AN APPROXIMATION METHOD TO PRICE VOLATILITY OPTIONS

by

You Wang

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____
You Wang, Candidate

ADVISORY COMMITTEE

____
Zhenyu Cui, Advisor   Date

____
Ionut Florescu, Reader   Date

STEVENS INSTITUTE OF TECHNOLOGY
Castle Point on Hudson
Hoboken, NJ 07030
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ABSTRACT

We propose an approximation method to value volatility options. This method is based on choosing models with closed form solution as an auxiliary model, and derive a mis-pricing formula between the true price and the auxiliary one, then apply Ito-Taylor expansions on the mis-pricing formula to create increasingly improved refinements. We propose an approach to evaluate volatility options under mean-reverting models, in which auxiliary model selection and expansion methods are explained. Method in this paper is applied to mean-reverting Constant elasticity of variance(CEV) model and double CEV models. Numerical results show that the proposed method is accurate and efficient.

Author: You Wang

Advisor: Zhenyu Cui

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Chapter 1

Introduction

The Volatility, as a measurement of the risk, is one of the most important element in the study of finance. Whaley (1993) proposes that by allowing investors to hedge directly against shifts in volatility, these securities enable investors to avoid the costs of dynamically adjusting positions for changes in volatility and serve to make the market more complete. During the past few years, derivatives written on volatility have developed rapidly, in which options are clearly the fundamental type of derivatives. However, unlike options on stocks, pricing volatility options can be challenging by its mean-revering property. On the basis of Cox et al. (1985), Hull and White (1987), Heston (1993), Grünbichler and Longstaff (1996) proposes Mean-Reverting Square Root(MRSR) model; Chan et al. (1992) then generalizes Mean-Reverting constant elasticity of variance(MRCEV) model and Gatheral (2008) proposes the Double CEV model to capture the dynamics of volatility. Grünbichler and Longstaff (1996) gives a closed-form solution for options under MRSR model, but there're no existing solutions to the following two models. The aim of our paper is to use the solution of square-root mean-reverting model to approximate the other two [1]models.

Heath and Platen (2002)(HP) first introduce a diffusion operator integral (DOI) method, he uses an auxiliary model and then apply Ito calculus on it to find unbiased variance reduction estimators for the true model, he also adapt DOI method to be employed in conjunction with PDE methods and test his method with Heston model. Similar to the idea of HP, Kristensen and Mele (2011)(KM) doesn't use auxiliary model to serve as a variance reduction estimator, but apply Ito-Taylor

[1]Strictly speaking, for double CEV model, we mainly consider Heston plus CEV model, that is $\gamma_1 = \frac{1}{2}$. Details are discussed in 3.2

expansions on the difference of auxiliary model and true model to create increasingly improved refinements. KM apply his method on several fields including bond pricing under CIR model and option pricing under stochastic volatility models.

Based on HP and KM's work, we extend their methods to price options under mean-revering models. This paper is organized as following, in chapter 2, we illustrate DOI method and KM's expansion method; In chapter 3, we discuss our method on approximating option prices under mean-reverting CEV model and Heston plus CEV model, including the auxiliary model selection, techniques of taking partial derivatives; In chapter 4, we show our numerical results and we put our conclusion in chapter 5.

Chapter 2

Method Description

In this section, we will discuss HP's DOI method and KM's approximation method. In section 2.1, we introduce HP's DOI method on the approximation of solutions of parabolic PDEs; In section 2.2, we illustrate complementary contents for KM's approximation method.

2.1 The DOI Variance Reduction Method

Consider a multi-factor model, in which a d-dimensional vector of state variables X_t on a filtered probability space$(\Omega, \mathcal{F}, \mathbb{Q})$ satisfies the following Stochastic Differential Equations(SDEs)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.1.1}$$

where $\mu(t, X_t)$ and $\sigma(t, X_t)$ are drift and diffusion functions under the risk-neutral measure $\mathbb{Q}$. HP imposes $\mu : [0, T] \times \Gamma \to \mathbb{R}^d$, $\sigma : [0, T] \times \Gamma \to \mathbb{R}^d$, in which Γ is assumed to be a bounded subset of $\mathbb{R}^d$. μ and σ also satisfies appropriate growth and Lipschitz conditions such that equation(2.1.1) admits a unique strong solution and is Markovian; W_t is a d-dimensional standard Brownian Motion and $t \in [0, T]$.

Let $w(t, x)$ be the valuation function satisfying $w : [0, T] \times \Gamma$, payoff function $h(t, x)$ satisfying $h : B \to \mathbb{R}, B \subseteq \mathbb{R}^d$. Define the infinitesimal generator L associated with equation (2.1.1) to be

$$Lw(t,x) = \frac{\partial w}{\partial t} + \sum_{i=1}^{d} \mu_i(t,x)\frac{\partial w}{\partial x} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(\sigma(t,x)\sigma^{\intercal}(t,x))_{i,j}\frac{\partial^2 w}{\partial x_i x_k} \tag{2.1.2}$$

Let $R(t,x)$ be the instantaneous short-term interest rate also satisfying $R : B \to \mathbb{R}$, combining with equation (2.1.1) and equation (2.1.2), the task is to find an approximation to the solution to the following partial differential equation(PDE)

$$Lw(x,t) = R(x,t)w(x,t) \tag{2.1.3}$$

with boundary condition $w(t,x) = h(t,x)$. It's easily seen that under risk neutral measure $\mathbb{Q}$, the instantaneous option price change is equal to the price gain in saving account.

Next we consider to use a m-dimensional($m \leq d$) process $\bar{X}(t)$ which is a simpler auxiliary model to approximate the price of option. $\bar{X}(t)$ satisfies the following SDE

$$d\bar{X}_t = \begin{cases} \bar{\mu}_i(t,\bar{X}_t)dt + \bar{\sigma}_i(t,\bar{X}_t)dW_t & 1 \leq i \leq m \\ 0 \cdot dt + 0 \cdot dW_t & m < i \leq d \end{cases} \tag{2.1.4}$$

where $\bar{\mu}(t,\bar{X}_t)$ and $\bar{\sigma}(t,\bar{X}_t)$ are drift and diffusion functions, and they are also assumed to satisfy appropriate conditions such that equation(2.1.4) admits a unique strong solution and is Markovian. In KM's method, it's automatically satisfied because he supposes there always exists closed-from solution to auxiliary model such that Ito-Taylor expansions can be applied further.

Let $\bar{w}(t,x)$ be the option price written on process $\bar{X}_t$, the infinitesimal generator $\bar{L}$ for option price $\bar{w}$ is the same as equation(2.1.2) but replacing $\mu(t,x)$, $\sigma(t,x)$ by

$\bar{\mu}(t,x)$ and $\bar{\sigma}(t,x)$. Therefore $\bar{w}(t,x)$ is a solution to

$$\bar{\mathcal{L}}\bar{w}(x,t) = R(x,t)\bar{w}(x,t) \tag{2.1.5}$$

Denote the price difference $\Delta w(t,x) = w(t,x) - \bar{w}(t,x)$, then Δw satisfies

$$L\Delta w(t,x) + (L - \bar{L})w(t,x) = R(x,t)\Delta w(t,x) \tag{2.1.6}$$

with boundary condition $\Delta w(t,x) = 0$. Next we define

$$\begin{aligned} \delta(t,x) &= (L - \bar{L})\bar{w}(t,x) \\ &= \sum_{i=1}^{d} \Delta\mu_i(t,x)\frac{\partial \bar{w}(t,x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} \Delta\sigma_{ij}^2(x,t)\frac{\partial^2 \bar{w}(x,t)}{\partial x_i \partial x_j} \end{aligned} \tag{2.1.7}$$

using Feynman-Kac representation leads to option price $w(t,x)$ taking the following form

$$w(t,x) = \bar{w}(t,x) + \int_t^T \mathbb{E}^{t,x}\left[\exp\left(-\int_t^s R(u,X_u)du\right)\delta(s,X_s)\right]ds \tag{2.1.8}$$

Finally, under the initial condition $Z_0 = w(0,x)$, the DOI estimator

$$Z_t = \bar{w}(t,x) + \int_t^T \exp\left(-\int_t^s R(u,X_u)du\right)\delta(s,X_s)ds \tag{2.1.9}$$

is an unbiased estimator for Z_0. And if a good auxiliary model is chosen, for example, when $\bar{L}$ is close to L, the variance of Z_t will be small.

2.2 KM's approximation Method based on the DOI method

Recall equation(2.1.8), this equation is the [1]Theorem 1(Asset Price Representation) in Kristensen and Mele (2011), KM names equation (2.1.7) the mispricing function. His method is to look for an approximation of the error term in order to adjust the price $\bar{w}(t,x)$ for the error due to the use of the auxiliary model to price options.

We first introduce Ito-Taylor expansion, for sufficiently smooth function $f(t,x)$, it is given by

$$\mathbb{E}^{t,x}[f(s,X_s)] = \sum_{N=0}^{J} \frac{(s-t)^N}{N!} L^N f(t,x) + \mathcal{R}_J \tag{2.2.1}$$

where L is the infinitesimal generator we have defined in equation (2.1.2), the remainder term $\mathcal{R}_J$ is given by

$$\mathcal{R}_J = \mathbb{E}^{t,x}\left[\int_t^s du_1 \int_t^{u_1} du_2 \cdots \int_t^{u_J} L^{J+1} f\left(u_{J+1}, X\left(u_{J+1}\right)\right) du_{J+1}\right] \tag{2.2.2}$$

For Ito-Taylor expansion to hold, KM imposes stronger assumptions for infinitesimal generator L:

1. First, L has a transition density $p_t(y|x)$ with respect to Lebesgue measure.

2. L has an invariant measure π satisfying: $\pi(x)p_t(y|x) = \pi(y)p_t(x|y)$.

The first condition requires the considered diffusion model to have a transition density, this is satisfied in most financial models; The second condition is a generalization

[1]Except from equation (2.1.7), KM also consider another mispricing function $d(t,x) = h(t,x) - \bar{h}(t,x)$ arising from using a wrong payoff function $\bar{h}(t,x)$, which is useful to apply his method on bond pricing. In this paper we mainly discuss option pricing thus we relax his condition here.

of time-reversibility. In particular, if the process is univariate and stationary, it is necessarily time-reversible and satisfies the second condition.

Assume closed form solution of auxiliary model $\bar{w}$ under auxiliary model is sufficiently smooth. In other words, for $N \geq 1$, assume $\delta(t,x)$ to be $2N$ times differentiable with respect to x, and to be N times differentiable with respect to t. By applying Ito-Taylor expansion to equation (2.1.8)

$$w(t,x) = \bar{w}(t,x) + \int_t^T \mathbb{E}^{t,x}\left[\exp\left(-\int_t^s R(u,X_u)du\right)\delta(s,X_s)\right]ds \tag{2.2.3}$$

Then KM proposes Definition 1(Asset Price Approximation) in his paper, it's given by

$$w_N(t,x) = \bar{w}(t,x) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!}\delta_n(t,x) \tag{2.2.4}$$

where δ_0 is the original mispricing function in equation (2.1.7), and

$$\begin{aligned} \delta_0(t,x) &= \delta(t,x) \\ \delta_n(t,x) &= L\delta_{n-1}(t,x) - R(t,x)\delta_{n-1}(t,x) \end{aligned} \tag{2.2.5}$$

KM refers in his paper that the Ito-Taylor in equation (2.2.4) can be calculated once for all, meaning once implemented, it requires no computation time. However, in practice, the number of terms in equation (2.2.4) grows exponentially as we keeping expansion. When the expansion order N is large, we must rely on symbolic languages to solve the approximation formula, the advantage of this method is constrained because process of solve for derivatives and further calculation can be very long. Example is discussed in our chapter 4.

At last, different from DOI method, KM's approximation method, based on the combination of price of auxiliary model and expansion of mispricing functions, leads to a nuisance parameter. This parameter does not affect the unknown price, but does enter the pricing formula for the auxiliary model. For example, if we use Black-Scholes model as auxiliary model to price options under Heston model,

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^1$$
$$dv_t = \kappa\left(\theta - v_t\right) dt + \xi\sqrt{v_t} dW_t^2$$

The mispricing function here is $\delta = \frac{1}{2}(v(t) - \sigma_0) S^2 \frac{\partial^2 w^{\mathrm{bs}}}{\partial S^2}$. Then the nuisance parameter is the instantaneous volatility σ_0 in Black-Shcoles model. KM proposes that when we approximate $w(x,t)$ with $w_N(x,t;\hat{\sigma_0})$, we consider

$$\hat{\sigma}_N(x,t) = \arg\min_{\sigma} \left(w_N(x,t;\sigma) - w_0(x,t;\sigma)\right)^2$$

where $w_0(x,t;\sigma) \equiv w^{\mathrm{bs}}(x,t;\sigma)$. Thus we can set instantaneous volatility to be the spot value of volatility in Heston model, or be the long-time mean of Heston model, that's is, $\sigma_0 = v(t)$ or $\sigma_0 = \theta$. In our case, the setting of nuisance parameter is specified in chapter 4.

Chapter 3

Approximations of volatility options

In this section, we apply KM's method to price options under several mean-reverting models.

3.1 Approximating options under mean-reverting CEV model

3.1.1 Auxiliary model selection

Chan et al. (1992) proposes the mean-reversion constant elasticity of variance(CEV) model, in which volatility follows

$$dV_t = (\alpha + \beta V_t)\, dt + \sigma V_t^{\gamma} dW_t$$

when β is negative, this model has mean-reverting property. We can rewrite it to be

$$dV_t = \kappa(m - V_t)dt + \sigma V_t^{\gamma} dW_t \tag{3.1.1}$$

where κ is the speed of mean-reversion, m is the long-term mean. A natural idea is follow KM's method, using Black-Scholes model as auxiliary model, then apply their method to approximate the volatility option price under mean-reverting CEV model. Denote L and L^{bs} to be infinitesimal generators of mean-reverting CEV model and Black-Scholes model respectively

$$\begin{aligned} Lw &= \frac{\partial w}{\partial t} + \kappa(m - V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V^{2\gamma}\frac{\partial^2 w}{\partial V^2} \\ L^{\mathrm{bs}}w &= \frac{\partial w}{\partial t} + rV\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V^2\frac{\partial^2 w}{\partial V^2} \end{aligned}$$

The mispricing function for using Black-Scholes model is then

$$\delta^{\mathrm{bs}} = (L - L^{\mathrm{bs}})w^{\mathrm{bs}} = (\kappa - r)V\frac{\partial w^{\mathrm{bs}}}{\partial t} + \kappa m \frac{\partial w^{\mathrm{bs}}}{\partial t} + \sigma^2(V - V^{2\gamma})\frac{\partial^2 w^{\mathrm{bs}}}{\partial V^2}$$

with the solution of Black-Scholes model w^{bs}. Note that δ^{bs} contains theta and gamma of option. Their differences in Black-Scholes model and mean-reverting model determines that we have to use other auxiliary models.

Take call option prices under $\gamma = \frac{1}{2}$ in model(3.1.1) as an example. This model is known as Square Root(SR) Mean-Reverting model proposed by Grünbichler and Longstaff (1996)(GL). GL also gives an explicit option price formula under this model, we will use it to illustrate our reason more clearly.

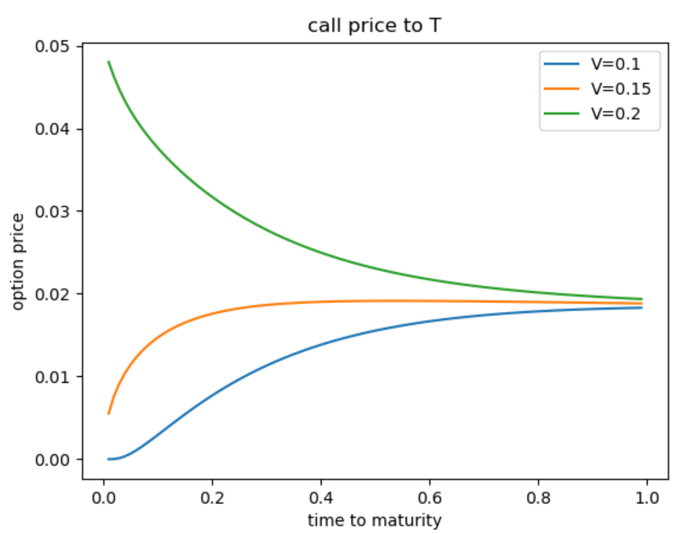


Figure 3.1: SR mean-reverting call price with regard to time to maturity

Figure 3.1 and 3.2 are option price changes with the change of time to maturity and underlying asset price correspondingly. From figure 3.1, we can find that in contrast to Black-Scholes model, the value of call option price under mean-reverting

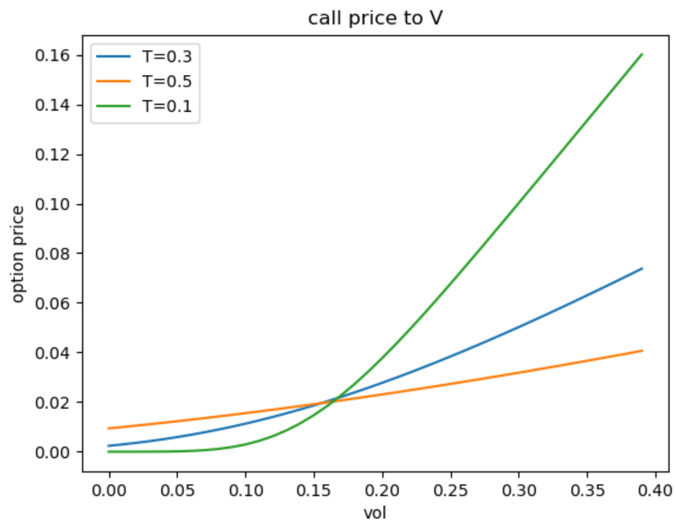


Figure 3.2: SR mean-reverting call price with regard to volatility

model is not always increasing as time to maturity increases; From figure 3.2, by contrast, the call option price does not converge to zero like option under Black-Scholes model as volatility goes to zero. In addition, GL also shows that V_t has less influence of the current value of the call option than in Black-Scholes model. For these reasons, we conclude that Black-Scholes model is not an appropriate auxiliary model.

Next we discuss that using the SR mean-reverting model as the auxiliary model, it's a special case of mean-reverting CEV model with $\gamma = \frac{1}{2}$

$$dV_t = \kappa(m - V_t)dt + \sigma\sqrt{V_t}dW_t \tag{3.1.2}$$

We are going to use it as our auxiliary model as it captures the mean-reverting property of general mean-reverting CEV models. GL gives an explicit option price formula to this model.

$$
\begin{aligned}
\bar{w} =& e^{-(\kappa+r)\tau} V Q(xK; \nu+4, \lambda) \\
& + m e^{-r\tau}(1-e^{-\kappa\tau}) Q(xK; \nu+2, \lambda) \\
& - e^{-r\tau} K Q(xK; \nu, \lambda)
\end{aligned}
\tag{3.1.3}
$$

where

$$
\begin{aligned}
\tau &= T - t \\
x &= \frac{4\kappa}{\sigma^2(1-e^{-\kappa\tau})} \\
\nu &= \frac{4\kappa m}{\sigma^2}, \\
\lambda &= e^{-\kappa\tau} x V
\end{aligned}
$$

and $Q(xK; \nu+i, \lambda)$ is the complementary distribution function for the non-central chi-squared density with $\nu+i$ degrees of freedom and non-centrality parameter λ.

Define the infinitesimal generators $\bar{L}$ for square root mean-reverting model and L for mean-reverting CEV model

$$
\begin{aligned}
Lw &= \frac{\partial w}{\partial t} + \kappa(m-V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V^{2\gamma}\frac{\partial^2 w}{\partial V^2} \\
\bar{L}w &= \frac{\partial w}{\partial t} + \kappa(m-V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2 w}{\partial V^2}
\end{aligned}
\tag{3.1.4}
$$

Subtract infinitesimal generators in equation(3.1.4), we get the mis-pricing formula for using square root mean-reverting model

$$
\delta = (L - \bar{L})\bar{w} = \frac{1}{2}\sigma^2(V^{2\gamma} - V)\frac{\partial^2 w}{\partial V^2}
$$

We can then use the approximation formula discussed in 2.2 to price call options under mean-reverting CEV model[1]

[1] Put options can be priced easily in the same way

$$w_N(t,x) = \bar{w}(t,x) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!}\delta_n(t,x) \tag{3.1.5}$$

where

$$\begin{aligned} \delta_0 = \delta &= \frac{1}{2}\sigma^2(V^{2\gamma} - V)\frac{\partial^2 w}{\partial V^2} \\ \delta_n(t,x) &= L\delta_{n-1}(t,x) - r\delta_{n-1}(t,x) \end{aligned} \tag{3.1.6}$$

The use of SR mean-reverting model keeps the mean-reverting property in our auxiliary model and provides a simpler mispricing function $\delta(t,x)$. However, call price of SR mean-reverting model, see equation (3.1.3), contains non-square chi-square distribution functions. Applying infinitesimal generator L on it can be challenging. In the next section we are going to talk about how to derive partial derivatives of distribution function $Q(xK; \nu + i, \lambda)$ with any order.

3.1.2 Derivatives of non-central chi-square distribution

In this section, we introduce our method to calculate derivatives of non-central chi-square distribution and further combine it with approximation methods. Our method is based on the recurrence relation of non-central chi-square distribution proposed by Cohen (1988), which is

$$\begin{aligned} \frac{\partial p(y;\nu,\lambda)}{\partial y} &= \frac{1}{2}[-p(y;\nu,\lambda) + p(y;\nu-2,\lambda)] \\ \frac{\partial p(y;\nu,\lambda)}{\partial \lambda} &= \frac{1}{2}[-p(y;\nu,\lambda) + p(y;\nu+2,\lambda)] \end{aligned} \tag{3.1.7}$$

where $p(y;\nu,\lambda)$ is the Probability Density Function(PDF) of non-central chi-square distribution with degrees of freedom ν and non-central parameter λ. From the relationship between Complementary Cumulative Distribution Function(CCDF) $Q(y;\nu,\lambda)$, Cumulative Distribution Function(CDF) $F(y;\nu,\lambda)$, and PDF we know that

$$\begin{aligned}\frac{\partial Q(y;\nu,\lambda)}{\partial y} &= \frac{\partial[1-F(y;\nu,\lambda)]}{\partial y}\\ &= -\frac{\partial F(y;\nu,\lambda)]}{\partial y}\\ &= -p(y;\nu,\lambda)\end{aligned} \tag{3.1.8}$$

Combine equation (3.1.7) and (3.1.8) we can derive the derivative of CCDF to non-central parameter λ. Rewrite the second equation in (3.1.7), we get

$$\begin{aligned}\frac{\partial p(y;\nu,\lambda)}{\partial \lambda} &= \frac{1}{2}[-p(y;\nu,\lambda)+p(y;\nu+2,\lambda)]\\ &= -\frac{1}{2}[-p(y;\nu+2,\lambda)+p(y;\nu,\lambda)]\\ &= -\frac{\partial p(y;\nu+2,\lambda)}{\partial y}\end{aligned} \tag{3.1.9}$$

Integrate both sides of (3.1.9) with respect to y, it's easy to see that

$$\begin{aligned}\frac{\partial}{\partial \lambda}F(y;\nu,\lambda) &= -\frac{\partial}{\partial y}F(y;\nu+2,\lambda)\\ &= -p(y;\nu+2,\lambda)\end{aligned} \tag{3.1.10}$$

It's obvious that derivative of CCDF to non-central parameter λ is given by

$$\begin{aligned}\frac{\partial Q(y;\nu,\lambda)}{\partial \lambda} &= \frac{\partial[1-F(y;\nu,\lambda)]}{\partial \lambda}\\ &= -\frac{\partial F(y;\nu,\lambda)]}{\partial \lambda}\\ &= p(y;\nu+2,\lambda)\end{aligned} \tag{3.1.11}$$

Until now we can summarize that the derivatives of CCDF and PDF are all combinations of PDFs with change of degrees of freedom. Without loss of accuracy, we make the degrees of freedom in PDF be similar with call option solution in (3.1.3), that is for $p(xK;\nu+i,\lambda)$, we let $i \in [2,6]$. Note that in GL's formula degrees of freedom $\nu \in [0,4]$, it we keep the same, it may cause some error when $\nu < 2$ in calculating the following derivative

$$\frac{\partial p(y;\nu,\lambda)}{\partial y} = \frac{1}{2}[-p(y;\nu,\lambda) + p(y;\nu-2,\lambda)]$$

because degrees of freedom must be larger than 0. Using the non-central chi-square property by Cohen (1988) we can do the following transformation

$$\begin{aligned} p(y;\nu-2,\lambda) &= \frac{\lambda}{y}p(y;\nu+2,\lambda) + \frac{v-2}{y}p(y;\nu,\lambda) \\ p(y;\nu+6,\lambda) &= \frac{y}{\lambda}p(y;\nu+2,\lambda) - \frac{\nu+2}{\lambda}p(y;\nu+4,\lambda) \end{aligned} \tag{3.1.12}$$

To calculate derivatives of call option price $\bar{w}$ under SR mean-reverting model, we further define some auxiliary derivatives. Recall the parameters y, ν and λ in GL's solution (3.1.1), where

$$\begin{aligned} y &= xK = \frac{4\kappa}{\sigma^2(1-e^{-\kappa\tau})}K \\ \nu &= \frac{4\kappa m}{\sigma^2}, \\ \lambda &= e^{-\kappa\tau}xV \end{aligned} \tag{3.1.13}$$

We define following auxiliary derivatives

$$\begin{aligned} \frac{\partial y}{\partial V} &= 0 \\ \frac{\partial \lambda}{\partial V} &= e^{-\kappa\tau}x \\ \frac{\partial y}{\partial t} &= \frac{-\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}} \cdot xK = \frac{-\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}}y \\ \frac{\partial \lambda}{\partial t} &= \frac{-\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}} \cdot xV \end{aligned} \tag{3.1.14}$$

Using chain rule, along with these auxiliary derivatives, derivatives of CCDF (3.1.8), (3.1.11), derivatives of PDF (3.1.7), all partial derivatives related to non-central distribution can be solved as

$$
\begin{aligned}
\frac{\partial Q(y;\nu,\lambda)}{\partial V} =& \frac{\partial Q}{\partial y}\frac{\partial y}{\partial V} + \frac{\partial Q}{\partial \lambda}\frac{\partial \lambda}{\partial V} \\
=& 0 + xe^{-\kappa\tau}p(y;\nu+2,\lambda) \\
=& xe^{-\kappa\tau}p(y;\nu+2,\lambda) \\
\frac{\partial Q(y;\nu,\lambda)}{\partial t} =& -\frac{\partial Q(y;\nu,\lambda)}{\partial \tau} \\
=& -\left(\frac{\partial Q}{\partial y}\frac{\partial y}{\partial \tau} + \frac{\partial Q}{\partial \lambda}\frac{\partial \lambda}{\partial \tau}\right) \\
=& -\frac{-\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}}xK\cdot[-p(y;\nu,\lambda)] - \frac{-\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}}xV\cdot p(y;\nu+2,\lambda) \\
=& \frac{\kappa x e^{-\kappa\tau}}{1-e^{-\kappa\tau}}\left[-Kp(y;\nu,\lambda) + Vp(y;\nu+2,\lambda)\right]
\end{aligned}
\tag{3.1.15}
$$

and

$$
\begin{aligned}
\frac{\partial p(y;\nu,\lambda)}{\partial V} =& \frac{\partial p}{\partial y}\frac{\partial y}{\partial V} + \frac{\partial p}{\partial \lambda}\frac{\partial \lambda}{\partial V} \\
=& 0 + xe^{-\kappa\tau}\cdot\frac{1}{2}[-p(y;\nu,\lambda) + p(y;\nu+2,\lambda)] \\
=& \frac{xe^{-\kappa\tau}}{2}[-p(y;\nu,\lambda) + p(y;\nu+2,\lambda)] \\
\frac{\partial p(y;\nu,\lambda)}{\partial t} =& -\frac{\partial p(y;\nu,\lambda)}{\partial \tau} \\
=& -\left(\frac{\partial p}{\partial y}\frac{\partial y}{\partial \tau} + \frac{\partial p}{\partial \lambda}\frac{\partial \lambda}{\partial \tau}\right) \\
=& -\frac{-\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}}xK\cdot\frac{1}{2}[-p(y;\nu,\lambda) + p(y;\nu-2,\lambda)] \\
& -\frac{-\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}}xV\cdot\frac{1}{2}[-p(y;\nu,\lambda) + p(y;\nu+2,\lambda)] \\
=& \frac{\kappa x e^{-\kappa\tau}}{2(1-e^{-\kappa\tau})}\left[Vp(y;\nu+2,\lambda) - (K+V)p(y;\nu,\lambda) + Kp(y;\nu-2,\lambda)\right]
\end{aligned}
\tag{3.1.16}
$$

Next we use our previous work to calculate Delta

$$\begin{aligned}\Delta_{\bar{w}} =& e^{-(\kappa+r)\tau}Q(xK;\nu+4,\lambda)+e^{-(\kappa+r)\tau}V\cdot xe^{-\kappa\tau}p(xK;\nu+6,\lambda)\\ &+me^{-r\tau}(1-e^{-\kappa\tau})\cdot xe^{-\kappa\tau}p(x;\nu+2,\lambda)-e^{-r\tau}K\cdot xe^{-\kappa\tau}p(x;\nu+2,\lambda)\end{aligned} \tag{3.1.17}$$

Obviously degrees of freedom in $p(y;\nu+6,\lambda)$ is out of our predefined bounds. We use (3.1.12) to substitute $p(y;\nu+6,\lambda)$ and simplify the equation

$$\begin{aligned}\Delta_{\bar{w}} =& e^{-(\kappa+r)\tau}Q(y;\nu+4,\lambda)\\ &+e^{-(\kappa+r)\tau}\lambda\left[\frac{y}{\lambda}p(y;\nu+2,\lambda)-\frac{\nu+2}{\lambda}p(y;\nu+4,\lambda)\right]\\ &+me^{-r\tau}(1-e^{-\kappa\tau})\cdot\frac{4\kappa}{\sigma^2(1-e^{-\kappa\tau})}e^{-\kappa\tau}p(y;\nu+2,\lambda)-e^{-(r+\kappa)\tau}Kp(y;\nu+2,\lambda)\\ =& e^{-(\kappa+r)\tau}[Q(y;\nu+4,\lambda)-2p(y;\nu+4,\lambda)]\end{aligned} \tag{3.1.18}$$

Similarly, Gamma of $\bar{w}$ can be calculated by

$$\begin{aligned}\Gamma_{\bar{w}} &= e^{-(\kappa+r)\tau}\left[xe^{-\kappa\tau}p(x;\nu+6,\lambda)-2\cdot\frac{xe^{-\kappa\tau}}{2}[-p(y;\nu+4,\lambda)+p(y;\nu+6,\lambda)]\right]\\ &= xe^{-(2\kappa+r)\tau}p(y;\nu+4,\lambda)\end{aligned} \tag{3.1.19}$$

We evaluate our method of calculating partial derivatives by comparing our formula with using finite difference on some range of option prices. Delta, Gamma, Theta calculated by finite difference is given by

$$\Delta^{fm} = \frac{\partial C}{\partial V} \approx \frac{C(V+\Delta) - C(V)}{\Delta V}$$
$$\Gamma^{fm} = \frac{\partial \Delta^{fm}}{\partial V} \approx \frac{\Delta^{fm}(V+\Delta) - \Delta^{fm}(V)}{\Delta V}$$
$$\Theta^{fm} = \frac{\partial C}{\partial t} \approx \frac{C(t+\Delta) - C(S)}{\Delta t}$$

We set step size of finite difference method to be 0.04, the parameters are $\theta = 0.15, \kappa = 4, \sigma = \sqrt{0.133}, r = 0.05, K = 0.15$. $T = 0.3$ for Delta and Gamma, $V = 0.2$ for Theta. Comparisons are shown in 3.3, we can find that the result of our method is right and very accurate.

From (3.1.6) we know that the mispricing function $\delta = \frac{1}{2}\sigma^2(V^{2\gamma} - V)\Gamma_{\bar{w}}$, where $\Gamma_{\bar{w}}$ has been solved before. An attractive feature of Delta and Gamma is that they are sufficiently neat, making it easy for us to apply Ito-Taylor expansions on mispricing function later. Additionally, in essence, to apply Ito-Taylor expansions on mispricing function, we use the following algorithm as used in calculating delta and gamma:

1. Use chain rule combining with (3.1.15) and (3.1.16)to apply infinitesimal generator on current mispricing function δ_n.

2. Substitute PDFs inside δ_n with noncentral parameter $\nu + i$ where $i \notin [2, 6]$.

3. Then we get δ_{n+1}, to calculate higher order approximation formula, go back to step 1.

Finally, we illustrate a method to implement KM's approximation method on volatility options under mean-reverting CEV model. Numerical results are shown in the next chapter.

3.2 Approximating options under Heston plus CEV model

Gatheral (2008) proposes volatility with double mean-reverting dynamics

$$dV_t = \kappa_1 \left(V_t' - V_t\right) dt + \sigma_1 V_t^{\gamma_1} dW_t$$

$$dV_t' = \kappa_2 \left(\theta - V_t'\right) dt + \sigma_2 V_t'^{\gamma_2} dW_t'$$

where $\gamma_1, \gamma_2 \in [\frac{1}{2}, 1]$.

- It's called Double Heston model in the case $\gamma_1 = \gamma_2 = \frac{1}{2}$.
- The case $\gamma_1 = \gamma_2 = 1$ Double Log-normal model.
- And the general Double CEV model.

We will discuss using our method to price options under double CEV model with $\gamma_1 = \frac{1}{2}, \gamma_2 \in [\frac{1}{2}, 1]$. That is

$$dV_t = \kappa_1 \left(V_t' - V_t\right) dt + \sigma_1 \sqrt{V_t} dW_t$$

$$dV_t' = \kappa_2 \left(\theta - V_t'\right) dt + \sigma_2 V_t'^{\gamma} dW_t'$$

The same auxiliary model is used to price options, it's given by

$$dV_t = \kappa(m - V_t)dt + \sigma\sqrt{V_t}dW_t$$

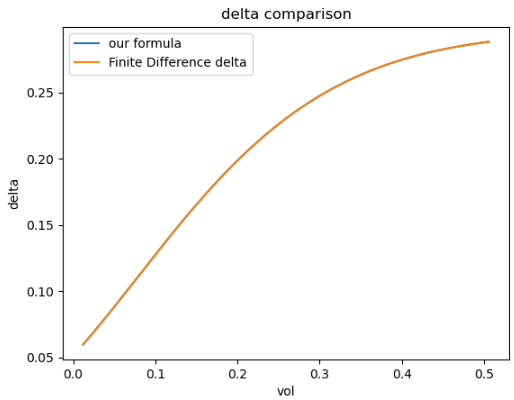
Define infinitesimal generator L for (3.2) and $\bar{L}$ for SR mean-reverting model

$$\begin{aligned}
Lw =& \frac{\partial w}{\partial t} + \kappa_1(V' - V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma_1^2 V \frac{\partial^2 w}{\partial V^2} \\
&+ \kappa_2(\theta - V')\frac{\partial w}{V'} + \frac{1}{2}\sigma_2^2 V \frac{\partial^2 w}{V'^2} + \rho\sigma_1\sigma_2\sqrt{V}V'^{\gamma_2}\frac{\partial^2 w}{\partial V \partial V'} \\
\bar{L}w =& \frac{\partial w}{\partial t} + \kappa(m - V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma_1 V \frac{\partial^2 w}{\partial V^2}
\end{aligned} \tag{3.2.1}$$

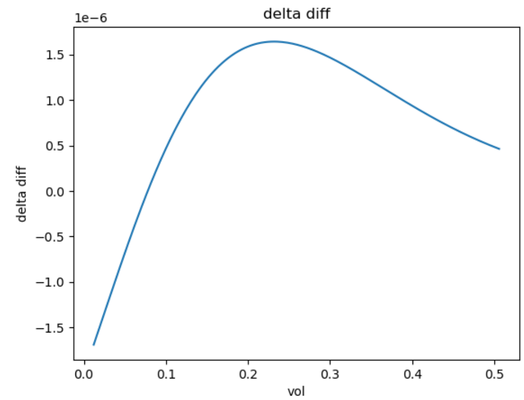
Mispricing function for it is then

$$\delta = (L - \bar{L})\bar{w} = \kappa(V' - m)\frac{\partial w}{\partial V}$$
$$= \kappa(V' - m)\Delta\bar{w}$$

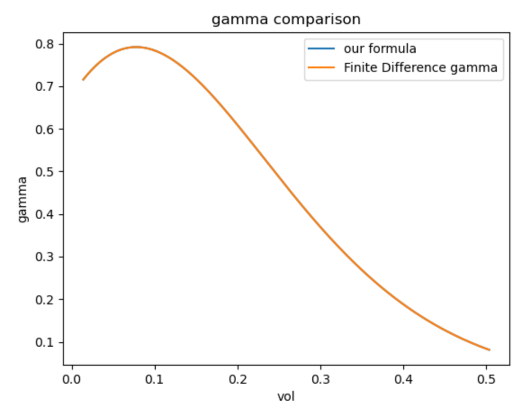
where $\Delta\bar{w}$ is given in equation (3.1.18). We can use the same method illustrated in 3.1 to apply Ito-Taylor expansions on it.



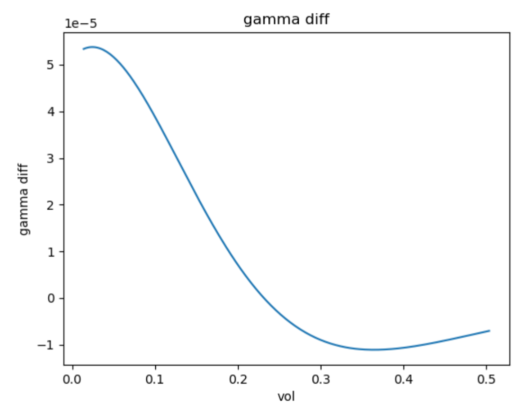
(a) Delta comparison



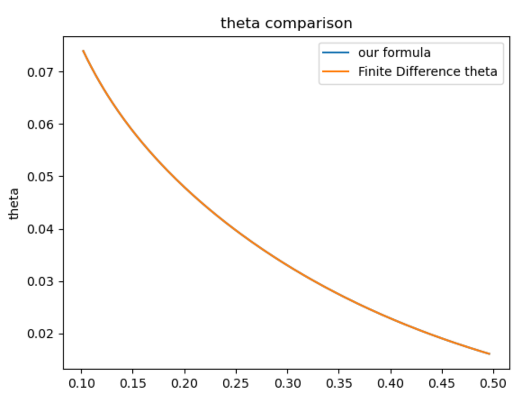
(b) relative error between our formula and finite difference method



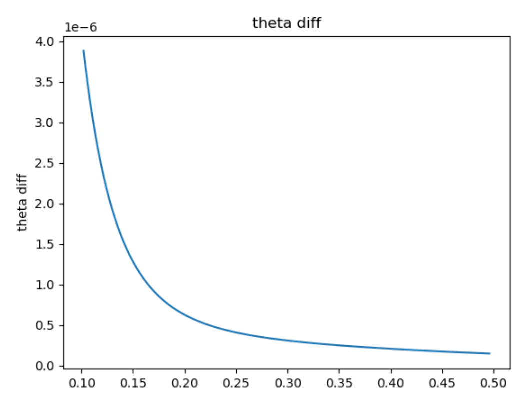
(c) Gamma comparison



(d) relative error between our formula and finite difference method



(e) Theta comparison



(f) relative error between our formula and finite difference method

Figure 3.3: Three Greeks comparison between our formula and finite difference method

Chapter 4

Numerical Results

In this section, we will show our approximation results for mean-reverting CEV model and Heston plus CEV model, expansion of mispricing functions is up to $N = 3$. We utilize a symbolic library of [1]python sympy to calculate expansions, and use Monte Carlo simulations with 200 steps and 100000 paths as our benchmark because there's no existing pricing formula for these models. We evaluate the accuracy of our results with two kinds of figures, the first kind is the direct comparison between benchmarks and our approximation results, the second one is the relative differences between benchmarks and our results. Besides, we also have attached detailed results in the Appendix A.

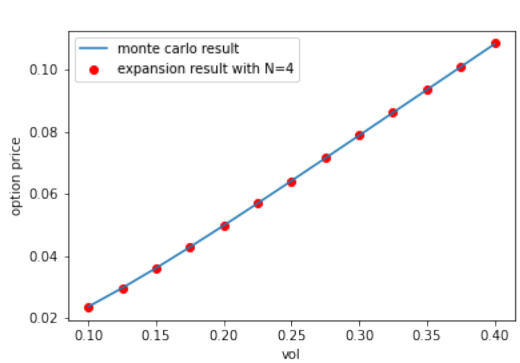
4.1 Volatility option prices under mean-reverting CEV model

For options under mean-reverting CEV model,

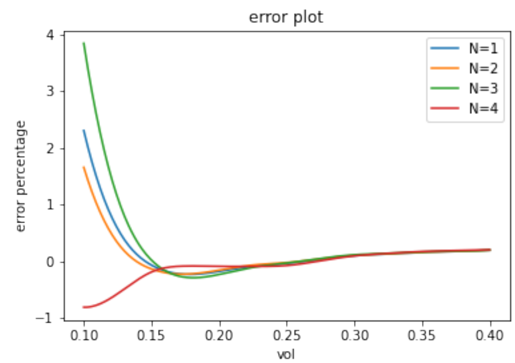
$$\begin{cases} dV_t = \kappa(\theta - V_t)dt + \sigma_{\text{CEV}} V_t^{\gamma} dW_t & \text{true model} \\ dV_t = \kappa(\theta - V_t)dt + \sigma_0 \sqrt{V_t} dW_t & \text{auxiliary model} \end{cases}$$

We use the same mean-reverting parameters as Grünbichler and Longstaff (1996) used in his model, the parameters are $\kappa = 4$, $\theta = 2$. Besides, we set the nuisance parameter $\sigma_0 = \sigma_{\text{CEV}} V_0^{\gamma - \frac{1}{2}}$, where $V_0 = V(t)$ is the initial value of volatility at start point t. We test our approximation method with different constant elasticity parameters, the main idea of setting these parameters is that for small γ, which enlarges the importance of V in the CEV part, we use a small σ; Whereas for large γ

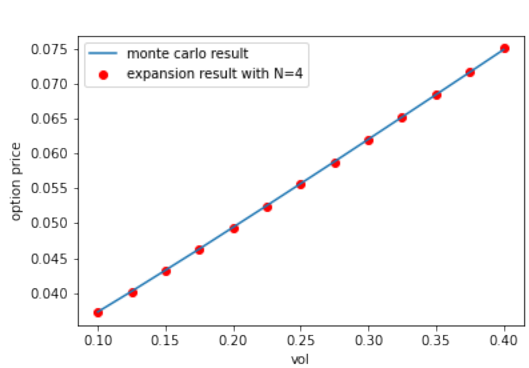
[1]Codes used for this paper can be accessed through my github



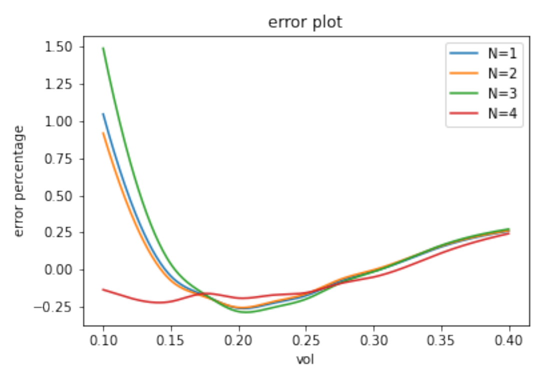
(a) price comparison, $T - t = 0.3$



(b) relative error, $T - t = 0.3$



(c) price comparison, $T - t = 0.5$

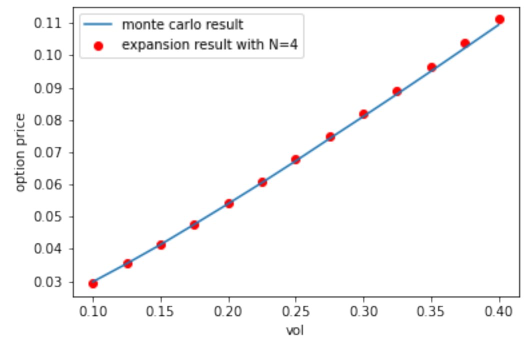


(d) relative error, $T - t = 0.5$

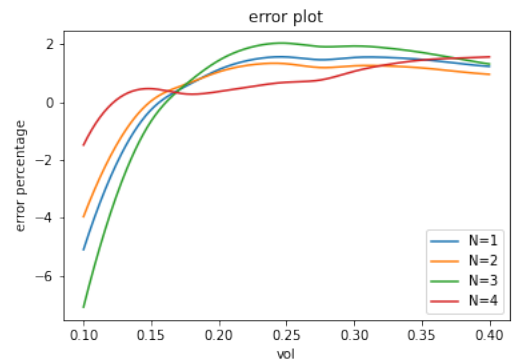
Figure 4.1: mean-reverting CEV model result 1
Parameters are $K = 0.15, \kappa = 4, \theta = 0.2, \sigma = 0.15, \gamma = 0.3$

we set a large σ.

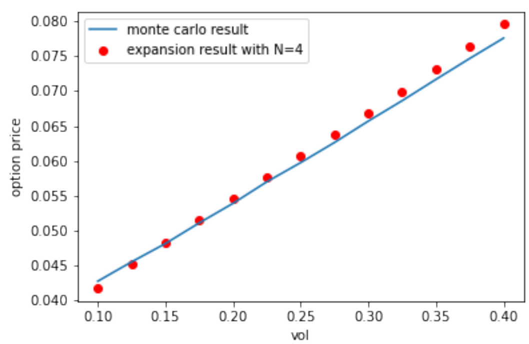
For figure 4.1, our parameters are $\sigma = 0.15$, $\gamma = 0.3$, and with different maturities, $\tau = T - t = 0.3$ for the first row and $\tau = T - t = 0.5$ for the second row. We can find that in 4.1a and 4.1c our approximation results with expansion order $N = 3$ are very accurate. Figure 4.1b and figure 4.1d show the relative error with different expansion orders. We can find that the results with $N = 3$ outperform other results, which implies that keep applying Ito-Taylor expansions on the mispricing functions can create increasingly improved refinements and provide us with more and more accurate results.



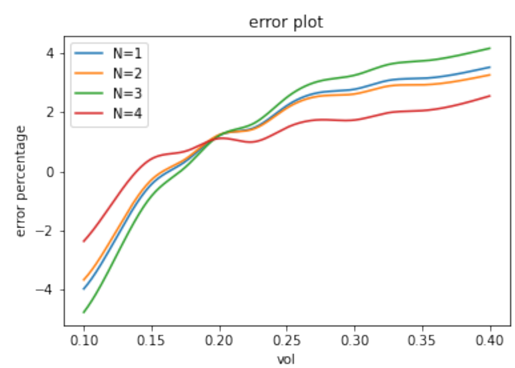
(a) price comparison, $T = 0.3$



(b) relative error, $T = 0.3$



(c) price comparison, $T = 0.5$



(d) relative error, $T = 0.5$

Figure 4.2: mean-reverting CEV model result 2
Parameters are $K = 0.15, \kappa = 4, \theta = 0.2, \sigma = 0.6, \gamma = 0.75$

Parameters for figure 4.2 are $\sigma = 0.6$, $\gamma = 0.75$, and with different maturities $\tau = 0.3$, $\tau = 0.5$. Similarly, our method still provide accurate results.

One may observe that when KM use Black-Scholes model as auxiliary model to price options under Heston model, for deep in-the-money options, relative differences always converge to 0 no matter how many orders of expansions are applied. That is because in his case delta of option is very close to 1 and vega is close to 0, which means option prices are mainly driven by underlying stocks' prices, and volatility has no influence on option prices. Besides, using Black-Scholes model makes mispricing function depend on gamma, while gamma of deep in-the-money options is also close to 0, meaning that their mispricing functions don't affect option prices at all. As a result, their figures show that all results' relative errors are converging to 0 as stock price increases.

However, in our model, underlying assets follow mean-reverting CEV model. Grünbichler and Longstaff (1996) mention that when volatility V is above its long-term mean, mean-reversion property implies the expected future value of V will be lower than its current value, making the expected payoff for a volatility call can be less than its current intrinsic value. The property of options under Black-Scholes world doesn't hold here, recall that before we set $\sigma_0 = \sigma_{\text{CEV}} V_0^{\gamma-\frac{1}{2}}$. Obviously when V_0 is large, $\sigma_{\text{CEV}} V_t^{\gamma} < \sigma_{\text{CEV}} V_0^{\gamma-\frac{1}{2}} \sqrt{V}_t$, causing the loss of accuracy in our auxiliary model. It gives an explanation why the relative error of our method is slightly larger than 0 for deep in-the-money options. Additionally, using our method to price deep out-of-money options can also be challenging. The loss of accuracy for approximating non-central chi-square distribution functions would be magnified when option price is very small.

4.2 Volatility option prices under Heston puls CEV model

For Heston plus CEV model,

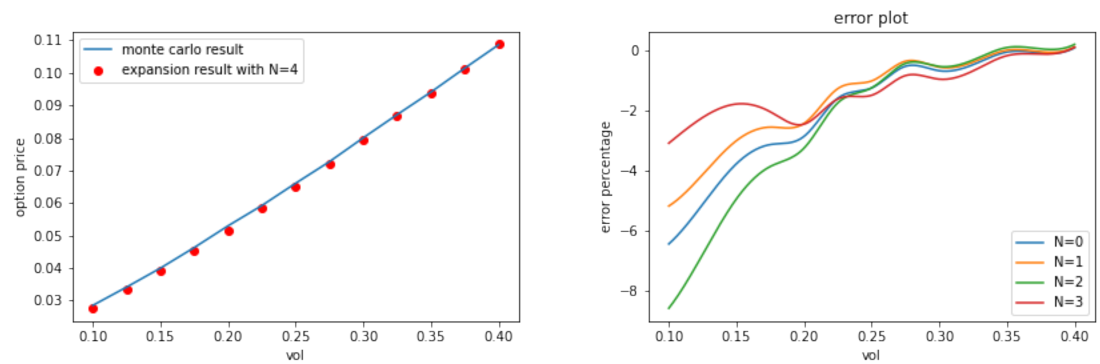
$$dV_t = \kappa_1 \left(V_t' - V_t\right) dt + \sigma_1 \sqrt{V_t} dW_t$$
$$dV_t' = \kappa_2 \left(\theta - V_t'\right) dt + \sigma_2 V_t'^{\gamma} dW_t'$$

and our auxiliary model, square root mean-reverting model,

$$dV_t = \kappa(\theta_0 - V_t)dt + \sigma_0 \sqrt{V_t} dW_t$$

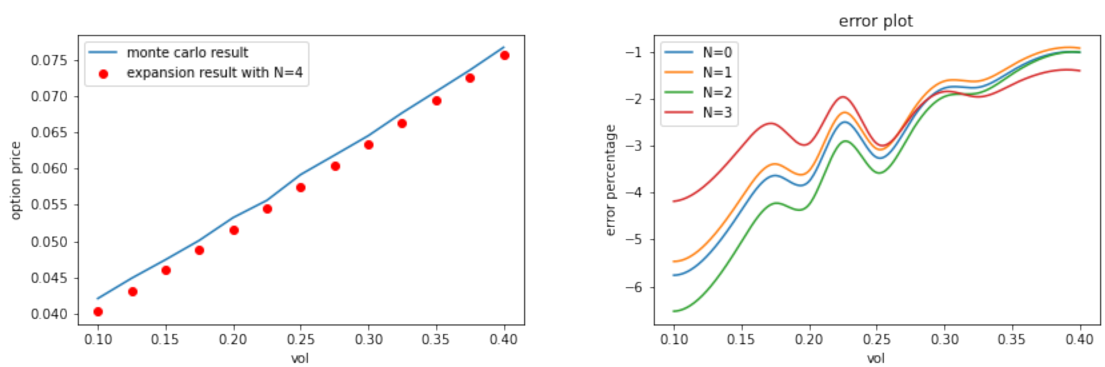
our parameters are $r = 0.05$, $K = 0.15$, $\kappa_1 = 4$, $\kappa_2 = 2$, $\theta = 0.2$, $\sigma_1 = 0.3$, $\sigma_2 = 0.8$,

$\gamma = 1.6$, $\rho = 0.5$ and different maturities $\tau = T - t = 0.3, \tau = T - t = 0.5$. We set the nuisance parameters $\theta_0 = V'(t)$, where $V'(t) = 0.2$ is the spot value for volatility of volatility at time t.



(a) price comparison, $T - t = 0.3$ (b) relative error, $T - t = 0.3$

Figure 4.3: Heston plus CEV model result 1



(a) price comparison, $T - t = 0.5$ (b) relative error, $T - t = 0.5$

Figure 4.4: Heston plus CEV model result 2

As is seen in 4.3a, applying our method under 2-dimensional model can still create relatively accurate results. Unlike mean-reverting CEV model, under Heston plus CEV model our relative differences are now converging to 0. This is because here the initial value of volatility doesn't enter mispricing function $\delta = \kappa_1(V_2 - \theta_2)\Delta_{\bar{w}}$.

Besides, we notice that for 4.4a when $\tau = 0.5$, our results seem not accurate enough. Though in figure 4.4b relative differences reduce as we apply higher further expansions. We may predict that if applying higher order expansions we could get more precise results. However, this raises a constraint of KM's method, number of terms grow exponentially in the final pricing formula as we keep expanding mispricing functions, causing the running time of calculation increasing dramatically, which is a trade-off between applying high order expansions and getting more accurate results. Moreover, if we still don't get a desired result with a threshold expansion order, say $N = 5$, we may need to reconsider our auxiliary model and nuisance parameters.

Chapter 5

Conclusion

In this paper, we introduce approximation method proposed by Heath and Platen (2002) and Kristensen and Mele (2011). Based on their work, we extend this method to price options under mean-reverting CEV model and Heston plus CEV models. Selections of auxiliary models and corresponding mis-pricing formula are discussed, we also illustrate techniques to calculate partial derivatives of non-central chi-square distribution functions when using square root mean-reverting as auxiliary model. Finally, we discuss our numerical results and explain the constraints of our method. In all, numerical results show that our method is efficient and accurate.

Appendix A

Appendix A Mean-Reverting CEV model Numerical Results

Appendices at the end of a dissertation are optional, and depend on the content of the dissertation. There can be one or more appendicies, however they should retain the page numbering requirements for dissertations. Any concerns about the formatting of an appendix should be brought to Doris Oliver, who can direct you how to format your appendix if you have questions.

Theoretical Dissertation Timeline		
Taskt	Time to Finish	Notes
Problem statement	10 hours	Initially very upbeat.
Research	3 days	Literature search to very previous studies.
Reformulation	4 hours	Presented and accepted by advisor
Research	20 days	Literature search to very previous studies.
Experiments	14 days	Do some experiments and get results.
Format	1 day	Understand format guidelines for paper.
Write	years	Write the paper.
Revise	not too long	Proof read, etc.
Format	1-3 days	Verify correct report format is used.
See Library	1 hour	Meet with Doris to verify formatting.
Defend	1 day	Defend your research.
Revise	0 hours	It was perfect the first time.
Submit	1 day	Submit final dissertation to the library.

Table A.1: T=0.3,K=0.15, kappa=4,m=0.2, sigma=0.15, gamma=0.3.csv

vol	mc	w1	w2	w3	w4
0.100	0.023570	0.024113	0.023960	0.024476	0.023380
0.125	0.029589	0.029760	0.029686	0.029921	0.029421
0.150	0.036036	0.036013	0.035989	0.036045	0.035969
0.175	0.042805	0.042709	0.042710	0.042686	0.042771
0.200	0.049793	0.049705	0.049716	0.049678	0.049750
0.225	0.056940	0.056892	0.056904	0.056881	0.056891
0.250	0.064213	0.064192	0.064201	0.064195	0.064168
0.275	0.071525	0.071552	0.071558	0.071564	0.071534
0.300	0.078864	0.078944	0.078948	0.078957	0.078941
0.325	0.086237	0.086351	0.086353	0.086361	0.086360
0.350	0.093613	0.093765	0.093765	0.093771	0.093778
0.375	0.101000	0.101181	0.101181	0.101184	0.101192
0.400	0.108382	0.108598	0.108598	0.108600	0.108606

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