AN APPROXIMATION METHOD TO PRICE VOLATILITY OPTIONS

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ABSTRACT

We propose an approximation method to value volatility options. This method is based on choosing models with closed form solution as an auxiliary model, and derive a mis-pricing formula between the true price and the auxiliary one, then apply Ito-Taylor expansions on the mis-pricing formula to create increasingly improved refinements. We propose an approach to evaluate volatility options under mean-reverting models, in which auxiliary model selection and expansion methods are explained. Method in this paper is applied to mean-reverting Constant elasticity of variance(CEV) model and double CEV models. Numerical results show that the proposed method is accurate and efficient.

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Introduction

Intro to be added

Chapter 2
Method Description

In this section, the origin DOI method and approximation method based on DOI method are described. In section 2.1,, we introduce the origin DOI method, seeHeath and Platen (Heath and Platen). In section 2.2, we illustrate the approximation method proposed by Kristensen and Mele (2011). In section 2.3, we discuss the selection of nuisance to improve efficiency and accuracy of approximation method.

2.1 The DOI Variance Reduction Method

Consider a multi-factor model, in which a d-dimensional vector of state variables $X(t)$ on a filtered probability space$(\Omega, \mathcal{F}, \mathbb{Q})$ satisfies the following Stochastic Differential Equations(SDEs)

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t) \tag{2.1.1}$$

where $\mu(t, X(t))$ and $\sigma(t, X(t))$ are drift and diffusion functions under the risk-neutral measure $\mathbb{Q}$, which also satisfies appropriate growth and Lipschitz conditions such that equation(2.1.1) admits a unique strong solution and is Markovian; $W(t)$ is a d-dimensional standard Brownian Motion and $t \in [0, T]$.

Let $w(t, x)$ be the value function of European option written on $X(T)$ with current state $X(t) = x$, $G(t, x)$ be the payoff function. We define the infinitesimal generator $\mathcal{L}$ associated with equation(2.1.1)to be

$$\mathcal{L}w(t,x) = \frac{\partial w}{\partial t} + \sum_{i=1}^{d} \mu_i(t,x)\frac{\partial w}{\partial x} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(\sigma(t,x)\sigma^{\intercal}(t,x))_{i,j}\frac{\partial^2 w}{\partial x_i x_k} \tag{2.1.2}$$

Let $R(t,x)$ be the instantaneous short-term interest rate, combining with equation(2.1.1) and equation(2.1.2), the price of European option V is a solution to the following partial differential equation(PDE)

$$\mathcal{L}w(x,t) = R(x,t)w(x,t) \tag{2.1.3}$$

with boundary condition $w(T, x(T)) = G(T, X(T))$. It's easily seen that under risk neutral measure $\mathbb{Q}$, the instantaneous option price change is equal to the price gain in saving account.

Next we consider to use a m-dimensional($m \leq d$) process $\bar{X}(t)$ which is a simpler auxiliary model to approximate the price of option. $\bar{X}(t)$ satisfies the following SDE

$$d\bar{X}(t) = \begin{cases} \bar{\mu}_i(t,\bar{X}(t))dt + \bar{\sigma}_i(t,\bar{X}(t))dW(t) & 1 \leq i \leq m \\ \bar{\mu}_i(t,\bar{X}(t)) = 0,\ \bar{\sigma}_i(t,\bar{X}(t)) = 0 & m < i \leq d \end{cases} \tag{2.1.4}$$

where $\bar{\mu}(t,\bar{X}(t))$ and $\bar{\sigma}(t,\bar{X}(t))$ are drift and diffusion functions, and they are also assumed to satisfy appropriate conditions such that equation(2.1.4) admits a unique strong solution and is Markovian.

Let $\bar{w}(t,x)$ be the option price written on process $\bar{X}(t)$ and assume $\bar{w}$ has closed form solution under this new process, the infinitesimal generator $\bar{\mathcal{L}}$ for option price $\bar{w}$ is the same as equation(2.1.2) but replacing $\mu(t,x)$, $\sigma(t,x)$ by $\bar{\mu}(t,x)$ and $\bar{\sigma}(t,x)$. Therefore $\bar{w}(t,x)$ is a solution to

$$\bar{\mathcal{L}}\bar{w}(x,t) = R(x,t)\bar{w}(x,t) \tag{2.1.5}$$

Denote the price difference $\Delta w(t,x) = w(t,x) - \bar{w}(t,x)$, by subtract equation(2.1.5 from equation(2.1.3)), $\Delta w(t,x)$ satisfies the following equation

$$\mathcal{L}\Delta w(t,x) + (\mathcal{L} - \bar{\mathcal{L}})w(t,x) = R(x,t)\Delta w(t,x) \tag{2.1.6}$$

with boundary condition $\Delta w(T,x) = G(T,x) - \bar{G}(T,x)$. We can find that the price difference arises from two parts:

- The use of a wrong payoff function $\bar{G}(t,x)$, it can be eliminated once we use the same payoff function in auxiliary model as it in general model
- The discrepancies between the auxiliary model and general model.

Define $\delta(t,x) = (\mathcal{L} - \bar{\mathcal{L}})w(t,x)$, $d(t,x) = G(t,x) - \bar{G}(t,x)$, under standard regularity conditions[1], we can derive the following formula by using the Feynman-Kac representation.

$$\begin{aligned} w(t,x) = \bar{w}(t,x) &+ \mathbb{E}_{t,x}\left[\exp\left(-\int_t^T R(s,X(s))ds\right) d(T,X(T))\right] \\ &+ \int_t^T \mathbb{E}_{t,x}\left[\exp\left(-\int_t^s R(u,X(u))du\right)\delta(s,X(s))\right] ds \end{aligned} \tag{2.1.7}$$

Finally, under the initial condition $Z_0 = w(0,x)$, the DOI estimator

$$\begin{aligned} Z_t = \bar{w}(t,x) &+ \exp\left(-\int_t^T R(s,X(s))ds\right) d(T,X(T)) \\ &+ \int_t^T \exp\left(-\int_t^s R(x(u),u)du\right)\delta(s,X(s))ds \end{aligned} \tag{2.1.8}$$

[1]See Appendix A in Kristensen and Mele (2011)

is an unbiased estimator for Z_0. And if a good auxiliary model is chosen, the variance of Z_t will be small.

2.2 Approximation Method based on the DOI method

Recall equation(2.1.7), instead of using it as an estimator to do simulations, Kristensen and Mele (2011) make some additional assumptions and use Ito-Taylor expansion to get closed form approximation formula.

For sufficiently smooth function $f(t,x)$,Ito-Taylor expansion is given by

$$\mathbb{E}^{t,x}[f(s,X(s))] = \sum_{N=0}^{J} \frac{(s-t)^N}{N!} (\mathcal{L})^N f(t,x) + \mathcal{R}_J \tag{2.2.1}$$

where the remainder term $\mathcal{R}_J$ is given by

$$\mathcal{R}_J = \mathbb{E}^{t,x}\left[\int_t^s du_1 \int_t^{u_1} du_2 \cdots \int_t^{u_J} (\mathcal{L})^{J+1} f\left(u_{J+1}, X\left(u_{J+1}\right)\right) du_{J+1}\right] \tag{2.2.2}$$

The process $X(t)$ here is defined in equation(2.1.1), and the infinitesimal generator $\mathcal{L}$ is defined in equation(2.1.2).

Assume closed form solution of option price $\bar{V}$ under auxiliary model and the difference of payoff function $d(t,x)$ is sufficiently smooth. In other words, for $N \geq 1$, assume $\delta(t,x)$ and $d(t,x)$ to be $2N$ times differentiable with respect to x, $\delta(t,x)$ to be N times differentiable with respect to t. By applying Ito-Taylor expansion to equation(2.1.7)

$$
\begin{aligned}
V(t,x) = \bar{V}(t,x) &+ \mathbb{E}_{t,x}\left[\exp\left(-\int_t^T R(s,X(s))ds\right) d(T,X(T))\right] \\
&+ \int_t^T \mathbb{E}_{t,x}\left[\exp\left(-\int_t^s R(u,X(u))du\right)\delta(s,X(s))\right] ds
\end{aligned}
\tag{2.2.3}
$$

We can get a closed-form approximation formula

$$
V_N(t,x) = \bar{V}(t,x) + \sum_{n=0}^{N} \frac{(T-t)^n}{n!} d_n(t,x) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!}\delta_n(t,x) \tag{2.2.4}
$$

where $d_0(t,x) = d(x)$, $\delta_0(t,x) = \delta(t,x)$, and

$$
\begin{aligned}
d_n(t,x) &= Ld_{n-1}(t,x) - R(t,x)d_{n-1}(t,x) \\
\delta_n(t,x) &= L\delta_{n-1}(t,x) - R(t,x)\delta_{n-1}(t,x)
\end{aligned}
\tag{2.2.5}
$$

Note that the terms in equation((2.2.4)) can be calculated once for all, meaning that it be computed much faster than simulation methods using estimator.

2.3 Nuisance parameter selection

As mentioned in 2.2, we use an auxiliary model and then expand the mis-pricing term, which leads to a nuisance parameter— a parameter that does not a¤ect the unknown price.

to be added

Chapter 3

Approximations of VIX options

3.1 Approximating options under mean-reverting CEV model

3.1.1 Drawbacks of using Black-Scholes model as an auxiliary model

Chan et al. (1992) proposes the mean-reversion CEV model, in which volatility follows

$$dV_t = (\alpha + \beta V_t)\,dt + \sigma V_t^{\gamma} dW_t$$

when β is negative, this model has mean-reverting property. We can rewrite it to be

$$dV_t = \kappa(m - V_t)dt + \sigma V_t^{\gamma} dW_t \tag{3.1.1}$$

where κ is the speed of mean-reversion, m is the long-run mean. A natural idea is to use Black-Scholes model as auxiliary model as mentioned in Kristensen and Mele (2011), then apply their method to approximate the VIX option price under mean-reverting CEV model. Denote $\mathcal{L}$ and $\mathcal{L}^{\mathrm{BS}}$ to be infinitesimal generators of mean-reverting CEV model and Black-Scholes model respectively

$$\begin{aligned}
\mathcal{L}w &= \frac{\partial w}{\partial t} + \kappa(m - V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V^{2\gamma}\frac{\partial^2 w}{\partial V^2} \\
\mathcal{L}^{\mathrm{BS}}w &= \frac{\partial w}{\partial t} + rV\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V^2\frac{\partial^2 w}{\partial V^2}
\end{aligned}$$

The mis-pricing term for using Black-Scholes model is then

$$\delta^{\mathrm{BS}} = (\mathcal{L} - \mathcal{L}^{\mathrm{BS}})w^{\mathrm{BS}} = (\kappa - r)V\frac{\partial w^{\mathrm{BS}}}{\partial t} + \kappa m\frac{\partial w^{\mathrm{BS}}}{\partial t} + \sigma^2(V - V^{2\gamma})\frac{\partial^2 w^{\mathrm{BS}}}{\partial V^2}$$

with the solution of Black-Scholes model w^{BS}. Note that δ^{BS} contains theta and gamma of option. Their differences in Black-Scholes model and mean-reverting model determines that we have to use other auxiliary models.

Take call option prices under $\gamma = \frac{1}{2}$ in model(3.1.1) as an example. This model is known as mean square root mean-reverting model proposed by Grünbichler and Longstaff (1996).

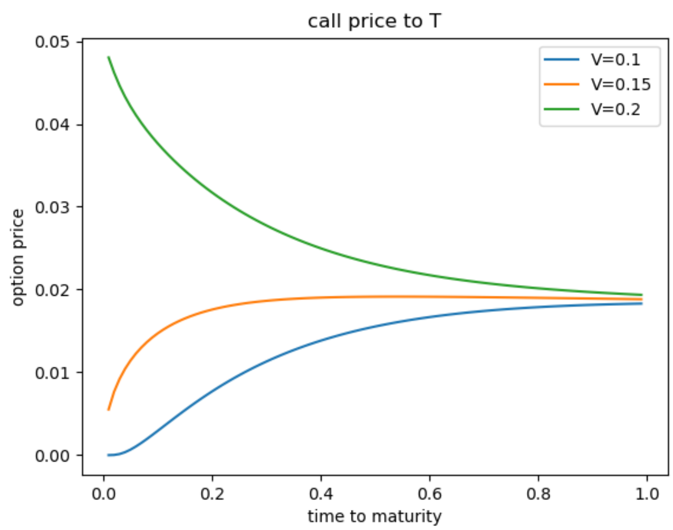

Figure 3.1: Call option price with regard to time to maturity

From figure 3.1, we can find that in contrast to Black-Scholes model, the value of call option price under mean-reverting model is not always increasing as time to maturity increases; From figure 3.2, by contrast, the call option price does not converge to zero as volatility goes to zero. In addition, Grünbichler and Longstaff (1996) also shows that V has less influence of the current value of the call option than in Black-Scholes model. For these reasons, we conclude that Black-Scholes model is not an appropriate auxiliary model and in the next section, we discuss that using the square root mean-reverting model as the auxiliary model.

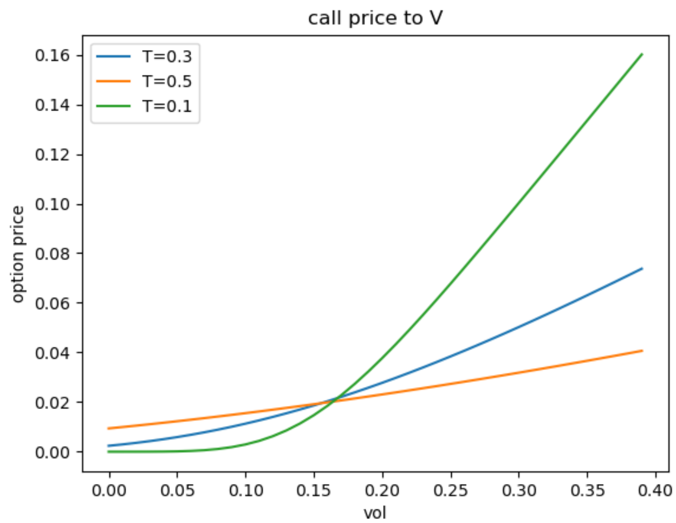

Figure 3.2: Call option price with regard to volatility

3.1.2 Using square root mean-reverting model as auxiliary model

Recall the mean-reverting CEV model with $\gamma = \frac{1}{2}$

$$dV_t = \kappa(m - V_t)dt + \sigma\sqrt{V_t}dW_t \tag{3.1.2}$$

We are going to use it as our auxiliary model as it captures the mean-reverting property of general mean-reverting CEV models. Grünbichler and Longstaff (1996) gives an explicit solution to this model. Denote the call option price $\bar{w}$ with strike K, constant risk-free rate r, time to maturity T and no expected premium for volatility risk is paid, its price is given by

$$\begin{aligned} \bar{w} =& e^{-(\kappa+r)T} VQ(xK; \nu + 4, \lambda) \\ &+ me^{-rT}(1 - e^{-\kappa T})Q(xK; \nu + 2, \lambda) \\ &- e^{-rT} KQ(xK; \nu, \lambda) \end{aligned} \tag{3.1.3}$$

where

$$
\begin{aligned}
x &= \frac{4\kappa}{\sigma^2(1-e^{-\kappa T})} \\
\nu &= \frac{4\kappa m}{\sigma^2}, \\
\lambda &= e^{-\kappa T} x V
\end{aligned}
$$

and $Q(xK; \nu + i, \lambda)$ is the complementary distribution function for the non-central chi-squared density with $\nu + i$ degrees of freedom and non-centrality parameter λ.

Define the infinitesimal generators $\bar{\mathcal{L}}$ for square root mean-reverting model and $\mathcal{L}$ for mean-reverting CEV model

$$
\begin{aligned}
\mathcal{L}w &= \frac{\partial w}{\partial t} + \kappa(m - V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V^{2\gamma}\frac{\partial^2 w}{\partial V^2} \\
\bar{\mathcal{L}}w &= \frac{\partial w}{\partial t} + \kappa(m - V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2 w}{\partial V^2}
\end{aligned}
\tag{3.1.4}
$$

Subtract infinitesimal generators in equation(3.1.4), we get the mis-pricing formula for using square root mean-reverting model

$$
\delta = (\mathcal{L} - \bar{\mathcal{L}})\bar{w} = \frac{1}{2}\sigma^2(V^{2\gamma} - V)\frac{\partial^2 w}{\partial V^2}
$$

We can then use the approximation formula discussed in 2.2 to price call options under mean-reverting CEV model[1]

$$
w_N(t, x) = \bar{w}(t, x) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!}\delta_n(t, x) \tag{3.1.5}
$$

where

$$
\begin{aligned}
\delta_0 = \delta &= \frac{1}{2}\sigma^2(V^{2\gamma} - V)\frac{\partial^2 w}{\partial V^2} \\
\delta_n(t, x) &= L\delta_{n-1}(t, x) - r\delta_{n-1}(t, x)
\end{aligned}
\tag{3.1.6}
$$

[1] Put options can be priced easily in the same way

Finally we get a closed form approximating formula for call options under mean-reverting CEV model. But notice that the call price (3.1.3) contains non-square chi square distribution functions, applying infinitesimal generator $\mathcal{L}$ on it can be a hard point and in the next section we are going to talk about how to derive partial derivatives of distribution function $Q(xK;\nu+i,\lambda)$.

3.1.3 Method to Calculate Derivatives In Expansions

In this section, methods to calculate closed-form partial derivatives of call option price $\bar{w}$ to time t and volatility V. Our method is based on the recurrence relation of non-central chi-square distribution proposed by Cohen (1988), which is

$$\begin{aligned} \frac{\partial p(xK;\nu,\lambda)}{\partial(xK)} &= \frac{1}{2}[-p(xK;\nu,\lambda)+p(xK;\nu-2,\lambda)] \\ \frac{\partial p(xK;\nu,\lambda)}{\partial\lambda} &= \frac{1}{2}[-p(xK;\nu,\lambda)+p(xK;\nu+2,\lambda)] \end{aligned} \tag{3.1.7}$$

where $p(xK;\nu,\lambda)$ is the Probability Density Function(PDF) of non-central chi-square distribution. From the relationship between Complementary Cumulative Distribution Function(CCDF) $Q(xK;\nu,\lambda)$, Cumulative Distribution Function(CDF) $F(xK;\nu,\lambda)$, and PDF we know that

$$\begin{aligned} \frac{\partial Q(xK;\nu,\lambda)}{\partial(xK)} &= \frac{\partial[1-F(xK;\nu,\lambda)]}{\partial(xK)} \\ &= -\frac{\partial F(xK;\nu,\lambda)]}{\partial(xK)} \\ &= -p(xK;\nu,\lambda) \end{aligned} \tag{3.1.8}$$

Rewrite the second equation in (3.1.7), we get

$$\begin{aligned}\frac{\partial p(xK;\nu,\lambda)}{\partial\lambda} &= \frac{1}{2}[-p(xK;\nu,\lambda)+p(xK;\nu+2,\lambda)]\\ &= -\frac{1}{2}[-p(xK;\nu+2,\lambda)+p(xK;\nu,\lambda)]\\ &= -\frac{\partial p(xK;\nu+2,\lambda)}{\partial(xK)}\end{aligned} \tag{3.1.9}$$

Integrate both sides of (3.1.9) with respect to xK and combine with (3.1.8), we can derive the partial derivative of CDF to non-central parameter λ

$$\begin{aligned}\frac{\partial}{\partial\lambda}F(xK;\nu,\lambda) &= -\frac{\partial}{\partial(xK)}F(xK;\nu+2,\lambda)\\ &= -p(xK;\nu+2,\lambda)\end{aligned} \tag{3.1.10}$$

Finally we get the partial derivative of CCDF to non-central parameter λ

$$\begin{aligned}\frac{\partial Q(xK;\nu,\lambda)}{\partial\lambda} &= \frac{\partial[1-F(xK;\nu,\lambda)]}{\partial\lambda}\\ &= -\frac{\partial F(xK;\nu,\lambda)]}{\partial\lambda}\\ &= p(xK;\nu+2,\lambda)\end{aligned} \tag{3.1.11}$$

Until now we can summarize that the derivatives of CCDF and PDF are all combinations of PDFs with change of degrees of freedom. Without loss of accuracy, we make the degrees of freedom in PDF be consistent with call option solution in (3.1.3), that is for $p(xK;\nu+i,\lambda)$, we let $i\in[0,4]$. Use the non-central chi-square property by Cohen (1988) to do the following transformation

$$\begin{aligned}p(xK;\nu-2,\lambda) &= \frac{\lambda}{xK}p(xK;\nu+2,\lambda)+\frac{\upsilon-2}{xK}p(xK;\nu,\lambda)\\ p(xK;\nu+6,\lambda) &= \frac{xK}{\lambda}p(xK;\nu+2,\lambda)-\frac{\nu+2}{\lambda}p(xK;\nu+4,\lambda)\end{aligned} \tag{3.1.12}$$

Next we use the results above to calculate delta and gamma of auxiliary call option price $\bar{w}$. Recall the parameter xK, ν and λ in (3.1.2), where

$$\begin{aligned} x &= \frac{4\kappa}{\sigma^2(1-e^{-\kappa T})} \\ \nu &= \frac{4\kappa m}{\sigma^2}, \\ \lambda &= e^{-\kappa T} xV \end{aligned} \tag{3.1.13}$$

Then we use chain rule calculate the following auxiliary derivatives

$$\begin{aligned} \frac{\partial Q(xK;\nu,\lambda)}{\partial V} &= \frac{\partial Q}{\partial x}\frac{\partial x}{\partial V} + \frac{\partial Q}{\partial \lambda}\frac{\partial \lambda}{\partial V} \\ &= 0 + xe^{-\kappa T} p(x;\nu+2,\lambda) \\ &= xe^{-\kappa T} p(x;\nu+2,\lambda) \end{aligned} \tag{3.1.14}$$

Thus delta is given by

$$\begin{aligned} \bar{w} =& e^{-(\kappa+r)T} Q(xK;\nu+4,\lambda) + e^{-(\kappa+r)T} V \cdot xe^{-\kappa T} p(xK;\nu+6,\lambda) \\ &+ me^{-rT}(1-e^{-\kappa T}) \cdot xe^{-\kappa T} p(x;\nu+2,\lambda) - e^{-rT} K \cdot xe^{-\kappa T} p(x;\nu+2,\lambda) \end{aligned} \tag{3.1.15}$$

Using (3.1.12) to substitute $p(xK;\nu+6,\lambda)$ and simplify the equation

$$\begin{aligned} \Delta_{\bar{w}} =& e^{-(\kappa+r)T} Q(xK;\nu+4,\lambda) \\ &+ e^{-(\kappa+r)T} \lambda \left[\frac{xK}{\lambda} p(xK;\nu+2,\lambda) - \frac{\nu+2}{\lambda} p(xK;\nu+4,\lambda) \right] \\ &+ me^{-rT}(1-e^{-\kappa T}) \cdot \frac{4\kappa}{\sigma^2(1-e^{-\kappa T})} e^{-\kappa T} p(x;\nu+2,\lambda) - e^{-(r+\kappa)T} K p(x;\nu+2,\lambda) \\ =& e^{-(\kappa+r)T} [Q(xK;\nu+4,\lambda) - 2p(xK;\nu+4,\lambda)] \end{aligned} \tag{3.1.16}$$

Similarly, we calculate another auxiliary derivative

$$\frac{\partial p(xK;\nu,\lambda)}{\partial V} = \frac{\partial p}{\partial(xK)}\frac{\partial(xK)}{\partial V} + \frac{\partial p}{\partial\lambda}\frac{\partial\lambda}{\partial V}$$
$$= \frac{xe^{-\kappa T}}{2}[-p(xK;\nu,\lambda) + p(xK;\nu+2,\lambda)] \tag{3.1.17}$$

As a result, gamma of $\bar{w}$ is then

$$\Gamma_{\bar{w}} = e^{-(\kappa+r)T}\left[xe^{-\kappa T}p(x;\nu+6,\lambda) - 2\cdot\frac{xe^{-\kappa T}}{2}[-p(xK;\nu,\lambda) + p(xK;\nu+2,\lambda)]\right]$$
$$= xe^{-(2\kappa+r)T}p(xK;nu+4,\lambda) \tag{3.1.18}$$

To apply infinitesimal generator on mis-pricing formula, we still need to calculate partial derivatives of PDF to time t. Define the following auxiliary functions

$$\frac{\partial(xK)}{\partial t} = \frac{-\kappa e^{-\kappa T}}{1-e^{-\kappa T}}\cdot xK$$
$$\frac{\partial\lambda}{\partial t} = \frac{-\kappa e^{-\kappa T}}{1-e^{-\kappa T}}\cdot xV \tag{3.1.19}$$

Then partial derivatives of PDF to t is given by

$$\frac{\partial p(xK;\nu,\lambda)}{\partial t} = \frac{\partial p}{\partial(xK)}\frac{\partial(xK)}{\partial t} + \frac{\partial p}{\partial\lambda}\frac{\partial\lambda}{\partial t}$$
$$= \frac{-\kappa xe^{-\kappa T}}{2(1-e^{-\kappa T})}\left[Vp(xK;\nu+2,\lambda) - (K+V)p(xK;\nu,\lambda) + Kp(xK;\nu-2,\lambda)\right] \tag{3.1.20}$$

From (3.1.6) we know that the mis-pricing formula $\delta = \frac{1}{2}\sigma^2(V^{2\gamma} - V)\Gamma_{\bar{w}}$, all terms in which have been solved from above. In essence, to apply Ito-Taylor expansions on δ, we use the following algorithm as used in calculating delta and gamma:

1. Combining previous auxiliary partial derivatives, use chain rule to apply in-

finitesimal generator on mis-pricing formula.

2. Substitute PDFs with noncentral parameter $\nu + i$ where $\nu \notin [0, 4]$.

3. Back to step 1, apply higher order infinitesimal generators.

Therefore, we illustrate a solution to implement approximation method on volatility options under mean-reverting CEV model. The expansions in approximating formula can be computed once for all, we can solve it manually or use symbolic language for higher orders. All terms in the result is explicit expect non-central chi-square PDFs, we plug $p(xK; \nu + i, \lambda)$ into the result at last.

3.2 Approximating options under double Heston model

Gatheral (2008) proposes volatility with double mean-reverting dynamics

$$dV_t = -\kappa \left(V_t - V'(t)\right) dt + \eta_1 V_t'^{\alpha} dW_1(t)$$
$$dV_t' = -c \left(V_t' - m\right) dt + \eta_2 V_t'^{\beta} dW_2(t)$$

where $\alpha, \beta \in [\frac{1}{2}, 1]$.

- It's called Double Heston model in the case $\alpha = \beta = \frac{1}{2}$.
- The case $\alpha = \beta = 1$ Double Log-normal model.
- And the general Double CEV model.

From our previous work, we can use the same auxiliary model to price options with V_t following heston dynamics and V_t' following any mean-reverting CEV process, we call it one Heston one CEV model. This model is given by

$$
\begin{aligned}
dV_t &= -\kappa \left(V_t - V'(t)\right) dt + \eta_1 \sqrt{V_t} dW_1(t) \\
dV'_t &= -c \left(V'_t - m\right) dt + \eta_2 V_t'^{\beta} dW_2(t)
\end{aligned}
\tag{3.2.1}
$$

Define infinitesimal generator $\mathcal{L}$ for (3.2.1) and $\bar{\mathcal{L}}$ for square root mean-reverting model

$$
\begin{aligned}
\mathcal{L}w &= \frac{\partial w}{\partial t} + \kappa(V' - V)\frac{\partial w}{\partial V} + \frac{1}{2}\eta_1^2 V \frac{\partial^2 w}{\partial V^2} \\
&+ \frac{\partial w}{\partial t} + c(m' - V')\frac{\partial w}{V'} + \frac{1}{2}\eta_2^2 V \frac{\partial^2 w}{V'^2} \\
\bar{\mathcal{L}}w &= \frac{\partial w}{\partial t} + \kappa(m - V)\frac{\partial w}{\partial V} + \frac{1}{2}\eta_1 V \frac{\partial^2 w}{\partial V^2}
\end{aligned}
\tag{3.2.2}
$$

Mis-pricing formula for it is then

$$
\begin{aligned}
\delta = (\mathcal{L} - \bar{\mathcal{L}})\bar{w} &= \kappa(V' - m)\frac{\partial w}{\partial V} \\
&= \kappa(V' - m)\Gamma_{\bar{w}}
\end{aligned}
$$

where $\Gamma_{\bar{w}}$ is given in (3.1.18).

Chapter 4

Numerical Results

In this section, we will show our pricing results for square root mean-reverting model and one heston one CEV model. We use Monte Carlo simulations as our benchmark as there's no existing pricing formulas for these two models.

4.1 Volatility option prices under mean-reverting CEV model

As mentioned in 2.3, we want to choose a nuisance parameter for our auxiliary model. In this case, our mis-pricing formula is given by

$$\delta = \frac{1}{2}\sigma^2(V^{2\gamma} - V)\frac{\partial^2 w}{\partial V^2} = \frac{1}{2}\sigma^2(V^{2\gamma-1} - 1)V\frac{\partial^2 w}{\partial V^2}$$

We consider to choose $\hat{\sigma}$ such that

$$\hat{\sigma}_N(x,t) = \arg\min_{\sigma}\left(w_N(x,t;\sigma) - w_0(x,t;\sigma)\right)^2$$

Therefore, we let $\sigma = \sigma_{CEV}$

Appendix A

Appendix A Title

Appendices at the end of a dissertation are optional, and depend on the content of the dissertation. There can be one or more appendicies, however they should retain the page numbering requirements for dissertations. Any concerns about the formatting of an appendix should be brought to Doris Oliver, who can direct you how to format your appendix if you have questions.

Theoretical Dissertation Timeline		
Taskt	Time to Finish	Notes
Problem statement	10 hours	Initially very upbeat.
Research	3 days	Literature search to very previous studies.
Reformulation	4 hours	Presented and accepted by advisor
Research	20 days	Literature search to very previous studies.
Experiments	14 days	Do some experiments and get results.
Format	1 day	Understand format guidelines for paper.
Write	years	Write the paper.
Revise	not too long	Proof read, etc.
Format	1-3 days	Verify correct report format is used.
See Library	1 hour	Meet with Doris to verify formatting.
Defend	1 day	Defend your research.
Revise	0 hours	It was perfect the first time.
Submit	1 day	Submit final dissertation to the library.

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