Induction, Recursive Definition

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Weak induction

Theorem

Suppose $P(n)$ is a predicate. If

1. (Base case) $P(0)$ is true, and
2. (Inductive step) for all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Strong induction

Theorem

Suppose $P(n)$ is a predicate. If

1. (Base case) $P(0)$ is true, and
2. (Inductive step) for all $k \in \mathbb{N}$, if $P(0), P(1), \ldots, P(k)$ are true, then $P(k+1)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Fundamental theorem of arithmetic

Theorem

Every positive integer has a unique prime factorization.

Connected Components

Theorem

Every graph with n nodes and c connected components must have at least $n - c$ edges. (Any connected graph has at least $n - 1$ edges.)

Equivalently,

$$|E| + (\# \text{ of connected components}) \geq |V|.$$

Equivalently,

Theorem

Every graph with n nodes and m edges must have at least $n - m$ connected components.

Proof: Induct on m.

Eulerian cycles

Theorem

A connected graph has an Eulerian cycle if and only if all vertices have even degree.

Proof: Induct on number of vertices.

Recurrences

A **recurrence** is a recursive description of a function.
Example:

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ f(n-1) + 1 & \text{otherwise} \end{cases}$$

In other words,

- Base case: $f(0) = 0$
- Recursive formula: $f(n) = f(n-1) + 1$ if $n \geq 1$.

Usually, we want a **closed form** solution, i.e., a non-recursive description of a function.

Solving recurrences

1. Guess the solution.
2. Prove your solution is correct using induction.

Series

What is...

- $2/3 + (2/3)^2 + (2/3)^3 + \cdots + (2/3)^{100}$?
- $3(2^0) + 3(2^1) + 3(2^2) + ... + 3(2^{100})$?
- $a + ar + ar^2 + \cdots + ar^n$?
- $a + ar + ar^2 + \ldots$?
- $2/3 + (2/3)^2 + (2/3)^3 + \ldots$?

Example

Suppose

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2f(n-1) + 1 & \text{otherwise} \end{cases}$$

What is $f(n)$?

Unrolling

Suppose

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 5f(n-1) + 1 & \text{otherwise} \end{cases}$$

What is $f(n)$?

Linear homogeneous recurrences

A **linear homogeneous recurrence** is of the form

$$f(n) = c_1 f(n-1) + c_2 f(n-2) + \cdots + c_k f(n-k).$$

where each c_i is a constant.
Example: $f(n) = f(n-1) + 3f(n-2) + 14f(n-3)$
Non-examples: $f(n) = f(n-1) + 1$
$f(n) = f(n-1) + n$

Linear homogeneous recurrences

Suppose

$$f(n) = \begin{cases} 3 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ 2f(n-1) + 3f(n-2) & \text{otherwise} \end{cases}$$

What is $f(n)$?

Linear homogeneous recurrences

If

$$f(n) - 2f(n-1) - 3f(n-2) = 0,$$

then the **characteristic equation** is

$$r^n - 2r^{n-1} - 3r^{n-2} \Leftrightarrow r^2 - 2r - 3 = 0,$$

and the **characteristic polynomial** is

$$r^2 - 2r - 3.$$

The roots of the characteristic polynomial are the bases in our solution.

Repeated roots

Suppose

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 6 & \text{if } n = 1 \\ 28 & \text{if } n = 2 \\ 6f(n-1) - 12f(n-2) + 8f(n-3) & \text{otherwise} \end{cases}$$

What is $f(n)$?

Another example

Suppose

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 9 & \text{if } n = 2 \\ 5f(n-1) - 8f(n-2) + 4f(n-3) & \text{otherwise} \end{cases}$$

What is $f(n)$?

Mergesort

Suppose

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2f(n/2) + n & \text{otherwise} \end{cases}$$

What is $f(n)$?

Proofs with recurrences

The Fibonacci numbers are defined as $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
Prove that

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$$

for $n \in \mathbb{Z}^+$.

Definitions

Definition

A **tree** is a simple acyclic connected graph.

Definition

A **rooted tree** has a special node called the **root**. We usually draw the root on top, and all edges leading away from the root go downward.

Definition

A **leaf** is a vertex of degree 1. An **internal node** is a non-leaf node.

Definition

In a rooted tree, a **parent** of a vertex is the vertex connected to it on the path to the root. A **child** of a vertex u is a vertex v such that u is the parent of v.

Definition

An m-ary tree is a rooted tree for which each vertex has at most n children.

Definition

An m-ary tree is full if every internal node has exactly 0 or n children.

Definition

The **height** of a node is the length of the path between itself and the root. The height of a tree is the maximum height of all of its leaves.

Definition

A tree is complete if all leaves have the same height.

How many nodes are there in a full m-ary tree with i internal nodes? How many nodes are in an m-ary tree with height h?

Induction on trees

- I want to prove that a statement P is true for all trees.
- I let $P(h)$ say that P is true for trees of height h.
- (Base Case). I prove $P(0)$.
- (Inductive Step): I assume that $P(k)$ is true. Suppose that T is a tree with height $k+1$ and root r. Then P is true for each of the subtrees rooted at the children of r.
  I use this to prove that P is true for T, so $P(k+1)$ is true.

Example

Proposition

In a binary tree of height h, the number of nodes is at most $2^{h+1} - 1$.

Recursion trees

Suppose

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2f(n/2) + n & \text{otherwise} \end{cases}$$

What is $f(n)$?

Recursion trees

Suppose

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ f(3n/4) + n & \text{otherwise} \end{cases}$$

What is $f(n)$?

Recursion trees

Suppose

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 3f(n/2) + n & \text{otherwise} \end{cases}$$

What is $f(n)$?

Context-free grammars

A CFG is a set of rules for rewriting strings. S is the starting string. Lowercase letters are terminals and cannot be rewritten.
Example:

- $S \to Sa$
- $S \to a \mid b \mid c$

A parse tree is a tree where each node is labeled with a string. The root is labeled with S and a node's children are labeled with the symbols that rewrite it.
Each string has its own parse tree.

Induction on CFGs

Example: