Recitation questions: 1(a):

$$\underbrace{80 \begin{array}{l} \overset{1-p}{\nearrow}\; y = 80u. \\ \underset{p}{\searrow}\; 70 = 80\cdot d. \end{array}}_{h} \qquad \begin{array}{l} r = 8\% \\ \delta = 3\% \\ h = 1. \end{array}$$

no arbitrage condition:

$$d < e^{(r-\delta)h} < u.$$

$$d = \frac{70}{80} = \frac{7}{8}, \quad u = \frac{y}{80}, \quad \underline{\frac{7}{8} < e^{(8\%-3\%)\cdot 1} < \frac{y}{80}}.$$

$$\Rightarrow y > 80\cdot e^{5\%} = 84.101877 =: b.$$

(b). assume $y = 83$. consider the following trading strategy

Transactions at time 0	at time 0	at time $h = 1$: If $S_t = 83$	at time $h = 1$: If $S_t = 70$
sell stock	$+80$	$-83\cdot e^{3\%\cdot 1}$	$-70\cdot e^{3\%\cdot 1}$
buy risk-free bond	-80	$80e^{8\%\cdot 1}$	$80e^{8\%\cdot 1}$
Profit/Loss	0	>0	>0

Q2: (a) stock price

$$\begin{array}{ccc} & & 120 \\ 100 & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\ & & 80 \end{array}$$

$u = 1.2$, $d = 0.8$

$\delta = 1\%$, $r = 11\%$.

$h = \frac{2}{12} = \frac{1}{6}$ year.

$K = 72$.

$$\begin{array}{ccc} & & 48 \\ C_0 & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\ & & 8 \end{array}$$

risk-neutral prob

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(11\%-1\%)\frac{1}{6}} - 0.8}{1.2 - 0.8}$$

$= 0.542$.

$1 - p^* = 0.458$.

$$C_0 = e^{-11\% \cdot \frac{1}{6}} (0.542 \times 48 + 0.458 \times 8)$$

$$= \underline{29.1408}.$$

(b)

$$\Delta = \left(\frac{C_u - C_d}{\cancel{S(u-d)}\ S_u - S_d} \right) e^{-\delta h} = \frac{48-8}{120-80} e^{-1\% \cdot \frac{1}{6}} = 0.998.$$

$$B = \frac{u \cdot C_d - d \cdot C_u}{u-d} \cdot e^{-rh} = \frac{1.2 \cdot 8 - 0.8 \cdot 48}{1.2 - 0.8} e^{-11\% \cdot \frac{1}{6}} = -70.692.$$

profit of holding hedging portfolio $(\Delta S + B)$ is $29.1408 - 28 =$

$\# \text{ of shares} = \frac{100}{1.1408}\Delta = 87.6578\Delta = 87.5$.

FE621. finite difference method for Hesten model / Monte Carlo.

for HW2 Q2: we need parity relations between knocked out and kno[cked in]

$(S_T-K)^+ \cdot \mathbb{1}\{\max_{0\le t\le T} S_t \le H\}$.

$+ \; (S_T-K)^+ \cdot \mathbb{1}\{\max_{0\le t\le T} S_t \ge H\}$.

$\mathbb{1}_A + \mathbb{1}_{A^c} = 1$

$= (S_T-K)^+$.

only Euro[pean]

put call parity: $(S_T-K)^+ - (K-S_T)^+ = S_T - K$.

$e^{-rT}E[(S_T-K)^+] - e^{-rT}E[(K-S_T)^+] = e^{-rT}E[S_T] -$

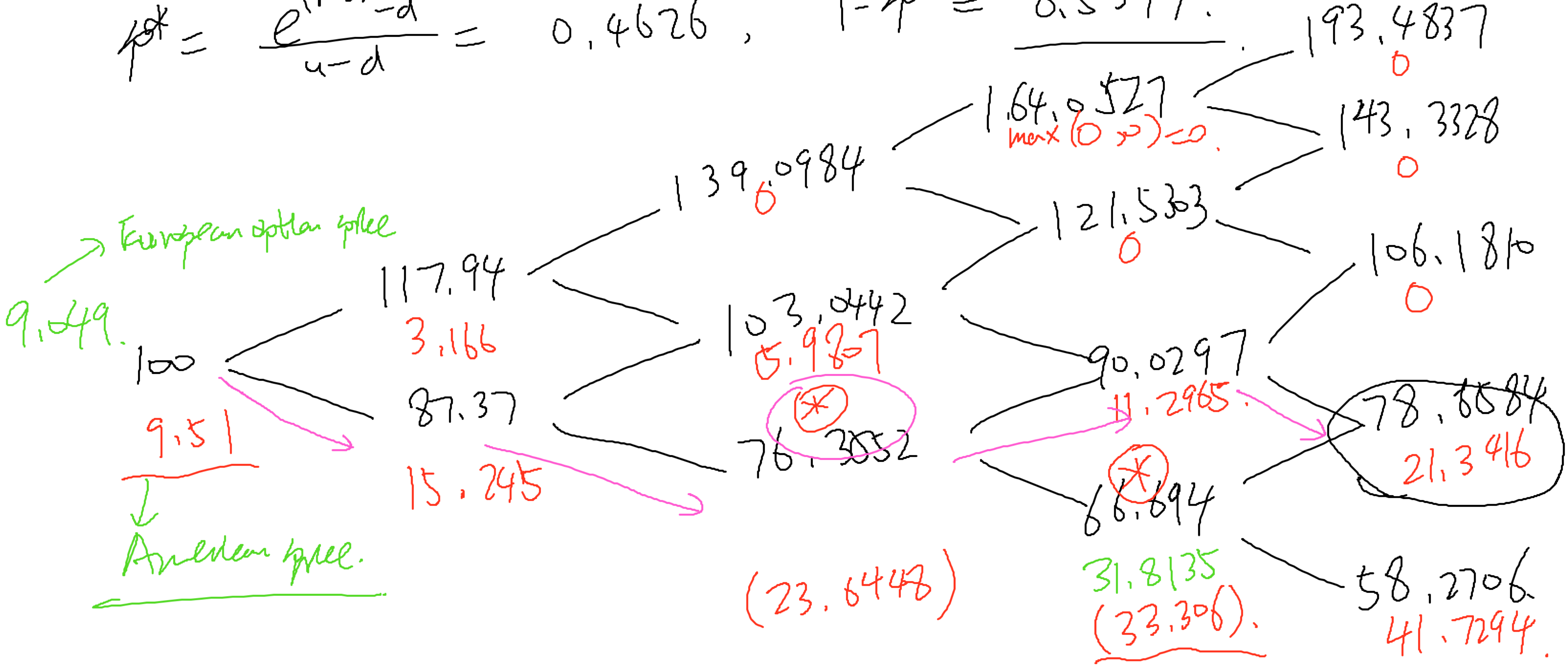
$C_0 - P_0 = S_0 - Ke^{-rT}$.

Q3 (a) and (b): Information; $r = 6\%$, $\sigma = 30\%$, $K = 100$, $\delta = 0$,

$h = \frac{3}{12} = \frac{1}{4}$ year, $u = e^{(r-\delta)h+\sigma\sqrt{h}} = 1.1794$

$d = e^{(r-\delta)h-\sigma\sqrt{h}} = 0.8737.$

$p^* = \dfrac{e^{(r-\delta)h} - d}{u-d} = 0.4626$, $1 - p^* = 0.5374.$



(c): hedging portfolio for American option:

at time 0, I hold $\Delta S + B$

$$\begin{cases} \Delta = e^{-\delta h}\left(\dfrac{C_u - C_d}{S_0(u-d)}\right) = -0.3951. \\ B = e^{-rh}\left(\dfrac{u \cdot C_d - d \cdot C_u}{u-d}\right) = 49.0178. \end{cases}$$

(d). The only chance to early exercise is at node $S_0 d^2$. Assume that we do not exercise, then value at that node is given by

$$0.9851 \cdot \left(p^* \times 11.29 + (1-p^*) \times 33.306\right) = 22.7768.$$

profit $(23.6448 - 22.7768) e^{6\% \cdot \frac{1}{2}} = 0.91$.

Stochastic volatility models: Heston model: $\underline{U = f(S_t, V_t)}$.

$$\begin{cases} \dfrac{dS_t}{S_t} = (r-q)\,dt + \sqrt{V_t}\,dW_t^{(1)} \\ dV_t = \kappa(\theta - V_t)\,dt + \sigma\sqrt{V_t}\,dW_t^{(2)}, \end{cases} \qquad E[dW_t^{(1)} dW_t^{(2)}] = \rho\,dt,$$

PDE is: Define $U(S, V, t)$ as the option value under Heston model.

$S = S_{T-t}$
$V = V_{T-t}$

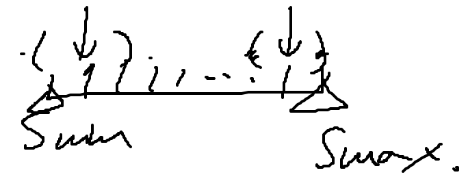
$$\frac{\partial U}{\partial t} = \frac{1}{2} V S^2 \frac{\partial^2 U}{\partial S^2} + \rho \cdot \sigma \cdot V \cdot S \frac{\partial^2 U}{\partial V \partial S} + \frac{1}{2}\sigma^2 V \frac{\partial^2 U}{\partial V^2} - rU + (r-q) \cdot S \cdot \frac{\partial U}{\partial S} + \kappa(\theta - V) \cdot \frac{\partial U}{\partial V}.$$

$t \longrightarrow$ time to maturity.

boundary condition: $U(S, V, 0) = (S-K)^+$.

Finite difference: create the grids: $t \in [0, T]$, $S \in [0, S_{max}]$, $V \in [0, V_{max}]$. S_{max}, V_{max} chosen by us.

$$\begin{cases} S_i = i \times ds, & i = 0,1,\dots,N_s. \\ V_j = j \times dv, & j = 0,1,\dots,N_v. \\ t_n = n \times dt, & n = 0,1,\dots,N_T. \end{cases}$$



here
$$ds = \frac{S_{max} - S_{min}}{N_s} = \frac{S_{max} - 0}{N_s},$$
$$dv = \frac{V_{max} - V_{min}}{N_v} = \frac{V_{max} - 0}{N_v}.$$
$$dt = \frac{t_{max} - t_{min}}{N_T} = \frac{T}{N_T}.$$

this u satisfies the discretized PDE.

Discretized value function is denoted as $U_{i,j}^{n} = U(S_i, V_j, t_n)$,

Above is a uniform grid; there also exist non-uniform grid.

Next step: use finite difference to approx partial derivatives:

- interior grid points $\rightarrow$ central difference.
- boundary grid points.

interior grid points:

$$\frac{\partial u}{\partial s}(s_i, v_j) \approx \frac{U^n_{i+1,j} - U^n_{i-1,j}}{s_{i+1} - s_{i-1}} .$$

(general including non-uniform grid)

(uniform grid) $$\frac{U^n_{i+1,j} - U^n_{i+1,j}}{2ds} .$$

$$\frac{\partial u}{\partial v}(s_i, v_j) \approx \frac{U^n_{i,j+1} - U^n_{i,j-1}}{v_{j+1} - v_{j-1}} .$$

uniform grid $$\frac{U^n_{i,j+1} - U^n_{i,j-1}}{2dv} .$$

$$\frac{\partial^2 u}{\partial s^2}(s_i, v_j) \approx \frac{U^n_{i-1,j}}{(s_i - s_{i-1})(s_{i+1} - s_{i-1})} - \frac{2U^n_{i,j}}{(s_i - s_{i-1})(s_{i+1} - s_i)} + \frac{U^n_{i+1,j}}{(s_{i+1} - s_i)(s_{i+1} - s_{i-1})}$$

uniform grid $$\frac{U^n_{i+1,j} - 4U^n_{i,j} + U^n_{i+1,j}}{2(ds)^2} .$$

boundary points:

boundary at maturity: $u(S_i, V_j, 0) = \max(0, S_i - K)$.

boundary at $S = S_{min}\ (=0)$, $u(0, V_j, t_n) = 0$.

boundary at $S = S_{max}$, $\frac{\partial u}{\partial S}(S_{max}, V_j, t_n) = 1$.

boundary at $V = V_{max}$, $\frac{\partial u}{\partial S}(S_i, V_{max}, t_n) = 1$.

boundary at $V = V_{min}\ (=0)$, idea is to approximate the simplified PDE by plugging in $v=0$.

$$\frac{\partial u}{\partial t} = -ru + (r-q) S \cdot \frac{\partial u}{\partial S} + \kappa\theta \cdot \frac{\partial u}{\partial v}.$$

solve above by finite differencing.

Target volatility option (TVO)

$$\frac{1}{RV_T}(S_T - K)^+.$$

realized volatility,

$$\frac{1}{VIX_T}(S_T - K)^+.$$

Uniform grid to discretize the Heston model: (explicit finite difference scheme).

$$
\begin{aligned}
U_{i,j}^{n+1} = {} & \left[1 - dt \cdot \left(i^2 j\, dV + \frac{\sigma^2 j}{dV} + r \right) \right] \cdot U_{i,j}^{n} \\
& + \left[\frac{i \cdot dt}{2} \left(i j\, dV - r + q \right) \right] U_{i-1,j}^{n} + \left[\frac{i\, dt}{2} \left(i j\, dV + r - q \right) \right] U_{i+1,j}^{n} \\
& + \left[\frac{dt}{2 dV} \left(\sigma^2 j - \kappa (\theta - j\, dV) \right) \right] U_{i,j-1}^{n} + \left[\frac{dt}{2 dV} \left(\sigma^2 j + \kappa (\theta - j\, dV) \right) \right] U_{i,j+1}^{n} \\
& + \frac{i j\, dt\, \sigma}{4} \left(U_{i+1,j+1}^{n} + U_{i-1,j-1}^{n} - U_{i-1,j+1}^{n} - U_{i+1,j-1}^{n} \right).
\end{aligned}
$$

$t \rightarrow$ time to maturity.

$t = N_t \cdot dt$, . . . $t = 2dt$ $t = dt$ $t = 0$

0 T.

Check the note on FDM on Heston model in Canvas.

PDE transformation methods:

(1) variable transformation $\longrightarrow$ change of variable,

(2) Laplace transformation

one-dimensional : BS PDE.

two-dimensional : Heston PDE.

PDE

$h(x,t)=0$

Method: separation of variables

principle: If $f(x)=g(t)$, where x and t are independent variables. then there exist a constant λ such that $f(x)=g(t)=\lambda$.

$$\begin{cases} f(x)=\lambda & \text{ODE in } x \\ g(t)=\lambda & \text{ODE in } t. \end{cases}$$

$u=u(x,t)=f(x)\,g(t).$

Heat equation: $\dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial^2 u}{\partial x^2}.$