The price impact of order book events: market orders, limit orders and cancellations

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While the long-ranged correlation of market orders and their impact on prices has been relatively well studied in the literature, the corresponding studies of limit orders and cancellations are scarce. We provide here an empirical study of the cross-correlation between all these different events, and their respective impact on future price changes. We define and extract from the data the "bare" impact these events would have, if they were to happen in isolation. For large tick stocks, we show that a model where the bare impact of all events is permanent and non-fluctuating is in good agreement with the data. For small tick stocks, however, bare impacts must contain a history dependent part, reflecting the internal fluctuations of the order book. We show that this effect can be accurately described by an autoregressive model on the past order flow. This framework allows us to decompose the impact of an event into three parts: an instantaneous jump component, the modification of the future rates of the different events, and the modification of the future gaps behind the best quotes. We compare in detail the present formalism with the temporary impact model that was proposed earlier to describe the impact of market orders when other types of events are not observed. Finally, we extend the model to describe the dynamics of the bid-ask spread.

Keywords: price impact, market orders, limit orders, cancellations, market microstructure, order flow

Contents

1. INTRODUCTION

The relation between order flow and price changes has attracted considerable attention in the recent years [1–6]. To the investors' dismay, trades on average impact the price in the direction of their transactions, i.e. buys push the price up and sells drive the price down. Although this sounds very intuitive, a little reflection shows that such a statement is far from trivial, for any buy trade in fact meets a sell trade, and vice-versa! On the other hand, there must indeed be a mechanism allowing information to be included into and reflected by prices. This is well illustrated by the Kyle model [7], where the trading of an insider progressively reveals his information by impactng the price. Traditionally, the above "one sell for one buy" paradox is resolved by arguing that there are two types of traders coexisting in the ecology of financial markets: (i) "informed" traders who place market orders for immediate execution, at the cost of paying half the bid-ask spread, and (ii) uninformed (or less informed) market makers who provide liquidity by placing limit orders on both sides of the order book, hoping to earn part of the bid-ask spread. In this setting, there is indeed an asymmetry between a buyer, placing a market order at the ask, and the corresponding seller with a limit order at the ask, and one can speak about a well defined impact of buy/sell (market) orders. The impact of market orders has therefore been empirically studied in great detail since the early nineties. As reviewed below, many surprising results have been obtained, such as a very weak dependence of impact on the volume of the market order, the long-range nature of the sign of the trades, and the resulting non-permanent, power-law decay of impact.

The conceptual problem is that the distinction between informed trader and market maker is no longer obvious in the present electronic markets, where each participant can place both limit and market orders, depending on his own strategies, the current state of the order book, etc. Although there is still an asymmetry between a buy market order and a sell limit order that enables one to define the direction of the trade, "informed" traders too may choose to place limit orders, aiming to decrease execution costs. Limit orders must therefore also have an impact: adding a buy limit order induces extra upwards pressure, and cancelling a buy limit order decreases this pressure. Surprisingly, there are very few quantitative studies of the impact of these orders – partly due to the fact that more detailed data, beyond trades and quotes, is often needed to conduct such studies.

The aim of the present paper is to provide a unified framework for the description of the impact of *all* order book events, at least at the best limits: market orders, limit orders and cancellations. We study the correlation between all events types and signs. Assuming a linear model of impact, we map out from empirical data (consisting purely of trades and quotes information) the average individual impact of these orders. We find that the impact of limit orders is similar (albeit somewhat smaller) to that of market orders.

We then compare these results to a simple model which assumes that all impact functions are permanent in time. This works well for large tick stocks, for which the bid-ask spread is nearly constant, with no gaps in the order book. The discrepancies between this simple model and small tick data are then scrutinized in detail and attributed to the *history dependence* of the impact function, which we are able to model successfully using a simple linear dependence of the gaps on the past order flow. Our final model is specified in Sec. 7, Eq. (31). This allows us to measure more accurately the average impact of all types of orders, and to assess precisely the importance of impact fluctuations due to changes in the gaps behind the best quotes.

The outline of this paper is as follows. We first review (Sec. 2) the relevant results on the impact of market orders and set the mathematical framework within which we will analyze our order book event data. We explain in particular why the market order impact function measured in previous studies is in fact "dressed" by the impact of other events (limit orders, cancellations), and by the history dependence of the impact. We then turn to the presentation of the data we have analyzed (Sec. 3), and of the various correlation functions that one can measure (Sec. 4). From these we determine the individual (or "bare"), lag-dependent impact functions of the different events occuring at the bid price or at the ask price (Sec. 5). We introduce a simplified model where these impact functions are constant in time, and show that this gives an good approximate account of our data for large tick stocks, while significant discrepancies appear for small tick stocks (Sec. 6). The systematic differences are explained by the dynamics of order flow deeper in the book, which can be modeled as a history dependent correction to the linear impact model (Sec. 7, see Eq. (31)). Our results are summarized in the conclusion, with open issues that would deserve more detailed investigation. In the Appendices we also show how the bid-ask spread dynamics can be accounted for within the framework introduced in the main text (Appendix A) and some supplementary information concerning the different empirical correlations that can be measured (Appendix B).

2. IMPACT OF MARKET ORDERS: A SHORT REVIEW

2.1. The transient impact model

Quantitative studies of the price impact of market orders have by now firmly established a number of stylized facts, some of which appear rather surprising at first sight. The salient points are (for a review, see [6]):

- Buy (sell) trades on average impact the price up (down). In other words, there is a strong correlation between price returns over a given time interval and the market order imbalance on the same interval.
- The impact curve as a function of the volume of the trade is strongly concave. In other words, large volumes impact the price only marginally more than small volumes.
- The sign of market orders is strongly autocorrelated in time. Despite this, the dynamics of the midpoint is very close to purely diffusive.

A simple model encapsulating these empirical facts assumes that the mid-point price p_t can be written at (trade) time t as a linear superposition of the impact of past trades [3, 4]:[1]

$$p_t = \sum_{t'<t} \left[G(t-t')\epsilon_{t'} v_{t'}^{\theta} + n_{t'} \right] + p_{-\infty}, \tag{1}$$

where $v_{t'}$ is the volume of the trade at time t', $\epsilon_{t'}$ the sign of that trade ($+$ for a buy, $-$ for a sell), and n_t is an independent noise term that models any price change not induced by trades (e.g. jumps due to news). The exponent θ is small; the dependence in v might in fact be logarithmic. The most important object in the above equation is the function $G(t-t')$ which describes the temporal evolution of the impact of a single trade, which can be called a 'propagator': how does the impact of the trade at time t' propagate, on average, up to time t?

An important result, derived in [3], is that $G(t-t')$ must decay with time in a very specific way, such as to off-set the autocorrelation of the trades, and maintain the (statistical) efficiency of markets. Clearly, if $G(t-t')$ did not decay at all, the returns would simply be proportional to the sign of the trades, and therefore would themselves be strongly autocorrelated in time. The resulting price dynamics would be highly predictable, which is not the case. Conversely, if $G(t-t')$ decayed to zero immediately, the price, as given by Eq. (1) would oscillate within a limited range, and the long-term volatility would be zero. The result of [3] is that if the correlation of signs $C(\ell) = \langle \epsilon_t \epsilon_{t+\ell} \rangle$ decays at large ℓ as $\ell^{-\gamma}$ with $\gamma < 1$ (as found empirically), then $G(t-t')$ must decay as $|t-t'|^{-\beta}$ with $\beta = (1-\gamma)/2$ for the price to be exactly diffusive at long times. The impact of single trades is therefore predicted to decay as a power-law (at least up to a certain time scale), at variance with simple models which assume that the impact decays exponentially to a non-zero "permanent" value. More generally, one can use the empirically observable impact function $\mathcal{R}(\ell)$, defined as:

$$\mathcal{R}(\ell) = \langle (p_{t+\ell} - p_t) \cdot \xi_t \rangle \tag{2}$$

and the correlation function $C(\ell)$ of $\xi_t = \epsilon_t v_t^{\theta}$ to map out, numerically, to complete shape of $G(t-t')$. This was done in [4], using:

$$\mathcal{R}(\ell) = \sum_{0<n\leq\ell} G(n)C(\ell-n) + \sum_{n>\ell} G(n)C(n-\ell) - \sum_{n>0} G(n)C(n). \tag{3}$$

This analysis is repeated in a more general setting below (see Sec. 5 and Eq. (13)). The above model, however, is approximate and incomplete in two, interrelated ways.

- First, Eq. (1) neglects the fluctuations of the impact: one expects that $\mathcal{G}(t' \to t)$, which is the impact of trade at some time t' measured until a later time t, to depend both on t and t' and not only on $t-t'$. Its formal definition is given by:

$$\mathcal{G}(t' \to t) = \frac{\partial p_t}{\partial \xi_{t'}}, \qquad \xi_t \equiv \epsilon_t v_t^{\theta}. \tag{4}$$

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[1] In the following, we only focus on price changes over small periods of time, so that the following additive model is adequate. For longer time scales, one should worry about multiplicative effects, which in this formalism would naturally arise from the fact that the bid-ask spread, and the gaps in the order book, are a fraction of the price. Therefore, the impact itself, $\mathcal{G}$, is expected to be proportional to the price. See [8] for a discussion of this point.

Impact can indeed be quite different depending on the state of the order book and the market conditions at t'. As a consequence, if one blindly uses Eq. (1) to compute the second moment of the price difference, $D(\ell) = \langle (p_{t+\ell} - p_t)^2 \rangle$, with G calibrated to reproduce the impact function $\mathcal{R}(\ell)$, the result clearly underestimates the empirical price variance: see Fig. 1.

- Second, other events of the order book can also change the mid-price, such as limit orders placed inside the bid-ask spread, or cancellations of all the volume at the bid or the ask. These events do indeed contribute to the price volatility and should be explicitly included in the description. A simplified description of price changes only in terms of market orders attempts to describe other events of the order book in an effective way, through the non-trivial time dependence of $G(\ell)$.

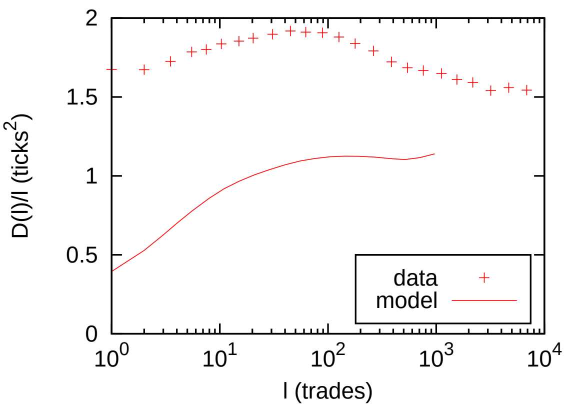

Figure 1: $D(\ell)/\ell$ and its approximation with the temporary impact model with only trades as events, with $n_t = 0$ and for small tick stocks. Results are shown when assuming that all trades have the same, non fluctuating, impact $G(\ell)$, calibrated to reproduce $\mathcal{R}(\ell)$. This simple model accounts for $\sim 2/3$ of the long term volatility. Other events and/or the fluctuations of impact must therefore contribute to the market volatility as well.

2.2. History dependence of the impact function

Let us make the above statements more transparent on toy-models. First, the assumption of a stationary impact function $\mathcal{G}(t' \to t') = G(t - t')$ is clearly an approximation. The past order flow ($< t'$) should affect the way the trade at time t' impacts the price, or, as argued by Lillo and Farmer, that liquidity may be history dependent [6, 9, 10]. Suppose for simplicity that the variable $\xi_t = \epsilon_t v_t^\theta$ is Gaussian (which turns out to be a good approximation) and that its impact is permanent but history dependent. If we assume that the past order flow has a small influence on the impact, we can expand $\mathcal{G}$ in powers of all past ξ's to get:

$$\mathcal{G}(t' \to t) = G_0 + G_1 \sum_{t_1 < t'} g_1(t' - t_1)\xi_{t_1} + G_2 \sum_{t_1, t_2 < t'} g_2(t' - t_1; t' - t_2)\xi_{t_1}\xi_{t_2} + \cdots . \tag{5}$$

Using the fact the ξ's are Gaussian with zero mean, one finds that the impact function $\mathcal{R}(\ell)$ within this toy-model is given by:

$$\mathcal{R}(\ell) = \sum_{0 < n \le \ell} C(n) \left[G_0 + G_2 \sum_{n_1, n_2 > 0} g_2(n_1; n_2) C(n_1 - n_2) \right] + 2G_2 \sum_{0 < n \le \ell} \sum_{n_1, n_2 > 0} g_2(n_1; n_2) C(n_1) C(n - n_2). \tag{6}$$

If one compares this expression with Eq. (3) to extract an effective propagator $G(\ell)$, it is clear that the resulting solution will have some non-trivial time dependence induced by the third term, proportional to G_2. Note that within this toy-model, G_1 does not contribute to the $\mathcal{R}(\ell)$ but contributes to the volatility $D(\ell)$.

2.3. The role of hidden events

Imagine now that two types of events are important for the dynamics of the price. Events of the first type are characterized by a random variable ξ_t (e.g., $\xi_t = \epsilon_t v_t^\theta$ in the above example), whereas events of the second type are charaterized by another random variable η_t. The "full" dynamical equation for the price is given by:

$$p_t = \sum_{t'<t} G_1(t-t')\xi_{t'} + \sum_{t'<t} G_2(t-t')\eta_{t'} + p_{-\infty}, \tag{7}$$

Imagine, however, that events of the second type are *not* observed. If for simplicity ξ and η's are correlated Gaussian random variables, one can always express the η's as linear superposition of past ξ's and find a model in terms of ξ's only, plus an uncorrelated 'noise' component n_t coming from the unobserved events:

$$p_t = \sum_{t'<t} \left[G_1(t-t')\xi_{t'} + G_2(t-t') \sum_{t''\le t'} \Xi(t'-t'')\xi_{t''} \right] + n_t + p_{-\infty}. \tag{8}$$

Ξ is the linear filter allowing to predict the η's in terms of the past ξ's. It can be expressed in a standard way in terms of the correlation function of the η's and the cross-correlation between ξ's and η's. Notice that the previous equation can be recast in the form of Eq. (1) plus noise, with an effective propagator "dressed" by the influence of the unobserved events:

$$G(\ell) = G_1(\ell) + \sum_{0<\ell'\le\ell} G_2(\ell')\Xi(\ell-\ell'). \tag{9}$$

From this equation, it is clear that a non-trivial dependence of G can arise even if the 'true' propagators G_1 and G_2 are time independent – in other words the decay of the impact of a single market order is in fact a consequence of the interplay of market and limit order flow. As a trivial example, suppose both bare propagators are equal and constant in time ($G_1(\ell) = G_2(\ell) = G$) and $\eta_t \equiv -\xi_t$, $\forall t$. This means that the two types of events impact the price but exactly cancel each other. Then, $\Xi(\ell) = \delta_{\ell,0}$ and $G(\ell) \equiv 0$, as it should: the dressed impact of events of the first type is zero. This is an idealized version of the asymmetric liquidity model of Lillo and Farmer mentioned above [9].

The aim of this paper is to investigate a model of impact similar to Eqs. (1) and (7), but where a wider class of order book events are explicitly taken into account. This will allow us to extract the corresponding single event impact functions, and study their time evolution. As a test for the completeness and accuracy of the model, the time behavior of other observables, such as the second moment of the price difference should be correctly accounted for. We start by presenting the data and extra notations which will be useful in the sequel. We then discuss the different correlation and response functions that can be measured on the data.

3. DATA AND NOTATIONS

In this paper we analyze data on 14 of the most liquid stocks traded at NASDAQ during the period 03/03/2008 – 19/05/2008, a total of 53 trading days. We only consider the usual trading time between 9:30–16:00, all other periods are discarded. We will always use ticks (0.01 US dollars) as the units of price. We will use the name "event" for any change that modifies the bid or ask price, or the volume quoted at these prices. Events deeper in the order book are unobserved and will not be described: although they do not have an immediate effect on the best quotes, our description will still be incomplete; in line with the previous section, we know that these unobserved events may "dress" the impact of the observed events.

Events will be used as the unit of time. This "event time" is similar, but more detailed than the notion of transaction time used in many recent papers. Since the dependence of impact on the volume of the trades is weak [6, 11], we have chosen to classify events not according to their volume but according to whether they change the mid-point or not. This strong dichotomy is another approximation to keep in mind. It leads to six possible types of events[2]:

- market orders that do not change the best price (noted MO^0) or that do (noted MO'),
- limit orders at the current bid or ask (LO^0) or inside the bid-ask spread so that they change the price (LO'),
- and cancellations at the bid or ask that do not remove all the volume quoted there (CA^0) or that do (CA').

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[2] Our data also included a small number ($\approx 0.3\%$) of marketable (or crossing) limit orders. In principle these could have been treated as a market order (and a consequent limit order for the remaining volume if there was any). Due to technical limitations we decided to instead remove these events and the related price changes.

π	event definition	event sign definition	gap definition ($\Delta_{\pi,\epsilon}$)
$\pi = \mathrm{MO}^0$	market order, volume < outstanding volume at the best	$\epsilon = \pm 1$ for buy/sell market orders	0
$\pi = \mathrm{MO}'$	market order, volume $\geq$ outstanding volume at the best	$\epsilon = \pm 1$ for buy/sell market orders	half of first gap behind the ask ($\epsilon = 1$) or bid ($\epsilon = -1$)
$\pi = \mathrm{CA}^0$	partial cancellation of the bid/ask queue	$\epsilon = \mp 1$ for buy/sell side cancellation	0
$\pi = \mathrm{LO}^0$	limit order at the current best bid/ask	$\epsilon = \pm 1$ for buy/sell limit orders	0
$\pi = \mathrm{CA}'$	complete cancellation of the best bid/ask	$\epsilon = \mp 1$ for buy/sell side cancellation	half of first gap behind the ask ($\epsilon = 1$) or bid ($\epsilon = -1$)
$\pi = \mathrm{LO}'$	limit order inside the spread	$\epsilon = \pm 1$ for buy/sell limit order	half distance of limit order from the earlier best quote on the same side

Table I: Summary of the 6 possible event types, the corresponding definitions of the event signs and gaps.

	ticker	$P(\mathrm{MO}^0)$	$P(\mathrm{MO}')$	$P(\mathrm{CA}^0)$	$P(\mathrm{LO}^0)$	$P(\mathrm{CA}')$	$P(\mathrm{LO}')$	mean spread (ticks)	mean price (USD)	time/event (sec)
large tick	AMAT	0.042	0.011	0.39	0.54	0.0018	0.013	1.11	17.45	0.16
	CMCSA	0.040	0.0065	0.41	0.53	0.0021	0.0087	1.12	20.29	0.15
	CSCO	0.051	0.0085	0.40	0.53	0.0010	0.0096	1.08	67.77	0.10
	DELL	0.042	0.0087	0.40	0.54	0.0019	0.011	1.10	20.22	0.17
	INTC	0.052	0.0073	0.40	0.54	0.00080	0.0081	1.08	19.43	0.12
	MSFT	0.054	0.0087	0.40	0.53	0.0012	0.010	1.09	27.52	0.098
	ORCL	0.050	0.0090	0.40	0.54	0.0012	0.010	1.09	20.86	0.16
small tick	AAPL	0.043	0.076	0.32	0.33	0.077	0.16	3.35	140.56	0.068
	AMZN	0.038	0.077	0.26	0.31	0.12	0.20	3.70	70.68	0.21
	APOL	0.042	0.080	0.24	0.33	0.11	0.20	3.78	55.24	0.40
	COST	0.054	0.069	0.27	0.36	0.082	0.16	2.62	67.77	0.39
	ESRX	0.042	0.074	0.24	0.32	0.12	0.20	4.12	60.00	0.63
	GILD	0.052	0.043	0.34	0.46	0.032	0.077	1.64	48.23	0.23

Table II: Summary statistics for all stocks, showing the probability of the different events, the mean spread in ticks, the mean price in dollars and the average time between events in seconds. The last column shows the total number of events in the sample.

The upper index ' ("prime") will thus denote that the event changed any of the best prices, and the upper index 0 that it did not. Abbreviations without the upper index (MO, CA, LO) refer to both the price changing and the non-price changing event type. The type of the event occuring at time t will be denoted by π_t.

Every event is given a sign ϵ_t according to its expected long-term effect on the price. For market orders this corresponds to usual order signs, i.e., $\epsilon_t = 1$ for buy market orders (at the ask price) and -1 for sell market orders (at the bid price). Cancelled sell limit orders and incoming buy limit orders both have $\epsilon_t = 1$, while others have $\epsilon_t = -1$. The above definitions are summarized in Table I. Note that the table also defines the gaps $\Delta_{\pi,\epsilon}$, which will be used later.

Our sample of stocks can be divided into two groups: large tick and small tick stocks. Large tick stocks are such that the bid-ask spread is almost always equal to one tick, whereas small tick stocks have spreads that are typically a few ticks.The behavior of the two groups is quite different, and this will be emphasized throughout the paper. For example, the events which change the best price have a relatively low probability for large tick stocks (about 3% altogether), but not for small tick stocks (up to 40%). Table II shows a summary of stocks, and some basic statistics. Note that there is a number of stocks with intermediate tick sizes, which to some extent possess the characteristics of both groups. Technically, they can be treated in exactly the same way as small tick stocks, and all our results remain valid. However, for the clarity of presentation, we will not consider them explicitly in this paper.

In the following calculations we will often rely on indicator variables denoted as $I(\pi_t = \pi)$. This expression is 1 if the event at t is of type π and zero otherwise. In other words, $I(\pi_t = \pi) = \delta_{\pi_t,\pi}$, where δ is the Kronecker-delta.

It will also be useful to define another sign variable corresponding to the "side" of the event at time t, which will be denoted by s_t. It indicates whether the event t took place at the bid ($s_t = -1$) or the ask ($s_t = 1$), thus:

$$s_t = \begin{cases} \epsilon_t & \text{if } \pi_t = \mathrm{MO}^0,\ \mathrm{MO}',\ \mathrm{CA}^0 \text{ or } \mathrm{CA}' \\ -\epsilon_t & \text{if } \pi_t = \mathrm{LO}^0 \text{ or } \mathrm{LO}' \end{cases} \tag{10}$$

The difference between ϵ and s is because limit orders correspond to the addition not the removal of volume, and thus they push prices away from the side of the book where they occur.

4. CORRELATION AND RESPONSE FUNCTIONS

In this section, we study the empirical temporal correlation of the different events defined above, and the response function to these events.

4.1. The autocorrelation of ϵ and s

We first investigate the autocorrelation function of the event signs, calculated as $\langle \epsilon_{t+\ell} \cdot \epsilon_t \rangle$. These are found to be short-ranged, see Fig. 2, where the correlation function dies out after 10-100 trades, corresponding to typically 10 seconds in real time. This is in contrast with several other papers [3, 4, 6, 12], where ϵ's are calculated for market orders only ($\pi = \mathrm{MO}^0, \mathrm{MO}'$), and those signs are known to be strongly persistent *among themselves*, with, as recalled in Sec. 2, a correlation decaying as a slow power law. However, the direction of incoming limit orders is negatively correlated with cancellations and market orders. Because the ϵ time series contains all types of events, the mixture of long-range positive and negative correlations balances such that only short-range persistence remains. Any other result would be incompatible with little predictability in price returns. As illustrated by the toy example of Sec. 2, Eq. (9), this mixing process in fact maintains statistical market efficiency, i.e. weak autocorrelation of price changes.

When limit orders and cancellations are included, one can independently analyze the persistence of the side s_t of the events. According to Eq. (10) this means flipping the event signs of limit orders in the ϵ time series, while keeping the rest unchanged. This change reverses the compensation mechanism discussed above, and s is found to have long-range correlations in time: $\langle s_{t+\ell} \cdot s_t \rangle$ is shown in Fig. 2 and decays as $\ell^{-\gamma}$ with $\gamma \approx 0.7$. This long range decay is akin to the long range persistence of market order signs discussed throughout the literature: since market orders tend to persistently hit one side of the book, one expects more limit orders and cancellations on the same side as well. Intuitively, if a large player splits his order and buys or sells using market orders for a long period of time, this will attract compensating limit orders on the same side of the book.

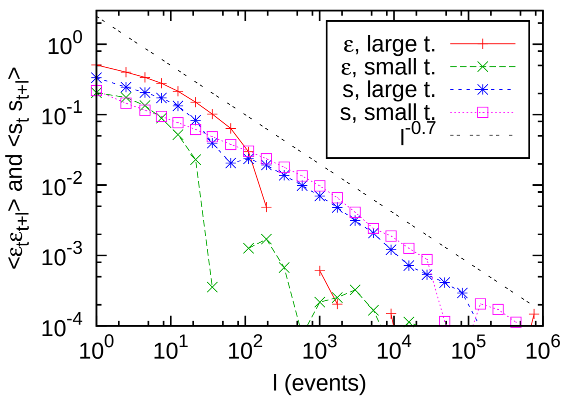

Figure 2: $\langle \epsilon_{t+\ell} \cdot \epsilon_t \rangle$ and $\langle s_{t+\ell} \cdot s_t \rangle$, averaged for large and small tick stocks.

4.2. The signed event-event correlation functions

We will see in the following, that for describing price impact the most important correlation functions are those defined between two (not necessarily different) signed event types. For some fixed π_1 and π_2 one can define a normalized

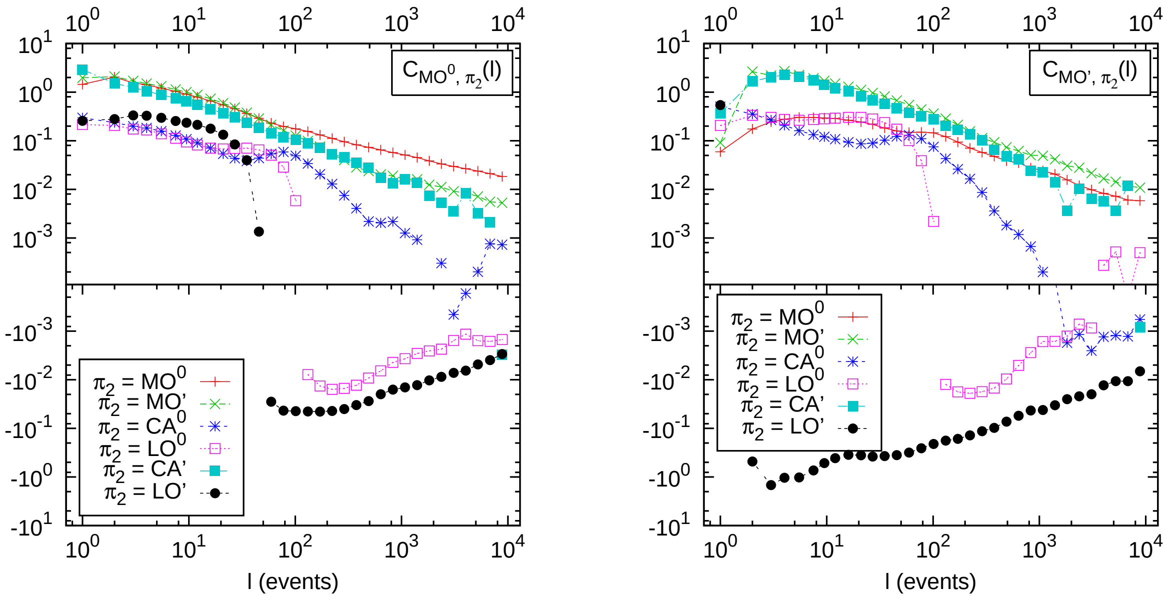

Figure 3: The normalized, signed event correlation functions $C_{\pi_1,\pi_2}(\ell)$, *(left)* $\pi_1 = \mathrm{MO}^0$, *(right)* $\pi_1 = \mathrm{MO}'$. The curves are labeled by their respective π_2's in the legend. The bottom panels show the negative values.

correlation as:

$$C_{\pi_1,\pi_2}(\ell) = \frac{\langle I(\pi_t = \pi_1)\epsilon_t I(\pi_{t+\ell} = \pi_2)\epsilon_{t+\ell}\rangle}{P(\pi_1)P(\pi_2)}. \tag{11}$$

Our convention is that the first index corresponds to the first event in chronological order. Because we have 6 event types, altogether there are $6^2 = 36$ of these event-event correlation functions. There are no clearly apparent, systematic differences between large and small tick stocks, hence we give results averaged over both groups in Fig. 3 for $\pi_1 = \mathrm{MO}^0$ and $\pi_1 = \mathrm{MO}'$. (Other correlation functions are plotted in Appendix B.) Trades among themselves and regardless of group are long range correlated as it is well known and was recalled above, and confirmed again in Fig. 3. For other cases, the sign of the correlations between event types varies and in many cases one observes a similarly slow decay that can be fitted by a power law with an exponent around 0.5. Furthermore, there are two distinctly different regimes. For $\ell \lesssim 100$ events which means up to $10-20$ seconds in real time returns are still autocorrelated (cf. Fig. 2). In this regime $C_{\mathrm{MO}^0,\pi_2}(\ell)$ is positive for any event type π_2, so small trades are followed by a ballistic move in the same direction by other trades and also by limit orders, while at the same time cancellations also push the price in the same direction. $C_{\mathrm{MO}',\pi_2}(\ell)$ is also positive except for LO$'$, where it is negative except for very small lags[3]. This means that if a market order removes a level, it is followed by further trades and cancellations in the same direction, but the level is refilled very quickly by incoming limit orders inside the spread. For longer times some correlation functions change sign. For example in Fig. 3(left) one can see this reversal for limit orders. Market orders "attract" limit orders, as noted in [4, 10, 13]. This "stimulated refill" process ensures a form of dynamic equilibrium: the correlated flow of market orders is offset by an excess inflow of opposing limit orders, such as to maintain the diffusive nature of the price. This is the same process causing the long-range correlations of s_t noted above.

In general, there are no reasons to expect time reversal symmetry, which would impose $C_{\pi_1,\pi_2}(\ell) = C_{\pi_2,\pi_1}(\ell)$. However, some pairs of events appear to obey this symmetry at least approximately, for example MO^0 and CA^0 or MO$'$ and CA$'$, see Fig. 4. On the other hand, for the pair MO$'$, LO$'$ one can see that limit orders that move the price are immediately followed by opposing market orders. The dual compensation, i.e. a stimulated refill of liquidity after a price moving market order MO$'$, only happens with some delay. MO^0 and limit orders also lead to some asymmetry, see Fig. 5; here we see that after a transient, non-aggressive market orders induce compensating limit orders more

[3] There is some sign of oscillations for small tick stocks.

efficiently than the reverse process.

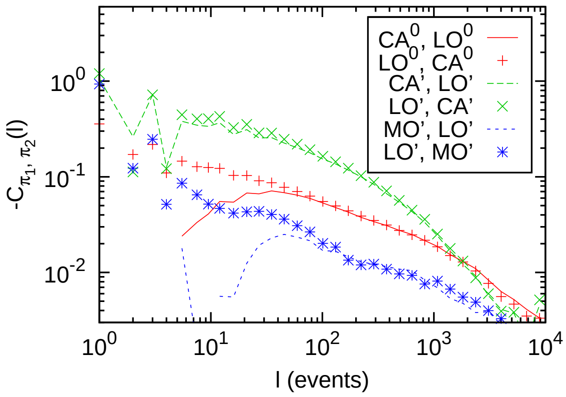

Figure 4: Examples for time reversal symmetry for normalized, signed event correlations for small tick stocks, note that it is $-C_{\pi_1,\pi_2}(\ell)$ plotted. Lines and points of the same color correspond to the same event pairs. The curves are labeled by their respective π_1's and π_2's in the legend.

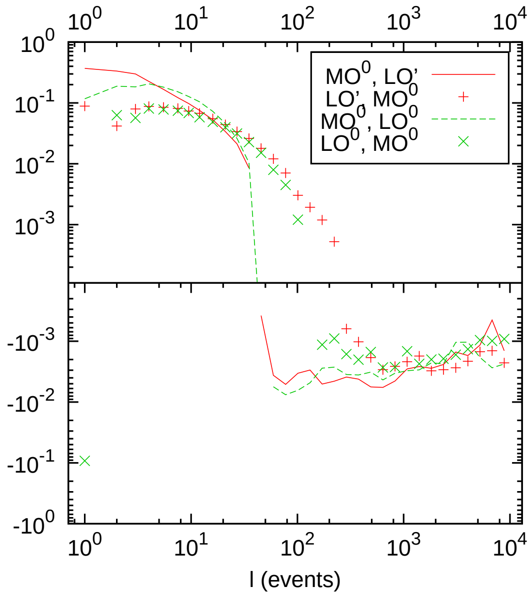

Figure 5: Examples for time reversal asymmetry for normalized, signed event correlations for small tick stocks. Lines and points of the same color correspond to the same event pairs. The curves are labeled by their respective π_1's and π_2's in the legend.

4.3. The indicator correlation functions

A similar definition of a correlation function is possible purely between indicators, without the signs:

$$\Pi_{\pi_1,\pi_2}(\ell) = \frac{\langle I(\pi_t = \pi_1) I(\pi_{t+\ell} = \pi_2)\rangle}{P(\pi_1)P(\pi_2)} - 1,$$

where we have removed 1 such as to make the function decay to zero at large times. Examples for averages over all stocks are plotted in Fig. 6. One finds that generally $\Pi_{\pi_1,\pi_2}(\ell)$ decays slower when both π_1 and π_2 move the price.

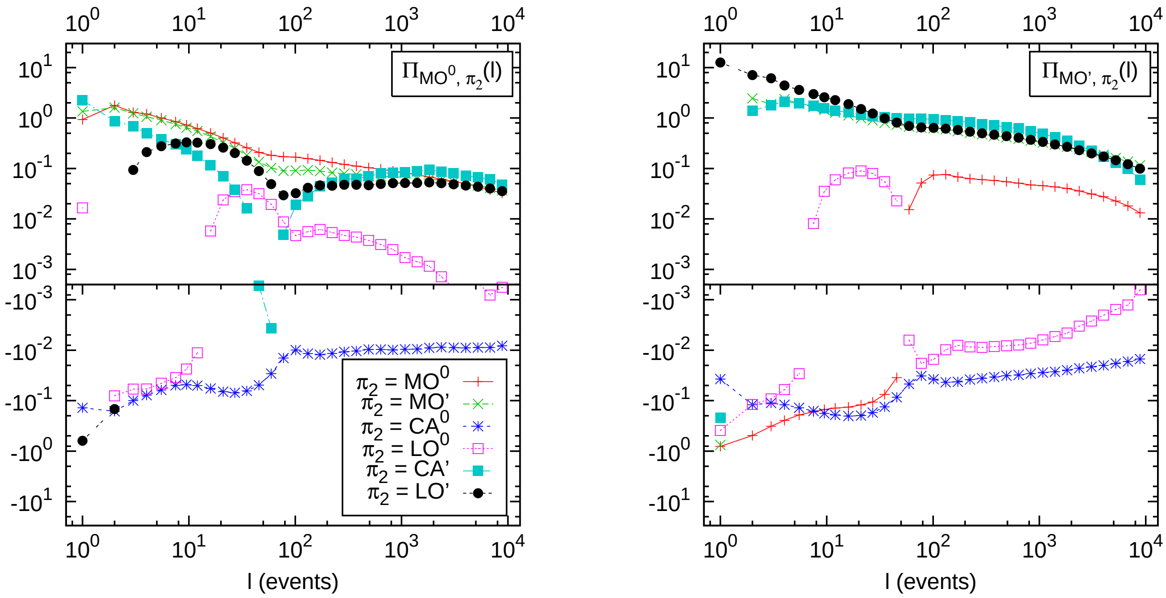

Figure 6: The normalized, unsigned event correlation functions $\Pi_{\pi_1,\pi_2}(\ell)$, *(left)* $\pi_1 = \mathrm{MO}^0$, *(right)* $\pi_1 = \mathrm{MO}'$. The curves are labeled by their respective π_2's in the legend. The bottom panels show the negative values.

This implies that events which change the best price are clustered in time: aggressive orders induce and reinforce each other.

4.4. The response function

Let us now turn to the response of the price to different types of orders. The average behavior of price after events of a particular type π defines the corresponding *response function* (or average impact function):

$$R_\pi(\ell) = \frac{\langle (p_{t+\ell} - p_t) \cdot \epsilon_t I(\pi_t = \pi) \rangle}{P(\pi)}. \tag{12}$$

This is a correlation function between the signed indicator $\epsilon_t I(\pi_t = \pi)$ at time t and the price change from t to $t + \ell$, normalized by the stationary probability of the event π, denoted as $P(\pi) = \langle I(\pi_t = \pi) \rangle$. This normalized response function gives the expected directional price change after an event π. Its behavior for all π's is shown in Fig. 7. We note that all type of events lead, on average, to a price change in the expected direction. Tautologically, $R_\pi(\ell = 1) > 0$ for price changing events and $R_\pi(\ell = 1) = 0$ for other events, but as the time lag ℓ increases, the impact of market orders grows significantly, specially for small tick stocks, whereas it remains roughly constant for limit orders/cancellations that do change the price. However, as emphasized in [3], the response function is hard to interpret directly since the correlations between events contribute to $R_\pi(\ell)$, see Eq. (3) above. A more intuitive quantity is the bare impact of an event, and we now attempt to deconvolute the effect of correlations to extract these quantities.

5. THE TEMPORARY IMPACT MODEL

Market orders move prices, but so do cancellations and limit orders. As reviewed in Sec. 2 above, one can try to describe the impact of all these events in an effective way in terms of a "dressed" propagator of market orders only, $G(\ell)$, as defined by Eq. (1). Let us extend this formalism to include any number of events in the following way. We assume, that after a lag of ℓ events, an event of type π has a remaining impact $G_\pi(\ell)$. The price is then expressed as

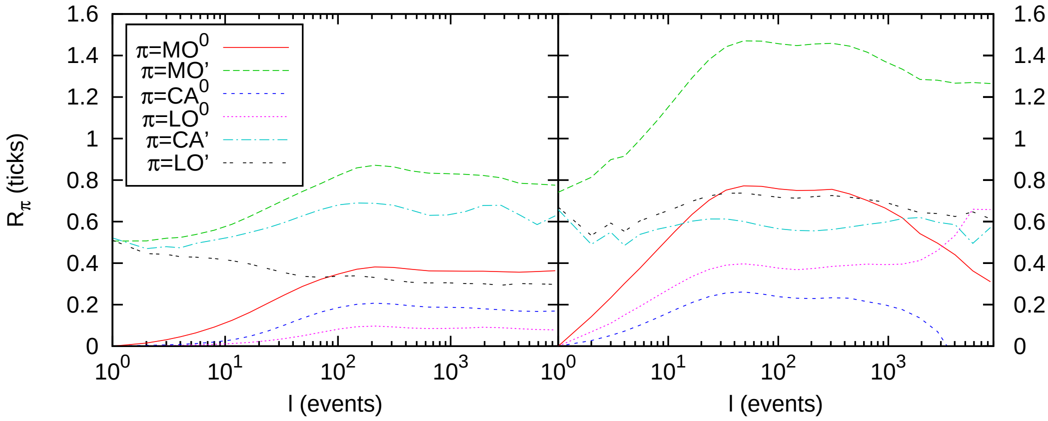

Figure 7: The normalized response function $R_\pi(\ell)$ for *(left)* large tick stocks and *(right)* small tick stocks. The curves are labeled according to π in the legend.

the sum of the impacts of *all* past events, plus some initial reference price:

$$p_t = \sum_{t'<t} \sum_{\pi} G_\pi(t-t') I(\pi_{t'} = \pi)\epsilon_{t'} + p_{-\infty}, \tag{13}$$

where the term with the indicators selects exactly one propagator for each t', the one corresponding to the particular event type at that time. After straightforward calculations, the response function (12) can be expressed through Eq. (13) as

$$R_{\pi_1}(\ell) = \sum_{\pi_2} P(\pi_2) \left[\sum_{0<n\le\ell} G_{\pi_2}(n) C_{\pi_1,\pi_2}(\ell-n) + \sum_{n>\ell} G_{\pi_2}(n) C_{\pi_2,\pi_1}(n-\ell) - \sum_{n>0} G_{\pi_2}(n) C_{\pi_2,\pi_1}(n) \right]. \tag{14}$$

This is a direct extension of Eq. (3), which was first obtained in Ref. [4]. One can invert the system of equations in (14), to evaluate the unobservable G_π's in terms of the observable R_π's and C_{π_1,π_2}'s. In order to do this, one rewrites the above in a matrix form, as

$$R_{\pi_1}(\ell) = \sum_{\pi_2} \sum_{n=0}^{\infty} A_{\ell,n}^{\pi_1,\pi_2} G_{\pi_2}(n), \tag{15}$$

where

$$A_{\ell,n}^{\pi_1,\pi_2} = P(\pi_2) \begin{cases} C_{\pi_1,\pi_2}(\ell-n) - C_{\pi_2,\pi_1}(n), & \text{if } 0 < n \le \ell \le L \\ C_{\pi_2,\pi_1}(n-\ell) - C_{\pi_2,\pi_1}(n), & \text{if } 0 < \ell < n \le L \end{cases} \tag{16}$$

and ∞ was replaced by a large enough cutoff L, convenient for numerical purposes. In the following, we use $L = 1000$, which allows to determine the functions G_π with a good precision up to $\ell \sim 300$, see Fig. 8.

As discussed in Sec. 2, the origin of the decay of market order price impact is that incoming limit orders maintain an equilibrium with market order flow. In order to keep prices diffusive, limit orders introduce a reverting force into prices, and this precisely off-sets the persistence in market order flow. However, our present extended formalism *explicitly* includes these limit orders (and also cancellations) as events. If *all* order book events were described, one naively expects that the G_π's should be lag-independent constants for events that change the price, and zero otherwise. Solving the above equation for G_π's, however, leads to functions that still depend on the lag ℓ, particularly for small tick stocks: see Fig. 8. We see that market orders that do not change the price immediately do impact the price on longer time scales. We also notice that the impact of single $\mathrm{MO}', \mathrm{MO}^0$ events first grows with lag and then decays slowly. The impact of limit orders seems to be significantly smaller than that of market orders, in particular for small tick stocks.

In the rest of the paper, we will try to understand in more detail where the lag dependence of G_π's comes from. The discussion of Sec. 2 already suggested that some history dependence of impact is responsible for this effect. Before

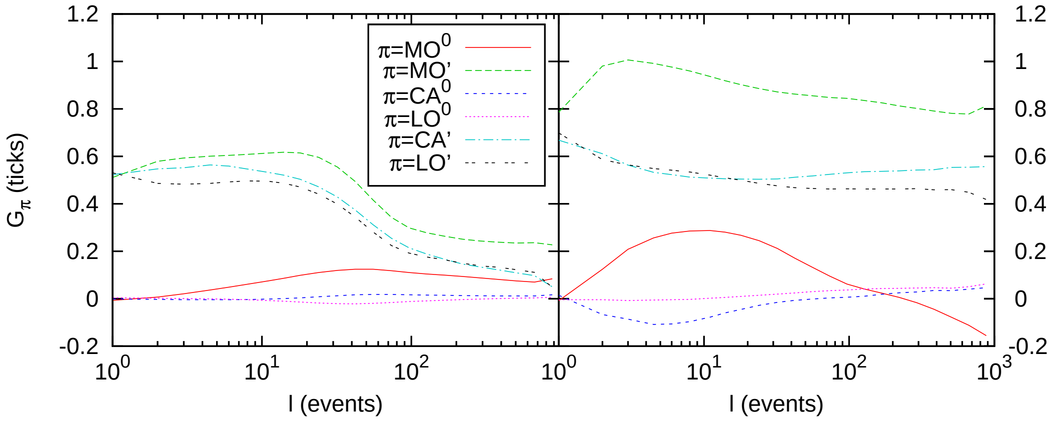

Figure 8: The Green's functions $G_\pi(\ell)$ in the temporary impact model for *(left)* large tick stocks and *(right)* small tick stocks.

dwelling into this, it is interesting to see how well the above augmented model predicts the volatility of the stocks once all the G_π's have been calibrated on the empirical R_π's. As just mentioned, the above Eq. (13) neglects the *fluctuations* of the impact, and we a priori expect some discrepancies. In order to make such a comparison, we first express exactly the variance of the price at lag ℓ, $D(\ell) = \langle (p_{t+\ell} - p_t)^2 \rangle$ in terms of the G's and the C's, generalizing the corresponding result obtained in [3]:

$$\begin{aligned} D(\ell) &= \langle (p_{t+\ell} - p_t)^2 \rangle = \sum_{0 \le n < \ell} \sum_{\pi_1} G_{\pi_1}(\ell - n)^2 P(\pi_1) + \sum_{n>0} \sum_{\pi_1} \left[G_{\pi_1}(\ell + n) - G_{\pi_1}(n) \right]^2 P(\pi_1) \\ &+ 2 \sum_{0 \le n < n' < \ell} \sum_{\pi_1, \pi_2} G_{\pi_1}(\ell - n) G_{\pi_2}(\ell - n') C_{\pi_1, \pi_2}(n' - n) \\ &+ 2 \sum_{0 < n < n' < \ell} \sum_{\pi_1, \pi_2} \left[G_{\pi_1}(\ell + n) - G_{\pi_1}(n) \right] \left[G_{\pi_2}(\ell + n') - G_{\pi_2}(n') \right] C_{\pi_1, \pi_2}(n - n') \\ &+ 2 \sum_{0 \le n < \ell} \sum_{n'>0} \sum_{\pi_1, \pi_2} G_{\pi_1}(\ell - n) \left[G_{\pi_2}(\ell + n') - G_{\pi_2}(n') \right] C_{\pi_2, \pi_1}(n' + n). \end{aligned} \tag{17}$$

The function $D(\ell)/\ell$, which should be constant for a strictly diffusive process, is plotted in Fig. 9, the symbols indicate the empirical data, and the dashed lines correspond to Eq. (17). Note that we fit both models to each stock separately, compute $D(\ell)/\ell$ in each case, and then average the results. We see that the overall agreement is fair for small tick stocks, but very bad for large tick stocks. The reason will turn out to be that for large ticks, a permanent, non fluctuating impact model accounts very well for the dynamics. This reflects that the spread and the gaps behind the best quotes are nearly constant in that case. But any small variation of G_π is amplified through the second term of Eq. (17) which is an infinite sum of positive terms. Hence it is much better to work backwards and test a model where the single event propagator is assumed to be strictly constant, as we will explain in the next section.

6. A CONSTANT IMPACT MODEL

In the above section we found that the single event propagators G_π appear to have a non-trivial time dependence. Another way to test this result is to invert the logic and assume that the G_π are time independent and see how well, or how badly, one can reproduce the response functions $R_\pi(\ell)$ and the price diffusion $D(\ell)$.

Let us start from the following *exact* formula for the midpoint price:

$$p_{t+\ell} = p_t + \sum_{t \le t' < t+\ell} \sum_{\pi} \Delta_{\pi, \epsilon_{t'}, t'} I(\pi_{t'} = \pi) \epsilon_{t'}. \tag{18}$$

Here $\Delta_{\pi,\epsilon_{t'},t'}$ denotes the price change at time t' if an event of type π happens. This Δ can also depend on the sign $\epsilon_{t'}$. For example, if $\pi = \mathrm{MO}'$ and $\epsilon_{t'} = -1$ this means that at a sell market order executed the total volume at the bid. The midquote price change is $-\Delta_{\mathrm{MO}',-1,t'}$, which usually means that the second best level was $b_{t'} - 2\Delta_{\mathrm{MO}',-1,t'}$, where $b_{t'}$ is the bid price before the event. The factor 2 is necessary, because the ask did not change, and the impact

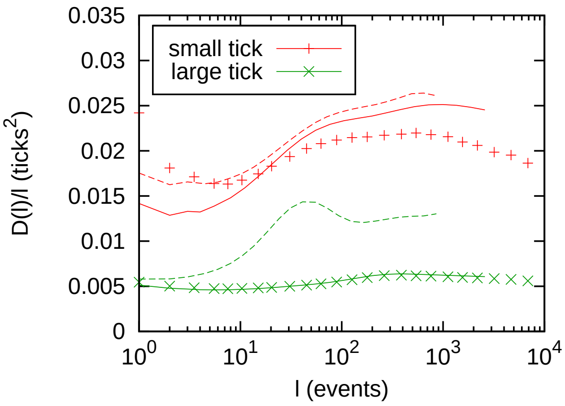

Figure 9: $D(\ell)/\ell$ and its approximations for the two groups of stocks. For small tick stocks the values were divided by 10. Symbols correspond to the empirical result. Dashed lines correspond to the temporary impact model with all 6 events and they are calculated from Eq. (17). The agreement is acceptable for small tick stocks, but very poor for large tick ones. Solid lines correspond to the constant impact model, see Eq. (20) below; in this case the agreement with large tick stocks in nearly perfect.

is defined by the change of the midquote. Hence $\Delta_{\mathrm{MO}'}$'s (and similarly $\Delta_{\mathrm{CA}'}$'s) correspond to half of the gap between the first and the second best quote *just before* the level was removed (see also Ref. [14]). Another example when $\pi = \mathrm{LO}'$ and $\epsilon_{t'} = -1$. This means that at t' a sell limit order was placed inside the spread. The midquote price change is $-\Delta_{\mathrm{LO}',-1,t'}$, which means that the limit order was placed at $a_{t'} - 2\Delta_{\mathrm{LO}',-1,t'}$, where $a_{t'}$ is the ask price. Thus $\Delta_{\mathrm{LO}'}$'s correspond to half of the gap between the first and the second best quote *right after* the limit order was placed. In the following we will call the Δ's *gaps*. Note that the events MO^0, CA^0 and LO^0 do not change the price, so their respective gaps are always zero: there are only three types of Δ's that are non-zero.

The permanent impact model is defined by replacing the time dependent Δ's by their average values. More precisely, let us introduce the average realized gap:

$$\Delta_\pi^{\mathrm{R}} = \frac{\langle I(\pi_t = \pi)\Delta_{\pi,\epsilon_t,t}\rangle}{P(\pi)}. \tag{19}$$

The multiplication by the indicator means that the gaps are sampled only when the price change corresponding to the gap is truly realized. Therefore, in general $\Delta_\pi^{\mathrm{R}} \neq \langle \Delta_{\pi,\epsilon_t,t}\rangle$, see Table III where one sees that the realized gap when a market order moves the price is in fact *larger* than the unconditional average. The logic is that the opening of a large gap behind the ask is a motivation for buying rapidly (or cancelling rapidly for sellers) before the price moves up.

Our approximate *constant impact* model then reads:

$$p_{t+\ell} = p_t + \sum_{t \le t' < t+\ell} \sum_\pi \Delta_\pi^{\mathrm{R}} I(\pi_{t'} = \pi)\epsilon_{t'}. \tag{20}$$

The response functions are then easily given by:

$$R_\pi(\ell) = \frac{\langle (p_{t+\ell} - p_t) I(\pi_t = \pi)\epsilon_t \rangle}{P(\pi)} = \sum_{0 \le t' < \ell} \sum_{\pi_1} \Delta_{\pi_1}^{\mathrm{R}} P(\pi_1) C_{\pi,\pi_1}(t'), \tag{21}$$

The formula (21) is quite simple to interpret. We fixed that the event that happened in t was of type π. Let us now express $C_{\pi,\pi_1}(\ell)$ as:

$$P(\pi)P(\pi_1)C_{\pi,\pi_1}(\ell) \propto P(\pi_{t+\ell} = \pi_1, \epsilon_{t+\ell} = \epsilon_t | \pi_t = \pi) - P(\pi_{t+\ell} = \pi_1, \epsilon_{t+\ell} = -\epsilon_t | \pi_t = \pi). \tag{22}$$

This represents the following: Given that the event at t was of type π and the event at $t + \ell$ is of type π_1, how much more is it probable, that the direction of the second event is the same as that of the first event? The total price response to some event can be understood as its own impact (lag zero), plus the sum of the biases in the course of future events, conditional to this initial event. These biases are multiplied by the average price change Δ^{R} that these induced future events cause. Of course, correlation does not mean causality, and we cannot a priori distinguish between events that are *induced* by the initial event, and those that merely *follow* the initial event (see [15] for a

	ticker	$2\Delta^{\rm R}_{\rm MO'}$	$2\Delta^{\rm R}_{\rm CA'}$	$2\Delta^{\rm R}_{\rm LO'}$	$2\langle\Delta_{\rm MO'}\rangle$
large tick	AMAT	1.02	1.04	1.02	1.00
	CMCSA	1.03	1.14	1.06	1.00
	CSCO	1.01	1.02	1.01	1.00
	DELL	1.01	1.05	1.02	1.00
	INTC	1.00	1.01	1.01	1.00
	MSFT	1.01	1.02	1.01	1.00
	ORCL	1.01	1.02	1.02	1.00
small tick	AAPL	1.31	1.27	1.27	1.14
	AMZN	1.51	1.22	1.30	1.17
	APOL	1.76	1.50	1.52	1.42
	COST	1.35	1.23	1.24	1.15
	ESRX	1.85	1.54	1.60	1.45
	GILD	1.11	1.13	1.11	1.03

Table III: Mean realized gaps and unconditional gaps in ticks for all stocks. All values were multiplied by 2, so that they correspond to the instantaneous change of the bid/ask and not of the midquote. Note that $\langle\Delta_{\rm MO'}\rangle = \langle\Delta_{\rm CA'}\rangle$, while $\langle\Delta_{\rm LO'}\rangle$ is not observable.

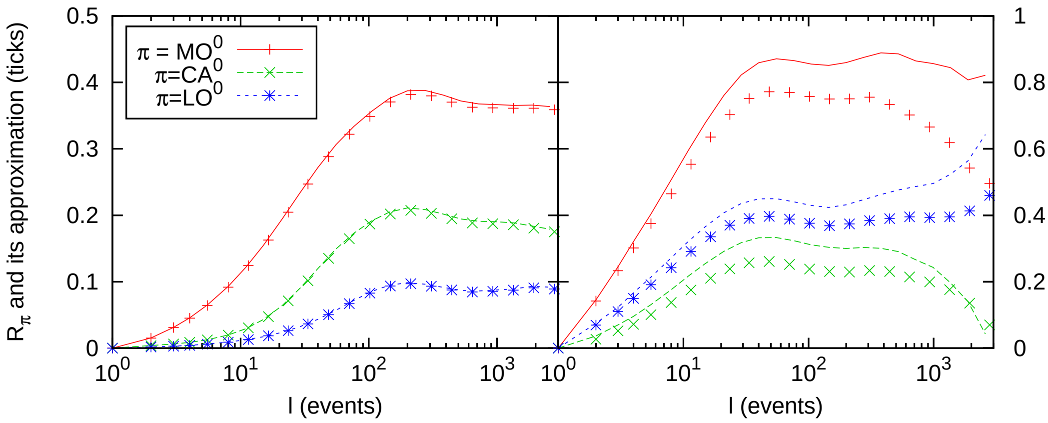

Figure 10: Comparison of true and approximated normalized response functions $R_\pi(\ell)$, using the constant gap model, for *(left)* large tick stocks and *(right)* small tick stocks, for events that *do not* change the price. Symbols correspond to the true value, and lines to the approximation. The data are labeled according to π in the legend.

related discussion). However, it seems reasonable to assume that there is a true causality chain between different types of events occuring on the same side of the book (i.e. a limit order refilling the best quote after a market order).

Let us now take Eq. (21), and check how well the true response functions are described by the above constant impact model. Figs. 10 and 11 show that the agreement is nearly perfect for large tick stocks, except when $\pi_1 = {\rm CA}'$, but these events are very rare (up to $\sim 0.2\%$). This agreement is expected because the order book is usually so dense that gaps hardly fluctuate at all; the small remaining discrepancies will in fact be cured below. The quality of the agreement suggests that the time dependence of the bare impact function G_π obtained in Sec. 5 above is partly a numerical artefact coming from the "brute force" inversion of Eq. (15).

For small ticks on the other hand, noticeable deviations are observed, and call for an extension of the model. This will be the focus of the next sections. One can extend the above model in yet another direction, by studying the dynamics of the spread rather than the dynamics of the mid-point, see Appendix A.

One can approximate the volatility within the same model as

$$D(\ell) = \left\langle (p_{t+\ell} - p_t)^2 \right\rangle \approx \sum_{t\le t',t''<t+\ell} \sum_{\pi_1}\sum_{\pi_2} P(\pi_1)P(\pi_2)C_{\pi_1,\pi_2}(t'-t'')\Delta^{\rm R}_{\pi_1}\Delta^{\rm R}_{\pi_2}. \tag{23}$$

As shown in Fig. 9, the constant gap model is again very precise for large tick stocks, but clear discrepancies are visible for small tick ones.

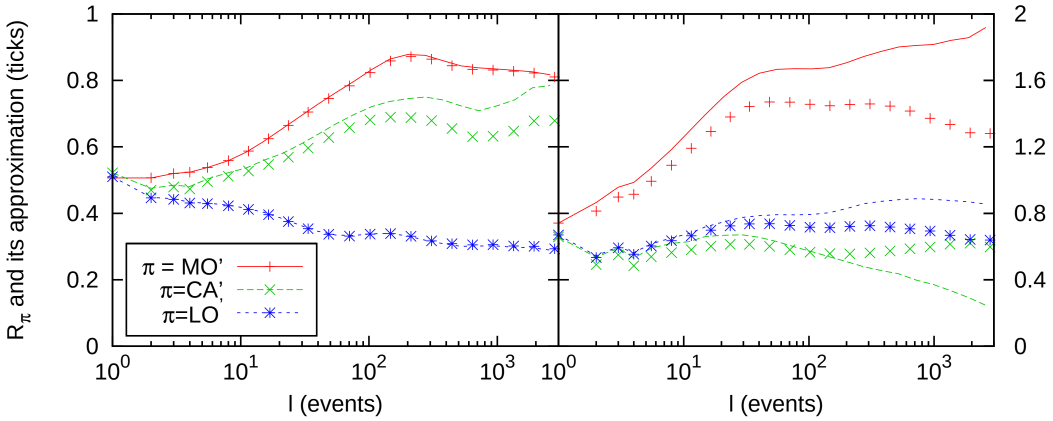

Figure 11: Comparison of true and approximated normalized response functions $R_\pi(\ell)$, using the constant gap model, for *(left)* large tick stocks and *(right)* small tick stocks, for events that change the price. Symbols correspond to the true value, and lines to the approximation. The data are labeled according to π in the legend.

7. THE GAP DYNAMICS OF SMALL TICK STOCKS

7.1. A decomposition of the gap fluctuations

Let us now try to better understand how gap fluctuations contribute to the response function, and why replacing the gap by its average realized value is not a good approximation for small tick stocks. By definition, without the constant gap approximation, the response function contains contributions which have the form

$$\left\langle \Delta_{\pi_2,\epsilon_{t+\ell},t+\ell} I(\pi_{t+\ell}=\pi_2)\epsilon_{t+\ell} I(\pi_t=\pi_1)\epsilon_t \right\rangle .$$

After using some basic properties of the event signs this quantity can be written as a sum over three contributions:

1. Firstly, there is the term from the constant gap approximation:

$$\Delta^{\mathrm{R}}_{\pi_2} \left\langle I(\pi_{t+\ell}=\pi_2)\epsilon_{t+\ell} I(\pi_t=\pi_1)\epsilon_t \right\rangle .$$

   This contains the highest order of the effect of event-event correlations.

2. There is a second term that we write as:

$$\frac{1}{2}\left\langle \underbrace{[\Delta_{\pi_2,+,t+\ell}-\Delta_{\pi_2,-,t+\ell}]\epsilon_t}_{(a)} I(\pi_{t+\ell}=\pi_2) I(\pi_t=\pi_1) \right\rangle .$$

   The sign of the expression is determined by (a), since the rest is non-negative. If (a) is positive, then after an upward price move consecutive upward moves are larger than downward ones, while if (a) is negative then they are smaller. This process can thus either accelerate or dampen the growth of the response function.

3. The third contribution is of the form

$$\frac{1}{2}\left\langle \underbrace{[\Delta_{\pi_2,+,t+\ell}+\Delta_{\pi_2,-,t+\ell}-2\Delta^{\mathrm{R}}_{\pi_2}]}_{(b)} \underbrace{\epsilon_t\epsilon_{t+\ell}}_{(c)} I(\pi_{t+\ell}=\pi_2) I(\pi_t=\pi_1) \right\rangle .$$

   Here (b) is positive, when the average of the two gaps (up and down) is greater than the time averaged realized value. (c) is positive, when the two events move the price in the same direction. Thus the full term gives a positive contribution to the response function, if two "parallel" events are correlated with larger gaps and hence decreased liquidity at the time of the second event, while opposing events correspond to increased liquidity at the time of the second event. The final effect of this term agrees with the previous one: If $(b) \times (c)$ is positive, then after an upward price move the consecutive upward moves become larger than downward ones and vice versa.

At this point it would be useful to introduce a dynamical model for the Δ's, to quantify the above correlations, but we are faced with the difficulty that $\Delta_{\pi,\epsilon,t}$ is only observed for $\pi = \pi_t$ and $\epsilon = \epsilon_t$. What we will do instead is to write a simple regression model directly for the quantity $\Delta_{\pi,\epsilon_t,t} I(\pi_t = \pi)\epsilon_t$. This can be evaluated from data, because the indicator is zero whenever the gap is unobservable. Then based on this knowledge we will revisit the influence of gap fluctuations on the price dynamics in Sec. 7.3.

7.2. A linear model for gap fluctuations

The correlation between events has a dynamical origin: market orders and cancellations attract replacement limit orders and vice versa. Eq. (18) is the exact time evolution of price written as a sum of the random variables $\Delta_{\pi,\epsilon_t,t} I(\pi_t = \pi)\epsilon_t$. The simplest possible linear dynamical model for these variables is:

$$\Delta_{\pi,\epsilon_t,t} I(\pi_t = \pi)\epsilon_t = \sum_{t'<t} \sum_{\pi_1} K_{\pi_1,\pi}(t-t') I(\pi_{t'} = \pi_1)\epsilon_{t'}. \tag{24}$$

It will also be useful to introduce a second regression model for the signed indicators only, that we write as:

$$\Delta^{\mathrm{R}}_{\pi} I(\pi_t = \pi)\epsilon_t = \sum_{t'<t} \sum_{\pi_1} \tilde{K}_{\pi_1,\pi}(t-t') I(\pi_{t'} = \pi_1)\epsilon_{t'}. \tag{25}$$

Both models can be calibrated to the data by using the same trick an in Sec. 5, forming expectation values on both sides and solving a set of linear equations between correlation functions, for example for K:

$$\left\langle \Delta_{\pi,\epsilon_{t+\ell},t+\ell} I(\pi_{t+\ell} = \pi)\epsilon_{t+\ell} I(\pi_t = \pi_1)\epsilon_t \right\rangle = \sum_{t'<t+\ell} \sum_{\pi_2} K_{\pi_2,\pi}(t+\ell-t') \left\langle I(\pi_{t'} = \pi_2)\epsilon_{t'} I(\pi_t = \pi_1)\epsilon_t \right\rangle, \tag{26}$$

except this time we have three separate solutions for $\pi = \mathrm{MO}'$, CA' and LO'. An example of the solution kernels K is given in Fig. 12; the sign of these kernels is expected from what we learnt in Sec. 4. We see for example that a MO event tends to make a future MO' more probable, and with an increased gap, which makes sense. The same can be repeated with respect to Eq. (25) to calculate the $\tilde{K}$'s.

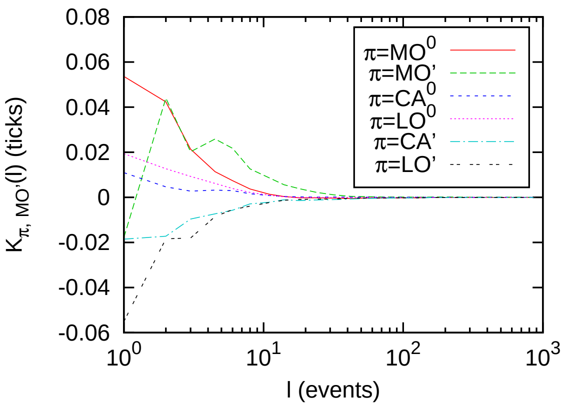

Figure 12: Estimates of $K_{\pi,\mathrm{MO}'}(t)$ for small ticks.

The first important finding is that these linear models perform surprisingly well in practice. With the fitted values of K's one can now simulate Eq. (24) as a model for predicting the rate and direction of various events. Fig. 13 shows that the actual value of the left hand side is a monotonic function of our prediction, and the relationship on average can be fitted with a straight line with slope 1, although small higher order (cubic) corrections seem to be present as well. Similar results can be found for $\tilde{K}$'s.

The relevance of these kernels can be understood through a simple argument. The sequence of events is characterized by the time series $\{\pi_t, \epsilon_t\}$ and together with the gaps this series defines the course of the price. How will the event π at time t affect the price at some later time $t + \ell$? This quantity, whose average value we will denote by $G^*_\pi(\ell)$, contains two contributions:

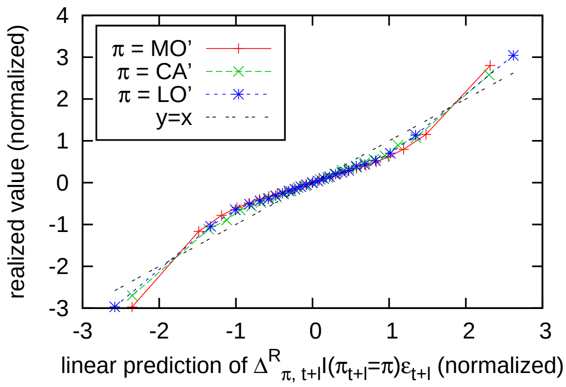

Figure 13: Performance of Eq. (24) for small ticks. Both axes normalized by standard deviation of predictor.

- A direct one: the immediate price change caused by the event, which is a constant (zero or non-zero depending on the value of the corresponding gap). For example, if right now a large buy market order is submitted, it will cause an immediate upward jump in the (ask) price. The average of such jumps due to events of type π is represented by the mean realized gap $\Delta^{\mathrm{R}}_{\pi}$.
- An induced, dynamic one: the change of the future event rates and their associated gaps. This modeled by Eq. (24), and quantified by the kernels K. To continue the above example of a large market order, it removes the best ask level, and hence we move into a denser part of the order book. The new first gap behind the ask is on average smaller here. So in effect, our initial event makes the ask gap shrink. In addition, some time after we submit our order, additional sell limit orders will arrive to compensate part of our upwards price pressure. These may move the (ask) price back downwards. If we decided not to submit our market order, these extra limit orders would not arrive either.

Within the linear model (24), the average price change at time $t+\ell$ attributed to an event of type π at time t, defined as

$$G^*_\pi(\ell) = \frac{1}{2}\left[\langle p_{t+\ell} | I(\pi_t = \pi); \epsilon_t = +1\rangle - \langle p_{t+\ell} | I(\pi_t = \pi); \epsilon_t = -1\rangle,\right] \tag{27}$$

is found to be

$$G^*_\pi(\ell) = \Delta^R_\pi + \sum_{0<t<\ell} \sum_{\pi_1} K_{\pi,\pi_1}(t). \tag{28}$$

Numerically, the G^*_π's are given in Fig. 14(left) for small tick stocks, very similar curves were found for large ticks but we will not detail those here.

What is the relation between G^*_π and the bare response function G_π needed as an input in Eq. (13) to reproduce the correct dynamics of the price? This can be guessed by decomposing further the dynamic contribution as the sum of two components. One is the change in the future event rates and signs, the other is the response of the size of the gaps. The first effect gives an average contribution to price change equal to $\sum_{0<t<\ell}\sum_{\pi_1} \tilde{K}_{\pi,\pi_1}(t)$, by definition of the kernels $\tilde{K}$. But this merely reflects the correlation between signed events, described by the correlation function C, and modelled above as a linear regression. This effect is therefore already accounted for within the constant gap model (see Eq. (21) above).

When the gaps are allowed to fluctuate, the extra contribution is captured by the remaining term as:

$$\delta G^*_\pi(\ell) = \sum_{0<t<\ell} \sum_{\pi_1} \left[K_{\pi,\pi_1}(t) - \tilde{K}_{\pi,\pi_1}(t)\right]. \tag{29}$$

One can thus improve upon the constant gap model by adding this fluctuating gap contribution to the average realized gaps, i.e. explore a model where the bare propagator reads:

$$G_\pi(\ell) = \Delta^R_\pi + \delta G^*_\pi(\ell), \tag{30}$$

which will be our final model, detailed in the next section.

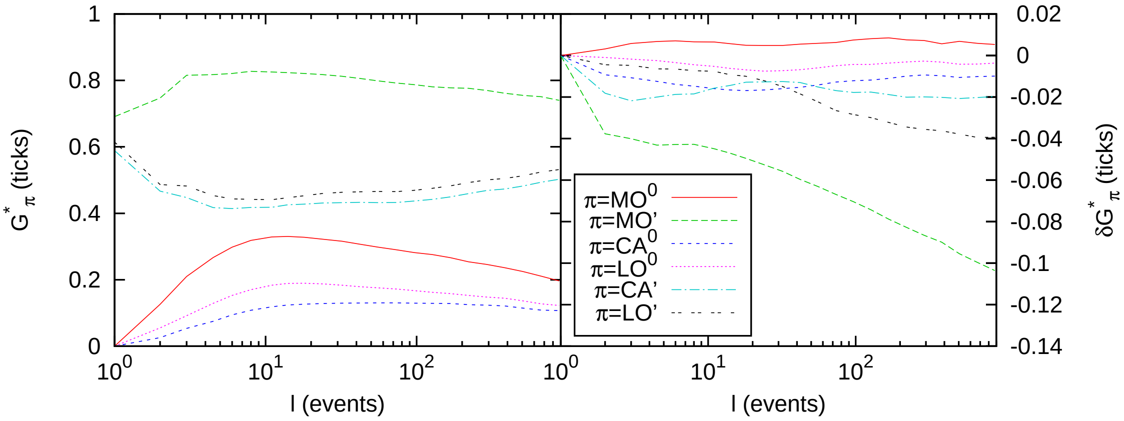

Figure 14: *(left)* Estimate of the average price change $G^*_\pi(\ell)$ due to an event π, based on Eq. (28), for small tick stocks. *(right)* Contribution of gap flexibility to the price change: $\delta G^*_\pi(\ell)$ calculated from Eq. (29) for small tick stocks. The curves are labeled according to π in the legend.

The new term describes the contribution of the gap "compressibility" to the impact of an event up to a time lag ℓ, and it is shown in Fig. 14(right). Perhaps surprisingly, it appears that a small market order MO^0 "softens" the book for small ticks: the gaps tend to grow on average and $\delta G^*_{\mathrm{MO}^0}$ is positive. Price changing events on the other hand "harden" the book, for all stocks the contribution is negative. Queue fluctuations (CA^0 and LO^0) seem less important, but for small ticks these types of events also harden the book. For large ticks δG^*'s are found to be about two orders of magnitude smaller, which confirms that gap fluctuations can be neglected to a good approximation in that case.

7.3. The final model for small ticks

The above analysis suggests a way to build and calibrate an impact model that describes in a consistent way (a) all types of events and (b) the history dependence of the gaps, as we argued to be necessary in Sec. 2. The discussion of the previous section motivates the following model:

$$p_{t+\ell} = p_t + \sum_{t \le t' < t+\ell} \sum_{\pi} \left[\Delta^R_\pi + \sum_{t''<t'} \sum_{\pi_2} \kappa_{\pi_2,\pi}(t'-t'') I(\pi_{t''} = \pi_2) \epsilon_{t'} \epsilon_{t''} \right] I(\pi_{t'} = \pi) \epsilon_{t'}, \tag{31}$$

where κ_{π_2,π_1} is a kernel that models the fluctuations of the gaps and their history dependence, which will be chosen such that the bare propagator of the model is given by Eq. (30) above.

The model specification, Eq. (31), is the central result of this paper. It can be seen as a permanent impact model with some history dependence, modeled as a linear regression on past events. By symmetry, this dependence should only include terms containing $\epsilon_{t'}\epsilon_{t''}$ since the influence of any past string of events on the ask must be the same as that of the mirror image of the string on the bid. More generally, one may expect higher order, non-linear correction terms of the form

$$\sum_{t_1,t_2,t_3<t'} \sum_{\pi_1,\pi_2,\pi_3} \kappa_{\pi_1,\pi_2,\pi_3;\pi}(t'-t_1, t'-t_2, t'-t_3) I(\pi_{t_1} = \pi_1) I(\pi_{t_2} = \pi_2) I(\pi_{t_3} = \pi_3) \epsilon_{t'} \epsilon_{t_1} \epsilon_{t_2} \epsilon_{t_3}, \tag{32}$$

or with a larger even number of ϵ's, but we will not explore such corrections further here, although Fig. 13 suggests these terms are present.

If we compute the bare propagator of this model, formally defined by Eq. (4), one finds:

$$G_\pi(\ell) = \left\langle \frac{\partial p_{t+\ell}}{\partial I(\pi_t = \pi)\epsilon_t} \right\rangle = \Delta^R_\pi + \sum_{t'>t}^{t+\ell} \sum_{\pi_2} \kappa_{\pi,\pi_2}(t'-t) \langle I(\pi_{t'} = \pi_2) | I(\pi_t = \pi) \rangle, \tag{33}$$

where the conditional expectation value can be expressed as $\langle I(\pi_{t'} = \pi_2) | I(\pi_t = \pi) \rangle = P(\pi_2)[1 + \Pi_{\pi,\pi_2}(t'-t)]$. Our final model is described by the required bare propagator given in Eq. (30) provided one makes the identification:

$$\kappa_{\pi,\pi_2}(t) P(\pi_2) [1 + \Pi_{\pi,\pi_2}(t)] = K_{\pi,\pi_2}(t) - \tilde{K}_{\pi,\pi_2}(t). \tag{34}$$

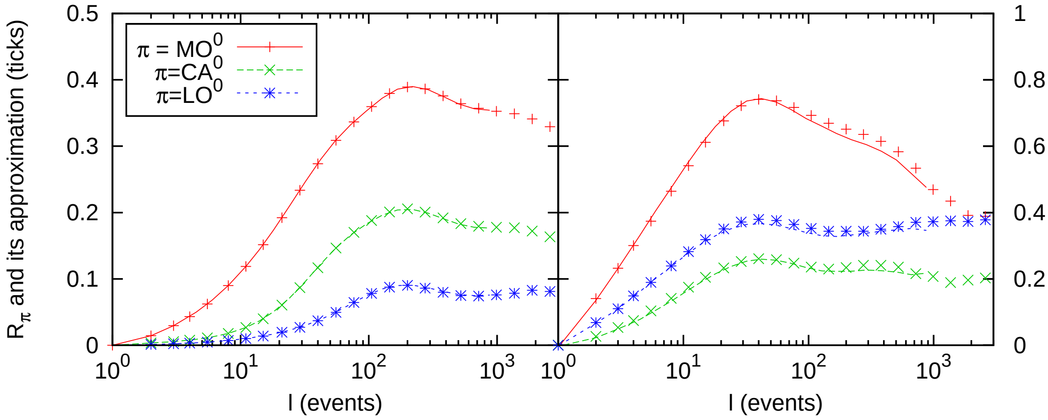

Figure 15: Comparison of true and approximated normalized response functions $R_\pi(\ell)$ of the final model for *(left)* large tick stocks and *(right)* small tick stocks, for events that do not change the price. Symbols correspond to the true value, and lines to the approximation. The inaccuracy for large ℓ is due to a finite size effect in matrix inversion. The data are labeled according to π in the legend.

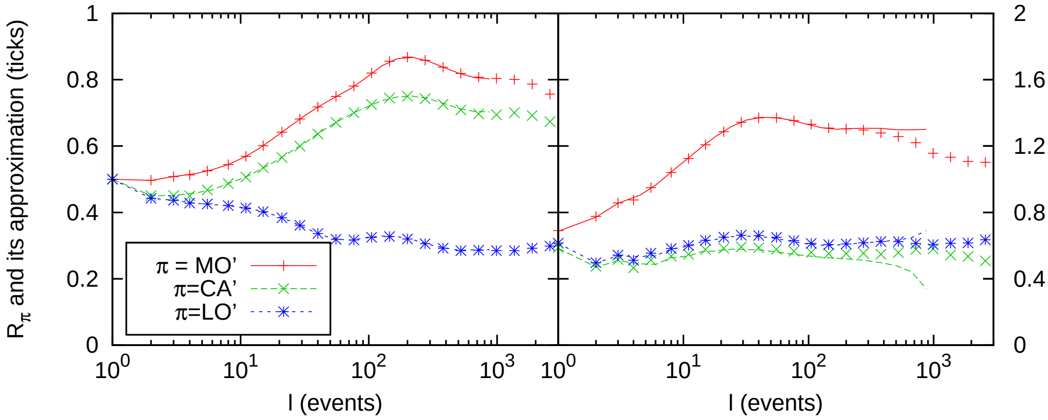

Figure 16: Comparison of true and approximated normalized response functions $R_\pi(\ell)$ of the final model for *(left)* large tick stocks and *(right)* small tick stocks, for events that change the price. The inaccuracy for large ℓ is due to a finite size effect in matrix inversion. Symbols correspond to the true value, and lines to the approximation. The data are labeled according to π in the legend.

We can now compute the average response functions $R_\pi(\ell)$ and the diffusion curve $D(\ell)$ within this model, and compare the results with empirical data.

For the response functions, the addition of the fluctuating gap term in Eq. (31) corrects the small discrepancies found within the constant impact model for large tick stocks. It also allows one to capture very satisfactorily the response function for small tick stocks, see Figs. 15 and 16.[4]

A more stringent test of the model is to check the behaviour of the diffusion curve $D(\ell)$. The exact calculation in

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[4] Note that in making these plots we neglected the first 30 and last 40 minutes of trading days, so they slightly differ from those in Sec. 6. The results of the constant gap model are essentially unchanged regardless of such an exclusion.

fact involves three and four-point correlation functions. A consistent approximation up to two-point level yields:

$$D(\ell) = \left\langle (p_{t+\ell} - p_t)^2 \right\rangle \approx \sum_{0 \le t', t'' < \ell} \sum_{\pi_1} \sum_{\pi_2} P(\pi_1) P(\pi_2) C_{\pi_1,\pi_2}(t' - t'') \Delta^{\mathrm{R}}_{\pi_1} \Delta^{\mathrm{R}}_{\pi_2} +$$
$$2 \sum_{-\ell < t < \ell} \sum_{\pi_2,\pi_3} \sum_{\tau > 0} (\ell - |t|) \Delta^{\mathrm{R}}_{\pi_3} \kappa^{+}_{\pi_2,\pi_3}(\tau, t) C_{\pi_2,\pi_3}(t + \tau) P(\pi_2) P(\pi_3) +$$
$$\sum_{-\ell < t < \ell} \sum_{\pi_2,\pi_4} \sum_{\tau,\tau' > 0} (\ell - |t|) \kappa^{++}_{\pi_2,\pi_4}(\tau, \tau', t) C_{\pi_2,\pi_4}(\tau - \tau' + t) P(\pi_2) P(\pi_4), \tag{35}$$

where

$$\kappa^{+}_{\pi_2,\pi_3}(\tau, t) = \sum_{\pi_1} \kappa_{\pi_2,\pi_1}(\tau) [I(t = 0) I(\pi_1 = \pi_3) + I(t \ne 0) P(\pi_1) + I(t = -\tau) P(\pi_1) \Pi_{\pi_2 \pi_1}(\tau)], \tag{36}$$

and, for $t \ge 0$,

$$\kappa^{++}_{\pi_2,\pi_4}(\tau, \tau', t) = \sum_{\pi_1,\pi_3} \kappa_{\pi_2,\pi_1}(\tau) \kappa_{\pi_4,\pi_3}(\tau') \{ I(t = \tau') I(\pi_1 = \pi_4) P(\pi_3) +$$
$$I(t \ne \tau') P(\pi_1) P(\pi_3) [\Pi_{\pi_1,\pi_3}(t) + 1] \}, \tag{37}$$

whereas for $t < 0$, we use $\kappa^{++}_{\pi_2,\pi_4}(\tau, \tau', -t) = \kappa^{++}_{\pi_4,\pi_2}(\tau', \tau, t)$. As Fig. 17 shows, for small tick stocks this approximation is a significant improvement for large ℓ when compared to the constant gap model. However, for small ℓ there is still some discrepancy coming from errors in the data that adds spurious high frequency white noise. To account for these, we add an effective, lag-independent constant to volatility, whose value was chosen as $D_0 = 0.04$ ticks squared. According to Fig. 17, this substantially improves the fit for short times, while leaving the long time contribution unaffected.

The conclusion is that our history dependent impact model reproduces both the empirical average response function and the diffusion constant is a rather accurate way. The remaining discrepancies are expected, since we have neglected several effects, including (i) all volume dependence, (ii) unobserved events deeper in the book and (iii) higher order, non-linear contributions to model history dependence.

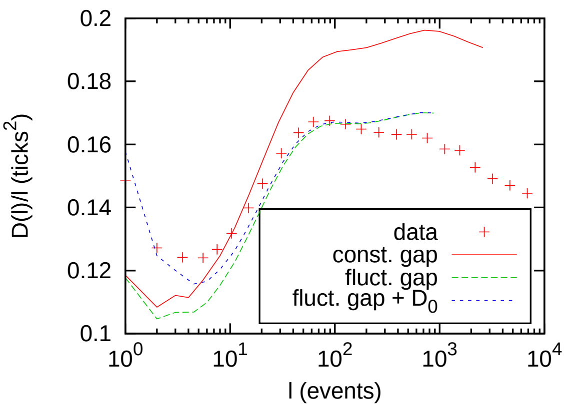

Figure 17: $D(\ell)/\ell$ and its approximations for small tick stocks. Symbols correspond to the true result excluding the beginning and the end of trading days, the red line corresponds to the permanent impact model with constant gaps, the green line to the fluctuating gap model, and the blue line to the fluctuating gap model plus constant. *Note:* Unlike the small tick data in Fig. 9, the vertical axis was not rescaled here.

8. CONCLUSIONS

Previous studies have focused on the impact of market orders only and have concluded that this impact decays in such a way to offset the correlation of the sign of the trades. The underlying mechanism is that market orders on one side of the book attract compensating limit orders, such that the conditional impact of a buy trade following other buy trades is smaller than the conditional impact of a sell trade following buy trades. Our study confirms that events happening on the same side of the book are long-range correlated, but the signed correlation function (that assigns an opposite sign to limit order and market order on the same side of the book) is short ranged, showing that the compensating effect alluded to above is indeed effective.

So this effect should be reflected in the limit order flow that "dresses" the bare impact of market orders. Treating all events on an equal footing should therefore, in a first approximation, make the impact of price changing events permanent: By including – besides the market orders – all limit orders and cancellations at the bid/ask, the price becomes a pure jump process. Every price change, whatever its cause (news, information or noise), can be attributed to exactly one of these events. The various event types can be approximated to have constant jump sizes that equal the average price change they cause. This simple picture works very well for large tick stocks, where both the average impact of all events and the volatility are quantitatively reproduced by a constant jump model. The situation is different for small tick stocks, the history dependence of these otherwise permanent jumps becomes important. Note that the effect discussed here is related to but different from the Lillo-Farmer model that connects the temporal decay of the dressed market order impact to the history dependent conditional impact of a new trade. Here, we are speaking of the history dependence in a framework where the impact of all events is accounted for.

Another important observation is that not only the jump sizes are history dependent, the events themselves also behave in an adaptive way. An event can induce further events that amplify or dampen its effect. As it is well known, the arrival of excess buy market orders is shortly followed by additional sell limit orders, but this is just one manifestation of such adaptive dynamics. For example, the reverse process, i.e. market orders arriving following an excess of limit orders is also present, albeit with some delay and a smaller intensity. Our description of these and similar mechanisms, also involving cancellations, is a generalization of the theory of market order price impact in the related literature.

In sum, the dynamics of prices consists of three processes: instantaneous jumps due to events, events inducing further events and thereby affecting the future jump *probabilities* (described by the correlation between events), and events exerting pressure on the gaps behind the best price and thereby affecting the future jump *sizes*. By approximating this third effect with a linear regression process, we have written down an explicit model, Eq. (31), that accounts very satisfactorily for most of our observations. We have shown how to calibrate such a model on empirical data using some auxiliary kernels K and $\tilde{K}$ defined by Eqs. (24) and (25). This way of extracting the bare propagator G_π, motivated by the above decomposition, seems to be less prone to numerical errors than the "brute force" inversion method used in Sec. 5.

The methods proposed in this work are rather simple and general, and can be adapted to measure the impact of any type of trade once a discrete categorization is adopted. One could for example subdivide the category MO^0 into small volumes and large volumes, or look at the impact of different option trades on the underlying, etc. Here, we have established that the bare impact of market orders is clearly larger than that of limit orders and that the bare impact of price-changing events shows only partial decay on the time scale that we are able to probe (1000 events only corresponds to a few minutes). It would certainly be interesting to study the long time behavior of these bare impact functions, as well as to understand how these impact functions behave overnight.

We hope to have provided here a consistent and complete framework to describe price fluctuations and impact at the finest possible scale. We believe that the interaction between market orders and limit orders, and the impact of these two types of orders, are crucial to understand the dynamics of the markets, the origin of volatility and the incipient instabilities that can arise when these counteracting forces are not on even keel. The interesting next step would be to analyze in detail these situations, where large liquidity fluctuations arise, and the above 'average' model breaks down. On a longer term, a worthwhile project is to construct a coarse-grained, continuous time model from the above microstructural bricks, and justify or reject the slew of models that have been proposed to describe financial time series (Lévy processes, GARCH, multifractal random walk, etc.).

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Appendix A: The dependence of the spread on the event flow

In this appendix we show how the framework introduced in the main text can be used to study the dynamics of the bid-ask spread. For the spread S_t, one can write an exact formula very similar to Eq. (18):

$$S_{t+\ell} = S_t + \sum_{t \le t' < t+\ell} \sum_{\pi} \overline{\Delta}_{\pi,\epsilon_{t'},t'} I(\pi_{t'} = \pi), \tag{A1}$$

where $\overline{\Delta}_\pi = \pm 2\Delta_\pi$ with the $+$ sign for $\pi = \mathrm{MO}', \mathrm{CA}'$ and the $-$ sign for $\pi = \mathrm{LO}'$. The other three $\overline{\Delta}$'s are zero, just as the respective Δ's were. The above equation is accurate because our model includes all the possible events that can change the best quotes, and thus all the possible events that can change the spread.

However, when formulating a permanent impact model for the spread dynamics in the same spirit as we did for the price, one should bear in mind that the spread is a mean-reverting quantity that oscillates around a mean value $\langle S \rangle$. In other words, the average value of $S_{t+\ell}$ when $\ell \to \infty$ is equal to $\langle S \rangle$, independently of the initial value S_t. Therefore:

$$\lim_{\ell\to\infty} \langle S_{t+\ell} - S_t | S_t \rangle = \lim_{\ell\to\infty} \sum_{t \le t' < t+\ell} \sum_{\pi_2} \left\langle \overline{\Delta}_{\pi_2,t',\epsilon_{t'}} I(\pi_{t'} = \pi_2) | S_t \right\rangle = \langle S \rangle - S_t. \tag{A2}$$

Since the right hand side obviously depends on S_t, the conditional value $\left\langle \overline{\Delta}_{\pi_2,t',\epsilon_{t'}} I(\pi_{t'} = \pi_2) | S_t \right\rangle$ also has to. This means that the event flow and possibly the gaps are correlated with the spread, and they adjust such that the spread mean reverts. If this were not true, the spread would follow an unbounded random walk. To illustrate this, the spread dependence of realized gaps is shown in Fig. 18.

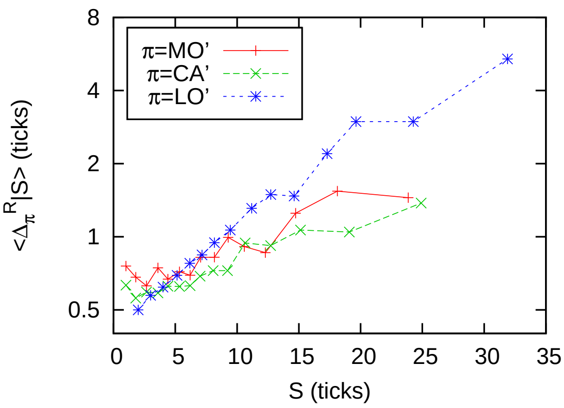

Figure 18: Realized gaps as a function of the spread (after removing the beginning and the end of the trading days).

A related study by Ponzi *et al.* [16] based on a selection of stocks from LSE comes to a similar conclusion. They find that both realized gaps and event rates are functions of the spread. In particular limit orders are placed deeper in the spread when the spread is larger. In addition, they show that the rate of transactions decreases with larger spreads, while the rates of cancellations and incoming limit orders increase sharply.

Such an adaptive behavior can be quantified through Eq. (A1), for $\ell = 1$ this reads

$$S_{t+1} - S_t = \overline{\Delta}_{\mathrm{MO}',\epsilon_t,t} I(\pi_t = \mathrm{MO}') + \overline{\Delta}_{\mathrm{CA}',\epsilon_t,t} I(\pi_t = \mathrm{CA}') - \left|\overline{\Delta}_{\mathrm{LO}',\epsilon_t,t}\right| I(\pi_t = \mathrm{LO}'), \tag{A3}$$

where we took the negative absolute value of $\overline{\Delta}_{\mathrm{LO}'}$ to emphasize that it is strictly negative, while all the other quantities in the equation are non-negative. The unconditional expectation value of the left hand side is zero, since

$\langle S_{t+1}\rangle = \langle S_t\rangle$. Thus on average the spread-altering effect of market orders, cancellations and limit orders balances out. If for example $S_t > \langle S\rangle$, then the spread mush shrink back to its mean, so the left hand side must be negative. The only way for the right hand side to become negative as well, is if the spread-opening contribution of market orders/cancellations decreases and/or the spread-closing effect of limit orders increases.

This is possible by the variation of both the gaps and the event rates, for our purposes it is enough to introduce a combined description the two. The simplest possible model of such adaptive dynamics is to assume that the conditional distribution of the random variable $\overline{\Delta}_{\pi,\epsilon_t,t} I(\pi_t = \pi)$ given the current spread value S_t can be approximated by that of

$$\overline{\Delta}_\pi^{\mathrm{R}} I(\pi_t = \pi) \left[1 + \frac{\alpha}{\overline{\Delta}_\pi^{\mathrm{R}}}(\langle S\rangle_\pi - S_t)\right],$$

where $I(\pi_t = \pi)$ now follows its unconditional distribution, $\langle S\rangle_\pi$ is the average value of the spread at the time of events of type π, and α is a constant parameter characterizing the strength of mean-reversion.

Even though the term in the brackets is understood to include contributions from both gap and rate adjustments, technically such a model only amounts to substituting

$$\overline{\Delta}_{\pi,t',\epsilon_{t'}} = \overline{\Delta}_\pi^{\mathrm{R}} + \alpha(\langle S\rangle_\pi - S_{t'}), \tag{A4}$$

for the gap dynamics in Eq. (A1), and this makes analytical calculations possible. One finds that the modified spread behavior is such that

$$S_{t+\ell} = S_t(1-\alpha)^\ell + \sum_{t\le t'<t+\ell}\sum_\pi (1-\alpha)^{t+\ell-t'+1} I(\pi_{t'} = \pi)(\overline{\Delta}_\pi^{\mathrm{R}} + \alpha\langle S\rangle_\pi), \tag{A5}$$

from which one deduces the spread response function:

$$\begin{aligned} R_{\pi_1}^S(\ell) &= \langle (S_{t+\ell} - S_t) I(\pi_t = \pi_1)\rangle / P(\pi_1) = \\ \left[\langle S\rangle - \langle S\rangle_{\pi_1}\right]\left[1-(1-\alpha)^\ell\right] &+ \sum_{t\le t'<t+\ell}\sum_{\pi_2}(1-\alpha)^{(t+\ell-1)-t'}\overline{\Delta}_{\pi_2}^{\mathrm{R}} P(\pi_2)\Pi_{\pi_1,\pi_2}(t'-t). \end{aligned} \tag{A6}$$

Eq. (A6) tells us that the dynamics of the spread is related to the autocorrelation of (unsigned) event types, just as the response function was related to the signed event autocorrelation functions (except for the inclusion of α to describe adjustment to the event flow).

To test this model on real data, in order to remove the effect of intraday periodicity and overnight effects, we will neglect the first 30 and last 40 minutes of trading days, and in all correlation functions we will only consider times when both events are within the same day. The spread response functions and their approximations *without* allowing for gap fluctuations ($\alpha = 0$) are shown in Figs. 19 and 20. The constant gap approximation works well for large tick stocks, but for small tick stocks only for short times, up to $l \approx 10 - 30$ events. This is in line with the findings of Sec. 6 for the response function of price.

One finds that the discrepancy for small tick stocks has two origins. First, due to the intraday non-stationarity of the spread the relationship

$$\lim_{\ell\to\infty} \frac{\overline{\Delta}_{\pi_2}^{\mathrm{R}} \langle I(\pi_t = \pi_1) I(\pi_{t+\ell} = \pi_2)\rangle}{\left\langle \overline{\Delta}_{\pi_2,\epsilon_{t+\ell},t+\ell} I(\pi_t = \pi_1) I(\pi_{t+\ell} = \pi_2)\right\rangle} = 1$$

no longer holds. After excluding the overnight contribution (when t and $t+\ell$ are in different days), the gaps in the denominator are no longer sampled from the first ℓ events of the day, where they are systematically larger than at the end of the day. Even after adjusting Π's to have the correct asymptotic value, one needs to introduce $\alpha > 0$ to find an approximately correct shape of the spread response functions. The results for one example stock (AAPL) are shown in Fig. 21.

Clearly this model is only intended as a first approximation, since it leads to an exponential decay of the spread autocorrelation function in constrast with the long memory found in the data, see Fig. 22. It is possible to give a more complete description of the spread along the lines of Sec. 7.3, but we will leave this for future research.

Appendix B: Plots of various correlation functions

In this appendix we show all the signed and some unsigned correlation functions, signed and unsigned, for small and large tick stocks separately, see Figs. 23-30.

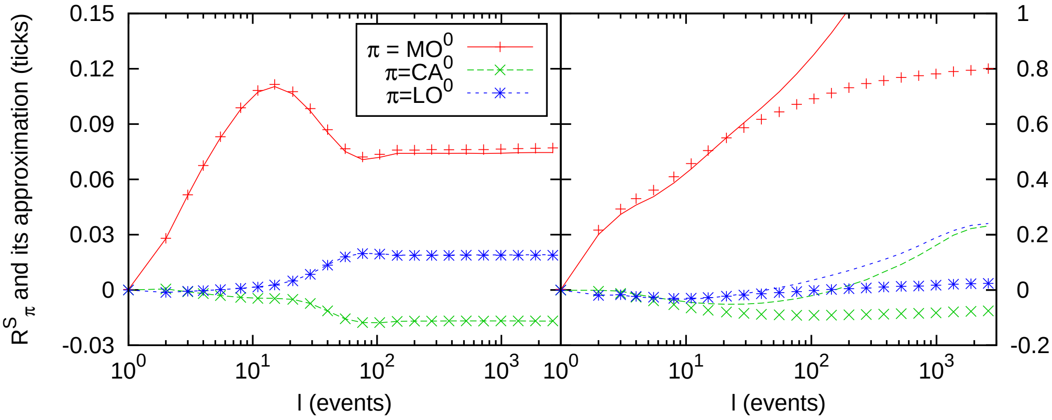

Figure 19: Comparison of true and approximated normalized spread response function $R^S_\pi(\ell)/P(\pi)$ for *(left)* large tick stocks and *(right)* small tick stocks for events that do not change the price. Symbols correspond to the true value, and lines to the approximation. The curves are labeled according to π in the legend.

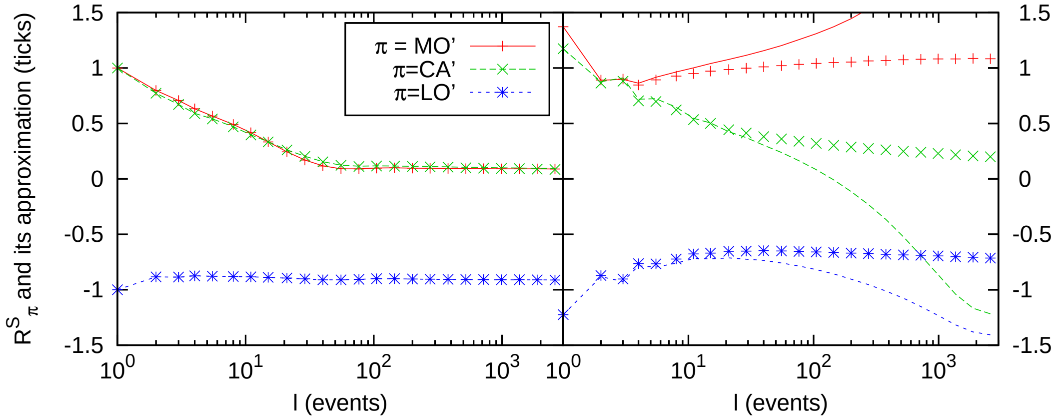

Figure 20: Comparison of true and approximated normalized spread response function $R^S_\pi(\ell)/P(\pi)$ for *(left)* large tick stocks and *(right)* small tick stocks for events that change the price. Symbols correspond to the true value, and lines to the approximation. The curves are labeled according to π in the legend.

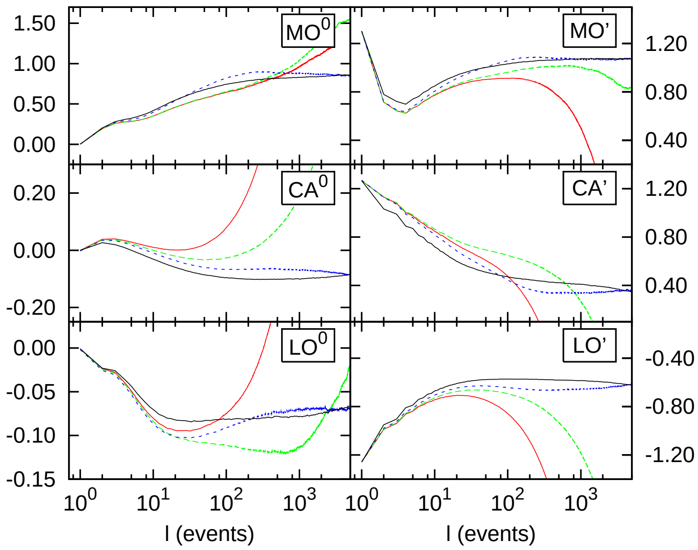

Figure 21: Comparison of the spread response functions of AAPL for four different cases: Eq. (A6) with $\alpha = 0$ (red), Eq. (A6) with adjusted Π's and $\alpha = 0$ (green), Eq. (A6) with adjusted Π's and $\alpha = 10^{-2}$ (blue), true response functions (black). Price in ticks.

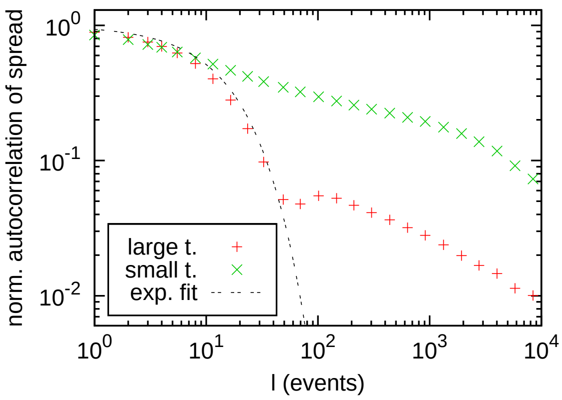

Figure 22: Autocorrelation function of the spread (after removing the beginning and the end of the trading days). One can see that the exponential decay suggested by our simplified model captures the short-time dynamics, but not the slow decay lasting for thousands of events.

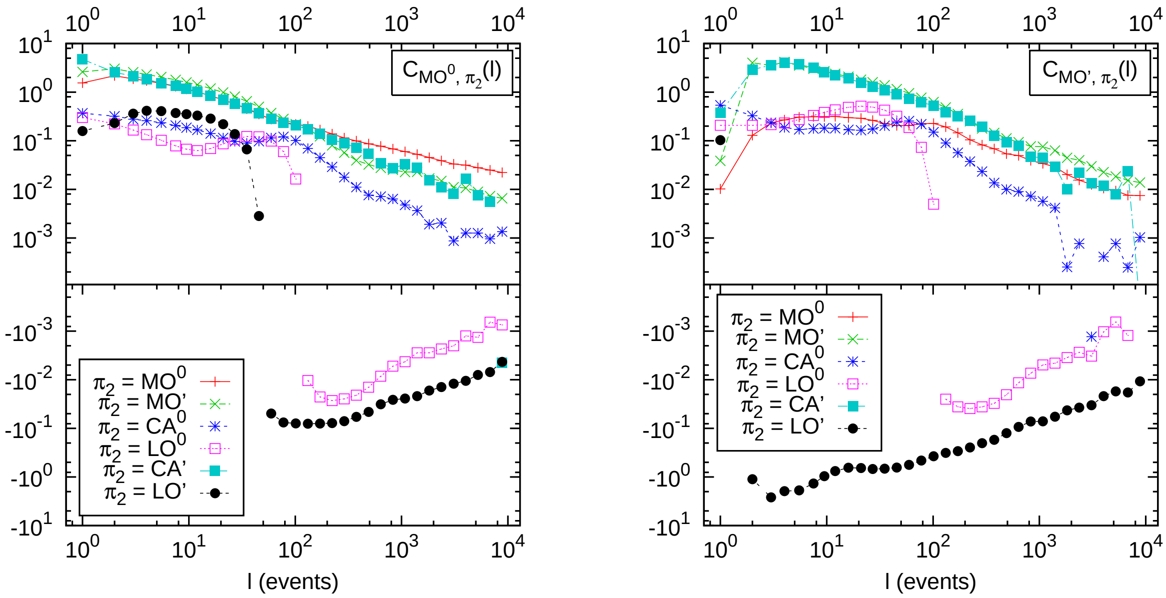

Figure 23: The normalized, signed event correlation functions $C_{\pi_1,\pi_2}(\ell)$ for large tick stocks, *(left)* $\pi_1 = \mathrm{MO}^0$, *(right)* $\pi_1 = \mathrm{MO}'$. The curves are labeled by their respective π_2's in the legend. The bottom panels show the negative values.

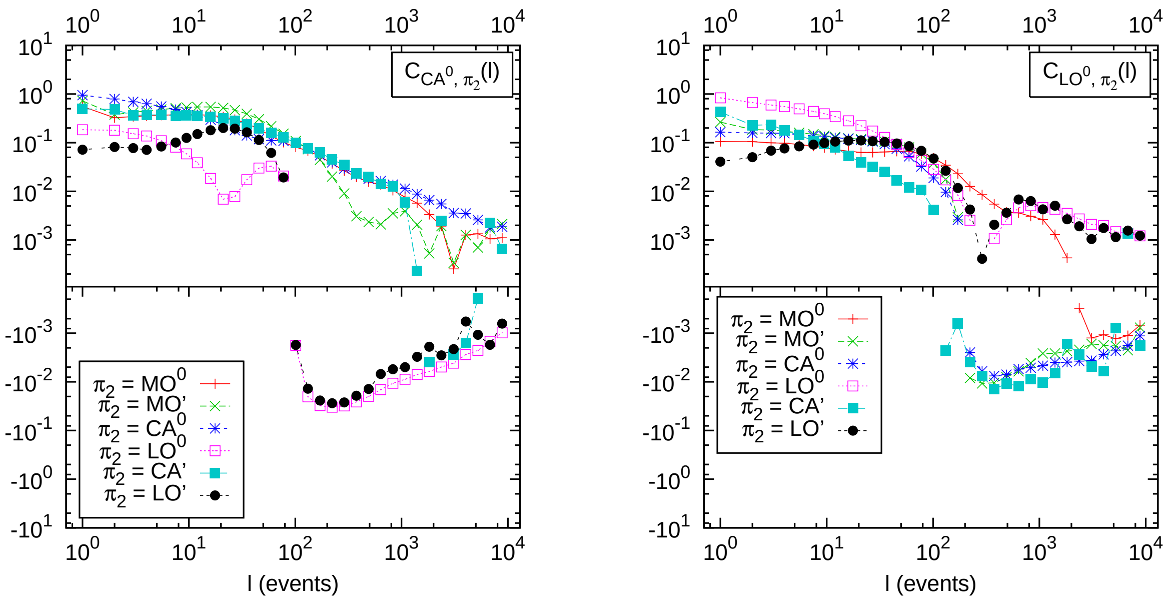

Figure 24: The normalized, signed event correlation functions $C_{\pi_1,\pi_2}(\ell)$ for large tick stocks, *(left)* $\pi_1 = \mathrm{CA}^0$, *(right)* $\pi_1 = \mathrm{LO}^0$. The curves are labeled by their respective π_2's in the legend. The bottom panels show the negative values.

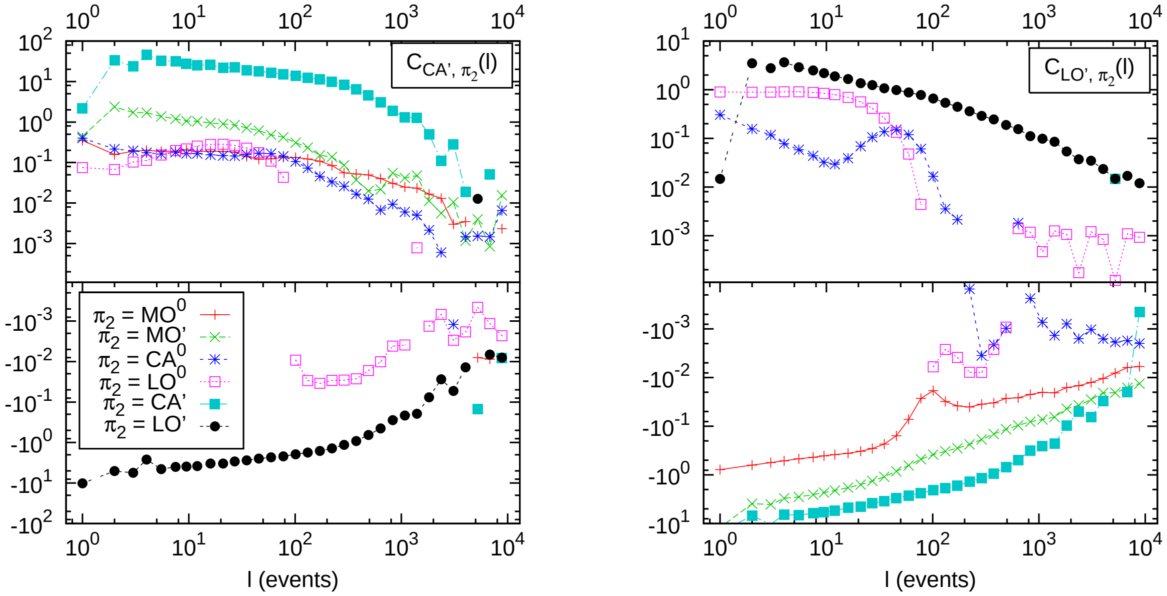

Figure 25: The normalized, signed event correlation functions $C_{\pi_1,\pi_2}(\ell)$ for large tick stocks, *(left)* $\pi_1 = \mathrm{CA}'$, *(right)* $\pi_1 = \mathrm{LO}'$. The curves are labeled by their respective π_2's in the legend. The bottom panels show the negative values.

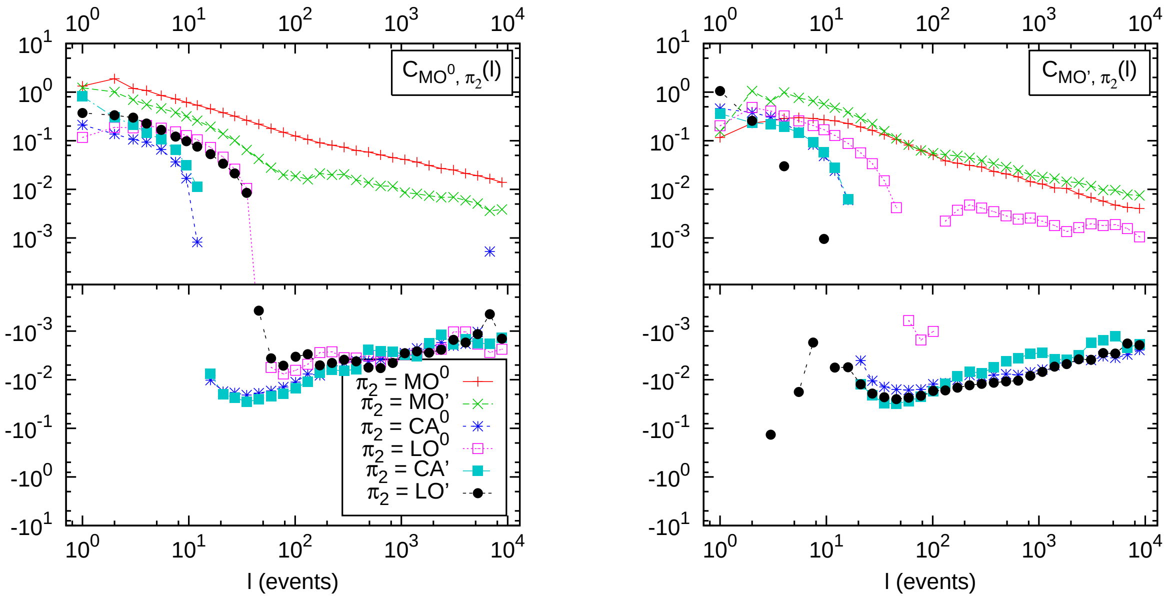

Figure 26: The normalized, signed event correlation functions $C_{\pi_1,\pi_2}(\ell)$ for small tick stocks, *(left)* $\pi_1 = \mathrm{MO}^0$, *(right)* $\pi_1 = \mathrm{MO}'$. The curves are labeled by their respective π_2's in the legend. The bottom panels show the negative values.

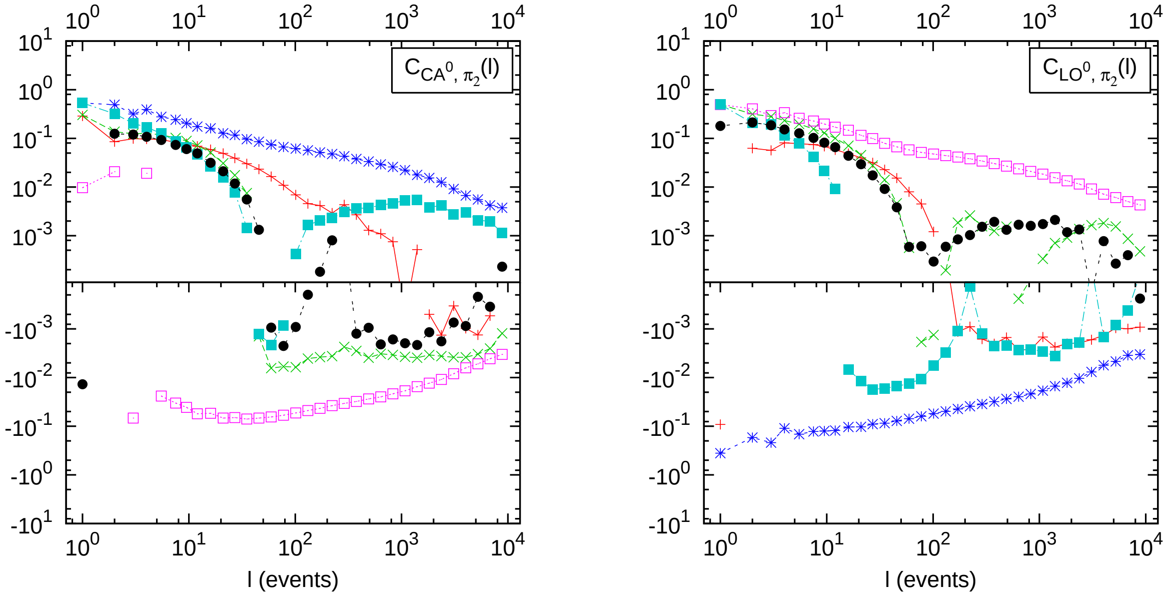

Figure 27: The normalized, signed event correlation functions $C_{\pi_1,\pi_2}(\ell)$ for small tick stocks, *(left)* $\pi_1 = \mathrm{CA}^0$, *(right)* $\pi_1 = \mathrm{LO}^0$. The curves correspond to the six possible values of π_2's, see the legend of Fig. 26 for details. The bottom panels show the negative values.

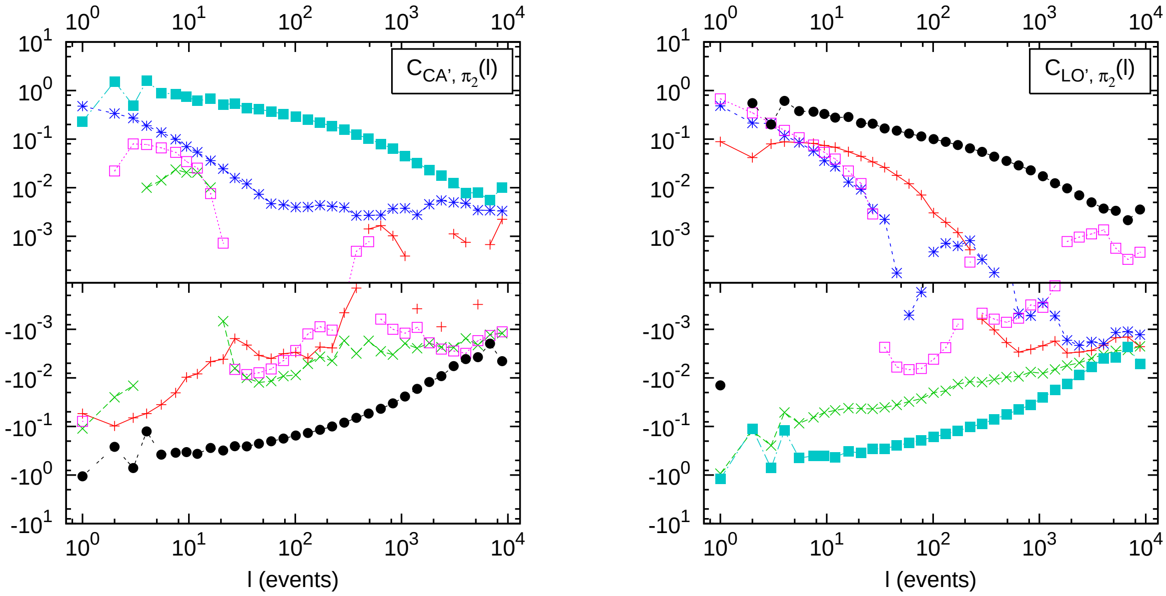

Figure 28: The normalized, signed event correlation functions $C_{\pi_1,\pi_2}(\ell)$ for small tick stocks, *(left)* $\pi_1 = \mathrm{CA}'$, *(right)* $\pi_1 = \mathrm{LO}'$. The curves correspond to the six possible values of π_2's, see the legend of Fig. 26 for details. The bottom panels show the negative values.

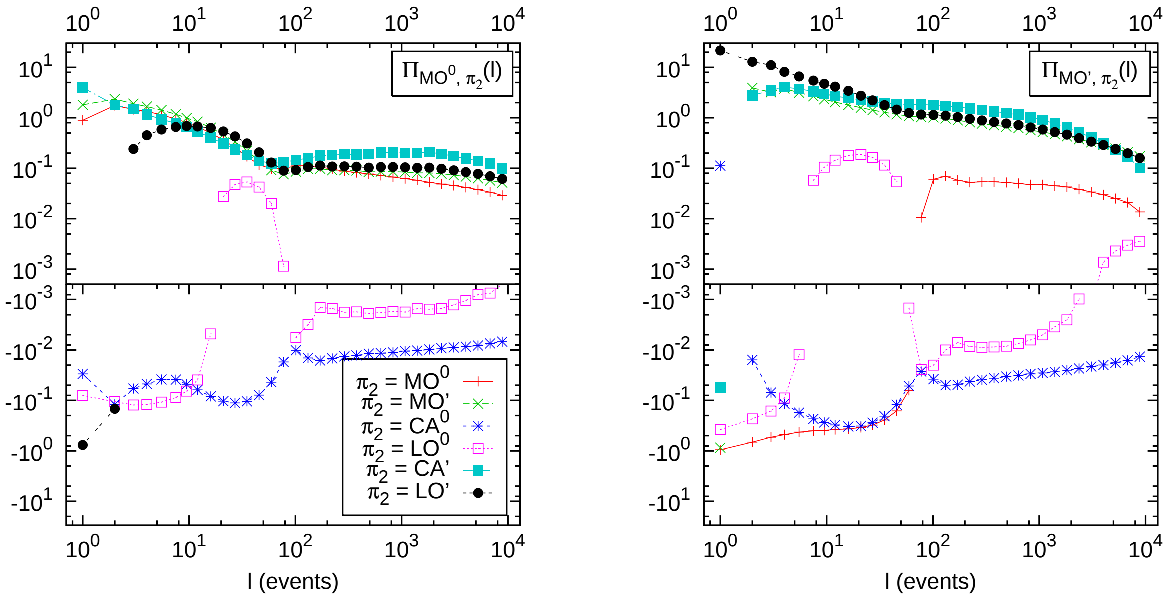

Figure 29: The normalized, unsigned event correlation functions $\Pi_{\pi_1,\pi_2}(\ell)$ for large tick stocks, *(left)* $\pi_1 = \mathrm{MO}^0$, *(right)* $\pi_1 = \mathrm{MO}'$. The curves are labeled by their respective π_2's in the legend. The bottom panels show the negative values.

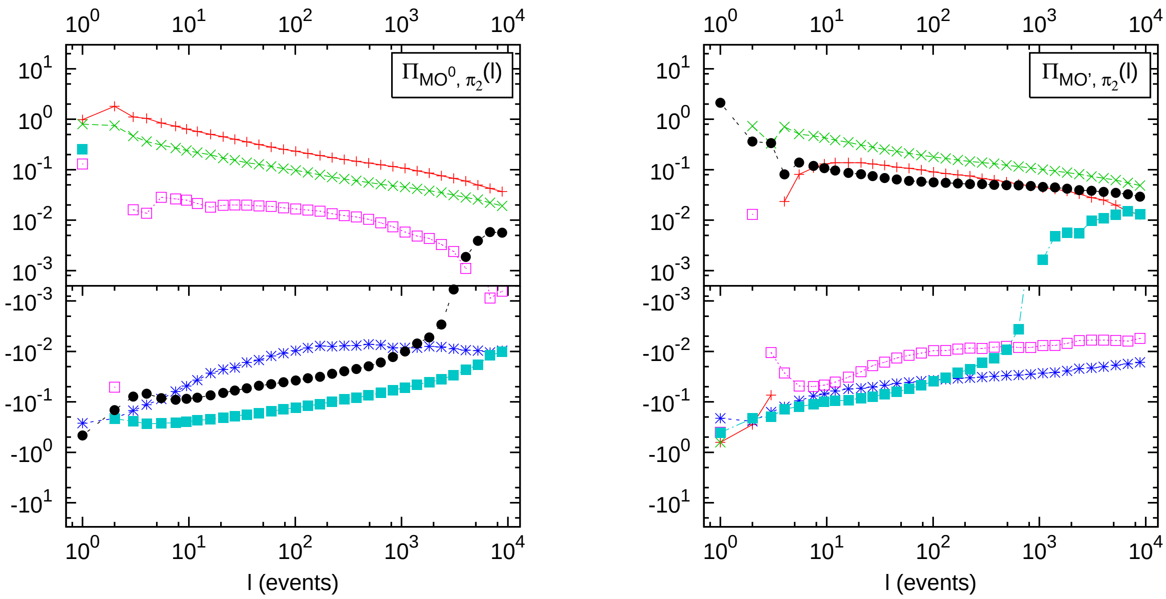

Figure 30: The normalized, unsigned event correlation functions $\Pi_{\pi_1,\pi_2}(\ell)$ for small tick stocks, *(left)* $\pi_1 = \mathrm{MO}^0$, *(right)* $\pi_1 = \mathrm{MO}'$. The curves correspond to the six possible values of π_2's, see the legend of Fig. 29 for details. The bottom panels show the negative values.