CS 173: Discrete Structures, Summer 2014
Homework 1

This homework contains 5 problems, each worth 10 points. It is due in class on Wednesday, June 25th. **Please follow the guidelines on the class web page about homework format and style.**

1. **Simplifying**

   Recall that $p \to q \equiv \neg p \lor q$. Use this equivalence and De Morgan's Laws to simplify the following formulas. (In other words, rewrite the following formulas using only p, q, r, $\neg$, $\lor$, and $\land$, such that each $\neg$ only applies to a single simple proposition.)

   (a) $(p \to q) \to r$

   Solution.

   $$\begin{aligned} (p \to q) \to r &\equiv (\neg p \lor q) \to r && (1) \\ &\equiv \neg(\neg p \lor q) \lor r && (2) \\ &\equiv (p \land \neg q) \lor r && (3) \end{aligned}$$

   $\square$

   (b) $p \to (q \to r)$

   Solution.

   $$\begin{aligned} p \to (q \to r) &\equiv \neg p \lor (q \to r) && (4) \\ &\equiv \neg p \lor \neg q \lor r && (5) \end{aligned}$$

   $\square$

2. **Casework**

   On a certain island, each person either always lies or always tells the truth.

   You meet two islanders. Let's call them Alice and Bob. In each of the following situations, what can we say about whether Alice and Bob are liars or truth-tellers?

   (a) Alice says that at least one of the two of them is a liar. Bob says nothing.

   Solution. Either Alice is telling the truth, or Alice is lying.
   Case 1: Suppose Alice is lying. In this case, neither one of them is a liar. This is impossible because Alice is a liar.
   Case 2: Suppose Alice is telling the truth. In this case, at least one of the two of them is a liar. Since Alice is not a liar, Bob must be a liar.
   Having gone through all the cases, we see that the only possibility is that Alice is a truth-teller and Bob is a liar.
   This possibility actually works: If Alice is a truth-teller and Bob is a liar, then at least one of the two of them is a liar, so Alice's statement is true. $\square$

(b) Alice says that they are both truth-tellers. Bob says that Alice is a liar.

Solution. Either Bob is telling the truth or Bob is lying.

Case 1: Suppose Bob is lying. In this case, Alice must be telling the truth, so both Alice and Bob are truth-tellers. This is impossible because Bob is a liar.

Case 2: Suppose Bob is telling the truth. In this case, Alice is a liar.

Having gone through all the cases, we see that the only possibility is that Alice is a liar and Bob is a truth-teller.

This possibility actually works: If Alice is a liar and Bob is a truth-teller, then it is false that they are both truth-tellers, so Alice's statement is false and Bob's is true. □

3. **Negation**

Negate the following propositions:

(a) Either all eleven-footed alligators are blue with orange stripes, or all eleven-footed alligators are black with white polka-dots, or both.

Solution. Let C be the set of all eleven-footed alligators, $E(x)$ mean that x is eleven-footed, $BO(x)$ mean that x is blue with orange stripes, and $BW(x)$ mean that x is black with white polka-dots. The negated proposition is then

$$\begin{aligned} &\neg[(\forall x \in C, BO(x)) \vee (\forall y \in C, BW(x))] && (6)\\ &\equiv \neg[\forall x \in C, BO(x)] \wedge \neg[\forall y \in C, BW(x)] && (7)\\ &\equiv [\exists x \in C, \neg BO(x)] \wedge [\exists y \in C, \neg BW(y)] && (8)\end{aligned}$$

In other words: "There exists an eleven-footed crocodile that is not blue with orange stripes, and there exists a crocodile (not necessarily the same one) that is not black with white polka-dots." □

(b) For every positive integer x, there exist positive integers y and z such that $x < yz$ and $y > 5$.

Solution. The negated proposition is

$$\begin{aligned} &\neg[\forall x \in \mathbb{Z}^+, \exists y \in \mathbb{Z}^+, \exists z \in \mathbb{Z}^+, ((x < yz) \wedge (y > 5))] && (9)\\ &\equiv \exists x \in \mathbb{Z}^+, \forall y \in \mathbb{Z}^+, \forall z \in \mathbb{Z}^+, \neg((x < yz) \wedge (y > 5)) && (10)\\ &\equiv \exists x \in \mathbb{Z}^+, \forall y \in \mathbb{Z}^+, \forall z \in \mathbb{Z}^+, (x \geq yz) \vee (y \leq 5) && (11)\end{aligned}$$

In other words: "There exists a positive integer x such that for all positive integers y and z, either $x \geq yz$ or $y \leq 5$, or both." □

4. **Direct proof**

Prove that if a, b, c, and d are positive real numbers and $a/b = c/d$, then

$$\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}.$$

Solution. Suppose a, b, c, and d are positive real numbers such that $a/b = c/d$. Then,

$$\begin{align}\frac{a}{b} &= \frac{c}{d} \tag{12}\\ \implies ad &= bc \tag{13}\\ \implies ad + cd &= bc + cd \tag{14}\\ \implies d(a+c) &= c(b+d) \tag{15}\\ \implies \frac{a+c}{b+d} &= \frac{c}{d}, \tag{16}\end{align}$$

where the last line is allowed because d and $b + d$ are positive. □

5. **Proof by contrapositive**

Prove that if x and y are positive real numbers and $2y^3 + 5xy^2 \leq 2x^3 + 3x^2y$, then $y \leq x$. Note: If you choose to prove the contrapositive instead, you must clearly state what the contrapositive is before proving it.

Solution. We can rewrite the statement to be proved as:

$$\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+, (2y^3 + 5xy^2 \leq 2x^3 + 3x^2y) \to (y \leq x).$$

The contrapositive of this statement is

$$\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+, (y > x) \to (2y^3 + 5xy^2 > 2x^3 + 3x^2y).$$

To prove this contrapositive, suppose that x and y are positive real numbers such that $y > x$. We know that

$$2y^3 > 2x^3$$

and

$$5xy^2 > 3xy^2 > 3x^2y.$$

Adding these two inequalities yields

$$2y^3 + 5xy^2 > 2x^3 + 3x^2y,$$

as desired. □