CS 173, Spring 2014
Midterm 2 Solutions, A Lecture

Problem 1: Checkbox (12 points)

Check the box that best characterizes each item. (2 points each)

$\exists a \in \mathbb{N}, \quad \forall (b, c) \in \mathbb{Z}^2,$
$(b = a) \ \wedge \ (c = -a)$

true ☐ **false** ☑

$f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{x}$ is

one-to-one but not onto ☐
onto but not one-to-one ☐
neither one-to-one nor onto ☐
bijective ☐
not a valid function ☑

The diameter of a C_{15} graph is

2 ☐ **3** ☐
7 ☑ **15** ☐

The chromatic number of a graph with a W_6 subgraph is

at least 3 ☑ **at most 3** ☐
exactly 3 ☐ **none of the above** ☐

$$\sum_{k=1}^{n} \frac{1}{2^k}$$

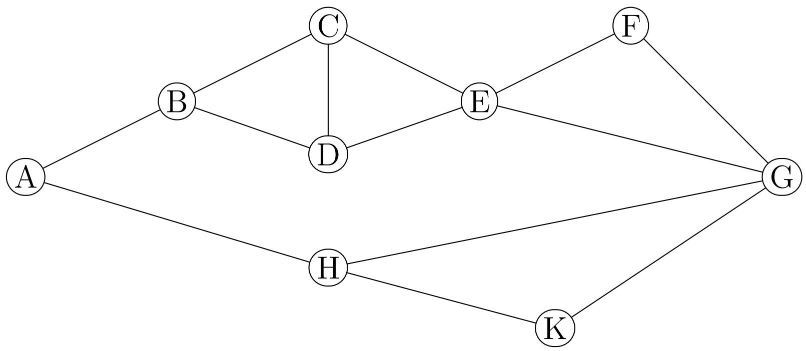
$1 - (\frac{1}{2})^n$ ☑ $2 - (\frac{1}{2})^n$ ☐ $2 - (\frac{1}{2})^{n+1}$ ☐
$1 - (\frac{1}{2})^k$ ☐ $2 - (\frac{1}{2})^k$ ☐ $2 - (\frac{1}{2})^{k+1}$ ☐

The minimum height of a binary tree with n nodes is

$n - 1$ ☐ $\log_2 n$ ☐
$\log_2 (n + 1)$ ☐ $\log_2 (n + 1) - 1$ ☑

Problem 2: Short answer (18 points)

(a) (10 points) Recall that a path never re-uses a node. How many paths are there from A to G in the following graph? Explain or show work.



Solution: The path has to go via B or via H.

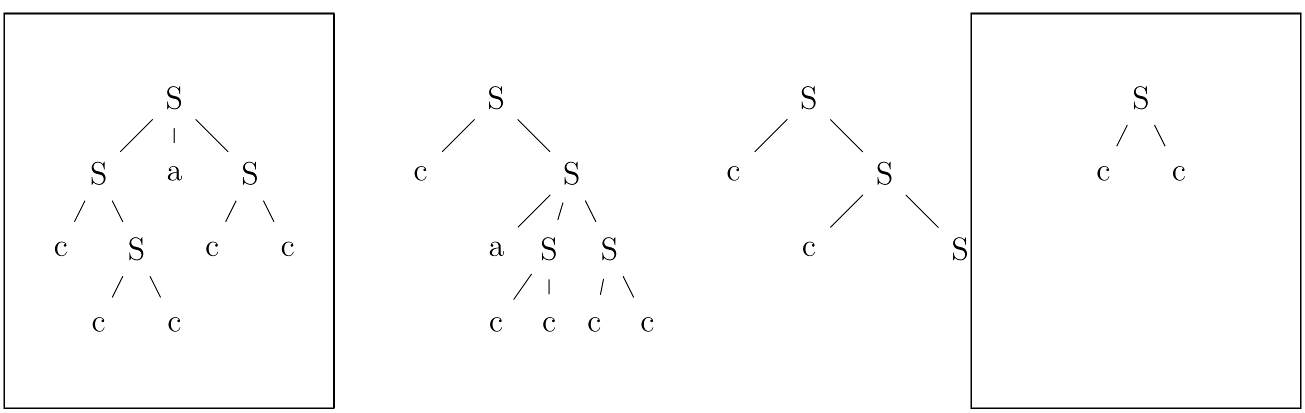
If the path goes through, H, there are two choices, depending on whether we go directly to G or via K.

If the path goes through B, then there are four ways to get from B to E, and then two ways to get from E to G. Thus, there are $4 \times 2 = 8$ ways to reach G.

So, in total, there are $2 + 8 = 10$ paths from A to G.

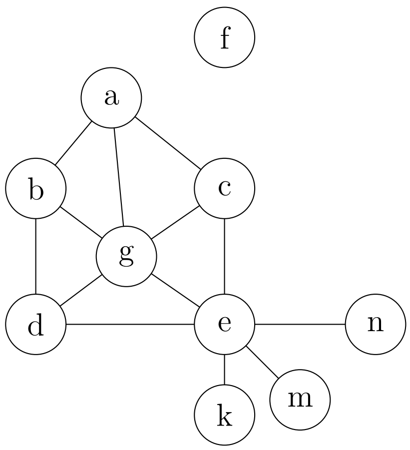
(b) (8 points) Here is a grammar (with start variable S and terminals a and c). Circle the trees that match the grammar.

$$S \rightarrow S\ a\ S \mid c\ S \mid c\ c$$



Problem 3: Short Answer (20 points)

(a) (10 points) Answer the questions about the graph on the left. (No need to justify.)



What is its chromatic number? **Solution: 4**

Has an Euler circuit? **Solution: no**

Is it bipartite? **Solution: no**

How many connected components? **Solution: 2**

What is the degree of e? **Solution: 6**

(b) (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (f(x) - y, 5y + 3)$. Prove that g is one-to-one.

Solution: Let (x, y) and (p, q) be two elements of $\mathbb{Z}^2$ and suppose that $g(x, y) = g(p, q)$. We need to show that $(x, y) = (p, q)$.

By the definition of g, $g(x, y) = g(p, q)$ implies that $(f(x) - y, 5y + 3) = (f(p) - q, 5q + 3)$. That is $f(x) - y = f(p) - q$ and $5y + 3 = 5q + 3$.

Since $5y + 3 = 5q + 3$, $5y = 5q$ and therefore $y = q$. Plugging this back into $f(x) - y = f(p) - q$, we find that $f(x) = f(p)$. But we know that f is one-to-one, so this implies that $x = p$.

Since $x = p$ and $y = q$, $(x, y) = (p, q)$, which is what we needed to show.

Problem 4: Recursion Tree (14 points)

Suppose that we are building a recursion tree for the function T, defined as follows:

$$T(1) = c \qquad \text{and} \qquad T(n) = nT(n/2) + n$$

Assume that the input n is a power of 2.

(a) How many nodes are there in level 2, i.e. two levels below the root? (3 points)

Solution: $n(n/2)$

(b) What is the value in each node in level 2? (2 points)

Solution: $n/4$

(c) What is the sum of the values in all the nodes at level k, where k is neither a root nor a leaf level (3 points)?

Solution: There are $n \cdot (n/2) \cdot (n/4) \ldots (n/2^{k-1})$ nodes, each of which contains the value $n/2^k$. So the sum is $(n/2^k)(n^k(\frac{1}{2})^{k(k-1)/2})$ which simplifies to $n^k(\frac{1}{2})^{k(k+1)/2}$.

(d) What is the level of the leaf nodes? (3 points)

Solution: $\log_2 n$

(e) What is the sum of all the values in the leaf nodes? (3 points)

Solution: Let $k = log_2 n$. Then the sum of the leaf values is $cn^k(\frac{1}{2})^{k(k-1)/2}$.

Problem 5: Tree Induction (18 points)

Let's define a Diagonal tree to be a binary tree containing 2D points such that:

- Each leaf node contains $(-1,-1)$, $(2,8)$, or $(3,11)$.
- An internal node with one child labelled (a,b) has label $(a+2,b+6)$.
- An internal node with two childen labelled (x,y) and (a,b) has label $(x+a,y+b-2)$.

Use strong induction to prove that the root node of any Diagonal tree has a label on the line $y = 3x+2$.

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): $h = 0$ for the base case, i.e. each tree consists of a single node that is both the root and a leaf. So the node must have label $(-1,-1)$, $(2,8)$, or $(3,11)$. All three of these points lie on the line $y = 3x+2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Any Diagonal tree of height less than k ($k \geq 1$) has a root node whose label is a point on the line $y = 3x+2$.

Inductive Step:

Let T be a Diagonal tree of height k. There are two cases:

Case 1: The root has a single child. Suppose the child has label (a,b). Then the root of T has label $(a+2,b+6)$. By the inductive hypothesis, (a,b) must lie on the line $y = 3x+2$. That is $b = 3x+2$. Then $b+6 = (3x+2)+6 = 3x+6+2 = 3(x+2)+2$. So the point $(a+2,b+6)$ lies on the line $y = 3x+2$, which is what we needed to show.

Case 2: The root has two children. Suppose that their labels are (p,q) and (a,b). Then the root of T has label $(p+a,q+b-2)$. By the inductive hypothesis, (p,q) and (a,b) must lie on the line $y = 3x+2$. That is $q = 3p+2$ and $b = 3a+2$.

Adding these two equations together, we get that $q+b = 3p+2+3a+2$. So $q+b-2 = 3(p+a)+2$. Therefore, the point $(p+a,q+b-2)$ lies on the line $y = 3x+2$, which is what we needed to show.

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Problem 6: Induction (18 points)

Let's define a function $g : \mathbb{Z}^+ \to \mathbb{N}$ as follows:

$$\begin{aligned} g(1) &= 2 \\ g(2) &= 4 \\ g(n) &= 2g(n-1) + 3g(n-2) - 4 \text{ for } n \geq 3 \end{aligned}$$

Use (strong) induction to prove that $g(n) = 3^{n-1} + 1$ for all $n \geq 1$. [Printed as $n \geq 1$ but corrected at the exam.]

Proof by induction on n

Base case(s): If $n = 1$, then $3^{n-1} + 1 = 3^0 + 1 = 2$. This is equal to $g(1)$, so the claim holds.

If $n = 2$, then $3^{n-1} + 1 = 3^1 + 1 = 4$. This is equal to $g(2)$, so the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $g(n) = 3^{n-1} + 1$ for $n = 1, 2, \ldots, k-1$, where $k \geq 3$.

Rest of the inductive step: We need to show that the claim holds for $n = k$.

$g(k) = 2g(k-1) + 3g(k-2) - 4$ by the definition of g.

By the inductive hypothesis, we know that $g(k-1) = 3^{k-2} + 1$ and $g(k-2) = 3^{k-3} + 1$. Substituting these facts into the equation above, we get

$$\begin{aligned} g(k) &= 2(3^{k-2} + 1) + 3(3^{k-3} + 1) - 4 \\ &= 2 \cdot 3^{k-2} + 2 + 3^{k-2} + 3 - 4 \\ &= 3 \cdot 3^{k-2} + 1 = 3^{k-1} + 1. \end{aligned}$$

So $g(k) = 3^{k-1} + 1$ which is what we needed to show.