Logic, Proofs

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Implication

- In $p \rightarrow q$, p is the **hypothesis** and q is the **conclusion**.
- "If the sky is blue, then grass is green" is true.
- "If the sky is blue, then $4 > 2$" is false.
- If the hypothesis is false, then the implication is true.
- "If pigs fly, then Urbana is in Illinois" is true.
- "If pigs fly, then I am the president of the U.S." is true.

Converse, contrapositive

- If I drank poison today, then I will be sick tomorrow.
- Converse: If I am sick tomorrow, then I drank poison today.
- Contrapositive: If I am not sick tomorrow, then I did not drink poison today.
- A statement is equivalent to its contrapositive, but not its converse.

Negation

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $\neg(p \to q) \equiv p \wedge \neg q$

Statement to negate

If I drop this plate, then I will go to the store and buy a new one.

Negation

Statement to negate

If I drop this plate, then I will go to the store and buy a new one.

- Let $D =$ "I drop this plate," $S =$ "I will go to the store," and $B =$ "I will buy a new plate."

$$\neg(D \rightarrow (S \wedge B)) \equiv D \wedge \neg(S \wedge B) \quad (1)$$
$$\equiv D \wedge (\neg S \vee \neg B) \quad (2)$$

"I will drop this plate, and I will either not go to the store or not buy a new plate."

Negation, Distributive Law

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- "I will go to the store and buy butter, margarine, or both" is equivalent to "Either I will go to the store and buy butter, or I will go to the store and buy margarine, or both"
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- "Either I will eat out or I will go home and make dinner" is equivalent to "I will either eat out or go home, and I will either eat out or make dinner"

Negation, Distributive Law

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Formula to negate

If I drop this plate, then I will go to the store and buy a new one.

$$\begin{aligned} \neg(D \rightarrow (S \wedge B)) &\equiv D \wedge \neg(S \wedge B) && (3) \\ &\equiv D \wedge (\neg S \vee \neg B) && (4) \\ &\equiv (D \wedge \neg S) \vee (D \wedge \neg B) && (5) \end{aligned}$$

"Either I will drop this plate and not go to the store, or I will drop this plate and not buy a new plate, or both"

Predicates

- A **predicate** is a statement that becomes true or false if you substitute in values for its variables. Predicates do not have truth values.
- Examples: "$x^4 \geq x + 10$,"
  "x is a plant".
- We can use letters to denote predicates.
- Examples: $P(x) =$ "$x^2 > 78$,"
  $Q(y) =$ "y is an animal"
  $R(x, y) =$ "x is the mother of y"
- $P(x)$ is true when $x = 10$ but false when $x = 0$.
  $Q(y)$ is true when y is a cheetah but false when y is bubble gum.

Quantifiers ($\forall, \exists$)

- We can make general statements about what happens when we substitute a variety of values for the variables.
- Examples: "For all integers x, we have $x^4 \geq x + 10$"
  "There exists an integer x such that $x^4 \geq x + 10$"
- $\forall$ means "for all"
- $\exists$ means "there exists"
- Examples: "$\forall x \in \mathbb{Z},\ x^4 \geq x + 10$"
  "$\exists x \in \mathbb{Z}, x^4 \geq x + 10$"
- ($\in$ means "in")
- ($\mathbb{Z}$ is the set of integers)

Quantifiers

- You must specify what values your variables can take.
- Example: $\forall x \in \mathbb{Z}^+, x^3 \geq 0$ is true,
  but $\forall x \in \mathbb{R}, x^3 \geq 0$ is false.
- ($\mathbb{Z}^+$ is the set of positive integers)
- ($\mathbb{R}$ is the set of real numbers)

Contrapositive

- $\forall x, P(x) \rightarrow Q(x) \equiv \forall x, \neg Q(x) \rightarrow \neg P(x)$
- "For any apartment, if it is infested with bedbugs, then I do not want to live in it."
- Contrapositive: "For any apartment, if I want to live in it, then it was not infested with bedbugs."

Multiple variables

- Example: "$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y = 0$" means that there exists a real number y such that for all real numbers x, it is true that $x + y = 0$.
- Example: "$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0$" means that for every real number x, there exists a real number y such that $x + y = 0$.
- Are these statements the same?

Negation

Suppose $p(x)$ is some property of x (say, "x is red" or "$x^2 > 21$").

- The opposite of "For all x, $P(x)$ is true" is "There exists x such that $P(x)$ is false."
- $\neg(\forall x, P(x)) \equiv \exists x, \neg P(x)$
- Examples: The negation of "All crocodiles are scary" is "At least one crocodile is not scary."
- The opposite of "There exists x such that $P(x)$ is true" is "For all x, $P(x)$ is false."
- $\neg(\exists x, p(x)) \equiv \forall x, \neg p(x)$
- The negation of "At least one crocodile is scary" is "All crocodiles are not scary."

Negation Example

Statement to negate

There is a crocodile such that if any person can handle the crocodile, then that person will be awarded a million dollars.

- Let $H(x, y) =$ "Person x can handle crocodile y."
- Let $A(x) =$ "Person x is awarded a million dollars."
- Let C be the set of all crocodiles.
- Let P be the set of all people.

Need to simplify

$$\neg(\exists y \in C, \forall x \in P, H(x, y) \rightarrow A(y))$$

Negation Example

- $\neg(\forall x, P(x)) \equiv \exists x, \neg P(x)$
- $\neg(\exists x, p(x)) \equiv \forall x, \neg p(x)$
- $\neg(p \rightarrow q) \equiv p \wedge \neg q$

$$\neg(\exists y \in C, \forall x \in P, H(x,y) \rightarrow A(y)) \tag{6}$$
$$\equiv \forall y \in C, \neg(\forall x \in P, H(x,y) \rightarrow A(y)) \tag{7}$$
$$\equiv \forall y \in C, \exists x \in P, \neg(H(x,y) \rightarrow A(y)) \tag{8}$$
$$\equiv \forall y \in C, \exists x \in P, H(x,y) \wedge \neg A(y) \tag{9}$$

"For each crocodile, there is a person such that that person can handle the crocodile but will not be awarded a million dollars."

Proving universal statements

Proposition

If k is an odd integer, then so is k^3. (For all integers k, if k is odd, then k^3 is odd.)

Proof:

1. Let k be an odd integer.
2. Since k is odd, we know that $k = 2n + 1$ for some integer n.
3. Thus,

$$\begin{aligned} k^3 &= (2n+1)^3 = (2n+1)(4n^2+4n+1) \\ &= 8n^3 + 12n^2 + 6n + 1 = 2(4n^3 + 6n^2 + 3n) + 1, \end{aligned}$$

which is odd.

General Procedure

1. State the hypothesis.
2. Apply technical definitions of terms like “odd” or “rational.”
3. Manipulate expressions until claim is verified.

More things to prove

- If a and b are odd, then so is ab.
- If k is rational, then k^2 is rational.
- If n is an integer, then $4(n^2 + n + 1) - 3n^2$ is a perfect square.
- The product of any two rationals is rational.

Proving existential statements

Proposition

There exists an integer n such that $n^2 \geq 2^n$.

Proof:
If $n = -1$, then $n^2 = 1 \geq 1/2 = 2^n$.

Disproving existential statements

Claim to disprove

There exists an integer n such that $n^2 - 4n + 4 < 0$.

- We must show that $\neg(\exists n \in \mathbb{Z}, n^2 - 4n + 4 < 0)$.
- ...or $\forall n \in \mathbb{Z}, \neg(n^2 - 4n + 4 < 0)$
- ...or $\forall n \in \mathbb{Z}, n^2 - 4n + 4 \geq 0$.

Proof:
For all integers n, we have $n^2 - 4n + 4 = (n-2)^2 \geq 0$.

Disproving universal statements

Claim to disprove

For any real number x, if $|x| > 5$, then $x^3 > x^2$.

Casework

Proposition

If x is real and $|x + 7| > 8$, then $|x| > 1$.

Proof:

1. Let x be a real number such that $|x + 7| > 8$.
2. If $|x + 7| > 8$, then either $x + 7 > 8$ or $x + 7 < -8$.
3. If $x + 7 > 8$, then $x > 1$, so $|x| > 1$.
4. If $x + 7 < -8$, then $x < -15$, so $|x| > 1$.
5. In all cases, $|x| > 1$.

Rephrasing claims

Proposition

There is no integer k such that k and is odd and k^2 is even.

- We must show $\neg(\exists k \in \mathbb{Z}, odd(k) \wedge even(k^2))$
- ...or $\forall k \in \mathbb{Z}, \neg(odd(k) \wedge even(k^2))$
- ...or $\forall k \in \mathbb{Z}, \neg odd(k) \vee \neg even(k^2))$
- ...or $\forall k \in \mathbb{Z}, even(k) \vee odd(k^2))$

Now it's a proof by cases (k is either even or odd...)