Midterm 1 Review

G. Carl Evans

University of Illinois

Summer 2013

Proof with concrete function

Define: $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ prove that f is bijective.

Solution

Define: $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ prove that f is bijective.

Let: $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$
To prove the f is bijective we must prove that f is one-to-one and onto.
Proof f is one-to-one:
Let $x, y \in \mathbb{R}$ s.t. $f(x) = f(y)$.

$$f(x) = f(y)$$

$$x^3 = y^3$$

Since the cubed root is a function we can take the cubed root of both sides.

$$x = y$$

We have now shown that f is one-to-one by definition. QED

Solution part 2

Define: $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ prove that f is bijective.

Proof f is onto:
Let $y \in \mathbb{R}$
Choose $x = \sqrt[3]{y}$
x is clearly a real number since cubed root is closed over the reals and $f(x) = (\sqrt[3]{y})^3 = y$. Thus you have found the required pre-image for f and f is onto by definition. QED
Since we have shown that f is both one-to-one and onto we have shown that f is bijective by definition. QED

Proof with abstract function

Define: A, B, and C are set and $f : B \to C$ and $g : A \to B$ are functions.

1. Prove that if $f \circ g$ is onto and f is one-to-one, then g is onto.
2. Give a concrete counter-example (involving small sets!) showing why the assumption that f is one-to-one is necessary above.

Part 1

Define: A, B, and C are set and $f : B \to C$ and $g : A \to B$ are functions.
Prove that if $f \circ g$ is onto and f is one-to-one, then g is onto.

Part 1 Solution

Suppose A, B, and C are set and $f : B \to C$ and $g : A \to B$ are functions such that $f \circ g$ is onto and f is one-to-one.
To show that g is onto. We need to show

$$\forall b \in B, \exists a \in A, g(a) = b$$

First we will prove that f is bijective. To do this we must show that f is one-to-one and onto. f is known to be one-to-one so we only need to show f is onto.
Let $x \in C$ Since $f \circ g$ is onto there exists an $a \in A$ such that $f(g(a)) = c$. From this and the definition of g, $g(a) \in B$ and since $f(g(a)) = c$ we have shown that f is onto.
Thus f is bijective.

Part 1 Solution page 2

Since f is bijective there is an inverse $f^{-1} : C \to B, f^{-1}(f(x)) = y$
Let b be an arbitrary element of B.
Let $c \in C \text{s.t.} c = f(b)$. Since f is a function from B to C this exists.
Since $f \circ g$ is onto there is an $a \in A$ such that $f(g(a)) = c$.

$$f^{-1}(f(g(a)) = f^{-1}(c)$$

$$g(a) = b$$

Thus for any $b \in B$ there exists an a such that $g(a) = b$ and we have proved that g is onto. QED

Number Theory and Contrapositive

Prove by contrapositive the following:
If $7 \nmid xy$ then $7 \nmid x$ and $7 \nmid y$

Solution

Prove by contrapositive the following:
If $7 \nmid xy$ then $7 \nmid x$ and $7 \nmid y$

The contrapositive is of the claim is If $7 \mid x$ or $7 \mid y$ then $7 \mid xy$.
Let x, y be integers.
WLOG $7 \mid x$
By definition of divides $x = 7n, n \in \mathbb{Z}$.

$$xy = 7ny$$

Since $7, n$ and y are integers $7 \mid xy$ by definition of divides. Thus the contrapositive hold and so does the original claim.

Equivalence Classes and Sets part (a)

Using the definition $a \equiv b \pmod{k}$ if and only if $a = b + nk$ for some $n \in \mathbb{Z}$ write $[x]_k$ in set builder notation. Your answer should be of the form

$$[x]_k = \{\, y \mid \ldots \}$$

Solution part (a)

Using the definition $a \equiv b \pmod{k}$ if and only if $a = b + nk$ for some $n \in \mathbb{Z}$ write $[x]_k$ in set builder notation. Your answer should be of the form

$$[x]_k = \{\, y \mid \ldots \}$$

Relations