CS 173: Discrete Structures, Summer 2014
Homework 5

This homework contains 4 problems and is due in class on Wednesday, July 23rd. **Please follow the guidelines on the class web page about homework format and style.**

In all questions, you must explain how you get your answers. Stating the answer with no supporting work will not receive full credit.

1. **Sums [10 points]**

   (a) Evaluate

   $$\sum_{i=1}^{n} \frac{i}{3^i}.$$

   What is the sum as $n \to \infty$?

   Solution. Let

   $$S = \sum_{i=1}^{n} \frac{i}{3^i}.$$

   Then,

   $$\frac{S}{3} = \sum_{i=1}^{n} \frac{i}{3^{i+1}} = \sum_{i=2}^{n+1} \frac{i-1}{3^i} = \sum_{i=1}^{n+1} \frac{i-1}{3^i} = \left(\sum_{i=1}^{n} \frac{i-1}{3^i}\right) + \frac{n}{3^{n+1}}.$$

   Subtracting these two equations yields

   $$\begin{aligned} \frac{2S}{3} &= \left(\sum_{i=1}^{n} \frac{1}{3^i}\right) - \frac{n}{3^{n+1}} && (1) \\ &= \frac{1}{3}\left(\sum_{i=0}^{n-1} \frac{1}{3^i}\right) - \frac{n}{3^{n+1}} && (2) \\ &= \frac{1}{3} \times \frac{1-(1/3^n)}{1-1/3} - \frac{n}{3^{n+1}} && (3) \\ &= \frac{1}{2}\left(1 - \frac{1}{3^n}\right) - \frac{n}{3^{n+1}} && (4) \end{aligned}$$

   Thus,

   $$\begin{aligned} S &= \frac{3}{2}\left(\frac{1}{2}\left(1 - \frac{1}{3^n}\right) - \frac{n}{3^{n+1}}\right) && (5) \\ &= \frac{3}{4}\left(1 - \frac{1}{3^n}\right) - \frac{n}{2 \times 3^n} && (6) \end{aligned}$$

   As $n \to \infty$, this approaches $3/4$. $\square$

(b) It is well known that

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}.$$

Use this equation to evaluate

$$\sum_{i=1}^{\infty} \frac{1}{(2i-1)^2}.$$

[Hint: First find $\sum_{i=1}^{\infty} \frac{1}{(2i)^2}$]

Solution. We have

$$\sum_{i=1}^{\infty} \frac{1}{(2i)^2} = \frac{1}{4} \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{1}{4} \times \frac{\pi^2}{6}.$$

On the other hand,

$$\sum_{i=1}^{\infty} \frac{1}{(2i)^2} + \sum_{i=1}^{\infty} \frac{1}{(2i-1)^2} = \sum_{i=1}^{\infty} \frac{1}{i^2},$$

so

$$\sum_{i=1}^{\infty} \frac{1}{(2i-1)^2} = \sum_{i=1}^{\infty} \frac{1}{i^2} - \sum_{i=1}^{\infty} \frac{1}{(2i)^2} = \frac{\pi^2}{6} - \frac{1}{4} \times \frac{\pi^2}{6} = \frac{3}{4} \times \frac{\pi^2}{6} = \frac{\pi^2}{8}$$

□

2. **Unrolling recurrences [20 points]**

It is well known that the sum of the first n positive integers is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

(a) [**5 points**] Use the above formula to find a closed form for

$$\sum_{i=1}^{k} (2i),$$

the sum of the first k even positive integers. [Warning: this is not the same as the sum of the even integers from 0 to k.]

Solution. We have

$$\sum_{i=1}^{k} (2i) = 2 \sum_{i=1}^{k} i = k(k+1).$$

□

(b) [**5 points**] Find a closed form for

$$\sum_{i=1}^{k}(2i-1),$$

the sum of the first k odd positive integers. [Warning: This is not the same as the sum of the odd integers from 1 to k.][Note: For this problem only, saying "We went over this in class" is not an acceptable justification. Please give an argument for why your answer is true.]

Solution. We have

$$\sum_{i=1}^{k}(2i-1) = 2\sum_{i=1}^{k} i - \sum_{i=1}^{k} 1 = k(k+1) - k = k^2 + k - k = k^2$$

$\square$

(c) [**10 points**] Unroll the recurrence

$$f(n) = \begin{cases} 42 & \text{if } n = 0 \\ 173 & \text{if } n = 1 \\ f(n-2) + n & \text{if } n \geq 2 \end{cases}$$

and guess (correctly) a solution to it. [Warning: See two previous warnings.] Check your solution for $n \in \{0, 1, 2, 3\}$. You do not need to prove that your solution is correct.

Solution. When we unroll the recurrence, we get

$$\begin{aligned}
f(n) &= f(n-2) + n && (7)\\
&= f(n-4) + (n-2) + n && (8)\\
&= f(n-6) + (n-4) + (n-2) + n && (9)\\
&\vdots && (10)\\
&= f(n-k) + (n-k+2) + (n-k+4) \cdots + n \quad (k \text{ is even}) && (11)\\
&\vdots && (12)\\
&= \begin{cases} f(0) + 2 + 4 + \cdots + n & : n \text{ is even} \\ f(1) + 3 + 5 + \cdots + n & : n \text{ is odd} \end{cases} && (13)\\
& && (14)
\end{aligned}$$

Thus, if n is even, then

$$f(n) = f(0) + \sum_{i=1}^{n/2}(2i) = f(0) + (n/2)(n/2+1) = 42 + \frac{n^2+2n}{4},$$

while if n is odd, then

$$f(n) = f(1) - 1 + \sum_{i=1}^{(n+1)/2} (2i-1) \tag{15}$$
$$= f(1) - 1 + \left(\frac{n+1}{2}\right)^2 \tag{16}$$
$$= 172 + \left(\frac{n+1}{2}\right)^2 \tag{17}$$
$$\tag{18}$$

Putting it all together,

$$f(n) = \begin{cases} 42 + \frac{n^2+2n}{4} & : n \text{ is even} \\ 172 + \left(\frac{n+1}{2}\right)^2 & : n \text{ is odd} \end{cases}$$

Let's check our solution for $n \in \{0, 1, 2, 3\}$.

$$f(0) = 42 = 42 + \frac{0^2 + 2(0)}{4} \tag{19}$$
$$f(1) = 173 = 172 + \left(\frac{1+1}{2}\right)^2 \tag{20}$$
$$f(2) = f(0) + 2 = 44 = 42 + \frac{8}{4} = 42 + \frac{2^2 + 2(2)}{4} \tag{21}$$
$$f(3) = f(1) + 3 = 176 = 172 + \left(\frac{4}{2}\right)^2 = 172 + \left(\frac{3+1}{2}\right)^2. \tag{22}$$

□

3. **Linear homogeneous recurrences [10 points]**

Consider the recurrence

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ f(n-1) + 8f(n-2) - 12f(n-3) & \text{if } n \geq 3 \end{cases}$$

(a) What are the solutions to the characteristic equation?

Solution. The characteristic equation is

$$r^n = r^{n-1} + 8r^{n-2} - 12r^{n-3} \tag{23}$$
$$\Longrightarrow r^3 = r^2 + 8r - 12 \tag{24}$$
$$\Longrightarrow r^3 - r^2 - 8r + 12 = 0 \tag{25}$$
$$\Longrightarrow (r-2)^2(r+3) = 0, \tag{26}$$

so the roots are 2 and -3. □

(b) What is the general form of the solution to the recurrence equation? (For example, if the solutions to the characteristic equation are $r = 2$ and $r = 3$, then the general solution is $f(n) = c_1 2^n + c_2 3^n$).

Solution. The general form of the solution is

$$f(n) = (c_1 + c_2 n)2^n + c_3(-3)^n,$$

where c_1, c_2, and c_3 are constants. □

(c) What is the final solution to the recurrence after you use the base cases to solve for the constants? Check your solution for $n \in \{0, 1, 2, 3\}$. You do not need to prove that your solution is correct.

Solution. The base cases tell us that

$$c_1 + c_3 = 0 \tag{27}$$
$$2c_1 + 2c_2 - 3c_3 = 1 \tag{28}$$
$$4c_1 + 8c_2 + 9c_3 = 2 \tag{29}$$

Multiplying (28) by 4 and subtracting (29) from the result yields

$$4c_1 - 21c_3 = 2$$

Multiplying (27) by 4 and subtracting (30) from the result yields.

$$25c_3 = -2 \implies c_3 = -2/25.$$

Plugging this into (27) yields

$$c_1 - 2/25 = 0 \implies c_1 = 2/25.$$

Plugging the values of c_1 and c_3 into (28) yields

$$4/25 + 2c_2 + 6/25 = 1 \implies 2c_2 = 15/25 \implies c_2 = 3/10.$$

Thus the solution is

$$f(n) = (2/25 + (3/10)n)2^n - (2/25)(-3)^n \tag{30}$$
$$= \frac{1}{50}((4 + 15n)2^n - 4(-3)^n) \tag{31}$$

Let's check our solution for $n \in \{0, 1, 2, 3\}$.

$$\begin{align}
f(0) &= 0 \tag{32}\\
&= \frac{1}{50}(4-4) \tag{33}\\
&= \frac{1}{50}((4+0)(1) - 4(1)) \tag{34}\\
&= \frac{1}{50}((4+15(0))2^0 - 4(-3)^0) \tag{35}\\
f(1) &= 1 \tag{36}\\
&= \frac{1}{50}(38+12) \tag{37}\\
&= \frac{1}{50}((4+15)(2) - 4(-3)) \tag{38}\\
&= \frac{1}{50}((4+15(1))2^1 - 4(-3)^1) \tag{39}\\
f(2) &= 2 \tag{40}\\
&= \frac{1}{50}(136-36) \tag{41}\\
&= \frac{1}{50}((4+30)(4) - 4(9)) \tag{42}\\
&= \frac{1}{50}((4+15(2))2^2 - 4(-3)^2) \tag{43}\\
f(3) &= f(2) + 8f(1) - 12f(0) \tag{44}\\
&= 2 + 8(1) - 12(0) \tag{45}\\
&= 10 \tag{46}\\
&= \frac{1}{50}(392+108) \tag{47}\\
&= \frac{1}{50}((4+45)(8) - 4(-27)) \tag{48}\\
&= \frac{1}{50}((4+15(3))2^3 - 4(-3)^3) \tag{49}
\end{align}$$

□

4. **Structural induction [10 points]** Use structural induction to show that any full binary tree with i internal nodes has $i+1$ leaves.

   Solution. The proof is by strong induction on the height of the tree. Define the predicate $P(h)$ by

   $P(h)$: Any full binary tree with height h and i internal nodes has $i+1$ leaves.

   Base Case: If $h = 0$, then the tree consists of a single node, which is a leaf. Since there are no internal nodes, i must be 0. Since there is one leaf, $P(0)$ is true.

Inductive Step: Suppose that $P(0), \ldots P(k)$ are all true. We want to show that $P(k+1)$ is true as well.

Let T be a full binary tree of height $k+1$ and with root r. Since $k \geq 0$, T has height at least 1, so r has at least one child. Since T is a full binary tree, r has two children. Let the children be x and y.

Let T_1 be the subtree of T rooted at x and T_2 be the subtree rooted at y. Suppose T_1 has i_1 internal nodes and T_2 has i_2 internal nodes. Both T_1 and T_2 have height strictly less than $k+1$, so by the inductive hypothesis, T_1 has i_1+1 leaves and T_2 has i_2+1 leaves.

The internal nodes of T are precisely the internal nodes of T_1 and T_2 along with r, so T has i_1+i_2+1 internal nodes. Thus, $i = i_1 + i_2$.

The leaves of T are precisely the leaves of T_1 and T_2, so T has

$$(i_1+1)+(i_2+1) = i_1+i_2+2 = i+1$$

leaves. $\square$