Midterm Exam FE621

Saturday, October 24th, 2020

Name:

- In this exam you may use any material you want. Please be aware the submission window closes 4 hours after it has been opened. So you need 4 continuous hours of work.
- Be very specific with your definitions and derivations. Showcase your work.
- Please submit a well written report containing all the answers to the exam questions. Submitting commented code is not appropriate and will earn 0 in this exam.
- Scanned handwritten pages are appropriate for certain problems requiring mathematical derivations.
- Communication with other students either physical or virtual is strictly forbidden.

For instructor's use only

Problem	Points	Score
1	20	
2	20	
3	30	
4	30	
Total	100	

Problem 1. Numerical integration

Consider the *Gauss–Laguerre quadrature* method. This method approximates the integral of a real–valued function $f : \mathbb{R} \to \mathbb{R}$ using:

$$\int_a^b f(x)dx \approx \sum_{k=1}^{N} w_k f(x_k).$$

In this expression the numbers $\{x_1, x_2, \ldots, x_N\}$ are the roots of the Laguerre polynomial $L_N(x)$ of order N:

$$L_N(x) := \sum_{k=0}^{N} \frac{(-1)^k}{k!} \binom{N}{k} x^k.$$

The weights w_k are obtained using the derivative of $L_N(x)$ evaluated at all the points $\{x_1, x_2, \ldots, x_N\}$:

$$w_k = \frac{e^{x_k}}{x_k} \left(\frac{N!}{L'_N(x_k)} \right)^2, \text{ for } k = 1, 2, \ldots, N.$$

a) Using $N = 2$ please approximate numerically the following integral using the Gauss–Laguerre quadrature:

$$\int_0^4 e^{-x^2/2} dx.$$

b) Let f be a real–valued continuous and integrable function. Assume you have the code for this function written and you can make calls to it by just writing $f(x)$. Write pseudo code that will approximate $\int_0^M f(x)dx$ using the Gauss–Laguerre quadrature, where M is a given constant. Also assume that you can call a function `polyroot` that computes the roots of any given polynomial.

Problem 2. Binomial tree

Here is the pseudo-code for the valuation of an European Put option using a multiplicative binomial tree. Please identify all the mistakes in the code below. Please indicate how to fix the mistakes.

```
Data: Parameters K, T, S, r, N, u, d
dt = T/N
p = (exp(r*dt)-d)/(u-d)
disc = exp(r*dt)
St[0] = S*d
for j = 1 to N do
 |  St[j] = St[j-1]*u/d
end
for j = 0 to N do
 |  C[j] = max (St[j] - K, 0.0)
end
for i = (N-1) down to 0 do
 |  for j = 0 to i do
 |   |  C[j] = disc * ( (1-p) * C[j] + p * C[j+1])
 |  end
end
European_put = C[0]
```

Problem 3. Option pricing using a trinomial tree

Construct a trinomial tree to price an American put option. To this end start with the following given parameters: $S_0 = 100$, $K = 120$, maturity $T = 8$ months, $r = 0$, $\delta = 0$, volatility $\sigma = 3\%$, time steps $N = 2$.

(a) What is a suitable choice for Δx to obtain a stable scheme? Calculate Δx

(b) With your choice of Δx calculate p_u, p_m and p_d.

(c) Calculate the American Put option price using the tree. It will be helpful to draw a diagram of the trinomial tree containing the probabilities and the stock values.

Problem 4.Finite Difference method for PDE

We know that an option price under a certain stochastic model satisfies the following PDE:

$$\frac{\partial V}{\partial t} + 2\tan(S)\frac{\partial V}{\partial S} + S^2\frac{\partial^2 V}{\partial S^2} - rV = 0.$$

Assume you have an equidistant grid with points of the form $(i, j) = (i\Delta t, j\Delta x)$, where $i \in \{1, 2, \ldots, N\}$ and $j \in \{-N_S, N_S\}$. Let $V_{i,j} = (i\Delta t, j\Delta x)$.

(a) Discretize the derivatives and give the finite difference equation for an Explicit scheme. Use the notation introduced above. Please simplify the final expressions so that the unknown quantity is on the left hand side and all the known quantities are on the right hand side.

(b) Derive the discretized equation for the Implicit scheme. Please simplify the final expressions so that the unknown quantities are on the left hand side and the known quantity is on the right hand side. You do not need to solve the corresponding matrix equation.