CS 173: Discrete Structures, Summer 2014
Homework 3

This homework contains 3 problems. It is due in class on Wednesday, July 9th. **Please follow the guidelines on the class web page about homework format and style.**

In all questions, you must explain how you get your answers. Stating the answer with no supporting work will not receive full credit.

1. **Counting [20 points]**

   (a) I roll three 6-sided dice. How many possible rolls are there such that at least one of the numbers that comes up is prime? Note that the dice are distinguishable (they are different colors, say). For example, rolling 1, 2, and 3 is different from rolling 3, 2, and 1.

   Solution. We will count the number of rolls in which none of the numbers that come up is prime, and subtract this from the total number of rolls.

   The total number of possible rolls is $6^3 = 216$.

   Since the non-primes from 1 to 6 are 1,4, and 6, the number of rolls in which none of the numbers that come up are prime is $3^3 = 27$.

   Thus, the answer is $216 - 27 = \boxed{189}$. □

   (b) I have 8 beads, all of different colors, that I want to string together into a necklace. Also, I do not want the red and green beads next to each other. How many different necklaces are possible? Note that necklaces can be rotated and flipped. For example, if I represent each bead by a letter, then the sequence $ABCDEFGH$ represents the same necklace as $EDCBAHGF$ does.

   Solution. First, let's pretend that rotating and flipping necklaces results in a different necklaces. We will imagine that the necklace is empty initially and add the beads one at a time.

   Initially, there are 8 places to place the red bead. After placing the red bead, there are 5 remaining places to place the green bead. After this, there are 6, 5, 4, 3, 2, and 1 places to place the remaining 6 beads.

   Thus, ignoring the fact that necklaces can be flipped and rotated, there are $8 \times 5 \times 6!$ different possible necklaces.

   Each necklace can be rotated into 8 different positions (including the one it is currently in). If we flip the necklace, there are again 8 different configurations. So, to take care of flips and rotations, we need to divide by 2×8.

   Thus, the answer is
   $$\frac{8 \times 5 \times 720}{2 \times 8} = \frac{5 \times 6!}{2} = \boxed{1800}.$$
   □

(c) In this problem, a **Boolean function on n variables** is a function whose domain is $\{0,1\}^n$ and whose co-domain is $\{0,1\}$. (In other words, it takes an n-tuple of zeros and ones and returns either 0 or 1). How many Boolean functions on n variables are there?

Solution. □

2. **Pigeonhole Principle [10 points]**

(a) Every point in a 2×2 square is colored either red or blue. Prove that there are two points that are a distance 1 from each other and are both the same color.

Solution. There exists an equilateral triangle with side length 1 such that all of the vertices of the triangle are in the 2×2 box.

Consider the vertices of this triangle, each of which are either red or blue. We have three vertices but only two colors, so by the pigeonhole principle, two of these vertices must be the same color. These two vertices are a distance 1 from each other and are both the same color. □

(b) I pick fifty-one of the first one hundred positive integers. Prove that I must have picked two integers a and b such that $a - b = 10$.

Solution. Consider the sets

$$\begin{array}{cccc} \{1,11\} & \{2,12\} & \dots & \{10,20\} \\ \{21,31\} & \{22,32\} & \dots & \{30,40\} \\ \vdots & \vdots & \ddots & \vdots \\ \{81,91\} & \{82,92\} & \dots & \{90,100\} \end{array}$$

Each of first one hundred positive integers falls into one of these 50 sets. Since I picked 51 numbers, two of them must fall in the same set. If a is the larger of those two numbers and b is the smaller, then $a - b = 10$ □

3. **Functions [20 points]**

(a) Find an $f : \mathbb{Z} \to \mathbb{Z}$ that is one-to-one but not onto. Prove that your function is one-to-one but not onto.

Solution. Let $f(x) = 2x$.

To see that f is one-to-one, let $x_1, x_2 \in \mathbb{Z}$ and $f(x_1) = f(x_2)$. We have

$$2x_1 = 2x_2 \implies x_1 = x_2.$$

To see that f is not onto, let $y = 1$. There is no integer x such that $2x = 1$, so there is no integer x such that $f(x) = y$. □

(b) Find an $f : \mathbb{Z} \to \mathbb{Z}$ that is onto but not one-to-one. Prove that your function is onto but not one-to-one.

Solution. Let $f(x) = \lfloor x/2 \rfloor$.
To see that f is onto, suppose $y \in \mathbb{Z}$. Let $x = 2y$. Then,

$$f(x) = \left\lfloor \frac{x}{2} \right\rfloor = \left\lfloor \frac{2y}{2} \right\rfloor = \lfloor y \rfloor = y,$$

since $y \in \mathbb{Z}$.
To see that f is not one-to-one, let $x_1 = 0$ and $x_2 = 1$. Then, $x_1 \neq x_2$ but

$$f(x_1) = f(0) = \left\lfloor \frac{0}{2} \right\rfloor = \lfloor 0 \rfloor = 0 = \left\lfloor \frac{1}{2} \right\rfloor = f(1) = f(x_2).$$

□

(c) Find an $f : \mathbb{Z} \to \mathbb{Z}$ that is neither one-to-one nor onto. You do not need to prove anything for this part; just write the function.

Solution. Let $f(x) = |x|$. □