

Recitation questions: 1(a):

$$80 \begin{cases} 1-p & y = 80u. \\ p & 70 = 80d. \end{cases} \quad \begin{array}{l} r = 3\% \\ \delta = 3\% \\ h = 1. \end{array}$$

no arbitrage condition:

$$d = \frac{70}{80} = \frac{7}{8}, \quad u = \frac{y}{80}, \quad \frac{7}{8} < e^{(r-\delta)h} < \frac{y}{80}. \quad (8\% - 3\%) \cdot 1$$

$$\Rightarrow y > 80 \cdot e^{5\%} = 84.1016817 =: b.$$

(b). assume  $y=83$ . consider the following trading strategy at time  $t=1$

Transactions at time 0	at time 0	If $S_t = 83$	If $S_t = 70$
sell stock	+80	$-83 \cdot e^{3\% \cdot 1}$	$-70 \cdot e^{3\% \cdot 1}$
buy risk-free bond	-80	$80 e^{8\% \cdot 1}$	$80 e^{8\% \cdot 1}$
Profit/Loss	0	$> 0$	$> 0$

Q2: (a) stock price  $100 \begin{cases} 120 \\ 80 \end{cases}$

$$u=1.2, d=0.8$$

$$\delta=1\%, r=11\%$$

$$h = \frac{2}{12} = \frac{1}{6} \text{ year}$$

$$K=72$$

$$C_0 \begin{cases} 48 \\ 8 \end{cases}$$

risk-neutral prob

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(11\%-1\%)\frac{1}{6}} - 0.8}{1.2 - 0.8}$$

$$= 0.542$$

$$1-p^* = 0.458$$

$$C_0 = e^{-11\% \cdot \frac{1}{6}} (0.542 \times 48 + 0.458 \times 8)$$

$$= 29.1408$$

(b)

$$\Delta = \left( \frac{C_u - C_d}{S_u - S_d} \right) e^{-\delta h} = \frac{48 - 8}{120 - 80} e^{-1\% \cdot \frac{1}{6}} = 0.998$$

$$B = \frac{u \cdot C_d - d \cdot C_u}{u-d} e^{-rh} = \frac{1.2 \cdot 8 - 0.8 \cdot 48}{1.2 - 0.8} e^{-11\% \cdot \frac{1}{6}} = -70.692$$

profit of holding hedging portfolio ( $\Delta S + B$ ) is  $29.1408 - 28 =$

$\# \text{ of shares} = \frac{100}{1.1408} \Delta = 87.6578 \Delta = 87.5$

FE021: finite difference method for Heston model / Monte Carlo  
for HW2 Q2: we need parity relations between knocked out and knocked in

$$+ \frac{(S_T - K)^+ \cdot \mathbb{1}_{\{\max_{0 \leq t \leq T} S_t \leq H\}}}{(S_T - K)^+ \cdot \mathbb{1}_{\{\max_{0 \leq t \leq T} S_t \geq H\}}}$$

✓  $\mathbb{1}_A + \mathbb{1}_{A^c} = 1$

$$= (S_T - K)^+$$

put call parity:  $\frac{(S_T - K)^+ - (K - S_T)^+}{e^{-rT} E[(S_T - K)^+]} - e^{-rT} E[(K - S_T)^+] = e^{-rT} E[S_T] -$  only Euro

$$\rightarrow C_0 - P_0 = S_0 - Ke^{-rT}$$

Q3 (a) and (b): Information;  $r=6\%$ ,  $\sigma=30\%$ ,  $K=100$ ,  $B=0$ ,  
 $h=\frac{3}{12}=\frac{1}{4}$  year,

$$u = e^{(r-\sigma)h + \sigma\sqrt{h}} = 1.1794$$

$$d = e^{(r-\sigma)h - \sigma\sqrt{h}} = 0.8737$$

$$p^* = \frac{e^{(r-\sigma)h} - d}{u - d} = 0.4626, \quad 1 - p^* = 0.5374.$$

→ European option price

9.449.

100

9.51



American price.

117.94

3.166

87.37

15.245

139.0984

103.0442

5.9807

76.3552

(23.6448)

164.0527

max(0, x) = 0.

121.533

90.0297

11.2965

66.694

31.8135  
(33.306).

193.4837

143.3328

106.1810

78.8584

21.3416

58.2706

41.7294

(c): hedging portfolio for American option:

at time 0, I hold  $\Delta S + B$

$$\begin{cases} \Delta = e^{-rh} \left( \frac{C_u - C_d}{S_0(u-d)} \right) = -0.3951, \end{cases}$$

$$\begin{cases} B = e^{-rh} \left( \frac{u \cdot C_d - d \cdot C_u}{u-d} \right) = 49.0178. \end{cases}$$

(d). The only chance to early exercise is at node  $S_0 d^2$ . Assume that we do not exercise, then

value at that node is given by

$$0.4851 \cdot (p^* \cdot 11.29 + (1-p^*) \cdot 33.306) = 22.7768.$$

$$\text{profit} = (23.6448 - 22.7768) e^{6\% \cdot \frac{1}{2}} = \underline{0.91}.$$

Stochastic volatility models; Heston model:  $U = f(S_t, V_t)$ .

$$\begin{cases} \frac{dS_t}{S_t} = (r - q)dt + \sqrt{V_t} dW_t^{(1)} \\ dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^{(2)} \end{cases}, \quad E[dW_t^{(1)} dW_t^{(2)}] = \rho dt$$

PDE is: Define  $U(S, V, t)$  as the option value under Heston model.

$$\frac{\partial U}{\partial t} = \frac{1}{2} \underline{V} S^2 \frac{\partial^2 U}{\partial S^2} + \rho \cdot \sigma \cdot \underline{V} \cdot S \frac{\partial^2 U}{\partial V \partial S} + \frac{1}{2} \sigma^2 \underline{V} \frac{\partial^2 U}{\partial V^2} - rU + (r - q) \cdot S \cdot \frac{\partial U}{\partial S} + \kappa(\theta - \underline{V}) \cdot \frac{\partial U}{\partial V}$$

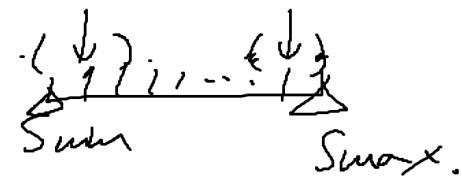
$t \rightarrow$  time to maturity.

boundary condition:  $U(S, V, 0) = (S - K)^+$

Finite difference: create the grids:

$t \in [0, T]$ ,  
 $S \in [0, S_{\max}]$ ,  
 $V \in [0, V_{\max}]$ .  
 chosen by us.

$$\begin{cases} S_i = i \times ds, & i = 0, 1, \dots, N_s. \\ V_j = j \times dv, & j = 0, 1, \dots, N_v. \\ t_n = n \times dt, & n = 0, 1, \dots, N_T. \end{cases}$$



here  $ds = \frac{S_{max} - S_{min}}{N_s} = \frac{S_{max} - 0}{N_s},$

$$dv = \frac{V_{max} - V_{min}}{N_v} = \frac{V_{max} - 0}{N_v}.$$

$$dt = \frac{t_{max} - t_{min}}{N_T} = \frac{T}{N_T}.$$

Discretized value function is denoted as  $U_{i,j}^n = U(S_i, V_j, t_n),$

Above is a uniform grid; there also exist non-uniform grid.

Next step: use finite difference to approx partial derivatives;

< interior grid points → central difference.  
boundary grid points.

this  $u$  satisfies the discretized PDE.

interior grid points:

$$\frac{\partial u}{\partial s}(s_i, v_j) \approx \frac{U_{i+1,j}^n - U_{i-1,j}^n}{s_{i+1} - s_{i-1}} \quad \text{(general including nonuniform grid)}$$

$$\text{(uniform grid)} \quad \frac{U_{i+1,j}^n - U_{i-1,j}^n}{2ds}$$

$$\frac{\partial u}{\partial v}(s_i, v_j) \approx \frac{U_{i,j+1}^n - U_{i,j-1}^n}{v_{j+1} - v_{j-1}}$$

$$\text{uniform grid} \quad \frac{U_{i,j+1}^n - U_{i,j-1}^n}{2dv}$$

$$\frac{\partial^2 u}{\partial s^2}(s_i, v_j) \approx \frac{U_{i-1,j}^n}{(s_i - s_{i-1})(s_{i+1} - s_{i-1})} - \frac{2U_{i,j}^n}{(s_i - s_{i-1})(s_{i+1} - s_i)} + \frac{U_{i+1,j}^n}{(s_{i+1} - s_i)(s_{i+1} - s_{i-1})}$$

$$\text{uniform grid} \quad \frac{U_{i+1,j}^n - 4U_{i,j}^n + U_{i-1,j}^n}{2(ds)^2}$$



boundary points:

boundary at maturity:  $u(s_i, v_i, 0) = \max(0, s_i - K)$

boundary at  $s = s_{\min} (=0)$ ,  $u(0, v_j, t_n) = 0$ .

boundary at  $s = s_{\max}$ ,  $\frac{\partial u}{\partial s}(s_{\max}, v_j, t_n) = 1$ .

boundary at  $v = v_{\max}$ ,  $\frac{\partial u}{\partial s}(s_i, v_{\max}, t_n) = 1$ .

boundary at  $v = v_{\min} (=0)$ , idea is to approximate

the simplified PDE by plugging in  $v=0$ .

$$\frac{\partial u}{\partial t} = -ru + (r-q)s \cdot \frac{\partial u}{\partial s} + K\theta \cdot \frac{\partial u}{\partial v}$$

solve above by finite differencing.

Target volatility  
options  
(TVO)

$\frac{1}{RV_T}(S_T - K)^+$   
 $\nearrow$  realized volatility,

$\frac{1}{VIX_T}(S_T - K)^+$

Uniform grid to discretize the Heston model: (explicit finite difference scheme).

$$\begin{aligned}
 U_{i,j}^{n+1} = & \left[ 1 - dt \cdot \left( i^2 j dv + \frac{\sigma^2 j}{dv} + r \right) \right] \cdot U_{i,j}^n \\
 & + \left[ \frac{i \cdot dt}{2} (i j dv - r + q) \right] U_{i+1,j}^n + \left[ \frac{i dt}{2} (i j dv + r - q) \right] U_{i-1,j}^n \\
 & + \left[ \frac{dt}{2 dv} (\sigma^2 j - \kappa (\theta - j dv)) \right] U_{i,j+1}^n + \left[ \frac{dt}{2 dv} (\sigma^2 j + \kappa (\theta - j dv)) \right] U_{i,j-1}^n \\
 & + \frac{i j dt \sigma}{4} \left( U_{i+1,j+1}^n + U_{i-1,j-1}^n - U_{i-1,j+1}^n - U_{i+1,j-1}^n \right).
 \end{aligned}$$

$t \rightarrow$  time to maturity.

$\downarrow$   
 $t = N_t \cdot dt, \dots, t = 2dt, t = dt, t = 0$

Check the note on FPA on Heston model in Canvas.

## PDE transformation methods:

(1) Variable transformation  $\rightarrow$  change of variable, ~~so~~

(2) Laplace transformation

one-dimensional : BS PDE.  
two-dimensional : Helmholtz PDE.

$$\hookrightarrow \text{PDE } h(x, t) = 0$$

Method: separation of variables

where  $x$  and  $t$  are independent variables. principle: If  $f(x) = g(t)$ , then there exist a constant  $\lambda$  such that  $f(x) = g(t) = \lambda$ .

$$\begin{cases} \underline{f(x) = \lambda} & \text{ODE in } x \\ \underline{g(t) = \lambda} & \text{ODE in } t \end{cases}$$

$$\underline{u = u(x, t) = f(x) g(t).}$$

$$\text{Heat equation: } \underline{\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}}.$$