Sets, Relations

June 25, 2014

Yesterday

Product Rule

If A and B are sets, then

$$|A \times B| = |A| \times |B|$$

Complementary Counting

If A is a set and U is the universe, then

$$|A| = |U| - |\overline{A}|$$

Principle of Inclusion-Exclusion (PIE)

Theorem

If A, B, and C are sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

PIE Examples

- 1000 people are asked whether they like Coke or Pepsi. 600 like Coke, 500 like Pepsi, and 200 like neither. How many like both?
- How many positive integers between 1 and 1,000,000, inclusive, are multiples of 2, 3, or 5?
- Suppose that license plate patterns consist of a sequence of three letters followed by a sequence of three digits. How many possible license plates contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left)?
- There are three baskets and six basketball players. Each player chooses a basket to throw his/her ball in. How many ways are there for all of the baskets to be occupied?

Subsets

Definition

If A and B are sets, then A is a **subset** of B if every element of A is an element of B, and we denote this by $A \subseteq B$.

- Example: $\{3, 5, 7\} \subseteq \{2, 3, 4, 5, 7\}$
- Example: For any set S, we have $\emptyset \subseteq S$ and $S \subseteq S$.

Proving that sets are subsets

Proposition

If A, B, and C are sets, then $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

To prove that $S \subseteq T$, take an arbitrary element $x \in S$ and show that $x \in T$.
Proof:

Subset Transitivity

Proposition

If A, B, C are sets such that $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof:

Set equality

Proposition

If A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

To prove that $S = T$, we must show that $S \subseteq T$ and $T \subseteq S$.
Proof:

Proofs with sets

Proposition

If A, B, C are non-empty sets and $A \times B \subseteq B \times C$, then $A \subseteq C$.

Proof:

Relations

Definition

A **relation** R on a set A is a set of ordered pairs of elements from A. If $(x, y) \in R$, then x is **related** to y and we denote this by xRy.

Example: If $A = \{1, 2, 3\}$ and R is the relation "strictly less than," then

$$R = \{(1, 2), (1, 3), (2, 3)\}.$$

Also, $1\,R\,2, 1\,R\,3$, and $2\,R\,3$.

Reflexivity

Definition

A relation is **reflexive** if for all $x \in A$, we have $x \, R \, x$. In other words, every element is related to itself.

Definition

A relation is **irreflexive** if for all $x \in A$, we have $x \not{R} \, x$. In other words, no element is related to itself.

Note that it is possible for a relation to be neither reflexive nor irreflexive.

Symmetry

Definition

A relation is **symmetric** if for all x and y A, we have that $x\,R\,y$ implies $y\,R\,x$.

Definition

A relation is **antisymmetric** if for all x and y in A, we have that $x\,R\,y$ implies $y \not R\, x$. Equivalently, $x\,R\,y$ and $y\,R\,x$ imply that $x = y$.

Proposition

If $A = \mathbb{Z}^+$ and $R = \{(x, y) \in A^2 \mid x = y^2\}$, then R is antisymmetric.

Proof:

Transitivity

Definition

A relation is **transitive** if for all $x, y, z \in A$, $x \, R \, y$ and $y \, R \, z$ together imply $x \, R \, z$.

Proposition

If $A = \mathbb{Z}^+$ and $R = \{(x, y) \in A^2 \mid x = y^2\}$, then R is not transitive.

Proof:

Transitive Closure

Definition

The **transitive closure** of a relation R is the smallest transitive relation R^* such that $R \subseteq R^*$.

Example: If $A = \mathbb{Z}$ and $R = \{(1,2),(2,3)\}$, then the transitive closure of R is $\{(1,2),(1,3),(2,3)\}$.
Example: If $A = \mathbb{Z}$ and
$R = \{(1,3),(2,1),(2,5),(3,2),(4,3),(5,3),(5,4)\}$, then what is the transitive closure of R?

Equivalence Relations

Definition

An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.

Proposition

Let m be a positive integer. If $A = \mathbb{Z}$ and $R = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv y \pmod{m}\}$, then R is an equivalence relation.

Proof:

Equivalence Classes

Definition

Given a set A, a relation R, and an element $a \in A$, the equivalence class of a is

$$\{x \in A \mid (x, a) \in R\}.$$

Example: In the previous proposition, if $m = 5$, then the equivalence class of 4 is $\{\cdots -6, -1, 4, 9, \dots\}$. We denote this by [4].

Proposition

For a given set A and relation R, the equivalence classes of R partition A. That is, the intersection of any two distinct equivalence classes is empty, and the union of all of the equivalence classes is A.

Proof:

Partial Orders

Definition

A **partial order** is a relation that is reflexive, antisymmetric, and transitive.

Proposition

If $A = \mathbb{Z}$ and $R = \{(x, y) \in \mathbb{Z}^2 : x \mid y\}$, then R is a partial order.

Proof:

Partial Orders

Definition

A **strict partial order** is a relation that is irreflexive, antisymmetric, and transitive.

Proposition

If $A = \mathbb{Z}$ and $R = \{(x, y) \in \mathbb{Z}^2 : x \mid y \wedge x \neq y\}$, then R is a strict partial order.

Proof:

Hasse Diagram

For a partially ordered set A, one represents each element of A as a vertex in the plane and draws a line segment or curve that goes upward from x to y whenever y covers x (that is, whenever $x \neq y$, $x\,R\,y$, and there is no z other than x or y such that $x\,R\,z$ and $z\,R\,y$).
Example: $A = \mathbb{Z}^+$ and $R = \{(x, y) \in \mathbb{Z}^2 : x \mid y\}$