

Chap8 — 5

多元复合函数的微分法

8.5.1 复合函数的偏导数

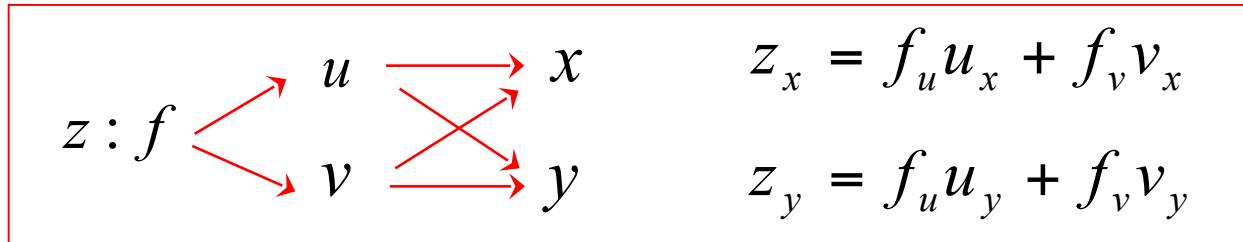
链法则

函数 $u = u(x, y), v = v(x, y)$ 在 (x, y) 存在偏导数，
 $z = f(u, v)$ 在相应的 (u, v) 处可微，则复合函数

$$z = f(u(x, y), v(x, y))$$

存在偏导数

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$



例 求 z_x, z_y

$$(1) \quad z = e^u \sin v, \text{ 而 } u = \frac{x^2}{y}, v = x^2 - xy + y^2$$

$$(2) \quad z = x^{xy}$$

例 设函数 $u = f(x, y, z)$ 可微, 而
 $x = x(t), y = y(t), z = z(t)$ 均可导; 试求复合函数 $u = f(x(t), y(t), z(t))$
对 t 的导数 (称为全导数)

例 $z = f(x, y, u, v)$ 可微, 而 $u = u(x, y), v = v(x, y)$
有二阶偏导数, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

例 设函数 $z = f(x - y, \frac{x}{y})$, f 是可微函数,
求偏导数 z_x, z_y

例 设 $z = f(x^2 - y^2, \varphi(xy))$, f, φ 可微, 求 $xz_x - yz_y$
(往年试题)

例 设函数 $z = f(x, y)$ 可微, 作变换 $x = r \cos \theta$,
 $y = r \sin \theta$, 试将

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

从变换关系看
宜对 r, θ 求导

化为以 r, θ 为变量的形式

例 函数 $z = f(xy, \frac{x}{y})$, f 有连续二阶偏导数, 求

$$\frac{\partial^2 z}{\partial x^2} \text{ 及 } \frac{\partial^2 z}{\partial x \partial y}$$

例 $F(x, y) = \int_a^{x+2y} f(t, t^2) dt$, f 可微, 求
 F_{xy}

例 设 $f(x, y) \in C^1$, $f(x, x^2) = x^3$, $f_x(x, x^2) = x^2 - 2x^4$

求 $f_y(x, x^2)$

(往年试题)

例 若 $f(x,y)$ 在 $(0,0)$ 可微, 且

$$\begin{aligned} & f(0,0)=0, \\ & f_x(0,0) = 1, f_y(0,0) = 2, \text{ 求 } \lim_{x \rightarrow 0} [1 + f(x, 2x)]^{\frac{1}{x}} \end{aligned}$$

(往年试题)

习题 8

21 (1) (4) 22 (1) (3)

23 (2) (3) 24 (2) (4)

25 (1) 26

27 (2) (3) 28 (2)

29 30 31 (1)

8.5.2 隐函数的偏导数

一. 隐函数及其偏导数

设函数 F 在 (x_0, y_0, z_0) 邻域内有连续偏导数，
且 $F(x_0, y_0, z_0) = 0, F_z(x_0, y_0) \neq 0$

则方程 $F(x, y, z) = 0$ 在 (x_0, y_0, z_0) 邻域内可确定唯一的
函数 $z = f(x, y)$, 满足

$$F(x, y, f(x, y)) \equiv 0, \quad z_0 = f(x_0, y_0)$$

且有

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

例 设 $z=z(x,y)$ 是由方程 $z = (3x + y)^{z+x}$
所确定的隐函数，求 z_x, z_y

例 设 $z=z(x,y)$ 是由方程 $x + y + z = e^{2z}$
所确定的隐函数，求 z_{xy}

例 设 $u=u(x,y)$ 是由方程 $u = \varphi(u) + \int_y^x p(t)dt$
所确定的隐函数，其 φ', p 连续, $\varphi'(u) \neq 1$ 函数
 $u=f(u)$, 求 $p(y)z_x + p(x)z_y$ (往年考研题)

二. 隐函数组及其偏导数

若函数 $F(x, y, u, v), G(x, y, u, v)$ 在点 $P_0(x_0, y_0, u_0, v_0)$ 某一邻域内有连续的偏导数；且 $F(x_0, y_0, u_0, v_0) = 0, G(x_0, y_0, u_0, v_0) = 0$ 行列式

$$J = \frac{\partial(F, G)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}$$

在点 P_0 不等于 0，则 $\begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases}$ 可唯一确定函数

$u = u(x, y), v = v(x, y)$ 满足此方程组及 $\begin{cases} u_0 = u(x_0, v_0) \\ v_0 = v(x_0, v_0) \end{cases}$

且有连续偏导数

$$\frac{\partial u}{\partial x} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(x, v)} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \\ \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \\ \end{vmatrix}}$$
$$\frac{\partial u}{\partial y} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(y, v)} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \\ \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \\ \end{vmatrix}}$$

$$\frac{\partial v}{\partial x} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(u, x)} = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \\ \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \\ \end{vmatrix}}$$
$$\frac{\partial u}{\partial y} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(u, y)} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \\ \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \\ \end{vmatrix}}$$

例 设函数 $u = u(x, y), v = v(x, y)$ 由方程组

$$\begin{cases} x^2 + y^2 - uv = 0 \\ xy - u^2 + v^2 = 0 \end{cases}$$

确定，求 $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ 及 $\frac{\partial v}{\partial y}$

例 函数 $y=y(x), z=z(x)$ 由方程组

$$\begin{cases} z = x + \varphi(x + y) \\ F(x, y, z) = 0 \end{cases}$$

其中 φ, F 均可微， $F_y + \varphi'F_z \neq 0$ ，求 y', z'

8.5.3 一阶全微分形式的不变性

函数 $z=f(u,v)$ 的全微分

$$dz = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv,$$

若 u, v 又是 x, y 的可微函数 $u = u(x, y), v = v(x, y)$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \quad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy.$$

复合函数 $z = f(u(x, y), v(x, y))$ 的全微分为

$$\begin{aligned} dz &= \left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \right) dx + \left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} \right) dy \\ &= \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv \end{aligned}$$

对 $f=f(u, v)$, 无论 u, v 是自变量或函数,

$$df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$$

多元函数全微分运算法则

$$(1) \quad d(u \pm v) = du \pm dv$$

$$(2) \quad d(uv) = udv + vdu$$

$$(3) \quad d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2} \quad (v \neq 0)$$

■ 在求隐函数（尤其是隐函数组）所有一阶偏导数时，利用微分形式不变性较简便

例 $z=f(x, y)$ 由方程 $z = x + y - xe^{x-y-z}$ 确定，
求 dz
(93 年考研题)

例 设 $v = x + u, y = u^2 + x^2, z = u^3 + x^3 + e^v$ 求由此方程
确定的函数的偏导数 z_x, z_y

H.W

习题 8

33 (1) (2) (3) (4)

34 (1) (3)

35 (1) 36 (2)

37 (1) (2) 39 (1) (3)