Relations, Functions

June 30, 2014

Relations

Definition

A **relation** R on a set A is a set of ordered pairs of elements from A. If $(x, y) \in R$, then x is **related** to y and we denote this by xRy.

Partial Orders

Definition

A relation is **reflexive** if for all $x \in A$, we have $x \, R \, x$.

Definition

A relation is **antisymmetric** if $x \, R \, y$ and $y \, R \, x$ imply that $x = y$.

Definition

A relation is **transitive** if for all $x, y, z \in A$, $x \, R \, y$ and $y \, R \, z$ together imply $x \, R \, z$.

Partial Orders

Definition

A **partial order** is a relation that is reflexive, antisymmetric, and transitive.

Proposition

If $A = \mathbb{N}$ and $R = \{(x, y) \in \mathbb{N}^2 : x \mid y\}$, then R is a partial order.

Proof:

Partial Orders

Definition

A **strict partial order** is a relation that is irreflexive, antisymmetric, and transitive.

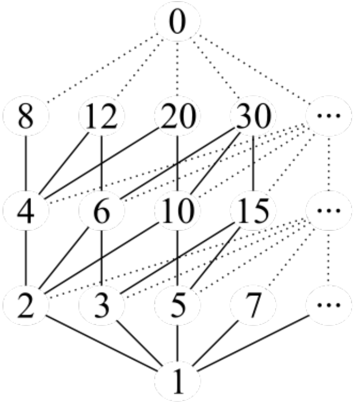
Proposition

If $A = \mathbb{N}$ and $R = \{(x, y) \in \mathbb{N}^2 : x \mid y \wedge x \neq y\}$, then R is a strict partial order.

Proof: Almost the same as for partial order.

Hasse Diagram

A line segment or curve goes upward from x to y whenever $x \neq y$, $x\,R\,y$, and there is no z other than x or y such that $x\,R\,z$ and $z\,R\,y$.



Functions

Definition

- A relation between two sets A and B is a set of ordered pairs (x, y) where $x \in A$ and $y \in B$.
- A **function** is a relation between a set of inputs (**domain**) and a set of permissible outputs (**co-domain**) such that each input is related to exactly one output.
- If f relates an input x to an output y, then f **maps** x to y and $f(x) = y$.
- If A is the domain and B is the co-domain, then the **function signature** is $f : A \to B$.

Examples of functions

- A is the set of people in this room, $B = \mathbb{Z}^+$, $f(x)$ is the age of person x.
- A is the set of people in this room, B is the set of all colors, $f(x)$ is the shirt color of x.
- $A = \mathbb{Z}$, $B = \mathbb{Z}$, and $f(x) = x^2$.

More random facts

- Each element in the domain must be mapped to exactly one element in the co-domain.
- Two functions $f : A \to B$ and $g : C \to D$ are equal if and only if $A = C$, $B = D$, and $f(x) = g(x)$ for every $x \in A$.

Image versus co-domain

- The co-domain is the set of *permissible* outputs (and appears in the signature).
- The **image** is the set of *actual* outputs.
- Example: If $f : \mathbb{Z} \to \mathbb{Z}$ and $f(x) = |x|$, then the $\mathbb{Z}$ is the co-domain and $\mathbb{N}$ is the image.

Definition

A function is **onto** if its image covers its entire co-domain. Equivalently, $f : A \to B$ is onto if every $y \in B$ is the output for some $x \in A$. In symbols,

$$\forall y \in B, \exists x \in A, f(x) = y.$$

Proving functions are onto

- Show that every element in the co-domain is also in the image.
- In other words, pick an arbitrary element $y \in B$ and show that there exists an $x \in A$ such that $f(x) = y$.
- Example: Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = 2x + 1$, then f is onto.
- Example: Prove that if $f : \mathbb{N}^2 \to \mathbb{Z}$ and $f(x, y) = x - y$, then f is onto.

Proving functions are not onto

- To show that a function is not onto, show that there are elements in the co-domain not in the image.
- A function is not onto if

$$\neg(\forall y \in B, \exists x \in A, f(x) = y) \tag{1}$$
$$\equiv \exists y \in B, \neg(\exists x \in A, f(x) = y) \tag{2}$$

- In other words, find some $y \in B$ such that there is no $x \in A$ satisfying $f(x) = y$.
- Example: Prove that if $f : \mathbb{Z} \to \mathbb{Z}$ and $f(x) = x^2$, then f is not onto.

Composition

Definition

If $f : A \to B$ and $g : B \to C$, then their **composition** is $g \circ f$ and is defined by $(g \circ f)(x) = g(f(x))$.

Example: If $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$, and $g(x) = 2x + 3$, then what is $(g \circ f)(x)$? $(f \circ g)(x)$?

Proposition

If $f : A \to B$, $g : B \to C$, and both f and g are onto, then $g \circ f$ is onto.

One-to-one

Definition

A function is **one-to-one** (as opposed to many-to-one) if it never maps two different inputs to the same output:

$$x \neq y \rightarrow f(x) \neq f(y).$$

In other words, it passes the "horizontal line test."

Proving that a function is one-to-one

- f is one-to-one if

$$x \neq y \rightarrow f(x) \neq f(y),$$

  which is equivalent to

$$f(x) = f(y) \rightarrow x = y$$

- Pick two arbitrary elements x_1 and x_2 in A. Show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- Example: Prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = 2x - 1$, then f is one-to-one.

Proving functions are not one-to-one

- Find two inputs that map to the same output.
- Example: Prove that if $f : \mathbb{Z} \to \mathbb{Z}$ and $f(x) = x^4$, then f is not one-to-one.

Composition

Definition

If $f : A \to B$ and $g : B \to C$, then their **composition** is $g \circ f$ and is defined by $(g \circ f)(x) = g(f(x))$.

Proposition

If $f : A \to B$, $g : B \to C$, and both f and g are one-to-one, then $g \circ f$ is one-to-one.

Strictly increasing functions

Definition

A function f is **strictly increasing** if $x < y$ implies $f(x) < f(y)$.

Proposition

Any strictly increasing function is one-to-one.

Pigeonhole Principle

Proposition

If you try to stuff n pigeons into k holes and $n > k$, then at least one hole will contain more than one pigeon.

Pigeonhole principle examples

- If there are 367 people in the room, then two must have the same birthday.
- If you pick 17 integers, then two of them will differ by a multiple of 16.
- If you pick five numbers from the integers 1 through 8, then two of them must add up to 9.
- If you throw five darts on a 2×2 square, then two darts will be within $\sqrt{2}$ of each other.
- At any party with two or more people, there must be at least two people who have the same number of friends (assuming that all of your friends consider you to be a friend as well).

Generalized Pigeonhole Principle

Proposition

If you try to stuff n pigeons into k holes, then at least one hole will contain at least n/k pigeons.

- "The maximum must be at least the average."
- Example: On New Year's, 300,000 people are in Times Square. At least $300000/366 = 819.7$ people will have the same birthday.

Bijections

Definition

A function $f : A \to B$ is a **bijection** if it is both one-to-one and onto.

Proposition

1. If $f : A \to B$ is onto, then $|A| \geq |B|$.
2. If $f : A \to B$ is one-to-one, then $|A| \leq |B|$.
3. If $f : A \to B$ is a bijection, then $|A| = |B|$.

Proof: By contrapositive and pigeonhole principle.

Inverse functions

Definition

If $f : A \to B$ is a bijection, then we define the inverse function $f^{-1} : B \to A$ by saying that $f^{-1}(y) = x$ if $f(x) = y$.

Since f is onto, there is at least one such x. Since f is one-to-one, there is only one such x.

Permutations

- There are 6 objects and I want to take 3 of them and arrange them in a row. How many arrangements are possible?
- There are n objects and I want to take k of them and arrange them in a row. How many arrangements are possible?
- Suppose $|A| = |B| = n$. How many bijections are there from A to B?
- Suppose $|A| = m$, $|B| = n$, and $m \leq n$. How many one-to-one functions are there from A to B?

More permutations

- How many ways can 7 people stand in line if Alice and Bob will not stand next to each other?
- How many ways can 7 people stand in line facing forward or backward if no two people standing next to each other can face one another?
- How many ways can 7 people stand in a circle?

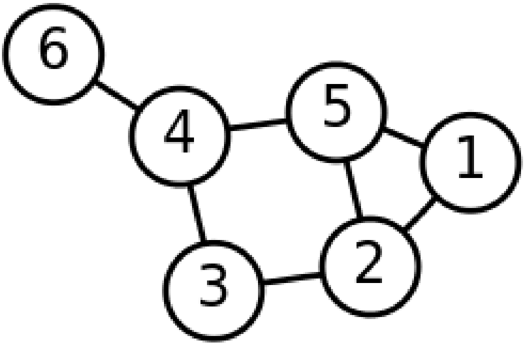
Multinomial coefficients

- How many possible distinct arrangements are there of the letters in the word “DOG”?
- How many possible distinct arrangements are there of the letters in the word “TATTER”?
- How many possible distinct arrangements are there of the letters in the word “MISSISSIPPI”?

Graphs

Definition

A **graph** is a set of objects (**nodes** or **vertices**) where some pairs of objects are connected by links (**edges**).



Graphs

- Edges are represented by ordered pairs of nodes.
- If V is the set of vertices and E is the set of edges, then the graph is G, where $G = (V, E)$.

Example: $G =$
$(\{1, 2, 3, 4, 5, 6\}, \{(1, 2), (1, 5), (2, 3), (2, 5), (3, 4), (4, 5), (4, 6)\})$

Kinds of graphs

- In a **directed graph**, the edges have a direction.
- In a **multigraph**, we allow multiple edges between the same pair of vertices.
- A **self-loop** is an edge connecting a vertex to itself.
- In a **simple graph**, there are no multiple edges or self-loops.
- **Important: Unless otherwise indicated, assume graphs are simple and undirected.**

Vertex degree

Definition

An edge is **incident** to a vertex v if it links v to another vertex u. We then say u and v are **adjacent** and that they are **neighbors**. The **degree** of a vertex v is denoted $\deg(v)$ and is the number of edges incident to it.

Proposition (Handshake Lemma)

In any graph, we have

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Special Graphs

Definition

The **complete graph** on n nodes is denoted K_n. It has n nodes, and every pair of nodes is linked by an edge. C_n is the cycle graph on n nodes, and W_n is the wheel graph with n spokes and $n+1$ nodes.

Proposition

K_n has $n(n-1)/2$ edges.

Graph isomorphism

Definition

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $f : V_1 \to V_2$ such that $(u, v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2$.

In other words, two graphs are isomorphic if we can relabel and move the vertices in one to get the other.

Subgraphs

Definition

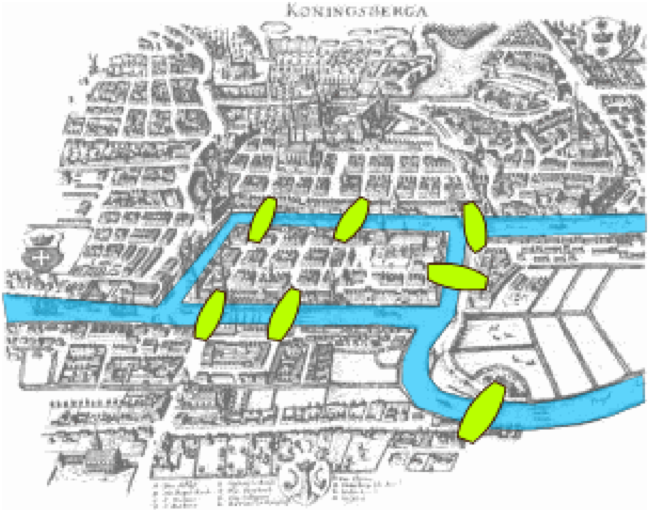
A graph $G_1 = (V_1, E_1)$ is a subgraph of $G_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$.

Proving graphs are not isomorphic

G and H are non-isomorphic if

- G and H have different numbers of vertices or edges.
- The degree sequences (i.e., list of degrees of the vertices) of G and H don't match.
- G contains a subgraph that H doesn't.
- Other things? Graph isomorphism is an unsolved problem...

Seven bridges of Konigsberg



Can you cross all seven bridges without crossing any bridge twice?