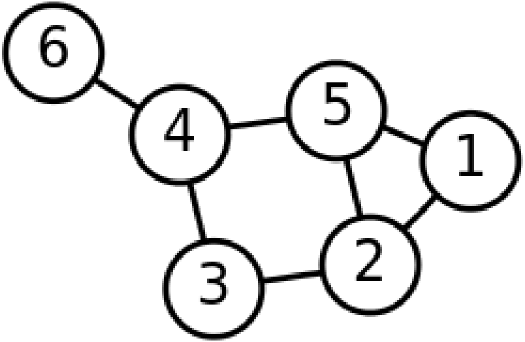
Graphs

July 10, 2014

Definition

A **graph** is a set of objects (**nodes** or **vertices**) where some pairs of objects are connected by links (**edges**).



Graphs

- Edges are represented by ordered pairs of nodes.
- If V is the set of vertices and E is the set of edges, then the graph is G, where $G = (V, E)$.
- $V(G)$ denotes the vertex set of a graph G, and $E(G)$ the edge set of G.

Example: $G = (\{1, 2, 3, 4, 5, 6\}, \{(1, 2), (1, 5), (2, 3), (2, 5), (3, 4), (4, 5), (4, 6)\})$

Kinds of graphs

- In a **directed graph**, the edges have a direction.
- In a **multigraph**, we allow multiple edges between the same pair of vertices.
- A **self-loop** is an edge connecting a vertex to itself.
- In a **simple graph**, there are no multiple edges or self-loops.

Vertex degree

Definition

An edge is **incident** to a vertex v if it links v to another vertex u. We then say u and v are **adjacent** and that they are **neighbors**. The **degree** of a vertex v is denoted $\deg(v)$ and is the number of edges incident to it (A self-loop is counted twice.).

Handshake lemma

Proposition (Degree-sum formula)

In any graph, we have

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Corollary (Handshake lemma)

Every finite undirected graph has an even number of vertices with odd degree.

At any party, the number of people who have shaken an odd number of people's hands is even.

Special Graphs

Definition

The **complete graph** on n nodes is denoted K_n. It has n nodes, and every pair of nodes is linked by an edge. C_n is the cycle graph on n nodes, and W_n is the wheel graph with n spokes and $n+1$ nodes.

Proposition

K_n has $n(n-1)/2$ edges.

Graph isomorphism

Definition

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $f : V_1 \to V_2$ such that

$$(u, v) \in E_1 \leftrightarrow (f(u), f(v)) \in E_2.$$

In other words, two graphs are isomorphic if we can relabel and move the vertices in one to get the other.

Subgraphs

Definition

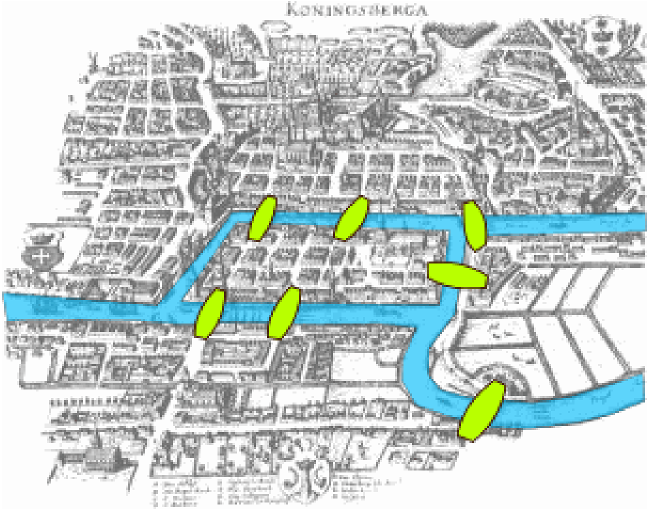
A graph $G_1 = (V_1, E_1)$ is a subgraph of $G_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$.

Proving graphs are not isomorphic

G and H are non-isomorphic if

- G and H have different numbers of vertices or edges.
- The degree sequences (i.e., list of degrees of the vertices) of G and H don't match.
- G contains a subgraph that H doesn't.

Seven bridges of Königsberg



Can you cross all seven bridges without crossing any bridge twice?

Walks, paths, cycles

Definition

A **walk** is an alternating sequence of vertices and edges, beginning and ending with a vertex, where each edge connects the vertex that comes before it to the vertex that comes after it.

Definition

A walk is **closed** if it starts and ends at the same vertex.

Definition

A walk is a **path** if it does not repeat vertices.

Definition

A walk is a **cycle** if it has at least three nodes, and starts and ends at the same vertex, but otherwise does not repeat vertices.

More definitions

Definition

A graph is **acyclic** if it does not contain a cycle as a subgraph.

Definition

A graph is **connected** if every pair of vertices has a path between them.

Definition

Given a vertex $v \in V(G)$, the **connected component** of G that contains v is the union of all paths in G that contain v.

Proposition

Any graph with n nodes and k connected components must have at least $n - k$ edges.

Proof later in the course (tomorrow? next week?)

Eulerian and Hamiltonian paths

Definition

An **Eulerian path** of a graph is a walk that uses each edge exactly once.

Definition

An **Eulerian cycle** of a graph is a closed walk that uses each edge exactly once.

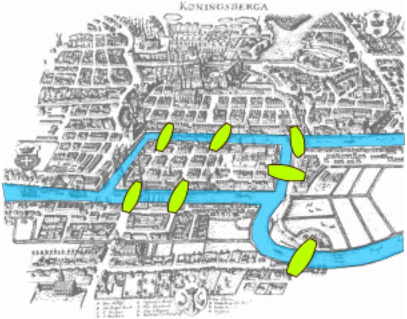
Definition

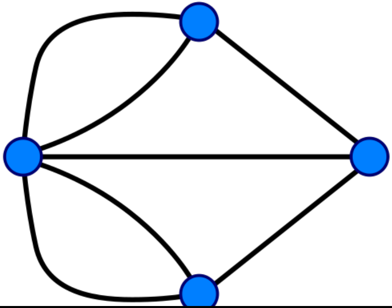
A **Hamiltonian path** of a graph is a path that uses each vertex exactly once.

Definition

A **Hamiltonian cycle** of a graph is a cycle that uses each vertex exactly once.

Bridges of Königsberg again





When do Eulerian cycles exist?

Theorem

A graph has an Eulerian path if and only if at most two of the vertices have odd degree.

Theorem

A graph has an Eulerian cycle if and only if all vertices have even degree.

Proof: "Only if" direction is simple. "If" direction later in the course.

Bipartite graphs

Definition

A graph G is bipartite if $V(G)$ and be divided into two disjoint sets L and R such that all edges connect a vertex in L to one in R.

Definition

In the **complete bipartite graph** $K_{m,n}$, we have $|L| = m$, $|R| = n$, and every pair of vertices (u, v) is also an edge if $u \in L$ and $v \in R$.

How many living, opposite-gender (current or former) sex partners (of the opposite gender) do men have in the United States when compared to women?

2-way bounding

- If A and B are sets, to show that $A = B$, we must show that $A \subseteq B$ and $B \subseteq A$.
- In a 2×2 square we can place at most 4 points such that all the points are at least $\sqrt{2}$ away from each other.

Definition

A graph is k-**colorable** if we can assign colors to the vertices such that we do not use more than k colors and no two adjacent vertices get the same color.

- W_5 is 4-colorable but not 3-colorable.