Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools

MODULE(S^2): Algebra for Secondary Mathematics Teaching

Adapted for University of Nebraska-Lincoln

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Set and Logical Notation

Set Notation

Definition 1.1. A **set** is a collection of objects, which are called the **elements** of the set.

$x \in D$	"x is an element of the set D" (a proposition about x and its *domain* D)
$P(x)$	A proposition about the variable x; may be true or false depending on x
$\{x \in D : P(x)\}$ $\{x \in D \mid P(x)\}$	The set of all elements of D for which $P(x)$ is true (a subset of D). Note that the "rule" for set membership may be given in many ways, not just an algebraic rule. For example, graphical, table, list, description.
$A \subseteq B$	"A is a subset of B" (a proposition about sets A and B)
$A \subsetneq B$	"A is a strict subset of B", i.e., "$A \subseteq B$ and $A \neq B$"
$A \supseteq B$	"A is a superset of B" or "B is a subset of A"
$A \supsetneq B$	"A is a strict superset of B" or "B is a strict subset of A", i.e., "$A \supseteq B$ and $A \neq B$"
$A \cap B$	The intersection of the sets A and B (a set)
$A \cup B$	The union of the sets A and B (a set)
$\varnothing$	The *empty set* (the set with no elements); also known as *null set*
$\lvert A \rvert$	The cardinality ("size") of A. When A is finite, $\lvert A \rvert$ is the number of elements in A.

Note. The notation for subset (without the bottom line) is ambiguous: some people use it to mean $A \subseteq B$ and others use it to mean $A \subsetneq B$. So we don't use it here.

Definition 1.2. Given sets A and B. We say A **is equal to** B if $A \subseteq B$ and $B \subseteq A$.
Notation: $A = B$.

Logical notation

$\neg P(x)$	The negation of $P(x)$
$\forall x, P(x)$	The proposition "For all values of x, $P(x)$ is true."
$\exists x : P(x)$	The proposition "There exists a value of x such that $P(x)$ is true."
$\forall x, P(x) \Rightarrow Q(x)$	The proposition "For all values of x, if $P(x)$ is true then $Q(x)$ is true."
$\forall x, P(x) \Leftrightarrow Q(x)$	The proposition "For all values of x, $P(x)$ is true if and only if $Q(x)$ is true."

Proof structures

To show that ...	Requires showing that ...
$x \in A$	x satisfies set membership rules for A
$x \notin A$	x does not satisfy at least one set membership rule of A
$A \subseteq B$	If $x \in A$, then $x \in B$
$A \subsetneq B$	(1) $A \subseteq B$ (2) there is an element of B that is not in A
$A = B$	(1) $A \subseteq B$ (2) $B \subseteq A$

Sets of numbers

$\mathbb{N}$	The set of *natural numbers* (positive whole numbers)
$\mathbb{Z}$	The set of *integers* (all whole numbers – positive, negative, and zero)
$\mathbb{Q}$	The set of *rational numbers* (all fractions)
$\mathbb{R}$	The set of *real numbers* (all numbers on the real line; equivalently, all decimal numbers)
$\mathbb{C}$	The set of *complex numbers* (all numbers of the form $a + bi$, where a and b are real)

Properties of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$

Operations are well-defined

Well-defined: There is an answer, and there isn't more than one answer.

Operations $+, -, \times$ on $\mathbb{R}$ and $\mathbb{C}$ are well-defined: This means that when we add two numbers, we get exactly one answer (we don't expect there two be two answers to "What is $a + b$?" and we expect that there is an answer); similarly, when we subtract one number from another, or multiply two numbers, we get exactly one answer.

Division by nonzero numbers is well-defined. (There is no good numerical answer to "What is $a/0$?")

Arithmetic Properties of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$

We state them below for $\mathbb{Z}$. They also hold for $\mathbb{R}$.

1	$a, b \in \mathbb{Z} \implies a + b \in \mathbb{Z}$	$\mathbb{Z}$ is closed under addition
2	$a, b, c \in \mathbb{Z} \implies a + (b + c) = (a + b) + c$	Addition in $\mathbb{Z}$ is associative
3	$a, b \in \mathbb{Z} \implies a + b = b + a$	Addition in $\mathbb{Z}$ is commutative
4	$a \in \mathbb{Z} \implies a + 0 = a = 0 + a$	0 is an additive identity in $\mathbb{Z}$
5	$\forall a \in \mathbb{Z}$, the equation $a + x = 0$ has a solution in $\mathbb{Z}$	Additive inverses exist in $\mathbb{Z}$
6	$a, b \in \mathbb{Z} \implies ab \in \mathbb{Z}$	$\mathbb{Z}$ is closed under multiplication
7	$a, b, c \in \mathbb{Z} \implies a(bc) = (ab)c$	Multiplication in $\mathbb{Z}$ is associative
8	$a, b, c \in \mathbb{Z} \implies a(b + c) = ab + ac$ and $(a + b)c = ac + bc$	Distributive property
9	$a, b \in \mathbb{Z} \implies ab = ba$	Multiplication in $\mathbb{Z}$ is commutative
10	$a \in \mathbb{Z} \implies a \cdot 1 = a = 1 \cdot a$	1 is a multiplicative identity in $\mathbb{Z}$
11	$a, b \in \mathbb{Z}, ab = 0 \implies a = 0$ or $b = 0$	$\mathbb{Z}$ has no zero divisors

Divides, Divisor, Factor

- Given $a, b \in \mathbb{Z}$, not both zero. We say *b divides a* if $a = bc$ for some integer c. Notation: $b \mid a$

  These all mean the same thing:

  - b divides a
  - b is a divisor of a
  - b is a factor of a
  - $b \mid a$

  If we want to say that b does not divide a, we write $b \nmid a$.
- A factor of a number is *trivial* if it is ± 1 or the $\pm$ number. A *nontrivial* factor that is not trivial.
- All nonzero natural numbers have a finite number of factors.
- Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Prime, Composite

- An integer p is *prime* if $p \neq 0, \pm 1$ if the only divisors of p are ± 1 and $\pm p$.

  An integer n is *composite* if $n \neq 0, \pm 1$, and it is not prime.
- Let $a \in \mathbb{Z}$. If p, q are primes such that $p \mid a$ and $q \mid a$, and $p \neq q$, then $pq \mid a$.

Even number

An integer n is even if it is divisible by 2.

Fundamental Theorem of Arithmetic

There is only one way to write any whole number as a product of positive primes (reordering doesn't count as a different way).

Sample handwritten proof

Let's use one of the proofs we did in class as an example. We begin with the typed up version and then show one way that this same proof might be handwritten.

Claim. If $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$, then $B \subsetneq A$.

Proof. Given $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$.

1. *Why $B \subseteq A$*: This means showing: if $x \in B$, then $x \in A$.
   Given $x \in B$. Then:

$$\begin{aligned} x &= 6k, k \in \mathbb{Z}, \text{ by definition of } B \\ &= 3 \cdot 2k \\ &= 3n, n \in \mathbb{Z}, \text{ because } 2 \in \mathbb{Z}, k \in \mathbb{Z}, \text{ and } \mathbb{Z} \text{ is closed under multiplication} \end{aligned}$$

   Therefore x satisfies set membership rules of A, implying $x \in A$.
   We have shown that if $x \in B$, then $x \in A$. Thus $B \subseteq A$, by definition of subset. $\boxed{1}$

2. *Why there is an element of A that is not in B.* We find an element of A that is not in B. If $x \in B$, then x is an even number because if $x = 6k$ for some $k \in \mathbb{Z}$, then x as $x = 2 \cdot (3k)$. Closure of multiplication in $\mathbb{Z}$ implies $3k \in \mathbb{Z}$, so x satisfies the definition of even number.
   However, some members of A are odd numbers: $3, 9, 15, \ldots$.
   Hence there are elements of A that are not in B. $\boxed{2}$

Why this means $B \subsetneq A$: (1) and (2) show that B and A satisfy the definition of strict subset, and we have $B \subsetneq A$. $\square$

Claim. $\left.\begin{array}{l} A = \{3n : n \in \mathbb{Z}\} \\ B = \{6n : n \in \mathbb{Z}\} \end{array}\right] \Rightarrow B \subsetneq A.$

Proof. Given $A = \{3n : n \in \mathbb{Z}\}$
$B = \{6n : n \in \mathbb{Z}\}$.

(1) $\boxed{B \subseteq A}$ We show: $x \in B \Rightarrow x \in A$.

Given $x \in B$.

$$(*)\quad \left[\begin{array}{ll} x = 6k, \ k \in \mathbb{Z} & \text{by defn of membership in } B \\ = 3 \cdot 2k & \\ = 3n, \ n \in \mathbb{Z}. & \text{by closure of mult in } \mathbb{Z} \ (2, k \in \mathbb{Z} \Rightarrow 2k \in \mathbb{Z}) \end{array}\right.$$

Hence x satisfies membership rules for A
$\Rightarrow x \in A$.
By defn of subset, $B \subseteq A$. $\boxed{1}$

(2) $\boxed{\exists\, x \in A \text{ s.t. } x \notin B}$ We find an elt of A not in B. Observe that all $x \in B$ are even (by defn of even):

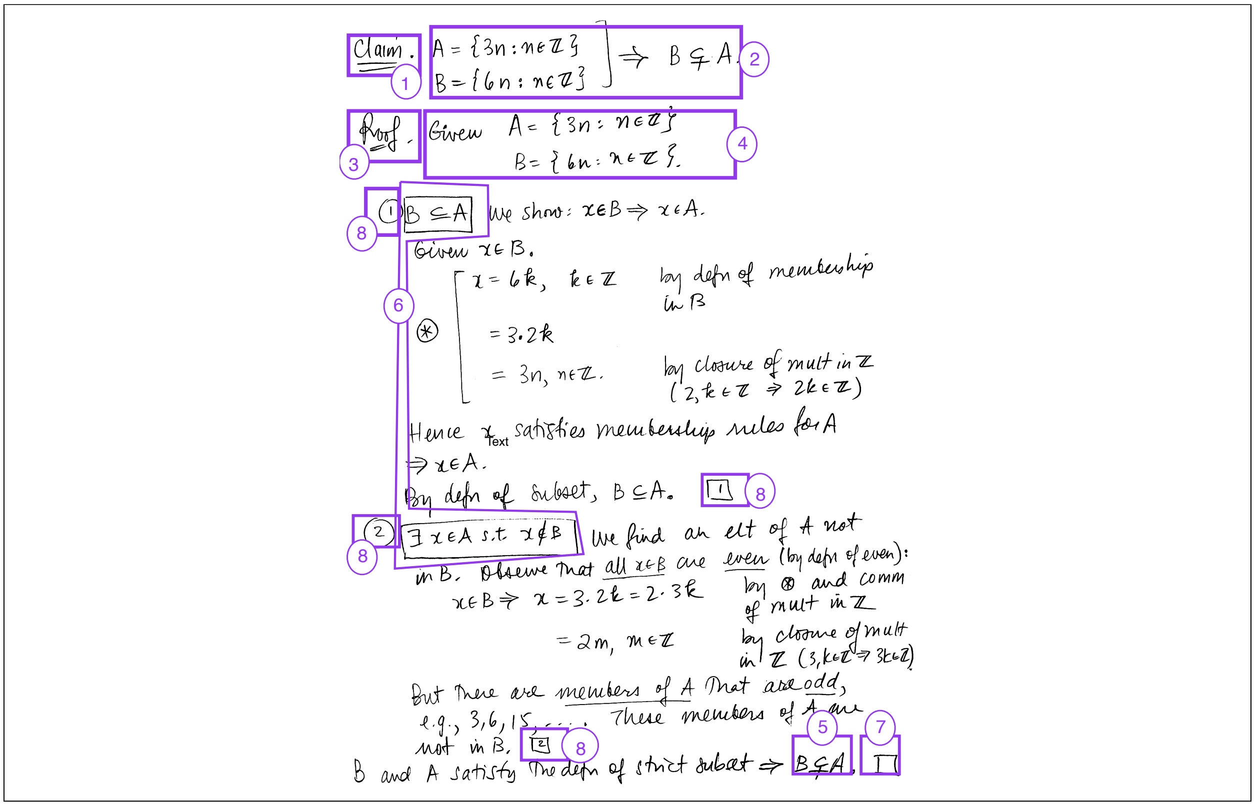
$$\begin{array}{ll} x \in B \Rightarrow x = 3 \cdot 2k = 2 \cdot 3k & \text{by } (*) \text{ and comm of mult in } \mathbb{Z} \\ \qquad\qquad = 2m, \ m \in \mathbb{Z} & \text{by closure of mult in } \mathbb{Z} \ (3, k \in \mathbb{Z} \Rightarrow 3k \in \mathbb{Z}) \end{array}$$

But there are members of A that are odd, e.g., $3, 6, 15, \ldots$. These members of A are not in B. $\boxed{2}$

B and A satisfy the defn of strict subset $\Rightarrow B \subsetneq A$. $\square$

Good proof communication

Here is the same proof, with key features pointed out. These features are explained at the bottom. In general, you want to incorporate most if not all of these features into any proof you write. Even though it might seem strange at first, you may find eventually that you learn math better when you develop the habits of incorporating these features into your own writing and being aware of these features in proofs you encounter.



Features of communicating proof well:
(**Essential features in bold**)

1. **Label the claim.**
2. **State the claim precisely.**
3. **Label the proof beginning.**
4. Begin a proof by reminding yourself and readers of the starting point: the conditions of the claim.
5. **End the proof with where you need to go: the conclusions of the claim.**
6. Summarize your approach to the reader.
7. **Label the proof end.** A traditional way is to use a box.
8. **Write up parts within a proof properly**. **Label when they begin and end.**
   - Give them a name (e.g., Claim A) if it is a proof within a proof
   - **Use labels like** [$\Rightarrow$] **and** [$\Leftarrow$] **if doing an if and only if proof.**
9. Diagrams are good only if you explain what you are showing. Give a caption.

1 Sets, Claims, Negations (Week 1) (Length: 2.5 hours)

Overview

Content

"Parent" relation, implicitly defined as a relation which assigns elements of $\mathbb{N}$ to its factors; used to examine subsets, mathematical statements and their negations, properties of $\mathbb{R}$ and $\mathbb{Z}$, and to engage in mathematical practices.

(Looking ahead:) The parent relation is used in Section 2 to introduce relations and inverse relations.

Subset, **superset**, **strict subset**, and **strict superset**; **equality of sets** A and B, defined as $A \subseteq B$ and $B \subseteq A$.

Mathematical statements, defined as those which can be evaluated as true or false; and

Negation of mathematical statement S, defined as a statement which is false if and only if S is true.

Properties of $\mathbb{R}$ and $\mathbb{Z}$ assumed. (These may have been introduced previously in an abstract algebra course.)

Proof Structures

To show that $x \in A$ means showing that x satisfies set membership rules for A; and **to show that $x \notin A$** means showing that x does not satisfy at least one set membership rule of A.

To show that $A \subseteq B$ requires showing that if $x \in A$, then $x \in B$.

To show that $A \subsetneq B$ requires showing that: (1) $A \subseteq B$; (2) there is an element of B that is not in A.

To show that $A = B$ requires showing that: (1) $A \subseteq B$; (2) $B \subseteq A$.

Mathematical/Teaching Practices

Clarifying mathematical questions, meaning to determine how different interpretations of question statements may have different mathematical consequences.

Conjecturing and being precise, in the sense of giving "satisfying" answers to mathematical questions

Communicating proofs well, which includes specifying claims, the body of the proof, and givens and conclusions explicitly, clearly, and correctly.

Summary

We introduce the "parent relation" as a context for engaging in mathematical practices as well as learning how to work with each other on exploratory tasks. The main tasks in this lesson are:

- *Which numbers have more than one pair of parents?*
- *Is one of these sets a subset of the other set? Check the mathematically correct statements. If you put a check in the $A \neq B$ column, list an element that is in one but not the other.*

	$A \subseteq B$	$A \subsetneq B$	$A \supseteq B$	$A \supsetneq B$	$A = B$	$A \neq B$	Neither is subset of the other
$A =$ multiples of 3, $B =$ multiples of 6							
$A =$ multiples of 6, $B =$ multiples of 9							
$A = \{n^2 : n \in \mathbb{N}, n > 0\}$, $B = \{1 + 3 + \cdots + (2n+1) : n \in \mathbb{N}\}$							
$A =$ functions of the form $x \mapsto 16^{ax}$, $B =$ functions of the form $x \mapsto 2^{ax}$							

Along the way we introduce notation for sets and subsets, discuss mathematical statements and their negations, and describe properties of $\mathbb{R}$ and $\mathbb{Z}$ assumed for now. There are also tasks in this lesson addressing these ideas.

Acknowledgements. The structure and some tasks of Set notation and Mathematical statements and their negations are from notes from Mira Bernstein and used with permission.

Opening inquiry: Number parents

We begin this lesson with the following inquiry:

Two numbers are parents of a child if the child is their product.

A child cannot be its own parent.

Which numbers have more than one pair of parents?

Child	Parents
6	2, 3
4	??
12	4, 3
12	2, 6

As we discussed this question, we learned some issues that arise when asking and answering mathematical questions:

- *Clarifying the question.* Let's assume that we are only working with natural numbers $(0, 1, 2, \ldots)$, and that 2, 2 is a set of parents for 4. So we are looking for natural numbers that have more than one pair of parents. We allow pairs of parents to repeat parents.
- *Finding and improving possible answers (conjecturing well).* Here are some possible answers (without explanations) to this question. Which is the most satisfying answer (without explanation)? Why?
    1. 12 has more than one pair of parents.
    2. 12, 18, 20, 28, 30, 42, 44 each have more than one pair of parents.
    3. Any number with at least least three different factors has more than one pair of parents.
    4. Any number with at least three different factors (that aren't itself or 1) has more than one pair of parents.
    5. Any number with at least three different factors (that aren't itself or 1) has more than one pair of parents. There are no other numbers with more than one pair of parents.

We concluded that an answer is satisfying when it gives the most complete and correct understanding of a situation. We also gave the analogy of answering a question that a child asks, and that the quality of being "satisfying" when giving an answer to a mathematical may well be similar to what makes an answer "satisfying" to a child.

- The first two are dissatisfying because they don't give any sort of pattern or big picture of what's going on. They raise the question: "Are those the only ones?"
- The third one is almost there, but is actually slightly incorrect. The fourth one is getting there, and it is correct. But still, neither answer the question of whether there are more answers.
- The fifth answer is the most satisfying because it provides the big picture of when a number works, and also says, yes, these are the only answers.

We also gave a name to the process of finding and improving answers to mathematical question: the practice of *conjecturing*. Before we get into proving or disproving our conjectures, we first talk about sets. This will give us a structure for addressing this inquiry more completely.

Sets, subsets, supersets, and set equality

Set notation

Definition 1.1. A **set** is a collection of objects, which are called the **elements** of the set.

$x \in D$	"x is an element of the set D" (a proposition about x and its *domain* D)
$P(x)$	A proposition about the variable x; may be true or false depending on x
$\{x \in D : P(x)\}$ $\{x \in D \mid P(x)\}$	The set of all elements of D for which $P(x)$ is true (a subset of D)
$A \subseteq B$	"A is a subset of B" (a proposition about sets A and B)
$A \subsetneq B$	"A is a strict subset of B", i.e., "$A \subseteq B$ and $A \neq B$"
$A \supseteq B$	"A is a superset of B" or "B is a subset of A"
$A \supsetneq B$	"A is a strict superset of B" or "B is a strict subset of A", i.e., "$A \supseteq B$ and $A \neq B$"
$A \cap B$	The intersection of the sets A and B (a set)
$A \cup B$	The union of the sets A and B (a set)
$\emptyset$	The *empty set* (the set with no elements); also known as *null set*
$\|A\|$	The cardinality ("size") of A. When A is finite, $\|A\|$ is the number of elements in A.

Note: The notation for subset (without the bottom line) is ambiguous: some people use it to mean $A \subseteq B$ and others use it to mean $A \subsetneq B$. So we don't use it here.

Definition 1.2. Given sets A and B. We say A **is equal to** B if $A \subseteq B$ and $B \subseteq A$. We denote equality with $A = B$.

1. Let $A = \{1, 2, \{3, 4\}, \{5\}\}$. Decide whether each of the following statements is true or false: (**Hint:** There are exactly six true statements.)

$$\begin{array}{rrrr} 1 \in A, & \{1,2\} \in A, & \{1,2\} \subseteq A, & \emptyset \in A, \\ 3 \in A, & \{3,4\} \in A, & \{3,4\} \subseteq A, & \emptyset \subseteq A, \\ \{1\} \in A, & \{1\} \subseteq A, & \{5\} \in A, & \{5\} \subseteq A. \end{array}$$

2. True or false? "All students in this class who are under 5 years old are also over 100 years old."

Solution.

1. (a) TRUE (b) false (c) TRUE (d) false
   (e) false (f) TRUE (g) false (h) TRUE
   (i) false (j) TRUE (k) TRUE (l) false

   Reasoning. There are four elements of the set A:
   - 1 (the number 1)
   - 2 (the number 2)
   - $\{3, 4\}$ (the set containing the numbers 3, 4)
   - $\{5\}$ (the set containing the number 5)

   The notation $\in$ means "is an element of" is . That's why (a), (f), (k) are TRUE and (b), (d), (e), (i) are false.

   The notation $\subseteq$ means "is a subset of". The set is a subset of A if each of its elements are also elements of A. That's why (c), (j) are TRUE and (g), (l) are false.

   Finally, (h) is TRUE on a technicality. It contains no elements. So all zero of its elements are part of A. The empty set is a subset of any set for this reason.
2. For most sections of mathematics courses at university level, this statement should be TRUE. ■

Note: One helpful metaphor may be thinking of the braces (the { and }) as permanent packaging, like gift wrap that doesn't come off. You can't take out what's inside the packaging. You can only hold the whole package. Even if only one thing is wrapped, you still can't hold the thing by itself, you can only hold it with its gift wrap. But if an object not wrapped, you can hold that object by itself.

Proof Structure: Showing set membership. To show that $x \in S$ means showing that x satisfies set membership rules for S; to show that $x \notin S$ means showing that x does not satisfy at least one set membership rule of A.

> Let $S = \{x \in \mathbb{Q} : x \text{ can be written as a fraction with denominator 2 and } |x| < 2\}$.
> True or false? $0.5 \in S$, $3.5 \in S$, $0.25 \in S$, $1 \in S$.

Solution. (Partial)

(a) $0.5 \in S$ is TRUE because it can be written as the fraction $\frac{1}{2}$ and $|0.5| < 2$. The number 0.5 satisfies all the rules of membership of S, so it is an element of S.

(b) $3.5 \in S$ is FALSE because even though it can be written as the fraction $\frac{7}{2}$, it does not satisfy the condition $|x| < 2$. The number 3.5 does not satisfy all the rules of membership of S, so it is not an element of S.

(c) $0.25 \in S$ is FALSE. (Why?)

(d) $1 \in S$ is TRUE. (Why? Hint: The fraction does not have to be in lowest terms ...) ■

Subset exploration

Is A a subset of B or vice versa? Complete this table with "yes" or "no" in each cell.

	$A \subseteq B$	$A \subsetneq B$	$A \supseteq B$	$A \supsetneq B$	$A = B$	$A \neq B$	Neither is subset of the other
A = multiples of 3, B = multiples of 6							
A = multiples of 6, B = multiples of 9							
$A = \{n^2 : n \in \mathbb{N}, n > 0\}$, $B = \{1+3+\cdots+(2n+1) : n \in \mathbb{N}\}$							
A = functions of the form $x \mapsto 16^{ax}$, B = functions of the form $x \mapsto 2^{ax}$							

Clarifying the question. We found that there were several ways that these questions needed to be clarified: In Row 1 and 2, we asked: what kind of multiples? We decided to consider only integer multiples. In Row 4, we asked: What is a? If $a \in \mathbb{Z}$, there are different consequences than when $a \in \mathbb{Q}$. We added this interpretation as a different row.

Making conjectures/observations and improving them. Possible conjectures about this table include:

- (set of integer multiples of 3) $\supseteq$ (set of integer multiples of 6)
- (set of integer multiples of 3) $\supsetneq$ (set of integer multiples of 6)
- (set of integer multiples of 6) and (set of integer multiples of 9) are not subsets of each other
- (set of perfect squares) = (set of sum of consecutive odd positive numbers)
- When $a \in \mathbb{Z}$, (set of functions of the form $x \mapsto 16^{ax}$) $\subsetneq$ (set of functions of the form $x \mapsto 2^{ax}$)
- When $a \in \mathbb{Q}$, (set of functions of the form $x \mapsto 16^{ax}$) = (set of functions of the form $x \mapsto 2^{ax}$)

Proving conjectures. We will use the properties listed in Section 0.2. We also use the following proof structures.

Proof Structure: Showing one set is a subset or strict subset of another.

- To show that $B \subseteq A$ requires showing: if $x \in B$, then $x \in A$.
- To show that $B \subsetneq A$ requires showing: (1) $B \subseteq A$; (2) there is an element of A that is not in B.

Proof Structure: Showing set equality.

- To show that $A = B$ requires showing: (1) $A \subseteq B$; (2) $B \subseteq A$.

Claim. If $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$, then $B \subseteq A$.

Proof. Given $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$. Showing that $B \subseteq A$ means showing: if $x \in B$, then $x \in A$.
Given $x \in B$. Then:

$$\begin{aligned} x &= 6k, k \in \mathbb{Z}, \text{ by definition of } B \\ &= 3 \cdot 2k \\ &= 3n, n \in \mathbb{Z}, \text{ because } 2 \in \mathbb{Z}, k \in \mathbb{Z}, \text{ and } \mathbb{Z} \text{ is closed under multiplication} \end{aligned}$$

Therefore x satisfies set membership rules of A, implying $x \in A$.
We have shown that if $x \in B$, then $x \in A$. Thus $B \subseteq A$, by definition of subset. □

Claim. If $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$, then $B \subsetneq A$.

Proof. Given $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$.

1. *Why $B \subseteq A$*: This means showing: if $x \in B$, then $x \in A$.
   Given $x \in B$. Then:

   $$\begin{aligned} x &= 6k, k \in \mathbb{Z}, \text{ by definition of } B \\ &= 3 \cdot 2k \\ &= 3n, n \in \mathbb{Z}, \text{ because } 2 \in \mathbb{Z}, k \in \mathbb{Z}, \text{ and } \mathbb{Z} \text{ is closed under multiplication} \end{aligned}$$

   Therefore x satisfies set membership rules of A, implying $x \in A$.
   We have shown that if $x \in B$, then $x \in A$. Thus $B \subseteq A$, by definition of subset. $\boxed{1}$
2. *Why there is an element of A that is not in B.* We find an element of A that is not in B. If $x \in B$, then x is an even number because if $x = 6k$ for some $k \in \mathbb{Z}$, then x as $x = 2 \cdot (3k)$. Closure of multiplication in $\mathbb{Z}$ implies $3k \in \mathbb{Z}$, so x satisfies the definition of even number.
   However, some members of A are odd numbers: $3, 9, 15, \ldots$.
   Hence there are elements of A that are not in B. $\boxed{2}$

Why this means $B \subsetneq A$: (1) and (2) show that B and A satisfy the definition of strict subset, and we have $B \subsetneq A$. □

Claim. If $A = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 16^{ax} : a \in \mathbb{Q}\}$ and $B = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 2^{ax} : a \in \mathbb{Q}\}$, then $A = B$.

Sketch of proof. Given $A = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 16^{ax} : a \in \mathbb{Q}\}$ and $B = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 2^{ax} : a \in \mathbb{Q}\}$.
We outline the steps of the proof for you to fill in.

1. *Why $A \subseteq B$:* $\boxed{1}$
2. *Why $B \subseteq A$:* $\boxed{2}$

Why the above means that $A = B$: □

Mathematical statements and their negations

Logical notation

$P(x)$	A proposition about the variable x; may be true or false depending on x
$\neg P(x)$	The negation of $P(x)$
$\forall x, P(x)$	The proposition "For all values of x, $P(x)$ is true."
$\exists x : P(x)$	The proposition "There exists a value of x such that $P(x)$ is true."
$\forall x, P(x) \Rightarrow Q(x)$	The proposition "For all values of x, if $P(x)$ is true then $Q(x)$ is true."
$\forall x, P(x) \Leftrightarrow Q(x)$	The proposition "For all values of x, $P(x)$ is true if and only if $Q(x)$ is true."

1. For each of the following statements, figure out what it means, and decide whether it is true, false, or neither.
   (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : y + x \in \{z \in \mathbb{Z} : z > 0\}$
   (b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : y + x \notin \mathbb{Z}$
   (c) $\forall g : \mathbb{R} \to \mathbb{R}, x \mapsto 2^{ax}, \exists h : \mathbb{R} \to \mathbb{R}, x \mapsto 4^{bx} : \forall x \in \mathbb{R}, g(x) = h(x)$
   (d) $\forall g : \mathbb{R} \to \mathbb{R}, x \mapsto 4^{ax}, \exists h : \mathbb{R} \to \mathbb{R}, x \mapsto 2^{bx} : \forall x \in \mathbb{R}, g(x) = h(x)$
2. Negate the following statements without using any negative words ("no", "not", "neither ... nor", etc.) Try to make your negation sound as much like normal English as possible.
   (a) Every word on this page starts with a consonant and ends with a vowel.
   (b) The set A is equal to the set B.
   (c) There is a book on this shelf in which every page has a word that starts and ends with a vowel.
   (d) The set A is a strict subset of the set B.

Solution. (*Partial*)

1. (a) For each real number x, there is a real number y so that $x + y$ is a positive integer. TRUE.
   Reasoning: If $x \in \mathbb{R}$, then take $y = 1 - x$. Then $x + y = 1$, which is a positive integer. Or take any positive integer n and take $y = n - x$.
   (b) What it means: ... (fill in the rest). FALSE.
   Reasoning: (Why?)
   (c) For each function $g(x) = 2^{ax}$ there is a function $f(x) = 4^{bx}$ so that $g(x) = h(x)$ on every possible real value of x. NEITHER.
   Reasoning: The truth of this statement depends on the possible values of a and b. If a and b must be integers, then there are some a where 2^{ax} cannot equal 4^{bx}. (All odd integers.) If a and b are rational or real, then for each a, we can take $b = \frac{a}{2}$, and then $2^{ax} = 4^{bx}$
   (d) What it means: ... (fill in the rest). TRUE.
   Reasoning: (Why?)
2. (a) THERE IS a word on this page that starts with a vowel OR ends with a consonant.
   (b) The set A has at least one element that is not in B OR the set B has at least one element that is not in A.
   (c) EVERY book on this shelf (... fill in the rest)
   (d) The set A equals B OR (... fill in the rest) ∎

Back to the opening inquiry

We have now spent some time discussing set notation and logical notation.

We began this class considering "parents" of numbers. We conjectured that:

> If a number has least three different factors (that are not itself or 1), then it has more than one pair of parents. There are no other numbers with more than one pair of parents.

One way of saying "factors of a number that are not itself or 1" is to say "nontrivial factors".

Applying set notation. Using set notation, we can interpret the conjecture as saying:

Conjecture 1.3 (Number parent conjecture, take 1)**.** Let

$$\begin{aligned} S &= \{n \in \mathbb{N} : n \text{ has at least three different non-trivial factors}\} \\ T &= \{n \in \mathbb{N} : n \text{ has more than one pair of parents}\} \end{aligned}$$

Then $S = T$.

How does this way of phrasing the conjecture match up with the original way?

- Look up the definition of set equality. What does $S = T$ mean by definition of set equality?
- Which part of set equality implies the first sentence ("If a number has least three different nontrivial factors, then it has more than one pair of parents.")?
- Which part of set quality implies the second sentence? ("There are no other numbers with more than one pair of parents")

Solution. By definition, $S = T$ means $S \subseteq T$ and $T \subseteq S$.

$S \subseteq T$ implies that "If a number has least three different nontrivial factors, then it has more than one pair of parents."

$T \subseteq S$ means that "there are no other numbers with more than one pair of parents." ■

Applying logical notation. There is another mathematically equivalent way of saying the conjecture using the logical notation we developed.

Conjecture 1.4 (Number parent conjecture, take 2)**.** $\forall n \in \mathbb{N}$, n has more than one pair of parents $\iff$ n has at least three nontrivial factors.

How does this way of phrasing the conjecture match up with the original way?

- What does "if and only if" mean?
- Which part of "iff" implies the first sentence ("If a number has least three different nontrivial factors, then it has more than one pair of parents.")? (An abbreviation for "if and only if" is "iff")
- Which part of "iff" implies the second sentence? ("There are no other numbers with more than one pair of parents")

Solution. By definition, P iff Q means that both $P \implies Q$ and $Q \implies P$ are true statements.

Given $n \in \mathbb{N}$, let the statement P be "n has more than one pair of parents", and the statement Q be statement "n has at least three nontrivial factors".

$Q \implies P$ being true implies that "if a number has least three different nontrivial factors, then it has more than one pair of parents."

$P \implies Q$ being true implies that "there are no other numbers with more than one pair of parents." ■

(The following is stated in two equivalent ways)

Proposition 1.5 (Number parent proposition)**.**

If $S = \{n \in \mathbb{N} : n$ has at least three different non-trivial factors$\}$ and $T = \{n \in \mathbb{N} : n$ has more than one pair of parents$\}$, then $S = T$.	*For all $n \in \mathbb{N}$, n has more than one pair of parents if and only if n has at least three nontrivial factors.*

Proof. Given $S = \{n \in \mathbb{N} : n \text{ has at least three different non-trivial factors}\}$ and $T = \{n \in \mathbb{N} : n \text{ has more than one pair of parents}\}$.

1. *Why $S \subseteq T$:*Let $n \in S$. Then there exist distinct $a, b, c \in \mathbb{N}$ such that $a \mid n$, $b \mid n$, and $c \mid n$. Either each of these are paired with another one of a, b, c to be a pair of parents of n or they are not. If they are not paired with any of each other, then n has at least three pairs of parents, which is more than one. If one of them is paired with another, there is still a third factor that cannot be paired with the other two (because they are already paired). So it is part of a second pair of parents. Thus $n \in T$.

   We have shown that if $n \in S$, then $n \in T$. By definition of subset, this shows $S \subseteq T$. $\boxed{1}$
2. *Why $T \subseteq S$:* Let $n \in T$. Then there exist at least two pairs $a, a' \in \mathbb{N}$ and $b, b' \in \mathbb{N}$ such that $aa' = n$ and $b, b' = n$, and $\{a, a'\} \neq \{b, b'\}$.

   If $a \neq a'$ and $b \neq b'$, then n has at least four factors, so $n \in S$.

It may be true that $a = a'$ or $b = b'$. If $a = a'$, though, then n is a perfect square and $b \neq b'$, since there is only one positive square root possible for every n. Similarly, if $b = b'$, then $a \neq a'$. In either case, n has at least three factors (either a, b, b' or a, a', b), so $n \in S$.

We have shown that if $n \in T$, then $n \in S$. By definition of subset, this shows $T \subseteq S$. $\boxed{2}$

We have shown that $S \subseteq T$ and $T \subseteq S$. By definition of set equality, we have shown $S = T$. $\square$

Summary of mathematical practices

Clarifying the question

- Make the best sense as you can of the question with what is available.
- Identify what is unambiguous, and then identify what is ambiguous.
- For the ambiguous parts, play around with different possibilities to see what is the most mathematically interesting possibility. Sometimes you may find that there are multiple interesting mathematical possibilities.

Conjecturing and Claim making

- Think of claims as an "I bet" statement.

  If you're the arbitrator for a bet between people, you would want to make absolutely sure that everyone knows exactly what the statement means, and also that everyone would agree on what evidence would count as showing the bet is true or not true!

  The same is true about mathematical statements. A mathematical statement needs to be crystal clear about what it means.
- Mathematical claims should either be true or false; if they "depend" on something, this means that there is often a better claim that can be made.
- The more general a claim, the better it is.

  For instance, "12 has more than one pair of parents" is a true claim, but a better claim is "All numbers with at least three distinct factors have more than one pair of parents" is an even better claim.
- The more "directions" a claim addresses, the better it is.

  For instance, "All numbers with at least three distinct factors have more than one pair of parents" is a true claim, but "A number has more than one pair of parents if and only if it has at least three distinct factors" goes even further to understanding the situation.

Exploring math: Our expectations

- Make claims.
- Try to prove them.
- If you get stuck, consider the negations of the claim.
- Try to prove those.
- Consider the "opposite direction" claim. (The "converse" of the claim.)
- Try to prove those.
- Aim to make the most satisfying claims possible.
- Rewrite, rewrite, rewrite! Use the rewriting process to help things get clear for yourself, your future students, and your future self, and your peers.

In-Class Resources

Opening Inquiry

Two numbers are parents of a child if the child is their product.

A child cannot be its own parent.

Which numbers have more than one pair of parents?

Child	Parents
6	2, 3
4	??
12	4, 3
12	2, 6

Clarifying what it means to be a pair of parents:

Notes on finding and improving answers to mathematical questions:

Getting to know set notation

1. Let $A = \{1, 2, \{3, 4\}, \{5\}\}$. Decide whether each of the following statements is true or false: (**Hint:** There are exactly six true statements.)

$$\begin{array}{rrrr} 1 \in A, & \{1,2\} \in A, & \{1,2\} \subseteq A, & \emptyset \in A, \\ 3 \in A, & \{3,4\} \in A, & \{3,4\} \subseteq A, & \emptyset \subseteq A, \\ \{1\} \in A, & \{1\} \subseteq A, & \{5\} \in A, & \{5\} \subseteq A. \end{array}$$

2. True or false? "All students in this class who are under 5 years old are also over 100 years old."

3. Let $S = \{x \in \mathbb{Q} : x \text{ can be written as a fraction with denominator 2 and } |x| < 2\}$.
   True or false? $0.5 \in S$, $3.5 \in S$, $0.25 \in S$, $1 \in S$.

Getting to know logical notation

1. For each of the following statements, figure out what it means, and decide whether it is true, false, or neither.
   (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : y + x \in \{z \in \mathbb{Z} : z > 0\}$
   (b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : y + x \notin \mathbb{Z}$

2. Negate the following statements without using any negative words ("no", "not", "neither ... nor", etc.) Try to make your negation sound as much like normal English as possible.
   (a) Every word on this page starts with a consonant and ends with a vowel.
   (b) The set A is equal to the set B.

Subset exploration

Is A a subset of B or vice versa? Complete this table with "yes" or "no" in each cell.

	$A \subseteq B$	$A \subsetneq B$	$A \supseteq B$	$A \supsetneq B$	$A = B$	$A \neq B$	Neither is subset of the other
$A =$ multiples of 3, $B =$ multiples of 6							
$A =$ multiples of 6, $B =$ multiples of 9							
$A = \{n^2 \mid n \in \mathbb{N}, n > 0\}$, $B = \{1 + 3 + \cdots + (2n+1) \mid n \in \mathbb{N}\}$							
$A =$ functions of the form $x \mapsto 16^{ax}$, $B =$ functions of the form $x \mapsto 2^{ax}$							

Back to the opening inquiry

We began this class considering "parents" of numbers. We conjectured that:

Applying set notation. Using set notation, we can interpret the conjecture as saying:

How does this way of phrasing the conjecture match up with the original way?

- Look up the definition of set equality. What does $S = T$ mean by definition of set equality?
- Which part of set equality implies the first sentence ("If a number has least three different nontrivial factors, then it has more than one pair of parents.")?
- Which part of set quality implies the second sentence? ("There are no other numbers with more than one pair of parents")

Applying logical notation. There is another mathematically equivalent way of saying the conjecture using the logical notation we developed:

How does this way of phrasing the conjecture match up with the original way?

- What does "if and only if" mean?
- Which part of "iff" implies the first sentence ("If a number has least three different nontrivial factors, then it has more than one pair of parents.")? (An abbreviation for "if and only if" is "iff")
- Which part of "iff" implies the second sentence? ("There are no other numbers with more than one pair of parents")

Homework

0. In this chapter, we learned about:
    - Showing that an element is or is not a member of a particular set
    - Showing that a set is a subset or strict subset of another set
    - The definition of set equality and how to show that two sets are equal
    - Number parents and children, in particular, that if
      $S = \{n \in \mathbb{N} : n \text{ has at least three different non-trivial factors}\}$ and
      $T = \{n \in \mathbb{N} : n \text{ has more than one pair of parents}\}$, then $S = T$.

   For each of these ideas:

   (a) Where in the text are these ideas located?

   (b) Review this section of the text. What definitions are most relevant? How do the examples use these definitions?

   (c) What questions or comments do you have about the ideas in this section?

1. Let $S = \{x \in \mathbb{Q} : x \text{ can be written as a fraction with denominator 2 and } |x| < 2\}$.
   Let $B = \{1 + 3 + \cdots + (2n + 1) : n \in \mathbb{N}\}$.
   Let C be the set of functions of the form $x \mapsto 27^{ax}$.
   Let D be the set of numbers with more than 5 parents.

   (a) Is "$0.25 \in S$" a true statement?

   (b) Is "$1 \in S$" a true statement?

   (c) Is 25 an element of B?

   (d) Is 24 an element of B?

   (e) When $a \in \mathbb{Z}$, is $x \mapsto 3^{5x}$ contained in C?

   (f) When $a \in \mathbb{Q}$, is $x \mapsto 3^{5x}$ contained in C?

   (g) Find a number that is a member of both B and D.

   After reading this problem, think about: What mathematical idea(s) listed in Problem 0 does this problem provide opportunities to understand? This is something helpful to think about for all the problems.
   In your responses, articulate clearly:
    - Whether you are showing that the element is or is not contained in the set;
    - How you used the definition of set membership to determine "yes" or "no"; and
    - Any definitions or ideas that you need in your reasoning.

2. (a) Complete this table with "True" or "False".

	$A \subseteq B$	$A \subsetneq B$	$A \supseteq B$	$A \supsetneq B$	$A = B$	$A \neq B$	Neither is subset of the other
$A =$ integer multiples of 14, $B =$ integer multiples of 21							
$A = \{n^2 : n \in \mathbb{N}, n > 0\}$, $B = \{1 + 3 + \cdots + (2n + 1) : n \in \mathbb{N}\}$							
$A =$ functions of the form $x \mapsto 27^{ax}$ $B =$ functions of the form $x \mapsto 3^{ax}$, $a \in \mathbb{Z}$							
$A =$ functions of the form $x \mapsto 27^{ax}$ $B =$ functions of the form $x \mapsto 3^{ax}$, $a \in \mathbb{Q}$							

   (b) Prove your "true" responses to Row 1.

(c) Prove your "true" responses to Row 2.

(d) Prove your "true" responses to Row 3.

(e) Prove your "true" responses to Row 4.

3. Read over the section containing Conjectures 1.3 and 1.4 and Proposition 1.5.

   (a) Write the following statements in your own words: Conjecture 1.3 and Conjecture 1.4. Then explain why they are mathematically equivalent.

   (b) In the proof of Proposition 1.5, in the section on showing $T \subseteq S$, the proof states, "Let $n \in T$. Then there exist at least two pairs $a, a' \in \mathbb{N}$ and $b, b' \in \mathbb{N}$ such that $\{a, a'\} \neq \{b, b'\}$." Why is the implication ("there exist at least two pairs ...") true?

   (c) In this section, the proof states,
   "If $a \neq a'$ and $b \neq b'$, then n has at least four factors, so $n \in S$.
   It may be true that $a = a'$ or $b = b'$. If $a = a'$, though, then n is a perfect square and $b \neq b'$, since there is only one positive square root possible for every n. Similarly, if $b = b'$, then $a \neq a'$. In either case, n has at least three factors (either a, b, b' or a, a', b), so $n \in S$."

      i. What is the negation of the statement "$a = a'$ or $b = b'$"?
      ii. Give an example of n such that "$a = a'$ or $b = b'$" is true.
      iii. Give an example of n such that "$a = a'$ or $b = b'$" is false.
      iv. Explain the meaning of the quoted passage using the examples you gave in (3(c)ii) and (3(c)iii).

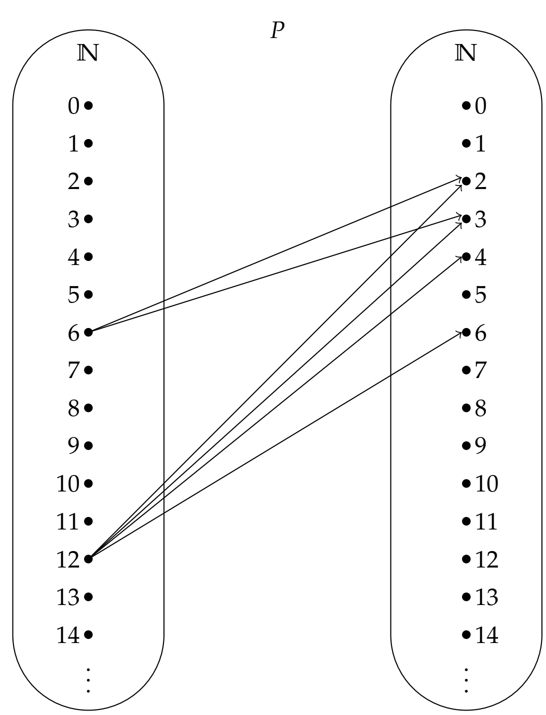
   (d) What is the logical reason for needing to show both that $S \subseteq T$ and $T \subseteq S$ to establish Proposition 1.5?

   (e) Why does the truth of the passage in 3b mean that that every $n \in T$ is also a member of S?

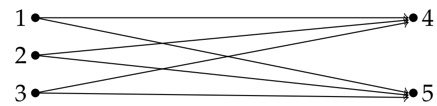
   (f) Walk through the steps of the entire proof of Proposition 1.5 using the example of $n = 12$.

   (g) Why does the first line of each part of the proof (why $S \subseteq T$ and why $T \subseteq S$) not apply to the case $n = 6$?

4. The diagram below shows P, a collection of arrows from a natural number to its parents. Some arrows below have been filled in, for example P assigns 6 to 2 and 3, and assigns 12 to $2, 3, 4, 6$. Draw in three more arrows from a natural number to its parents.



5. Let D and R be sets. The **Cartesian product** of D and R, which we will work with more next time, is defined as the set of ordered pairs $\{(x, y) : x \in D, y \in R\}$. For example, if $D = \{1, 2, 3\}$ and $R = \{4, 5\}$, then $D \times R$ is the set $\{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6)\}$. One way to think about it is that it is all the pairs you can get by drawing all possible arrows from elements of D to elements of R when you draw the elements lined up in parallel to each other:



Each arrow represents an ordered pair, with the starting point of the arrow being the first coordinate of the ordered pair and the ending point of the arrow being the second coordinate of the ordered pair.

Let $A = \{5, 6, 10\}$ and $B = \{-1, -2, -3\}$. Draw an arrow representation of the Cartesian product $A \times B$ and list its elements as ordered pairs.

Part II

Relations and Functions

2 Relations (Week 2) (Length: ˜3 hours)

Overview

Content

Cartesian product of two sets A and B, denoted $A \times B$, defined as the set of ordered pairs $\{(a, b) : a \in A, b \in B\}$.

Relation from a set D to set C, defined from three different perspectives: the "middle school", "high school", and "university"; and their mathematically equivalence.

- The "middle school" version is described in terms of a set of arrows between an input and output space.
- The "high school" version formalizes arrows to assignments.
- The "university" version defines a relation as a subset of the Cartesian product $D \times R$.

We call these definitions the middle school, high school, and university versions to refer to when they most likely arise.

Inverse of a relation, defined from these three perspectives; their mathematical equivalence.

Composition of relations $P : D \to D$ then $Q : D \to D$, defined as the relation that assigns x to z whenever there is a $y \in D$ such that P assigns $x \mapsto y$ and Q assigns $y \mapsto z$. (See p. **??** for why we only consider the case $P : D \to D$ and $Q : D \to D$.)

Graph of a (real) relation, defined as the set of points $(x, y) \in \mathbb{R}^2$ such that the relation assigns x to y.

Graph of an (real) equation in variables x and y, defined as the set of points $(x, y) \in \mathbb{R}^2$ such that evaluating the equation at x and y results in a true statement.

Function from a set D to a set R, defined as a relation from D to R such that each input in D is assigned to no more than one output in R; how this definition can be interpreted from the three perspectives for relation.

Proof Structures	Mathematical/Teaching Practices
To show that a point (x, y) is on a graph of a relation means showing that the relation assigns x to y. **To show that a point (x, y) is on the graph of an equation** means showing that evaluating the equation at x and y results in a true statement.	**Connecting mathematically equivalent definitions**, meaning to understand how different but equivalent definitions can serve different pedagogical and mathematical purposes. **Connecting different mathematical representations** of the same concept, meaning to think about different ways of drawing and describing the same mathematical idea.

Summary

One goal of this lesson is to introduce relations and functions from an advanced perspective. However, more importantly, the goal is to connect the advanced perspective to high school and middle school perspectives, so that teachers have a sense of where the math can go.

Using the parent relation as an opening example, we define *relation* in the three (mathematically equivalent) described above. We then define *domain*, *range*, *image* of a point and set, and *preimage* of a point and set.

To highlight the universality of these concepts throughout high school and middle school mathematics, we use examples from algebra (from story problems and also graphs of relations such as $x = y^2$), trigonometry (the relation from $[0, 360)$ to $\mathbb{R}$ defined by equivalence of angle measure in degrees), and geometry (rigid motions).

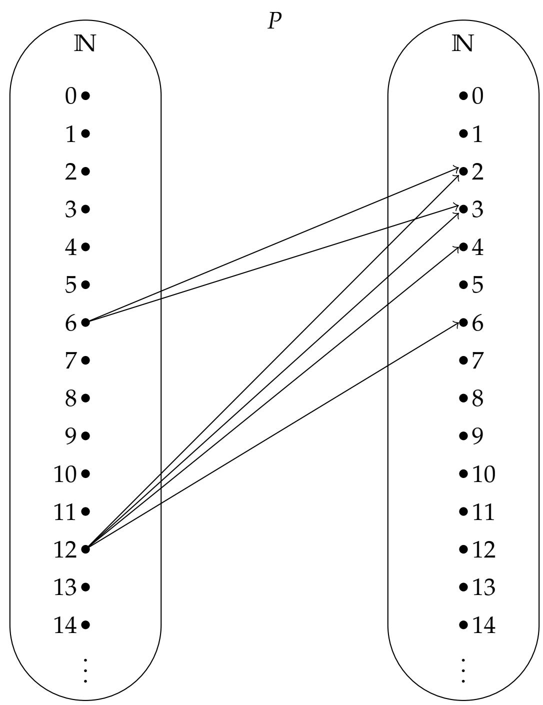
We then introduce *inverse relations*, *functions*, *graphs of functions*, and *compositions of functions*. For each concept in this lesson, we ask teachers to consider how they might explain the connection between the concept and the middle school, high school, and university conceptions of relation, as well as how they might explain how different representations denote mathematically equivalent ideas.

Opening example: Parent relation

We learned about natural number parents and children last time.

1. What is the definition of a parent of a natural number child?
2. Let P assign a natural number to each of its parents. We can represent P as a set of arrows from $\mathbb{N}$ to $\mathbb{N}$. Some arrows below have been filled in, for example P assigns 6 to 2 and 3, and assigns 12 to $2, 3, 4, 6$. Draw in more arrows.
3. Consider this statement: "Some children have no parents, some children have exactly one parent, and some children have multiple parents."

   Is this statement true or false? Why?
4. How about this statement? "Some numbers have no children, and some numbers have multiple children."



Solution. (*Partial*) Given a number $n \in \mathbb{N}$, a parent of n is a nontrivial factor of n.

The first statement is true:

- n has no parents when n is 0, 1, or prime
- n has exactly one parent when n is a perfect square of a prime number
- n has multiple parents otherwise

These are represented by no arrows starting at a number, exactly one arrow starting at a number, and multiple arrows starting at a number.

The second statement is also true. 0 and 1 have no children. All other numbers have multiple children (infinitely many, in fact). These are represented by arrows ending at a number or not. ■

Defining relations

Cartesian products

Let's discuss Cartesian products, which you first saw in your homework from last week.

Definition 2.1. Let D and R be sets. The **Cartesian product** of D and R is defined as the set of ordered pairs $\{(x, y) : x \in D, y \in R\}$. It is denoted $D \times R$.

Let $A = \{5, 6, 10\}$, $B = \{-1, -2, -3\}$, $C = \{-1, 1\}$. Let $\mathbb{N}$ denote natural numbers, $\mathbb{Z}$ the integers, and $\mathbb{R}$ the real numbers.

List the elements of the following Cartesian products:

- $A \times B$
- $A \times C$
- $\mathbb{Z} \times C$
- $C \times \mathbb{Z}$
- $\mathbb{N} \times \mathbb{N}$.

Which of the above sets contains the element $(6, -1)$? How about $(-1, 10)$?

How would you describe $\mathbb{R} \times \mathbb{R}$?

How about $\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$?

Solution. (*Partial*) $(6, -1) \in A \times B, \mathbb{Z} \times C, \mathbb{N} \times \mathbb{N}$. It is not an element of any of the other sets.

$(-1, 10) \in C \times \mathbb{Z}, \mathbb{N} \times \mathbb{N}$. It is not an element of any of the other sets.

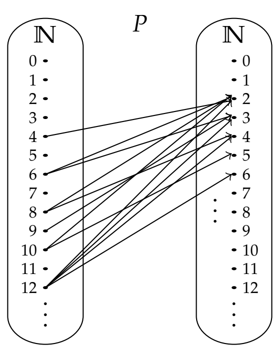
$\mathbb{R} \times \mathbb{R}$ is the coordinate plane.

$\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$ can be thought of as all the coordinates of 3-space. ■

Note: The Cartesian product is sometimes referred to as the "cross" or "cross product" or "direct product" of two sets. The term "cross product of two vectors" also shows up in linear algebra, but this meaning is independent of the Cartesian product. It's an unfortunate coincidence that the same term is used.

Relations

In middle school, if relations are introduced, they are often done so in the form of a cloud diagram, such as drawn in the opening task. (*Question:* What do the ". . . " mean in the below diagram?)



In our example, a relation P maps numbers in $\mathbb{N}$ to numbers in $\mathbb{N}$, and the map is represented by arrows connecting input numbers to output numbers.

Definition 2.2 (Relation: Middle school version)**.** A **relation** from a set D to a set R is a set of arrows going from elements of D to elements of R.

If there is an arrow from and element x to an element y, we say the relation **maps** or **assigns** x to y.

We use the notation $r : D \to R$ to mean a relation from D to R called r.

Definition 2.3 (Parent relation)**.** The parent relation $P : \mathbb{N} \to \mathbb{N}$ is the set of arrows from each element of $\mathbb{N}$ to its nontrivial factors.

Note: A relation $P : D \to R$ may map an element of D to no elements of R, exactly one element of R, or multiple elements of R.

An element of R may have no elements of D mapping to it, exactly one element of D mapping to it, or multiple elements of D mapping to it.

Definition 2.4. For a relation $r : D \to R$, we say that

- D is the **candidate domain**;
- R is the **candidate range** (or **codomain**);
- the **image** of an element $x \in D$ is the set of elements of R that x is mapped to. Similarly, the **image** of a subset of $S \subseteq D$ is the subset of R containing the images of all elements of S.
- When a element in D maps to no element in R, we say it has **empty image**. If an element in D does map to at least one element in R, we say it has **nonempty image**.
- the **domain** of r is the subset $D' \subseteq D$ of elements with nonempty image.
- the **preimage** of an element of R is the set of elements of D that map to that element of R. Similarly, the **preimage** of a subset $T \subseteq R$ is the subset of D containing the preimages of all elements of T.
- When a element in R has no element in D mapping to it, we say it has **empty preimage**. If an element in R does have an element in D mapping to it, we say it has **nonempty preimage**.
- the **range** (or **image**) of the relation r is the subset $R' \subseteq R$ with nonempty preimage.

Note: In these materials, we will use the terms "codomain" and "candidate range" interchangeably. We will also use the terms "range" and "image" interchangeably. The terms "codomain" and "image" typically do not show up in K-12 materials; they are typically introduced in university or graduate mathematics. The term "range" is standard to middle school and high school materials, though sometimes "range" is used to mean "candidate range" and other times it is used to mean "the set of elements with nonempty preimage". In these materials, "range" only refers to the latter.

We will watch a short video of teaching by Ms. Barbara Shreve of San Lorenzo High School. The video shows her teaching an intervention class called Algebra Success. The students in this class have been previously unsuccessful in Algebra 1. They are working on finding intercepts of equations to get ready for working with quadratics.

As you watch the video, it may be tempting to think about what you personally think is good or not as good about the teaching, or what you might have done differently. But before getting to these kinds of judgments, it is more important to simply observe what is going on, what the students' reasoning is, and what the story line is. (This is just like when working with students, as we will see later in this class and you will learn in your methods class: before evaluating students' work, we must first observe and understand students' work without judgment.) Here, we will practice observing the interactions between teachers and

As you watch the video, think about the following questions:

- How does the teacher emphasize to students to explain their reasoning?
- How does the teacher help students feel comfortable sharing their reasoning?
- How was the definition of x-intercept or y-intercept used?

Here is a link to the video: `http://www.insidemathematics.org/classroom-videos/public-lessons/9th-11th-grade-math-quadratic-functions/introduction-part-b`

Again, our discussion will be about the viewing questions. We will have time for general comments later. Let's take the viewing questions one at a time.

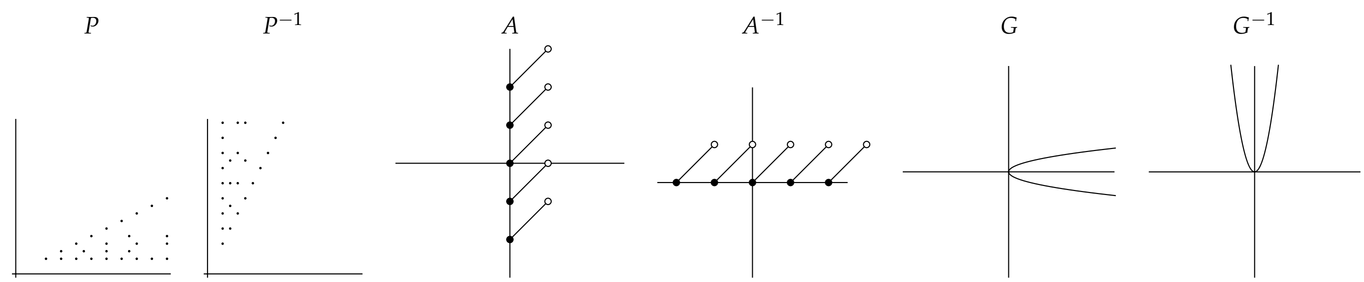
When addressing the viewing questions, be specific about your evidence from the video to support what you are saying.

Now that we have discussed the viewing questions, what other thoughts or questions come to mind?

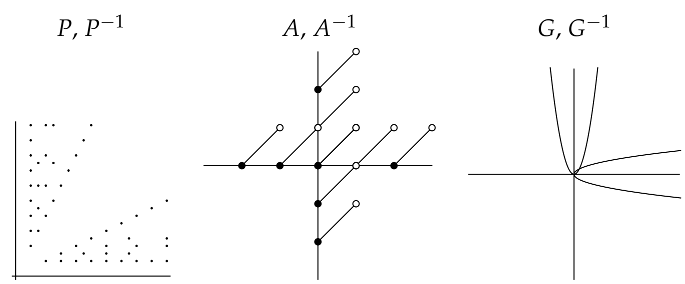
Putting it all together: Investigating graphs of inverses

To finish this chapter, let's investigate the examples we have worked with and try to generalize.

Here are some graphs that we've seen:



And here are those graphs again, this time pairing relations and their inverse relations.



What do you notice about these graphs? Find a way to fold this page so that when you hold up the folded page to the light, the graphs of P and P^{-1} are lined up with each other. Find a way to fold the paper in this way for A and A^{-1}, and then G and G^{-1}.

When you fold the coordinate plane in this way, where does $(1, 2)$ go? Where goes $(-4, 3)$ go? Where goes $(-100, -100001)$ go?

In general, when you fold the coordinate plane in this way, where does the point (a, b) go? Why does this make sense?

Explain why it makes sense that folding the plane this way should always bring a graph of a relation to the graph of its inverse. Do this in two ways:

- First, explain this discovery and why it makes sense using P and P^{-1}, including some specific examples. Make sure to use specific examples and also explain why no other fold will work.
- Then, explain this discovery and why it makes sense in general. Your explanation here should rely only

on the definition of relation and its graph, and not depend on any particular examples.

In teaching, it is often useful to be able to provide these two kinds of explanation: specific and general. Specific explanations have to do with a particular example and may help the students be able to keep an image in mind. General explanations help students understand why the reasoning for a specific example applies to a larger class of objects.

Summary

Content

This chapter had a lot going on! We defined relation in three different ways, which we called the middle school, high school, and university ways. We then talked about various properties of relations, such as its domain and range, as well as the image and preimage of points and subsets. We ended by talking about graphs of relations and equations.

Throughout this discussion, we saw algebraic, graphical, and cloud diagram ways of representing relations.

We then discussed inverse relations and compositions of relations, which also can be understood in terms of these different representations.

Another common thread was Cartesian products, which is how we defined ordered pairs. This allowed us to define relations the university way, and it also allowed us to talk meaningfully about graphs of relations. The graph of a relation from $\mathbb{R}$ to $\mathbb{R}$ lives in the space given by the Cartesian product $\mathbb{R} \times \mathbb{R}$, otherwise known as $\mathbb{R}^2$.

The two explorations we did tied together representations and the concepts we discussed:

- Given any relation r, we discovered that the graph of the relation $r^{-1} \circ r$ always contains all points of the form (a, a) where a is in the domain of r.
- Given any relation r, we discovered that the graph of the relation r^{-1} can be obtained by reflecting (folding) the graph of r about the line $y = x$.

This last one may seem familiar: in high school we often teach this statement with the graph of functions. But as you learned, this statement applies to relations in general! You will examine this from a teaching perspective for homework, as well as finish the proofs of these explorations.

In the proofs, you will use the two proof structures we learned:

- To show that a point (x, y) is on a graph of a relation means showing that the relation assigns x to y.
- To show that a point (x, y) is on the graph of an equation means showing that evaluating the equation at x and y results in a true statement.

Connecting mathematically equivalent definitions

- Explain how they appear different and how they appear the same.
- Explain why they are mathematically equivalent.
- Analyze the mathematical and pedagogical purposes of each definition and why they may be appropriate for different levels of mathematical study, or why one version makes more sense after having worked with another version.

Connecting different mathematical representations

- Explain how representations appear different and how they appear the same.
- Identify how the representations can be used highlight different features.
- Go back and forth between the representations: where are the features of one representation located in the other representation?

For the teaching practices of connecting mathematically equivalent definitions and connecting different mathematical representations, it is helpful to be able to provide both specific and general explanations.

- Specific explanations have to do with a particular example and may help the students be able to keep an image in mind.
- General explanations help students understand why the reasoning for a specific example applies to a larger class of objects.

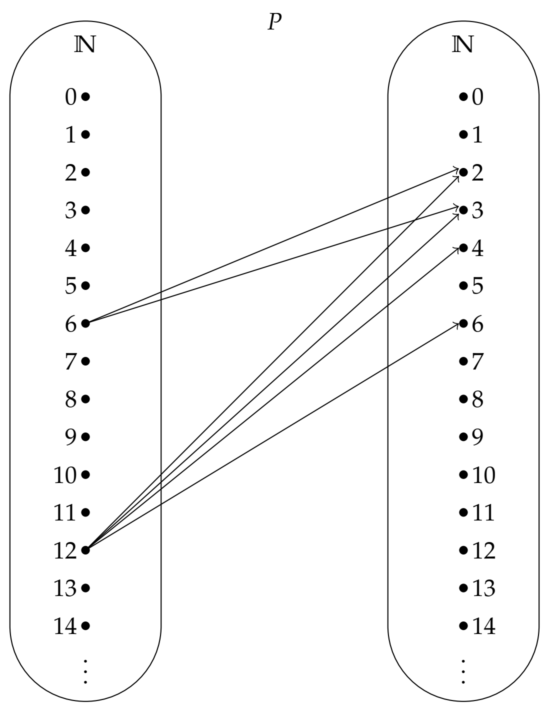
In-Class Resources

Opening Example: Parent Relation

We learned about natural number parents and children last time.

1. What is the definition of a parent of a natural number child?
2. Let P assign a natural number to each of its parents. We can represent P as a set of arrows from $\mathbb{N}$ to $\mathbb{N}$. Some arrows below have been filled in, for example P assigns 6 to 2 and 3, and assigns 12 to $2, 3, 4, 6$. Draw in more arrows.
3. Consider this statement: "Some children have no parents, some children have exactly one parent, and some children have multiple parents."

   Is this statement true or false? Why?
4. How about this statement? "Some numbers have no children, and some numbers have multiple children."



Getting familiar with relations and associated concepts

Cartesian products

(We will define A, B C on the board.)

1. (a) List the elements of the following Cartesian products:
   - $A \times B$
   - $A \times C$
   - $\mathbb{Z} \times C$
   - $C \times \mathbb{Z}$
   - $\mathbb{N} \times \mathbb{N}$.

   (b) Which of the above sets contains the element $(6, -1)$? How about $(-1, 10)$?

   (c) How would you describe $\mathbb{R} \times \mathbb{R}$?

   (d) How about $\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$?

Domain, range, image, preimage

Do #2. Do not proceed to #3 or #4 yet.

2. (a) What are the domain and range of the car-owner relation and the room-course relation?

   (b) Suppose this table contains course assignments to rooms at 1pm. What is the image of Math Bldg Room 100? What is the preimage of Math 996, Math 405, Math 100, and Math 221 under the room-course relation?

Room	Course in room at 1pm
Math Bldg room 100	Math 996
Math Bldg room 104	Math 100
Engineering Bldg room 750	Math 405
not being offered this term	Math 221

   (c) Let T be the relation that maps each day of the year 2030 to the its average temperature in $^\circ F$ that day. Describe a possible candidate domain, domain, candidate range, and range of this relation.

   (d) Let A be the relation that maps each degree in the interval $[0^\circ, 360^\circ)$ to all degrees in the interval $(-\infty, \infty)$ that give an equivalent angle measure. What is the preimage of 361°? What is the image of 0°?

   (e) Let ρ be the relation that maps a point in the plane to its rotation about the origin by 90°. (This means 90° *counter*clockwise.) What is the image of the point $(1, 0)$? What is the preimage of the point $(-2, 0)$?

   (f) Let G be the relation that maps x to every y such that $x = y^2$. What is the image of 4? What is the preimage of -6?

Connecting different representations

3. Interpret the definitions of candidate domain, domain, image, preimage, candidate range, and range in terms of arrows and their start and end points.
4. Interpret the definitions of these same concepts in terms of the graph of a relation. Use the graphs of A and P to illustrate what you mean.

Mathematical/teaching practice: Connecting different representations

- Explain how representations appear different and how they appear the same.
- Identify how the representations can be used highlight different features.
- Go back and forth between the representations: where are the features of one representation located in the other representation?

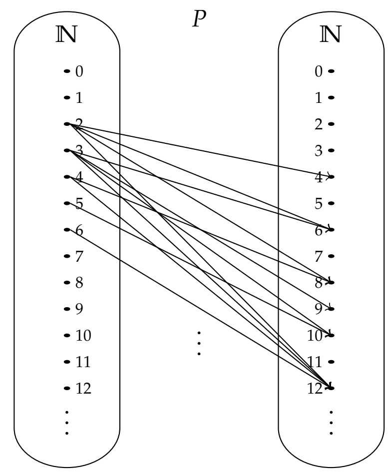
Getting familiar with inverses of relations

Back to the opening example: Inverting the parent relation

The inverse of the parent relation could be represented like this:

What other arrows does the inverse of the parent relation contain?

What might be a good name for this relation?



Connecting different but mathematically equivalent definitions

Discuss the three versions of the definition of inverse of a relation. What do they each say? How would you represent them? What makes them mathematically equivalent?

Mathematical/teaching practice: Connecting different but mathematically equivalent definitions

- Explain how they appear different and how they appear the same.
- Explain why they are mathematically equivalent.
- Analyze the mathematical and pedagogical purposes of each definition and why they may be appropriate for different levels of mathematical study, or why one version makes more sense after having worked with another version.

Working with compositions algebraically

Let P be the parent relation and let $t : \mathbb{N} \to \mathbb{N}, x \mapsto 2x$.

1. (a) For the relation $t \circ P$, what is the image of 6? Of 12? Of 9?
   (b) For the relation $t \circ P$, what is the preimage of 4? Of the set $\{1, 3, 5\}$? Of the set $\{4, 14\}$?
   (c) What are the domain and range of the relations $t \circ P$?

2. How would you represent $P \circ t$? What are its domain and range?

3. (a) Sketch a representation of $A^{-1} \circ A$.
   (b) What are its domain and range?
   (c) What is the image of 45°?
   (d) What is the preimage of 45°?

We often think about inverses as "undoing" something. How well does this analogy work in the case of relations? What goes well with the analogy? What goes wrong with the analogy?

GRAPHS OF RELATIONS

Proof structures:

To show that a point (a, b) is on a graph of a relation means showing that the relation assigns a to b.

To show that a point (x, y) is on the graph of an equation means showing that evaluating the equation at $x = a$ and $y = b$ results in a true statement.

1. What are all the points on the graph of A with x-coordinate $45°$? With y-coordinate $45°$?

2. What are all the points on the graph of A^{-1} with x-coordinate $60°$? With y-coordinate $60°$?
   Is the point $(60, 400)$ on the graph of A^{-1}? How about $(430, 70)$? $(70, 430)$? $(10, 200)$? Why or why not?
   What is the graph of A^{-1}? How do you know you have graphed all points and not graphed any extra points?

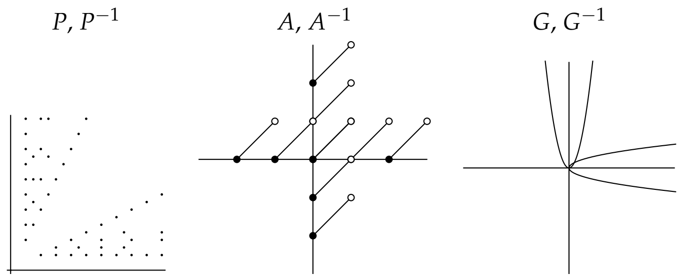
3. Show that for any relation $r : D \to D$, if $x \in D$ is in the domain of r, then (x, x) lies on the graph of $r^{-1} \circ r$.

4. Is the point $(1, 2)$ on the graph of $x^2 + \frac{y^2}{4} = 1$? How about the point $(\frac{1}{2}, \frac{2}{3})$?
   Find a point that is on the graph of $y = x^2 + 1$ and the graph of $y = 6x - 4$. Is that the only point? Are there more points? How do you know?
   Describe the graph of the equation $x = 0$. Describe the graph of the equation $y = 0$. How do you know that these graphs look this way?
   Where does $x^2 + \frac{y^2}{4} = 1$ intersect the x-axis? Where does $x^2 + \frac{y^2}{4} = 1$ intersect the y-axis?
   What is a point that is on the graph of $y = x^2 + 1$ and the graph of $y = 5$. Then solve for x when in $y = x^2 + 1$ when y is 5. Explain the numerical coincidence.

5. What are all the x-intercepts of the graphs of A and A^{-1}?
   What are all the y-intercepts of the graphs of A and A^{-1}?
   What is the intersection of the graphs A and A^{-1}?

CLOSING INQUIRY

To finish this chapter, let's investigate the examples of relations and their inverses. Here are some graphs that we've seen:



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When you fold the coordinate plane in this way, where does $(1, 2)$ go? Where goes $(-4, 3)$ go? Where goes $(-100, -100001)$ go?

In general, when you fold the coordinate plane in this way, where does the point (a, b) go? Why does this make sense?

Explain why it makes sense that folding the plane this way should always bring a graph of a relation to the graph of its inverse. Do this in two ways:

- First, explain this discovery and why it makes sense using P and P^{-1}, including some specific examples. Make sure to use specific examples and also explain why no other fold will work.
- Then, explain this discovery and why it makes sense in general. Your explanation here should rely only on the definition of relation and its graph, and not depend on any particular examples.

REFERENCE: RELATIONS

Definition 2.1. Let D and R be sets. The **Cartesian product** of D and R is defined as the set of ordered pairs $\{(x, y) : x \in D, y \in R\}$. It is denoted $D \times R$.

Definitions 2.2, ??, ??:

Middle school version	A **relation** from a set D to a set R is a set of arrows going from elements of D to elements of R.
High school version	A **relation** P from a set D to a set R a set of assignments from elements of D, called inputs, to elements of R, called outputs.
University version	A relation $r : D \to R$ is a subset of $D \times R$, i.e., $r \subseteq D \times R$.

Notation: $r : D \to R$ refers to a relation from D to R; and $x \mapsto y$ refers to an assignment from $x \in D$ to $y \in R$

A relation may map an element of D to 0, exactly 1, or multiple elements of R. An element of R may have 0, exactly 1, or multiple elements of D mapping to it.

Definition 2.4. For a relation $r : D \to R$, we say that

- D is the **candidate domain**;
- R is the **candidate range** (or **codomain**);
- the **image** of an element $x \in D$ is the set of elements of R that x is mapped to. Similarly, the **image** of a subset of $S \subseteq D$ is the subset of R containing the images of all elements of S.
- When a element in D maps to no element in R, we say it has **empty image**. Otherwise it has **nonempty image**.
- the **domain** of r is the subset $D' \subseteq D$ of elements with nonempty image.
- the **preimage** of an element of R is the set of elements of D that map to R. Similarly, the **preimage** of a subset $T \subseteq R$ is the subset of D containing the preimages of all elements of T.
- When a element in R has no element in D mapping to it, we say it has **empty preimage**. Otherwise it has **nonempty preimage**.
- the **range** (or **image**) of the relation r is the subset $R \subseteq D$ with nonempty preimage.

Definitions ??, ??, ??:

Middle school version	If r is a relation from a set D to R, then the **inverse relation** of r is the relation that swaps the direction of the arrows of r. The arrows of the inverse go from elements of R to elements of D.
High school version	Given a relation $r : D \to R$, the **inverse relation** of r is the set of assignments $y \mapsto x$ such that $x \mapsto y$ is an assignment of r.
High school version	Given a relation $r : D \to R$, the **inverse relation** of r is defined as the subset $\{(y, x) \subseteq R \times D : (x, y) \in r\}$.

Notation: r^{-1}.

Definition ??. Given two relations $P : D \to D$ and $Q : D \to D$, we define the **composition** of P then Q as the relation that assigns x to z whenever there is a $y \in D$ such that P assigns $x \mapsto y$ and Q assigns $y \mapsto z$.

Definition ??. The **graph of a relation** $r : D \to R$ is defined as the set of points $(a, b) \in \mathbb{R}^2$ such that r assigns $a \mapsto b$.

Definition ??. The **graph of an equation** in x and y is defined as the set of points $(a, b) \in \mathbb{R}^2$ such that evaluating the equation at $x = a$ and $y = b$ results in a true statement.

Definition ??. Given an equation in x and y and its graph, all points $(a, 0)$ on the graph are called **x-intercepts** of the graph. All points $(0, b)$ on the graph are called **y-intercepts** of the graph. A graph may have 0, 1, or multiple x-intercepts and y-intercepts.

Definition ??. Two graphs **intersect** each other at (a, b) when (a, b) is contained in both graphs. Graphs may intersect at zero points, one point, multiple points, and sometimes even infinitely many points.

Proof structures:

- To show that a point (a, b) is on a graph of a relation means showing that the relation assigns a to b.
- To show that a point (x, y) is on the graph of an equation means showing that evaluating the equation at $x = a$ and $y = b$ results in a true statement.

Homework

0. In this chapter, we learned about:

   - Defining:
     - Relation, in three equivalent ways; candidate domain, domain, candidate range, range, preimage of a point and of a subset, image of a point and of a subset
     - Inverse relation
     - Composition of relations
     - Graph of a (real) relation; x- and y-intercepts; intersection of graphs
   - Showing that a point (x, y) is on the graph of a relation
   - Showing that a point (x, y) is on the graph of an equation
   - The mathematical/teaching practices of:
     - Connecting mathematically equivalent definitions
     - Connecting different mathematical representations

   For each of these ideas:

   (a) Where in the text are these ideas located?

   (b) Review this section of the text. What definitions and results were important? How do examples use these definitions and results?

   (c) What questions or comments do you have about the ideas in this section?

1. Describe the following concepts in terms of the middle school and university versions of the definition of relation:

   (a) candidate domain, domain

   (b) image of a point, image of a subset

   (c) candidate range, range

   (d) preimage of a point, preimage of a subset

   (e) x- and y-intercepts

   (f) intersection of the graphs of two relations

   (g) inverse relation

   (h) composition of two relations

   In your work for each,

   (i) address both the middle school and university versions of the definition of relation

   (ii) include diagrams

   (iii) describe the concepts using a specific example, using a relation such as the parent relation P or angle relation A as defined in Example **??**.

   (iv) then phrase the descriptions so that they can apply without any changes to any possible example of a relation

   (v) explain why the descriptions in terms of both versions are mathematically equivalent

In questions 2, 3, and 9, we examine relations which are functions. This means that each input in the domain is assigned to only one output element, so the notation $f(x)$ makes sense: there is only one element that f sends x to. We will also examine this definition more in question 11.

2. (a) Graph the following and their inverse relations. (You may need to look up the relations in a high school resource online or elsewhere.) We will use these examples in the next chapter.
   - sine function ($x \mapsto \sin(x)$ for $x \in \mathbb{R}$)
   - cosine function ($x \mapsto \cos(x)$ for $x \in \mathbb{R}$)
   - absolute value function ($\mathrm{Abs} : \mathbb{R} \to \mathbb{R}, x \mapsto |x|$)

- squaring function ($\mathrm{Sq} : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$)
- cubing function ($\mathrm{Cu} : \mathbb{R} \to \mathbb{R}, x \mapsto x^3$)

(b) In each of the graphs of the given functions, label any maxima or minima with its coordinate. In the graphs of their inverses, label any points that are leftmost or rightmost with its coordinate. Use exact form (this means to not use approximations, including decimal approximations; use symbols such as π as needed).

(c) What are the following images and preimages?

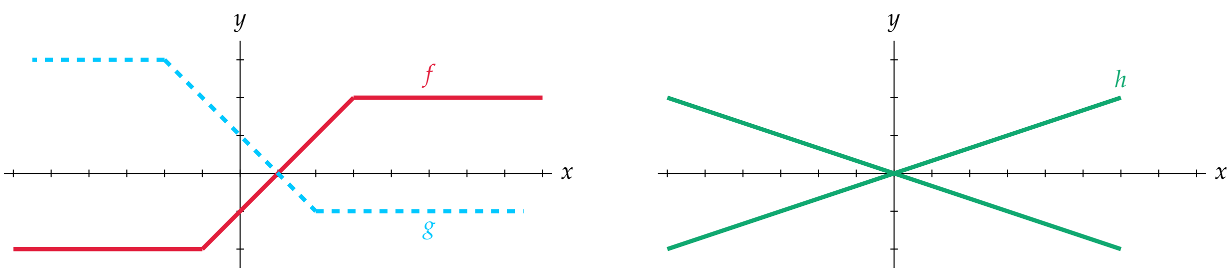
i. What is the image of $[0, \pi]$ for the sine function? How about for the cosine function?
ii. What is the preimage of $[0.5, 1]$ for the sine function? How about for the cosine function?
iii. What is the image of $[0.5, 1] \cup [-3, -1] \cup [3, \infty]$ for Abs?
iv. What is the preimage of $[100, 144]$ for Abs?
v. What is the image of $[0.5, 1] \cup [-3, -1] \cup [3, \infty]$ for Sq?
vi. What is the preimage of $[100, 144]$ for Sq?
vii. What is the image of $[0.5, 1] \cup [-3, -1] \cup [3, \infty]$ for Cu? (for this question, you may use approximations)
viii. What is the preimage of $[100, 144]$ for Cu? (for this question, you may use approximations)

3. Below are graphs of the relations f, g, h. The pieces of these graphs are lines and line segments, and their turning points are integer coordinate points. Consecutive tick marks on the axes are distance 1 from each other.

(a) Find the values of $g \circ f(-4)$, $g \circ f(-1)$, $g \circ f(0)$, $g \circ f(3)$, $g \circ f(4)$.

(b) Where does $h \circ g$ map each of $0, 2, 4$?

(c) Where does $g \circ h$ map each of $0, 2, 4$?



4. In this task, you will watch a short video of teaching by Ms. Barbara Shreve of San Lorenzo High School. The video shows her teaching an intervention class called Algebra Success. The students in this class have been previously unsuccessful in Algebra 1. They are working on finding intercepts of equations to get ready for working with quadratics.

As you watch the video, it may be tempting to think about what you personally think is good or not as good about the teaching, or what you might have done differently. But before getting to these kinds of judgments, it is more important to simply observe what is going on, what the students' reasoning is, and what the story line is. (This is just like when working with students, as we will see later in this class and you will learn in your methods class: before evaluating students' work, we must first observe and understand students' work without judgment.) Here, we will practice observing the interactions between teachers and

As you watch the video, think about the following questions:

(a) How does the teacher emphasize to students to explain their reasoning?

(b) How does the teacher help students feel comfortable sharing their reasoning?

(c) How was the definition of x-intercept or y-intercept used?

Here is a link to the video: `http://www.insidemathematics.org/classroom-videos/public-lessons/9th-11th-grade-math-quadratic-functions/introduction-part-b`

Talk through your responses to the discussion questions with a fellow teacher in our class. As you talk, be sure to point to the evidence that you are basing your ideas on. Then write down your response to these discussion questions. Describe the evidence you are using by selecting quotes from the transcript (found on the video site).

5. (This task can be thought of as a followup to question 4.)
Suppose that you are teaching about intercepts of graphs and you are going over a solution to the problem: *Find the y-intercept of the graph of the equation* $y = (x-3)^2$. Your class has the following conversation:

    You: How did you start this problem?
    Student A: I put in a 0 for the x-value.
    You: Let's talk about what Student A did. If we're finding a y-intercept, why do we start by putting a 0 for the x? Anyone have an idea?
    Student B: Because zero's the easiest thing.
    Student C: Because zero is where the line crosses.
    Student B: Wait, it's because you want to cancel it out.

    (a) Solve the problem that the class is working on.
    (b) Explain the logic of student B's thinking.
    (c) Explain the logic of student C's thinking.
    (d) In the equivalent of at most 4 tweets, explain why " If we're finding a y-intercept, we start by putting a 0 for the x.". (A tweet is 140 characters.) Your explanation should tie together the definition of graph and the definition of y-intercept.
    (e) What questions might you pose to students to get at the ideas in this explanation?

6. Suppose you are teaching about how to find the intersection of graphs of quadratic and linear equations, and you plan for your class to work on this exploratory task:

    How many points do the graphs of $y = x^2$ *and* $y = x$ *intersect at? What are those points?*
    How many points do the graphs of $y = x^2 + a$ *and* $y = x$ *intersect at? What are those points? How does the answer depend on a?*

    An important initial step of planning to teach a task is to solve that task yourself.

    (a) Keeping in mind what we have learned about satisfying answers to mathematical questions: What is a good answer to this task? Describe your conjecture.
    (b) Prove your conjecture.

    Another step in planning is to figure out how you might explain how key ideas are used to solve the task.

    (c) Explain as you would to high school students in this class how the definition of intersection of graph is used in finding solutions to the exploration.

7. Show that for any relation $r : D \to D$, if $x \in D$ is in the domain of r, then (x, x) lies on the graph of $r^{-1} \circ r$.

8. (Based on Chazan (1993)[1]). Suppose you are teaching algebra and a student asks, " Why do we call 'x' a variable in equations like $6x + 5 = 10$ when it stands for just one number?"

    (a) What is the student thinking? How might they have arrived at this question?
    (b) What are you sure that the student understands? What are you unsure that the student understands?
    (c) What is the mathematical issue here?
    (d) How do you respond?

9. Suppose that you are introducing the idea that one way to obtain the graph of an inverse of a relation is to reflect the graph of the relation over the line $y = x$.

    (a) To help students understand this reflection, you use examples such as "Where would the point $(3, 0)$ go if you reflected it about the line $y = x$? What about the point $(0, 15)$? $(3, 15)$?" You then follow up with other examples.
    Using the examples $(3, 0)$, $(0, 15)$, and $(3, 15)$, explain geometrically why it makes sense that point $(3, 0)$ is mapped to the point $(0, 3)$ by reflection over the line $y = x$, and similarly for $(0, 15)$ to $(15, 0)$, and $(3, 15)$ to $(15, 3)$.
    Then explain, why, in general, the point (a, b) is mapped to the point (b, a) when reflected over the line $y = x$. This part of your explanation should apply to any possible example of a point without referring specifically to any examples.

[1] Chazan, D. (1993). F(x)=G(x)?: An approach to modeling with algebra. *For the Learning of Mathematics, 13*, 22-26.

(b) You then use the example of $f(x) = 5x$. What is the equation of its inverse relation f^{-1}?

(c) Using the example of $f(x) = 5x$, explain why it makes sense that f^{-1} must have the graph of the equation you found in (b). In your explanation, draw on the definition of inverse relation, the definition of graph of a relation, and the definition of graph of an equation, as well as your general explanation in part (a). Be explicit about where you use these definitions.

(d) Then explain why, in general, it is true that one way to obtain the graph of an inverse of a relation is to reflect the graph of the relation over the line $y = x$.

10. We can think of definite integrals such as $\int_0^x t^2\,dt$ as a function with input variable x.

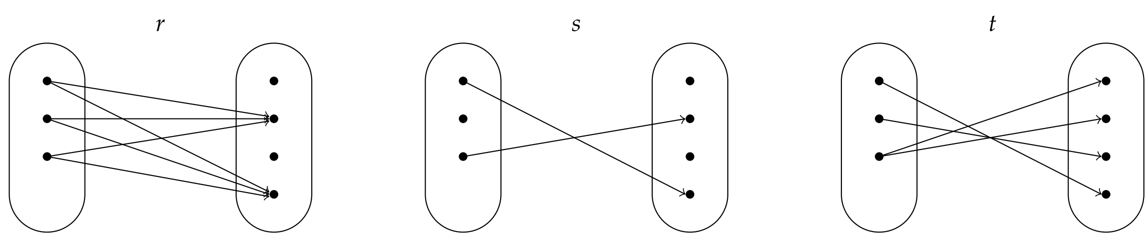
    Using the definition of graph of an equation, explain why when you shift every point on the graph of $y = \int_0^x t^2\,dt$ down by $\frac{1}{3}$, you obtain the graph of $y = \int_1^x t^2\,dt$.

11. Read the following definition:

    Definition. A *function* from D to R is defined as a relation from D to R where each input in D is assigned to no more than one output in R.

    Complete the following table by placing checkmarks to indicate: which of the relations below are also functions?

	r	r^{-1}	s	s^{-1}	t	t^{-1}
is a function						
is NOT a function						



3 Functions: Correspondence View (Week 3) and Covariation View (Week 4) (Length: ~6 hours)

Overview

This lesson focuses on honing mathematical/teaching practices in the following content.

Content

Functions, defined as a relation from D to R where each input in D is assigned to no more than one output in R.

Invertible function, defined as a function whose inverse relation is a function; **non-invertible function**, defined as a function whose inverse relation is not a function.

Partial inverse of a function f, defined implicitly as a function g that is an inverse of f on a restricted domain of f, and where the domain of g equals the range of f.

Additionally, we revisit **composition** and **inverse** so that we can use them to compare and contrast correspondence and covariation views. By these terms, we mean:

- (Correspondence) Conceiving of functions and their behavior primarily in terms of maps from individual elements of the domain to individual elements of the range.
- (Covariation) Conceiving of functions and their behavior primarily in terms of coordinating how changes in the value of one variable impact the value of the other variable.

Mathematical/Teaching Practices

Introducing a definition, involving introductory examples and non-examples, a precise statement of the definition, and interpreting the definition and its terminology using examples, non-examples, and various representations.

Explaining a mathematical "test" of a property, meaning to introduce what the test does, how it works, how to pass or fail the test, and why the test works.

Noticing student thinking, meaning to observe what the student may be thinking, interpreting what the student may understand or not, as well as what you are not sure whether they understand, and responding.

Recognizing and explaining correspondence and covariation views, meaning to attend to which view or combination of views is being used, and constructing explanations from both views.

Summary

This chapter comes in three parts: reviewing key examples from Homework 2, working with functions from a correspondence point of view, and working with functions from a covariation point of view.

Review of Homework 2. We review how graphs can be used to represent relations (and hence functions) using Homework 2 Problem 3. We also review the functions used in Homework 2 Problem 2 so that the discussion of partial inverses can focus on how to construct partial inverses as opposed to details about the functions themselves, particularly sine and cosine graphs.

Functions and the correspondence view. After defining *functions*, and reviewing Homework 2 Problem 11, we introduce the teaching practices of *introducing a definition* and *explaining a mathematical "test" of a property*. The latter uses the example of the *vertical line test*. We then define *invertible function*. We continue practicing how to explain a mathematical test of a property by looking at the *horizontal line test*. We then define *partial inverses of functions* and discuss *how to construct viable partial inverses of functions*. We construct partial inverses for sine and cosine functions.

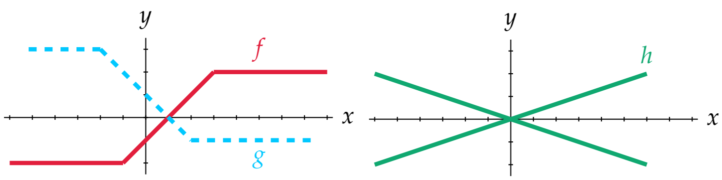
Covariational view. We use the Morgan Minicase to introduce the difference between correspondence and covariational views. By covariational view, we mean understanding how changing the value of one variable impacts the value of the other variable, and learning to coordinate changes in one variable with changes in the other. This minicase introduces the teaching practice of *recognizing and explaining correspondence and covariation views*. We revisit composition and inverse to compare and contrast these views and provide an opportunity engage in this teaching practice.

Acknowledgements. The Morgan Minicase activities are based on materials in progress for the Content Knowledge for Teaching Minicases project of the Educational Testing Service and is used with permission. The Morgan Minicase is one of a suite of items developed by the Measures of Effective Teaching project and examined through a collaborative grant between the Educational Testing Service and University of Nebraska-Lincoln (NSF #DGE-1445551/1445630).

Review of key examples

Using the definition of graph of a relation

In the previous homework, you were given the graphs of two functions f and g and asked to find values such as $g(f(1))$ or $g(g(f(2)))$. Let us now consider how graphs can be used to find these values, and how the reasoning relies on the definition of graph of a relation.

> Compare your responses to Homework 2 Problem 3. How did you use the definition of graph in your reasoning? Use $g \circ f(3)$ as an example to illustrate your use of these definitions. Then illustrate your reasoning using $h \circ g$ and where it maps 2.
>
> 
>

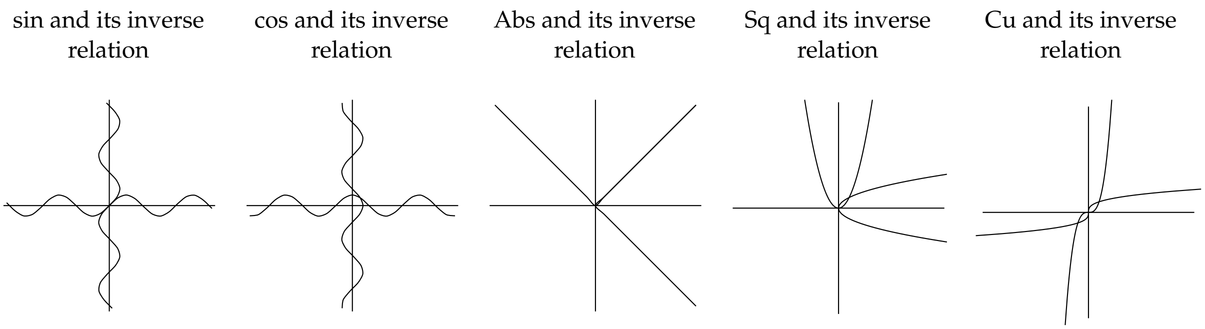
Solution. (*Partial.*) By definition of a graph of an equation, (a, b) is on the graph of $y = f(x)$ if and only if $b = f(a)$. There is only one coordinate on the graph of f with the x-value 3, so that coordinate's y-value must be $f(3)$. We then find where g sends the input $f(3)$ using similar logic.

For finding where $h \circ g$ maps 2, we first seek where g maps 2. That coordinate's y-value must be $g(2)$. There are multiple possible values for where $h \circ g$ maps 2, because the graph of h contains multiple coordinate points whose x-value is $g(2)$. ■

Some functions and relations we will examine further today

In the homework, you worked with the graphs of $y = \sin(x)$, $y = \cos(x)$, $y = |x|$, $y = x^2$, and $y = x^3$ and their inverse relations. We also learned last time that a point (a, b) is on the graph of a relation r if and only if (b, a) is on the graph of its inverse relation r^{-1}.

You may have obtained graphs that looked like this:



Functions and the correspondence view

As you saw in the previous homework Problem 11 (p. 39), a function is a special kind of relation, where every element of the domain is assigned to exactly one element of the range. Here are two ways to think about functions:

Definition 3.1 (Function: Middle and High school version)**.** A **function** f from D to R is a relation from D to R where each input in D is assigned to no more than one output in R.

Definition 3.2 (Function: University version)**.** A **function** is a relation $f : D \to R$, such that if $(x, y), (x, y') \in f$, then $y = y'$.

With relations in general, we can't say exactly where an input is mapped to, because it may be mapped to multiple outputs such as in the parent function. However, with functions, an input determines the output uniquely in the sense that it is either undefined or it is exactly one value. It is because of this property that we have function notation. The notation $f(x)$ means "the at most one value that f maps x to".

Here is a table summarizing how we can use this notation to interpret concepts associated to relations:

	Relations in general	When the relation is a function
Domain	All elements $a \in D$ with nonempty image	All elements $a \in D$ such that $f(a)$ is defined
Range	All element $b \in R$ with nonempty preimage	All elements $b \in R$ such that $b = f(x)$ has a solution in x
Composition $g \circ f$	The relation contains all assignments $x \to z$ such that there is a y where f maps x to y and g maps y to z	This relation maps x to $f(g(x))$. It is also a function.
Graph of f	The point (a, b) is on the graph of f if and only if f maps a to b	The point (a, b) is on the graph of f if and only if $b = f(a)$. *Or*: The point (a, b) is on the graph of f if and only if a is a solution to $b = f(x)$.

The above table uses the following definitions:

Definition 3.3. Given a function $f : D \to R$, we say that $f(x)$ is **defined** if x has nonempty image. Otherwise, if x has empty image, we say $f(x)$ is **undefined**.

Given $b \in R$, We say that the equation $b = f(x)$ **has a solution in** x if there is an $a \in D$ such that $b = f(a)$ is a true statement. Otherwise, if there is no $a \in D$ such that $b = f(a)$ is true, then we say the equation **has no solution**.

TEACHING GRAPHS OF FUNCTIONS AS A CASE OF TEACHING DEFINITIONS

Functions and relations are fundamental concepts in middle and high school. When you are teaching the function and relation, you will mostly likely find it useful to discuss their definitions in the context of different representations, such as cloud diagrams, T-charts (table of input and output values) and graphs. As we have been experiencing, teaching definitions often contains these components:

Teaching definitions

- Introductory examples and/or non-examples of the definition
- Precise statement of the definition
- Interpreting the precise statement, especially any new terminology or key rules, in terms of the introductory example and/or non-example
- Interpreting the terminology and rules in terms of the introductory examples, often using different representations that students will continue to encounter.

If you were teaching the definition of function, you might use examples such as the ones we have used today, or other ones that build on or review what your students are familiar with.

Review the definition of the graph of a relation. We can use this definition for functions, too, because all functions are relations.

Graph the relations $f(x) = \frac{1}{x}$, $g(x) = \sqrt{x}$ and Abs^{-1} on separate graphs. For each, how could you use the graph to understand the definition of function?

Suppose your students are seeing graphs of functions for the very first time. How would you explain the definition of a function using a graph, in general terms? (meaning, not relying on any particular example; an explanation whose wording would apply to all possible cases)

Remember that your students haven't seen anything called a "vertical line test", so they don't know this. However, the above might be a good way of getting to the idea of the "vertical line test."

Teaching the vertical line test as a case of explaining a mathematical "test" of a property

A common part of high school lessons on graphs is discussing the "vertical line test". This is a "test" in the sense that it will "test" a mathematical thing (a graph of a relation) for a particular property (whether the relation is a function). There are many "tests" like this in high school mathematics; for instance, you may remember "convergence tests" from calculus. In general, here are some elements that may be useful to keep in mind when explaining a mathematical "test" of a property, using the vertical line test as a way to illustrate a way to teach mathematical "tests" of properties.

Explaining a Mathematical "Test" of a Property

Introduce *what*: • Name the test. • What is the test supposed to tell us? (Be precise!) • What are you testing? (Be precise!)	*Example*: The Vertical Line Test • Today we will learn something called the **<u>Vertical Line Test</u>**. • This test is a way of telling **<u>whether a relation is a function</u>**. • We will test the **<u>graph of a relation</u>** to tell this.
Describe *how*: • How do you do the test? • How do you tell whether the thing passes or fails the test?	Here is how we do the Vertical Line Test: • Graph the relation. • Think about all vertical lines in the plane. Look at whether the vertical lines intersect the graph of the relation. Here is how to pass or fail: • If all of the vertical lines cross the graph zero or one times and no more, then the graph passes. Otherwise the graph fails.
Deliver the *punchline*: What happens when the thing "passes" the test? What happens when the thing "fails" the test?	If the graph of a relation passes, then the relation is a function. If the graph of a relation fails, then the relation is not a function.
Explain *why* the test "works":	[This explanation is for your homework]

To warm up to understand this table, we first discussed the following:

What does the "vertical line test" do? How do you perform this "test"? How does a graph of a relation to pass or fail the "vertical line test"? What does it mean about the relation if its graph passes or fails? Why is this conclusion true?

As we discussed these issues, we talked about the elements of the table.

Based on the discussion we have had, how would you do the following?

Explaining a Mathematical "Test" of a Property: The Vertical Line Test

Introduce *what*: • Name the test. • What is the test supposed to tell us? (Be precise!) • What are you testing? (Be precise!)	
Describe *how*: • How do you do the test? • How do you tell whether the thing passes or fails the test?	
Deliver the *punchline*: What happens when the thing "passes" the test? What happens when the thing "fails" the test?	
Explain *why* the test "works":	

What do you think of these elements? What might the purpose of each element be as far as teaching and high school students go?

INVERTIBLE FUNCTIONS AND THE HORIZONTAL LINE TEST

As we learned in the previous chapter, all functions have inverse *relations*.

This is because all relations have inverse relations, and functions are relations.

Only some functions have inverse *functions*, meaning inverse relations that happen to be functions.

Definition 3.4. A function $f : D \to R$ is **invertible function** if its inverse relation $f^{-1} : R \to D$ is a function.

A function $f : D \to R$ is **non-invertible function** if its inverse relation $f^{-1} : R \to D$ is a not a function.

Examples of functions we have seen include sine, cosine, Abs, Sq, Cu, $x \mapsto \frac{1}{x}$, and $x \mapsto \sqrt{x}$.

How would you use these examples as a way to teach the definitions of *invertible function* and *non-invertible function*? As you think about this, focus on this element of explaining definitions: **interpreting the terminology and rules in terms of the introductory examples**, using representations that students will continue to encounter.

When you are teaching invertible functions to students who are seeing it for the first time, they may not have heard of anything called a "horizontal line test." It is good practice to try to base explanations of invertibility as much as possible on the definitions of function and invertible/non-invertible functions as a way to help students understand the ideas conceptually. This provides a more solid foundation for ultimately understanding the horizontal line test.

What is the "horizontal line test"? Explain this test of a mathematical property.

Explaining ______________________________

Introduce *what*: • Name the test. • What is the test supposed to tell us? (Be precise!) • What are you testing? (Be precise!)	
Describe *how*: • How do you do the test? • How do you tell whether the thing passes or fails the test?	
Deliver the *punchline*: What happens when the thing "passes" the test? What happens when the thing "fails" the test?	
Explain *why* the test "works":	

We have discussed previously that inverse often means "undoing". In the case of invertible functions, the "undoing" metaphor works well.

Theorem 3.5 (Non-invertible may not undo, invertible always undos)**.** *Let $f : D \to R$ be a function and f^{-1} be its inverse relation.*

- *When f is non-invertible, $f^{-1} \circ f$ may map an element x to an element other than x.*
- *When f is invertible, we have $f^{-1} \circ f(x) = x$ for all x in the domain of f.*

We can see this in our examples of non-invertible functions, such as Sq, sin, or cos. The proof below summarizes the general phenomenon.

Proof. Given $f : D \to R$ is a function and f^{-1} is its inverse relation. Let a be an element of the domain of f, and suppose $f(a) = b$. When f is non-invertible, f^{-1} may map a to multiple elements, so $f^{-1} \circ f$ may then map a to multiple elements. When f is invertible, $f^{-1}(b) = a$, so $f^{-1} \circ f(a) = a$. □

This brings us to the definition of the inverse of an invertible function seen most often in high school.

Definition 3.6 (Inverse of a function: High school version)**.** If $f : D \to R$ is not an invertible function, then it does not have an inverse function.

If $f : D \to R$ is an invertible function, then the **inverse function** of f is the function f^{-1} such that for all x in the domain of f, we have

$$f^{-1} \circ f(x) = x.$$

Suppose f and g are invertible functions. Is $g \circ f$ invertible? If so, what is its inverse?

Constructing partial inverses of functions (aka "fake inverses")

There are two procedures that are often discussed in high school related to inverse of a function:

- To find the graph of the inverse of a function, reflect the graph of the function over the line $y = x$
- To find the formula for the inverse of a function, switch the y's and x's then solve for y.

We previously discussed why the first procedure makes sense. It is because reflecting over the line $y = x$ always sends the coordinate (a, b) to the coordinate (b, a). The coordinates represent assignments, and inverses switch input with output. We can use similar reasoning to explain why the second procedure makes sense.

Explain why the following procedure works: *To find the formula for the inverse of a function, switch the y's and x's then solve for y.* Your explanation should consist of two parts: specific and general. For the specific version, you might use an example such as $f(x) = x^3 + 1$. The general version should not rely on any example and instead apply to all functions.

Solution. [Homework.] ■

What if a function is non-invertible but we would like to be able to create some sort of inverse anyway?

The function $\mathrm{Sq} : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ does not have an inverse function. However, suppose you wanted to construct an function that is like an inverse, given that we know we can't have a true inverse. Rank these from best to worst as candidates for an "inverse" for Sq: (Remember that $\sqrt{x}$ is defined as the zero or positive root of x, when a real root exists.)

- $\mathrm{Rt}_1(x) = -\sqrt{x}$
- $\mathrm{Rt}_2(x) = \sqrt{x}$
- $\mathrm{Rt}_3(x)$ is defined to be $-\sqrt{x}$ when $x \in [0, 1)$ and $\sqrt{x}$ when $x \in [1, \infty)$.
- $\mathrm{Rt}_4(x)$ is defined to be $-\sqrt{-x}$ when $x \leq 0$ and $\sqrt{x}$ when $x \geq 0$.

Each of the functions $\mathrm{Rt}_1, \mathrm{Rt}_2, \mathrm{Rt}_3, \mathrm{Rt}_4$ are called "partial inverses". They are like inverse functions, except that they don't work everywhere; they only work some of the time. So they are "fake" in the sense that they don't work on the entire domain, but they are still an "inverse" in the sense that they will be an inverse in a restricted subset of the domain.

What is the domain of Sq?

On what subset of the domain of Sq do each of $\mathrm{Rt}_1, \mathrm{Rt}_2, \mathrm{Rt}_3, \mathrm{Rt}_4$ serve as a true inverse? (They satisfy the equation in Definition 3.6.)

On what subset of the domain of Sq are the functions $\mathrm{Rt}_1, \mathrm{Rt}_2, \mathrm{Rt}_3, \mathrm{Rt}_4$ serve as a fake inverse? (They do not satisfy the equation in Definition 3.6.).

Finally, let's apply the ideas of partial inverse to a trigonometric function:

Construct at least three candidates for partial inverses ("fake inverses") for $f(x) = \sin(x)$.

Rank the candidates you have constructed from best to worst "inverse" for the sine function.

On what subset of the domain of sine do each of your candidates serve as a true inverse? On what subset of the domain of sine do your candidates serve as a fake inverse?

Construct at least three candidates for partial inverses ("fake inverses") for $f(x) = \cos(x)$.

Rank the candidates you have constructed from best to worst "inverse" for the cosine function.

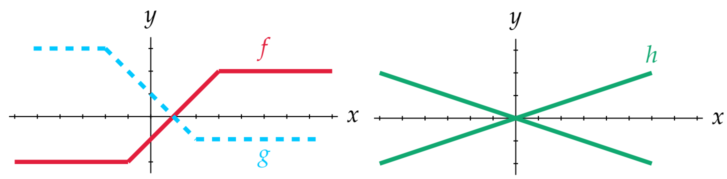
On what subset of the domain of cosine do each of your candidates serve as a true inverse? On what subset of the domain of cosine do your candidates serve as a fake inverse?

You may have concluded that the "best" inverse is the candidate whose domain of being a true inverse is $[\frac{\pi}{2}, \frac{\pi}{2}]$. This is in fact how the function that you may have seen before, with the (unfortunate) notation $\sin^{-1}$, is defined. (An alternative notation is arcsin.) In general, when working with functions for which we would like to have an inverse but the functions are not actually invertible, we can define partial inverses ("fake inverses") for them which we take to be standard. Examples of this include $x \mapsto \sqrt{x}$ for Sq, arcsin for sine, and arccosine for cosine. They usually are the candidate where if you looked at the alternatives, you would agree that there's something about that candidate that seems to work better: they are continuous, the subset of the domain on which they are a "true" inverse is close to zero or symmetric around zero, etc.

In-Class Resources

Opener: Review of key examples

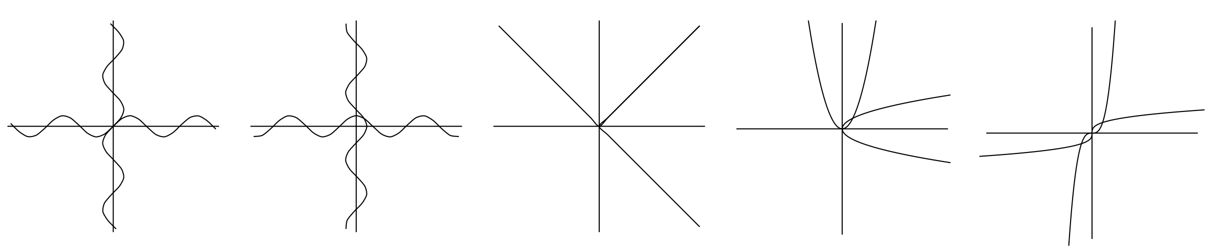
Compare your responses to Homework 2 Problem 3. How did you use the definition of graph in your reasoning? Use $g \circ f(3)$ as an example to illustrate your use of these definitions. Then illustrate your reasoning using $h \circ g$ and where it sends 2.



In the homework, you worked with the graphs of $y = \sin(x)$, $y = \cos(x)$, $y = |x|$, $y = x^2$, and $y = x^3$ and their inverse relations. We also learned last time that a point (a, b) is on the graph of a relation r if and only if (b, a) is on the graph of its inverse relation r^{-1}.

You may have obtained graphs that looked like this:

sin and its inverse relation	cos and its inverse relation	Abs and its inverse relation	Sq and its inverse relation	Cu and its inverse relation

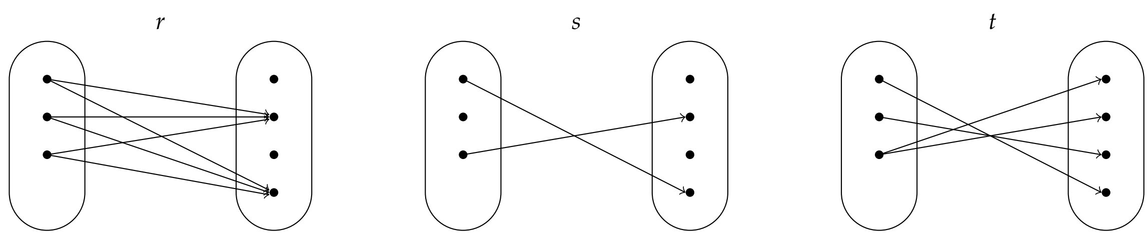


What comments or questions do you have about these graphs?

Compare your response to problem 11 from the last homework with someone next to you. What comments or questions do you have about the definition of function?

Definition. A *function* from D to R is defined as a relation from D to R where each input in D is assigned to no more than one output in R.

	r	r^{-1}	s	s^{-1}	t	t^{-1}
is a function						
is NOT a function						



TEACHING DEFINITIONS: GRAPHS OF FUNCTIONS

Teaching definitions

- Introductory examples and/or non-examples of the definition
- Precise statement of the definition
- Interpreting the precise statement, especially any new terminology or key rules, in terms of the introductory example and/or non-example
- Interpreting the terminology and rules in terms of the introductory examples, often using different representations that students will continue to encounter.

Review the definition of the graph of a relation. (We can use this definition for functions, too, because all functions are relations.)

Graph the relations $f(x) = \frac{1}{x}$, $g(x) = \sqrt{x}$ and Abs^{-1} on separate graphs. For each, how could you use the graph to understand the definition of function?

Suppose your students are seeing graphs of functions for the very first time. How would you explain the definition of a function using a graph, in general terms? (meaning, not relying on any particular example; an explanation whose wording would apply to all possible cases)

Explaining a mathematical “test” of a property: The Vertical Line Test

What does the “vertical line test” do? How do you perform this “test”? How does a graph of a relation to pass or fail the “vertical line test”? What does it mean about the relation if its graph passes or fails? Why is this conclusion true?

Based on discussion of the above questions, how would you do the following?

Explaining a Mathematical “Test” of a Property: The Vertical Line Test

<table>
<tr><td>Introduce what:
Name the test.
What is the test supposed to tell us? (Be precise!)
What are you testing? (Be precise!)</td><td></td></tr>
<tr><td>Describe how:
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<tr><td>Deliver the punchline: What happens when the thing “passes” the test? What happens when the thing “fails” the test?</td><td></td></tr>
<tr><td>Explain why the test “works”:</td><td></td></tr>
</table>

Teaching definitions: Invertible function

Definition 3.4. A function $f : D \to R$ is **invertible function** if its inverse relation $f^{-1} : R \to D$ is a function.
A function $f : D \to R$ is **non-invertible function** if its inverse relation $f^{-1} : R \to D$ is a not a function.

Examples of functions we have seen include sine, cosine, Abs, Sq, Cu, $x \mapsto \frac{1}{x}$, and $x \mapsto \sqrt{x}$.

How would you use these examples as a way to teach the definitions of *invertible function* and *non-invertible function*? As you think about this, focus on interpreting the terminology and rules in terms of the introductory examples.

EXPLAINING A MATHEMATICAL "TEST" OF A PROPERTY: THE HORIZONTAL LINE TEST

What is the "horizontal line test"? Explain this test of a mathematical property.

Explaining ________________________________

Introduce *what*: • Name the test. • What is the test supposed to tell us? (Be precise!) • What are you testing? (Be precise!)	
Describe *how*: • How do you do the test? • How do you tell whether the thing passes or fails the test?	
Deliver the *punchline*: What happens when the thing "passes" the test? What happens when the thing "fails" the test?	
Explain *why* the test "works":	

Inverses and partial inverses

Suppose f and g are invertible functions. Is $g \circ f$ invertible? If so, what is its inverse?

Explain why the following procedure works:

> To find the formula for the inverse of a function, switch the y's and x's then solve for y.

Your explanation should consist of two parts: specific and general. For the specific version, you might use an example such as $f(x) = x^3 + 1$. The general version should not rely on any example and instead apply to all functions.

What if a function is non-invertible but we would like to be able to create some sort of inverse anyway? The function $\mathrm{Sq} : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ does not have an inverse function. However, suppose you wanted to construct an function that is like an inverse, given that we know we can't have a true inverse.

Rank these from best to worst as candidates for an "inverse" for Sq:

(Remember that $\sqrt{x}$ is defined as the zero or positive root of x, when a real root exists.)

- $\mathrm{Rt}_1(x) = -\sqrt{x}$
- $\mathrm{Rt}_2(x) = \sqrt{x}$
- $\mathrm{Rt}_3(x)$ is defined to be $-\sqrt{x}$ when $x \in [0, 1)$ and $\sqrt{x}$ when $x \in [1, \infty)$.
- $\mathrm{Rt}_4(x)$ is defined to be $-\sqrt{-x}$ when $x \leq 0$ and $\sqrt{x}$ when $x \geq 0$.

PARTIAL INVERSES, CONTINUED

Construct at least three candidates for partial inverses ("fake inverses") for $f(x) = \sin(x)$.

Rank the candidates you have constructed from best to worst "inverse" for the sine function.

On what subset of the domain of sine do each of your candidates serve as a true inverse? On what subset of the domain of sine do your candidates serve as a fake inverse?

Construct at least three candidates for partial inverses ("fake inverses") for $f(x) = \cos(x)$.

Rank the candidates you have constructed from best to worst "inverse" for the cosine function.

On what subset of the domain of cosine do each of your candidates serve as a true inverse? On what subset of the domain of cosine do your candidates serve as a fake inverse?

4 Comparing Correspondence and Covariation Views of Functions

Introducing covariation

So far we have been looking at things from a *correspondence view*, meaning that we primarily think of functions in terms of individual inputs and the images to which they are assigned. In other words, we focus on how inputs and outputs correspond. This perspective can be very useful for defining things like domain, range, composition, or inverse. However, this perspective can be at a distinct disadvantage when it comes to looking at the behavior of a function or of understanding how changes in the input and output variables influence each other.

By *covariational view*, we mean understanding how changing the value of one variable impacts the value of the other variable, and learning to coordinate changes in one variable with changes in the other. We saw this already in Homework 3 Problems 6 and 7. We use the Morgan Minicase[2] to dig deeper into the difference between correspondence and covariational views. This minicase also introduces the teaching practice of *recognizing and explaining correspondence and covariation views*.

Ms. Morgan's class

During a lesson on writing equations of linear functions represented in tables, Ms. Morgan asked her students to write the equation of the linear function represented in the table below, and to explain how they found their answers.

x	y
1	6
2	11
3	16
4	21

Students found the correct equation, but they gave different explanations of how they found their answers:

Student A:	Each time the value of x goes up by 1, the value of y goes up by 5, so the slope is 5. And if x goes down by 1, then y will have to go down by 5, so the y-intercept is 1. That means the equation is $y = 5x + 1$.
Student B:	I just looked at the value of y and saw that it kept increasing by 5, so $m = 5$. Then I subtracted that number from the first value of y in the table, so $b = 1$. You always put m times x and add the b, so the equation is $y = 5x + 1$.
Student C:	For this function, I saw that you can always multiply the value of x by 5 and then add 1 to get the value of y, so the equation is $y = 5x + 1$.

Read through the student responses in Ms. Morgan's class.

- **Observe**: What is the student thinking? How might they have arrived at each step of their solution?
- **Interpret**: What are you sure that each student understands? What are you sure that each student does not understand? What are you unsure that each student understands? Based on what evidence?

  Here are some concepts to consider analyzing for students' understanding:
  - y-intercept
  - Constant rate of change
  - Form of a linear equation
  - How changes in one variable impact changes in the other variable
  - Definition of graph of a function
- **Interpret, continued:** Are the explanations mathematically complete? Why or why not?

Observe: What may Student A/B/C be thinking?

Interpret:

[2](c) 2013, Educational Testing Service, used with permission

I am sure that Student A/B/C understands ...	I am sure that Student A/B/C does NOT understand ...	I am unsure whether Student A/B/C understands ...

What is complete or incomplete about Student A/B/C's explanation?

There are two useful views on functions: covariation and correspondence. Use this space to take notes on what these mean.

Correspondence:

Covariation:

Then discuss: How do covariation and correspondence views come up in the students' thinking?

Solution. (*Partial*)

Ways that the covariation and correspondence views arise in the Morgan Minicase include the following:

- How changes in one variable impact changes in the other variable: Covariation view; and
- Definition of graph of a function: Correspondence view.

Student A (does provide a mathematically complete explanation)

- understands concept of constant rate of change, y-intercept, form of linear function
- may not understand correspondence view of function

Student B (does *not* provide a mathematically complete explanation)

- does know that linear functions have the form $y = mx + b$, where $m, b \in \mathbb{R}$
- does not understand constant rate of change
- may not understand y-intercept
- does not necessarily understand correspondence view of function

Student C (does provide a mathematically complete explanation)

- does know that linear functions have the form $y = mx + b$, where $m, b \in \mathbb{R}$
- does understand correspondence view of function
- may not understand constant rate of change, y-intercept.

■

NOTICING STUDENT THINKING AND RECOGNIZING CORRESPONDENCE AND COVARIATION VIEWS

In any teaching, it is important to attend to student work so that you can notice the student thinking. Some things to keep in mind for this are:

Noticing student thinking

- First **observe** what the student's thinking is, **without judgment** as to what they understand or do not understand.
- Then, **interpret** what they understand, may not understand, or what you are unsure of whether they understand. Always base this interpretation on the evidence of the student thinking you have, and be sure that you know what evidence you are drawing upon.

  Only after this, you might interpret the completeness or correctness of the student's thinking.
- From here, you might **respond** to the student based on what you have observed and interpreted.

The reason to split up observing without judgement, interpreting, and responding is that it interpretations tend to be more accurate after we have taken a step back to observe what the student may be thinking, without judgment.

In the case of teaching functions, it is helpful to be able to *recognize and explain correspondence and covariation viewpoints*. The Morgan Minicase gave us an opportunity to recognize how both can show up in response to the same problem as well as how differently they can appear. In the next part of this lesson, we will practice explanations from a covariation view. Throughout this minicase and in the next section, we keep in mind the following.

Recognizing and explaining correspondence and covariation viewpoints

Correspondence and covariation views can be thought of as the following:

- (Correspondence) Conceiving of functions and their behavior primarily in terms of maps from individual elements of the domain to individual elements of the range.
- (Covariation) Conceiving of functions and their behavior primarily in terms of coordinating how changes in the value of one variable impact the value of the other variable.

When introducing ideas in class, coming up with examples, or giving explanations, it is helpful to think about whether you are working with a correspondence or covariation view, and then to see what an explanation in the other view might look like.

When noticing student thinking, it can be helpful to interpret whether they are taking a correspondence or covariation view.

EXPLAINING FROM CORRESPONDENCE AND COVARIATION VIEWS: BUILDING FUNCTIONS

We now revisit our ways of building functions to see how the covariation view can come into play.

Let's begin by looking at compositions of functions that high school students are likely to encounter: linear functions and quadratic functions.

Suppose that f is a linear function whose constant rate of change is 5 output units per input unit; g is a linear function whose constant rate of change is -3 output units per input unit; and h is a linear function whose constant rate of change is 0 output units per input unit.

What is the rate of change of $g \circ f$? Circle your response.

2   -2 15 -15 8 -8 it is not constant   none of the above

What about for $f \circ g$? What is your reasoning?

What about for $h \circ f$? $f \circ h$? What is your reasoning?

Correspondence view. There are many ways to address this problem; for instance, we might find the answer using algebra, saying that if $f(x) = 5x + b$ and $g = -3x + c$ then $g \circ f(x) = -15x + d$, where d is a constant. This kind of explanation would be more from a correspondence viewpoint than a covariation one, because it looks at how inputs are assigned to outputs of each function and performs the composition for an individual element. It is not an explanation from a covariation view because it does not take into account how changes in one variable (input or output) of $g \circ f$ would impact the value of the other variable. Through similar reasoning, we can find that $f \circ g$ must have slope -15, and that $h \circ f$ and $f \circ h$ will both have slope 0. (Try this yourself to make sure that you see why.)

Covariation view. Each time the value of x changes by $+1$ unit, the value of $f(x)$ changes by $+5$ units, and the value of $g(x)$ changes by -3 units. Now let us look at what changing the value of x by $+1$ unit does to the output of $g(f(x))$. So if we change x by $+1$, then $f(x)$ changes by $+5$ units, which means $g(f(x))$ changes by $-3 \cdot 5$ units. This is true for all x, so the rate of change for $g \circ f$ is -15 output units per input unit. We can use similar reasoning to solve the other cases. (Try this yourself.)

Suppose that L is a linear function, and Q is a quadratic function. What is $Q \circ L$? Circle your response.

linear quadratic cubic something else

What is your reasoning? How would you explain this from a covariation view?

Let sin be the sine function. What is $\sin \circ L$? Circle your response.

linear sinusoidal something else

(We can think of sinusoidal as behaving like sine or cosine: it moves "up" and "down" in a period fashion, always to the same maximum and minimum value.)

What is your reasoning? How would you explain this from a covariation view?

Correspondence view. The function Q can be expressed as $Q(x) = ax^2 + bx + c$, and L can be expressed $L = mx + d$. Then $Q \circ L(x) = a(mx + d)^2 + b(mx + d) + c$, which is still quadratic.

Covariation view. We can think of a quadratic function as one whose rate of change of rate of change is constant. If we change the input variable to Q linearly, as we would for $Q \circ L$, then the rate of change of the rate of change would multiply by the linear rate of change. It would still be constant. So the composition is still quadratic.

Combining points of view. The function $\sin \circ L$ can be expressed as $x \mapsto \sin(mx + d)$. It still has a regular period (though the periods are now $1/m$ of the length they were before since the inputs are moving m times as fast through sin) and the maxima and minima stay the same. So $\sin \circ L$ is still sinusoidal.

We have so far looked at examples of composition. Now we examine inverses.

Review: What is the high school definition of inverse of an invertible function? (Look this up to check whether you remembered correctly.)

Suppose that f is a linear function with constant rate of change 5 output units per input unit.

What is f^{-1}? Circle your response.

a linear function with constant rate of change -5

a linear function with constant rate of change 5

a linear function with constant rate of change $\frac{1}{5}$

a linear function with constant rate of change $-\frac{1}{5}$

none of the above

What is your reasoning?

How does this make sense in terms of the high school definition of inverse of an invertible function?

How would you explain this from a covariation view?

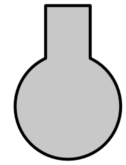
Correspondence view. If $f(x) = 5x + b$, the inverse function of f maps x to y where $x = 5y + b$. Solving for y, we obtain $y = \frac{x-b}{5}$, which is a linear function with constant rate of change $\frac{1}{5}$.

Covariation view. If f is changing at a rate of 5 output units to 1 input unit, and we want to compose this with something that will get us back to 1 output unit per input unit (to satisfy the equation $f^{-1} \circ f(x) = x$), then we need f^{-1} to slow the input $f(x)$ down by $\frac{1}{5}$ output units per input unit. So f^{-1} is a linear function with with constant rate of change $\frac{1}{5}$.

Bottle Problem

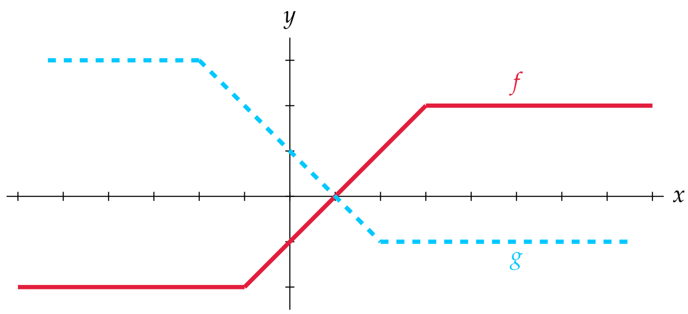
Sketch a graph of the volume of water in the bottle as a function of the height, as the bottle is being filled up with water.

Then sketch volume as a function of height.



Revisiting a key example

Let us look back at a key example from the beginning of class.



In the below, we ask about "sections" of a function. We are using this word informally and here use it to refer to places where the rate of change is constant.

What is the rate of change of each section of f? What is the domain of each section? What is the image of each section?

What is the rate of change of each section of g? What is the domain of each section? What is the image of each section?

How many sections are there of $g \circ f$? What are they? What is the rate of change of each section?

How would you use the above information to graph $g \circ f$?

How did you use correspondence and covariation views in your reasoning?

Solution. (*Partial.*) Sections of f:

section (domain)	rate of change	image
$(-\infty, -1)$	0	$\{-2\}$
$(-1, 3)$	1	$(-2, 2)$
$(3, \infty)$	0	$\{2\}$

Sections of g:

section (domain)	rate of change	image
$(-\infty, -2)$	0	$\{3\}$
$(-2, 2)$	-1	$(3, -1)$
$(2, \infty)$	0	$\{-1\}$

There are three sections of $g \circ f$, corresponding to the sections of f. These sections go nicely into the sections of g. The rate of change of each section is $0, 1 \cdot -1, 0$.

Summary of mathematical/teaching practices

Teaching definitions

- Introductory examples and/or non-examples of the definition
- Precise statement of the definition
- Interpreting the precise statement, especially any new terminology or key rules, in terms of the introductory example and/or non-example
- Interpreting the terminology and rules in terms of the introductory examples, often using different representations that students will continue to encounter.

Explaining a Mathematical "Test" of a Property

- Introduce *what*:
  - Name the test.
  - What is the test supposed to tell us? (Be precise!)
  - What are you testing? (Be precise!)
- Describe *how*:
  - How do you do the test?
  - How do you tell whether the thing passes or fails the test?
- Deliver the *punchline*: What happens when the thing "passes" the test? What happens when the thing "fails" the test?
- Explain *why* the test "works":

Noticing student thinking

- **Observe**: What is the student thinking? How might they have arrived at each step of their solution?
- **Interpret**: What are you sure that each student understands? What are you sure that each student does not understand? What are you unsure that each student understands? Based on what evidence?
- **Interpret, continued:** Are the explanations mathematically complete? Why or why not?
- **Respond**: Based on the above, determine how to respond.

Recognizing and explaining correspondence and covariation views

Correspondence and covariation views can be thought of as the following:

- (Correspondence) Conceiving of functions and their behavior primarily in terms of maps from individual elements of the domain to individual elements of the range.
- (Covariation) Conceiving of functions and their behavior primarily in terms of coordinating how changes in the value of one variable impact the value of the other variable.

When introducing ideas in class, coming up with examples, or giving explanations, it is helpful to think about whether you are working with a correspondence or covariation view, and then to see what an explanation in the other view might look like.

When noticing student thinking, it can be helpful to interpret whether they are taking a correspondence or covariation view.

In-Class Resources

MORGAN MINICASE

Ms. Morgan's class

During a lesson on writing equations of linear functions represented in tables, Ms. Morgan asked her students to write the equation of the linear function represented in the table below, and to explain how they found their answers.

x	y
1	6
2	11
3	16
4	21

Students found the correct equation, but they gave different explanations of how they found their answers:

Student A:	Each time the value of x goes up by 1, the value of y goes up by 5, so the slope is 5. And if x goes down by 1, then y will have to go down by 5, so the y-intercept is 1. That means the equation is $y = 5x + 1$.
Student B:	I just looked at the value of y and saw that it kept increasing by 5, so $m = 5$. Then I subtracted that number from the first value of y in the table, so $b = 1$. You always put m times x and add the b, so the equation is $y = 5x + 1$.
Student C:	For this function, I saw that you can always multiply the value of x by 5 and then add 1 to get the value of y, so the equation is $y = 5x + 1$.

Read through the student responses in Ms. Morgan's class.

- **Observe**: What is the student thinking? How might they have arrived at each step of their solution?
- **Interpret**: What are you sure that each student understands? What are you sure that each student does not understand? What are you unsure that each student understands? Based on what evidence?

  Here are some concepts to consider analyzing for students' understanding:

  - y-intercept
  - Constant rate of change
  - Form of a linear equation
  - How changes in one variable impact changes in the other variable
  - Definition of graph of a function

- **Interpret, continued:** Are the explanations mathematically complete? Why or why not?

Record your thinking on the next page.

Observe: What may Student A be thinking?

Interpret:

I am sure that Student A understands …	I am sure that Student A does NOT understand …	I am unsure whether Student A understands …

Observe: What may Student B be thinking?

Interpret:

I am sure that Student B understands …	I am sure that Student B does NOT understand …	I am unsure whether Student B understands …

Observe: What may Student C be thinking?

Interpret:

I am sure that Student C understands …	I am sure that Student C does NOT understand …	I am unsure whether Student C understands …

Practicing Correspondence and Covariational Explanations

There are two useful views on functions: covariation and correspondence. Use this space to take notes on what these mean.

Correspondence:

Covariation:

Then discuss: How do covariation and correspondence views come up in the students' thinking in Ms. Morgan's class?

1. Suppose that f is a linear function whose constant rate of change is 5 output units per input unit; g is a linear function whose constant rate of change is -3 output units per input unit; and h is a linear function whose constant rate of change is 0 output units per input unit.

   What is the rate of change of $g \circ f$? Circle your response.

   2 $\quad -2 \quad$ 15 $\quad -15 \quad$ 8 $\quad -8 \quad$ it is not constant $\quad$ none of the above

   What about for $f \circ g$? What is your reasoning?

   What about for $h \circ f$? $f \circ h$? What is your reasoning?

2. Suppose that L is a linear function, and Q is a quadratic function. What is $Q \circ L$? Circle your response.

   linear $\quad$ quadratic $\quad$ cubic $\quad$ something else

   What is your reasoning? How would you explain this from a covariation view?

   Let sin be the sine function. What is $\sin \circ L$? Circle your response.

   linear $\quad$ sinusoidal $\quad$ something else

   (We can think of sinusoidal as behaving like sine or cosine: it moves "up" and "down" in a period fashion, always to the same maximum and minimum value.)

   What is your reasoning? How would you explain this from a covariation view?

3. Review: What is the high school definition of inverse of an invertible function? (Look this up to check whether you remembered correctly.)

4. Suppose that f is a linear function with constant rate of change 5 output units per input unit.

   What is f^{-1}? Circle your response.

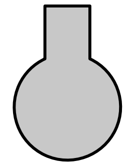
   a linear function with constant rate of change -5
   a linear function with constant rate of change 5
   a linear function with constant rate of change $\frac{1}{5}$
   a linear function with constant rate of change $-\frac{1}{5}$
   none of the above

   What is your reasoning?

   How does this make sense in terms of the high school definition of inverse of an invertible function?

   How would you explain this from a covariation view?

Bottle Problem

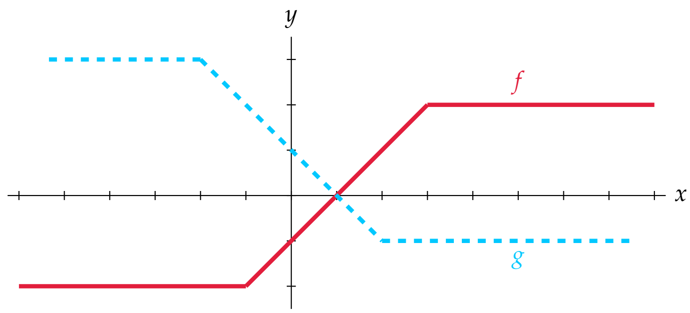


Sketch a graph of the volume of water in the bottle as a function of the height, as the bottle is being filled up with water.

Then sketch volume as a function of height.

Revisiting a key example

Let us look back at a key example from the beginning of class.



1. What is the rate of change of each section of f? What is the domain of each section? What is the image of each section?
2. What is the rate of change of each section of g? What is the domain of each section? What is the image of each section?
3. How many sections are there of $g \circ f$? What are they? What is the rate of change of each section?
4. How would you use the above information to graph $g \circ f$?
5. How did you use correspondence and covariation views in your reasoning?

Homework for Chapters 3-4

0. In these chapters, we learned about:

   - Functions and invertible functions
   - Partial inverse of a function and how to construct them
   - Correspondence and covariation views for inverse and composition
   - The mathematical/teaching practices of:
     - Introducing a definition
     - Explaining a mathematical "test" of a property
     - Noticing student thinking
     - Recognizing and explaining correspondence and covariation views.

   For each of these ideas:

   (a) Where in the text are these ideas located?

   (b) Review this section of the text. What definitions and results were important? How do examples use these definitions and results?

   (c) What questions or comments do you have about the ideas in this section?

CHAPTER 3

1. Explain why the following procedure works:

   > To find the formula for the inverse of an invertible function, switch the y's and x's then solve for y.

   Explain why this procedure works in terms of the example:

   (a) $f(x) = 5x$

   (b) $f(x) = x^3 - 1$

   (c) $f(x) = \frac{1}{x-3}$

   Then:

   (d) Explain why this procedure in general terms.

   (e) Explain why this procedure is equivalent to reflecting the graph of the function about the line $y = x$.

   (f) Explain what would work and what would not work if you were to use this procedure on non-invertible functions.

2. (This problem comes from Mason, Burton, and Stacey (2010, p. 203)[3]) Under what conditions can you rotate the graph of a function about the origin, and still have the resulting graph being the graph of a function? If the graph of a function cannot be rotated about the origin without ceasing to be the graph of a function, might there be other points which could act as center of rotation and preserve the property of being the graph of a function?

3. (a) Let f and g be two invertible functions. Explain why $g \circ f$ is invertible in terms of the middle school and university versions of the definition of relation.

   (b) Explain why the inverse of $g \circ f$ should be $(f^{-1}) \circ (g^{-1})$ in terms of the middle school version of the definition of composition.

   (c) Explain why the inverse of $g \circ f$ should be $(f^{-1}) \circ (g^{-1})$ in terms of the university version of the definition of invertible function.

4. (a) Construct three candidates for partial inverses for the sine function.

   (b) On what subset of the domain of sine do each of your candidates serve as a true inverse? On what subset of the domain of sine do your candidates serve as only a partial inverse?

   (c) Construct three candidates for partial inverses for the cosine function?

[3] Mason, J., Burton, L., Stacey, K. (2010). *Thinking Mathematically*. Essex, England: Pearson Education Limited.

(d) On what subset of the domain of cosine do each of your candidates serve as a true inverse? On what subset of the domain of cosine do your candidates serve as only a partial inverse?

5. Explain the following tests of a mathematical property using the structure discussed in Section 3.3.2 (beginning p. 43). In communicating your explanation, place the different sections of the structure separately from each other and label them.

   (a) Vertical line test.

   (b) Horizontal line test.

   For the following tests of mathematical properties, in the section on *why* it works, provide an explanation in two parts, first in terms of a specific example, and second as a general explanation.

   (c) Testing points on a graph of a linear inequality to see which side to shade (e.g., http://www.wtamu.edu/academic/anns/mps/math/mathlab/beg_algebra/beg_alg_tut24_ineq.htm). *Note:* The definition of graph of an inequality in x and y is similar to that of graph of an equation: It is the set of points (a, b) that if you evaluate the inequality at $x = a$ and $y = b$, you get a true statement. The property being tested here is of a half-plane, and whether the points in that half-plane satisfy a given inequality.

   (d) Direct comparison test (for convergence of a series).

   (e) Ratio test (for convergence of a series).

6. Examine the table to the right. Based on the given information, discuss how change in x appears to impact change in y.

x	y
3	16
5	26
6	31
9	46

   (a) If x changes by ± 1 units, how does y appear to change?

   (b) If x changes by $\pm h$ units, how does y appear to change, in terms of h?

   (c) Based on how change in x impacts change in y, and using the given data, describe how you would find a plausible value of y when x is 0.

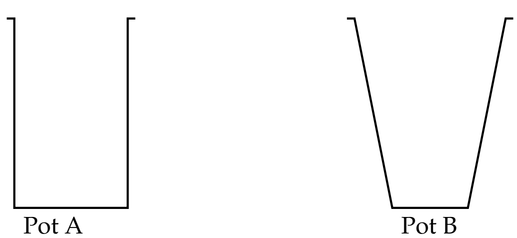
7. Pot A and Pot B have the radial cross section shown below. (This means that to get the shapes of Pot A and Pot B, you can rotate this cross section around a central axis.) The sides of Pot A are vertical. Both pots have a 1 gallon capacity.

   (a) Water is being poured into Pot A at an unsteady pace. Draw a graph that represents the relationship between volume of water and height of the water, with volume as input variable, height as output variable.

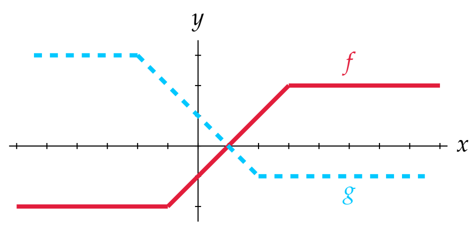
   (b) Draw a graph that represents the same relationship for Pot A, but this time with height as an input variable and volume as output variable.

   (c) Water is being poured into Pot B at an unsteady pace. Draw a graph that represents the relationship between volume of water and height of the water, with volume as input variable, height as output variable.

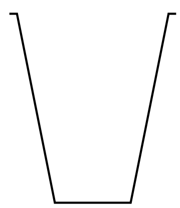
   (d) Draw a graph that represents the same relationship for Pot B, but this time with height as an input variable and volume as output variable.



8. Suppose you are teaching the idea of inverse of a composition of invertible functions in a high school class. You want to explain why the inverse of $g \circ f$ should be $(f^{-1}) \circ (g^{-1})$ with the example of $f(x) = x + 5$ and $g(x) = 3x$.
   (a) Give an explanation from a correspondence view.
   (b) Give an explanation from a covariation view.
9. Below are graphs of the relations f and g. The pieces of these graphs are lines and line segments, and their turning points are integer coordinate points. Consecutive tick marks on the axes are distance 1 from each other.
   (a) What is the rate of change of each section of f? What is the domain of each section? What is the image of each section?
   (b) What is the rate of change of each section of g? What is the domain of each section? What is the image of each section?
   (c) How many sections are there of $g \circ f$? What are they? What is the rate of change of each section?
   (d) How would you use the above information to graph $g \circ f$? Cite specifically where you use each piece of information from (a), (b), and (c).
   (e) How did you use correspondence and covariation views in your reasoning?



Problems 10 and 11 use the diagram below. Pot B has the radial cross section shown and has a 1 gallon capacity.



Pot B

10. Let B be the function that maps volume of water in Pot B to height of water, when Pot B starts empty.
    Let C be the function that maps volume of water in Pot B to height of water, when Pot B starts with 4 cups of water already in it. (Look up how many cups are in a gallon.)
    Let v represent volume and h represent height.
    (a) Graph $h = B(v)$ and $h = C(v)$ on the same set of v-h axes.
    (b) How do the graphs of B and C relate? Why does this make sense?
11. Pot D fills up at twice the rate as Pot B, meaning that the rate of change of height with respect to volume for Pot D is twice that of Pot B.
    (a) If Pot D and Pot B are equally tall, how much volume does Pot D hold?
    (b) Let D be the function that maps volume of water in Pot D to height of water, when Pot D starts empty. Graph $h = B(v)$ and $h = D(v)$ on the same set of v-h axes.
    (c) Which of the following best captures the relationship between $B(v)$ and $D(v)$?

$$B(v) = 2D(v) \qquad B(v) = D(2v) \qquad B(v) = \frac{1}{2}D(v) \qquad B(v) = D(\frac{1}{2}v)$$

Simulation of Practice: Concept of Inverse

Suppose that you are teaching high school pre-calculus and you are introducing the concept of inverse of an invertible function. The students have already learned the definition of invertible function.

Here is a task that you plan to use:

> A pot has straight vertical sides, stands 6 inches tall, and has 2 gallon capacity. Water is being poured into this pot.
>
> (a) Draw a picture of this pot.
>
> (b) Find the relationship between v, for a volume of water poured into the pot, and h, the height of water in the pot at that volume. Graph this relationship.
>
> (c) If there is $\frac{3}{4}$ gallon of water in the pot, how high is the water level?
>
> (d) If the water level in the pot is 3 inches high, how much water is in the pot?
>
> (e) If the water level is 3.5 inches high, how much water is in the pot?

You would like your class to understand both of the following definitions, and how these definitions can be interpreted using different representations, especially algebraic, graphical, and verbal.

> **Definition 1.** Given an invertible function f, the inverse of f is the function that maps $y \mapsto x$ whenever $x \mapsto y$ is an assignment of f. The inverse function is denoted f^{-1}.
>
> **Definition 2.** Given an invertible function f, the inverse of f is the function such that for all x in the domain of f, we have $f^{-1} \circ f(x) = x$.

Break down your plan into the following sections. In communicating this plan, place these sections separately from each other and label them. The page limit for this plan is 2 pages (meaning 1 page double-sided or 2 pages single-sided.)

- **Goals for the lesson**. Write this to be consistent with the scenario described above as well as the sections below. You may need to revise/reword this part as you work through the rest of the planning. (This is a very normal thing to happen when planning a lesson.)
- **Solution to the task**. Include at least least three different solutions to (e).
- **Motivating example.** Write down what you would say after the class has completed the activity to introduce the concept of inverse of a function. In this description, work in the notation you would use to refer the function being inverted, and how you would define this function and its inverse.
- **Key terminology.** State the order that you would bring in each of Definitions 1 and 2. Then list the phrases or terminology in these definitions that you think students would benefit from discussion to understand.
- **Illustrating concept with multiple representations.** Describe how you would use the problem context, algebraic notation, and graph to help students make sense of these phrases or terminology, and then the definition.
    - What are 2-3 questions you would ask to lead a discussion on this?
    - For each question, what would you anticipate students to respond?
    - For each question, what would an ideal response be?
- **Mathematical equivalence of definitions.** Suppose you ask the class the question: "Why are these two definitions saying the same thing?"
    - What would you anticipate students to respond to this question?
    - Describe two ideal responses to this question, one using the example, and one that is a general explanation.
- **Summary.** Write down a summary sentence or two that you would use to conclude the discussion. This sentences should be short enough to be easy to say, and drive home the main mathematical point of the lesson.
- **Follow up.** Write a task you would assign for homework to launch a discussion on how to understand the following statement: if f is an invertible function and $y = f(x)$, then $y = f^{-1}(x)$.

FEEDBACK CHART

Descriptor	Meets Expectations	Does Not Meet Expectations
Are anticipated solutions realistic?	Both correct and incorrect solutions are generated using a variety of methods.	Solutions are not reasonable and/or do not include a variety of method and/or no correct solution is provided.
Is the whole class discussion plan reasonable?	Discussion questions are included that follow a logical path and will move student thinking forward.	Discussion questions are not logically sequenced and/or not appropriate to move thinking forward.
Are anticipated student responses for the whole class discussion reasonable?	Anticipated student responses are reasonable.	Anticipated student responses are unreasonable or not included.
Will the task posed move thinking toward understanding the statement, *if f is an invertible function and $y = f(x)$, then $y = f^{-1}(x)$*?	Task provided will move students toward understanding statement.	Task provided does not logically follow from previous and/or will not move toward understanding the statement.

REFLECTION PROMPT (TO BE COMPLETED AFTER RECEIVING FEEDBACK)

1. What are some take-aways for you about using student thinking in moving toward a learning goal?
2. When you teach this concept in the future, what will you change? What will you keep? For what reasons?

Simulation of Practice: Concepts of Linear Equations and Graphs

(This simulation of practice is an adaptation of the Allen Minicase, part of the Content Knowledge for Teaching Minicases project of the Educational Testing Service.)

Suppose that you are working with your students to review for an end of year Algebra I exam. You include the following problem on a worksheet of practice problems that they completed for homework.

> While visiting New York City, John kept track of the amount of money he spent on transportation by recording the distance he traveled by taxi and the cost of the ride.
>
> | Distance, d, in miles | Cost, C, in dollars |
> |---|---|
> | 3 | 8.25 |
> | 5 | 12.75 |
> | 11 | 26.25 |
>
> Show that the data can be represented by the linear function $C = 2.25d + 1.5$.

Your students used different methods to solve the problem; two solutions that you would like to go over with the class are that of Jing's and Matt's. You notice that one of these solutions is correct, and the other one has a mathematical error.

Jing	Matt
$\frac{12.75 - 8.25}{5 - 3} = \frac{4.50}{2} = 2.25$ $\begin{aligned} C - 26.25 &= 2.25(d - 11) \\ C - 26.25 &= 2.25d - 24.75 \\ C &= 2.25d + 1.5 \end{aligned}$	$\begin{aligned} 2.25(3) + 1.5 &= 8.25 \\ 2.25(5) + 1.5 &= 12.75 \\ 2.25(11) + 1.5 &= 26.25 \end{aligned}$

You are planning what you might say to the class in response to these methods.

Record a video of yourself providing a response to students with these solution methods

1. Summarize what Jing and Matt may be thinking.
2. Say what is worthwhile about Jing's thinking and Matt's thinking.
3. Help the students complete their thinking (if there are gaps in the thinking), prompt the students to investigate an error, or help the students move forward in their thinking.
4. Give specific feedback to students that they would be able to use in their future work in mathematics.

Feedback Chart

Descriptor	Meets Expectations	Does Not Meet Expectations
Is the summary of student thinking reasonable?	Summary points to reasonable explanation of student responses.	Summary does not attend to what students might have been thinking.
Is the response to student 1 reasonable?	Response to student 1 appropriately helps the student complete their thinking, prompts the student to investigate an error, or helps the student move forward in their thinking.	Response to student 1 does not accurately assess student understanding and move the student in a reasonable direction.
Is the response to student 2 reasonable?	Response to student 2 appropriately helps the student complete their thinking, prompts the student to investigate an error, or helps the student move forward in their thinking.	Response to student 2 does not accurately assess student understanding and move the student in a reasonable direction.
Is the mathematical language used appropriate?	Oral description of mathematical ideas uses accurate mathematical language.	Oral description of mathematical ideas does not use accurate mathematical language.

Reflection Prompt (to be completed after receiving feedback)

1. What are some take-aways for you about using student thinking in moving toward a learning goal?
2. In light of this experience, what are the mathematical points you will make sure to highlight when you teach this in the future? What will you deemphasize? For what reasons?

Part III

Transformations of Functions

5 Defining transformations (Weeks 5-6) (Length: ~6 hours)

Overview

Content
Thing 1, defined as ...
Thing 2, defined as ...

Proof Structures	**Mathematical/Teaching Practices**
To show ... means ...	**Practice name**, meaning ...

Summary

Prep for next lesson: Make sure to assign mini lesson for homework between 5 and 6.

This set of lessons is shorter so that you can have time to conduct a review session and/or have a buffer for the lessons 1-4 taking longer than expected.

Many ways to compose to the same function

```
Jeremy:
> On p. 81, the scenario is unclear. What is the input quantity, x? Does (6,600) mean that after 6 hours,
> the factory was making 600 candies per hour? Or... Do you mean that C is a function that outputs the
> number of candies made per hour, or is C the name of a function that outputs the total number of candies
> made after making candies for x hours?
I haven't looked at this yet but will put it into the notes for that Chapter to finalize later.
```

Define:

$$A : \mathbb{R} \to \mathbb{R}, x \to 3x$$

$$B : \mathbb{R} \to \mathbb{R}, x \to x + 4$$

$$E : \mathbb{R} \to \mathbb{R}, x \to x + 12.$$

Write $x \mapsto 3x + 12$ as a composition of A, B, E in two different ways. (You do not have to use all of them in either of the ways.)

Solution. $3x + 12$ can be obtained by multiplying by 3 first, then adding 12; or adding 4 first, then multiplying by 3. ∎

Observations based on problem:

- To make more complicated functions, we can sometimes compose simpler functions.
- One of the more common patterns is decomposing expressions like $3x + 12$ into multiplication/division then addition/subtraction, or addition/subtraction then multiplication/division.

$$x \overset{\times 3}{\rightsquigarrow} 3x \overset{+12}{\rightsquigarrow} (3x) + 12 \quad \text{or} \quad x \overset{+4}{\rightsquigarrow} x + 4 \overset{\times 3}{\rightsquigarrow} 3(x + 4)$$

1. Find four different ways to start from x and obtain the expression $2\sin(\frac{1}{3}x + 12) - 10$ through compositions.
2. Let $A : \mathbb{R} \to \mathbb{R}, x \mapsto 3x, \quad B : \mathbb{R} \to \mathbb{R}, x \mapsto x + 4, \quad C : \mathbb{R} \to \mathbb{R}, x \mapsto 2x, \quad D : \mathbb{R} \to \mathbb{R}, x \mapsto x + 5$. Write the map $f : \mathbb{R} \to \mathbb{R}, x \mapsto 2\sin(3x + 12) + 10$ in terms of A, B, C, D and sin.

Solution.

1. Use different types of multiplication/division and addition/subtraction combinations on the inside and outside of the sin function.
2. $f(x) = C(D(\sin(A(B(x)))))$

∎

Observations based on problem:

- We can express the above multiplication/division and addition/subtraction combinations as function compositions.
- This leads into the idea of input transformations.

Input transformations

Definition 5.1. Given a function f, a function that can be expressed in the form

$$f \circ T$$

is said to be an **input transformation** of f. (The assignment rule is $f(T(x))$ instead of $f(x)$).

On January 1, 2101, Wall-E the robot wakes up at 9am, and at noon, he begins shelving away a large pile of metal toy blocks, one by one. He decides to put away 1 block every 6 minutes in the the first hour as he is figuring out his system for these blocks. Then, in the second hour, he puts away 1 block every 4 minutes. From there on, he puts away 1 block every minute. When will he have put away 144 blocks?

Now let's represent some different scenarios with graphs. In each of the graphs:

- Put a green dot at the point in the graph representing when Wall-E begins his work.
- Put a red dot at the point in the graph representing when Wall-E has put away 144 blocks.

Let W be the function of Wall-E's total blocks put away versus time.

1. On the same set of axes, graph each of the following functions of Wall-E's total blocks put away versus time.
   - W, the function of Wall-E's total blocks put away versus time.
   - The function if Wall-E had started at 11:30am instead of noon.
   - The function if Wall-E had started at 10:30am instead of noon.
   - The function if Wall-E had started at 2pm instead of noon.
2. On the same set of axes, graph each of the following functions of Wall-E's total blocks put away versus time.
   - The function in the original situation.
   - The function if Wall-E's internal clock is running 2 times slower (so what he thinks is an hour is actually 2 hours).
   - The function is Wall-E's internal clock is running 1.5 times fast (so what he thinks is an hour is actually 40 minutes.)

Example. Let W be the Wall-E function in the opening problem. What is T for:

(a) The scenario where Wall-E begins at 11:30am? 10:30am? 2pm?

(b) The scenario where Wall-E moves 2 times slower? 1.5 times faster?

(c) What if Wall-E began at 10:30am and moves 1.5 times faster?

(d) What would be a scenario that is represented by $W(3x+1)$? How about $W(3(x+1))$? (Here assume the unit of time are in hours).

Principles:

- If you transform the input to move faster per time, the graph of the function will appear to move faster per time. The graph will look as though you took the original graph and you squished the input coordinates of the coordinate plane, because you are moving faster through time.
- If you transform the input to have a head start, the graph will look as though you pushed the origin forward, because you are effectively starting before the zero mark.
- If you transform the input to move slower per time, ... [homework].
- If you transform the input to start later, ... [homework].

Illustrating principles through word problems

Common types of word problem contexts in high school and middle include:

- Races (on foot, by car, by bike, etc.). Input is typically time, output is typically distance.

- Candle burning. Input is typically time, output is typically height of the candle.
- Cell phone plan costs (solo plan, family plan, other incentives). Input is typically total duration of calls, output is typically cost.
- Mileage (of a motorcycle, car, truck, etc.). Input is typically distance, output is typically cost of gas or volume of gas used.
- Ferris wheel. Input is typically time, output is typically height from the ground.

Each of these word problems can be used to illustrate the principles of input transformations. The strategy is typically to compare and contrast a function (which represents an situation) and a transformation of the function (which represents a competitor or alternate possibility).

Characteristics of using a word problem to represent a transformation principle

- Use realistic input and output units.
- Use realistic input and output quantities.
- Clear problem statement.
- Ideal solution to the problem, in which you define the original function and its transformation with different variables.
- Commentary about the solution where you graph the function and its transformation and then discuss why the differences in the graph make sense in terms of the problem context and its solution. This part should lead sensibly to a statement of the principle(s) you are illustrating.
- State the principle(s) you are illustrating.

6 Output transformations and explaining definitions

Today our agenda is:

- Mini lesson
- Output transformations
- Explanations with definition of graph

Mini lesson

As you view the demo mini lesson, give feedback on:

Are units and quantities realistic?	Input units and quantities realistic/not realistic Output units and quantities realistic/not realistic
How clear is the problem statement?	Clear / sort-of clear / unclear
How accurate is the solution to the problem? How clear is the solution to the problem?	Accurate / sort-of accurate / not accurate Clear / sort-of clear / unclear
Is the explanation of the graph strictly procedural (e.g., "you move the graph by 1 unit to the right") or does it combine the procedure with explanations both in terms of the definition of graph (in terms of input and output) and the problem context? ("Combine" means "connection is not implied, it is explicit")	Strictly procedural / not strictly procedural Connects explicitly with definition of graph / does not do so Connects explicitly with problem context / does not do so
How accurate are the statement of the principles? How clear are they?	Accurate / sort-of accurate / not accurate Clear / sort-of clear / unclear

Output transformations

Definition 6.1. Given a function f, a function that can be expressed in the form $T \circ f$ is said to be an **output transformation** of f. (T takes as input the outputs of f.)

Example. Wall-E encounters another stack of metal toy blocks to put away. This time, the toy boxes are glued in pairs, so in each move, he is actually moving 2 blocks rather than one block. He otherwise makes the same moves as he did before.

(a) When does he put away 144 blocks?

(b) What does the graph of blocks put away versus time looks like?

(c) Express this function of blocks versus time in this scenario as a transformation of W.

Example. Wall-E has already put away 144 blocks, and he is on to his next pile of metal blocks. . He makes the same moves as he did before.

(a) What does the graph of total blocks put away versus time look like, including the 144 blocks in the count?

(b) Express this function of blocks versus time in this scenario as a transformation of W.

Principles:

- If you transform the output by a factor, the graph will be stretched by that factor.
- If you transform the output by a constant up or down, graph will be moved up and down by that factor.

Explaining transformation principles in terms of the definition of graph

Recall that graph of an equation $y = f(x)$ contains the point (a, b) if and only if when you evaluate the equation $y = f(x)$ at (a, b), you obtain a true statement. I.e., The point (a, b) is on the graph if and only if "$b = f(a)$" is a true statement.

Suppose the point (x, y) is on the graph $y = C(x)$, where C represents candies made per hour at a candy factory.

(a) Let $T(x) = 2x$. Does $C(T(x))$ represent more candies made or fewer candies made, per hour?

(b) If $(6, 600)$ is on the graph of $y = C(x)$, what is a coordinate on the graph of $y = C(T(x))$?

(c) Now let $T(x) = \frac{1}{3}x$. Write two scenarios, one described by $T(C(x))$, and the other described by $C(T(x))$.

(d) If $(6, 600)$ is on the graph of $y = C(x)$, what is a coordinate on the graph of $y = T(C(x))$? What is a coordinate on the graph of $y = C(T(x))$?

Use this task to explain input and output principles, incorporating the feedback that we discussed in the mini lesson. We will have some of you present so we can practice giving and receiving further feedback.

In-Class Resources

MANY WAYS TO COMPOSE TO THE SAME FUNCTION

Define:

$$A : \mathbb{R} \to \mathbb{R}, x \to 3x$$
$$B : \mathbb{R} \to \mathbb{R}, x \to x + 4$$
$$C : \mathbb{R} \to \mathbb{R}, x \mapsto 2x$$
$$D : \mathbb{R} \to \mathbb{R}, x \mapsto x + 5$$
$$E : \mathbb{R} \to \mathbb{R}, x \to x + 12.$$

1. Write $x \mapsto 3x + 12$ as a composition of A, B, E in two different ways. (You do not have to use all of them in either of the ways.)

2. (a) Find four different ways to start from x and obtain the expression $2\sin(\frac{1}{3}x + 12) - 10$ through compositions.
   (b) Write the map $f : \mathbb{R} \to \mathbb{R}, x \mapsto 2\sin(3x + 12) + 10$ in terms of A, B, C, D and sin.

Takeaways:

Solve the following problem:

> On January 1, 2101, Wall-E the robot wakes up at 9am. At noon, Wall-E begins shelving away a large pile of metal toy blocks, one by one. He decides to put away 1 block every 6 minutes in the the first hour as he is figuring out his system for these blocks. Then, in the second hour, he puts away 1 block every 4 minutes. From there on, he puts away 1 block every minute. When will he have put away 144 blocks?

Input Transformations

Now let's represent some different scenarios with graphs. In each of the graphs:

- Put a green dot at the point in the graph representing when Wall-E begins his work.
- Put a red dot at the point in the graph representing when Wall-E has put away 144 blocks.

Let W be the function of Wall-E's total blocks put away versus time.

1. On one pair of axes, graph each of the following functions of Wall-E's total blocks put away versus time.
   - W, the function of Wall-E's total blocks put away versus time.
   - The function if Wall-E had started at 11:30am instead of noon.
   - The function if Wall-E had started at 10:30am instead of noon.
   - The function if Wall-E had started at 2pm instead of noon.

2. On one pair of axes, graph each of following functions of Wall-E's total blocks put away versus time.
   - The function in the original situation.
   - The function if Wall-E's internal clock is running 2 times slower (so what he thinks is an hour is actually 2 hours).
   - The function is Wall-E's internal clock is running 1.5 times fast (so what he thinks is an hour is actually 40 minutes.)

3. Let W be the function of Wall-E's total blocks put away versus time, as defined previously.
   (a) Describe a modification to the scenario in which the function of total blocks put away versus time would be $W(3x+1)$.
   (b) Describe a modification to the scenario in which the function of total blocks put away versus time would be $W(3(x+1))$.
   (c) In each of the above scenarios you wrote, highlight in pink where the "3" comes in, and highlight in green where the "+1" comes in.
   (d) On the same pair of axes, graph $y = W(3x+1)$ and $y = W(3(x+1))$. Explain the differences you see in terms of the scenario.

Mini lesson feedback

Your name:

Presenters' names:

Feedback:

Are units and quantities realistic?	Input units and quantities realistic/not realistic Output units and quantities realistic/not realistic
How clear is the problem statement?	Clear / sort-of clear / unclear
How accurate is the solution to the problem? How clear is the solution to the problem?	Accurate / sort-of accurate / not accurate Clear / sort-of clear / unclear
Is the explanation of the graph strictly procedural (e.g., "you move the graph by 1 unit to the right") or does it combine the procedure with explanations both in terms of the definition of graph (in terms of input and output) and the problem context? ("Combine" means "connection is not implied, it is explicit")	Strictly procedural / not strictly procedural Connects explicitly with definition of graph / does not Connects explicitly with problem context / does not
How accurate are the statement of the principles? How clear are they?	Accurate / sort-of accurate / not accurate Clear / sort-of clear / unclear

Other comments:

Output transformations

Defintion. Given a function f, a function that can be expressed in the form $T \circ f$ is said to be an **output transformation** of f. (T takes as input the outputs of f.)

1. Wall-E encounters another stack of metal toy blocks to put away. This time, the toy boxes are glued in pairs, so in each move, he is actually moving 2 blocks rather than one block. He otherwise makes the same moves as he did before.

   (a) When does he put away 144 blocks?

   (b) What does the graph of blocks put away versus time looks like?

   (c) Express this function of blocks versus time in this scenario as a transformation of W.

2. Wall-E has already put away 144 blocks, and he is one to his next pile of metal blocks. He makes the same moves as he did before.

   (a) What does the graph of total blocks put away versus time look like, including the 144 blocks in the count?

   (b) Express this function of blocks versus time in this scenario as a transformation of W.

Explaining transformation principles in terms of the definition of graph

Recall that graph of an equation $y = f(x)$ contains the point (x_0, y_0) if and only if when you evaluate the equation $y = f(x)$ at (x_0, y_0), you obtain a true statement. I.e., the point (x_0, y_0) is on the graph if and only if "$y_0 = f(x_0)$" is a true statement.

1. Suppose the point (x, y) is on the graph $y = C(x)$, where C represents candies made per hour at a candy factory.
   (a) Let $T(x) = 2x$. Does $C(T(x))$ represent more candies made or fewer candies made, per hour?
   (b) If $(6, 600)$ is on the graph of $y = C(x)$, what is a coordinate on the graph of $y = C(T(x))$?
   (c) Now let $T(x) = \frac{1}{3}x$. Write two scenarios:
   one described by $T(C(x))$
   and the other described by $C(T(x))$.
   (d) If $(6, 600)$ is on the graph of $y = C(x)$, what is a coordinate on the graph of $y = T(C(x))$? What is a coordinate on the graph of $y = C(T(x))$?

2. Given a function f.
   (a) State *how* the graphs of $y = f(ax)$, and $y = f(-ax)$ can be obtained as transformations of the graph of $y = f(x)$
   (b) Prove *why* your answer to (a) works. Your proof should use the definition of graph.

Explaining with definition of graph: Feedback

Your name:

Presenters' names:

Feedback:

Are units and quantities realistic?	Input units and quantities realistic/not realistic Output units and quantities realistic/not realistic
How clear is the problem statement?	Clear / sort-of clear / unclear
How accurate is the solution to the problem? How clear is the solution to the problem?	Accurate / sort-of accurate / not accurate Clear / sort-of clear / unclear
Is the explanation of the graph strictly procedural (e.g., "you move the graph by 1 unit to the right") or does it combine the procedure with explanations both in terms of the definition of graph (in terms of input and output) and the problem context? ("Combine" means "connection is not implied, it is explicit")	Strictly procedural / not strictly procedural Connects explicitly with definition of graph / does not Connects explicitly with problem context / does not
How accurate are the statement of the principles? How clear are they?	Accurate / sort-of accurate / not accurate Clear / sort-of clear / unclear

Other comments:

Homework

1. Let W be the function of Wall-E's total blocks put away versus time, as defined in the Lesson 5 Explorations. *Hint: It may be helpful for this homework problem to put the origin at 9am instead of at noon.*

   (a) Describe a modification to the scenario in which the function of total blocks put away versus time would be $W(3x+1)$.

   (b) Describe a modification to the scenario in which the function of total blocks put away versus time would be $W(3(x+1))$.

   (c) In each of the above scenarios you wrote, highlight in pink where the "3" comes in, and highlight in green where the "+1" comes in.

   (d) On the same pair of axes, graph $y = W(3x+1)$ and $y = W(3(x+1))$. Explain the differences you see in terms of the scenario.

2. Complete the following:

   Principles of graph transformation

   - If you transform the input to move faster per time, the graph of the function will appear to move faster per time. The graph will look as though you took the original graph and you squished the input coordinates of the coordinate plane, because you are moving faster through time.
   - If you transform the input to have a head start, the graph will look as though you pushed the origin forward, because you are effectively starting before the zero mark.
   - If you transform the input to move slower per time:
   - If you transform the input to start later:

3. Read over Section 5.4 on illustrating principles of transformations.

   - Prepare a 7-minute lesson in which you explain the solution to a problem that combines a faster/slower per time transformation with a head start/delayed start transformation.
   - Assume that you had assigned the problem for homework, and most students have done the problem correctly. You are now reviewing the problem so as to reinforce the principles.
   - The problem should use one of the common contexts listed in Section 5.4.
   - Your explanation should adhere to the characteristics of a good explanation listed in that section.
   - Your lesson should include some comprehension questions about the graphs that tie directly into the statement of the principles.

Here is the feedback chart that will be used for the lesson:

Are units and quantities realistic?	Input units and quantities realistic/not realistic Output units and quantities realistic/not realistic
How clear is the problem statement?	Clear / sort-of clear / unclear
How accurate is the solution to the problem? How clear is the solution to the problem?	Accurate / sort-of accurate / not accurate Clear / sort-of clear / unclear
Is the explanation of the graph strictly procedural (e.g., "you move the graph by 1 unit to the right") or does it combine the procedure with explanations both in terms of the definition of graph (in terms of input and output) and the problem context? ("Combine" means "connection is not implied, it is explicit")	Strictly procedural / not strictly procedural Connects explicitly with definition of graph / does not do so Connects explicitly with problem context / does not do so
How accurate are the statement of the principles? How clear are they?	Accurate / sort-of accurate / not accurate Clear / sort-of clear / unclear