of Coal is priced at \$94 million, that of Electric at \$85 million, and that of Steel at \$100 million. ■

Balancing Chemical Equations

Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns, the propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) and water (H_2O), according to an equation of the form

$$(x_1)C_3H_8 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O \tag{4}$$

To "balance" this equation, a chemist must find whole numbers $x_1, \ldots, x_4$ such that the total numbers of carbon (C), hydrogen (H), and oxygen (O) atoms on the left match the corresponding numbers of atoms on the right (because atoms are neither destroyed nor created in the reaction).

A systematic method for balancing chemical equations is to set up a vector equation that describes the numbers of atoms of each type present in a reaction. Since equation (4) involves three types of atoms (carbon, hydrogen, and oxygen), construct a vector in $\mathbb{R}^3$ for each reactant and product in (4) that lists the numbers of "atoms per molecule," as follows:

$$C_3H_8: \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, \; O_2: \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \; CO_2: \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \; H_2O: \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{Carbon} \\ \leftarrow \text{Hydrogen} \\ \leftarrow \text{Oxygen} \end{array}$$

To balance equation (4), the coefficients $x_1, \ldots, x_4$ must satisfy

$$x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} = x_3\begin{bmatrix}1\\0\\2\end{bmatrix} + x_4\begin{bmatrix}0\\2\\1\end{bmatrix}$$

To solve, move all the terms to the left (changing the signs in the third and fourth vectors):

$$x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} + x_3\begin{bmatrix}-1\\0\\-2\end{bmatrix} + x_4\begin{bmatrix}0\\-2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Row reduction of the augmented matrix for this equation leads to the general solution

$$x_1 = \tfrac{1}{4}x_4, \; x_2 = \tfrac{5}{4}x_4, \; x_3 = \tfrac{3}{4}x_4, \; \text{with } x_4 \text{ free}$$

Since the coefficients in a chemical equation must be integers, take $x_4 = 4$, in which case $x_1 = 1$, $x_2 = 5$, and $x_3 = 3$. The balanced equation is

$$C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$$

The equation would also be balanced if, for example, each coefficient were doubled. For most purposes, however, chemists prefer to use a balanced equation whose coefficients are the smallest possible whole numbers.

Network Flow

Systems of linear equations arise naturally when scientists, engineers, or economists study the flow of some quantity through a network. For instance, urban planners and traffic engineers monitor the pattern of traffic flow in a grid of city streets. Electrical engineers calculate current flow through electrical circuits. Economists analyze the distribution of products from manufacturers to consumers through a network of wholesalers and retailers. For many networks, the systems of equations involve hundreds or even thousands of variables and equations.

A *network* consists of a set of points called *junctions*, or *nodes*, with lines or arcs called *branches* connecting some or all of the junctions. The direction of flow in each branch is indicated, and the flow amount (or rate) is either shown or is denoted by a variable.

The basic assumption of network flow is that the total flow into the network equals the total flow out of the network and that the total flow into a junction equals the total flow out of the junction. For example, Figure 1 shows 30 units flowing into a junction through one branch, with x_1 and x_2 denoting the flows out of the junction through other branches. Since the flow is "conserved" at each junction, we must have $x_1 + x_2 = 30$. In a similar fashion, the flow at each junction is described by a linear equation. The problem of network analysis is to determine the flow in each branch when partial information (such as the flow into and out of the network) is known.

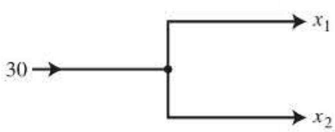


FIGURE 1
A junction or node.

EXAMPLE 2 The network in Figure 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

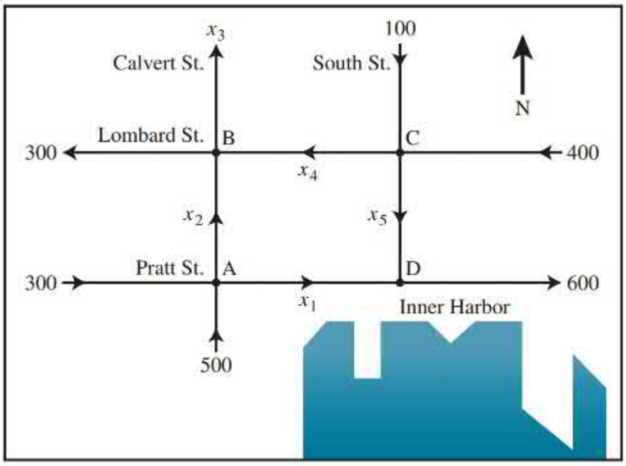


FIGURE 2 Baltimore streets.

SOLUTION Write equations that describe the flow, and then find the general solution of the system. Label the street intersections (junctions) and the unknown flows in the branches, as shown in Figure 2. At each intersection, set the flow in equal to the flow out.

Intersection	Flow in		Flow out
A	$300 + 500$	$=$	$x_1 + x_2$
B	$x_2 + x_4$	$=$	$300 + x_3$
C	$100 + 400$	$=$	$x_4 + x_5$
D	$x_1 + x_5$	$=$	600

Also, the total flow into the network $(500 + 300 + 100 + 400)$ equals the total flow out of the network $(300 + x_3 + 600)$, which simplifies to $x_3 = 400$. Combine this equation with a rearrangement of the first four equations to obtain the following system of equations:

$$\begin{aligned} x_1 + x_2 \qquad\qquad\qquad &= 800 \\ x_2 - x_3 + x_4 \qquad &= 300 \\ x_4 + x_5 &= 500 \\ x_1 \qquad\qquad\qquad + x_5 &= 600 \\ x_3 \qquad\qquad &= 400 \end{aligned}$$

Row reduction of the associated augmented matrix leads to

$$\begin{aligned} x_1 \qquad\qquad + x_5 &= 600 \\ x_2 \qquad - x_5 &= 200 \\ x_3 \qquad &= 400 \\ x_4 + x_5 &= 500 \end{aligned}$$

The general flow pattern for the network is described by

$$\begin{cases} x_1 = 600 - x_5 \\ x_2 = 200 + x_5 \\ x_3 = 400 \\ x_4 = 500 - x_5 \\ x_5 \text{ is free} \end{cases}$$

A negative flow in a network branch corresponds to flow in the direction opposite to that shown on the model. Since the streets in this problem are one way, none of the variables here can be negative. This fact leads to certain limitations on the possible values of the variables. For instance, $x_5 \leq 500$ because x_4 cannot be negative. Other constraints on the variables are considered in Practice Problem 2. ■

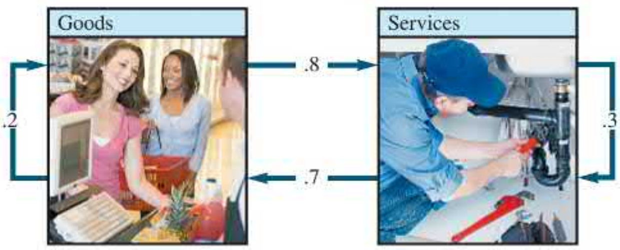
Practice Problems

1. Suppose an economy has three sectors: Agriculture, Mining, and Manufacturing. Agriculture sells 5% of its output to Mining and 30% to Manufacturing, and retains the rest. Mining sells 20% of its output to Agriculture and 70% to Manufacturing, and retains the rest. Manufacturing sells 20% of its output to Agriculture and 30% to Mining, and retains the rest. Determine the exchange table for this economy, where the columns describe how the output of each sector is exchanged among the three sectors.

2. Consider the network flow studied in Example 2. Determine the possible range of values of x_1 and x_2. [*Hint:* The example showed that $x_5 \leq 500$. What does this imply about x_1 and x_2? Also, use the fact that $x_5 \geq 0$.]

1.6 Exercises

1. Suppose an economy has only two sectors, Goods and Services. Each year, Goods sells 80% of its output to Services and keeps the rest, while Services sells 70% of its output to Goods and retains the rest. Find equilibrium prices for the annual outputs of the Goods and Services sectors that make each sector's income match its expenditures.



2. Find another set of equilibrium prices for the economy in Example 1. Suppose the same economy used Japanese yen instead of dollars to measure the value of the various sectors' outputs. Would this change the problem in any way? Discuss.

3. Consider an economy with three sectors, Chemicals & Metals, Fuels & Power, and Machinery. Chemicals sells 30% of its output to Fuels and 50% to Machinery and retains the rest. Fuels sells 80% of its output to Chemicals and 10% to Machinery and retains the rest. Machinery sells 40% to Chemicals and 40% to Fuels and retains the rest.

   a. Construct the exchange table for this economy.

   b. Develop a system of equations that leads to prices at which each sector's income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices.

   c. Find a set of equilibrium prices when the price for the Machinery output is 100 units.

4. Suppose an economy has four sectors, Agriculture (A), Energy (E), Manufacturing (M), and Transportation (T). Sector A sells 10% of its output to E and 25% to M and retains the rest. Sector E sells 30% of its output to A, 35% to M, and 25% to T and retains the rest. Sector M sells 30% of its output to A, 15% to E, and 40% to T and retains the rest. Sector T sells 20% of its output to A, 10% to E, and 30% to M and retains the rest.

   a. Construct the exchange table for this economy.

   b. Find a set of equilibrium prices for the economy.

Balance the chemical equations in Exercises 5–10 using the vector equation approach discussed in this section.

5. Boron sulfide reacts violently with water to form boric acid and hydrogen sulfide gas (the smell of rotten eggs). The unbalanced equation is

   $B_2S_3 + H_2O \rightarrow H_3BO_3 + H_2S$

   [For each compound, construct a vector that lists the numbers of atoms of boron, sulfur, hydrogen, and oxygen.]

6. When solutions of sodium phosphate and barium nitrate are mixed, the result is barium phosphate (as a precipitate) and sodium nitrate. The unbalanced equation is

   $Na_3PO_4 + Ba(NO_3)_2 \rightarrow Ba_3(PO_4)_2 + NaNO_3$

   [For each compound, construct a vector that lists the numbers of atoms of sodium (Na), phosphorus, oxygen, barium, and nitrogen. For instance, barium nitrate corresponds to $(0, 0, 6, 1, 2)$.]

7. Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide (gas):

$$NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + H_2O + CO_2$$

8. The following reaction between potassium permanganate ($KMnO_4$) and manganese sulfate in water produces manganese dioxide, potassium sulfate, and sulfuric acid:

$$KMnO_4 + MnSO_4 + H_2O \rightarrow MnO_2 + K_2SO_4 + H_2SO_4$$

[For each compound, construct a vector that lists the numbers of atoms of potassium (K), manganese, oxygen, sulfur, and hydrogen.]

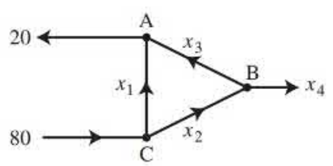
T 9. If possible, use exact arithmetic or rational format for calculations in balancing the following chemical reaction:

$$PbN_6 + CrMn_2O_8 \rightarrow Pb_3O_4 + Cr_2O_3 + MnO_2 + NO$$

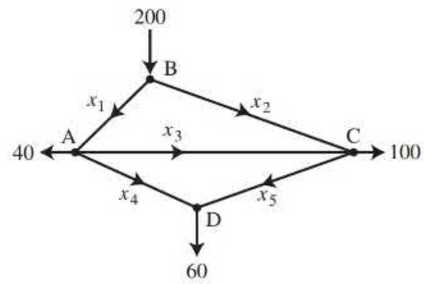
T 10. The chemical reaction below can be used in some industrial processes, such as the production of arsene (AsH_3). Use exact arithmetic or rational format for calculations to balance this equation.

$$MnS + As_2Cr_{10}O_{35} + H_2SO_4 \rightarrow HMnO_4 + AsH_3 + CrS_3O_{12} + H_2O$$

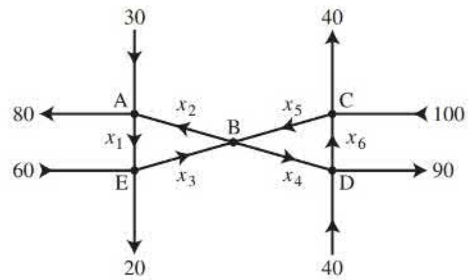
11. Find the general flow pattern of the network shown in the figure. Assuming that the flows are all nonnegative, what is the largest possible value for x_3?



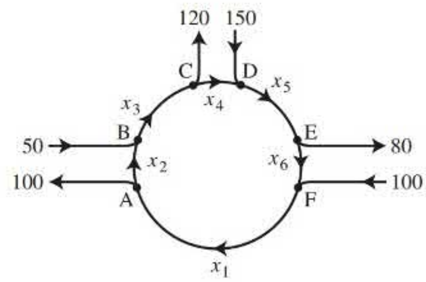
12. a. Find the general traffic pattern in the freeway network shown in the figure. (Flow rates are in cars/minute.)
    b. Describe the general traffic pattern when the road whose flow is x_4 is closed.
    c. When $x_4 = 0$, what is the minimum value of x_1?



13. a. Find the general flow pattern in the network shown in the figure.
    b. Assuming that the flow must be in the directions indicated, find the minimum flows in the branches denoted by x_2, x_3, x_4, and x_5.



14. Intersections in England are often constructed as one-way "roundabouts," such as the one shown in the figure. Assume that traffic must travel in the directions shown. Find the general solution of the network flow. Find the smallest possible value for x_6.



Solutions to Practice Problems

1. Write the percentages as decimals. Since all output must be taken into account, each column must sum to 1. This fact helps to fill in any missing entries.

Distribution of Output from			
Agriculture	**Mining**	**Manufacturing**	**Purchased by**
.65	.20	.20	Agriculture
.05	.10	.30	Mining
.30	.70	.50	Manufacturing

Solutions to Practice Problems (Continued)

2. Since $x_5 \leq 500$, the equations D and A for x_1 and x_2 imply that $x_1 \geq 100$ and $x_2 \leq 700$. The fact that $x_5 \geq 0$ implies that $x_1 \leq 600$ and $x_2 \geq 200$. So, $100 \leq x_1 \leq 600$, and $200 \leq x_2 \leq 700$.

1.7 Linear Independence

The homogeneous equations in Section 1.5 can be studied from a different perspective by writing them as vector equations. In this way, the focus shifts from the unknown solutions of $A\mathbf{x} = \mathbf{0}$ to the vectors that appear in the vector equations.

For instance, consider the equation

$$x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix} + x_3\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \tag{1}$$

This equation has a trivial solution, of course, where $x_1 = x_2 = x_3 = 0$. As in Section 1.5, the main issue is whether the trivial solution is the *only one*.

DEFINITION

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{2}$$

Equation (2) is called a **linear dependence relation** among $\mathbf{v}_1, \dots, \mathbf{v}_p$ when the weights are not all zero. An indexed set is linearly dependent if and only if it is not linearly independent. For brevity, we may say that $\mathbf{v}_1, \dots, \mathbf{v}_p$ are linearly dependent when we mean that $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a linearly dependent set. We use analogous terminology for linearly independent sets.

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}4\\5\\6\end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$.

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

SOLUTION

a. We must determine if there is a nontrivial solution of equation (1) above. Row operations on the associated augmented matrix show that

$$\begin{bmatrix}1&4&2&0\\2&5&1&0\\3&6&0&0\end{bmatrix} \sim \begin{bmatrix}1&4&2&0\\0&-3&-3&0\\0&0&0&0\end{bmatrix}$$

Clearly, x_1 and x_2 are basic variables, and x_3 is free. Each nonzero value of x_3 determines a nontrivial solution of (1). Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent (and not linearly independent).

b. To find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, completely row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \qquad - 2x_3 &= 0 \\ x_2 + \ \ x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free. Choose any nonzero value for x_3—say, $x_3 = 5$. Then $x_1 = 10$ and $x_2 = -5$. Substitute these values into equation (1) and obtain

$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$$

This is one (out of infinitely many) possible linear dependence relations among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. ■

Linear Independence of Matrix Columns

Suppose that we begin with a matrix $A = [\,\mathbf{a}_1 \ \cdots \ \mathbf{a}_n\,]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. Thus we have the following important fact.

> The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

EXAMPLE 2 Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

SOLUTION To study $A\mathbf{x} = \mathbf{0}$, row reduce the augmented matrix:

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

At this point, it is clear that there are three basic variables and no free variables. So the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and the columns of A are linearly independent. ■

Sets of One or Two Vectors

A set containing only one vector—say, $\mathbf{v}$—is linearly independent if and only if $\mathbf{v}$ is not the zero vector. This is because the vector equation $x_1\mathbf{v} = \mathbf{0}$ has only the trivial solution when $\mathbf{v} \neq \mathbf{0}$. The zero vector is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

The next example will explain the nature of a linearly dependent set of two vectors.

EXAMPLE 3 Determine if the following sets of vectors are linearly independent.

a. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ b. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

SOLUTION

a. Notice that $\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$, namely $\mathbf{v}_2 = 2\mathbf{v}_1$. Hence $-2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$, which shows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent.

b. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are certainly *not* multiples of one another. Could they be linearly dependent? Suppose c and d satisfy

$$c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}$$

If $c \neq 0$, then we can solve for $\mathbf{v}_1$ in terms of $\mathbf{v}_2$, namely $\mathbf{v}_1 = (-d/c)\mathbf{v}_2$. This result is impossible because $\mathbf{v}_1$ is *not* a multiple of $\mathbf{v}_2$. So c must be zero. Similarly, d must also be zero. Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set. ■

The arguments in Example 3 show that you can always decide *by inspection* when a set of two vectors is linearly dependent. Row operations are unnecessary. Simply check whether at least one of the vectors is a scalar times the other. (The test applies only to sets of *two* vectors.)

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

In geometric terms, two vectors are linearly dependent if and only if they lie on the same line through the origin. Figure 1 shows the vectors from Example 3.

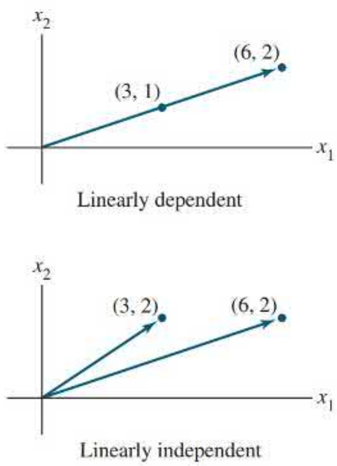


FIGURE 1

Sets of Two or More Vectors

The proof of the next theorem is similar to the solution of Example 3. Details are given at the end of this section.

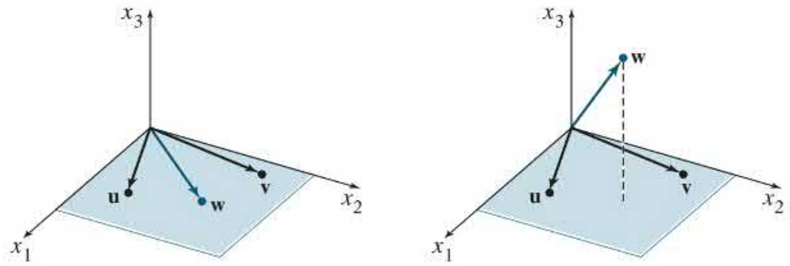
THEOREM 7 **Characterization of Linearly Dependent Sets**

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

Warning: Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors. See Practice Problem 1(c).

EXAMPLE 4 Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by $\mathbf{u}$ and $\mathbf{v}$, and explain why a vector $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

SOLUTION The vectors **u** and **v** are linearly independent because neither vector is a multiple of the other, and so they span a plane in $\mathbb{R}^3$. (See Section 1.3.) In fact, Span $\{\mathbf{u}, \mathbf{v}\}$ is the x_1x_2-plane (with $x_3 = 0$). If **w** is a linear combination of **u** and **v**, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, by Theorem 7. Conversely, suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. By Theorem 7, some vector in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linear combination of the preceding vectors (since $\mathbf{u} \neq \mathbf{0}$). That vector must be **w**, since **v** is not a multiple of **u**. So **w** is in Span $\{\mathbf{u}, \mathbf{v}\}$. See Figure 2. ■



Linearly dependent, **w** in Span $\{\mathbf{u}, \mathbf{v}\}$

Linearly independent, **w** *not* in Span $\{\mathbf{u}, \mathbf{v}\}$

FIGURE 2 Linear dependence in $\mathbb{R}^3$.

Example 4 generalizes to any set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^3$ with **u** and **v** linearly independent. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ will be linearly dependent if and only if **w** is in the plane spanned by **u** and **v**.

The next two theorems describe special cases in which the linear dependence of a set is automatic. Moreover, Theorem 8 will be a key result for work in later chapters.

THEOREM 8

> If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

FIGURE 3
If $p > n$, the columns are linearly dependent.

PROOF Let $A = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_p]$. Then A is $n \times p$, and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of n equations in p unknowns. If $p > n$, there are more variables than equations, so there must be a free variable. Hence $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and the columns of A are linearly dependent. See Figure 3 for a matrix version of this theorem. ■

Warning: Theorem 8 says nothing about the case in which the number of vectors in the set does *not* exceed the number of entries in each vector.

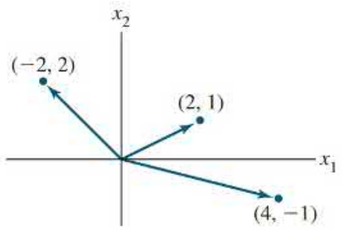


FIGURE 4
A linearly dependent set in $\mathbb{R}^2$.

EXAMPLE 5 The vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are linearly dependent by Theorem 8, because there are three vectors in the set and there are only two entries in each vector. Notice, however, that none of the vectors is a multiple of one of the other vectors. See Figure 4. ■

THEOREM 9

> If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

PROOF By renumbering the vectors, we may suppose $\mathbf{v}_1 = \mathbf{0}$. Then the equation $1\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p = \mathbf{0}$ shows that S is linearly dependent. ■

EXAMPLE 6 Determine by inspection if the given set is linearly dependent.

a. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$ b. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$ c. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

SOLUTION

a. The set contains four vectors, each of which has only three entries. So the set is linearly dependent by Theorem 8.

b. Theorem 8 does not apply here because the number of vectors does not exceed the number of entries in each vector. Since the zero vector is in the set, the set is linearly dependent by Theorem 9.

c. Compare the corresponding entries of the two vectors. The second vector seems to be $-3/2$ times the first vector. This relation holds for the first three pairs of entries, but fails for the fourth pair. Thus neither of the vectors is a multiple of the other, and hence they are linearly independent. ■

In general, you should read a section thoroughly *several* times to absorb an important concept such as linear independence. The notes in the *Study Guide* for this section will help you learn to form mental images of key ideas in linear algebra. For instance, the following proof is worth reading carefully because it shows how the definition of linear independence can be *used*.

PROOF OF THEOREM 7 (Characterization of Linearly Dependent Sets)
If some $\mathbf{v}_j$ in S equals a linear combination of the other vectors, then $\mathbf{v}_j$ can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight (-1) on $\mathbf{v}_j$. [For instance, if $\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, then $\mathbf{0} = (-1)\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\mathbf{v}_4 + \cdots + 0\mathbf{v}_p$.] Thus S is linearly dependent.

Conversely, suppose S is linearly dependent. If $\mathbf{v}_1$ is zero, then it is a (trivial) linear combination of the other vectors in S. Otherwise, $\mathbf{v}_1 \neq \mathbf{0}$, and there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Let j be the largest subscript for which $c_j \neq 0$. If $j = 1$, then $c_1\mathbf{v}_1 = \mathbf{0}$, which is impossible because $\mathbf{v}_1 \neq \mathbf{0}$. So $j > 1$, and

$$c_1\mathbf{v}_1 + \cdots + c_j\mathbf{v}_j + 0\mathbf{v}_{j+1} + \cdots + 0\mathbf{v}_p = \mathbf{0}$$

$$c_j\mathbf{v}_j = -c_1\mathbf{v}_1 - \cdots - c_{j-1}\mathbf{v}_{j-1}$$

$$\mathbf{v}_j = \left(-\frac{c_1}{c_j}\right)\mathbf{v}_1 + \cdots + \left(-\frac{c_{j-1}}{c_j}\right)\mathbf{v}_{j-1} \quad ■$$

Practice Problems

1. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$.

a. Are the sets $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{u}, \mathbf{w}\}$, $\{\mathbf{u}, \mathbf{z}\}$, $\{\mathbf{v}, \mathbf{w}\}$, $\{\mathbf{v}, \mathbf{z}\}$, and $\{\mathbf{w}, \mathbf{z}\}$ each linearly independent? Why or why not?

b. Does the answer to Part (a) imply that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ is linearly independent?

c. To determine if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ is linearly dependent, is it wise to check if, say, $\mathbf{w}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{z}$?

d. Is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ linearly dependent?

2. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^n$ and $\mathbf{v}_4$ is a vector in $\mathbb{R}^n$. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also a linearly dependent set.

1.7 Exercises

In Exercises 1–4, determine if the vectors are linearly independent. Justify each answer.

1. $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}$

2. $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

4. $\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

In Exercises 5–8, determine if the columns of the matrix form a linearly independent set. Justify each answer.

5. $\begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}$

6. $\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -3 & 3 & -2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$

In Exercises 9 and 10, (a) for what values of h is $\mathbf{v}_3$ in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, and (b) for what values of h is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly *dependent*? Justify each answer.

9. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 10 \\ -6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ h \end{bmatrix}$

10. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -10 \\ h \end{bmatrix}$

In Exercises 11–14, find the value(s) of h for which the vectors are linearly *dependent*. Justify each answer.

11. $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$

12. $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$

13. $\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$

Determine by inspection whether the vectors in Exercises 15–20 are linearly *independent*. Justify each answer.

15. $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

16. $\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$

17. $\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}$

18. $\begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

19. $\begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$

20. $\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 21–28, mark each statement True or False (**T/F**). Justify each answer on the basis of a careful reading of the text.

21. **(T/F)** The columns of a matrix A are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

22. **(T/F)** Two vectors are linearly dependent if and only if they lie on a line through the origin.

23. **(T/F)** If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

24. **(T/F)** If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

25. **(T/F)** The columns of any 4×5 matrix are linearly dependent.

26. **(T/F)** If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\mathbf{z}$ is in Span $\{\mathbf{x}, \mathbf{y}\}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

27. **(T/F)** If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in Span $\{\mathbf{x}, \mathbf{y}\}$.

28. **(T/F)** If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more vectors than there are entries in each vector.

In Exercises 29–32, describe the possible echelon forms of the matrix. Use the notation of Example 1 in Section 1.2.

29. A is a 3×3 matrix with linearly independent columns.

30. A is a 2×2 matrix with linearly dependent columns.

31. A is a 4×2 matrix, $A = [\mathbf{a}_1 \quad \mathbf{a}_2]$, and $\mathbf{a}_2$ is not a multiple of $\mathbf{a}_1$.

32. A is a 4×3 matrix, $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$, such that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent and $\mathbf{a}_3$ is not in Span $\{\mathbf{a}_1, \mathbf{a}_2\}$.

33. How many pivot columns must a 7×5 matrix have if its columns are linearly independent? Why?

34. How many pivot columns must a 5×7 matrix have if its columns span $\mathbb{R}^5$? Why?

35. Construct 3×2 matrices A and B such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

36. a. Fill in the blank in the following statement: "If A is an $m \times n$ matrix, then the columns of A are linearly independent if and only if A has _____ pivot columns."

b. Explain why the statement in (a) is true.

Exercises 37 and 38 should be solved *without performing row operations*. [*Hint:* Write $A\mathbf{x} = \mathbf{0}$ as a vector equation.]

37. Given $A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$, observe that the third column is the sum of the first two columns. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

38. Given $A = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix}$, observe that the first column plus twice the second column equals the third column. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

Each statement in Exercises 39–44 is either true (in all cases) or false (for at least one example). If false, construct a specific example to show that the statement is not always true. Such an example is called a *counterexample* to the statement. If a statement is true, give a justification. (One specific example cannot explain why a statement is always true. You will have to do more work here than in Exercises 21–28.)

39. **(T/F-C)** If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

40. **(T/F-C)** If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3 = \mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

41. **(T/F-C)** If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^4$ and $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

42. **(T/F-C)** If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3$ is *not* a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.

43. **(T/F-C)** If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.

44. **(T/F-C)** If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are linearly independent vectors in $\mathbb{R}^4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent. [*Hint:* Think about $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0 \cdot \mathbf{v}_4 = \mathbf{0}$.]

45. Suppose A is an $m \times n$ matrix with the property that for all $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. Use the definition of linear independence to explain why the columns of A must be linearly independent.

46. Suppose an $m \times n$ matrix A has n pivot columns. Explain why for each $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. [*Hint:* Explain why $A\mathbf{x} = \mathbf{b}$ cannot have infinitely many solutions.]

T In Exercises 47 and 48, use as many columns of A as possible to construct a matrix B with the property that the equation $B\mathbf{x} = \mathbf{0}$ has only the trivial solution. Solve $B\mathbf{x} = \mathbf{0}$ to verify your work.

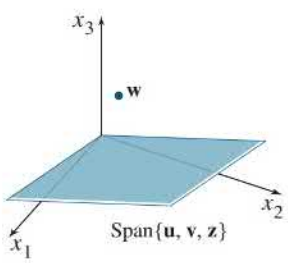
47. $A = \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix}$

48. $A = \begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$

T 49. With A and B as in Exercise 47 select a column $\mathbf{v}$ of A that was not used in the construction of B and determine if $\mathbf{v}$ is in the set spanned by the columns of B. (Describe your calculations.)

T 50. Repeat Exercise 49 with the matrices A and B from Exercise 48. Then give an explanation for what you discover, assuming that B was constructed as specified.

STUDY GUIDE offers additional resources for mastering the concept of linear independence.



Solutions to Practice Problems

1. a. Yes. In each case, neither vector is a multiple of the other. Thus each set is linearly independent.

b. No. The observation in Part (a), by itself, says nothing about the linear independence of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$.

c. No. When testing for linear independence, it is usually a poor idea to check if one selected vector is a linear combination of the others. It may happen that the selected vector is not a linear combination of the others and yet the whole set of vectors is linearly dependent. In this practice problem, $\mathbf{w}$ is not a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{z}$.

d. Yes, by Theorem 8. There are more vectors (four) than entries (three) in them.

2. Applying the definition of linearly dependent to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ implies that there exist scalars c_1, c_2, and c_3, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Adding $0\,\mathbf{v}_4 = \mathbf{0}$ to both sides of this equation results in

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\,\mathbf{v}_4 = \mathbf{0}.$$

Since c_1, c_2, c_3 and 0 are not *all* zero, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ satisfies the definition of a linearly dependent set.

1.8 Introduction to Linear Transformations

The difference between a matrix equation $A\mathbf{x} = \mathbf{b}$ and the associated vector equation $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ is merely a matter of notation. However, a matrix equation $A\mathbf{x} = \mathbf{b}$ can arise in linear algebra (and in applications such as computer graphics and signal processing) in a way that is not directly connected with linear combinations of vectors. This happens when we think of the matrix A as an object that "acts" on a vector $\mathbf{x}$ by multiplication to produce a new vector called $A\mathbf{x}$.

For instance, the equations

$$\underset{\overset{\uparrow}{A}}{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}} \underset{\overset{\uparrow}{\mathbf{x}}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} = \underset{\overset{\uparrow}{\mathbf{b}}}{\begin{bmatrix} 5 \\ 8 \end{bmatrix}} \quad \text{and} \quad \underset{\overset{\uparrow}{A}}{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}} \underset{\overset{\uparrow}{\mathbf{u}}}{\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}} = \underset{\overset{\uparrow}{\mathbf{0}}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

say that multiplication by A transforms $\mathbf{x}$ into $\mathbf{b}$ and transforms $\mathbf{u}$ into the zero vector. See Figure 1.

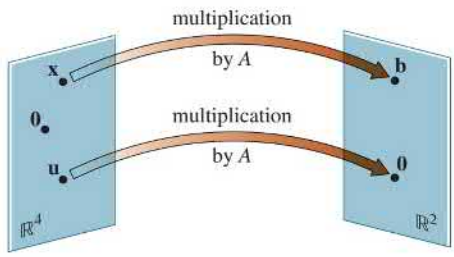


FIGURE 1 Transforming vectors via matrix multiplication.

From this new point of view, solving the equation $A\mathbf{x} = \mathbf{b}$ amounts to finding all vectors $\mathbf{x}$ in $\mathbb{R}^4$ that are transformed into the vector $\mathbf{b}$ in $\mathbb{R}^2$ under the "action" of multiplication by A.

The correspondence from $\mathbf{x}$ to $A\mathbf{x}$ is a *function* from one set of vectors to another. This concept generalizes the common notion of a function as a rule that transforms one real number into another.

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$