2. Convergence of Power Method.

(d) part 1: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A-\lambda I) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+b)\lambda + (ad-bc) = 0.$

Since we know trace of $A = a+b$, $\det A = ad-bc$.

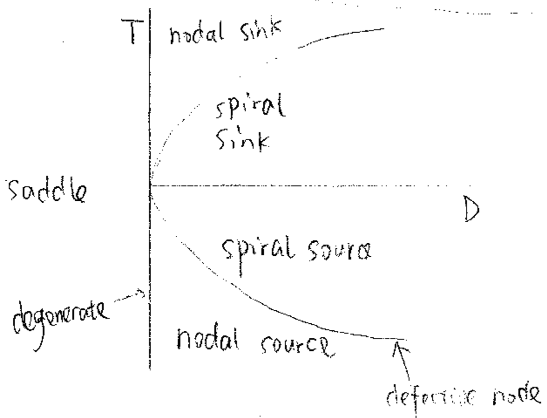
Let $T = \operatorname{tr} A$, $D = \det A$. if 2×2 matrix A has eigenvalue λ_1 and λ_2,

then $T = \lambda_1 + \lambda_2$ $\quad D = \lambda_1 \lambda_2$.

$\Rightarrow \det(A-\lambda I) = \lambda^2 - T\lambda + D \Rightarrow \lambda = \frac{T \pm \sqrt{T^2-4D}}{2}$ (By Quadratic Formula)

$\Rightarrow$ If $T^2-4D \geq 0$, λ will be real $\Rightarrow$ The curve of determinant-trace graph is

if $T^2-4D<0$, λ will be complex. given by function $D = \frac{T^2}{4}$



In our plot:

if $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 > 0$, we have source

if $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 < 0$, we have sink

if $\lambda_1 \neq \lambda_2$, $\lambda_1 > 0, \lambda_2 < 0$, or $\lambda_1 < 0, \lambda_2 > 0$. we have saddle.

If $T<0$, $T^2-4D>0$, and

if $D>0$, $\lambda_1, \lambda_2 < 0$. sink.

if $D<0$, $\lambda_1>0, \lambda_2<0$, or $\lambda_1<0, \lambda_2>0$. saddle

if $D=0$. $\lambda_1>0, \lambda_2<0$ or $\lambda_1<0, \lambda_2>0$.

Cited: (Thomas W. Judson,
Department of Mathematics, Harvard University, Spring 2008)

for $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d-\lambda & -b \\ -c & a-\lambda \end{bmatrix}$, $\det(A^{-1}-\lambda I) = \frac{1}{\det A}\left[\lambda^2 - (a+b)\lambda + (ab+bc)\right] = 0.$

$\Rightarrow \frac{1}{T}(\lambda^2 - T\lambda + D) = 0 \Rightarrow \lambda^2 - T\lambda + D = 0.$ (here, T and D are based on A)

$\Rightarrow$ We can tell that for A and A^{-1}, they have same quadratic equation to solve for λ_1 and λ_2, and they also have the same equation $D = \frac{T^2}{4}$ to draw the determinant-trace graph.

Therefore, for A^{-1} and A, they have similar D-T graph.