

AN APPROXIMATION METHOD TO PRICE VOLATILITY OPTIONS

by

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ABSTRACT

We propose an approximation method to value volatility options. This method is based on choosing models with closed form solution as an auxiliary model, and derive a mis-pricing formula between the true price and the auxiliary one, then apply Ito-Taylor expansions on the mis-pricing formula to create increasingly improved refinements. We propose an approach to evaluate volatility options under mean-reverting models, in which auxiliary model selection and expansion methods are explained. Method in this paper is applied to mean-reverting Constant elasticity of variance(CEV) model and double CEV models. Numerical results show that the proposed method is accurate and efficient.

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Table of Contents

Abstract	iii
Dedication	iv
Acknowledgments	iv
List of Tables	vii
List of Figures	viii
1 Introduction	1
2 Method Description	3
2.1 The DOI Variance Reduction Method	3
2.2 KM's approximation Method based on the DOI method	6
3 Approximations of VIX options	9
3.1 Approximating options under mean-reverting CEV model	9
3.1.1 Drawbacks of using Black-Scholes model as an auxiliary model	9
3.1.2 Using square root mean-reverting model as auxiliary model	11
3.1.3 Method to Calculate Derivatives In Expansions	13
3.2 Approximating options under double Heston model	19
4 Numerical Results	21
4.1 Volatility option prices under mean-reverting CEV model	21
4.2 Volatility option prices under Heston puls CEV model	24

5 Conclusion	27
A Appendix A Mean-Reverting CEV model Numerical Results	1
Bibliography	3

List of Tables

A.1	T=0.3,K=0.15, kappa=4,m=0.2, sigma=0.15, gamma=0.3.csv	2
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List of Figures

3.1	Call option price with regard to time to maturity	10
3.2	Call option price with regard to volatility	11
3.3	Deltas are calculated by our formula, and finite difference method. The parameters used are $T = 0.3, \alpha = 0.60, \beta = 4.00, \sigma = 0.133, r = 0.05$, and $K = 0.15$.	16
3.4	gammas are calculated by our formula, and finite difference method. The parameters used are $T = 0.3, \alpha = 0.60, \beta = 4.00, \sigma = 0.133, r = 0.05$, and $K = 0.15$.	17
4.1	mean-reverting CEV model result 1	22
4.2	mean-reverting CEV model result 2	23
4.3	Heston plus CEV model result 1	25
4.4	Heston plus CEV model result 2	25

Chapter 1

Introduction

The Volatility, as a measurement of the risk, is one of the most important element in the study of finance. Whaley (1993) proposes that by allowing investors to hedge directly against shifts in volatility, these securities enable investors to avoid the costs of dynamically adjusting positions for changes in volatility and serve to make the market more complete. During the past few years, derivatives written on volatility have developed rapidly, in which options are clearly the fundamental type of derivatives. However, unlike options on stocks, pricing volatility options can be challenging by its mean-reverting property. On the basis of Cox et al. (1985), Hull and White (1987), Heston (1993), Grünbichler and Longstaff (1996) proposes Mean-Reverting Square Root(MRSR) model; Chan et al. (1992) then generalizes Mean-Reverting constant elasticity of variance(MRCEV) model and Gatheral (2008) proposes the Double CEV model to capture the dynamics of volatility. Grünbichler and Longstaff (1996) gives a closed-form solution for options under MRSR model, but there're no existing solutions to the following two models. The aim of our paper is to use the solution of square-root mean-reverting model to approximate the other two ¹models.

Heath and Platen (2002)(HP) first introduce a diffusion operator integral (DOI) method, he uses an auxiliary model and then apply Ito calculus on it to find unbiased variance reduction estimators for the true model, he also adapt DOI method to be employed in conjunction with PDE methods and test his method with Heston model. Similar to the idea of HP, Kristensen and Mele (2011)(KM) doesn't use auxiliary model to serve as a variance reduction estimator, but apply Ito-Taylor

¹Strictly speaking, for double CEV model, we mainly consider Heston plus CEV model, that is $\gamma_1 = \frac{1}{2}$. Details are discussed in 3.2

expansions on the difference of auxiliary model and true model to create increasingly improved refinements. KM apply his method on several fields including bond pricing under CIR model and option pricing under stochastic volatility models.

Based on HP and KM's work, we extend their methods to price options under mean-reverting models. This paper is organized as following, in chapter 2, we illustrate DOI method and KM's expansion method; In chapter 3, we discuss our method on approximating option prices under mean-reverting CEV model and Heston plus CEV model, including the auxiliary model selection, techniques of taking partial derivatives; In chapter 4, we show our numerical results and we put our conclusion in chapter 5.

Chapter 2

Method Description

In this section, we will discuss HP's DOI method and KM's approximation method. In section 2.1, we introduce HP's DOI method on the approximation of solutions of parabolic PDEs; In section 2.2, we illustrate complementary contents for KM's approximation method.

2.1 The DOI Variance Reduction Method

Consider a multi-factor model, in which a d -dimensional vector of state variables X_t on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ satisfies the following Stochastic Differential Equations(SDEs)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \quad (2.1.1)$$

where $\mu(t, X_t)$ and $\sigma(t, X_t)$ are drift and diffusion functions under the risk-neutral measure \mathbb{Q} . HP imposes $\mu : [0, T] \times \Gamma \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \Gamma \rightarrow \mathbb{R}^d$, in which Γ is assumed to be a bounded subset of \mathbb{R}^d . μ and σ also satisfies appropriate growth and Lipschitz conditions such that equation(2.1.1) admits a unique strong solution and is Markovian; W_t is a d -dimensional standard Brownian Motion and $t \in [0, T]$.

Let $w(t, x)$ be the valuation function satisfying $w : [0, T] \times \Gamma$, payoff function $h(t, x)$ satisfying $h : B \rightarrow \mathbb{R}$, $B \subseteq \mathbb{R}^d$. Define the infinitesimal generator L associated with equation (2.1.1) to be

$$Lw(t, x) = \frac{\partial w}{\partial t} + \sum_{i=1}^d \mu_i(t, x) \frac{\partial w}{\partial x} + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d (\sigma(t, x) \sigma^\top(t, x))_{i,j} \frac{\partial^2 w}{\partial x_i \partial x_j} \quad (2.1.2)$$

Let $R(t, x)$ be the instantaneous short-term interest rate also satisfying $R : B \rightarrow \mathbb{R}$, combining with equation (2.1.1) and equation (2.1.2), the task is to find an approximation to the solution to the following partial differential equation(PDE)

$$Lw(x, t) = R(x, t)w(x, t) \quad (2.1.3)$$

with boundary condition $w(t, x) = h(t, x)$. It's easily seen that under risk neutral measure \mathbb{Q} , the instantaneous option price change is equal to the price gain in saving account.

Next we consider to use a m -dimensional($m \leq d$) process $\bar{X}(t)$ which is a simpler auxiliary model to approximate the price of option. $\bar{X}(t)$ satisfies the following SDE

$$d\bar{X}_t = \begin{cases} \bar{\mu}_i(t, \bar{X}_t)dt + \bar{\sigma}_i(t, \bar{X}_t)dW_t & 1 \leq i \leq m \\ 0 \cdot dt + 0 \cdot dW_t & m < i \leq d \end{cases} \quad (2.1.4)$$

where $\bar{\mu}(t, \bar{X}_t)$ and $\bar{\sigma}(t, \bar{X}_t)$ are drift and diffusion functions, and they are also assumed to satisfy appropriate conditions such that equation(2.1.4) admits a unique strong solution and is Markovian. In KM's method, it's automatically satisfied because he supposes there always exists closed-form solution to auxiliary model such that Ito-Taylor expansions can be applied further.

Let $\bar{w}(t, x)$ be the option price written on process \bar{X}_t , the infinitesimal generator \bar{L} for option price \bar{w} is the same as equation(2.1.2) but replacing $\mu(t, x)$, $\sigma(t, x)$ by

$\bar{\mu}(t, x)$ and $\bar{\sigma}(t, x)$. Therefore $\bar{w}(t, x)$ is a solution to

$$\bar{\mathcal{L}}\bar{w}(x, t) = R(x, t)\bar{w}(x, t) \quad (2.1.5)$$

Denote the price difference $\Delta w(t, x) = w(t, x) - \bar{w}(t, x)$, then Δw satisfies

$$L\Delta w(t, x) + (L - \bar{L})w(t, x) = R(x, t)\Delta w(t, x) \quad (2.1.6)$$

with boundary condition $\Delta w(t, x) = 0$. Next we define

$$\begin{aligned} \delta(t, x) &= (L - \bar{L})\bar{w}(t, x) \\ &= \sum_{i=1}^d \Delta\mu_i(t, x) \frac{\partial \bar{w}(t, x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d \Delta\sigma_{ij}^2(x, t) \frac{\partial^2 \bar{w}(x, t)}{\partial x_i \partial x_j} \end{aligned} \quad (2.1.7)$$

using Feynman-Kac representation leads to option price $w(t, x)$ taking the following form

$$w(t, x) = \bar{w}(t, x) + \int_t^T \mathbb{E}^{t,x} \left[\exp \left(- \int_t^s R(u, X_u) du \right) \delta(s, X_s) \right] ds \quad (2.1.8)$$

Finally, under the initial condition $Z_0 = w(0, x)$, the DOI estimator

$$Z_t = \bar{w}(t, x) + \int_t^T \exp \left(- \int_t^s R(u, X_u) du \right) \delta(s, X_s) ds \quad (2.1.9)$$

is an unbiased estimator for Z_0 . And if a good auxiliary model is chosen, for example, when \bar{L} is close to L , the variance of Z_t will be small.

2.2 KM's approximation Method based on the DOI method

Recall equation(2.1.8), this equation is the ¹Theorem 1(Asset Price Representation) in Kristensen and Mele (2011), KM names equation (2.1.7) the mispricing function. His method is to look for an approximation of the error term in order to adjust the price $\bar{w}(t, x)$ for the error due to the use of the auxiliary model to price options.

We first introduce Ito-Taylor expansion, for sufficiently smooth function $f(t, x)$, it is given by

$$\mathbb{E}^{t,x}[f(s, X_s)] = \sum_{N=0}^J \frac{(s-t)^N}{N!} L^N f(t, x) + \mathcal{R}_J \quad (2.2.1)$$

where L is the infinitesimal generator we have defined in equation (2.1.2), the remainder term \mathcal{R}_J is given by

$$\mathcal{R}_J = \mathbb{E}^{t,x} \left[\int_t^s du_1 \int_t^{u_1} du_2 \cdots \int_t^{u_J} L^{J+1} f(u_{J+1}, X(u_{J+1})) du_{J+1} \right] \quad (2.2.2)$$

For Ito-Taylor expansion to hold, KM imposes stronger assumptions for infinitesimal generator L :

1. First, L has a transition density $p_t(y|x)$ with respect to Lebesgue measure.
2. L has an invariant measure π satisfying: $\pi(x)p_t(y|x) = \pi(y)p_t(x|y)$.

The first condition requires the considered diffusion model to have a transition density, this is satisfied in most financial models; The second condition is a generalization

¹Except from equation (2.1.7), KM also consider another mispricing function $d(t, x) = h(t, x) - \bar{h}(t, x)$ arising from using a wrong payoff function $\bar{h}(t, x)$, which is useful to apply his method on bond pricing. In this paper we mainly discuss option pricing thus we relax his condition here.

of time-reversibility. In particular, if the process is univariate and stationary, it is necessarily time-reversible and satisfies the second condition.

Assume closed form solution of auxiliary model \bar{w} under auxiliary model is sufficiently smooth. In other words, for $N \geq 1$, assume $\delta(t, x)$ to be $2N$ times differentiable with respect to x , and to be N times differentiable with respect to t . By applying Ito-Taylor expansion to equation (2.1.8)

$$w(t, x) = \bar{w}(t, x) + \int_t^T \mathbb{E}^{t,x} \left[\exp \left(- \int_t^s R(u, X_u) du \right) \delta(s, X_s) \right] ds \quad (2.2.3)$$

Then KM proposes Definition 1 (Asset Price Approximation) in his paper, it's given by

$$w_N(t, x) = \bar{w}(t, x) + \sum_{n=0}^N \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(t, x) \quad (2.2.4)$$

where δ_0 is the original mispricing function in equation (2.1.7), and

$$\begin{aligned} \delta_0(t, x) &= \delta(t, x) \\ \delta_n(t, x) &= L\delta_{n-1}(t, x) - R(t, x)\delta_{n-1}(t, x) \end{aligned} \quad (2.2.5)$$

KM refers in his paper that the Ito-Taylor in equation (2.2.4) can be calculated once for all, meaning once implemented, it requires no computation time. However, in practice, the number of terms in equation (2.2.4) grows exponentially as we keep expanding. When the expansion order N is large, we must rely on symbolic languages to solve the approximation formula, the advantage of this method is constrained because process of solve for derivatives and further calculation can be very long. Example is discussed in our chapter 4.

At last, different from DOI method, KM's approximation method, based on the combination of price of auxiliary model and expansion of mispricing functions, leads to a nuisance parameter. This parameter does not affect the unknown price, but does enter the pricing formula for the auxiliary model. For example, if we use Black-Scholes model as auxiliary model to price options under Heston model,

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^1 \\ dv_t &= \kappa (\theta - v_t) dt + \xi \sqrt{v_t} dW_t^2 \end{aligned}$$

The mispricing function here is $\delta = \frac{1}{2}(v(t) - \sigma_0)S^2 \frac{\partial^2 w^{\text{bs}}}{\partial S^2}$. Then the nuisance parameter is the instantaneous volatility σ_0 in Black-Scholes model. KM proposes that when we approximate $w(x, t)$ with $w_N(x, t; \hat{\sigma}_0)$, we consider

$$\hat{\sigma}_N(x, t) = \arg \min_{\sigma} (w_N(x, t; \sigma) - w_0(x, t; \sigma))^2$$

where $w_0(x, t; \sigma) \equiv w^{\text{bs}}(x, t; \sigma)$. Thus we can set instantaneous volatility to be the spot value of volatility in Heston model, or be the long-time mean of Heston model, that's is, $\sigma_0 = v(t)$ or $\sigma_0 = \theta$. In our case, the setting of nuisance parameter is specified in chapter 4.

Chapter 3

Approximations of VIX options

3.1 Approximating options under mean-reverting CEV model

3.1.1 Drawbacks of using Black-Scholes model as an auxiliary model

Chan et al. (1992) proposes the mean-reversion CEV model, in which volatility follows

$$dV_t = (\alpha + \beta V_t) dt + \sigma V_t^\gamma dW_t$$

when β is negative, this model has mean-reverting property. We can rewrite it to be

$$dV_t = \kappa(m - V_t)dt + \sigma V_t^\gamma dW_t \quad (3.1.1)$$

where κ is the speed of mean-reversion, m is the long-run mean. A natural idea is to use Black-Scholes model as auxiliary model as mentioned in Kristensen and Mele (2011), then apply their method to approximate the VIX option price under mean-reverting CEV model. Denote \mathcal{L} and \mathcal{L}^{BS} to be infinitesimal generators of mean-reverting CEV model and Black-Scholes model respectively

$$\begin{aligned} \mathcal{L}w &= \frac{\partial w}{\partial t} + \kappa(m - V) \frac{\partial w}{\partial V} + \frac{1}{2} \sigma^2 V^{2\gamma} \frac{\partial^2 w}{\partial V^2} \\ \mathcal{L}^{\text{BS}}w &= \frac{\partial w}{\partial t} + rV \frac{\partial w}{\partial V} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 w}{\partial V^2} \end{aligned}$$

The mis-pricing term for using Black-Scholes model is then

$$\delta^{\text{BS}} = (\mathcal{L} - \mathcal{L}^{\text{BS}})w^{\text{BS}} = (\kappa - r)V \frac{\partial w^{\text{BS}}}{\partial t} + \kappa m \frac{\partial w^{\text{BS}}}{\partial V} + \sigma^2(V - V^{2\gamma}) \frac{\partial^2 w^{\text{BS}}}{\partial V^2}$$

with the solution of Black-Scholes model w^{BS} . Note that δ^{BS} contains theta and gamma of option. Their differences in Black-Scholes model and mean-reverting model determines that we have to use other auxiliary models.

Take call option prices under $\gamma = \frac{1}{2}$ in model(3.1.1) as an example. This model is known as mean square root mean-reverting model proposed by Grünbichler and Longstaff (1996).

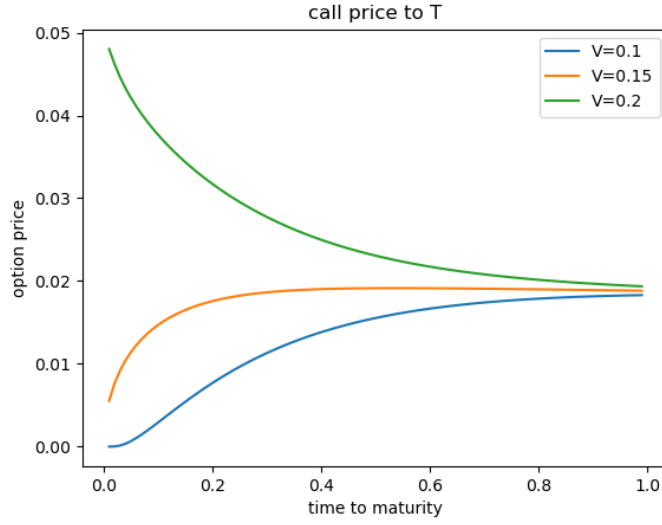


Figure 3.1: Call option price with regard to time to maturity

From figure 3.1, we can find that in contrast to Black-Scholes model, the value of call option price under mean-reverting model is not always increasing as time to maturity increases; From figure 3.2, by contrast, the call option price does not converge to zero as volatility goes to zero. In addition, Grünbichler and Longstaff (1996) also shows that V has less influence of the current value of the call option than in Black-Scholes model. For these reasons, we conclude that Black-Scholes model is not an appropriate auxiliary model and in the next section, we discuss that using the square root mean-reverting model as the auxiliary model.

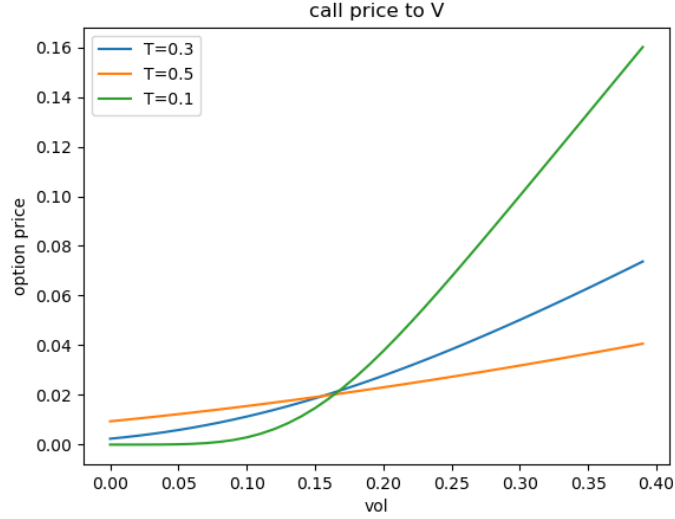


Figure 3.2: Call option price with regard to volatility

3.1.2 Using square root mean-reverting model as auxiliary model

Recall the mean-reverting CEV model with $\gamma = \frac{1}{2}$

$$dV_t = \kappa(m - V_t)dt + \sigma\sqrt{V_t}dW_t \quad (3.1.2)$$

We are going to use it as our auxiliary model as it captures the mean-reverting property of general mean-reverting CEV models. Grünbichler and Longstaff (1996) gives an explicit solution to this model. Denote the call option price \bar{w} with strike K , constant risk-free rate r , time to maturity T and no expected premium for volatility risk is paid, its price is given by

$$\begin{aligned} \bar{w} = & e^{-(\kappa+r)T} V Q(xK; \nu + 4, \lambda) \\ & + m e^{-rT} (1 - e^{-\kappa T}) Q(xK; \nu + 2, \lambda) \\ & - e^{-rT} K Q(xK; \nu, \lambda) \end{aligned} \quad (3.1.3)$$

where

$$\begin{aligned}
x &= \frac{4\kappa}{\sigma^2(1 - e^{-\kappa T})} \\
\nu &= \frac{4\kappa m}{\sigma^2}, \\
\lambda &= e^{-\kappa T} x V
\end{aligned}$$

and $Q(xK; \nu + i, \lambda)$ is the complementary distribution function for the non-central chi-squared density with $\nu + i$ degrees of freedom and non-centrality parameter λ .

Define the infinitesimal generators $\bar{\mathcal{L}}$ for square root mean-reverting model and \mathcal{L} for mean-reverting CEV model

$$\begin{aligned}
\mathcal{L}w &= \frac{\partial w}{\partial t} + \kappa(m - V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V^{2\gamma} \frac{\partial^2 w}{\partial V^2} \\
\bar{\mathcal{L}}w &= \frac{\partial w}{\partial t} + \kappa(m - V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V \frac{\partial^2 w}{\partial V^2}
\end{aligned} \tag{3.1.4}$$

Subtract infinitesimal generators in equation(3.1.4), we get the mis-pricing formula for using square root mean-reverting model

$$\delta = (\mathcal{L} - \bar{\mathcal{L}})\bar{w} = \frac{1}{2}\sigma^2(V^{2\gamma} - V)\frac{\partial^2 w}{\partial V^2}$$

We can then use the approximation formula discussed in 2.2 to price call options under mean-reverting CEV model¹

$$w_N(t, x) = \bar{w}(t, x) + \sum_{n=0}^N \frac{(T - t)^{n+1}}{(n + 1)!} \delta_n(t, x) \tag{3.1.5}$$

where

$$\begin{aligned}
\delta_0 &= \delta = \frac{1}{2}\sigma^2(V^{2\gamma} - V)\frac{\partial^2 w}{\partial V^2} \\
\delta_n(t, x) &= L\delta_{n-1}(t, x) - r\delta_{n-1}(t, x)
\end{aligned} \tag{3.1.6}$$

¹Put options can be priced easily in the same way

Finally we get a closed form approximating formula for call options under mean-reverting CEV model. But notice that the call price (3.1.3) contains non-square chi square distribution functions, applying infinitesimal generator \mathcal{L} on it can be a hard point and in the next section we are going to talk about how to derive partial derivatives of distribution function $Q(xK; \nu + i, \lambda)$.

3.1.3 Method to Calculate Derivatives In Expansions

In this section, methods to calculate closed-form partial derivatives of call option price \bar{w} to time t and volatility V . Our method is based on the recurrence relation of non-central chi-square distribution proposed by Cohen (1988), which is

$$\begin{aligned}\frac{\partial p(xK; \nu, \lambda)}{\partial(xK)} &= \frac{1}{2}[-p(xK; \nu, \lambda) + p(xK; \nu - 2, \lambda)] \\ \frac{\partial p(xK; \nu, \lambda)}{\partial \lambda} &= \frac{1}{2}[-p(xK; \nu, \lambda) + p(xK; \nu + 2, \lambda)]\end{aligned}\tag{3.1.7}$$

where $p(xK; \nu, \lambda)$ is the Probability Density Function(PDF) of non-central chi-square distribution. From the relationship between Complementary Cumulative Distribution Function(CCDF) $Q(xK; \nu, \lambda)$, Cumulative Distribution Function(CDF) $F(xK; \nu, \lambda)$, and PDF we know that

$$\begin{aligned}\frac{\partial Q(xK; \nu, \lambda)}{\partial(xK)} &= \frac{\partial[1 - F(xK; \nu, \lambda)]}{\partial(xK)} \\ &= -\frac{\partial F(xK; \nu, \lambda)}{\partial(xK)} \\ &= -p(xK; \nu, \lambda)\end{aligned}\tag{3.1.8}$$

Rewrite the second equation in (3.1.7), we get

$$\begin{aligned}
\frac{\partial p(xK; \nu, \lambda)}{\partial \lambda} &= \frac{1}{2}[-p(xK; \nu, \lambda) + p(xK; \nu + 2, \lambda)] \\
&= -\frac{1}{2}[-p(xK; \nu + 2, \lambda) + p(xK; \nu, \lambda)] \\
&= -\frac{\partial p(xK; \nu + 2, \lambda)}{\partial(xK)}
\end{aligned} \tag{3.1.9}$$

Integrate both sides of (3.1.9) with respect to xK and combine with (3.1.8), we can derive the partial derivative of CDF to non-central parameter λ

$$\begin{aligned}
\frac{\partial}{\partial \lambda} F(xK; \nu, \lambda) &= -\frac{\partial}{\partial(xK)} F(xK; \nu + 2, \lambda) \\
&= -p(xK; \nu + 2, \lambda)
\end{aligned} \tag{3.1.10}$$

Finally we get the partial derivative of CCDF to non-central parameter λ

$$\begin{aligned}
\frac{\partial Q(xK; \nu, \lambda)}{\partial \lambda} &= \frac{\partial[1 - F(xK; \nu, \lambda)]}{\partial \lambda} \\
&= -\frac{\partial F(xK; \nu, \lambda)}{\partial \lambda} \\
&= p(xK; \nu + 2, \lambda)
\end{aligned} \tag{3.1.11}$$

Until now we can summarize that the derivatives of CCDF and PDF are all combinations of PDFs with change of degrees of freedom. Without loss of accuracy, we make the degrees of freedom in PDF be consistent with call option solution in (3.1.3), that is for $p(xK; \nu + i, \lambda)$, we let $i \in [0, 4]$. Use the non-central chi-square property by Cohen (1988) to do the following transformation

$$\begin{aligned}
p(xK; \nu - 2, \lambda) &= \frac{\lambda}{xK} p(xK; \nu + 2, \lambda) + \frac{\nu - 2}{xK} p(xK; \nu, \lambda) \\
p(xK; \nu + 6, \lambda) &= \frac{xK}{\lambda} p(xK; \nu + 2, \lambda) - \frac{\nu + 2}{\lambda} p(xK; \nu + 4, \lambda)
\end{aligned} \tag{3.1.12}$$

Next we use the results above to calculate delta and gamma of auxiliary call

option price \bar{w} . Recall the parameter xK , ν and λ in (3.1.2), where

$$\begin{aligned} x &= \frac{4\kappa}{\sigma^2(1 - e^{-\kappa T})} \\ \nu &= \frac{4\kappa m}{\sigma^2}, \\ \lambda &= e^{-\kappa T} xV \end{aligned} \tag{3.1.13}$$

Then we use chain rule calculate the following auxiliary derivatives

$$\begin{aligned} \frac{\partial Q(xK; \nu, \lambda)}{\partial V} &= \frac{\partial Q}{\partial x} \frac{\partial x}{\partial V} + \frac{\partial Q}{\partial \lambda} \frac{\partial \lambda}{\partial V} \\ &= 0 + xe^{-\kappa T} p(x; \nu + 2, \lambda) \\ &= xe^{-\kappa T} p(x; \nu + 2, \lambda) \end{aligned} \tag{3.1.14}$$

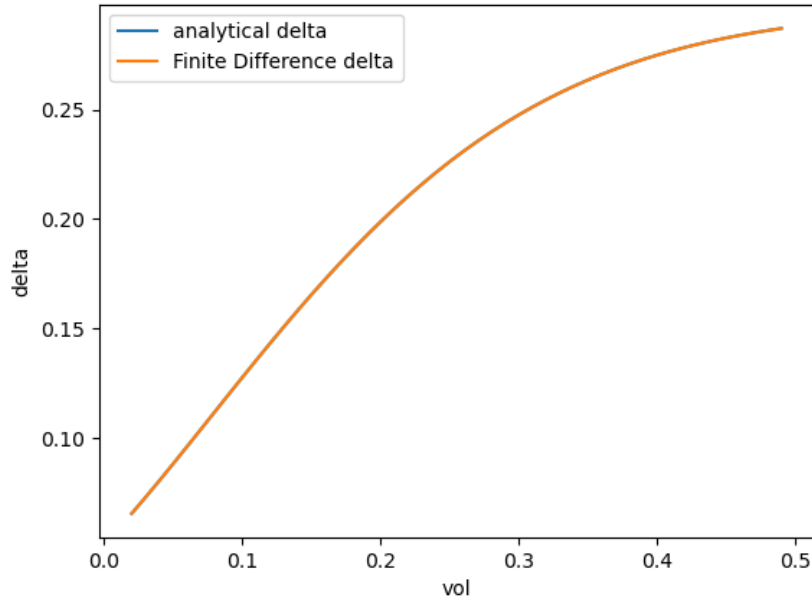
Thus delta is given by

$$\begin{aligned} \Delta_{\bar{w}} &= e^{-(\kappa+r)T} Q(xK; \nu + 4, \lambda) + e^{-(\kappa+r)T} V \cdot xe^{-\kappa T} p(xK; \nu + 6, \lambda) \\ &\quad + me^{-rT} (1 - e^{-\kappa T}) \cdot xe^{-\kappa T} p(x; \nu + 2, \lambda) - e^{-rT} K \cdot xe^{-\kappa T} p(x; \nu + 2, \lambda) \end{aligned} \tag{3.1.15}$$

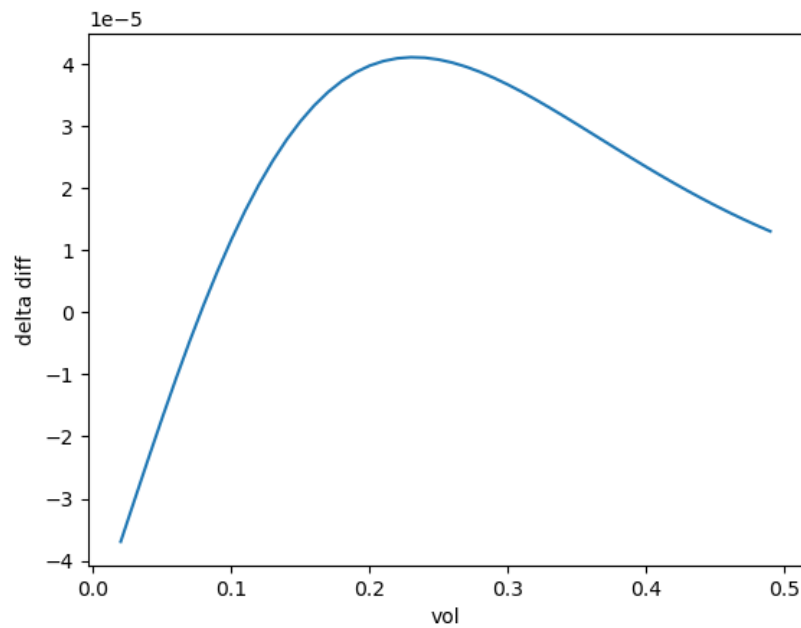
Using (3.1.12) to substitute $p(xK; \nu + 6, \lambda)$ and simplify the equation

$$\begin{aligned} \Delta_{\bar{w}} &= e^{-(\kappa+r)T} Q(xK; \nu + 4, \lambda) \\ &\quad + e^{-(\kappa+r)T} \lambda \left[\frac{xK}{\lambda} p(xK; \nu + 2, \lambda) - \frac{\nu + 2}{\lambda} p(xK; \nu + 4, \lambda) \right] \\ &\quad + me^{-rT} (1 - e^{-\kappa T}) \cdot \frac{4\kappa}{\sigma^2(1 - e^{-\kappa T})} e^{-\kappa T} p(x; \nu + 2, \lambda) - e^{-(r+\kappa)T} K p(x; \nu + 2, \lambda) \\ &= e^{-(\kappa+r)T} [Q(xK; \nu + 4, \lambda) - 2p(xK; \nu + 4, \lambda)] \end{aligned} \tag{3.1.16}$$

Similarly, we calculate another auxiliary derivative

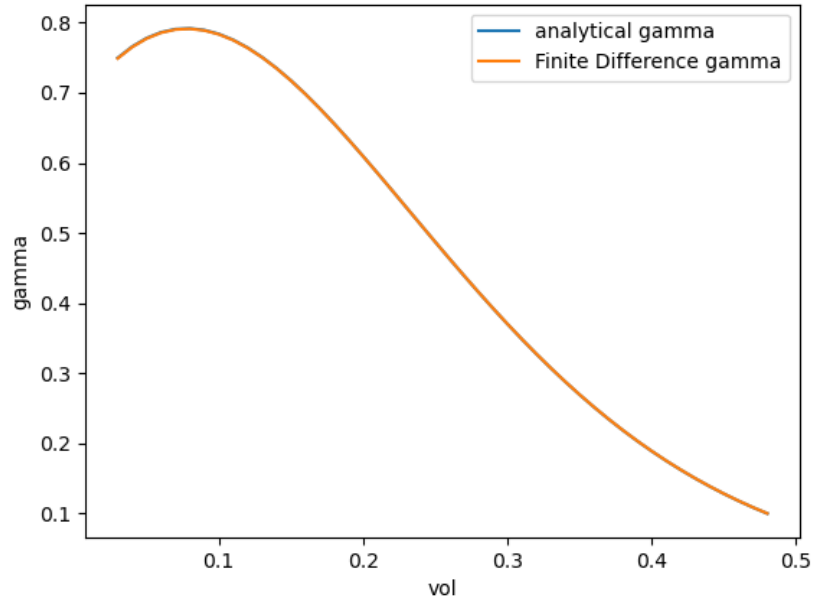


(a) delta comparison

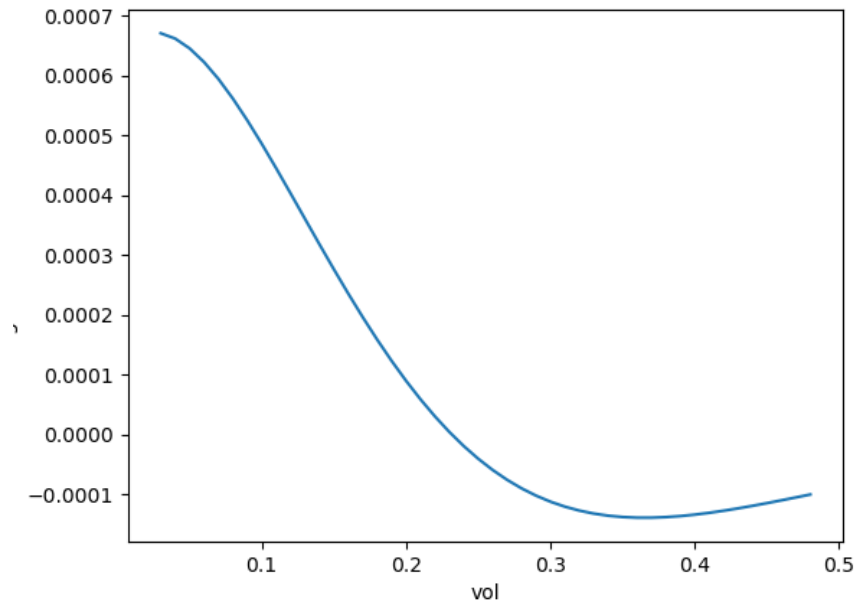


(b) delta differences between two methods

Figure 3.3: Deltas are calculated by our formula, and finite difference method. The parameters used are $T = 0.3, \alpha = 0.60, \beta = 4.00, \sigma = 0.133, r = 0.05$, and $K = 0.15$.



(a) gamma comparison



(b) gamma differences between two methods

Figure 3.4: gammas are calculated by our formula, and finite difference method. The parameters used are $T = 0.3, \alpha = 0.60, \beta = 4.00, \sigma = 0.133, r = 0.05$, and $K = 0.15$.

$$\begin{aligned}
\frac{\partial p(xK; \nu, \lambda)}{\partial V} &= \frac{\partial p}{\partial(xK)} \frac{\partial(xK)}{\partial V} + \frac{\partial p}{\partial \lambda} \frac{\partial \lambda}{\partial V} \\
&= \frac{xe^{-\kappa T}}{2} [-p(xK; \nu, \lambda) + p(xK; \nu + 2, \lambda)]
\end{aligned} \tag{3.1.17}$$

As a result, gamma of \bar{w} is then

$$\begin{aligned}
\Gamma_{\bar{w}} &= e^{-(\kappa+r)T} \left[xe^{-\kappa T} p(x; \nu + 6, \lambda) - 2 \cdot \frac{xe^{-\kappa T}}{2} [-p(xK; \nu, \lambda) + p(xK; \nu + 2, \lambda)] \right] \\
&= xe^{-(2\kappa+r)T} p(xK; \nu + 4, \lambda)
\end{aligned} \tag{3.1.18}$$

To apply infinitesimal generator on mis-pricing formula, we still need to calculate partial derivatives of PDF to time t . Define the following auxiliary functions

$$\begin{aligned}
\frac{\partial(xK)}{\partial t} &= \frac{-\kappa e^{-\kappa T}}{1 - e^{-\kappa T}} \cdot xK \\
\frac{\partial \lambda}{\partial t} &= \frac{-\kappa e^{-\kappa T}}{1 - e^{-\kappa T}} \cdot xV
\end{aligned} \tag{3.1.19}$$

Then partial derivatives of PDF to t is given by

$$\begin{aligned}
\frac{\partial p(xK; \nu, \lambda)}{\partial t} &= \frac{\partial p}{\partial(xK)} \frac{\partial(xK)}{\partial t} + \frac{\partial p}{\partial \lambda} \frac{\partial \lambda}{\partial t} \\
&= \frac{-\kappa x e^{-\kappa T}}{2(1 - e^{-\kappa T})} [Vp(xK; \nu + 2, \lambda) - (K + V)p(xK; \nu, \lambda) + Kp(xK; \nu - 2, \lambda)]
\end{aligned} \tag{3.1.20}$$

From (3.1.6) we know that the mis-pricing formula $\delta = \frac{1}{2}\sigma^2(V^{2\gamma} - V)\Gamma_{\bar{w}}$, all terms in which have been solved from above. In essence, to apply Ito-Taylor expansions on δ , we use the following algorithm as used in calculating delta and gamma:

1. Combining previous auxiliary partial derivatives, use chain rule to apply in-

finitesimal generator on mis-pricing formula.

2. Substitute PDFs with noncentral parameter $\nu + i$ where $\nu \notin [0, 4]$.
3. Back to step 1, apply higher order infinitesimal generators.

Therefore, we illustrate a solution to implement approximation method on volatility options under mean-reverting CEV model. The expansions in approximating formula can be computed once for all, we can solve it manually or use symbolic language for higher orders. All terms in the result is explicit expect non-central chi-square PDFs, we plug $p(xK; \nu + i, \lambda)$ into the result at last.

3.2 Approximating options under double Heston model

? proposes volatility with double mean-reverting dynamics

$$\begin{aligned} dV_t &= -\kappa (V_t - V'(t)) dt + \eta_1 V_t'^{\alpha} dW_1(t) \\ dV_t' &= -c (V_t' - m) dt + \eta_2 V_t'^{\beta} dW_2(t) \end{aligned}$$

where $\alpha, \beta \in [\frac{1}{2}, 1]$.

- It's called Double Heston model in the case $\alpha = \beta = \frac{1}{2}$.
- The case $\alpha = \beta = 1$ Double Log-normal model.
- And the general Double CEV model.

From our previous work, we can use the same auxiliary model to price options with V_t following heston dynamics and V_t' following any mean-reverting CEV process, we call it one Heston one CEV model. This model is given by

$$\begin{aligned}
dV_t &= -\kappa(V_t - V'(t))dt + \eta_1 \sqrt{V_t} dW_1(t) \\
dV'_t &= -c(V'_t - m)dt + \eta_2 V_t'^{\beta} dW_2(t)
\end{aligned} \tag{3.2.1}$$

Define infinitesimal generator \mathcal{L} for (3.2.1) and $\bar{\mathcal{L}}$ for square root mean-reverting model

$$\begin{aligned}
\mathcal{L}w &= \frac{\partial w}{\partial t} + \kappa(V' - V)\frac{\partial w}{\partial V} + \frac{1}{2}\eta_1^2 V \frac{\partial^2 w}{\partial V^2} \\
&\quad + \frac{\partial w}{\partial t} + c(m' - V')\frac{\partial w}{\partial V'} + \frac{1}{2}\eta_2^2 V \frac{\partial^2 w}{\partial V'^2} \\
\bar{\mathcal{L}}w &= \frac{\partial w}{\partial t} + \kappa(m - V)\frac{\partial w}{\partial V} + \frac{1}{2}\eta_1 V \frac{\partial^2 w}{\partial V^2}
\end{aligned} \tag{3.2.2}$$

Mis-pricing formula for it is then

$$\begin{aligned}
\delta &= (\mathcal{L} - \bar{\mathcal{L}})\bar{w} = \kappa(V' - m)\frac{\partial w}{\partial V} \\
&= \kappa(V' - m)\Gamma_{\bar{w}}
\end{aligned}$$

where $\Gamma_{\bar{w}}$ is given in (3.1.18).

Chapter 4

Numerical Results

In this section, we will show our approximation results for mean-reverting CEV model and Heston plus CEV model, expansion of mis-pricing terms is up to $N = 3$. We utilize a symbolic library of ¹python sympy to calculate expansions, and use Monte Carlo simulations with 200 steps and 100000 paths as our benchmark because there's no existing pricing formula for these models. We evaluate the accuracy of our results with two kinds of figures, the first kind is the direct comparison between benchmarks and our approximation results, the second one is the relative differences between benchmarks and our results. Besides, we also have attached detailed results in the Appendix A.

4.1 Volatility option prices under mean-reverting CEV model

For options under mean-reverting CEV model,

$$\begin{cases} dV_t = \kappa(\theta - V_t)dt + \sigma_{\text{CEV}}V_t^\gamma dW_t & \text{true model} \\ dV_t = \kappa(\theta - V_t)dt + \sigma_0\sqrt{V_t}dW_t & \text{auxiliary model} \end{cases}$$

We use the same mean-reverting parameters as Grünbichler and Longstaff (1996) used in his model, the parameters are $\kappa = 4$, $\theta = 2$. Besides, we set the nuisance parameter $\sigma_0 = \sigma_{\text{CEV}}V_0^{\gamma-\frac{1}{2}}$, where $V_0 = V(t)$ is the initial value of volatility at start point t . We test our approximation method with different constant elasticity parameters, the main idea of setting these parameters is that for small γ , which enlarges the importance of V in the CEV part, we use a small σ ; Whereas for large γ

¹Codes used for this paper can be accessed through my github

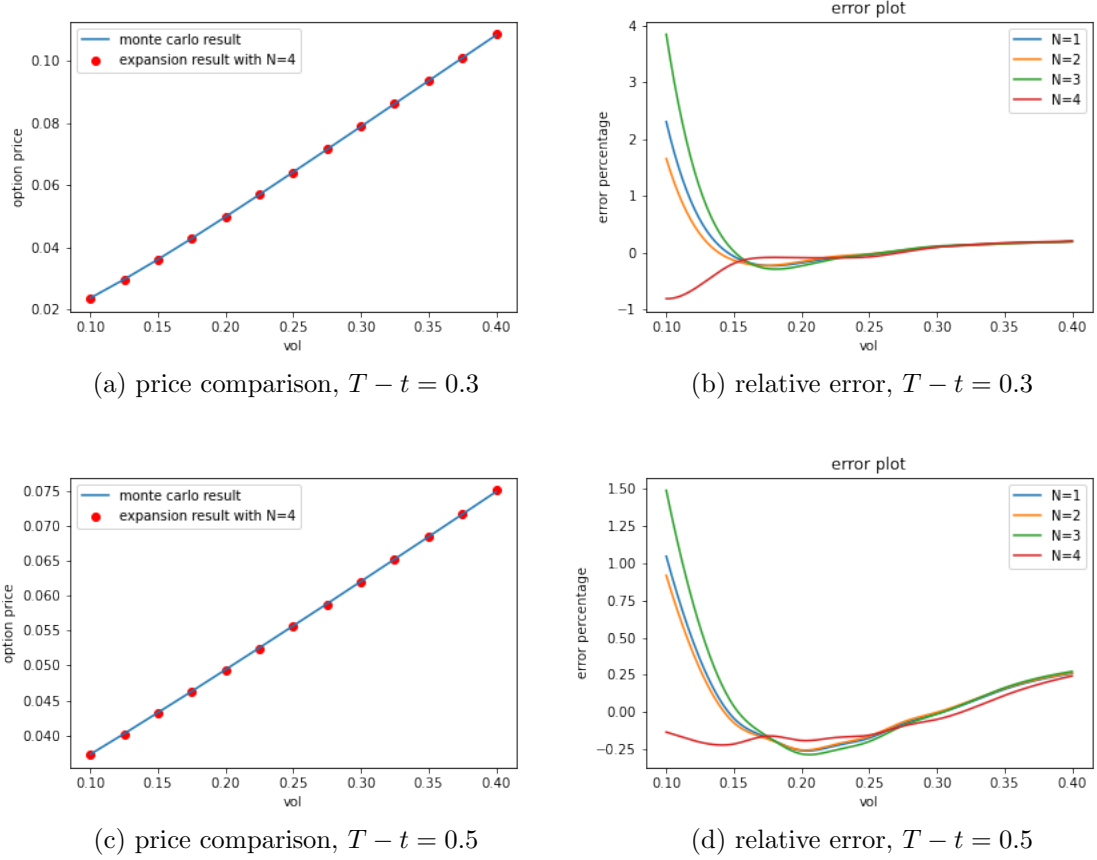


Figure 4.1: mean-reverting CEV model result 1
Parameters are $K = 0.15, \kappa = 4, \theta = 0.2, \sigma = 0.15, \gamma = 0.3$

we set a large σ .

For figure 4.1, our parameters are $\sigma = 0.15, \gamma = 0.3$, and with different maturities, $\tau = T - t = 0.3$ for the first row and $\tau = T - t = 0.5$ for the second row. We can find that in 4.1a and 4.1c our approximation results with expansion order $N = 3$ are very accurate. Figure 4.1b and figure 4.1d show the relative error with different expansion orders. We can find that the results with $N = 3$ outperform other results, which implies that keep applying Ito-Taylor expansions on the mis-pricing terms can create increasingly improved refinements and provide us with more and more accurate results.

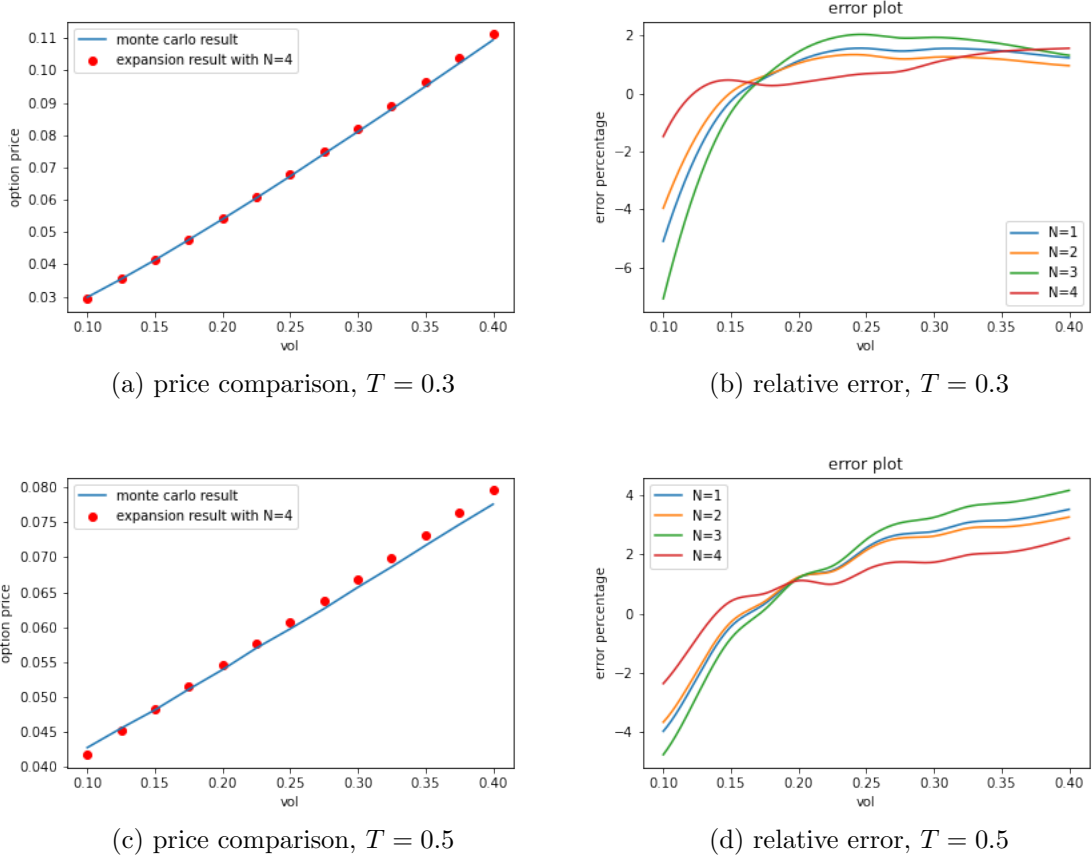


Figure 4.2: mean-reverting CEV model result 2
Parameters are $K = 0.15, \kappa = 4, \theta = 0.2, \sigma = 0.6, \gamma = 0.75$

Parameters for figure 4.2 are $\sigma = 0.6, \gamma = 0.75$, and with different maturities $\tau = 0.3, \tau = 0.5$. Similarly, our method still provide accurate results.

One may observe that when KM use Black-Scholes model as auxiliary model to price options under Heston model, for deep in-the-money options, relative differences always converge to 0 no matter how many orders of expansions are applied. That is because in his case delta of option is very close to 1 and vega is close to 0, which means option prices are mainly driven by underlying stocks' prices, and volatility has no influence on option prices. Besides, using Black-Scholes model makes mis-pricing term depend on gamma, while gamma of deep in-the-money options is also close to

0, meaning that their mis-pricing terms don't affect option prices at all. As a result, their figures show that all results' relative errors are converging to 0 as stock price increases.

However, in our model, underlying assets follow mean-reverting CEV model. Grünbichler and Longstaff (1996) mention that when volatility V is above its long-term mean, mean-reversion property implies the expected future value of V will be lower than its current value, making the expected payoff for a volatility call can be less than its current intrinsic value. The property of options under Black-Scholes world doesn't hold here, recall that before we set $\sigma_0 = \sigma_{\text{CEV}} V_0^{\gamma - \frac{1}{2}}$. Obviously when V_0 is large, $\sigma_{\text{CEV}} V_t^\gamma < \sigma_{\text{CEV}} V_0^{\gamma - \frac{1}{2}} \sqrt{V_t}$, causing the loss of accuracy in our auxiliary model. It gives an explanation why the relative error of our method is slightly larger than 0 for deep in-the-money options. Additionally, using our method to price deep out-of-the-money options can also be challenging. The loss of accuracy for approximating non-central chi-square distribution functions would be magnified when option price is very small.

4.2 Volatility option prices under Heston plus CEV model

For Heston plus CEV model,

$$\begin{aligned} dV_t &= \kappa_1 (V'_t - V_t) dt + \sigma_1 \sqrt{V_t} dW_t \\ dV'_t &= \kappa_2 (\theta - V'_t) dt + \sigma_2 V_t'^{\gamma} dW'_t \end{aligned}$$

and our auxiliary model, square root mean-reverting model,

$$dV_t = \kappa(\theta_0 - V_t)dt + \sigma_0 \sqrt{V_t} dW_t$$

our parameters are $r = 0.05$, $K = 0.15$, $\kappa_1 = 4$, $\kappa_2 = 2$, $\theta = 0.2$, $\sigma_1 = 0.3$, $\sigma_2 = 0.8$,

$\gamma = 1.6$, $\rho = 0.5$ and different maturities $\tau = T - t = 0.3, \tau = T - t = 0.5$. We set the nuisance parameters $\theta_0 = V'(t)$, where $V'(t) = 0.2$ is the spot value for volatility of volatility at time t .

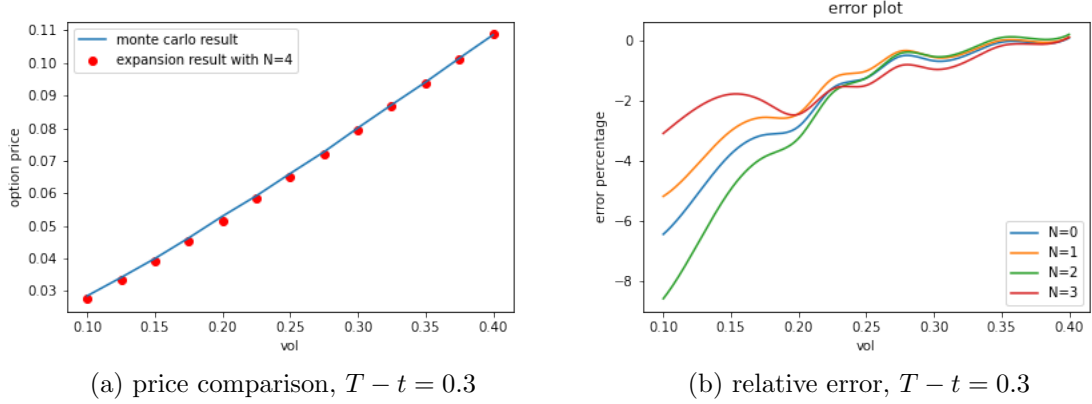


Figure 4.3: Heston plus CEV model result 1

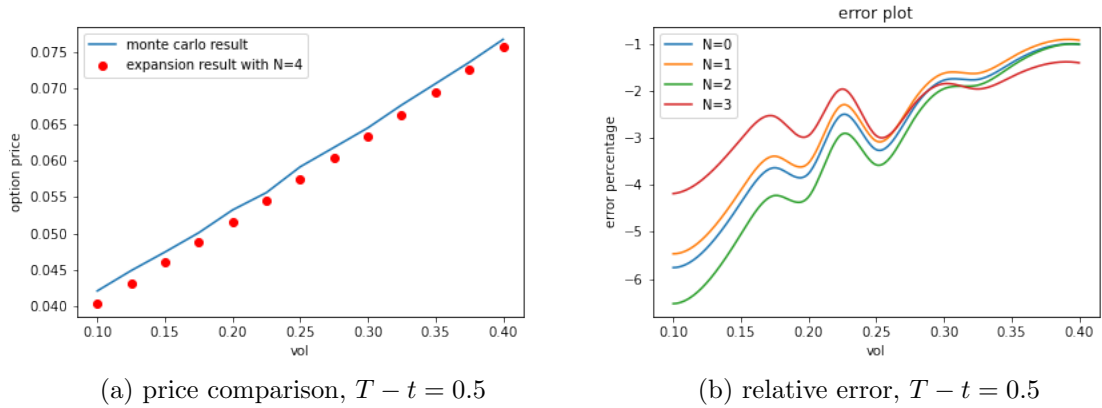


Figure 4.4: Heston plus CEV model result 2

As is seen in 4.3a, applying our method under 2-dimensional model can still create relatively accurate results. Unlike mean-reverting CEV model, under Heston plus CEV model our relative differences are now converging to 0. This is because here the initial value of volatility doesn't enter mis-pricing term $\delta = \kappa_1(V_2 - \theta_2)\Delta_{\bar{w}}$.

Besides, we notice that for 4.4a when $\tau = 0.5$, our results seem not accurate enough. Though in figure 4.4b relative differences reduce as we apply higher further expansions. We may predict that if applying higher order expansions we could get more precise results. However, this raises a constraint of KM's method, number of terms grow exponentially in the final pricing formula as we keep expanding mis-pricing terms, causing the running time of calculation increasing dramatically, which is a trade-off between applying high order expansions and getting more accurate results. Moreover, if we still don't get a desired result with a threshold expansion order, say $N = 5$, we may need to reconsider our auxiliary model and nuisance parameters.

Chapter 5

Conclusion

In this paper, we introduce approximation method proposed by ? and Kristensen and Mele (2011). Based on their work, we extend this method to price options under mean-reverting CEV model and Heston plus CEV models. Selections of auxiliary models and corresponding mis-pricing formula are discussed, we also illustrate techniques to calculate partial derivatives of non-central chi-square distribution functions when using square root mean-reverting as auxiliary model. Finally, we discuss our numerical results and explain the constraints of our method. In all, numerical results show that our method is efficient and accurate.

Appendix A

Appendix A Mean-Reverting CEV model Numerical Results

Appendices at the end of a dissertation are optional, and depend on the content of the dissertation. There can be one or more appendices, however they should retain the page numbering requirements for dissertations. Any concerns about the formatting of an appendix should be brought to Doris Oliver, who can direct you how to format your appendix if you have questions.

Theoretical Dissertation Timeline		
Taskt	Time to Finish	Notes
Problem statement	10 hours	Initially very upbeat.
Research	3 days	Literature search to very previous studies.
Reformulation	4 hours	Presented and accepted by advisor
Research	20 days	Literature search to very previous studies.
Experiments	14 days	Do some experiments and get results.
Format	1 day	Understand format guidelines for paper.
Write	years	Write the paper.
Revise	not too long	Proof read, etc.
Format	1-3 days	Verify correct report format is used.
See Library	1 hour	Meet with Doris to verify formatting.
Defend	1 day	Defend your research.
Revise	0 hours	It was perfect the first time.
Submit	1 day	Submit final dissertation to the library.

Table A.1: $T=0.3, K=0.15, \kappa=4, m=0.2, \sigma=0.15, \gamma=0.3$.csv

vol	mc	w1	w2	w3	w4
0.100	0.023570	0.024113	0.023960	0.024476	0.023380
0.125	0.029589	0.029760	0.029686	0.029921	0.029421
0.150	0.036036	0.036013	0.035989	0.036045	0.035969
0.175	0.042805	0.042709	0.042710	0.042686	0.042771
0.200	0.049793	0.049705	0.049716	0.049678	0.049750
0.225	0.056940	0.056892	0.056904	0.056881	0.056891
0.250	0.064213	0.064192	0.064201	0.064195	0.064168
0.275	0.071525	0.071552	0.071558	0.071564	0.071534
0.300	0.078864	0.078944	0.078948	0.078957	0.078941
0.325	0.086237	0.086351	0.086353	0.086361	0.086360
0.350	0.093613	0.093765	0.093765	0.093771	0.093778
0.375	0.101000	0.101181	0.101181	0.101184	0.101192
0.400	0.108382	0.108598	0.108598	0.108600	0.108606

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