CS 173: Discrete Structures, Summer 2014
Homework 3

This homework contains 5 problems and is due in class on Wednesday, July 16th. **Please follow the guidelines on the class web page about homework format and style.**

In all questions, you must explain how you get your answers. Stating the answer with no supporting work will not receive full credit.

1. **Hypercubes [15 points]**

   The **n-dimensional hypercube graph** is denoted Q_n and is defined as follows:

   - The vertex set of Q_n is $\{0,1\}^n$. In other words, each vertex is an n-tuple whose entries are all 0 or 1.
   - Given two vertices u and v, there is an edge connecting u and v if u and v differ in exactly one entry. For example, in Q_3, there is an edge between $(0,0,0)$ and $(0,0,1)$, but no edge between $(0,0,0)$ and $(0,1,1)$.

   For example, the vertices of Q_2 are $(0,0),(0,1),(1,0)$, and $(1,1)$. If we let these vertices be a,b,c, and d, respectively, then the edges in Q_2 are $(a,b),(b,d),(d,c)$, and (c,a).

   In this problem, you may find it helpful to omit commas and parentheses when writing down vertices. For example, instead of writing $(0,1,0)$, you can just write 010 instead.

   (a) We can easily see that Q_n has 2^n nodes. But how many edges does Q_n have? [Hint: Do not solve the recurrence in section 12.4 in the textbook. Use the degree-sum formula instead.]

   Solution. Suppose $v \in V(Q_n)$. Node v is adjacent to any node u such that u and v differ in one entry. Since v has n entries and there are only two possible values for each entry, v has n neighbors, so $\deg(v) = n$.
   By the degree-sum formula,

$$
\begin{aligned}
2|E(Q_n)| &= \sum_{v \in V(Q_n)} \deg(v) && (1)\\
&= \sum_{v \in V(Q_n)} n && (2)\\
&= |V(Q_n)|n && (3)\\
&= n2^n. && (4)
\end{aligned}
$$

   Thus, $|E(Q_n)| = n2^{n-1}$. $\square$

   (b) Show that Q_n is 2-colorable.

Solution. Let the colors be red and blue. For any v, if the sum of the entries of v is odd, we color it red. If the sum is even, we color it blue.

We need to show that no two adjacent vertices are given the same color. Suppose u and v are two adjacent vertices. Then, they must differ in exactly one entry, say the i-th entry.

Without loss of generality, suppose the i-th entry of u is 0 and the i-th entry of v is 1 (otherwise, just swap u and v). Then, the sum of the entries of v is one more than the sum of the entries of u, so the two sums must have different parities, and thus u and v are given different colors. □

2. **Connectivity [10 points]**

Let G be a simple graph with n nodes, where n is even. Prove or disprove: If every node of G has degree at least $n/2$, then G is connected.

Solution. The claim is true. Suppose every node in G has degree at least $n/2$, and suppose u and v are two vertices in G. We must show that there is a path from u to v in G. There are two cases.

Case 1: Node u is adjacent to node v. In this case, $u, (u, v), v$ is a path from u to v in G.

Case 2: Nodes u and v are not adjacent. Let S be the set of nodes adjacent to u and T be the set of nodes adjacent to v. Then, S contains neither u nor v, and T contains neither u nor v. That is, $|S \cup T| \leq n - 2$.

Both u and v have degree at least $n/2$, so $|S|, |T| \geq n/2$. By the principle of inclusion-exclusion,

$$|S \cap T| = |S| + |T| - |S \cup T| \geq n/2 + n/2 - (n - 2) = 2.$$

This implies that there exists a vertex w that is in both S and T and is thus adjacent to both u and v. Hence, $u, (u, w), w, (w, v), v$ is a path from u to v.

Since □

3. **Horses [5 points]**

Here is a proof by induction that all horses are the same color.

Proof. Let

$$P(n): \text{ In any group of } n \text{ horses, all the horses are the same color.}$$

We want to prove that $P(n)$ is true for any $n \in \mathbb{Z}^+$.

Base Case ($n = 1$): If there is only one horse in the group, then all horses in the group are the same color. Thus, $P(1)$ is true.

Inductive Step: Assume that $P(k)$ is true. We want to prove that $P(k + 1)$ is true as well.

Suppose we have a group of $k + 1$ horses. Let's say the horses are $h_1, \ldots, h_{k+1}$.

By the inductive hypothesis, horses $h_1, \ldots, h_k$ are all the same color, and horses $h_2, \ldots, h_{k+1}$ are all the same color.

Thus, horse h_1 is the same color as horses $h_2, \ldots, h_k$, which are all the same color as horse h_{k+1}, so all of the $k+1$ horses are the same color. Hence, $P(k+1)$ is true. $\square$

Briefly explain what is wrong with this proof. A sentence or two should be sufficient.

Solution. The inductive step fails when $k = 1$. In particular, it does not make sense to say that horse h_1 is the same color as horses $h_2, \ldots, h_k$, which is the same color as horse h_{k+1}. $\square$

4. **Fibonacci numbers [10 points]**

The Fibonacci numbers are defined as follows:

$$F_0 = 0, F_1 = 1 \tag{5}$$
$$F_n = F_{n-1} + F_{n-2} \text{ if } n \geq 2. \tag{6}$$

For example, $F_2 = 1, F_3 = 2, F_4 = 3$, $F_5 = 5$, and $F_6 = 8$.

Use induction to prove that if $n \in \mathbb{Z}^+$, then

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

Solution. Let

$$P(n) : \sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

We will prove using induction that $P(n)$ is true for all $n \in \mathbb{Z}^+$.

<u>Base case ($n = 1$)</u>: We have

$$\sum_{i=1}^{1} F_i^2 = 1^2 = (1)(1) = F_1 F_2,$$

so $P(1)$ is true.

<u>Inductive step</u>: Assume that $P(k)$ is true. We have

$$\begin{aligned}
\sum_{i=1}^{k+1} F_i^2 &= \sum_{i=1}^{k} F_i^2 + F_{k+1}^2 && (7)\\
&= F_k F_{k+1} + F_{k+1}^2 && (8)\\
&= F_{k+1}(F_k + F_{k+1}) && (9)\\
&= F_{k+1} F_{k+2}, && (10)
\end{aligned}$$

so $P(k+1)$ is true as well. $\square$

5. **Circles [10 points]**

I draw n circles, not necessarily of the same size, in a plane in order to divide it into as many regions as possible. Prove using induction that I can make (at least) $n^2 - n + 2$ regions. [In fact, $n^2 - n + 2$ regions is the most possible if $n \geq 1$, but you don't have to prove this.]

Solution. Fix some point in the plane and call it the origin. Let R_n be the maximum number of regions that the plane can be divided into when I draw n unit circles (i.e., circles with radius 1) all containing the origin inside, and let

$$P(n) : R_n \geq n^2 - n + 2$$

We will prove using induction that $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Base case ($n = 1$): A single circle containing the origin splits the plane into 2 regions, so

$$R_n \geq 2 = 1^2 - 1 + 2.$$

Thus, $P(1)$ is true.

Inductive step: Assume that $R_k \geq k^2 - k + 2$, where $k \geq 1$. Imagine that k unit circles containing the origin have already been drawn, where $k \geq 1$, and I am trying to draw the $(k+1)$-st circle C^*.

For each existing circle C, the circle C^* will intersect C twice, since C and C^* are unit circles both containing the origin (If you don't believe this, it is explained at the end of the solution). Since there are k existing circles, the circle C^* can intersect the existing circles $2k$ times. We can assume all of these intersections are distinct (otherwise, perturb the position of C^* so that they are). These intersections divide C^* into $2k$ arcs, each of which lie in a different region defined by the k existing circles. Each of these $2k$ regions, and no others, is then split in two by C^*.

Thus, drawing C^* will cause the number of regions to increase by $2k$, and so

$$\begin{aligned}
C_{k+1} &\geq C_k + 2k && (11)\\
&\geq k^2 - k + 2 + 2k && (12)\\
&\geq k^2 + k + 2 && (13)\\
&\geq k^2 + 2k + 1 - k - 1 + 2 && (14)\\
&\geq (k+1)^2 - (k+1) + 2. && (15)
\end{aligned}$$

Thus, $P(k+1)$ is true. This completes the inductive step.

Before, we claimed that if two unit circles both contain the origin inside, then they must intersect each other twice. We will prove this claim now.

Let O_1 be the center of the first unit circle, O_2 the center of the second, and O be the origin. Then, $OO_1 < 1$ and $OO_2 < 1$, so

$$O_1O_2 \leq OO_1 + OO_2 < 1 + 1 = 2.$$

Since the distance between the centers of the two circles is less than 2, the circles must intersect. □