AN APPROXIMATION METHOD TO PRICE VOLATILITY OPTIONS

by

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ABSTRACT

We propose an approximation method to value volatility options. This method is based on choosing models with closed form solution as an auxiliary model, and derive a mis-pricing formula between the true price and the auxiliary one, then apply Ito-Taylor expansions on the mis-pricing formula to create increasingly improved refinements. We propose an approach to evaluate volatility options under mean-reverting models, in which auxiliary model selection and expansion methods are explained. Method in this paper is applied to mean-reverting Constant elasticity of variance(CEV) model and double CEV models. Numerical results show that the proposed method is accurate and efficient.

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Chapter 1

Introduction

The Volatility, as a measurement of the risk, is one of the most important element in the study of finance. Whaley (1993) proposes that by allowing investors to hedge directly against shifts in volatility, these securities enable investors to avoid the costs of dynamically adjusting positions for changes in volatility and serve to make the market more complete. During the past few years, derivatives written on volatility have developed rapidly, in which options are clearly the fundamental type of derivatives. However, unlike options on stocks, pricing volatility options can be challenging by its mean-revering property. On the basis of Cox et al. (1985), Hull and White (1987), Heston (1993), Grünbichler and Longstaff (1996) proposes Mean-Reverting Square Root(MRSR) model; Chan et al. (1992) then generalizes Mean-Reverting constant elasticity of variance(MRCEV) model and Gatheral (2008) proposes the Double CEV model to capture the dynamics of volatility. Grünbichler and Longstaff (1996) gives a closed-form solution for options under MRSR model, but there're no existing solutions to the following two models. The aim of our paper is to use the solution of square-root mean-reverting model to approximate the other two [1]models.

Heath and Platen (2002)(HP) first introduce a diffusion operator integral (DOI) method, he uses an auxiliary model and then apply Ito calculus on it to find unbiased variance reduction estimators for the true model, he also adapt DOI method to be employed in conjunction with PDE methods and test his method with Heston model. Similar to the idea of HP, Kristensen and Mele (2011)(KM) doesn't use auxiliary model to serve as a variance reduction estimator, but apply Ito-Taylor

[1]Strictly speaking, for double CEV model, we mainly consider Heston plus CEV model, that is $\gamma_1 = \frac{1}{2}$. Details are discussed in 3.2

expansions on the difference of auxiliary model and true model to create increasingly improved refinements. KM apply his method on several fields including bond pricing under CIR model and option pricing under stochastic volatility models.

Based on HP and KM's work, we extend their methods to price options under mean-revering models. This paper is organized as following, in chapter 2, we illustrate DOI method and KM's expansion method; In chapter 3, we discuss our method on approximating option prices under mean-reverting CEV model and Heston plus CEV model, including the auxiliary model selection, techniques of taking partial derivatives; In chapter 4, we show our numerical results and we put our conclusion in chapter 5.

Chapter 2

Method Description

In this section, we will discuss HP's DOI method and KM's approximation method. In section 2.1, we introduce HP's extension to the approximation of solutions of parabolic PDEs; In section 2.2, we illustrate complementary contents for KM's approximation method.

2.1 The DOI Variance Reduction Method

Consider a multi-factor model, in which a d-dimensional vector of state variables X_t on a filtered probability space$(\Omega, \mathcal{F}, \mathbb{Q})$ satisfies the following Stochastic Differential Equations(SDEs)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.1.1}$$

where $\mu(t, X_t)$ and $\sigma(t, X_t)$ are drift and diffusion functions under the risk-neutral measure $\mathbb{Q}$. HP imposes $\mu : [0, T] \times \Gamma \to \mathbb{R}^d$, $\sigma : [0, T] \times \Gamma \to \mathbb{R}^d$, in which Γ is assumed to be a bounded subset of $\mathbb{R}^d$. μ and σ also satisfies appropriate growth and Lipschitz conditions such that equation(2.1.1) admits a unique strong solution and is Markovian; W_t is a d-dimensional standard Brownian Motion and $t \in [0, T]$.

Let $w(t, x)$ be the valuation function satisfying $w : [0, T] \times \Gamma$, payoff function $h(t, x)$ satisfying $h : B \to \mathbb{R}, B \subseteq \mathbb{R}^d$. Define the infinitesimal generator L associated with equation (2.1.1) to be

$$Lw(t,x) = \frac{\partial w}{\partial t} + \sum_{i=1}^{d} \mu_i(t,x)\frac{\partial w}{\partial x} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(\sigma(t,x)\sigma^{\intercal}(t,x))_{i,j}\frac{\partial^2 w}{\partial x_i x_k} \tag{2.1.2}$$

Let $R(t,x)$ be the instantaneous short-term interest rate also satisfying $R : B \to \mathbb{R}$, combining with equation (2.1.1) and equation (2.1.2), the task is to find an approximation to the solution to the following partial differential equation(PDE)

$$Lw(x,t) = R(x,t)w(x,t) \tag{2.1.3}$$

with boundary condition $w(t,x) = h(t,x)$. It's easily seen that under risk neutral measure $\mathbb{Q}$, the instantaneous option price change is equal to the price gain in saving account.

Next we consider to use a m-dimensional($m \leq d$) process $\bar{X}(t)$ which is a simpler auxiliary model to approximate the price of option. $\bar{X}(t)$ satisfies the following SDE

$$d\bar{X}_t = \begin{cases} \bar{\mu}_i(t,\bar{X}_t)dt + \bar{\sigma}_i(t,\bar{X}_t)dW_t & 1 \leq i \leq m \\ 0 \cdot dt + 0 \cdot dW_t & m < i \leq d \end{cases} \tag{2.1.4}$$

where $\bar{\mu}(t,\bar{X}_t)$ and $\bar{\sigma}(t,\bar{X}_t)$ are drift and diffusion functions, and they are also assumed to satisfy appropriate conditions such that equation(2.1.4) admits a unique strong solution and is Markovian. In KM's method, it's automatically satisfied because he supposes there exists closed-from solution to auxiliary model such that he can apply his approximation method.

Let $\bar{w}(t,x)$ be the option price written on process $\bar{X}_t$, the infinitesimal generator $\bar{L}$ for option price $\bar{w}$ is the same as equation(2.1.2) but replacing $\mu(t,x)$, $\sigma(t,x)$ by

$\bar{\mu}(t,x)$ and $\bar{\sigma}(t,x)$. Therefore $\bar{w}(t,x)$ is a solution to

$$\bar{\mathcal{L}}\bar{w}(x,t) = R(x,t)\bar{w}(x,t) \tag{2.1.5}$$

Denote the price difference $\Delta w(t,x) = w(t,x) - \bar{w}(t,x)$, then Δw satisfies

$$L\Delta w(t,x) + (L - \bar{L})w(t,x) = R(x,t)\Delta w(t,x) \tag{2.1.6}$$

with boundary condition $\Delta w(t,x) = 0$. Define $\delta(t,x) = (L - \bar{L})\bar{w}(t,x)$, using Feynman-Kac representation leads to the following form

$$w(t,x) = \bar{w}(t,x) + \int_t^T \mathbb{E}_{t,x}\left[\exp\left(-\int_t^s R(u,X_u)du\right)\delta(s,X_s)\right]ds \tag{2.1.7}$$

Finally, under the initial condition $Z_0 = w(0,x)$, the DOI estimator is then

$$Z_t = \bar{w}(t,x) + \int_t^T \exp\left(-\int_t^s R(u,X_u)du\right)\delta(s,X_s)ds \tag{2.1.8}$$

is an unbiased estimator for Z_0. And if a good auxiliary model is chosen, for example, when $\bar{L}$ is close to L, the variance of Z_t will be small.

2.2 Approximation Method based on the DOI method

Recall equation(2.1.7), instead of using it as an estimator to do simulations, Kristensen and Mele (2011) make some additional assumptions and use Ito-Taylor expansion to get closed form approximation formula.

For sufficiently smooth function $f(t,x)$,Ito-Taylor expansion is given by

$$\mathbb{E}^{t,x}[f(s,X(s))] = \sum_{N=0}^{J} \frac{(s-t)^N}{N!} (\mathcal{L})^N f(t,x) + \mathcal{R}_J \tag{2.2.1}$$

where the remainder term $\mathcal{R}_J$ is given by

$$\mathcal{R}_J = \mathbb{E}^{t,x}\left[\int_t^s du_1 \int_t^{u_1} du_2 \cdots \int_t^{u_J} (\mathcal{L})^{J+1} f\left(u_{J+1}, X\left(u_{J+1}\right)\right) du_{J+1}\right] \tag{2.2.2}$$

The process $X(t)$ here is defined in equation(2.1.1), and the infinitesimal generator $\mathcal{L}$ is defined in equation(2.1.2).

Assume closed form solution of option price $\bar{V}$ under auxiliary model and the difference of payoff function $d(t,x)$ is sufficiently smooth. In other words, for $N \geq 1$, assume $\delta(t,x)$ and $d(t,x)$ to be $2N$ times differentiable with respect to x, $\delta(t,x)$ to be N times differentiable with respect to t. By applying Ito-Taylor expansion to equation(2.1.7)

$$\begin{aligned} V(t,x) = \bar{V}(t,x) &+ \mathbb{E}_{t,x}\left[\exp\left(-\int_t^T R(s,X(s))ds\right) d(T,X(T))\right] \\ &+ \int_t^T \mathbb{E}_{t,x}\left[\exp\left(-\int_t^s R(u,X(u))du\right)\delta(s,X(s))\right] ds \end{aligned} \tag{2.2.3}$$

We can get a closed-form approximation formula

$$V_N(t,x) = \bar{V}(t,x) + \sum_{n=0}^{N} \frac{(T-t)^n}{n!} d_n(t,x) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(t,x) \tag{2.2.4}$$

where $d_0(t,x) = d(x)$, $\delta_0(t,x) = \delta(t,x)$, and

$$d_n(t,x) = Ld_{n-1}(t,x) - R(t,x)d_{n-1}(t,x)$$
$$\delta_n(t,x) = L\delta_{n-1}(t,x) - R(t,x)\delta_{n-1}(t,x) \tag{2.2.5}$$

Note that the terms in equation((2.2.4)) can be calculated once for all, meaning that it be computed much faster than simulation methods using estimator.

2.3 Nuisance parameter selection

As mentioned in 2.2, we use an auxiliary model and then expand the mis-pricing term, which leads to a nuisance parameter— a parameter that does not a¤ect the unknown price.

to be added

Chapter 3

Approximations of VIX options

3.1 Approximating options under mean-reverting CEV model

3.1.1 Drawbacks of using Black-Scholes model as an auxiliary model

Chan et al. (1992) proposes the mean-reversion CEV model, in which volatility follows

$$dV_t = (\alpha + \beta V_t)\,dt + \sigma V_t^{\gamma} dW_t$$

when β is negative, this model has mean-reverting property. We can rewrite it to be

$$dV_t = \kappa(m - V_t)dt + \sigma V_t^{\gamma} dW_t \tag{3.1.1}$$

where κ is the speed of mean-reversion, m is the long-run mean. A natural idea is to use Black-Scholes model as auxiliary model as mentioned in Kristensen and Mele (2011), then apply their method to approximate the VIX option price under mean-reverting CEV model. Denote $\mathcal{L}$ and $\mathcal{L}^{\mathrm{BS}}$ to be infinitesimal generators of mean-reverting CEV model and Black-Scholes model respectively

$$\begin{aligned}\mathcal{L}w &= \frac{\partial w}{\partial t} + \kappa(m - V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V^{2\gamma}\frac{\partial^2 w}{\partial V^2}\\ \mathcal{L}^{\mathrm{BS}}w &= \frac{\partial w}{\partial t} + rV\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V^2\frac{\partial^2 w}{\partial V^2}\end{aligned}$$

The mis-pricing term for using Black-Scholes model is then

$$\delta^{\mathrm{BS}} = (\mathcal{L} - \mathcal{L}^{\mathrm{BS}})w^{\mathrm{BS}} = (\kappa - r)V\frac{\partial w^{\mathrm{BS}}}{\partial t} + \kappa m\frac{\partial w^{\mathrm{BS}}}{\partial t} + \sigma^2(V - V^{2\gamma})\frac{\partial^2 w^{\mathrm{BS}}}{\partial V^2}$$

with the solution of Black-Scholes model w^{BS}. Note that δ^{BS} contains theta and gamma of option. Their differences in Black-Scholes model and mean-reverting model determines that we have to use other auxiliary models.

Take call option prices under $\gamma = \frac{1}{2}$ in model(3.1.1) as an example. This model is known as mean square root mean-reverting model proposed by Grünbichler and Longstaff (1996).

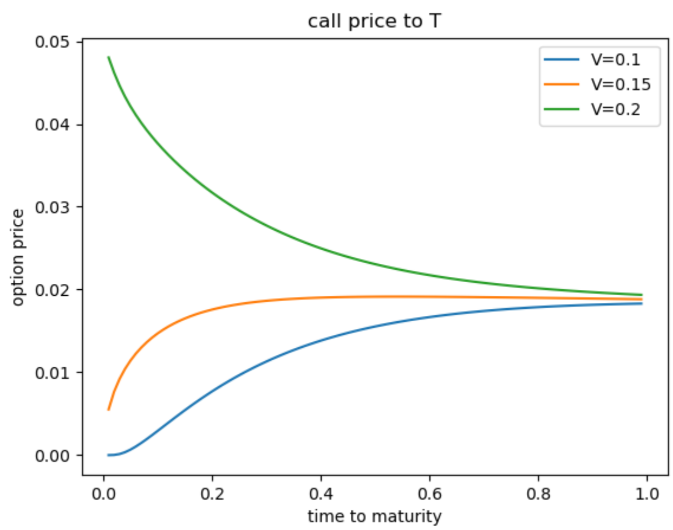


Figure 3.1: Call option price with regard to time to maturity

From figure 3.1, we can find that in contrast to Black-Scholes model, the value of call option price under mean-reverting model is not always increasing as time to maturity increases; From figure 3.2, by contrast, the call option price does not converge to zero as volatility goes to zero. In addition, Grünbichler and Longstaff (1996) also shows that V has less influence of the current value of the call option than in Black-Scholes model. For these reasons, we conclude that Black-Scholes model is not an appropriate auxiliary model and in the next section, we discuss that using the square root mean-reverting model as the auxiliary model.

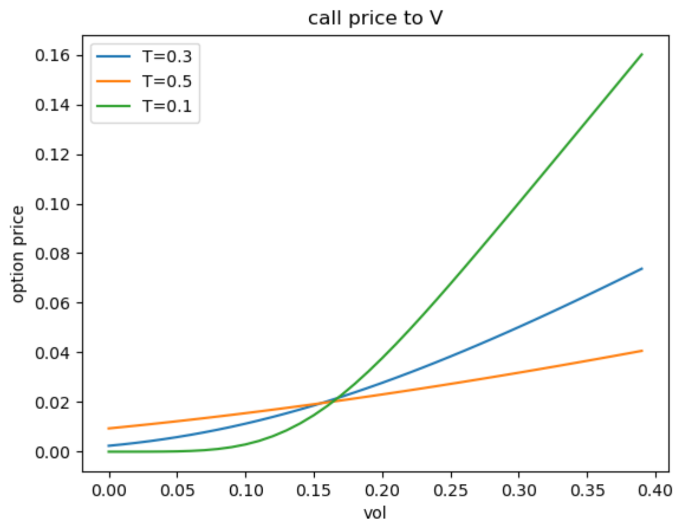


Figure 3.2: Call option price with regard to volatility

3.1.2 Using square root mean-reverting model as auxiliary model

Recall the mean-reverting CEV model with $\gamma = \frac{1}{2}$

$$dV_t = \kappa(m - V_t)dt + \sigma\sqrt{V_t}dW_t \tag{3.1.2}$$

We are going to use it as our auxiliary model as it captures the mean-reverting property of general mean-reverting CEV models. Grünbichler and Longstaff (1996) gives an explicit solution to this model. Denote the call option price $\bar{w}$ with strike K, constant risk-free rate r, time to maturity T and no expected premium for volatility risk is paid, its price is given by

$$\begin{aligned}\bar{w} =& e^{-(\kappa+r)T} VQ(xK; \nu + 4, \lambda) \\ &+ me^{-rT}(1 - e^{-\kappa T})Q(xK; \nu + 2, \lambda) \\ &- e^{-rT} KQ(xK; \nu, \lambda)\end{aligned} \tag{3.1.3}$$

where

$$x = \frac{4\kappa}{\sigma^2(1-e^{-\kappa T})}$$
$$\nu = \frac{4\kappa m}{\sigma^2},$$
$$\lambda = e^{-\kappa T} x V$$

and $Q(xK; \nu + i, \lambda)$ is the complementary distribution function for the non-central chi-squared density with $\nu + i$ degrees of freedom and non-centrality parameter λ.

Define the infinitesimal generators $\bar{\mathcal{L}}$ for square root mean-reverting model and $\mathcal{L}$ for mean-reverting CEV model

$$\begin{aligned}
\mathcal{L}w &= \frac{\partial w}{\partial t} + \kappa(m-V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V^{2\gamma}\frac{\partial^2 w}{\partial V^2} \\
\bar{\mathcal{L}}w &= \frac{\partial w}{\partial t} + \kappa(m-V)\frac{\partial w}{\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2 w}{\partial V^2}
\end{aligned} \tag{3.1.4}$$

Subtract infinitesimal generators in equation(3.1.4), we get the mis-pricing formula for using square root mean-reverting model

$$\delta = (\mathcal{L} - \bar{\mathcal{L}})\bar{w} = \frac{1}{2}\sigma^2(V^{2\gamma} - V)\frac{\partial^2 w}{\partial V^2}$$

We can then use the approximation formula discussed in 2.2 to price call options under mean-reverting CEV model[1]

$$w_N(t,x) = \bar{w}(t,x) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!}\delta_n(t,x) \tag{3.1.5}$$

where

$$\begin{aligned}
\delta_0 = \delta &= \frac{1}{2}\sigma^2(V^{2\gamma} - V)\frac{\partial^2 w}{\partial V^2} \\
\delta_n(t,x) &= L\delta_{n-1}(t,x) - r\delta_{n-1}(t,x)
\end{aligned} \tag{3.1.6}$$

[1]Put options can be priced easily in the same way

Finally we get a closed form approximating formula for call options under mean-reverting CEV model. But notice that the call price (3.1.3) contains non-square chi square distribution functions, applying infinitesimal generator $\mathcal{L}$ on it can be a hard point and in the next section we are going to talk about how to derive partial derivatives of distribution function $Q(xK; \nu + i, \lambda)$.

3.1.3 Method to Calculate Derivatives In Expansions

In this section, methods to calculate closed-form partial derivatives of call option price $\bar{w}$ to time t and volatility V. Our method is based on the recurrence relation of non-central chi-square distribution proposed by Cohen (1988), which is

$$\begin{aligned} \frac{\partial p(xK;\nu,\lambda)}{\partial(xK)} &= \frac{1}{2}[-p(xK;\nu,\lambda) + p(xK;\nu-2,\lambda)] \\ \frac{\partial p(xK;\nu,\lambda)}{\partial\lambda} &= \frac{1}{2}[-p(xK;\nu,\lambda) + p(xK;\nu+2,\lambda)] \end{aligned} \tag{3.1.7}$$

where $p(xK; \nu, \lambda)$ is the Probability Density Function(PDF) of non-central chi-square distribution. From the relationship between Complementary Cumulative Distribution Function(CCDF) $Q(xK; \nu, \lambda)$, Cumulative Distribution Function(CDF) $F(xK; \nu, \lambda)$, and PDF we know that

$$\begin{aligned} \frac{\partial Q(xK;\nu,\lambda)}{\partial(xK)} &= \frac{\partial[1 - F(xK;\nu,\lambda)]}{\partial(xK)} \\ &= -\frac{\partial F(xK;\nu,\lambda)]}{\partial(xK)} \\ &= -p(xK;\nu,\lambda) \end{aligned} \tag{3.1.8}$$

Rewrite the second equation in (3.1.7), we get

$$
\begin{aligned}
\frac{\partial p(xK;\nu,\lambda)}{\partial\lambda} &= \frac{1}{2}[-p(xK;\nu,\lambda)+p(xK;\nu+2,\lambda)] \\
&= -\frac{1}{2}[-p(xK;\nu+2,\lambda)+p(xK;\nu,\lambda)] \\
&= -\frac{\partial p(xK;\nu+2,\lambda)}{\partial(xK)}
\end{aligned}
\tag{3.1.9}
$$

Integrate both sides of (3.1.9) with respect to xK and combine with (3.1.8), we can derive the partial derivative of CDF to non-central parameter λ

$$
\begin{aligned}
\frac{\partial}{\partial\lambda}F(xK;\nu,\lambda) &= -\frac{\partial}{\partial(xK)}F(xK;\nu+2,\lambda) \\
&= -p(xK;\nu+2,\lambda)
\end{aligned}
\tag{3.1.10}
$$

Finally we get the partial derivative of CCDF to non-central parameter λ

$$
\begin{aligned}
\frac{\partial Q(xK;\nu,\lambda)}{\partial\lambda} &= \frac{\partial[1-F(xK;\nu,\lambda)]}{\partial\lambda} \\
&= -\frac{\partial F(xK;\nu,\lambda)]}{\partial\lambda} \\
&= p(xK;\nu+2,\lambda)
\end{aligned}
\tag{3.1.11}
$$

Until now we can summarize that the derivatives of CCDF and PDF are all combinations of PDFs with change of degrees of freedom. Without loss of accuracy, we make the degrees of freedom in PDF be consistent with call option solution in (3.1.3), that is for $p(xK;\nu+i,\lambda)$, we let $i\in[0,4]$. Use the non-central chi-square property by Cohen (1988) to do the following transformation

$$
\begin{aligned}
p(xK;\nu-2,\lambda) &= \frac{\lambda}{xK}p(xK;\nu+2,\lambda)+\frac{v-2}{xK}p(xK;\nu,\lambda) \\
p(xK;\nu+6,\lambda) &= \frac{xK}{\lambda}p(xK;\nu+2,\lambda)-\frac{\nu+2}{\lambda}p(xK;\nu+4,\lambda)
\end{aligned}
\tag{3.1.12}
$$

Next we use the results above to calculate delta and gamma of auxiliary call option price $\bar{w}$. Recall the parameter xK, ν and λ in (3.1.2), where

$$\begin{aligned} x &= \frac{4\kappa}{\sigma^2(1-e^{-\kappa T})} \\ \nu &= \frac{4\kappa m}{\sigma^2}, \\ \lambda &= e^{-\kappa T} x V \end{aligned} \tag{3.1.13}$$

Then we use chain rule calculate the following auxiliary derivatives

$$\begin{aligned} \frac{\partial Q(xK;\nu,\lambda)}{\partial V} &= \frac{\partial Q}{\partial x}\frac{\partial x}{\partial V} + \frac{\partial Q}{\partial \lambda}\frac{\partial \lambda}{\partial V} \\ &= 0 + xe^{-\kappa T} p(x;\nu+2,\lambda) \\ &= xe^{-\kappa T} p(x;\nu+2,\lambda) \end{aligned} \tag{3.1.14}$$

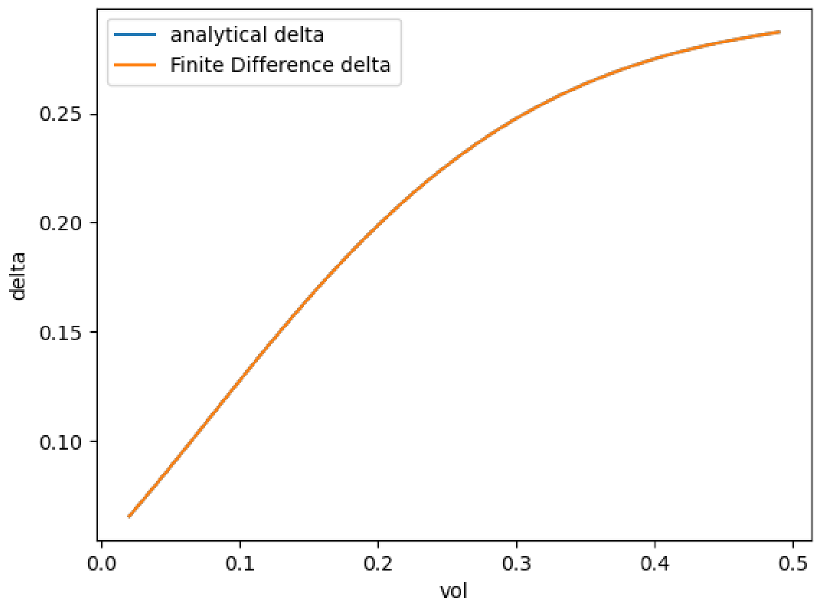
Thus delta is given by

$$\begin{aligned} \Delta_{\bar{w}} =& e^{-(\kappa+r)T} Q(xK;\nu+4,\lambda) + e^{-(\kappa+r)T} V \cdot xe^{-\kappa T} p(xK;\nu+6,\lambda) \\ &+ me^{-rT}(1-e^{-\kappa T}) \cdot xe^{-\kappa T} p(x;\nu+2,\lambda) - e^{-rT} K \cdot xe^{-\kappa T} p(x;\nu+2,\lambda) \end{aligned} \tag{3.1.15}$$

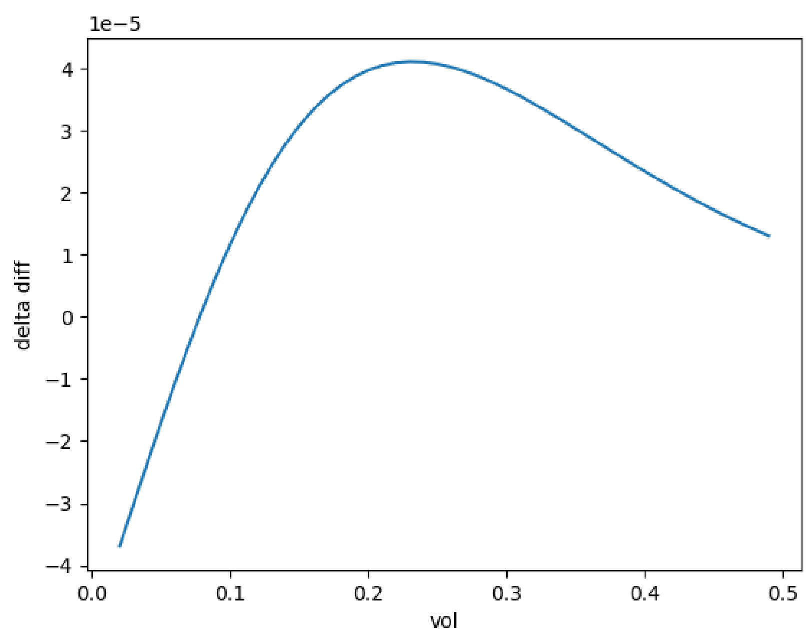
Using (3.1.12) to substitute $p(xK;\nu+6,\lambda)$ and simplify the equation

$$\begin{aligned} \Delta_{\bar{w}} =& e^{-(\kappa+r)T} Q(xK;\nu+4,\lambda) \\ &+ e^{-(\kappa+r)T}\lambda\left[\frac{xK}{\lambda}p(xK;\nu+2,\lambda) - \frac{\nu+2}{\lambda}p(xK;\nu+4,\lambda)\right] \\ &+ me^{-rT}(1-e^{-\kappa T}) \cdot \frac{4\kappa}{\sigma^2(1-e^{-\kappa T})} e^{-\kappa T} p(x;\nu+2,\lambda) - e^{-(r+\kappa)T} K p(x;\nu+2,\lambda) \\ =& e^{-(\kappa+r)T}[Q(xK;\nu+4,\lambda) - 2p(xK;\nu+4,\lambda)] \end{aligned} \tag{3.1.16}$$

Similarly, we calculate another auxiliary derivative

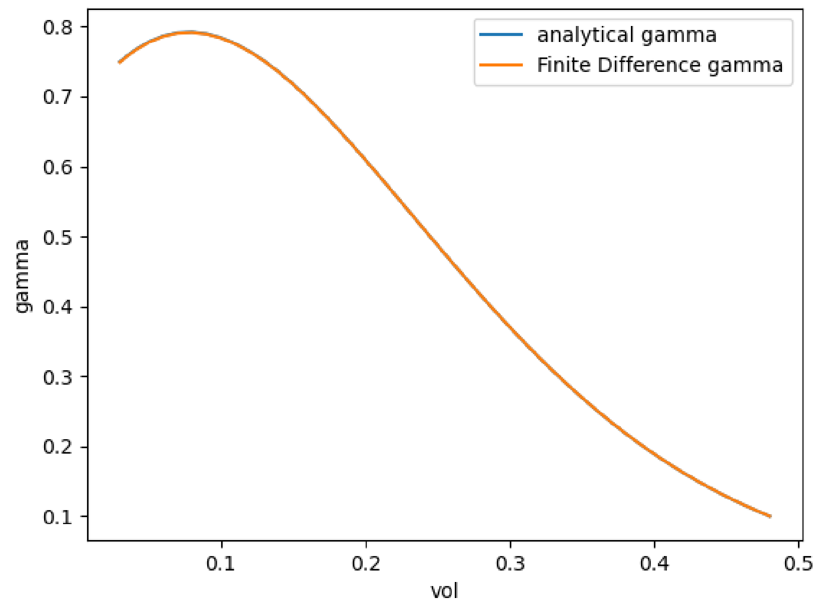


(a) delta comparison

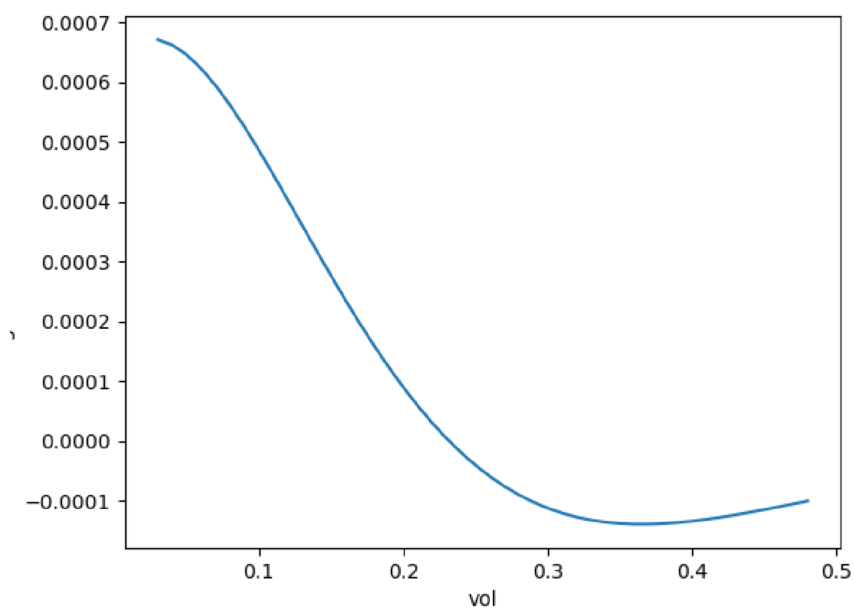


(b) delta differences between two methods

Figure 3.3: Deltas are calculated by our formula, and finite difference method. The parameters used are $T = 0.3$,$\alpha = 0.60$, $\beta = 4.00$, $\sigma = 0.133$, $r = 0.05$, and $K = 0.15$.



(a) gamma comparison



(b) gamma differences between two methods

Figure 3.4: gammas are calculated by our formula, and finite difference method. The parameters used are $T = 0.3$,$\alpha = 0.60$, $\beta = 4.00$, $\sigma = 0.133$, $r = 0.05$, and $K = 0.15$.

$$\begin{aligned}\frac{\partial p(xK;\nu,\lambda)}{\partial V} &= \frac{\partial p}{\partial (xK)}\frac{\partial (xK)}{\partial V} + \frac{\partial p}{\partial \lambda}\frac{\partial \lambda}{\partial V} \\ &= \frac{xe^{-\kappa T}}{2}[-p(xK;\nu,\lambda) + p(xK;\nu+2,\lambda)]\end{aligned} \tag{3.1.17}$$

As a result, gamma of $\bar{w}$ is then

$$\begin{aligned}\Gamma_{\bar{w}} &= e^{-(\kappa+r)T}\left[xe^{-\kappa T}p(x;\nu+6,\lambda) - 2\cdot\frac{xe^{-\kappa T}}{2}[-p(xK;\nu,\lambda)+p(xK;\nu+2,\lambda)]\right] \\ &= xe^{-(2\kappa+r)T}p(xK;nu+4,\lambda)\end{aligned} \tag{3.1.18}$$

To apply infinitesimal generator on mis-pricing formula, we still need to calculate partial derivatives of PDF to time t. Define the following auxiliary functions

$$\begin{aligned}\frac{\partial (xK)}{\partial t} &= \frac{-\kappa e^{-\kappa T}}{1-e^{-\kappa T}}\cdot xK \\ \frac{\partial \lambda}{\partial t} &= \frac{-\kappa e^{-\kappa T}}{1-e^{-\kappa T}}\cdot xV\end{aligned} \tag{3.1.19}$$

Then partial derivatives of PDF to t is given by

$$\begin{aligned}\frac{\partial p(xK;\nu,\lambda)}{\partial t} &= \frac{\partial p}{\partial (xK)}\frac{\partial (xK)}{\partial t} + \frac{\partial p}{\partial \lambda}\frac{\partial \lambda}{\partial t} \\ &= \frac{-\kappa x e^{-\kappa T}}{2(1-e^{-\kappa T})}\left[Vp(xK;\nu+2,\lambda) - (K+V)p(xK;\nu,\lambda) + Kp(xK;\nu-2,\lambda)\right]\end{aligned} \tag{3.1.20}$$

From (3.1.6) we know that the mis-pricing formula $\delta = \frac{1}{2}\sigma^2(V^{2\gamma} - V)\Gamma_{\bar{w}}$, all terms in which have been solved from above. In essence, to apply Ito-Taylor expansions on δ, we use the following algorithm as used in calculating delta and gamma:

1. Combining previous auxiliary partial derivatives, use chain rule to apply in-

finitesimal generator on mis-pricing formula.

2. Substitute PDFs with noncentral parameter $\nu + i$ where $\nu \notin [0, 4]$.

3. Back to step 1, apply higher order infinitesimal generators.

Therefore, we illustrate a solution to implement approximation method on volatility options under mean-reverting CEV model. The expansions in approximating formula can be computed once for all, we can solve it manually or use symbolic language for higher orders. All terms in the result is explicit expect non-central chi-square PDFs, we plug $p(xK; \nu + i, \lambda)$ into the result at last.

3.2 Approximating options under double Heston model

? proposes volatility with double mean-reverting dynamics

$$dV_t = -\kappa \left(V_t - V'(t)\right) dt + \eta_1 V_t'^{\alpha} dW_1(t)$$
$$dV_t' = -c \left(V_t' - m\right) dt + \eta_2 V_t'^{\beta} dW_2(t)$$

where $\alpha, \beta \in [\frac{1}{2}, 1]$.

- It's called Double Heston model in the case $\alpha = \beta = \frac{1}{2}$.
- The case $\alpha = \beta = 1$ Double Log-normal model.
- And the general Double CEV model.

From our previous work, we can use the same auxiliary model to price options with V_t following heston dynamics and V_t' following any mean-reverting CEV process, we call it one Heston one CEV model. This model is given by

$$
\begin{aligned}
dV_t &= -\kappa\left(V_t - V'(t)\right)dt + \eta_1\sqrt{V_t}dW_1(t) \\
dV'_t &= -c\left(V'_t - m\right)dt + \eta_2 V_t'^{\beta} dW_2(t)
\end{aligned}
\tag{3.2.1}
$$

Define infinitesimal generator $\mathcal{L}$ for (3.2.1) and $\bar{\mathcal{L}}$ for square root mean-reverting model

$$
\begin{aligned}
\mathcal{L}w &= \frac{\partial w}{\partial t} + \kappa(V' - V)\frac{\partial w}{\partial V} + \frac{1}{2}\eta_1^2 V\frac{\partial^2 w}{\partial V^2} \\
&\quad + \frac{\partial w}{\partial t} + c(m' - V')\frac{\partial w}{V'} + \frac{1}{2}\eta_2^2 V\frac{\partial^2 w}{V'^2} \\
\bar{\mathcal{L}}w &= \frac{\partial w}{\partial t} + \kappa(m - V)\frac{\partial w}{\partial V} + \frac{1}{2}\eta_1 V\frac{\partial^2 w}{\partial V^2}
\end{aligned}
\tag{3.2.2}
$$

Mis-pricing formula for it is then

$$
\begin{aligned}
\delta = (\mathcal{L} - \bar{\mathcal{L}})\bar{w} &= \kappa(V' - m)\frac{\partial w}{\partial V} \\
&= \kappa(V' - m)\Gamma_{\bar{w}}
\end{aligned}
$$

where $\Gamma_{\bar{w}}$ is given in (3.1.18).

Chapter 4

Numerical Results

In this section, we will show our approximation results for mean-reverting CEV model and Heston plus CEV model, expansion of mis-pricing terms is up to $N = 3$. We utilize a symbolic library of [1]python sympy to calculate expansions, and use Monte Carlo simulations with 200 steps and 100000 paths as our benchmark because there's no existing pricing formula for these models. We evaluate the accuracy of our results with two kinds of figures, the first kind is the direct comparison between benchmarks and our approximation results, the second one is the relative differences between benchmarks and our results. Besides, we also have attached detailed results in the Appendix A.

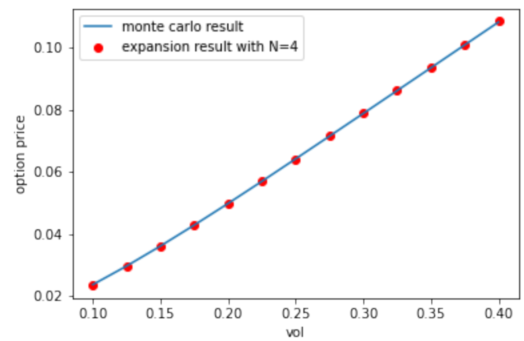
4.1 Volatility option prices under mean-reverting CEV model

For options under mean-reverting CEV model,

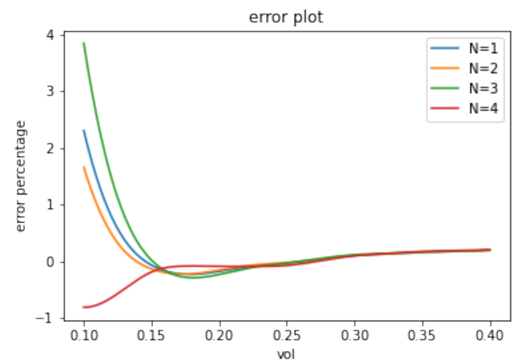
$$\begin{cases} dV_t = \kappa(\theta - V_t)dt + \sigma_{\text{CEV}} V_t^{\gamma} dW_t & \text{true model} \\ dV_t = \kappa(\theta - V_t)dt + \sigma_0 \sqrt{V_t} dW_t & \text{auxiliary model} \end{cases}$$

We use the same mean-reverting parameters as Grünbichler and Longstaff (1996) used in his model, the parameters are $\kappa = 4$, $\theta = 2$. Besides, we set the nuisance parameter $\sigma_0 = \sigma_{\text{CEV}} V_0^{\gamma - \frac{1}{2}}$, where $V_0 = V(t)$ is the initial value of volatility at start point t. We test our approximation method with different constant elasticity parameters, the main idea of setting these parameters is that for small γ, which enlarges the importance of V in the CEV part, we use a small σ; Whereas for large γ

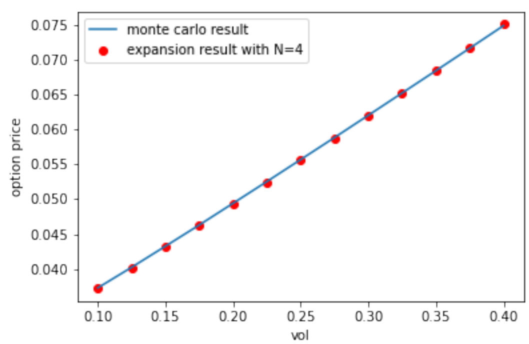
[1]Codes used for this paper can be accessed through my github



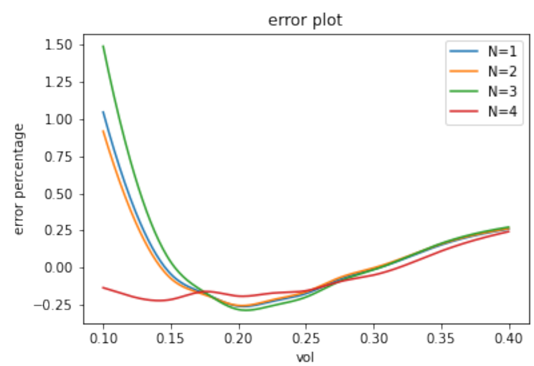
(a) price comparison, $T - t = 0.3$



(b) relative error, $T - t = 0.3$



(c) price comparison, $T - t = 0.5$

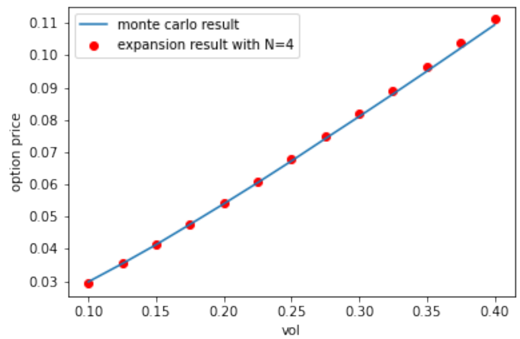


(d) relative error, $T - t = 0.5$

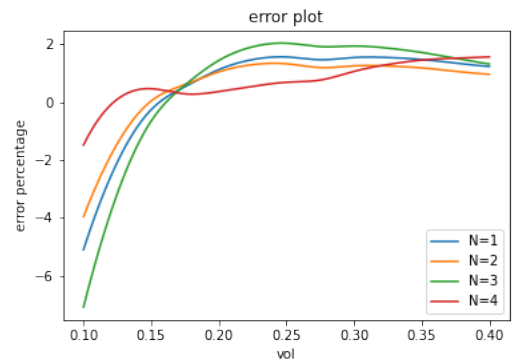
Figure 4.1: mean-reverting CEV model result 1
Parameters are $K = 0.15, \kappa = 4, \theta = 0.2, \sigma = 0.15, \gamma = 0.3$

we set a large σ.

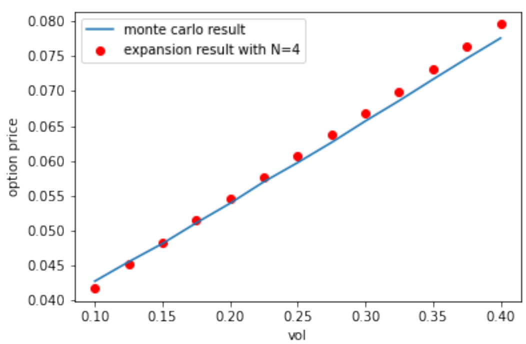
For figure 4.1, our parameters are $\sigma = 0.15$, $\gamma = 0.3$, and with different maturities, $\tau = T - t = 0.3$ for the first row and $\tau = T - t = 0.5$ for the second row. We can find that in 4.1a and 4.1c our approximation results with expansion order $N = 3$ are very accurate. Figure 4.1b and figure 4.1d show the relative error with different expansion orders. We can find that the results with $N = 3$ outperform other results, which implies that keep applying Ito-Taylor expansions on the mis-pricing terms can create increasingly improved refinements and provide us with more and more accurate results.



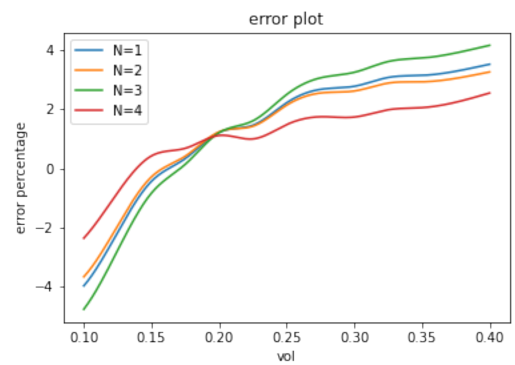
(a) price comparison, $T = 0.3$



(b) relative error, $T = 0.3$



(c) price comparison, $T = 0.5$



(d) relative error, $T = 0.5$

Figure 4.2: mean-reverting CEV model result 2
Parameters are $K = 0.15, \kappa = 4, \theta = 0.2, \sigma = 0.6, \gamma = 0.75$

Parameters for figure 4.2 are $\sigma = 0.6$, $\gamma = 0.75$, and with different maturities $\tau = 0.3$, $\tau = 0.5$. Similarly, our method still provide accurate results.

One may observe that when KM use Black-Scholes model as auxiliary model to price options under Heston model, for deep in-the-money options, relative differences always converge to 0 no matter how many orders of expansions are applied. That is because in his case delta of option is very close to 1 and vega is close to 0, which means option prices are mainly driven by underlying stocks' prices, and volatility has no influence on option prices. Besides, using Black-Scholes model makes mis-pricing term depend on gamma, while gamma of deep in-the-money options is also close to

0, meaning that their mis-pricing terms don't affect option prices at all. As a result, their figures show that all results' relative errors are converging to 0 as stock price increases.

However, in our model, underlying assets follow mean-reverting CEV model. Grünbichler and Longstaff (1996) mention that when volatility V is above its long-term mean, mean-reversion property implies the expected future value of V will be lower than its current value, making the expected payoff for a volatility call can be less than its current intrinsic value. The property of options under Black-Scholes world doesn't hold here, recall that before we set $\sigma_0 = \sigma_{\text{CEV}} V_0^{\gamma - \frac{1}{2}}$. Obviously when V_0 is large, $\sigma_{\text{CEV}} V_t^{\gamma} < \sigma_{\text{CEV}} V_0^{\gamma - \frac{1}{2}} \sqrt{V}_t$, causing the loss of accuracy in our auxiliary model. It gives an explanation why the relative error of our method is slightly larger than 0 for deep in-the-money options. Additionally, using our method to price deep out-of-money options can also be challenging. The loss of accuracy for approximating non-central chi-square distribution functions would be magnified when option price is very small.

4.2 Volatility option prices under Heston puls CEV model

For Heston plus CEV model,

$$dV_t = \kappa_1 \left(V_t' - V_t \right) dt + \sigma_1 \sqrt{V_t} dW_t$$

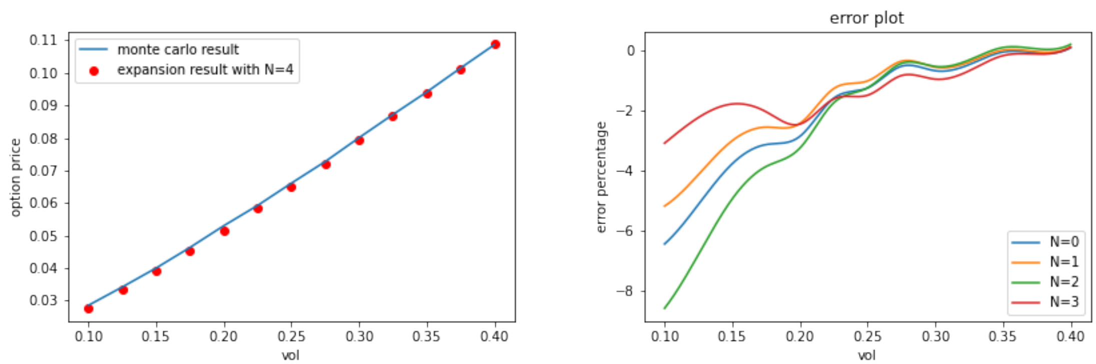
$$dV_t' = \kappa_2 \left(\theta - V_t' \right) dt + \sigma_2 V_t'^{\gamma} dW_t'$$

and our auxiliary model, square root mean-reverting model,

$$dV_t = \kappa(\theta_0 - V_t)dt + \sigma_0 \sqrt{V_t} dW_t$$

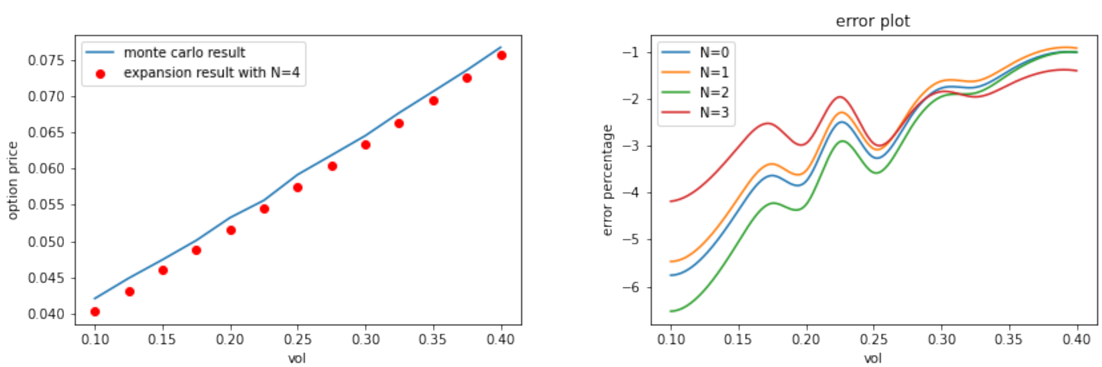
our parameters are $r = 0.05$, $K = 0.15$, $\kappa_1 = 4$, $\kappa_2 = 2$, $\theta = 0.2$, $\sigma_1 = 0.3$, $\sigma_2 = 0.8$,

$\gamma = 1.6$, $\rho = 0.5$ and different maturities $\tau = T - t = 0.3, \tau = T - t = 0.5$. We set the nuisance parameters $\theta_0 = V'(t)$, where $V'(t) = 0.2$ is the spot value for volatility of volatility at time t.



(a) price comparison, $T - t = 0.3$ (b) relative error, $T - t = 0.3$

Figure 4.3: Heston plus CEV model result 1



(a) price comparison, $T - t = 0.5$ (b) relative error, $T - t = 0.5$

Figure 4.4: Heston plus CEV model result 2

As is seen in 4.3a, applying our method under 2-dimensional model can still create relatively accurate results. Unlike mean-reverting CEV model, under Heston plus CEV model our relative differences are now converging to 0. This is because here the initial value of volatility doesn't enter mis-pricing term $\delta = \kappa_1(V_2 - \theta_2)\Delta_{\bar{w}}$.

Besides, we notice that for 4.4a when $\tau = 0.5$, our results seem not accurate enough. Though in figure 4.4b relative differences reduce as we apply higher further expansions. We may predict that if applying higher order expansions we could get more precise results. However, this raises a constraint of KM's method, number of terms grow exponentially in the final pricing formula as we keep expanding mis-pricing terms, causing the running time of calculation increasing dramatically, which is a trade-off between applying high order expansions and getting more accurate results. Moreover, if we still don't get a desired result with a threshold expansion order, say $N = 5$, we may need to reconsider our auxiliary model and nuisance parameters.

Chapter 5

Conclusion

In this paper, we introduce approximation method proposed by **?** and Kristensen and Mele (2011). Based on their work, we extend this method to price options under mean-reverting CEV model and Heston plus CEV models. Selections of auxiliary models and corresponding mis-pricing formula are discussed, we also illustrate techniques to calculate partial derivatives of non-central chi-square distribution functions when using square root mean-reverting as auxiliary model. Finally, we discuss our numerical results and explain the constraints of our method. In all, numerical results show that our method is efficient and accurate.

Appendix A

Appendix A Mean-Reverting CEV model Numerical Results

Appendices at the end of a dissertation are optional, and depend on the content of the dissertation. There can be one or more appendicies, however they should retain the page numbering requirements for dissertations. Any concerns about the formatting of an appendix should be brought to Doris Oliver, who can direct you how to format your appendix if you have questions.

Theoretical Dissertation Timeline		
Taskt	Time to Finish	Notes
Problem statement	10 hours	Initially very upbeat.
Research	3 days	Literature search to very previous studies.
Reformulation	4 hours	Presented and accepted by advisor
Research	20 days	Literature search to very previous studies.
Experiments	14 days	Do some experiments and get results.
Format	1 day	Understand format guidelines for paper.
Write	years	Write the paper.
Revise	not too long	Proof read, etc.
Format	1-3 days	Verify correct report format is used.
See Library	1 hour	Meet with Doris to verify formatting.
Defend	1 day	Defend your research.
Revise	0 hours	It was perfect the first time.
Submit	1 day	Submit final dissertation to the library.

Table A.1: T=0.3,K=0.15, kappa=4,m=0.2, sigma=0.15, gamma=0.3.csv

vol	mc	w1	w2	w3	w4
0.100	0.023570	0.024113	0.023960	0.024476	0.023380
0.125	0.029589	0.029760	0.029686	0.029921	0.029421
0.150	0.036036	0.036013	0.035989	0.036045	0.035969
0.175	0.042805	0.042709	0.042710	0.042686	0.042771
0.200	0.049793	0.049705	0.049716	0.049678	0.049750
0.225	0.056940	0.056892	0.056904	0.056881	0.056891
0.250	0.064213	0.064192	0.064201	0.064195	0.064168
0.275	0.071525	0.071552	0.071558	0.071564	0.071534
0.300	0.078864	0.078944	0.078948	0.078957	0.078941
0.325	0.086237	0.086351	0.086353	0.086361	0.086360
0.350	0.093613	0.093765	0.093765	0.093771	0.093778
0.375	0.101000	0.101181	0.101181	0.101184	0.101192
0.400	0.108382	0.108598	0.108598	0.108600	0.108606

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