

# CS 173, Spring 2013, B Lecture

## Midterm 1, 19 February 2013

**NAME:**

**NETID** (e.g. hpotter23, not 314159265):

**Circle your discussion:**

**Th 2      Th 3      Th 4      Th 5      Fri 9      Fri 10      Fri 11**

You will lose a point if don't circle the discussion you are officially registered for. Check the rosters at the podium if you aren't sure. Check here ☐ if you have changed section within the last week.

Problem	1	2	3	4	5	6	7	Total
Possible	15	15	12	14	12	15	17	100
Score								

**We will be checking photo ID's during the exam. Have your ID handy.**  
(Forgot your ID? See us at the end of the exam.)

**Turn in your exam at the front when you are done.**

**You have 75 minutes to finish the exam.**

## INSTRUCTIONS (read carefully)

- There are 7 problems, each on a separate page. Make sure you have a complete exam. The point value of each problem is indicated next to the problem, and on the cover page table.
- Points may be deducted for solutions which are correct, but hard to read, hard to understand, poorly explained, or excessively complicated.
- When writing proofs, use your best mathematical style, e.g. introduce variables and assumptions, justify important or non-obvious steps, put your steps in logical order.
- Brief explanations and/or showing work, even when not explicitly asked for, may increase partial credit for buggy answers. Partial credit for multiple-choice questions is very rare.
- It is not necessary to simplify or calculate out complex constant expressions such as  $(0.7)^3(0.3)^5$ ,  $\frac{0.15}{3.75}$ ,  $3^{17}$ , and  $7!$ , unless it is explicitly indicated to completely simplify.
- It is wise to skim all problems and point values first, to best plan your time.
- This is a closed book exam.  
Turn off your cell phone now.  
No notes or electronic devices of any kind are allowed.  
These should be secured in your bag and out of reach during the exam.
- Do all work in the space provided, using the backs of sheets if necessary.  
If your work is on the backside then you must clearly indicate so on the problem.  
See the proctor if you need more paper.
- Please bring any apparent bugs or ambiguity to the attention of the proctors.
- After the midterm is over, you may discuss its contents with other students **only** after verifying that they have also taken the exam (e.g. they aren't about to take a makeup exam).

## Problem 1: Multiple choice (15 points)

Check the most appropriate box for each statement. Check only one box per statement. If you change your answer, make sure it's easy to tell which box is your final selection.

$$\neg p \rightarrow \neg q \equiv \neg q \vee p$$

True ☒ False ☐

$$\forall x \in \mathbb{R}, |x| < -3 \rightarrow x < -5$$

true ☒ undefined truth value ☐

false ☐

$$-7 \equiv 13 \pmod{6}$$

true ☐ false ☒

$$\text{lcm}(5, 15)$$

5 ☐ 15 ☒

45 ☐ 75 ☐

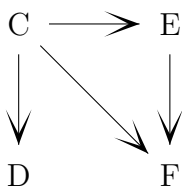
$$\forall p, q, k \in \mathbb{Z}^+, \gcd(pq, qk) =$$

$q$  ☐  $pq$  ☐

$pqk$  ☐  $q \gcd(p, k)$  ☒

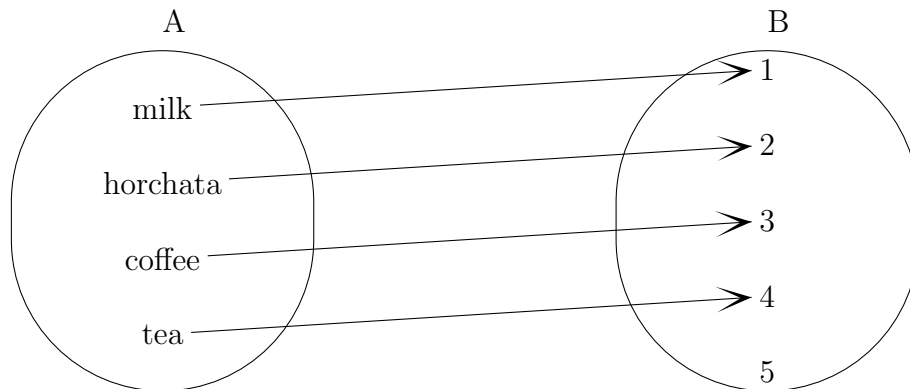
## Problem 2: Short answer (15 points)

(a) (10 points) Check all boxes that correctly characterize this relation on the set  $\{A, B, C, D, E, F\}$



Reflexive:	<input type="checkbox"/>	Irreflexive:	<input type="checkbox"/>
Symmetric:	<input type="checkbox"/>	Antisymmetric:	<input checked="" type="checkbox"/>
Transitive:	<input checked="" type="checkbox"/>		

(b) (5 points) The following picture shows the contents of a set A. Complete it to make an example of a function from A to B that is one-to-one but not onto. To complete the picture, add elements to set B (represent each element in B with an integer) and draw arrows showing how input values map to output values.



### Problem 3: Short answer (12 points)

- (a) (6 points) Let  $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = (x - 1)^2$  and  $g : \mathbb{R} \rightarrow \mathbb{R}, g(x) = 2x$ . Write the expression for  $(g \circ f)(x)$ , and compute  $(g \circ f)(5)$ . Show your work.

$$(g \circ f)(x) = g(f(x)) = g((x - 1)^2) = 2(x - 1)^2$$

$$(g \circ f)(5) = 2(5 - 1)^2 = 32$$

- (b) (6 points) Suppose we have the following sets:

$$A = \{\text{fox}, \text{cat}\}$$

$$B = \{3, 4\}$$

$$C = \{3, 7\}$$

$$A \times (B \cap C) = \{\text{fox}, \text{cat}\} \times \{3\} = \{(\text{fox}, 3), (\text{cat}, 3)\}$$

$$\{\text{red}\} \times (B \cup C) = \{\text{red}\} \times \{3, 4, 7\} = \{(\text{red}, 3), (\text{red}, 4), (\text{red}, 7)\}$$

$$A \cap C \cap B = \emptyset$$

#### Problem 4: Short answer (14 points)

- (a) (6 points) State the contrapositive of the following claim. Your answer should be in words, with all negations (e.g. “not”) on individual predicates.

For every cat  $c$ , if  $c$  is large, then  $c$  is cuddly and  $c$  is a pet.

*Solution:* For every cat  $c$ , if  $c$  is not cuddly or  $c$  is not a pet, then  $c$  is not large.

- (b) (8 points) In  $\mathbb{Z}_{11}$ , find the value of  $[7]^{15}$ . You must show your work, keeping all numbers in your calculations small. **You may not use a calculator.** You must express your final answer as  $[n]$ , where  $0 \leq n \leq 10$ .

*Solution:*

$$[7]^{15} = ([7]^2)^7 [7] = [49^7][7] = [5^7][7] = [(5^2)^3][5][7] = [3^3][5][7] = [5][2] = [10]$$

### Problem 5: Relations (12 points)

- (a) (6 points) A relation is a partial order if it has which three properties? (Naming the properties is sufficient. You don't have to define them.)

*Solution:* antisymmetric, transitive, reflexive

- (b) (6 points) Suppose that  $R$  is the relation on the set of integers such that  $aRb$  if and only if  $\gcd(a,b)=3$ . Is  $R$  transitive? Informally explain why it is, or give a concrete counter-example showing that it is not.

*Solution:* No,  $R$  is not transitive. For example,  $\gcd(6, 9) = 3$  and  $\gcd(9, 12) = 3$  but  $\gcd(6, 12) = 6$ . So  $6R9$  and  $9R12$  but not  $6R12$ .

### Problem 6: Proof (15 points)

Recall that  $[s, t]$  is a closed interval of the real line. You can assume that  $s \leq t$  for any closed interval. Let  $I$  be the set containing all closed intervals  $[s, t]$ . Finally, the relation  $F$  on  $I$  is defined by

$$[s, t]F[p, q] \text{ if and only if } q \leq s$$

There are two equivalent definitions for what it means for  $R$  defined on set  $A$  to be antisymmetric:

Definition 1:  $\forall x, y \in A$ , if  $x \neq y$  and  $xRy$ , then  $y \not Rx$

Definition 2:  $\forall x, y \in A$ , if  $xRy$  and  $yRx$ , then  $x = y$

Prove that  $F$  is antisymmetric using one of these definitions.

*Solution:* We need to show that if  $[s, t]F[p, q]$  and  $[p, q]F[s, t]$ , then  $[s, t] = [p, q]$ . Suppose  $[s, t]F[p, q]$  and  $[p, q]F[s, t]$ . By definition of closed interval,  $s \leq t$  and  $p \leq q$ . Also, by definition of  $F$ ,  $q \leq s$  and  $t \leq p$ . So  $s \leq t \leq p$  and  $p \leq q \leq s$ . Since  $s \leq p$  and  $p \leq s$ ,  $s = p$ . Therefore, since  $s \leq t \leq p$ ,  $t = s = p$ . Similarly, since  $p \leq q \leq s$ ,  $p = q = s$ . So  $[s, t] = [p, q]$ . So  $F$  is antisymmetric by definition. QED.



Write your netID, in case this page gets pulled off:

### Problem 7: Proof (17 points)

$$S = \{(a, b) \in \mathbb{Z}^2 \mid a \equiv 4 \pmod{7} \text{ and } b \equiv 3 \pmod{7}\}$$

$$T = \{(p, q) \in \mathbb{Z}^2 \mid 2p + 3q \equiv 3 \pmod{7}\}$$

Congruence mod  $k$  can be defined as follows: if  $s, t, k$  are integers,  $k$  positive, then  $s \equiv t \pmod{k}$  if and only if  $s = t + nk$  for some integer  $n$ . Using this definition, prove that  $S \subseteq T$ .

Use your best mathematical style, e.g. introduce variables and assumptions, justify important or non-obvious steps, put your steps in logical order. Use only the given definition and algebra, not any facts you might know about modular arithmetic.

*Solution:* I need to show that an arbitrary element of  $S$  must be an element of  $T$ . Let  $(a, b)$  be an arbitrary element of  $S$ . Then, by definition of  $S$ ,  $a \equiv 4 \pmod{7}$  and  $b \equiv 3 \pmod{7}$ . So  $a = 7k + 4$  for some integer  $k$ , and  $b = 7j + 3$  for some integer  $j$  by definition of congruence mod  $k$ . So, by substitution,  $2a + 3b = 14k + 8 + 21j + 9 = 14k + 21j + 17 = 3 + (2k + 3j + 2)7$ .  $2k + 3j + 2$  is integer because  $k$  and  $j$  are integers. So  $2a + 3b \equiv 3 \pmod{7}$  by definition of congruent mod  $k$ . So  $(a, b)$  is an element of  $T$  by definition of  $T$ . QED.