Induction, Recursive Definition

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Recurrences

A **recurrence** is a recursive description of a function.
Example:

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ f(n-1)+1 & \text{otherwise} \end{cases}$$

In other words,

- Base case: $f(0) = 0$
- Recursive formula: $f(n) = f(n-1) + 1$ if $n \geq 1$.

Usually, we want a **closed form** solution, i.e., a non-recursive description of a function.

Solving recurrences

1. Guess the solution.
2. Prove your solution is correct using induction.

Linear homogeneous recurrences

A **linear homogeneous recurrence** is of the form

$$f(n) = c_1 f(n-1) + c_2 f(n-2) + \cdots + c_k f(n-k).$$

where each c_i is a constant.
Example: $f(n) = f(n-1) + 3f(n-2) + 14f(n-3)$
Non-examples: $f(n) = f(n-1) + 1$
$f(n) = f(n-1) + n$

Linear homogeneous recurrences

Suppose

$$f(n) = \begin{cases} 3 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ 2f(n-1) + 3f(n-2) & \text{otherwise} \end{cases}$$

What is $f(n)$?

Linear homogeneous recurrences

If

$$f(n) - 2f(n-1) - 3f(n-2) = 0,$$

then the **characteristic equation** is

$$r^n - 2r^{n-1} - 3r^{n-2} \Leftrightarrow r^2 - 2r - 3 = 0,$$

and the **characteristic polynomial** is

$$r^2 - 2r - 3.$$

The roots of the characteristic polynomial are the bases in our solution.

Repeated roots

Suppose

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 6 & \text{if } n = 1 \\ 28 & \text{if } n = 2 \\ 6f(n-1) - 12f(n-2) + 8f(n-3) & \text{otherwise} \end{cases}$$

What is $f(n)$?

Another example

Suppose

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 9 & \text{if } n = 2 \\ 5f(n-1) - 8f(n-2) + 4f(n-3) & \text{otherwise} \end{cases}$$

What is $f(n)$?

Mergesort

Suppose

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2f(n/2) + n & \text{otherwise} \end{cases}$$

What is $f(n)$?

Proofs with recurrences

The Fibonacci numbers are defined as $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
Prove that

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$$

for $n \in \mathbb{Z}^+$.