Functions, Graphs

July 1, 2014

General homework issues

- **Make sure your homework is neat and understandable.**
- Show your work.
- Don't leave eraser marks.
- Tell me what you're doing. Use complete sentences and introduce variables before using them (e.g., "Let x be..." or "There exists x such that..."). Don't just spit out a bunch of equations and formulas.

3a: Translating words into symbols

- "Either all alligators are orange or all alligators are black."
- (All alligators are orange) $\vee$ (All alligators are white)
- The "all" suggests using $\forall$
- $[\forall x \in C, OR(x)] \vee [\forall y \in C, W(y)]$.

3b: Translating symbols into words

- $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x < y$.
- Is it "There exists a real number x such that for all reals y, $x < y$."?
- Or is it "There exists a real x, for all reals y, such that $x < y$."?

Translating symbols into words

In general:

- $\exists x$ becomes "there exists x such that"
- $\forall x$ becomes "for all x" or "for all x, we have"

5: Backwards proofs

Prove that if $x > y > 0$, then $x^2y > xy^2$.
This is backwards and unacceptable:

Proof.

Suppose $x > y > 0$. Then,

$$x^2y > xy^2 \tag{1}$$
$$\implies x^2y - xy^2 > 0 \tag{2}$$
$$\implies xy(x-y) > 0, \tag{3}$$

which is true. □

5: Backwards proofs

Prove that if $x > y > 0$, then $x^2y > xy^2$.
This is ok, but not recommended:

Proof.

Suppose $x > y > 0$. We need to show

$$x^2y > xy^2 \tag{4}$$
$$\Leftrightarrow x^2y - xy^2 > 0 \tag{5}$$
$$\Leftrightarrow xy(x - y) > 0, \tag{6}$$

Since $x > y$, $x - y$ is positive. Since x, y, and $x - y$ are all positive, (6) is true. □

5: Backwards proofs

Prove that if $x > y > 0$, then $x^2y > xy^2$.
This is recommended:

Proof.

Suppose $x > y > 0$. Since $x, y > 0$, we know that $xy > 0$, so we can multiply both sides of

$$x > y$$

by xy to get

$$x^2y > xy^2.$$

which is what we wanted. □

Important equations are set off on their own lines, and the relationship between equations and statements is clear.

5: Backwards proofs

Prove that if $x > y > 0$, then $x^2y > xy^2$.
This is also ok:

Proof.

Suppose $x > y > 0$. Then,

$$xy(x - y) > 0, \tag{7}$$
$$\implies x^2y - xy^2 > 0 \tag{8}$$
$$\implies x^2y > xy^2 \tag{9}$$

which is what we wanted. □

Pigeonhole Principle

Proposition

If you try to stuff n pigeons into k holes and $n > k$, then at least one hole will contain more than one pigeon.

Generalized Pigeonhole Principle

Proposition

If you try to stuff n pigeons into k holes, then at least one hole will contain at least n/k pigeons.

- "The maximum must be at least the average."
- Example: On New Year's, 300,000 people are in Times Square. At least $300000/366 = 819.7$ people will have the same birthday.

Bijections

Definition

A function $f : A \to B$ is a **bijection** if it is both one-to-one and onto.

Proposition

1. If $f : A \to B$ is onto, then $|A| \geq |B|$.
2. If $f : A \to B$ is one-to-one, then $|A| \leq |B|$.
3. If $f : A \to B$ is a bijection, then $|A| = |B|$.

Proof: By contrapositive and pigeonhole principle.

Inverse functions

Definition

If $f : A \to B$ is a bijection, then we define the inverse function $f^{-1} : B \to A$ by saying that $f^{-1}(y) = x$ if $f(x) = y$.

Since f is onto, there is at least one such x. Since f is one-to-one, there is only one such x.

Permutations

- There are 6 objects and I want to take 3 of them and arrange them in a row. How many arrangements are possible?
- There are n objects and I want to take k of them and arrange them in a row. How many arrangements are possible?
- Suppose $|A| = |B| = n$. How many bijections are there from A to B?
- Suppose $|A| = m$, $|B| = n$, and $m \leq n$. How many one-to-one functions are there from A to B?

More permutations

- How many ways can 7 people stand in line if Alice and Bob will not stand next to each other?
- How many ways can 7 people stand in line facing forward or backward if no two people standing next to each other can face one another?
- How many ways can 7 people stand in a circle?

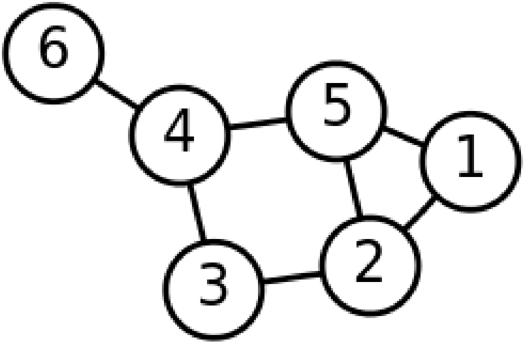
Multinomial coefficients

- How many possible distinct arrangements are there of the letters in the word "DOG"?
- How many possible distinct arrangements are there of the letters in the word "TATTER"?
- How many possible distinct arrangements are there of the letters in the word "MISSISSIPPI"?

Graphs

Definition

A **graph** is a set of objects (**nodes** or **vertices**) where some pairs of objects are connected by links (**edges**).



Graphs

- Edges are represented by ordered pairs of nodes.
- If V is the set of vertices and E is the set of edges, then the graph is G, where $G = (V, E)$.

Example: $G =$
$(\{1, 2, 3, 4, 5, 6\}, \{(1, 2), (1, 5), (2, 3), (2, 5), (3, 4), (4, 5), (4, 6)\})$

Kinds of graphs

- In a **directed graph**, the edges have a direction.
- In a **multigraph**, we allow multiple edges between the same pair of vertices.
- A **self-loop** is an edge connecting a vertex to itself.
- In a **simple graph**, there are no multiple edges or self-loops.
- **Important: Unless otherwise indicated, assume graphs are simple and undirected.**

Vertex degree

Definition

An edge is **incident** to a vertex v if it links v to another vertex u. We then say u and v are **adjacent** and that they are **neighbors**. The **degree** of a vertex v is denoted $\deg(v)$ and is the number of edges incident to it.

Proposition (Degree-sum formula)

In any graph, we have

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Corollary (Handshake Lemma)

At any party, the number of people who have shaken an odd number of people's hands is even.

Special Graphs

Definition

The **complete graph** on n nodes is denoted K_n. It has n nodes, and every pair of nodes is linked by an edge. C_n is the cycle graph on n nodes, and W_n is the wheel graph with n spokes and $n+1$ nodes.

Proposition

K_n has $n(n-1)/2$ edges.

Graph isomorphism

Definition

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $f : V_1 \to V_2$ such that $(u, v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2$.

In other words, two graphs are isomorphic if we can relabel and move the vertices in one to get the other.

Subgraphs

Definition

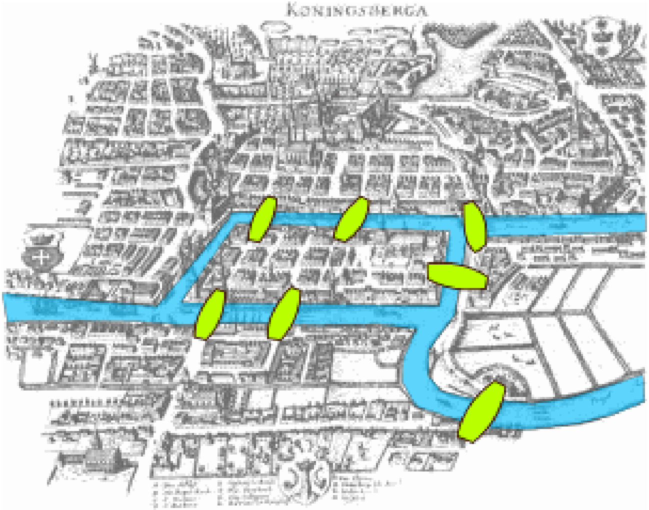
A graph $G_1 = (V_1, E_1)$ is a subgraph of $G_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$.

Proving graphs are not isomorphic

G and H are non-isomorphic if

- G and H have different numbers of vertices or edges.
- The degree sequences (i.e., list of degrees of the vertices) of G and H don't match.
- G contains a subgraph that H doesn't.
- Other things? Graph isomorphism is an unsolved problem...

Seven bridges of Konigsberg



Can you cross all seven bridges without crossing any bridge twice?