CS 173: Discrete Structures, Spring 2013
Exam 1 Review Solutions

1. **Set Operations**

   Suppose you were given the following sets:

$$\begin{aligned}
\mathbf{A} &= \{\text{Piano}, (\text{Violin}, \text{Viola}, \text{Cello}), \text{Guitar}\} \\
\mathbf{B} &= \{(\text{Flute, Piccolo}), \text{Cymbals}\} \\
\mathbf{C} &= \{\text{Piano}, \text{Flute}\} \\
\mathbf{D} &= \{(\text{Violin, Viola, Cello}), (\text{Flute, Piccolo})\}
\end{aligned}$$

   List the elements of the set or find the values for the following expressions:

   (a) $|A|$
   Solution: 3

   (b) $A \cup D$
   Solution: { Piano, (Violin, Viola, Cello), Guitar, (Flute,Piccolo) }

   (c) $A \cap C$
   Solution: { Piano }

   (d) $B \cap C$
   Solution: $\emptyset$

   (e) $A - (B - C)$
   Solution: $(B - C) =$ { (Flute, Piccolo), Cymbals }
   $A - (B - C) =$ {Piano, (Violin, Viola, Cello), Guitar}

   (f) $(B \cap D) \times C$
   Solution: $(B \cap D) =$ {(Flute, Piccolo)}
   $(B \cap D) \times C$ = { ((Flute, Piccolo), Piano), ((Flute, Piccolo), Flute) }
   = { (Flute, Piccolo, Piano), (Flute, Piccolo, Flute) }

   (g) $A \times \emptyset$
   Solution: $\emptyset$

   (h) $C \times \{\emptyset\}$
   Solution: $\{(\text{Piano}, \emptyset), (\text{Flute}, \emptyset)\}$

2. **Counting with sets**

   In our role-playing game, an evil character may be an elf or a troll, it may be red, green, brown, or black, and it may have scales or hair. A good character may be an elf or a human or a lion, it may be green, brown, or blue, and it has hair or fur. When we look at a character, we can't see whether it is good or evil. How many choices do we have for its appearance?

   Solution: There are $2 \cdot 4 \cdot 2 = 16$ types of evil characters. There are $3 \cdot 3 \cdot 2 = 18$ types of good characters. But there are 2 types of characters that could be good or evil. So we have a total of $16 + 18 - 2 = 32$ possible appearances.

   Suppose we have a 26 character alphabet. How many 6-letter strings start with PRE or end in TH?

   Solution: There are 26^3 strings starting with PRE, 26^4 strings ending in TH, and 26 strings that start with PRE and end in TH. Thus we have a total of $26^3 + 26^4 - 26 = 26(26^2 + 26^3 - 1)$ total strings.

3. **Euclidean algorithm**

   Trace the execution of the Euclidean algorithm for computing GCD on the inputs $a = 837$ and $b = 2015$. That is, give a table showing the values of the main variables (x, y, r) for each pass through the loop. Explicitly indicate what the output value is.

   Solution:

x	y	r
837	2015	837
2015	837	341
837	341	155
341	155	31
155	31	0
31	0	

   Therefore, the algorithm outputs GCD(837, 2015) = 31. Note that the algorithm terminates when y = 0, **not** when r = 0.

4. **Direct Proof Using Congruence mod k**

   In the book, you will find several equivalent ways to define congruence mod k. For this problem, use the following definition: for any integers x and y and any positive integer m, $x \equiv y \pmod{m}$ if there is an integer k such that $x = y + km$.

   Using this definition prove that, for all integers a, b, c, p, q where p and q are positive, if $a \equiv b \pmod{p}$ and $c \equiv b \pmod{q}$ and $q|p$, then $a - 2c \equiv (-b) \pmod{q}$.

   Solution:
   Let a, b, c, p, q be integers, where p and q are positive. Suppose that $a \equiv b \pmod{p}$ and $c \equiv b \pmod{q}$ and $q|p$. By the given definition of congruence, $a = b + pr$ and

$c = b + qt$, where r and t are integers. Since $q|p$, we know that $p = qu$, where u is an integer.

Therefore, by substituting $b + pr$ for a and $b + qt$ for c:

$$a - 2c = b + pr - 2(b + qt)$$

By substituting qu for p, we get:

$$\begin{aligned} a - 2c &= b + qur - 2(b + qt) \\ &= b + qur - 2b - 2qt \\ &= (-b) + q(ur - 2t) \\ &= (-b) + qw \end{aligned}$$

where $w = ur - 2t$. By closure, w must be an integer. Therefore, by the definition given for congruence, $a - 2c \equiv (-b) \pmod{q}$.

5. **Equivalence classes**

   Let $A = \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} - \{(0,0)\}$, i.e. pairs of non-negative reals in which no more than one of the two numbers is zero.

   Consider the equivalence relation $\sim$ on A defined by

   $$(x, y) \sim (p, q) \quad \text{iff} \quad (xy)(p + q) = (pq)(x + y)$$

   (a) List four elements of $[(3, 1)]$. Hint: what equation do you get if you set (x, y) to $(3, 1)$ and $q = 2p$?

   (b) Give two other distinct equivalence classes that are not equal to $[(3, 1)]$.

   (c) Describe the members of $[(0, 4)]$.

   Solutions:

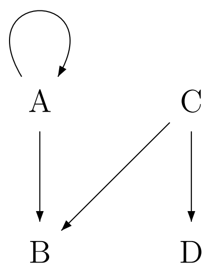
   (a) $(3, 1)$, $(1, 3)$, $(\frac{9}{8}, \frac{9}{4})$, $(\frac{9}{4}, \frac{9}{8})$. You can find a range of other elements by setting q to other multiples of p.

   (b) For example, $[(3, 2)]$, $[(3, 4)]$

   (c) All pairs of the form $(0, y)$ or $(x, 0)$.

   If $(x, y) = (0, 4)$, then the equation $(xy)(p+q) = (pq)(x+y)$ reduces to $0(p+q) = (pq)4$. So this means either p or q must also be zero and, then, it doesn't matter what value we give to the other.

6. **Relation properties**



Reflexive: $\square$ **Irreflexive**: $\square$

Symmetric: $\square$ **Antisymmetric**: $\boxed{\surd}$

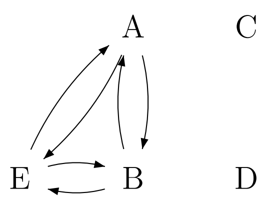
Transitive: $\boxed{\surd}$

$\sim$ is the relation on $\mathbb{R}$ such that $x \sim y$ if and only if $xy = 1$

Reflexive: $\square$ **Irreflexive**: $\square$

Symmetric: $\boxed{\surd}$ **Antisymmetric**: $\square$

Transitive: $\square$



Reflexive: $\square$ **Irreflexive**: $\boxed{\surd}$

Symmetric: $\boxed{\surd}$ **Antisymmetric**: $\square$

Transitive: $\square$

7. **Proofs on Relations**

(a) Define a relation $\sim$ on the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ by the rule $f \sim g$ if and only if $\exists k \in \mathbb{R}$ such that $f(x) = g(x)$ for every $x \geq k$. Prove that $\sim$ is an equivalence relation. **Hint**: each part of this proof is quite brief.

Solution: Let f, g, and h be functions from $\mathbb{R}$ to $\mathbb{R}$.

Reflexive: Note that f is equal to itself on all inputs, so, in particular, $f(x) = f(x)$ for all $x \geq 0$. Thus, $f \sim f$, and the relation is reflexive.

Symmetric: Suppose $f \sim g$. Then there exists $k \in \mathbb{R}$ such that $f(x) = g(x)$ for all $x \geq k$. But then $g(x) = f(x)$ for all $x \geq k$. So $g \sim f$, and the relation is reflexive.

Transitive: Suppose $f \sim g$ and $g \sim h$. Then there exist real numbers k and k' such that $f(x) = g(x)$ for all $x \geq k$ and $g(x) = h(x)$ for all $x \geq k'$. Let $K = \max(k, k')$. Then, we see that $f(x) = g(x) = h(x)$ for all $x \geq K$. So $f(x) = h(x)$ for all $x \geq K$. Thus, $f \sim h$, and the relation is transitive.

Since the relation is reflexive, symmetric, and transitive, it is an equivalence relation.