CS 173: Discrete Structures, Summer 2014
Homework 7 Solution Sketches

This homework will not be graded and is never due. You do not need to hand anything in. Solution sketches will be posted on Monday, August 4th.

Some of these questions will appear on the final exam.

1. **Proof by contradiction**

   (a) Alice is trying to prove that $\sqrt{4}$ is irrational using the method that we used in class to prove that $\sqrt{2}$ is irrational. Where does Alice's proof go wrong?

   Solution. Just because a^2 is a multiple of 4 does not mean a is as well. □

   (b) Prove that $\sqrt{2} + \sqrt{3}$ is irrational. You will probably need to show that $\sqrt{6}$ is irrational first.

   Solution. First note that $6 \nmid a$ implies $6 \nmid a^2$. You can show this using a proof by cases (a is congruent to either 1,2,3,4, or 5 $\pmod 6$).
   The contrapositive of the above claim is that $6 \mid a^2$ implies $6 \mid a$. Using this, we can show that $\sqrt{6}$ is irrational, following the same outline as the proof that $\sqrt{2}$ is irrational.
   Now suppose, for sake of contradiction, that $\sqrt{2} + \sqrt{3}$ is rational. Then, for some integers a, b,

   $$\sqrt{2} + \sqrt{3} = \frac{a}{b} \tag{1}$$
   $$\Longrightarrow 5 + 2\sqrt{6} = \frac{a^2}{b^2} \tag{2}$$
   $$\Longrightarrow \sqrt{6} = \frac{a^2 - 5b^2}{2b^2} \tag{3}$$

   This is a contradiction because $\sqrt{6}$ is irrational. □

2. **Combinations**

   A standard deck of 52 cards has four suits and thirteen ranks. Assume that a hand is made up of 5 cards.

   (a) A straight flush is a hand that contains five cards in sequence, all of the same suit. How many possible straight flushes are there?

   Solution. 4×9, assuming that aces can only be played with value 1 and not including royal flushes. □

(b) A four-of-a-kind is a hand that contains all four cards of one rank and any other card. How many four-of-a-kinds are there?

Solution. $13 \times (12 \times 4)$ □

(c) A full house is a hand that contains three matching cards of one rank and two matching cards of another rank. How many full houses are there?

Solution. $(13 \times \binom{4}{3}) \times (12 \times \binom{4}{2})$ □

(d) A flush is a hand where all five cards are of the same suit but not in sequence. How many flushes are there?

Solution. $4 \times (\binom{13}{5} - 9)$, including royal flushes. □

(e) A straight is a hand that contains five cards of sequential rank in at least two different suits. How many straights are there?

Solution. $9 \times 4^5 - 4 \times 9$, assuming that aces can only be played with value 1. □

(f) A three-of-a-kind is a hand that contains three cards of the same rank, plus two cards not of this rank nor the same as each other. How many three-of-a-kinds are there?

Solution. $(13 \times \binom{4}{3}) \times (\binom{12}{2} \times 4 \times 4)$ □

(g) A two-pair is a hand that contains two cards of the same rank, plus two cards of another rank (that match each other but not the first pair), plus any card not of either rank. How many two-pairs are there?

Solution. $(\binom{13}{2} \times \binom{4}{2} \times \binom{4}{2}) \times (11 \times 4)$ □

(h) A one-pair is a hand that contains two cards of one rank, plus three cards which are not of this rank nor the same as each other. How many one-pairs are there?

Solution. $(13 \times \binom{4}{2}) \times (\binom{12}{3} \times 4^3)$. □

3. **Stars and bars**

(a) How many ordered 4-tuples (x_1, x_2, x_3, x_4) satisfy

$$x_1 + x_2 + x_3 + x_4 = 100$$

if $x_1, \ldots, x_4$ must be positive integers?

Solution. $\binom{99}{3}$ □

(b) How many ordered 4-tuples (x_1, x_2, x_3, x_4) satisfy

$$x_1 + x_2 + x_3 + x_4 = 100$$

if $x_1, \ldots, x_4$ must be natural numbers?

Solution. $\binom{103}{3}$ □

(c) How many ordered 4-tuples (x_1, x_2, x_3, x_4) satisfy

$$x_1 + x_2 + x_3 + x_4 \leq 100$$

if $x_1, \ldots, x_4$ must be natural numbers?

Solution. $\binom{104}{4}$ □

(d) How many ordered 4-tuples (x_1, x_2, x_3, x_4) satisfy

$$x_1 + x_2 + x_3 + x_4 \leq 100$$

if $x_1, \ldots, x_4$ must be positive integers?

Solution. $\binom{100}{4}$ □

4. **Combinatorial identities**

(a) Expand $(1+1)^n$ by the binomial theorem to show that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Also, find a combinatorial proof of the above equation.

Solution. The first part is straightforward. For the second part, both sides count the number of subsets of a set S of n elements. The left side does this by counting the number of such subsets of size k for each k. To see that the right side does this as well, note that to specify a subset of S, we can go through each of the n elements of S and specify whether or not it is in the subset. □

(b) Use Pascal's identity and induction on n to show that

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Solution. This is straightforward. □

(c) Prove that

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}$$

by showing that both sides are actually counting the same thing.

Solution. The left side counts the number of ways to pick k objects from a pool of $m+n$ objects. To see that the right side does too, split the pool of $m+n$ objects into a pool of size m and one of size n. Picking k objects from the original pool is the same thing as picking i objects from the pool of size m and $k-i$ objects from the pool of size n, as i varies. □

5. **Binomial theorem**

   (a) What is the coefficient of x^2y^{14} in $(3x+2y^2)^8$?

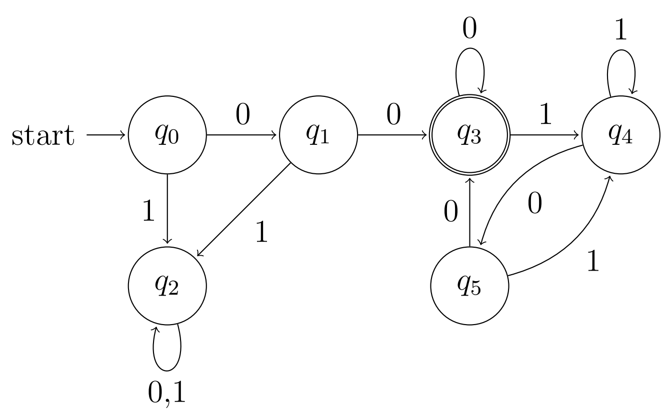
   Solution. 0 □

   (b) What is the coefficient of x^3y^{10} in $(3x+2y^2)^8$?

   Solution. $\binom{8}{3}3^3 2^5$ □

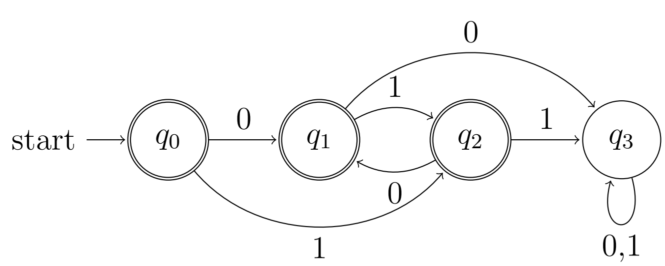
6. **Deterministic finite automata**

   (a) Design a DFA that accepts all strings that start with two 0s and end with two 0s. It should accept strings like 00 and 000.



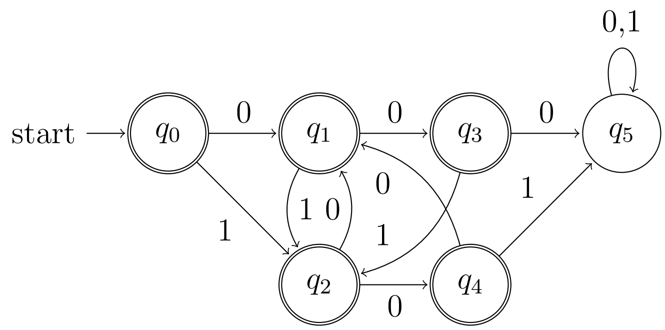
   Solution. □

   (b) Design a DFA that accepts all strings that alternate between 0s and 1s.



   Solution. □

   (c) Design a DFA that accepts all strings that contain neither three consecutive 1s nor three consecutive 0s.



Solution. □

7. **Countability** Which of the following sets are countable?

   (a) the set of all C programs.

   Solution. Countable □

   (b) the set of all functions $f : \mathbb{N} \to \{0, 1\}$.

   Solution. Not countable □

   (c) the set of all pairs of integers.

   Solution. Countable □

   (d) the set of all finite sets of integers.

   Solution. Countable □

   (e) $\mathbb{P}(\mathbb{N})$.

   Solution. Not countable □

   (f) the set of all finite-length bit strings.

   Solution. Countable □

   (g) the set of all bit strings, including those of infinite length.

   Solution. Not countable □

8. **Another false proof**

   Here is a proof using diagonalization that the set of rational numbers between 0 and 1 is uncountable:

Proof. The proof is by contradiction. Suppose that there is a one-to-one correspondence between the natural numbers and the rational numbers:

$$0 \leftrightarrow 0.d_{11}d_{12}d_{13}\ldots \tag{4}$$
$$1 \leftrightarrow 0.d_{21}d_{22}d_{23}\ldots \tag{5}$$
$$2 \leftrightarrow 0.d_{31}d_{32}d_{33}\ldots \tag{6}$$
$$\vdots \tag{7}$$

(If a rational number terminates, then we just pad the end with 0s.)

Now form the number $0.e_1e_2e_3\ldots$, where $e_i \neq d_{ii}$ for all i. This number does not appear in the above table. Therefore, the table is not a one-to-one correspondence, which is a contradiction. □

Explain why the above proof is wrong.

Solution. $0.e_1e_2e_3\ldots$ is not rational. □