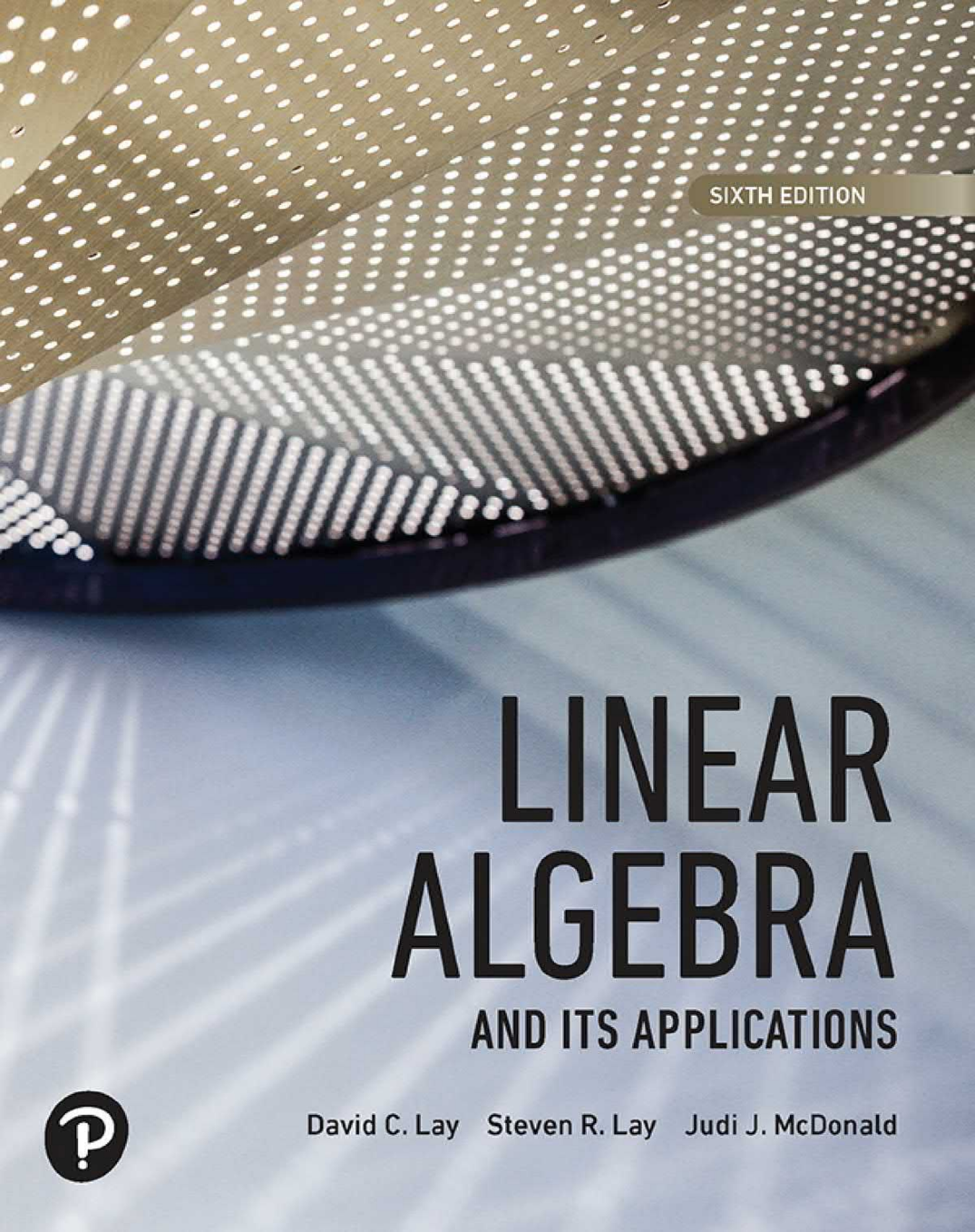


SIXTH EDITION

LINEAR ALGEBRA

AND ITS APPLICATIONS

David C. Lay Steven R. Lay Judi J. McDonald

2 Matrix Algebra

Introductory Example

COMPUTER MODELS IN AIRCRAFT DESIGN

To design the next generation of commercial and military aircraft, engineers at Boeing's Phantom Works use 3D modeling and computational fluid dynamics (CFD). They study the airflow around a virtual airplane to answer important design questions before physical models are created. This has drastically reduced design cycle times and cost—and linear algebra plays a crucial role in the process.

The virtual airplane begins as a mathematical "wire-frame" model that exists only in computer memory and on graphics display terminals. (Model of a Boeing 747 is shown.) This mathematical model organizes and influences each step of the design and manufacture of the airplane—both the exterior and interior. The CFD analysis concerns the exterior surface.

Although the finished skin of a plane may seem smooth, the geometry of the surface is complicated. In addition to wings and a fuselage, an aircraft has nacelles, stabilizers, slats, flaps, and ailerons. The way air flows around these structures determines how the plane moves through the sky. Equations that describe the airflow are complicated, and they must account for engine intake, engine exhaust, and the wakes left by the wings of the plane. To study the airflow, engineers need a highly refined description of the plane's surface.

A computer creates a model of the surface by first superimposing a three-dimensional grid of "boxes" on the original wire-frame model. Boxes in this grid lie either completely inside or completely outside the plane, or they intersect the surface of the plane. The computer selects the boxes that intersect the surface and subdivides them, retaining only the smaller boxes that still intersect the surface. The subdividing process is repeated until the grid is extremely fine. A typical grid can include more than 400,000 boxes.

The process for finding the airflow around the plane involves repeatedly solving a system of linear equations $A\mathbf{x} = \mathbf{b}$ that may involve up to 2 million equations and variables. The vector $\mathbf{b}$ changes each time, based on data from the grid and solutions of previous equations. Using the fastest computers available commercially, a Phantom Works team can spend from a few hours to several days setting up and solving a single airflow problem. After the team analyzes the solution, they may make small changes to the airplane surface and begin the whole process again. Thousands of CFD runs may be required.

This chapter presents two important concepts that assist in the solution of such massive systems of equations:

- *Partitioned matrices:* A typical CFD system of equations has a "sparse" coefficient matrix with mostly zero entries. Grouping the variables correctly leads to a partitioned matrix with many zero blocks. Section 2.4 introduces such matrices and describes some of their applications.

- *Matrix factorizations:* Even when written with partitioned matrices, the system of equations is complicated. To further simplify the computations, the CFD software at Boeing uses what is called an LU factorization of the coefficient matrix. Section 2.5 discusses LU and other useful matrix factorizations. Further details about factorizations appear at several points later in the text.

To analyze a solution of an airflow system, engineers want to visualize the airflow over the surface of the plane. They use computer graphics, and linear algebra provides the engine for the graphics. The wire-frame model of the plane's surface is stored as data in many matrices. Once the image has been rendered on a computer screen, engineers can change its scale, zoom in or out of small regions, and rotate the image to see parts that may be hidden from view. Each of these operations is accomplished by appropriate matrix multiplications. Section 2.7 explains the basic ideas.

TU-Delft and Air France-KLM are investigating a flying V aircraft design because of its potential for significantly better fuel economy.

Our ability to analyze and solve equations will be greatly enhanced when we can perform algebraic operations with matrices. Furthermore, the definitions and theorems in this chapter provide some basic tools for handling the many applications of linear algebra that involve two or more matrices. For $n \times n$ matrices, the Invertible Matrix Theorem in Section 2.3 ties together most of the concepts treated earlier in the text. Sections 2.4 and 2.5 examine partitioned matrices and matrix factorizations, which appear in most modern uses of linear algebra. Sections 2.6 and 2.7 describe two interesting applications of matrix algebra: to economics and to computer graphics. Sections 2.8 and 2.9 provide readers enough information about subspaces to move directly into Chapters 5, 6, and 7, without covering Chapter 4. You may want to omit these two sections if you plan to cover Chapter 4 before moving to Chapter 5.

2.1 Matrix Operations

If A is an $m \times n$ matrix—that is, a matrix with m rows and n columns—then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i, j)-entry of A. See Figure 1. For instance, the $(3, 2)$-entry is the number a_{32} in the third row, second column. Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$. Often, these columns are denoted by $\mathbf{a}_1, \dots, \mathbf{a}_n$, and the matrix A is written as

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

Observe that the number a_{ij} is the ith entry (from the top) of the jth column vector $\mathbf{a}_j$.

The **diagonal entries** in an $m \times n$ matrix $A = [\, a_{ij} \,]$ are $a_{11}, a_{22}, a_{33}, \dots$, and they form the **main diagonal** of A. A **diagonal matrix** is a square $n \times n$ matrix whose nondiagonal entries are zero. An example is the $n \times n$ identity matrix, I_n. An $m \times n$ matrix whose entries are all zero is a **zero matrix** and is written as 0. The size of a zero matrix is usually clear from the context.

$$\begin{array}{rc} & \begin{matrix} \phantom{a_{11}} & & \text{Column} & & \phantom{a_{1n}} \\ & & j & & \end{matrix} \\ \text{Row } i & \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A \\ & \begin{matrix} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{matrix} \end{array}$$

FIGURE 1 Matrix notation.

Sums and Scalar Multiples

The arithmetic for vectors described earlier has a natural extension to matrices. We say that two matrices are **equal** if they have the same size (i.e., the same number of rows and the same number of columns) and if their corresponding columns are equal, which amounts to saying that their corresponding entries are equal. If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B. Since vector addition of the columns is done entrywise, each entry in $A + B$ is the sum of the corresponding entries in A and B. The sum $A + B$ is defined only when A and B are the same size.

EXAMPLE 1 Let

$$A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$$

but $A + C$ is not defined because A and C have different sizes. ■

If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the corresponding columns in A. As with vectors, $-A$ stands for $(-1)A$, and $A - B$ is the same as $A + (-1)B$.

EXAMPLE 2 If A and B are the matrices in Example 1, then

$$2B = 2\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$$

$$A - 2B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}$$ ■

It was unnecessary in Example 2 to compute $A - 2B$ as $A + (-1)2B$ because the usual rules of algebra apply to sums and scalar multiples of matrices, as the following theorem shows.

THEOREM 1

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$
b. $(A + B) + C = A + (B + C)$
c. $A + 0 = A$
d. $r(A + B) = rA + rB$
e. $(r + s)A = rA + sA$
f. $r(sA) = (rs)A$

Each equality in Theorem 1 is verified by showing that the matrix on the left side has the same size as the matrix on the right and that corresponding columns are equal. Size is no problem because A, B, and C are equal in size. The equality of columns follows immediately from analogous properties of vectors. For instance, if the jth columns of A, B, and C are $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{c}_j$, respectively, then the jth columns of $(A+B)+C$ and $A+(B+C)$ are

$$(\mathbf{a}_j + \mathbf{b}_j) + \mathbf{c}_j \quad \text{and} \quad \mathbf{a}_j + (\mathbf{b}_j + \mathbf{c}_j)$$

respectively. Since these two vector sums are equal for each j, property (b) is verified.

Because of the associative property of addition, we can simply write $A+B+C$ for the sum, which can be computed either as $(A+B)+C$ or as $A+(B+C)$. The same applies to sums of four or more matrices.

Matrix Multiplication

When a matrix B multiplies a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(B\mathbf{x})$. See Figure 2.

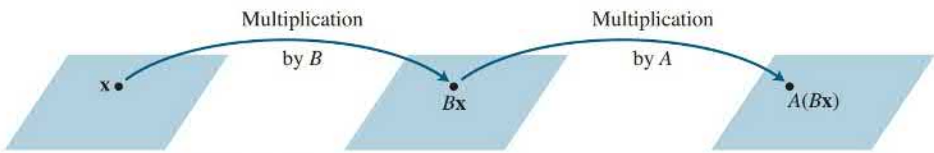


FIGURE 2 Multiplication by B and then A.

Thus $A(B\mathbf{x})$ is produced from $\mathbf{x}$ by a *composition* of mappings—the linear transformations studied in Section 1.8. Our goal is to represent this composite mapping as multiplication by a single matrix, denoted by AB, so that

$$A(B\mathbf{x}) = (AB)\mathbf{x} \tag{1}$$

See Figure 3.

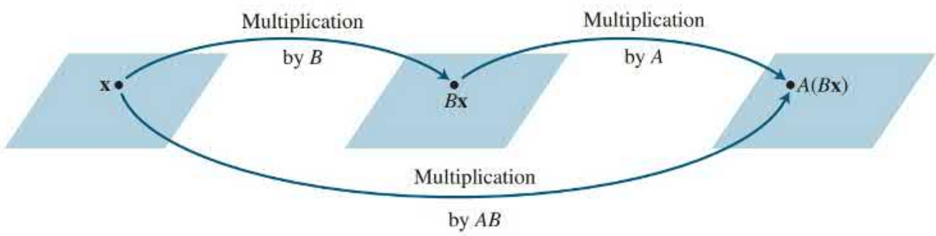


FIGURE 3 Multiplication by AB.

If A is $m \times n$, B is $n \times p$, and $\mathbf{x}$ is in $\mathbb{R}^p$, denote the columns of B by $\mathbf{b}_1, \dots, \mathbf{b}_p$ and the entries in $\mathbf{x}$ by $x_1, \dots, x_p$. Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + \cdots + x_p\mathbf{b}_p$$

By the linearity of multiplication by A,

$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1) + \cdots + A(x_p\mathbf{b}_p) \\ &= x_1 A\mathbf{b}_1 + \cdots + x_p A\mathbf{b}_p \end{aligned}$$

The vector $A(B\mathbf{x})$ is a linear combination of the vectors $A\mathbf{b}_1, \dots, A\mathbf{b}_p$, using the entries in $\mathbf{x}$ as weights. In matrix notation, this linear combination is written as

$$A(B\mathbf{x}) = [\, A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p \,]\mathbf{x}$$

Thus multiplication by $[\, A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p \,]$ transforms $\mathbf{x}$ into $A(B\mathbf{x})$. We have found the matrix we sought!

DEFINITION

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \dots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \dots, A\mathbf{b}_p$. That is,

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

This definition makes equation (1) true for all $\mathbf{x}$ in $\mathbb{R}^p$. Equation (1) proves that the composite mapping in Figure 3 is a linear transformation and that its standard matrix is AB. *Multiplication of matrices corresponds to composition of linear transformations.*

EXAMPLE 3 Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.

SOLUTION Write $B = [\, \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \,]$, and compute:

$$A\mathbf{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad A\mathbf{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad A\mathbf{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -1 \end{bmatrix} \qquad = \begin{bmatrix} 0 \\ 13 \end{bmatrix} \qquad = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

Then

$$AB = A[\, \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \,] = \begin{array}{c} \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{array} \end{array}$$

■

Notice that since the first column of AB is $A\mathbf{b}_1$, this column is a linear combination of the columns of A using the entries in $\mathbf{b}_1$ as weights. A similar statement is true for each column of AB.

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Obviously, the number of columns of A must match the number of rows in B in order for a linear combination such as $A\mathbf{b}_1$ to be defined. Also, the definition of AB shows that AB *has the same number of rows as* A *and the same number of columns as* B.

EXAMPLE 4 If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?

SOLUTION Since A has 5 columns and B has 5 rows, the product AB is defined and is a 3×2 matrix:

$$\underset{3\times 5}{\overset{A}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}}} \underset{5\times 2}{\overset{B}{\begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \\ * & * \end{bmatrix}}} = \underset{3\times 2}{\overset{AB}{\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}}}$$

Match

Size of AB

The product BA is *not* defined because the 2 columns of B do not match the 3 rows of A. ■

The definition of AB is important for theoretical work and applications, but the following rule provides a more efficient method for calculating the individual entries in AB when working small problems by hand.

ROW–COLUMN RULE FOR COMPUTING AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

To verify this rule, let $B = [\,\mathbf{b}_1 \ \cdots \ \mathbf{b}_p\,]$. Column j of AB is $A\mathbf{b}_j$, and we can compute $A\mathbf{b}_j$ by the row–vector rule for computing $A\mathbf{x}$ from Section 1.4. The ith entry in $A\mathbf{b}_j$ is the sum of the products of corresponding entries from row i of A and the vector $\mathbf{b}_j$, which is precisely the computation described in the rule for computing the (i, j)-entry of AB.

EXAMPLE 5 Use the row–column rule to compute two of the entries in AB for the matrices in Example 3. An inspection of the numbers involved will make it clear how the two methods for calculating AB produce the same matrix.

SOLUTION To find the entry in row 1 and column 3 of AB, consider row 1 of A and column 3 of B. Multiply corresponding entries and add the results, as shown below:

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \square & \square & 2(6)+3(3) \\ \square & \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & \square & \square \end{bmatrix}$$

For the entry in row 2 and column 2 of AB, use row 2 of A and column 2 of B:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & 1(3)+-5(-2) & \square \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & 13 & \square \end{bmatrix}$$

■

EXAMPLE 6 Find the entries in the second row of AB, where

$$A = \begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

SOLUTION By the row–column rule, the entries of the second row of AB come from row 2 of A (and the columns of B):

$$\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \square & \square \\ -4+21-12 & 6+3-8 \\ \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square \\ 5 & 1 \\ \square & \square \\ \square & \square \end{bmatrix}$$ ■

Notice that since Example 6 requested only the second row of AB, we could have written just the second row of A to the left of B and computed

$$\begin{bmatrix} -1 & 3 & -4 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \end{bmatrix}$$

This observation about rows of AB is true in general and follows from the row–column rule. Let $\text{row}_i(A)$ denote the ith row of a matrix A. Then

$$\text{row}_i(AB) = \text{row}_i(A) \cdot B \tag{2}$$

Properties of Matrix Multiplication

The following theorem lists the standard properties of matrix multiplication. Recall that I_m represents the $m \times m$ identity matrix and $I_m\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^m$.

THEOREM 2

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)
b. $A(B + C) = AB + AC$ (left distributive law)
c. $(B + C)A = BA + CA$ (right distributive law)
d. $r(AB) = (rA)B = A(rB)$ for any scalar r
e. $I_mA = A = AI_n$ (identity for matrix multiplication)

PROOF Properties (b)–(e) are considered in the exercises. Property (a) follows from the fact that matrix multiplication corresponds to composition of linear transformations (which are functions), and it is known (or easy to check) that the composition of functions is associative. Here is another proof of (a) that rests on the "column definition" of the product of two matrices. Let

$$C = [\,\mathbf{c}_1 \quad \cdots \quad \mathbf{c}_p\,]$$

By the definition of matrix multiplication,

$$\begin{aligned} BC &= [\,B\mathbf{c}_1 \quad \cdots \quad B\mathbf{c}_p\,] \\ A(BC) &= [\,A(B\mathbf{c}_1) \quad \cdots \quad A(B\mathbf{c}_p)\,] \end{aligned}$$

Recall from equation (1) that the definition of AB makes $A(B\mathbf{x}) = (AB)\mathbf{x}$ for all $\mathbf{x}$, so

$$A(BC) = [\,(AB)\mathbf{c}_1 \quad \cdots \quad (AB)\mathbf{c}_p\,] = (AB)C$$ ■

The associative and distributive laws in Theorems 1 and 2 say essentially that pairs of parentheses in matrix expressions can be inserted and deleted in the same way as in the algebra of real numbers. In particular, we can write ABC for the product, which can be computed either as $A(BC)$ or as $(AB)C$.[1] Similarly, a product $ABCD$ of four matrices can be computed as $A(BCD)$ or $(ABC)D$ or $A(BC)D$, and so on. It does not matter how we group the matrices when computing the product, so long as the left-to-right order of the matrices is preserved.

The left-to-right order in products is critical because AB and BA are usually not the same. This is not surprising, because the columns of AB are linear combinations of the columns of A, whereas the columns of BA are constructed from the columns of B. The position of the factors in the product AB is emphasized by saying that A is *right-multiplied* by B or that B is *left-multiplied* by A. If $AB = BA$, we say that A and B **commute** with one another.

EXAMPLE 7 Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Show that these matrices do not commute. That is, verify that $AB \neq BA$.

SOLUTION

$$\begin{aligned} AB &= \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix} \\ BA &= \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix} \end{aligned}$$ ■

Example 7 illustrates the first of the following list of important differences between matrix algebra and the ordinary algebra of real numbers. See Exercises 9–12 for examples of these situations.

***Warnings*:**

1. In general, $AB \neq BA$.
2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$. (See Exercise 10.)
3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$. (See Exercise 12.)

[1] When B is square and C has fewer columns than A has rows, it is more efficient to compute $A(BC)$ than $(AB)C$.

Powers of a Matrix

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = \underbrace{A \cdots A}_{k}$$

If A is nonzero and if $\mathbf{x}$ is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying $\mathbf{x}$ by A repeatedly k times. If $k = 0$, then $A^0\mathbf{x}$ should be $\mathbf{x}$ itself. Thus A^0 is interpreted as the identity matrix. Matrix powers are useful in both theory and applications (Sections 2.6, 5.9, and later in the text).

The Transpose of a Matrix

Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE 8 Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$$

Then

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$$ ■

THEOREM 3

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$
b. $(A + B)^T = A^T + B^T$
c. For any scalar r, $(rA)^T = rA^T$
d. $(AB)^T = B^TA^T$

Proofs of (a)–(c) are straightforward and are omitted. For (d), see Exercise 41. Usually, $(AB)^T$ is not equal to A^TB^T, even when A and B have sizes such that the product A^TB^T is defined.

The generalization of Theorem 3(d) to products of more than two factors can be stated in words as follows:

The transpose of a product of matrices equals the product of their transposes in the *reverse* order.

The exercises contain numerical examples that illustrate properties of transposes.

Artificial intelligence (AI) involves having a computer learn to recognize important information about anything that can be presented in a digitized format. One important area of AI is identifying whether the object in a picture matches a chosen object such as a number, fingerprint, or face.

In the next example, matrix transposition and matrix multiplication are used to tell whether or not a 2×2 block of colored squares matches the chosen checkerboard pattern in Figure 4.

EXAMPLE 9 In order to feed a 2×2 colored block into the computer, it first gets converted into a 4×1 vector by assigning a 1 to each block that is blue and a 0 to each block that is white. Then, the computer converts the block of numbers into a vector by placing the numbers in each column below the numbers in the column to its left.

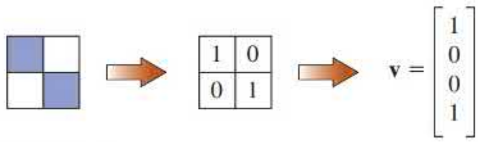
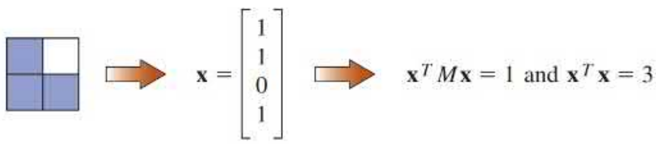


FIGURE 4

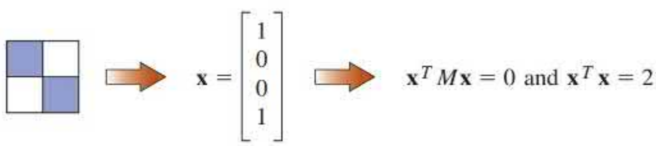
Let $M = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$.

Notice that $\mathbf{v}^T M \mathbf{v} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0,$

and $\mathbf{w}^T M \mathbf{w} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$, where $\mathbf{w}$ is the vector generated by a 2×2 block of all white squares. It can be verified that for any other vector $\mathbf{x}$ generated from a 2×2 block of white and blue squares, if $\mathbf{x}$ is not $\mathbf{v}$ or $\mathbf{w}$, then the product $\mathbf{x}^T M \mathbf{x}$ is nonzero. Thus, if a computer checks the value of $\mathbf{x}^T M \mathbf{x}$ and finds it is nonzero, the computer knows that the pattern corresponding to $\mathbf{x}$ is not the checkerboard with a blue square in the top left corner.



This pattern is not the checkerboard pattern since $\mathbf{x}^T M \mathbf{x} \neq 0$.



This pattern is the checkerboard pattern since $\mathbf{x}^T M \mathbf{x} = 0$, but $\mathbf{x}^T \mathbf{x} \neq 0$.

FIGURE 5

However, if the computer finds that $\mathbf{x}^T M\mathbf{x} = 0$, then $\mathbf{x}$ could be either $\mathbf{v}$ or $\mathbf{w}$. To distinguish between the two, the computer can calculate the product $\mathbf{x}^T\mathbf{x}$, for $\mathbf{x}^T\mathbf{x}$ is zero if and only if $\mathbf{x}$ is $\mathbf{w}$.[2] Thus, to conclude that $\mathbf{x}$ is equal to $\mathbf{v}$, the computer must have $\mathbf{x}^T M\mathbf{x} = 0$ and $\mathbf{x}^T\mathbf{x} \neq 0$. ■

Example 5 of Section 6.3 illustrates one way to choose a matrix M so that matrix multiplication and transposition can be used to identify a particular pattern of colored squares.

Another important aspect of AI starts even before the data is fed to the machine. In Section 1.9, it is illustrated how matrix multiplication can be used to move vectors around in space. In the next example, matrix multiplication is used to *scrub* data and prepare it for processing.

EXAMPLE 10 The dates of ground crew accidents for January and February of 2020 are listed in the columns of matrix T for Toronto Pearson Airport and matrix C for Chicago O'Hare Airport:

$$\text{Toronto: } T = \begin{bmatrix} 1 & 12 & 14 & 15 & 21 & 22 & 23 & 1 & 2 & 3 & 12 & 15 & 17 & 19 & 26 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

$$\text{Chicago: } C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 11 & 22 & 23 & 24 & 1 & 2 & 5 & 20 & 21 \end{bmatrix}$$

Clearly the data is listed differently in the two matrices. Canada and the United states have different traditions for whether the month or day comes first when writing a date. For matrix T, the day is listed in the first row and the month is listed in the second row. For matrix C, the month is listed in the first row and the day is listed in the second row. In order to use this data, the first and second rows need to be swapped in one of the matrices. Reviewing the effects of matrix multiplication in Table 1 of Section 1.9, notice that the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ switches the x_1 and x_2 coordinates of any vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ it is applied to and indeed

$$AT = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 12 & 14 & 15 & 21 & 22 & 23 & 1 & 2 & 3 & 12 & 15 & 17 & 19 & 26 \end{bmatrix}$$

has the data listed in the same order as it is listed in matrix C. The matrices AT and C can now be fed into the same machine. ■

In Exercises 51 and 52 you will be asked to *scrub* further data for this project.[3]

[2] To see why $\mathbf{x}^T\mathbf{x}$ is zero if and only if $\mathbf{x}$ is $\mathbf{w}$, let $\mathbf{x}^T = [x_1\ x_2\ x_3\ x_4]$. Then $\mathbf{x}^T\mathbf{x} = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and this sum is zero if and only if the coordinates of $\mathbf{x}$ are all zero. That is, if and only if $\mathbf{x} = \mathbf{w}$.

[3] Although the data in this example and the corresponding exercises are fictitious, Data Analytics students at Washington State University identified scrubbing the data they received as an important first step in their actual analysis of ground crew accidents at three major airports in the United States.

Numerical Notes

1. The fastest way to obtain AB on a computer depends on the way in which the computer stores matrices in its memory. The standard high-performance algorithms, such as in LAPACK, calculate AB by columns, as in our definition of the product. (A version of LAPACK written in C++ calculates AB by rows.)
2. The definition of AB lends itself well to parallel processing on a computer. The columns of B are assigned individually or in groups to different processors, which independently and hence simultaneously compute the corresponding columns of AB.

Practice Problems

1. Since vectors in $\mathbb{R}^n$ may be regarded as $n \times 1$ matrices, the properties of transposes in Theorem 3 apply to vectors, too. Let
$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$
Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$, and $\mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined?
2. Let A be a 4×4 matrix and let $\mathbf{x}$ be a vector in $\mathbb{R}^4$. What is the fastest way to compute $A^2\mathbf{x}$? Count the multiplications.
3. Suppose A is an $m \times n$ matrix, all of whose rows are identical. Suppose B is an $n \times p$ matrix, all of whose columns are identical. What can be said about the entries in AB?

2.1 Exercises

In Exercises 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1. $-2A$, $B - 2A$, AC, CD
2. $A + 2B$, $3C - E$, CB, EB

In the rest of this exercise set and in those to follow, you should assume that each matrix expression is defined. That is, the sizes of the matrices (and vectors) involved "match" appropriately.

3. Let $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$. Compute $3I_2 - A$ and $(3I_2)A$.
4. Compute $A - 5I_3$ and $(5I_3)A$, when
$$A = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -3 \\ -4 & 1 & 8 \end{bmatrix}.$$

In Exercises 5 and 6, compute the product AB in two ways: (a) by the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately, and (b) by the row–column rule for computing AB.

5. $A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$
6. $A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$
7. If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?
8. How many rows does B have if BC is a 3×4 matrix?
9. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. What value(s) of k, if any, will make $AB = BA$?
10. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.
11. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3×3 matrix B, not the identity matrix or the zero matrix, such that $AB = BA$.

12. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

13. Let $\mathbf{r}_1, \dots, \mathbf{r}_p$ be vectors in $\mathbb{R}^n$, and let Q be an $m \times n$ matrix. Write the matrix $[\, Q\mathbf{r}_1 \cdots Q\mathbf{r}_p \,]$ as a *product* of two matrices (neither of which is an identity matrix).

14. Let U be the 3×2 cost matrix described in Example 6 of Section 1.8. The first column of U lists the costs per dollar of output for manufacturing product B, and the second column lists the costs per dollar of output for product C. (The costs are categorized as materials, labor, and overhead.) Let $\mathbf{q}_1$ be a vector in $\mathbb{R}^2$ that lists the output (measured in dollars) of products B and C manufactured during the first quarter of the year, and let $\mathbf{q}_2, \mathbf{q}_3$, and $\mathbf{q}_4$ be the analogous vectors that list the amounts of products B and C manufactured in the second, third, and fourth quarters, respectively. Give an economic description of the data in the matrix UQ, where $Q = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4]$.

Exercises 15–24 concern arbitrary matrices A, B, and C for which the indicated sums and products are defined. Mark each statement True or False **(T/F)**. Justify each answer.

15. **(T/F)** If A and B are 2×2 with columns $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}_1, \mathbf{b}_2$, respectively, then $AB = [\mathbf{a}_1\mathbf{b}_1 \quad \mathbf{a}_2\mathbf{b}_2]$.

16. **(T/F)** If A and B are 3×3 and $B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]$, then $AB = [A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3]$.

17. **(T/F)** Each column of AB is a linear combination of the columns of B using weights from the corresponding column of A.

18. **(T/F)** The second row of AB is the second row of A multiplied on the right by B.

19. **(T/F)** $AB + AC = A(B + C)$

20. **(T/F)** $A^T + B^T = (A + B)^T$

21. **(T/F)** $(AB)C = (AC)B$

22. **(T/F)** $(AB)^T = A^T B^T$

23. **(T/F)** The transpose of a product of matrices equals the product of their transposes in the same order.

24. **(T/F)** The transpose of a sum of matrices equals the sum of their transposes.

25. If $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$, determine the first and second columns of B.

26. Suppose the first two columns, $\mathbf{b}_1$ and $\mathbf{b}_2$, of B are equal. What can you say about the columns of AB (if AB is defined)? Why?

27. Suppose the third column of B is the sum of the first two columns. What can you say about the third column of AB? Why?

28. Suppose the second column of B is all zeros. What can you say about the second column of AB?

29. Suppose the last column of AB is all zeros, but B itself has no column of zeros. What can you say about the columns of A?

30. Show that if the columns of B are linearly dependent, then so are the columns of AB.

31. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A cannot have more columns than rows.

32. Suppose $AD = I_m$ (the $m \times m$ identity matrix). Show that for any $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution. [*Hint:* Think about the equation $AD\mathbf{b} = \mathbf{b}$.] Explain why A cannot have more rows than columns.

33. Suppose A is an $m \times n$ matrix and there exist $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. Prove that $m = n$ and $C = D$. [*Hint:* Think about the product CAD.]

34. Suppose A is a $3 \times n$ matrix whose columns span $\mathbb{R}^3$. Explain how to construct an $n \times 3$ matrix D such that $AD = I_3$.

In Exercises 35 and 36, view vectors in $\mathbb{R}^n$ as $n \times 1$ matrices. For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, called the **scalar product**, or **inner product**, of $\mathbf{u}$ and $\mathbf{v}$. It is usually written as a single real number without brackets. The matrix product $\mathbf{u}\mathbf{v}^T$ is an $n \times n$ matrix, called the **outer product** of $\mathbf{u}$ and $\mathbf{v}$. The products $\mathbf{u}^T\mathbf{v}$ and $\mathbf{u}\mathbf{v}^T$ will appear later in the text.

35. Let $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Compute $\mathbf{u}^T\mathbf{v}$, $\mathbf{v}^T\mathbf{u}$, $\mathbf{u}\mathbf{v}^T$, and $\mathbf{v}\mathbf{u}^T$.

36. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$, how are $\mathbf{u}^T\mathbf{v}$ and $\mathbf{v}^T\mathbf{u}$ related? How are $\mathbf{u}\mathbf{v}^T$ and $\mathbf{v}\mathbf{u}^T$ related?

37. Prove Theorem 2(b) and 2(c). Use the row–column rule. The (i, j)-entry in $A(B + C)$ can be written as

$$a_{i1}(b_{1j} + c_{1j}) + \cdots + a_{in}(b_{nj} + c_{nj}) \text{ or } \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$

38. Prove Theorem 2(d). [*Hint:* The (i, j)-entry in $(rA)B$ is $(ra_{i1})b_{1j} + \cdots + (ra_{in})b_{nj}$.]

39. Show that $I_m A = A$ when A is an $m \times n$ matrix. You can assume $I_m\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^m$.

40. Show that $AI_n = A$ when A is an $m \times n$ matrix. [*Hint:* Use the (column) definition of AI_n.]

41. Prove Theorem 3(d). [*Hint:* Consider the jth row of $(AB)^T$.]

42. Give a formula for $(AB\mathbf{x})^T$, where $\mathbf{x}$ is a vector and A and B are matrices of appropriate sizes.

T 43. Use a web search engine such as Google to find documentation for your matrix program, and write the commands that will produce the following matrices (without keying in each entry of the matrix).

a. A 5×6 matrix of zeros

b. A 3×5 matrix of ones

c. The 6×6 identity matrix

d. A 5×5 diagonal matrix, with diagonal entries 3, 5, 7, 2, 4

A useful way to test new ideas in matrix algebra, or to make conjectures, is to make calculations with matrices selected at random. Checking a property for a few matrices does not prove that the property holds in general, but it makes the property more believable. Also, if the property is actually false, you may discover this when you make a few calculations.

T 44. Write the command(s) that will create a 6×4 matrix with random entries. In what range of numbers do the entries lie? Tell how to create a 3×3 matrix with random integer entries between -9 and 9. [*Hint:* If x is a random number such that $0 < x < 1$, then $-9.5 < 19(x - .5) < 9.5$.]

T 45. Construct a random 4×4 matrix A and test whether $(A + I)(A - I) = A^2 - I$. The best way to do this is to compute $(A + I)(A - I) - (A^2 - I)$ and verify that this difference is the zero matrix. Do this for three random matrices. Then test $(A + B)(A - B) = A^2 - B^2$ the same way for three pairs of random 4×4 matrices. Report your conclusions.

T 46. Use at least three pairs of random 4×4 matrices A and B to test the equalities $(A + B)^T = A^T + B^T$ and $(AB)^T = A^T B^T$. (See Exercise 45.) Report your conclusions. [*Note:* Most matrix programs use A' for A^T.]

T 47. Let

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute S^k for $k = 2, \ldots, 6$.

T 48. Describe in words what happens when you compute A^5, A^{10}, A^{20}, and A^{30} for

$$A = \begin{bmatrix} 1/6 & 1/2 & 1/3 \\ 1/2 & 1/4 & 1/4 \\ 1/3 & 1/4 & 5/12 \end{bmatrix}$$

T 49. The matrix M can detect a particular 2×2 colored pattern like in Example 9. Create a nonzero 4×1 vector $\mathbf{x}$ by choosing each entry to be a zero or one. Test to see if $\mathbf{x}$ corresponds to the right pattern by calculating $\mathbf{x}^T M \mathbf{x}$. If $\mathbf{x}^T M \mathbf{x} = 0$, then $\mathbf{x}$ is the pattern identified by M. If $\mathbf{x}^T M \mathbf{x} \neq 0$, try a different nonzero vector of zeros and ones. You may want to be systematic in the way that you choose each $\mathbf{x}$ in order to avoid testing the same vector twice. You are using "guess and check" to determine which pattern of 2×2 colored squares the matrix M detects.

$$M = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T 50. Repeat Exercise 49 with the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

T 51. Use the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to switch the first and second rows of the matrix M containing dates of accidents at the Montreal Trudeau Airport.

Montreal:

$$M = \begin{bmatrix} 2 & 3 & 16 & 24 & 25 & 26 & 6 & 7 & 19 & 26 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}$$

This data in matrix M has been scrubbed in matrix AM and can be fed into the same machine as the other data from Example 10.

T 52. Use the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ to remove the last row from the matrix N containing dates of accidents at the New York JFK Airport.

New York:

$$N = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 12 & 21 & 22 & 3 & 20 & 21 \\ 2020 & 2020 & 2020 & 2020 & 2020 & 2020 & 2020 \end{bmatrix}$$

The data in matrix N has been scrubbed in matrix BN and can be fed into the same machine as the other data from Example 10.

Solutions to Practice Problems

1. $A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. So $(A\mathbf{x})^T = \begin{bmatrix} -4 & 2 \end{bmatrix}$. Also,

$$\mathbf{x}^T A^T = \begin{bmatrix} 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \end{bmatrix}.$$

The quantities $(A\mathbf{x})^T$ and $\mathbf{x}^T A^T$ are equal, by Theorem 3(d). Next,

$$\mathbf{x}\mathbf{x}^T = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix}$$

$$\mathbf{x}^T\mathbf{x} = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = [\,25 + 9\,] = 34$$

A 1×1 matrix such as $\mathbf{x}^T\mathbf{x}$ is usually written without the brackets. Finally, $A^T\mathbf{x}^T$ is not defined, because $\mathbf{x}^T$ does not have two rows to match the two columns of A^T.

2. The fastest way to compute $A^2\mathbf{x}$ is to compute $A(A\mathbf{x})$. The product $A\mathbf{x}$ requires 16 multiplications, 4 for each entry, and $A(A\mathbf{x})$ requires 16 more. In contrast, the product A^2 requires 64 multiplications, 4 for each of the 16 entries in A^2. After that, $A^2\mathbf{x}$ takes 16 more multiplications, for a total of 80.
3. First observe that by the definition of matrix multiplication,

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_1 \quad \cdots \quad A\mathbf{b}_1],$$

so the columns of AB are identical. Next, recall that $\text{row}_i(AB) = \text{row}_i(A) \cdot B$. Since all the rows of A are identical, all the rows of AB are identical. Putting this information about the rows and columns together, it follows that all the entries in AB are the same.

2.2 The Inverse of a Matrix

Matrix algebra provides tools for manipulating matrix equations and creating various useful formulas in ways similar to doing ordinary algebra with real numbers. This section investigates the matrix analogue of the reciprocal, or multiplicative inverse, of a nonzero number.

Recall that the multiplicative inverse of a number such as 5 is 1/5 or 5^{-1}. This inverse satisfies the equations

$$5^{-1}(5) = 1 \quad \text{and} \quad 5(5^{-1}) = 1$$

The matrix generalization requires *both* equations and avoids the slanted-line notation (for division) because matrix multiplication is not commutative. Furthermore, a full generalization is possible only if the matrices involved are square.[1]

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

where $I = I_n$, the $n \times n$ identity matrix. In this case, C is an **inverse** of A. In fact, C is uniquely determined by A, because if B were another inverse of A, then $B = BI = B(AC) = (BA)C = IC = C$. This unique inverse is denoted by A^{-1}, so that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

A matrix that is *not* invertible is sometimes called a **singular matrix**, and an invertible matrix is called a **nonsingular matrix**.

[1] One could say that an $m \times n$ matrix A is invertible if there exist $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. However, these equations imply that A is square and $C = D$. Thus, A is invertible as defined above. See Exercises 31–33 in Section 2.1.

EXAMPLE 1 If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, then

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $C = A^{-1}$. ■

Here is a simple formula for the inverse of a 2×2 matrix, along with a test to tell if the inverse exists.

THEOREM 4

> Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and
>
> $$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
>
> If $ad - bc = 0$, then A is not invertible.

The simple proof of Theorem 4 is outlined in Exercises 35 and 36. The quantity $ad - bc$ is called the **determinant** of A, and we write

$$\det A = ad - bc$$

Theorem 4 says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$.

EXAMPLE 2 Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

SOLUTION Since $\det A = 3(6) - 4(5) = -2 \neq 0$, A is invertible, and

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 6/(-2) & -4/(-2) \\ -5/(-2) & 3/(-2) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$ ■

Invertible matrices are indispensable in linear algebra—mainly for algebraic calculations and formula derivations, as in the next theorem. There are also occasions when an inverse matrix provides insight into a mathematical model of a real-life situation, as in Example 3.

THEOREM 5

> If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

PROOF Take any $\mathbf{b}$ in $\mathbb{R}^n$. A solution exists because if $A^{-1}\mathbf{b}$ is substituted for $\mathbf{x}$, then $A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}$. So $A^{-1}\mathbf{b}$ is a solution. To prove that the solution is unique, show that if $\mathbf{u}$ is any solution, then $\mathbf{u}$, in fact, must be $A^{-1}\mathbf{b}$. Indeed, if $A\mathbf{u} = \mathbf{b}$, we can multiply both sides by A^{-1} and obtain

$$A^{-1}A\mathbf{u} = A^{-1}\mathbf{b}, \quad I\mathbf{u} = A^{-1}\mathbf{b}, \quad \text{and} \quad \mathbf{u} = A^{-1}\mathbf{b}$$ ■

EXAMPLE 3 A horizontal elastic beam is supported at each end and is subjected to forces at points 1, 2, and 3, as shown in Figure 1. Let $\mathbf{f}$ in $\mathbb{R}^3$ list the forces at these points, and let $\mathbf{y}$ in $\mathbb{R}^3$ list the amounts of deflection (that is, movement) of the beam at the three points. Using Hooke's law from physics, it can be shown that

$$\mathbf{y} = D\mathbf{f}$$

where D is a *flexibility matrix*. Its inverse is called the *stiffness matrix*. Describe the physical significance of the columns of D and D^{-1}.

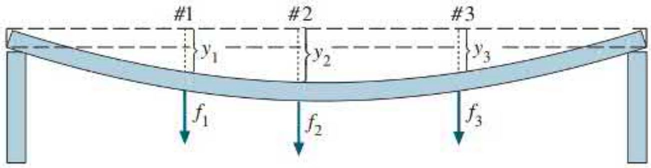


FIGURE 1 Deflection of an elastic beam.

SOLUTION Write $I_3 = [\,\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3\,]$ and observe that

$$D = DI_3 = [\,D\mathbf{e}_1 \quad D\mathbf{e}_2 \quad D\mathbf{e}_3\,]$$

Interpret the vector $\mathbf{e}_1 = (1, 0, 0)$ as a unit force applied downward at point 1 on the beam (with zero force at the other two points). Then $D\mathbf{e}_1$, the first column of D, lists the beam deflections due to a unit force at point 1. Similar descriptions apply to the second and third columns of D.

To study the stiffness matrix D^{-1}, observe that the equation $\mathbf{f} = D^{-1}\mathbf{y}$ computes a force vector $\mathbf{f}$ when a deflection vector $\mathbf{y}$ is given. Write

$$D^{-1} = D^{-1}I_3 = [\,D^{-1}\mathbf{e}_1 \quad D^{-1}\mathbf{e}_2 \quad D^{-1}\mathbf{e}_3\,]$$

Now interpret $\mathbf{e}_1$ as a deflection vector. Then $D^{-1}\mathbf{e}_1$ lists the forces that create the deflection. That is, the first column of D^{-1} lists the forces that must be applied at the three points to produce a unit deflection at point 1 and zero deflections at the other points. Similarly, columns 2 and 3 of D^{-1} list the forces required to produce unit deflections at points 2 and 3, respectively. In each column, one or two of the forces must be negative (point upward) to produce a unit deflection at the desired point and zero deflections at the other two points. If the flexibility is measured, for example, in inches of deflection per pound of load, then the stiffness matrix entries are given in pounds of load per inch of deflection. ■

The formula in Theorem 5 is seldom used to solve an equation $A\mathbf{x} = \mathbf{b}$ numerically because row reduction of $[\,A \quad \mathbf{b}\,]$ is nearly always faster. (Row reduction is usually more accurate, too, when computations involve rounding off numbers.) One possible exception is the 2×2 case. In this case, mental computations to solve $A\mathbf{x} = \mathbf{b}$ are sometimes easier using the formula for A^{-1}, as in the next example.

EXAMPLE 4 Use the inverse of the matrix A in Example 2 to solve the system

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7 \end{aligned}$$

SOLUTION This system is equivalent to $A\mathbf{x} = \mathbf{b}$, so

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$ ■

The next theorem provides three useful facts about invertible matrices.

THEOREM 6

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

PROOF To verify statement (a), find a matrix C such that

$$A^{-1}C = I \quad \text{and} \quad CA^{-1} = I$$

In fact, these equations are satisfied with A in place of C. Hence A^{-1} is invertible, and A is its inverse. Next, to prove statement (b), compute:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

A similar calculation shows that $(B^{-1}A^{-1})(AB) = I$. For statement (c), use Theorem 3(d), read from right to left, $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$. Similarly, $A^T(A^{-1})^T = I^T = I$. Hence A^T is invertible, and its inverse is $(A^{-1})^T$. ■

Remark: Part (b) illustrates the important role that definitions play in proofs. The theorem claims that $B^{-1}A^{-1}$ is the inverse of AB. The proof establishes this by showing that $B^{-1}A^{-1}$ satisfies the definition of what it means to be the inverse of AB. Now, the inverse of AB is a matrix that when multiplied on the left (or right) by AB, the product is the identity matrix I. So the proof consists of showing that $B^{-1}A^{-1}$ has this property.

The following generalization of Theorem 6(b) is needed later.

The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

There is an important connection between invertible matrices and row operations that leads to a method for computing inverses. As we shall see, an invertible matrix A is row equivalent to an identity matrix, and we can find A^{-1} by *watching the row reduction of A to I.*

Elementary Matrices

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix. The next example illustrates the three kinds of elementary matrices.

EXAMPLE 5 Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A.

SOLUTION Verify that

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}, \quad E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$

$$E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

Addition of -4 times row 1 of A to row 3 produces E_1A. (This is a row replacement operation.) An interchange of rows 1 and 2 of A produces E_2A, and multiplication of row 3 of A by 5 produces E_3A. ■

Left-multiplication (that is, multiplication on the left) by E_1 in Example 5 has the same effect on any $3 \times n$ matrix. It adds -4 times row 1 to row 3. In particular, since $E_1 \cdot I = E_1$, we see that E_1 *itself* is produced by this same row operation on the identity. Thus Example 5 illustrates the following general fact about elementary matrices. See Exercises 37 and 38.

> If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

Since row operations are reversible, as shown in Section 1.1, elementary matrices are invertible, for if E is produced by a row operation on I, then there is another row operation of the same type that changes E back into I. Hence there is an elementary matrix F such that $FE = I$. Since E and F correspond to reverse operations, $EF = I$, too.

> Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

EXAMPLE 6 Find the inverse of $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

SOLUTION To transform E_1 into I, add $+4$ times row 1 to row 3. The elementary matrix that does this is

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +4 & 0 & 1 \end{bmatrix}$$ ■

The following theorem provides the best way to "visualize" an invertible matrix, and the theorem leads immediately to a method for finding the inverse of a matrix.

THEOREM 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Remark: The comment on the proof of Theorem 11 in Chapter 1 noted that "P if and only if Q" is equivalent to two statements: (1) "If P then Q" and (2) "If Q then P." The second statement is called the *converse* of the first and explains the use of the word *conversely* in the second paragraph of this proof.

PROOF Suppose that A is invertible. Then, since the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ (Theorem 5), A has a pivot position in every row (Theorem 4 in Section 1.4). Because A is square, the n pivot positions must be on the diagonal, which implies that the reduced echelon form of A is I_n. That is, $A \sim I_n$.

Now suppose, conversely, that $A \sim I_n$. Then, since each step of the row reduction of A corresponds to left-multiplication by an elementary matrix, there exist elementary matrices $E_1, \dots, E_p$ such that

$$A \sim E_1 A \sim E_2(E_1 A) \sim \cdots \sim E_p(E_{p-1} \cdots E_1 A) = I_n$$

That is,

$$E_p \cdots E_1 A = I_n \tag{1}$$

Since the product $E_p \cdots E_1$ of invertible matrices is invertible, (1) leads to

$$\begin{aligned}(E_p \cdots E_1)^{-1}(E_p \cdots E_1)A &= (E_p \cdots E_1)^{-1} I_n \\ A &= (E_p \cdots E_1)^{-1}\end{aligned}$$

Thus A is invertible, as it is the inverse of an invertible matrix (Theorem 6). Also,

$$A^{-1} = [\,(E_p \cdots E_1)^{-1}\,]^{-1} = E_p \cdots E_1$$

Then $A^{-1} = E_p \cdots E_1 I_n$, which says that A^{-1} results from applying $E_1, \dots, E_p$ successively to I_n. This is the same sequence in (1) that reduced A to I_n. ■

An Algorithm for Finding A^{-1}

If we place A and I side by side to form an augmented matrix $[\,A \quad I\,]$, then row operations on this matrix produce identical operations on A and on I. By Theorem 7, either there are row operations that transform A to I_n and I_n to A^{-1} or else A is not invertible.

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[\,A \quad I\,]$. If A is row equivalent to I, then $[\,A \quad I\,]$ is row equivalent to $[\,I \quad A^{-1}\,]$. Otherwise, A does not have an inverse.

EXAMPLE 7 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

SOLUTION

$$[\,A \quad I\,] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

Theorem 7 shows, since $A \sim I$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

■

Reasonable Answers

Once you have found a candidate for the inverse of a matrix, you can check that your answer is correct by finding the product of A with A^{-1}. For the inverse found for matrix A in Example 7, notice

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

confirming that answer is correct. It is not necessary to check that $A^{-1}A = I$ since A is invertible.

Another View of Matrix Inversion

Denote the columns of I_n by $\mathbf{e}_1, \dots, \mathbf{e}_n$. Then row reduction of $[\,A \quad I\,]$ to $[\,I \quad A^{-1}\,]$ can be viewed as the simultaneous solution of the n systems

$$A\mathbf{x} = \mathbf{e}_1, \quad A\mathbf{x} = \mathbf{e}_2, \quad \dots, \quad A\mathbf{x} = \mathbf{e}_n \tag{2}$$

where the "augmented columns" of these systems have all been placed next to A to form $[\,A \quad \mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n\,] = [\,A \quad I\,]$. The equation $AA^{-1} = I$ and the definition of matrix multiplication show that the columns of A^{-1} are precisely the solutions of the systems in (2). This observation is useful because some applied problems may require finding only one or two columns of A^{-1}. In this case, only the corresponding systems in (2) need to be solved.

Numerical Note

In practical work, A^{-1} is seldom computed, unless the entries of A^{-1} are needed. Computing both A^{-1} and $A^{-1}\mathbf{b}$ takes about three times as many arithmetic operations as solving $A\mathbf{x} = \mathbf{b}$ by row reduction, and row reduction may be more accurate.

Practice Problems

1. Use determinants to determine which of the following matrices are invertible.

a. $\begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 4 & -9 \\ 0 & 5 \end{bmatrix}$ c. $\begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix}$

2. Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if it exists.

3. If A is an invertible matrix, prove that $5A$ is an invertible matrix.

2.2 Exercises

Find the inverses of the matrices in Exercises 1–4.

1. $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$ 2. $\begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 8 & 3 \\ -7 & -3 \end{bmatrix}$ 4. $\begin{bmatrix} 3 & -2 \\ 7 & -4 \end{bmatrix}$

5. Verify that the inverse you found in Exercise 1 is correct.

6. Verify that the inverse you found in Exercise 2 is correct.

7. Use the inverse found in Exercise 1 to solve the system

$$\begin{aligned} 8x_1 + 3x_2 &= 2 \\ 5x_1 + 2x_2 &= -1 \end{aligned}$$

8. Use the inverse found in Exercise 2 to solve the system

$$\begin{aligned} 3x_1 + x_2 &= -2 \\ 7x_1 + 2x_2 &= 3 \end{aligned}$$

9. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, and $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

   a. Find A^{-1}, and use it to solve the four equations $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, $A\mathbf{x} = \mathbf{b}_3$, $A\mathbf{x} = \mathbf{b}_4$

   b. The four equations in part (a) can be solved by the same set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (a) by row reducing the augmented matrix $\begin{bmatrix} A & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}$

10. Use matrix algebra to show that if A is invertible and D satisfies $AD = I$, then $D = A^{-1}$.

In Exercises 11–20, mark each statement True or False **(T/F)**. Justify each answer.

11. **(T/F)** In order for a matrix B to be the inverse of A, both equations $AB = I$ and $BA = I$ must be true.

12. **(T/F)** A product of invertible $n \times n$ matrices is invertible, and the inverse of the product is the product of their inverses in the same order.

13. **(T/F)** If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.

14. **(T/F)** If A is invertible, then the inverse of A^{-1} is A itself.

15. **(T/F)** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible.

16. **(T/F)** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad = bc$, then A is not invertible.

17. **(T/F)** If A is an invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *each* $\mathbf{b}$ in $\mathbb{R}^n$.

18. **(T/F)** If A can be row reduced to the identity matrix, then A must be invertible.

19. **(T/F)** Each elementary matrix is invertible.

20. **(T/F)** If A is invertible, then the elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n.

21. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Show that the equation $AX = B$ has a unique solution $A^{-1}B$.

22. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Explain why $A^{-1}B$ can be computed by row reduction:

$$\text{If } [A \;\; B] \sim \cdots \sim [I \;\; X], \text{ then } X = A^{-1}B.$$

If A is larger than 2×2, then row reduction of $[A \;\; B]$ is much faster than computing both A^{-1} and $A^{-1}B$.

23. Suppose $AB = AC$, where B and C are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true, in general, when A is not invertible?

24. Suppose $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Show that $B = C$.

25. Suppose A, B, and C are invertible $n \times n$ matrices. Show that ABC is also invertible by producing a matrix D such that $(ABC)\,D = I$ and $D\,(ABC) = I$.

26. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible. [*Hint:* Let $C = AB$, and solve this equation for A.]

27. Solve the equation $AB = BC$ for A, assuming that A, B, and C are square and B is invertible.

28. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

29. If A, B, and C are $n \times n$ invertible matrices, does the equation $C^{-1}(A + X)B^{-1} = I_n$ have a solution, X? If so, find it.

30. Suppose A, B, and X are $n \times n$ matrices with A, X, and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B \qquad (3)$$

a. Explain why B is invertible.

b. Solve (3) for X. If you need to invert a matrix, explain why that matrix is invertible.

31. Explain why the columns of an $n \times n$ matrix A are linearly independent when A is invertible.

32. Explain why the columns of an $n \times n$ matrix A span $\mathbb{R}^n$ when A is invertible. [*Hint:* Review Theorem 4 in Section 1.4.]

33. Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A has n pivot columns and A is row equivalent to I_n. By Theorem 7, this shows that A must be invertible. (This exercise and Exercise 34 will be cited in Section 2.3.)

34. Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why A must be invertible. [*Hint:* Is A row equivalent to I_n?]

Exercises 35 and 36 prove Theorem 4 for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

35. Show that if $ad - bc = 0$, then the equation $A\mathbf{x} = \mathbf{0}$ has more than one solution. Why does this imply that A is not invertible? [*Hint:* First, consider $a = b = 0$. Then, if a and b are not both zero, consider the vector $\mathbf{x} = \begin{bmatrix} -b \\ a \end{bmatrix}$.]

36. Show that if $ad - bc \neq 0$, the formula for A^{-1} works.

Exercises 37 and 38 prove special cases of the facts about elementary matrices stated in the box following Example 5. Here A is a 3×3 matrix and $I = I_3$. (A general proof would require slightly more notation.)

37. a. Use equation (1) from Section 2.1 to show that $\text{row}_i(A) = \text{row}_i(I) \cdot A$, for $i = 1, 2, 3$.

b. Show that if rows 1 and 2 of A are interchanged, then the result may be written as EA, where E is an elementary matrix formed by interchanging rows 1 and 2 of I.

c. Show that if row 3 of A is multiplied by 5, then the result may be written as EA, where E is formed by multiplying row 3 of I by 5.

38. Show that if row 3 of A is replaced by $\text{row}_3(A) - 4\text{row}_1(A)$, the result is EA, where E is formed from I by replacing $\text{row}_3(I)$ by $\text{row}_3(I) - 4\text{row}_1(I)$.

Find the inverses of the matrices in Exercises 39–42, if they exist. Use the algorithm introduced in this section.

39. $\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$

40. $\begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix}$

41. $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

42. $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$

43. Use the algorithm from this section to find the inverses of

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let A be the corresponding $n \times n$ matrix, and let B be its inverse. Guess the form of B, and then prove that $AB = I$ and $BA = I$.

44. Repeat the strategy of Exercise 43 to guess the inverse of $A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$. Prove that your guess is correct.

45. Let $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$. Find the third column of A^{-1} without computing the other columns.

T 46. Let $A = \begin{bmatrix} -25 & -9 & -27 \\ 546 & 180 & 537 \\ 154 & 50 & 149 \end{bmatrix}$. Find the second and third columns of A^{-1} without computing the first column.

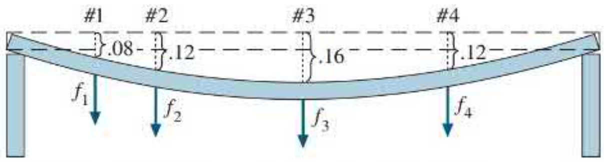
47. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}$. Construct a 2×3 matrix C (by trial and error) using only 1, -1, and 0 as entries, such that $CA = I_2$. Compute AC and note that $AC \neq I_3$.

48. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Construct a 4×2 matrix D using only 1 and 0 as entries, such that $AD = I_2$. Is it possible that $CA = I_4$ for some 4×2 matrix C? Why or why not?

49. Let $D = \begin{bmatrix} .005 & .002 & .001 \\ .002 & .004 & .002 \\ .001 & .002 & .005 \end{bmatrix}$ be a flexibility matrix, with flexibility measured in inches per pound. Suppose that forces of 30, 50, and 20 lb are applied at points 1, 2, and 3, respectively, in Figure 1 of Example 3. Find the corresponding deflections.

T 50. Compute the stiffness matrix D^{-1} for D in Exercise 49. List the forces needed to produce a deflection of .04 in. at point 3, with zero deflections at the other points.

T 51. Let $D = \begin{bmatrix} .0040 & .0030 & .0010 & .0005 \\ .0030 & .0050 & .0030 & .0010 \\ .0010 & .0030 & .0050 & .0030 \\ .0005 & .0010 & .0030 & .0040 \end{bmatrix}$ be a flexibility matrix for an elastic beam with four points at which force is applied. Units are centimeters per newton of force. Measurements at the four points show deflections of .08, .12, .16, and .12 cm. Determine the forces at the four points.



Deflection of elastic beam in Exercises 51 and 52.

T 52. With D as in Exercise 51, determine the forces that produce a deflection of .24 cm at the second point on the beam, with zero deflections at the other three points. How is the answer related to the entries in D^{-1}? [*Hint:* First answer the question when the deflection is 1 cm at the second point.]

Solutions to Practice Problems

1. a. $\det \begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix} = 3 \cdot 6 - (-9) \cdot 2 = 18 + 18 = 36$. The determinant is nonzero, so the matrix is invertible.

b. $\det \begin{bmatrix} 4 & -9 \\ 0 & 5 \end{bmatrix} = 4 \cdot 5 - (-9) \cdot 0 = 20 \neq 0$. The matrix is invertible.

c. $\det \begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix} = 6 \cdot 6 - (-9)(-4) = 36 - 36 = 0$. The matrix is not invertible.

2. $$[\,A \quad I\,] \sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix}$$

So $[\,A \quad I\,]$ is row equivalent to a matrix of the form $[\,B \quad D\,]$, where B is square and has a row of zeros. Further row operations will not transform B into I, so we stop. A does not have an inverse.

3. Since A is an invertible matrix, there exists a matrix C such that $AC = I = CA$. The goal is to find a matrix D so that $(5A)D = I = D(5A)$. Set $D = 1/5\,C$. Applying Theorem 2 from Section 2.1 establishes that $(5A)(1/5\,C) = (5)(1/5)(AC) = 1\,I = I$, and $(1/5\,C)(5A) = (1/5)(5)(CA) = 1\,I = I$. Thus $1/5\,C$ is indeed the inverse of A, proving that A is invertible.

2.3 Characterizations of Invertible Matrices

This section provides a review of most of the concepts introduced in Chapter 1, in relation to systems of n linear equations in n unknowns and to *square* matrices. The main result is Theorem 8.

THEOREM 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

$$\begin{tikzcd}
& \text{(a)} \arrow[dr, Rightarrow] & \\
\text{(b)} \arrow[ur, Rightarrow] & & \text{(j)} \arrow[d, Rightarrow] \\
\text{(c)} \arrow[u, Rightarrow] & & \text{(d)} \arrow[ll, Rightarrow]
\end{tikzcd}$$

FIGURE 1

First, we need some notation. If the truth of statement (a) always implies that statement (j) is true, we say that (a) *implies* (j) and write (a) $\Rightarrow$ (j). The proof will establish the "circle" of implications shown in Figure 1. If any one of these five statements is true, then so are the others. Finally, the proof will link the remaining statements of the theorem to the statements in this circle.

PROOF If statement (a) is true, then A^{-1} works for C in (j), so (a) $\Rightarrow$ (j). Next, (j) $\Rightarrow$ (d) by Exercise 31 in Section 2.1. (Turn back and read the exercise.) Also, (d) $\Rightarrow$ (c) by Exercise 33 in Section 2.2. If A is square and has n pivot positions, then the pivots must lie on the main diagonal, in which case the reduced echelon form of A is I_n. Thus (c) $\Rightarrow$ (b). Also, (b) $\Rightarrow$ (a) by Theorem 7 in Section 2.2. This completes the circle in Figure 1.

$$\begin{tikzcd}
& \text{(k)} \arrow[dr, Rightarrow] & \\
\text{(a)} \arrow[ur, Rightarrow] & & \text{(g)} \arrow[ll, Rightarrow]
\end{tikzcd}$$

Next, (a) $\Rightarrow$ (k) because A^{-1} works for D. Also, (k) $\Rightarrow$ (g) by Exercise 32 in Section 2.1, and (g) $\Rightarrow$ (a) by Exercise 34 in Section 2.2. So (k) and (g) are linked to the circle. Further, (g), (h), and (i) are equivalent for any matrix, by Theorem 4 in Section 1.4 and Theorem 12(a) in Section 1.9. Thus, (h) and (i) are linked through (g) to the circle.

$$\text{(g)} \Leftrightarrow \text{(h)} \Leftrightarrow \text{(i)}$$

$$\text{(d)} \Leftrightarrow \text{(e)} \Leftrightarrow \text{(f)}$$

$$\text{(a)} \Leftrightarrow \text{(l)}$$

Since (d) is linked to the circle, so are (e) and (f), because (d), (e), and (f) are all equivalent for *any* matrix A. (See Section 1.7 and Theorem 12(b) in Section 1.9.) Finally, (a) $\Rightarrow$ (l) by Theorem 6(c) in Section 2.2, and (l) $\Rightarrow$ (a) by the same theorem with A and A^T interchanged. This completes the proof. ■

Because of Theorem 5 in Section 2.2, statement (g) in Theorem 8 could also be written as "The equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for each $\mathbf{b}$ in $\mathbb{R}^n$." This statement certainly implies (b) and hence implies that A is invertible.

The next fact follows from Theorem 8 and Exercise 10 in Section 2.2.

Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.

The Invertible Matrix Theorem divides the set of all $n \times n$ matrices into two disjoint classes: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices. Each statement in the theorem describes a property of every $n \times n$ invertible matrix. The *negation* of a statement in the theorem describes a property of every $n \times n$ singular matrix. For instance, an $n \times n$ singular matrix is *not* row equivalent to I_n, does *not* have n pivot positions, and has linearly *dependent* columns. Negations of other statements are considered in the exercises.

EXAMPLE 1 Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

SOLUTION

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

So A has three pivot positions and hence is invertible, by the Invertible Matrix Theorem, statement (c). ■

STUDY GUIDE offers an expanded table for the Invertible Matrix Theorem.

The power of the Invertible Matrix Theorem lies in the connections it provides among so many important concepts, such as linear independence of columns of a matrix A and the existence of solutions to equations of the form $A\mathbf{x} = \mathbf{b}$. It should be emphasized, however, that the Invertible Matrix Theorem *applies only to square matrices*. For example, if the columns of a 4×3 matrix are linearly independent, we cannot use the Invertible Matrix Theorem to conclude anything about the existence or nonexistence of solutions to equations of the form $A\mathbf{x} = \mathbf{b}$.

Invertible Linear Transformations

Recall from Section 2.1 that matrix multiplication corresponds to composition of linear transformations. When a matrix A is invertible, the equation $A^{-1}A\mathbf{x} = \mathbf{x}$ can be viewed as a statement about linear transformations. See Figure 2.

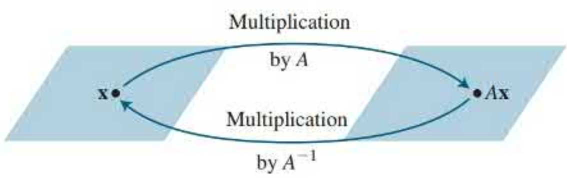


FIGURE 2 A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{1}$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{2}$$

The next theorem shows that if such an S exists, it is unique and must be a linear transformation. We call S the **inverse** of T and write it as T^{-1}.

THEOREM 9

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equations (1) and (2).

Remark: See the comment on the proof of Theorem 7.

PROOF Suppose that T is invertible. Then (2) shows that T is onto $\mathbb{R}^n$, for if $\mathbf{b}$ is in $\mathbb{R}^n$ and $\mathbf{x} = S(\mathbf{b})$, then $T(\mathbf{x}) = T(S(\mathbf{b})) = \mathbf{b}$, so each $\mathbf{b}$ is in the range of T. Thus A is invertible, by the Invertible Matrix Theorem, statement (i).

Conversely, suppose that A is invertible, and let $S(\mathbf{x}) = A^{-1}\mathbf{x}$. Then, S is a linear transformation, and S obviously satisfies (1) and (2). For instance,

$$S(T(\mathbf{x})) = S(A\mathbf{x}) = A^{-1}(A\mathbf{x}) = \mathbf{x}$$

Thus T is invertible. The proof that S is unique is outlined in Exercise 47. ■

EXAMPLE 2 What can you say about a one-to-one linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^n$?

SOLUTION The columns of the standard matrix A of T are linearly independent (by Theorem 12 in Section 1.9). So A is invertible, by the Invertible Matrix Theorem, and T maps $\mathbb{R}^n$ *onto* $\mathbb{R}^n$. Also, T is invertible, by Theorem 9. ■

Numerical Notes

In practical work, you might occasionally encounter a "nearly singular" or **ill-conditioned** matrix—an invertible matrix that can become singular if some of its entries are changed ever so slightly. In this case, row reduction may produce fewer than n pivot positions, as a result of roundoff error. Also, roundoff error can sometimes make a singular matrix appear to be invertible.

Some matrix programs will compute a **condition number** for a square matrix. The larger the condition number, the closer the matrix is to being singular. The condition number of the identity matrix is 1. A singular matrix has an infinite condition number. In extreme cases, a matrix program may not be able to distinguish between a singular matrix and an ill-conditioned matrix.

Exercises 49–53 show that matrix computations can produce substantial error when a condition number is large.

Practice Problems

1. Determine if $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{bmatrix}$ is invertible.
2. Suppose that for a certain $n \times n$ matrix A, statement (g) of the Invertible Matrix Theorem is *not* true. What can you say about equations of the form $A\mathbf{x} = \mathbf{b}$?
3. Suppose that A and B are $n \times n$ matrices and the equation $AB\mathbf{x} = \mathbf{0}$ has a nontrivial solution. What can you say about the matrix AB?

2.3 Exercises

Unless otherwise specified, assume that all matrices in these exercises are $n \times n$. Determine which of the matrices in Exercises 1–10 are invertible. Use as few calculations as possible. Justify your answers.

1. $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$

2. $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$

4. $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$

7. $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

T 9. $\begin{bmatrix} 4 & 0 & -7 & -7 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 3 & -1 \end{bmatrix}$

T 10. $\begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & -5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix}$

In Exercises 11–20, the matrices are all $n \times n$. Each part of the exercises is an *implication* of the form "If 'statement 1', then 'statement 2'." Mark an implication as True if the truth of "statement 2" *always* follows whenever "statement 1" happens to be true. An implication is False if there is an instance in which "statement 2" is false but "statement 1" is true. Justify each answer.

11. **(T/F)** If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.

12. **(T/F)** If there is an $n \times n$ matrix D such that $AD = I$, then there is also an $n \times n$ matrix C such that $CA = I$.

13. **(T/F)** If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent.

14. **(T/F)** If the columns of A are linearly independent, then the columns of A span $\mathbb{R}^n$.

15. **(T/F)** If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

16. **(T/F)** If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then the solution is unique for each $\mathbf{b}$.

17. **(T/F)** If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions.

18. **(T/F)** If the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, then A has n pivot positions.

19. **(T/F)** If A^T is not invertible, then A is not invertible.

20. **(T/F)** If there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one.

21. An $m \times n$ **upper triangular matrix** is one whose entries *below* the main diagonal are 0's (as in Exercise 8). When is a square upper triangular matrix invertible? Justify your answer.

22. An $m \times n$ **lower triangular matrix** is one whose entries *above* the main diagonal are 0's (as in Exercise 3). When is a square lower triangular matrix invertible? Justify your answer.

23. Can a square matrix with two identical columns be invertible? Why or why not?

24. Is it possible for a 5×5 matrix to be invertible when its columns do not span $\mathbb{R}^5$? Why or why not?

25. If A is invertible, then the columns of A^{-1} are linearly independent. Explain why.

26. If C is 6×6 and the equation $C\mathbf{x} = \mathbf{v}$ is consistent for every $\mathbf{v}$ in $\mathbb{R}^6$, is it possible that for some $\mathbf{v}$, the equation $C\mathbf{x} = \mathbf{v}$ has more than one solution? Why or why not?

27. If the columns of a 7×7 matrix D are linearly independent, what can you say about solutions of $D\mathbf{x} = \mathbf{b}$? Why?

28. If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute. Explain why.

29. If the equation $G\mathbf{x} = \mathbf{y}$ has more than one solution for some $\mathbf{y}$ in $\mathbb{R}^n$, can the columns of G span $\mathbb{R}^n$? Why or why not?

30. If the equation $H\mathbf{x} = \mathbf{c}$ is inconsistent for some $\mathbf{c}$ in $\mathbb{R}^n$, what can you say about the equation $H\mathbf{x} = \mathbf{0}$? Why?

31. If an $n \times n$ matrix K cannot be row reduced to I_n, what can you say about the columns of K? Why?

32. If L is $n \times n$ and the equation $L\mathbf{x} = \mathbf{0}$ has the trivial solution, do the columns of L span $\mathbb{R}^n$? Why?

33. Verify the boxed statement preceding Example 1.

34. Explain why the columns of A^2 span $\mathbb{R}^n$ whenever the columns of A are linearly independent.

35. Show that if AB is invertible, so is A. You cannot use Theorem 6(b), because you cannot *assume* that A and B are invertible. [*Hint:* There is a matrix W such that $ABW = I$. Why?]

36. Show that if AB is invertible, so is B.

37. If A is an $n \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution for some $\mathbf{b}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is

not one-to-one. What else can you say about this transformation? Justify your answer.

38. If A is an $n \times n$ matrix and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, what else can you say about this transformation? Justify your answer.

39. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Without using Theorems 5 or 8, explain why each equation $A\mathbf{x} = \mathbf{b}$ has in fact exactly one solution.

40. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Without using the Invertible Matrix Theorem, explain directly why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

In Exercises 41 and 42, T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$. Show that T is invertible and find a formula for T^{-1}.

41. $T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$

42. $T(x_1, x_2) = (6x_1 - 8x_2, -5x_1 + 7x_2)$

43. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Explain why T is both one-to-one and onto $\mathbb{R}^n$. Use equations (1) and (2). Then give a second explanation using one or more theorems.

44. Let T be a linear transformation that maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Show that T^{-1} exists and maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Is T^{-1} also one-to-one?

45. Suppose T and U are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(U\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Is it true that $U(T\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$? Why or why not?

46. Suppose a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(\mathbf{u}) = T(\mathbf{v})$ for some pair of distinct vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Can T map $\mathbb{R}^n$ onto $\mathbb{R}^n$? Why or why not?

47. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation, and let S and U be functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ such that $S(T(\mathbf{x})) = \mathbf{x}$ and $U(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Show that $U(\mathbf{v}) = S(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^n$. This will show that T has a unique inverse, as asserted in Theorem 9. [*Hint:* Given any $\mathbf{v}$ in $\mathbb{R}^n$, we can write $\mathbf{v} = T(\mathbf{x})$ for some $\mathbf{x}$. Why? Compute $S(\mathbf{v})$ and $U(\mathbf{v})$.]

48. Suppose T and S satisfy the invertibility equations (1) and (2), where T is a linear transformation. Show directly that S is a linear transformation. [*Hint:* Given $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^n$, let $\mathbf{x} = S(\mathbf{u})$, $\mathbf{y} = S(\mathbf{v})$. Then $T(\mathbf{x}) = \mathbf{u}$, $T(\mathbf{y}) = \mathbf{v}$. Why? Apply S to both sides of the equation $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$. Also, consider $T(c\mathbf{x}) = cT(\mathbf{x})$.]

T **49.** Suppose an experiment leads to the following system of equations:

$$\begin{aligned} 4.5x_1 + 3.1x_2 &= 19.249 \\ 1.6x_1 + 1.1x_2 &= 6.843 \end{aligned} \tag{3}$$

a. Solve system (3), and then solve system (4), below, in which the data on the right have been rounded to two decimal places. In each case, find the *exact* solution.

$$\begin{aligned} 4.5x_1 + 3.1x_2 &= 19.25 \\ 1.6x_1 + 1.1x_2 &= 6.84 \end{aligned} \tag{4}$$

b. The entries in (4) differ from those in (3) by less than .05%. Find the percentage error when using the solution of (4) as an approximation for the solution of (3).

c. Use your matrix program to produce the condition number of the coefficient matrix in (3).

Exercises 50–52 show how to use the condition number of a matrix A to estimate the accuracy of a computed solution of $A\mathbf{x} = \mathbf{b}$. If the entries of A and $\mathbf{b}$ are accurate to about r significant digits and if the condition number of A is approximately 10^k (with k a positive integer), then the computed solution of $A\mathbf{x} = \mathbf{b}$ should usually be accurate to at least $r - k$ significant digits.

T **50.** Find the condition number of the matrix A in Exercise 9. Construct a random vector $\mathbf{x}$ in $\mathbb{R}^4$ and compute $\mathbf{b} = A\mathbf{x}$. Then use your matrix program to compute the solution $\mathbf{x}_1$ of $A\mathbf{x} = \mathbf{b}$. To how many digits do $\mathbf{x}$ and $\mathbf{x}_1$ agree? Find out the number of digits your matrix program stores accurately, and report how many digits of accuracy are lost when $\mathbf{x}_1$ is used in place of the exact solution $\mathbf{x}$.

T **51.** Repeat Exercise 50 for the matrix in Exercise 10.

T **52.** Solve an equation $A\mathbf{x} = \mathbf{b}$ for a suitable $\mathbf{b}$ to find the last column of the inverse of the *fifth-order Hilbert matrix*

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix}$$

How many digits in each entry of $\mathbf{x}$ do you expect to be correct? Explain. [*Note:* The exact solution is (630, −12600, 56700, −88200, 44100).]

T **53.** Some matrix programs, such as MATLAB, have a command to create Hilbert matrices of various sizes. If possible, use an inverse command to compute the inverse of a twelfth-order or larger Hilbert matrix, A. Compute AA^{-1}. Report what you find.

STUDY GUIDE offers additional resources for reviewing and reflecting on what you have learned.

Solutions to Practice Problems

1. The columns of A are obviously linearly dependent because columns 2 and 3 are multiples of column 1. Hence, A cannot be invertible (by the Invertible Matrix Theorem).

Solutions to Practice Problems (Continued)

2. If statement (g) is *not* true, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for at least one $\mathbf{b}$ in $\mathbb{R}^n$.
3. Apply the Invertible Matrix Theorem to the matrix AB in place of A. Then statement (d) becomes: $AB\mathbf{x} = \mathbf{0}$ has only the trivial solution. This is not true. So AB is not invertible.

2.4 Partitioned Matrices

A key feature of our work with matrices has been the ability to regard a matrix A as a list of column vectors rather than just a rectangular array of numbers. This point of view has been so useful that we wish to consider other **partitions** of A, indicated by horizontal and vertical dividing rules, as in Example 1 below. Partitioned matrices appear in most modern applications of linear algebra because the notation highlights essential structures in matrix analysis, as in the chapter introductory example on aircraft design. This section provides an opportunity to review matrix algebra and use the Invertible Matrix Theorem.

EXAMPLE 1 The matrix

$$A = \left[\begin{array}{rrr|rr|r} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]$$

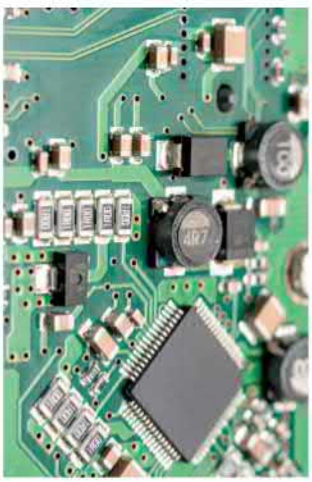
can also be written as the 2×3 **partitioned** (or **block**) **matrix**

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

whose entries are the *blocks* (or *submatrices*)

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -4 \end{bmatrix}$$ ■



EXAMPLE 2 When a matrix A appears in a mathematical model of a physical system such as an electrical network, a transportation system, or a large corporation, it may be natural to regard A as a partitioned matrix. For instance, if a microcomputer circuit board consists mainly of three VLSI (very large-scale integrated) microchips, then the matrix for the circuit board might have the general form

$$A = \left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right]$$

The submatrices on the "diagonal" of A—namely A_{11}, A_{22}, and A_{33}—concern the three VLSI chips, while the other submatrices depend on the interconnections among those microchips. ■

Addition and Scalar Multiplication

If matrices A and B are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum $A + B$. In this

Solution to Practice Problem

Assemble the matrices right-to-left for the three operations. Using $\mathbf{p} = (-2, 6)$, $\cos(-30°) = \sqrt{3}/2$, and $\sin(-30°) = -.5$, we have

$$\underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Translate back by } p} \underbrace{\begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Rotate around the origin}} \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Translate by } -p}$$

$$= \begin{bmatrix} \sqrt{3}/2 & 1/2 & \sqrt{3}-5 \\ -1/2 & \sqrt{3}/2 & -3\sqrt{3}+5 \\ 0 & 0 & 1 \end{bmatrix}$$

2.8 Subspaces of $\mathbb{R}^n$

This section focuses on important sets of vectors in $\mathbb{R}^n$ called *subspaces*. Often subspaces arise in connection with some matrix A, and they provide useful information about the equation $A\mathbf{x} = \mathbf{b}$. The concepts and terminology in this section will be used repeatedly throughout the rest of the book.[1]

DEFINITION

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a. The zero vector is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

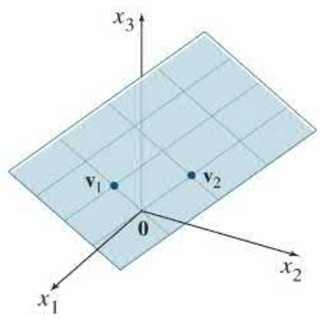


FIGURE 1 Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ as a plane through the origin.

In words, a subspace is *closed* under addition and scalar multiplication. As you will see in the next few examples, most sets of vectors discussed in Chapter 1 are subspaces. For instance, a plane through the origin is the standard way to visualize the subspace in Example 1. See Figure 1.

EXAMPLE 1 If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^n$ and $H = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$, then H is a subspace of $\mathbb{R}^n$. To verify this statement, note that the zero vector is in H (because $0\mathbf{v}_1 + 0\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$). Now take two arbitrary vectors in H, say,

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

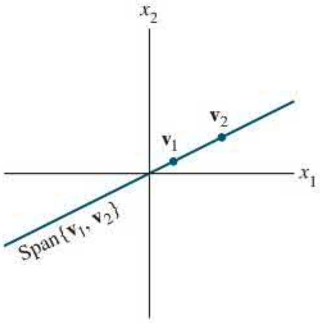
Then

$$\mathbf{u} + \mathbf{v} = (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2$$

which shows that $\mathbf{u} + \mathbf{v}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and hence is in H. Also, for any scalar c, the vector $c\mathbf{u}$ is in H, because $c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2$. ■

If $\mathbf{v}_1$ is not zero and if $\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ simply span a *line* through the origin. So a line through the origin is another example of a subspace.

[1] Sections 2.8 and 2.9 are included here to permit readers to postpone the study of most or all of the next two chapters and to skip directly to Chapter 5, if so desired. *Omit* these two sections if you plan to work through Chapter 4 before beginning Chapter 5.



$\mathbf{v}_1 \neq \mathbf{0}$, $\mathbf{v}_2 = k\mathbf{v}_1$.

EXAMPLE 2 A line L *not* through the origin is *not* a subspace, because it does not contain the origin, as required. Also, Figure 2 shows that L is not closed under addition or scalar multiplication. ■

$\mathbf{u} + \mathbf{v}$ is not on L $\qquad$ $2\mathbf{w}$ is not on L

FIGURE 2

EXAMPLE 3 For $\mathbf{v}_1, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is a subspace of $\mathbb{R}^n$. The verification of this statement is similar to the argument given in Example 1. We shall now refer to Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ as **the subspace spanned** (or **generated**) by $\mathbf{v}_1, \dots, \mathbf{v}_p$. ■

Note that $\mathbb{R}^n$ is a subspace of itself because it has the three properties required for a subspace. Another special subspace is the set consisting of only the zero vector in $\mathbb{R}^n$. This set, called the **zero subspace**, also satisfies the conditions for a subspace.

Column Space and Null Space of a Matrix

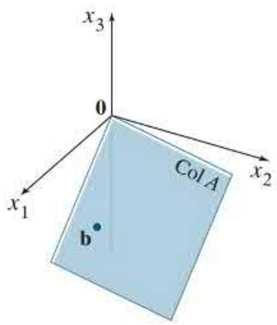
Subspaces of $\mathbb{R}^n$ usually occur in applications and theory in one of two ways. In both cases, the subspace can be related to a matrix.

DEFINITION

The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A.

If $A = [\,\mathbf{a}_1 \ \cdots \ \mathbf{a}_n\,]$, with the columns in $\mathbb{R}^m$, then Col A is the same as Span$\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Example 4 shows that the **column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$**. Note that Col A equals $\mathbb{R}^m$ only when the columns of A span $\mathbb{R}^m$. Otherwise, Col A is only part of $\mathbb{R}^m$.

EXAMPLE 4 Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. Determine whether $\mathbf{b}$ is in the column space of A.



SOLUTION The vector $\mathbf{b}$ is a linear combination of the columns of A if and only if $\mathbf{b}$ can be written as $A\mathbf{x}$ for some $\mathbf{x}$, that is, if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution. Row reducing the augmented matrix $[\,A \ \ \mathbf{b}\,]$,

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we conclude that $A\mathbf{x} = \mathbf{b}$ is consistent and $\mathbf{b}$ is in Col A. ■

The solution of Example 4 shows that when a system of linear equations is written in the form $A\mathbf{x} = \mathbf{b}$, the column space of A is the set of all $\mathbf{b}$ for which the system has a solution.

DEFINITION

The **null space** of a matrix A is the set Nul A of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

When A has n columns, the solutions of $A\mathbf{x} = \mathbf{0}$ belong to $\mathbb{R}^n$, and the null space of A is a subset of $\mathbb{R}^n$. In fact, Nul A has the properties of a *subspace* of $\mathbb{R}^n$.

THEOREM 12

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

PROOF The zero vector is in Nul A (because $A\mathbf{0} = \mathbf{0}$). To show that Nul A satisfies the other two properties required for a subspace, take any $\mathbf{u}$ and $\mathbf{v}$ in Nul A. That is, suppose $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Then, by a property of matrix multiplication,

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

Thus $\mathbf{u} + \mathbf{v}$ satisfies $A\mathbf{x} = \mathbf{0}$, and so $\mathbf{u} + \mathbf{v}$ is in Nul A. Also, for any scalar c, $A(c\mathbf{u}) = c(A\mathbf{u}) = c(\mathbf{0}) = \mathbf{0}$, which shows that $c\mathbf{u}$ is in Nul A. ■

To test whether a given vector $\mathbf{v}$ is in Nul A, just compute $A\mathbf{v}$ to see whether $A\mathbf{v}$ is the zero vector. Because Nul A is described by a condition that must be checked for each vector, we say that the null space is defined *implicitly*. In contrast, the column space is defined *explicitly*, because vectors in Col A can be constructed (by linear combinations) from the columns of A. To create an explicit description of Nul A, solve the equation $A\mathbf{x} = \mathbf{0}$ and write the solution in parametric vector form. (See Example 6.)[2]

Basis for a Subspace

Because a subspace typically contains an infinite number of vectors, some problems involving a subspace are handled best by working with a small finite set of vectors that span the subspace. The smaller the set, the better. It can be shown that the smallest possible spanning set must be linearly independent.

DEFINITION

A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

EXAMPLE 5 The columns of an invertible $n \times n$ matrix form a basis for all of $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem. One such matrix is the $n \times n$ identity matrix. Its columns are denoted by $\mathbf{e}_1, \ldots, \mathbf{e}_n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$. See Figure 3. ■

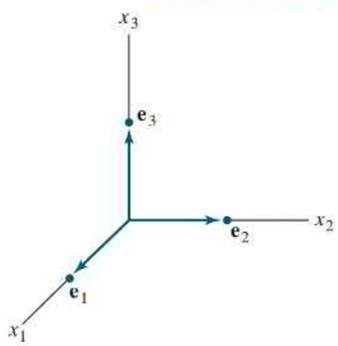


FIGURE 3
The standard basis for $\mathbb{R}^3$.

[2] The contrast between Nul A and Col A is discussed further in Section 4.2.

The next example shows that the standard procedure for writing the solution set of $A\mathbf{x} = \mathbf{0}$ in parametric vector form actually identifies a basis for Nul A. This fact will be used throughout Chapter 5.

EXAMPLE 6 Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

SOLUTION First, write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 - 2x_2 \qquad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2, x_4, and x_5 free.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underset{\uparrow \atop \mathbf{u}}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\uparrow \atop \mathbf{v}}{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + x_5 \underset{\uparrow \atop \mathbf{w}}{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w} \tag{1}$$

Equation (1) shows that Nul A coincides with the set of all linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. That is, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ generates Nul A. In fact, this construction of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ automatically makes them linearly independent, because equation (1) shows that $\mathbf{0} = x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$ only if the weights x_2, x_4, and x_5 are all zero. (Examine entries 2, 4, and 5 in the vector $x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$.) So $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a *basis* for Nul A. ■

Finding a basis for the column space of a matrix is actually less work than finding a basis for the null space. However, the method requires some explanation. Let's begin with a simple case.

EXAMPLE 7 Find a basis for the column space of the matrix

$$B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

SOLUTION Denote the columns of B by $\mathbf{b}_1, \dots, \mathbf{b}_5$ and note that $\mathbf{b}_3 = -3\mathbf{b}_1 + 2\mathbf{b}_2$ and $\mathbf{b}_4 = 5\mathbf{b}_1 - \mathbf{b}_2$. The fact that $\mathbf{b}_3$ and $\mathbf{b}_4$ are combinations of the pivot columns means that any combination of $\mathbf{b}_1, \dots, \mathbf{b}_5$ is actually just a combination of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_5$. Indeed, if $\mathbf{v}$ is any vector in Col B, say,

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 + c_4\mathbf{b}_4 + c_5\mathbf{b}_5$$

then, substituting for $\mathbf{b}_3$ and $\mathbf{b}_4$, we can write $\mathbf{v}$ in the form

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3(-3\mathbf{b}_1 + 2\mathbf{b}_2) + c_4(5\mathbf{b}_1 - \mathbf{b}_2) + c_5\mathbf{b}_5$$

which is a linear combination of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_5$. So $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5\}$ spans Col B. Also, $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_5$ are linearly independent, because they are columns from an identity matrix. So the pivot columns of B form a basis for Col B. ■

The matrix B in Example 7 is in reduced echelon form. To handle a general matrix A, recall that linear dependence relations among the columns of A can be expressed in the form $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x}$. (If some columns are not involved in a particular dependence relation, then the corresponding entries in $\mathbf{x}$ are zero.) When A is row reduced to echelon form B, the columns are drastically changed, but the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same set of solutions. That is, the columns of A have *exactly the same linear dependence relationships* as the columns of B.

EXAMPLE 8 It can be verified that the matrix

$$A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_5\,] = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$$

is row equivalent to the matrix B in Example 7. Find a basis for Col A.

SOLUTION From Example 7, the pivot columns of A are columns 1, 2, and 5. Also, $\mathbf{b}_3 = -3\mathbf{b}_1 + 2\mathbf{b}_2$ and $\mathbf{b}_4 = 5\mathbf{b}_1 - \mathbf{b}_2$. Since row operations do not affect linear dependence relations among the columns of the matrix, we should have

$$\mathbf{a}_3 = -3\mathbf{a}_1 + 2\mathbf{a}_2 \quad \text{and} \quad \mathbf{a}_4 = 5\mathbf{a}_1 - \mathbf{a}_2$$

Check that this is true! By the argument in Example 7, $\mathbf{a}_3$ and $\mathbf{a}_4$ are not needed to generate the column space of A. Also, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ must be linearly independent, because any dependence relation among $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_5$ would imply the same dependence relation among $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_5$. Since $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5\}$ is linearly independent, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ is also linearly independent and hence is a basis for Col A. ■

The argument in Example 8 can be adapted to prove the following theorem.

THEOREM 13

The pivot columns of a matrix A form a basis for the column space of A.

Warning: Be careful to use *pivot columns of A itself* for the basis of Col A. The columns of an echelon form B are often not in the column space of A. (For instance, in Examples 7 and 8, the columns of B all have zeros in their last entries and cannot generate the columns of A.)

Practice Problems

1. Let $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$. Is $\mathbf{u}$ in Nul A? Is $\mathbf{u}$ in Col A? Justify each answer.

2. Given $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, find a vector in Nul A and a vector in Col A.

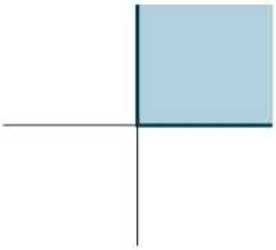
3. Suppose an $n \times n$ matrix A is invertible. What can you say about Col A? About Nul A?

STUDY GUIDE offers additional resources for mastering the concepts of subspace, column space, and null space.

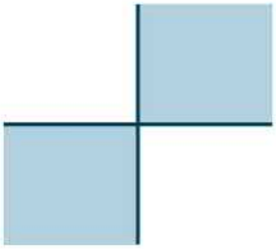
2.8 Exercises

Exercises 1–4 display sets in $\mathbb{R}^2$. Assume the sets include the bounding lines. In each case, give a specific reason why the set H is *not* a subspace of $\mathbb{R}^2$. (For instance, find two vectors in H whose sum is *not* in H, or find a vector in H with a scalar multiple that is not in H. Draw a picture.)

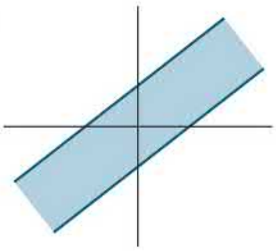
1.



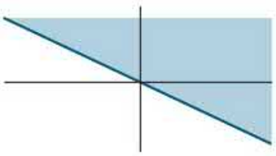
2.



3.



4.



5. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 8 \\ 2 \\ -9 \end{bmatrix}$. Determine if $\mathbf{w}$ is in the subspace of $\mathbb{R}^3$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

6. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -7 \\ 9 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ -8 \\ 6 \\ 5 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} -4 \\ 10 \\ -7 \\ -5 \end{bmatrix}$. Determine if $\mathbf{u}$ is in the subspace of $\mathbb{R}^4$ generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

7. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$, and $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$.

a. How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. How many vectors are in Col A?

c. Is $\mathbf{p}$ in Col A? Why or why not?

8. Let $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -6 \\ 3 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 1 \\ 14 \\ -9 \end{bmatrix}$. Determine if $\mathbf{p}$ is in Col A, where $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$.

9. With A and $\mathbf{p}$ as in Exercise 7, determine if $\mathbf{p}$ is in Nul A.

10. With $\mathbf{u} = (-2, 3, 1)$ and A as in Exercise 8, determine if $\mathbf{u}$ is in Nul A.

In Exercises 11 and 12, give integers p and q such that Nul A is a subspace of $\mathbb{R}^p$ and Col A is a subspace of $\mathbb{R}^q$.

11. $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}$

13. For A as in Exercise 11, find a nonzero vector in Nul A and a nonzero vector in Col A.

14. For A as in Exercise 12, find a nonzero vector in Nul A and a nonzero vector in Col A.

Determine which sets in Exercises 15–20 are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify each answer.

15. $\begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 10 \\ -3 \end{bmatrix}$

16. $\begin{bmatrix} -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

17. $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}$

18. $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$

19. $\begin{bmatrix} 3 \\ -8 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -5 \end{bmatrix}$

20. $\begin{bmatrix} 1 \\ -6 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$

In Exercises 21–30, mark each statement Ture or False **(T/F)**. Justify each answer.

21. **(T/F)** A subspace of $\mathbb{R}^n$ is any set H such that (i) the zero vector is in H, (ii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ are in H, and (iii) c is a scalar and $c\mathbf{u}$ is in H.

22. **(T/F)** A subset H of $\mathbb{R}^n$ is a subspace if the zero vector is in H.

23. **(T/F)** If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the same as the column space of the matrix $[\mathbf{v}_1 \ \dots \ \mathbf{v}_p]$.

24. **(T/F)** Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of these vectors is a subspace of $\mathbb{R}^n$.

25. **(T/F)** The set of all solutions of a system of m homogeneous equations in n unknowns is a subspace of $\mathbb{R}^m$.

26. **(T/F)** The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

27. **(T/F)** The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

28. **(T/F)** The column space of a matrix A is the set of solutions of $A\mathbf{x} = \mathbf{b}$.

29. **(T/F)** Row operations do not affect linear dependence relations among the columns of a matrix.

30. **(T/F)** If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col A.

Exercises 31–34 display a matrix A and an echelon form of A. Find a basis for Col A and a basis for Nul A.

31. $A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

32. $A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

33. $A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

34. $A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -1 & 7 & 0 & 6 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

35. Construct a nonzero 3×3 matrix A and a nonzero vector $\mathbf{b}$ such that $\mathbf{b}$ is in Col A, but $\mathbf{b}$ is not the same as any one of the columns of A.

36. Construct a nonzero 3×3 matrix A and a vector $\mathbf{b}$ such that $\mathbf{b}$ is *not* in Col A.

37. Construct a nonzero 3×3 matrix A and a nonzero vector $\mathbf{b}$ such that $\mathbf{b}$ is in Nul A.

38. Suppose the columns of a matrix $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_p]$ are linearly independent. Explain why $\{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ is a basis for Col A.

In Exercises 39–44, respond as comprehensively as possible, and justify your answer.

39. Suppose F is a 5×5 matrix whose column space is not equal to $\mathbb{R}^5$. What can you say about Nul F?

40. If R is a 6×6 matrix and Nul R is *not* the zero subspace, what can you say about Col R?

41. If Q is a 4×4 matrix and Col $Q = \mathbb{R}^4$, what can you say about solutions of equations of the form $Q\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ in $\mathbb{R}^4$?

42. If P is a 5×5 matrix and Nul P is the zero subspace, what can you say about solutions of equations of the form $P\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ in $\mathbb{R}^5$?

43. What can you say about Nul B when B is a 5×4 matrix with linearly independent columns?

44. What can you say about the shape of an $m \times n$ matrix A when the columns of A form a basis for $\mathbb{R}^m$?

In Exercises 45 and 46, construct bases for the column space and the null space of the given matrix A. Justify your work.

T **45.** $A = \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix}$

T **46.** $A = \begin{bmatrix} 5 & 2 & 0 & -8 & -8 \\ 4 & 1 & 2 & -8 & -9 \\ 5 & 1 & 3 & 5 & 19 \\ -8 & -5 & 6 & 8 & 5 \end{bmatrix}$

Solutions to Practice Problems

1. To determine whether $\mathbf{u}$ is in Nul A, simply compute

$$A\mathbf{u} = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}\begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The result shows that $\mathbf{u}$ is in Nul A. Deciding whether $\mathbf{u}$ is in Col A requires more work. Reduce the augmented matrix $[A \quad \mathbf{u}]$ to echelon form to determine whether the equation $A\mathbf{x} = \mathbf{u}$ is consistent:

$$\begin{bmatrix} 1 & -1 & 5 & -7 \\ 2 & 0 & 7 & 3 \\ -3 & -5 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 5 & -7 \\ 0 & 2 & -3 & 17 \\ 0 & -8 & 12 & -19 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 5 & -7 \\ 0 & 2 & -3 & 17 \\ 0 & 0 & 0 & 49 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has no solution, so $\mathbf{u}$ is not in Col A.

2. In contrast to Practice Problem 1, finding a vector in Nul A requires more work than testing whether a specified vector is in Nul A. However, since A is already in reduced echelon form, the equation $A\mathbf{x} = \mathbf{0}$ shows that if $\mathbf{x} = (x_1, x_2, x_3)$, then $x_2 = 0$, $x_3 = 0$, and x_1 is a free variable. Thus, a basis for Nul A is $\mathbf{v} = (1, 0, 0)$. Finding just one vector in Col A is trivial, since each column of A is in Col A. In this particular case, the same vector $\mathbf{v}$ is in both Nul A *and* Col A. For most $n \times n$ matrices, the zero vector of $\mathbb{R}^n$ is the only vector in both Nul A and Col A.

3. If A is invertible, then the columns of A span $\mathbb{R}^n$, by the Invertible Matrix Theorem. By definition, the columns of any matrix always span the column space, so in this case Col A is all of $\mathbb{R}^n$. In symbols, Col $A = \mathbb{R}^n$. Also, since A is invertible, the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This means that Nul A is the zero subspace. In symbols, Nul $A = \{\mathbf{0}\}$.

2.9 Dimension and Rank

This section continues the discussion of subspaces and bases for subspaces, beginning with the concept of a coordinate system. The definition and example below should make a useful new term, *dimension*, seem quite natural, at least for subspaces of $\mathbb{R}^3$.

Coordinate Systems

The main reason for selecting a basis for a subspace H, instead of merely a spanning set, is that each vector in H can be written in only one way as a linear combination of the basis vectors. To see why, suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ is a basis for H, and suppose a vector $\mathbf{x}$ in H can be generated in two ways, say,

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p \quad \text{and} \quad \mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_p\mathbf{b}_p \tag{1}$$

Then, subtracting gives

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_p - d_p)\mathbf{b}_p \tag{2}$$

Since $\mathcal{B}$ is linearly independent, the weights in (2) must all be zero. That is, $c_j = d_j$ for $1 \le j \le p$, which shows that the two representations in (1) are actually the same.

DEFINITION

Suppose the set $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ is a basis for a subspace H. For each $\mathbf{x}$ in H, the **coordinates of x relative to the basis** $\mathcal{B}$ are the weights $c_1, \dots, c_p$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$, and the vector in $\mathbb{R}^p$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the **coordinate vector of x (relative to** $\mathcal{B}$**)** or the $\mathcal{B}$**-coordinate vector of x**.[1]

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

SOLUTION If $\mathbf{x}$ is in H, then the following vector equation is consistent:

$$c_1\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

The scalars c_1 and c_2, if they exist, are the $\mathcal{B}$-coordinates of $\mathbf{x}$. Row operations show that

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $c_1 = 2$, $c_2 = 3$, and $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The basis $\mathcal{B}$ determines a "coordinate system" on H, which can be visualized by the grid shown in Figure 1. ■

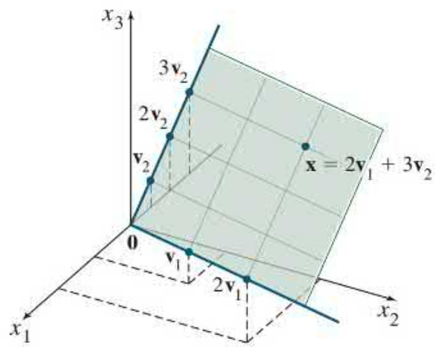


FIGURE 1 A coordinate system on a plane H in $\mathbb{R}^3$.

Notice that although points in H are also in $\mathbb{R}^3$, they are completely determined by their coordinate vectors, which belong to $\mathbb{R}^2$. The grid on the plane in Figure 1

[1] It is important that the elements of $\mathcal{B}$ are numbered because the entries in $[\mathbf{x}]_{\mathcal{B}}$ depend on the order of the vectors in $\mathcal{B}$.

makes H "look" like $\mathbb{R}^2$. The correspondence $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ is a one-to-one correspondence between H and $\mathbb{R}^2$ that preserves linear combinations. We call such a correspondence an *isomorphism*, and we say that H is *isomorphic* to $\mathbb{R}^2$.

In general, if $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ is a basis for H, then the mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one correspondence that makes H look and act the same as $\mathbb{R}^p$ (even though the vectors in H themselves may have more than p entries). (Section 4.4 has more details.)

The Dimension of a Subspace

It can be shown that if a subspace H has a basis of p vectors, then every basis of H must consist of exactly p vectors. (See Exercises 35 and 36.) Thus the following definition makes sense.

DEFINITION The **dimension** of a nonzero subspace H, denoted by $\dim H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.[2]

The space $\mathbb{R}^n$ has dimension n. Every basis for $\mathbb{R}^n$ consists of n vectors. A plane through $\mathbf{0}$ in $\mathbb{R}^3$ is two-dimensional, and a line through $\mathbf{0}$ is one-dimensional.

EXAMPLE 2 Recall that the null space of the matrix A in Example 6 in Section 2.8 had a basis of 3 vectors. So the dimension of Nul A in this case is 3. Observe how each basis vector corresponds to a free variable in the equation $A\mathbf{x} = \mathbf{0}$. Our construction always produces a basis in this way. So, to find the dimension of Nul A, simply identify and count the number of free variables in $A\mathbf{x} = \mathbf{0}$. ■

DEFINITION The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A.

Since the pivot columns of A form a basis for Col A, the rank of A is just the number of pivot columns in A.

EXAMPLE 3 Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

SOLUTION Reduce A to echelon form:

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

The matrix A has 3 pivot columns, so rank $A = 3$. ■

[2] The zero subspace has *no* basis (because the zero vector by itself forms a linearly dependent set).

The row reduction in Example 3 reveals that there are two free variables in $A\mathbf{x} = \mathbf{0}$, because two of the five columns of A are *not* pivot columns. (The nonpivot columns correspond to the free variables in $A\mathbf{x} = \mathbf{0}$.) Since the number of pivot columns plus the number of nonpivot columns is exactly the number of columns, the dimensions of Col A and Nul A have the following useful connection. (See the Rank Theorem in Section 4.6 for additional details.)

THEOREM 14

The Rank Theorem

If a matrix A has n columns, then rank A + dim Nul $A = n$.

The following theorem is important for applications and will be needed in Chapters 5 and 6. The theorem (proved in Section 4.5) is certainly plausible, if you think of a p-dimensional subspace as isomorphic to $\mathbb{R}^p$. The Invertible Matrix Theorem shows that p vectors in $\mathbb{R}^p$ are linearly independent if and only if they also span $\mathbb{R}^p$.

THEOREM 15

The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

Rank and the Invertible Matrix Theorem

The various vector space concepts associated with a matrix provide several more statements for the Invertible Matrix Theorem. They are presented below to follow the statements in the original theorem in Section 2.3.

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.
n. Col $A = \mathbb{R}^n$
o. rank $A = n$
p. dim Nul $A = 0$
q. Nul $A = \{\mathbf{0}\}$

PROOF Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning. The other four statements are linked to the earlier ones of the theorem by the following chain of almost trivial implications:

$$(g) \Rightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (q) \Rightarrow (d)$$

Statement (g), which says that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, implies statement (n), because Col A is precisely the set of all $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent. The implications (n) $\Rightarrow$ (o) $\Rightarrow$ (p) follow from the definitions of *dimension* and *rank*. If the rank of A is n, the number of columns of A, STUDY GUIDE offers an expanded Invertible Matrix Theorem Table.

then dim Nul $A = 0$, by the Rank Theorem, and so Nul $A = \{\mathbf{0}\}$. Thus (p) $\Rightarrow$ (q). Also, statement (q) implies that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, which is statement (d). Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete. ■

Numerical Notes

Many algorithms discussed in this text are useful for understanding concepts and making simple computations by hand. However, the algorithms are often unsuitable for large-scale problems in real life.

Rank determination is a good example. It would seem easy to reduce a matrix to echelon form and count the pivots. But unless exact arithmetic is performed on a matrix whose entries are specified exactly, row operations can change the apparent rank of a matrix. For instance, if the value of x in the matrix $\begin{bmatrix} 5 & 7 \\ 5 & x \end{bmatrix}$ is not stored exactly as 7 in a computer, then the rank may be 1 or 2, depending on whether the computer treats $x - 7$ as zero.

In practical applications, the effective rank of a matrix A is often determined from the singular value decomposition of A, to be discussed in Section 7.4.

Practice Problems

1. Determine the dimension of the subspace H of $\mathbb{R}^3$ spanned by the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. (First, find a basis for H.)

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -7 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 6 \\ -7 \end{bmatrix}$$

2. Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ .2 \end{bmatrix}, \begin{bmatrix} .2 \\ 1 \end{bmatrix} \right\}$$

for $\mathbb{R}^2$. If $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, what is $\mathbf{x}$?

3. Could $\mathbb{R}^3$ possibly contain a four-dimensional subspace? Explain.

2.9 Exercises

In Exercises 1 and 2, find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$. Illustrate your answer with a figure, as in the solution of Practice Problem 2.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2. $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

In Exercises 3–6, the vector $\mathbf{x}$ is in a subspace H with a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.

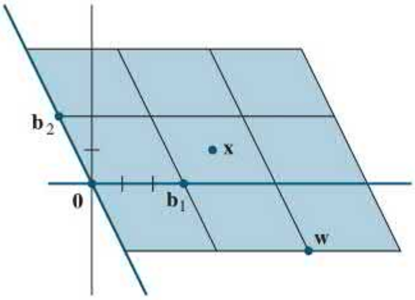
3. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

4. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

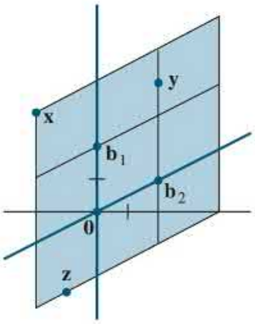
5. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -7 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 10 \\ -7 \end{bmatrix}$

6. $\mathbf{b}_1 = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 11 \\ 0 \\ 7 \end{bmatrix}$

7. Let $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{w}]_\mathcal{B}$ and $[\mathbf{x}]_\mathcal{B}$. Confirm your estimate of $[\mathbf{x}]_\mathcal{B}$ by using it and $\{\mathbf{b}_1, \mathbf{b}_2\}$ to compute $\mathbf{x}$.



8. Let $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} -1 \\ -2.5 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{x}]_\mathcal{B}$, $[\mathbf{y}]_\mathcal{B}$, and $[\mathbf{z}]_\mathcal{B}$. Confirm your estimates of $[\mathbf{y}]_\mathcal{B}$ and $[\mathbf{z}]_\mathcal{B}$ by using them and $\{\mathbf{b}_1, \mathbf{b}_2\}$ to compute $\mathbf{y}$ and $\mathbf{z}$.



Exercises 9–12 display a matrix A and an echelon form of A. Find bases for Col A and Nul A, and then state the dimensions of these subspaces.

9. $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 13 and 14, find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

13. $\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -7 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 9 \\ -5 \end{bmatrix}$

15. Suppose a 3×5 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$? Is Nul $A = \mathbb{R}^2$? Explain your answers.

16. Suppose a 4×7 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$? What is the dimension of Nul A? Explain your answers.

In Exercises 17–26, mark each statement True or False **(T/F)**. Justify each answer. Here A is an $m \times n$ matrix.

17. **(T/F)** If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for a subspace H and if $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then $c_1, \dots, c_p$ are the coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$.

18. **(T/F)** If $\mathcal{B}$ is a basis for a subspace H, then each vector in H can be written in only one way as a linear combination of the vectors in $\mathcal{B}$.

19. **(T/F)** Each line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$.

20. **(T/F)** If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for a subspace H of $\mathbb{R}^n$, then the correspondence $\mathbf{x} \mapsto [\mathbf{x}]_\mathcal{B}$ makes H look and act the same as $\mathbb{R}^p$.

21. **(T/F)** The dimension of Col A is the number of pivot columns of A.

22. **(T/F)** The dimension of Nul A is the number of variables in the equation $A\mathbf{x} = \mathbf{0}$.

23. **(T/F)** The dimensions of Col A and Nul A add up to the number of columns of A.

24. **(T/F)** The dimension of the column space of A is rank A.

25. **(T/F)** If a set of p vectors spans a p-dimensional subspace H of $\mathbb{R}^n$, then these vectors form a basis for H.

26. **(T/F)** If H is a p-dimensional subspace of $\mathbb{R}^n$, then a linearly independent set of p vectors in H is a basis for H.

In Exercises 27–32, justify each answer or construction.

27. If the subspace of all solutions of $A\mathbf{x} = \mathbf{0}$ has a basis consisting of three vectors and if A is a 5×7 matrix, what is the rank of A?

28. What is the rank of a 4×5 matrix whose null space is three-dimensional?

29. If the rank of a 7×6 matrix A is 4, what is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?

30. Show that a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5\}$ in $\mathbb{R}^n$ is linearly dependent when dim Span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5\} = 4$.

31. If possible, construct a 3×4 matrix A such that dim Nul $A = 2$ and dim Col $A = 2$.

32. Construct a 4×3 matrix with rank 1.

33. Let A be an $n \times p$ matrix whose column space is p-dimensional. Explain why the columns of A must be linearly independent.

34. Suppose columns 1, 3, 5, and 6 of a matrix A are linearly independent (but are not necessarily pivot columns) and the rank of A is 4. Explain why the four columns mentioned must be a basis for the column space of A.

35. Suppose vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p$ span a subspace W, and let $\{\mathbf{a}_1, \ldots, \mathbf{a}_q\}$ be any set in W containing more than p vectors. Fill in the details of the following argument to show that $\{\mathbf{a}_1, \ldots, \mathbf{a}_q\}$ must be linearly dependent. First, let $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ and $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_q]$.

a. Explain why for each vector $\mathbf{a}_j$, there exists a vector $\mathbf{c}_j$ in $\mathbb{R}^p$ such that $\mathbf{a}_j = B\mathbf{c}_j$.

b. Let $C = [\mathbf{c}_1 \ \cdots \ \mathbf{c}_q]$. Explain why there is a nonzero vector $\mathbf{u}$ such that $C\mathbf{u} = \mathbf{0}$.

c. Use B and C to show that $A\mathbf{u} = \mathbf{0}$. This shows that the columns of A are linearly dependent.

36. Use Exercise 35 to show that if $\mathcal{A}$ and $\mathcal{B}$ are bases for a subspace W of $\mathbb{R}^n$, then $\mathcal{A}$ cannot contain more vectors than $\mathcal{B}$, and, conversely, $\mathcal{B}$ cannot contain more vectors than $\mathcal{A}$.

T 37. Let $H = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Show that $\mathbf{x}$ is in H, and find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$, when

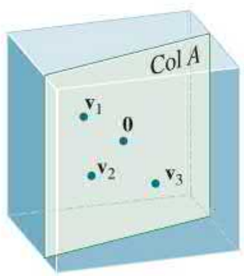
$$\mathbf{v}_1 = \begin{bmatrix} 11 \\ -5 \\ 10 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 14 \\ -8 \\ 13 \\ 10 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 19 \\ -13 \\ 18 \\ 15 \end{bmatrix}$$

T 38. Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Show that $\mathcal{B}$ is a basis for H and $\mathbf{x}$ is in H, and find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$, when

$$\mathbf{v}_1 = \begin{bmatrix} -6 \\ 4 \\ -9 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 8 \\ -3 \\ 7 \\ -3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -9 \\ 5 \\ -8 \\ 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 7 \\ -8 \\ 3 \end{bmatrix}$$

STUDY GUIDE offers additional resources for mastering the concepts of dimension and rank.

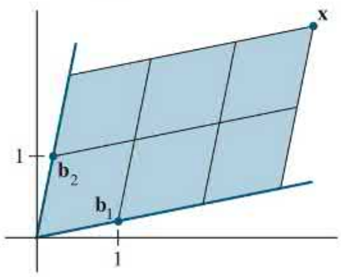
Solutions to Practice Problems



1. Construct $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ so that the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is the column space of A. A basis for this space is provided by the pivot columns of A.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -8 & -7 & 6 \\ 6 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & -10 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The first two columns of A are pivot columns and form a basis for H. Thus dim $H = 2$.



2. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $\mathbf{x}$ is formed from a linear combination of the basis vectors using weights 3 and 2:

$$\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2 = 3\begin{bmatrix} 1 \\ .2 \end{bmatrix} + 2\begin{bmatrix} .2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 2.6 \end{bmatrix}$$

The basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ determines a *coordinate system* for $\mathbb{R}^2$, illustrated by the grid in the figure. Note how $\mathbf{x}$ is 3 units in the $\mathbf{b}_1$-direction and 2 units in the $\mathbf{b}_2$-direction.

3. A four-dimensional subspace would contain a basis of four linearly independent vectors. This is impossible inside $\mathbb{R}^3$. Since any linearly independent set in $\mathbb{R}^3$ has no more than three vectors, any subspace of $\mathbb{R}^3$ has dimension no more than 3. The space $\mathbb{R}^3$ itself is the only three-dimensional subspace of $\mathbb{R}^3$. Other subspaces of $\mathbb{R}^3$ have dimension 2, 1, or 0.

CHAPTER 2 PROJECTS

Chapter 2 projects are available online at `bit.ly/30IM8gT`.

A. *Other Matrix Products*: This project introduces two new operations on square matrices called the Jordan product and the commutator product and their properties are explored.

B. *Adjacency Matrices*: The purpose of this project is to show how powers of a matrix may be used to investigate graphs.

C. *Dominance Matrices*: The purpose of this project is to apply matrices and their powers to questions concerning various forms of competition between individuals and groups.

D. *Condition Numbers*: The purpose of this project is to show how a condition number of a matrix A may be defined, and how its value affects the accuracy of solutions to systems of equations $A\mathbf{x} = \mathbf{b}$.

E. *Equilibrium Temperature Distributions*: The purpose of this project is to discuss a physical situation in which solving a system of linear equations becomes necessary: that of determining the equilibrium temperature of a thin plate.

F. *The LU and QR Factorizations*: The purpose of this project is to explore a relationship between two matrix factorizations: the LU factorization and the QR factorization.

G. *The Leontief Input–Output Model*: The purpose of this project is to provide three more examples of the Leontief input–output model in action.

H. *The Art of Linear Transformations*: This project illustrates how to graph a polygon and then use linear transformations to move it around in the plane.

CHAPTER 2 SUPPLEMENTARY EXERCISES

Assume that the matrices mentioned in Exercises 1–15 below have appropriate sizes. Mark each statement True or False **(T/F)**. Justify each answer.

1. **(T/F)** If A and B are $m \times n$, then both AB^T and A^TB are defined.

2. **(T/F)** If $AB = C$ and C has 2 columns, then A has 2 columns.

3. **(T/F)** Left-multiplying a matrix B by a diagonal matrix A, with nonzero entries on the diagonal, scales the rows of B.

4. **(T/F)** If $BC = BD$, then $C = D$.

5. **(T/F)** If $AC = 0$, then either $A = 0$ or $C = 0$.

6. **(T/F)** If A and B are $n \times n$, then $(A + B)(A - B) = A^2 - B^2$.

7. **(T/F)** An elementary $n \times n$ matrix has either n or $n + 1$ nonzero entries.

8. **(T/F)** The transpose of an elementary matrix is an elementary matrix.

9. **(T/F)** An elementary matrix must be square.

10. **(T/F)** Every square matrix is a product of elementary matrices.

11. **(T/F)** If A is a 3×3 matrix with three pivot positions, there exist elementary matrices $E_1, \dots, E_p$ such that $E_p \cdots E_1 A = I$.

12. **(T/F)** If $AB = I$, then A is invertible.

13. **(T/F)** If A and B are square and invertible, then AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$.

14. **(T/F)** If $AB = BA$ and if A is invertible, then $A^{-1}B = BA^{-1}$.

15. **(T/F)** If A is invertible and if $r \neq 0$, then $(rA)^{-1} = rA^{-1}$.

16. Find the matrix C whose inverse is $C^{-1} = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$.

17. Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Show that $A^3 = 0$. Use matrix algebra to compute the product $(I - A)(I + A + A^2)$.

18. Suppose $A^n = 0$ for some $n > 1$. Find an inverse for $I - A$.

19. Suppose an $n \times n$ matrix A satisfies the equation $A^2 - 2A + I = 0$. Show that $A^3 = 3A - 2I$ and $A^4 = 4A - 3I$.