FE-621 Homework3

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Problem 1

(a) For the explicit method the final discretized equation is:

$$-\frac{V_{i+1,j}-V_{i,j}}{\Delta t}=\nu\frac{V_{i+1,j+1}-V_{i+1,j-1}}{2\Delta x}+\frac{1}{2}\sigma^2\frac{V_{i+1,j+1}-2V_{i+1,j}+V_{i+1,j-1}}{\Delta x^2}-rV_{i+1,j}$$

where $\nu=r-q-\frac{\sigma^2}{2}$

Rearrange the equation we get:

$$\begin{aligned}
V_{i,j}&=p_{\mathrm{u}}V_{i+1.j+1}+p_{\mathrm{m}}V_{i+1.j}+p_{\mathrm{d}}V_{i+1,j-1}\\
p_{\mathrm{u}}&=\Delta t\left(\frac{\sigma^2}{2\Delta x^2}+\frac{\nu}{2\Delta x}\right)\\
p_{\mathrm{m}}&=1-\Delta t\frac{\sigma^2}{\Delta x^2}-r\Delta t\\
p_{\mathrm{d}}&=\Delta t\left(\frac{\sigma^2}{2\Delta x^2}-\frac{\nu}{2\Delta x}\right)
\end{aligned}$$

At last, the boundary conditions:

$$V_{N_j,j}=\begin{cases}V_{N_j-1,j+1} & for\ calls\\ 0 & for\ puts\end{cases}$$

$$V_{-N_j,j}=\begin{cases}0 & for\ calls\\ V_{-N_j+1,j+1} & for\ puts\end{cases}$$

Then we can implement explicit method in our program:

1. First of all, we define a `payoff` function to calculate payoffs:

```
def payoff(op_type, s, k):
    if op_type == 'c':
        return np.maximum(s - k, 0)
    elif op_type == 'p':
        return np.maximum(k - s, 0)
    else:
        raise ValueError("undefined option type")
```

2. Next define the explicit function as `e_fdm`:

   - In the function, we first precomputes constants $\Delta t, \nu, p_u, p_m, p_d$.

   - Then use `np.arange` to generate a list from N_j to $-N_j$ with step 1:

$$l = (N_j, N_j - 1 \cdots, 0, \cdots - N_j + 1, -N_j)$$

$$S_t = S_0 e^{l \cdot \Delta x}$$

   - Next we define a `backward` function to do backward calculation, and here use `np.roll` to do vectorized calculation instead of looping; Besides in the function we apply the boundary and early exercise (if it's an American option) condition.

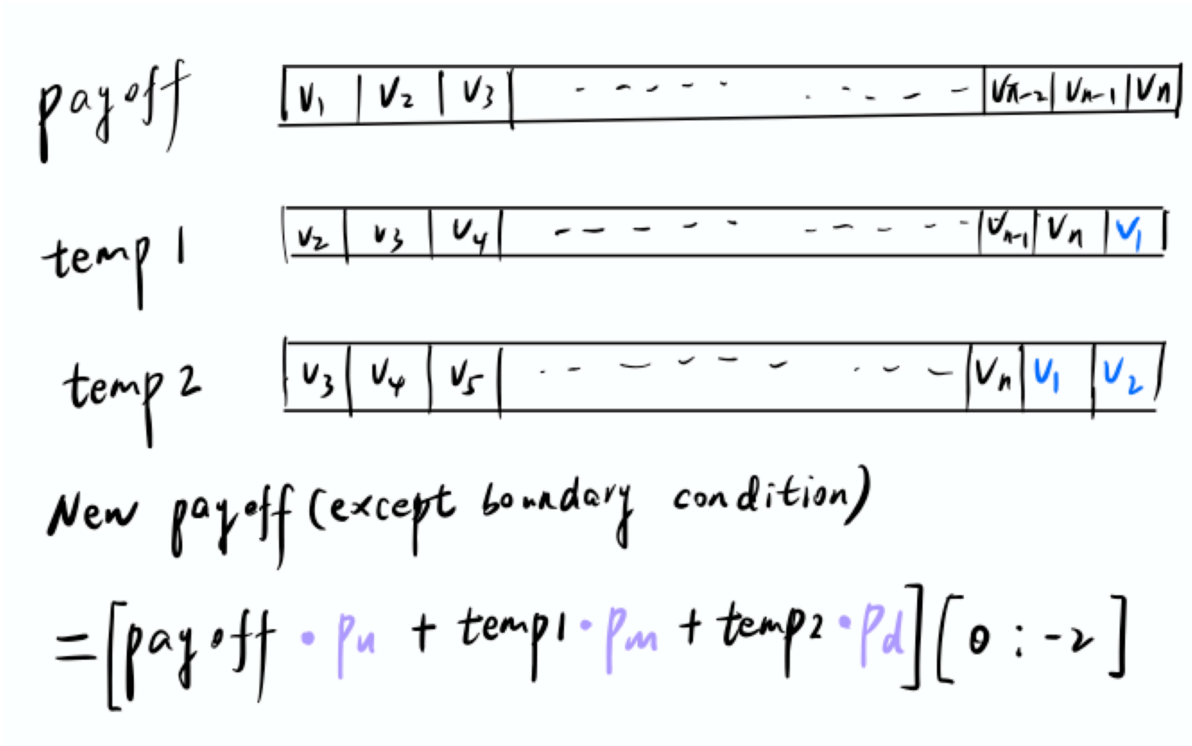

Figure 1: Basic idea of backward calculation

- At last of `e_fdm` function, do `backward` for N times and return the result.

Codes is in the following block:

```
def e_fdm(S, K, T, r, sigma, q, N, Nj, dx, op_type, style):
# precompute
dt = T / N
nu = r - q - sigma ** 2 / 2
pu = 0.5 * dt * ((sigma / dx) ** 2 + nu / dx)
pm = 1 - dt * (sigma / dx) ** 2 - r * dt
pd = 0.5 * dt * ((sigma / dx) ** 2 - nu / dx)

# stock price and payoff at maturity
st = np.arange(Nj, -Nj - 1, -1)
st = np.exp(st * dx) * S
p = payoff(op_type, st, K)

def backward(p):
    temp1 = np.roll(p, -1)
    temp2 = np.roll(p, -2)
```

```
    temp3 = p * pu + temp1 * pm + temp2 * pd
    p[1:-1] = temp3[0:-2]
    if op_type == 'c':
        p[0] = p[1] + (st[0] - st[1])
        p[-1] = p[-2]
    elif op_type == 'p':
        p[0] = p[1]
        p[-1] = p[-2] + (st[-2] - st[-1])
    if style == 'a':
        p = np.maximum(p, payoff(op_type, st, K))

for i in range(N):
    backward(p)

return p[N]
```

(b)

$$-\frac{V_{i+1,j} - V_{i,j}}{\Delta t} = \frac{1}{2}\sigma^2 \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta x^2} + \nu \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta x} - rV_{i,j}$$

It can be written as:

$$\begin{aligned}
V_{i+1,i} &= p_{\mathrm{u}} V_{i,i+1} + p_{\mathrm{m}} V_{i,j} + p_{\mathrm{d}} V_{i,i-1} \\
p_{\mathrm{u}} &= -\frac{1}{2}\Delta t \left(\frac{\sigma^2}{\Delta x^2} + \frac{\nu}{\Delta x} \right) \\
p_{\mathrm{m}} &= 1 + \Delta t \frac{\sigma^2}{\Delta x^2} + r\Delta t \\
p_d &= -\frac{1}{2}\Delta t \left(\frac{\sigma^2}{\Delta x^2} - \frac{\nu}{\Delta x} \right)
\end{aligned}$$

The boundary conditions are the same as we talked before in Problem a.

So it can be expressed in the matrix form: $Ax = b$

$$
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & \dots & 0 \\
p_u & p_m & p_d & 0 & 0 & \dots & 0 \\
0 & p_u & p_m & p_d & 0 & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & p_m & p_d & 0 \\
0 & 0 & 0 & \ddots & p_u & p_m & p_d \\
0 & 0 & 0 & \dots & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
V_{i,N_j} \\ V_{i,N_j-1} \\ V_{i,N_j-2} \\ \vdots \\ \vdots \\ V_{i,-N_j+1} \\ V_{i,-N_j}
\end{bmatrix}
=
\begin{bmatrix}
\lambda_U \\ V_{i+1,N_j-1} \\ V_{i+1,N_j-2} \\ \vdots \\ \vdots \\ V_{i+1,-N_j+1} \\ \lambda_L
\end{bmatrix}
$$

To implement it in python:

1. First precomputes constants.
2. Then use `numpy` functions to construct matrix A.
3. Next calculate λ_U, λ_L.
4. In the `backward` function, we substitute the first and last element of payoff array with λ_U and λ_L. Then use `np.solve` to solve the equation $Ax = b$.
5. Do `backward` for N times.

```python
def i_fdm(S, K, T, r, sigma, q, N, Nj, dx, op_type, style):
    # precompute constants
    dt = T / N
    nu = r - q - sigma ** 2 / 2
    pu = - 0.5 * dt * ((sigma / dx) ** 2 + nu / dx)
    pm = 1 + dt * (sigma / dx) ** 2 + r * dt
    pd = - 0.5 * dt * ((sigma / dx) ** 2 - nu / dx)

    # construct tridiagnal matrix
    l1 = np.zeros((1, 2 * Nj + 1))
    l2 = np.zeros((1, 2 * Nj + 1))
    l1[0][0] = 1
    l1[0][1] = -1
    l2[0][-1] = -1
    l2[0][-2] = 1
    rows = 2 * Nj - 1
    cols = 2 * Nj + 1
    a = np.eye(rows, cols, 0) * pu \
        + np.eye(rows, cols, 1) * pm \
        + np.eye(rows, cols, 2) * pd
    a = np.r_[l1, a, l2]
```

```python
# stock price and payoff at maturity
st = np.arange(Nj, -Nj - 1, -1)
st = np.exp(st * dx) * S
p = payoff(op_type, st, K)

# lambda
if op_type == 'c':
    lambda_u = st[0] - st[1]
    lambda_l = 0
elif op_type == 'p':
    lambda_u = 0
    lambda_l = st[-2] - st[-1]

# backward calculation
def backward(p):
    b = np.append(lambda_u, p[1:-1])
    b = np.append(b, lambda_l)
    x = np.linalg.solve(a, b)
    print(x.shape)
    return x

for i in range(N):
    p = backward(p)

return p[N]
```

(c) The Crank-Nicolson finite different method averages the space derivatives at i and $i+1$:

$$
\begin{aligned}
&-\frac{V_{i+1,j}-V_{i,j}}{\Delta t} \\
&= \frac{1}{2}\sigma^2\left(\frac{(V_{i+1,j+1}-2V_{i+1,j}+V_{i+1,j-1})+(V_{i,j+1}-2V_{i,j}+V_{i,j-1})}{2\Delta x^2}\right) \\
&\quad +\nu\left(\frac{(V_{i+1,j+1}-V_{i+1,j-1})+\left(V_{l,j+1}-V_{i,j-1}\right)}{4\Delta x}\right)-r\left(\frac{V_{i+1,j}+V_{i,j}}{2}\right)
\end{aligned}
$$

Which can be rewritten as:

$$
\begin{aligned}
p_{\mathrm{u}} V_{i,j+1} + p_{\mathrm{m}} V_{i,j} + p_{\mathrm{d}} V_{i,j-1} &= -p_{\mathrm{n}} V_{i+1,j+1} - (p_{\mathrm{m}} - 2) V_{i+1,j} - p_{\mathrm{d}} V_{i+1,j-1} \\
p_{\mathrm{u}} &= -\frac{1}{4}\Delta t \left(\frac{\sigma^2}{\Delta x^2} + \frac{\nu}{\Delta x} \right) \\
p_{\mathrm{m}} &= 1 + \Delta t \frac{\sigma^2}{2\Delta x^2} + \frac{r\Delta t}{2} \\
p_{\mathrm{d}} &= -\frac{1}{4}\Delta t \left(\frac{\sigma^2}{\Delta x^2} - \frac{\nu}{\Delta x} \right)
\end{aligned}
$$

So to implement it in python, the only difference is the p_u, p_m p_d and the matrix A:

```
def cn_fdm(S, K, T, r, sigma, q, N, Nj, dx, op_type, style):
    # precompute constants
    dt = T / N
    nu = r - q - sigma ** 2 / 2
    pu = - 0.25 * dt * ((sigma / dx) ** 2 + nu / dx)
    pm = 1 + 0.5 * dt * (sigma / dx) ** 2 + 0.5 * r * dt
    pd = - 0.25 * dt * ((sigma / dx) ** 2 - nu / dx)

    # construct tridiagnal matrix
    l1 = np.zeros((1, 2 * Nj + 1))
    l2 = np.zeros((1, 2 * Nj + 1))
    l1[0][0] = 1
    l1[0][1] = -1
    l2[0][-1] = -1
    l2[0][-2] = 1
    rows = 2 * Nj - 1
    cols = 2 * Nj + 1
    a = np.eye(rows, cols, 0) * pu \
        + np.eye(rows, cols, 1) * pm \
        + np.eye(rows, cols, 2) * pd
    a = np.r_[l1, a, l2]

    # stock price and payoff at maturity
    st = np.arange(Nj, -Nj - 1, -1)
    st = np.exp(st * dx) * S
    p = payoff(op_type, st, K)

    # lambda
    if op_type == 'c':
        lambda_u = st[0] - st[1]
        lambda_l = 0
    elif op_type == 'p':
```

```python
        lambda_u = 0
        lambda_l = st[-2] - st[-1]

    # backward calculation
    def backward(p):
        temp1 = np.roll(p, -1)
        temp2 = np.roll(p, -2)
        temp3 = -p * pu - temp1 * (pm-2) - temp2 * pd
        p[1:-1] = temp3[0:-2]
        b = np.append(lambda_u, p[1:-1])
        b = np.append(b, lambda_l)
        x = np.linalg.solve(a, b)
        return x

    for i in range(N):
        p = backward(p)

    return p[N]
```

(d) The order of convergence for explicit algorithm and implicit algorithm are both $O(\Delta x^2 + \Delta t)$, and we know that the best choice for $\Delta x = \sigma\sqrt{3\Delta t}$. Besides, if the process follows the usual geometric Brownian motion, then one needs to cover at least the range $(-3\sigma\sqrt{T}, 3\sigma\sqrt{T})$ which is the range containing 99.7% of the possible values for the return.

To estimate $\Delta t, \Delta x$, we approximate that:

$$\begin{cases} \Delta x^2 + \Delta t = \epsilon \\ \Delta x = \sigma\sqrt{3\Delta t} \end{cases}$$

Solve it we can get:

$$\Delta t = \frac{\epsilon}{3\sigma^2 + 1}$$
$$\Delta x = \sigma\sqrt{\frac{3\epsilon}{3\sigma^2 + 1}}$$

For N and N_j:

$$N = \frac{T}{\Delta t} = \frac{3\sigma^2 + 1}{\epsilon} \cdot T$$
$$N_j = 0.5 \cdot (\frac{6\sigma\sqrt{T}}{\Delta x} - 1) = \sqrt{3N} - 0.5$$

(e) Use the result we calculated in d:

```
import numpy as np
import matplotlib.pyplot as plt
from finite_diff_methods import *

# params
S = 100
K = 100
T = 1
sigma = 0.2
r = 0.06
q = 0.02

epsilon = 0.0001
dt = epsilon / (3 * sigma ** 2 + 1)
dx = sigma * np.sqrt(3 * dt)
N = int(np.ceil(T / dt))
Nj = int(np.ceil((2 * np.sqrt(3 * N) - 1) / 2))

# part e
print("result of part e:")
print("dt: ", dt)
print("dx: ", dx)
print("N: ", N)
print("Nj: ", Nj)

ec = e_fdm(S, K, T, r, sigma, q, N, Nj, dx, 'c', 'e')
ep = e_fdm(S, K, T, r, sigma, q, N, Nj, dx, 'p', 'e')
print("call of explicit method is: {0}, "
      "put of explicit method is: {1}".format(ec, ep))
ic = i_fdm(S, K, T, r, sigma, q, N, Nj, dx, 'c', 'e')
ip = i_fdm(S, K, T, r, sigma, q, N, Nj, dx, 'p', 'e')
print("call of implicit method is: {0}, "
      "put of implicit method is: {1}".format(ic, ip))
cc = cn_fdm(S, K, T, r, sigma, q, N, Nj, dx, 'c', 'e')
cp = cn_fdm(S, K, T, r, sigma, q, N, Nj, dx, 'p', 'e')
```

```
print("call of Crank-Nicolson method is: {0}, "
      "put of Crank-Nicolson method is: {1}".format(cc, cp))
```

Table of $\Delta t, \Delta x, N, N_j$:

Δt	Δx	N	N_j
$8.9286 \cdot 10^{-5}$	0.003273	11200	183

Options' prices using 3 methods: (It's also the result of part h)

type	explicit	implicit	Crank-Nicolson
call	9.7285	9.7283	9.7284
put	5.8850	5.8848	8.8849

(f) First define the `BS_formula` function:

```
def BS_formula(Type, S, K, T, sigma, r):
    d1 = (np.log(S / K) + (r + sigma ** 2 / 2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    if Type == 'c':
        return norm.cdf(d1) * S - norm.cdf(d2) * K * np.exp(-r * T)
    elif Type == 'p':
        return K * np.exp(-r * T) * norm.cdf(-d2) - norm.cdf(-d1) * S
    else:
        raise TypeError("Type must be 'c' for call, 'p' for put")
```

Next define `get_iter` function to get the number of iterations:

```
def get_iter(S, K, T, r, sigma, q, op_type, method):
    N = 50
    dt = T / N
    dx = sigma * np.sqrt(3 * dt)
    Nj = int(np.ceil((2 * np.sqrt(3 * N) - 1) / 2))
    fd_price = method(S, K, T, r, sigma, q, N, Nj, dx, op_type, 'e')
    bs_price = BS_formula(op_type, S, K, T, sigma, r, q)
    iter = 0
    while abs(fd_price - bs_price) > epsilon:
        N += 100
```

```
        dt = T / N
        dx = sigma * np.sqrt(3 * dt)
        Nj = int(np.ceil((2 * np.sqrt(3 * N) - 1) / 2))
        fd_price = method(S, K, T, r, sigma, q, N, Nj, dx, op_type, 'e')
        bs_price = BS_formula(op_type, S, K, T, sigma, r, q)
        iter += 1
    return iter
```

At last we use the function above:

```
esc = get_iter(S, K, T, r, sigma, q, 'c', e_fdm)
esp = get_iter(S, K, T, r, sigma, q, 'p', e_fdm)
print("step1 of explicit method is: {0}, "
      "step2 of explicit method is: {1}".format(esc, esp))
isc = get_iter(S, K, T, r, sigma, q, 'c', i_fdm)
isp = get_iter(S, K, T, r, sigma, q, 'p', i_fdm)
print("step1 of explicit method is: {0}, "
      "step2 of explicit method is: {1}".format(isc, isp))
csc = get_iter(S, K, T, r, sigma, q, 'c', cn_fdm)
csp = get_iter(S, K, T, r, sigma, q, 'p', cn_fdm)
print("step1 of Crank-Nicolson method is: {0}, "
      "step2 of Crank-Nicolson method is: {1}".format(csc, csp))
```

Note the iterations calculated in this problem is relative iterations, if the time step increases by one, the running time would be to long. So here I let time step increase by 100, and other parameters changes correspondingly.

Result of iterations for three methods:

method	call	put
EFD	76	154
IFD	193	321
CNFD	139	237

From the table we can find implicit finite difference method takes the most iterations and explicit finite difference method uses the least iterations; Additionally, all three methods takes more iterations to get put price with error less than epsilon we choose.

(g) Use a `prob` function to calculate probs:

```python
def prob(T, r, sigma, q, N, dx):
    dt = T / N
    nu = r - q - sigma ** 2 / 2
    pu = - 0.5 * dt * ((sigma / dx) ** 2 + nu / dx)
    pm = 1 + dt * (sigma / dx) ** 2 + r * dt
    pd = - 0.5 * dt * ((sigma / dx) ** 2 - nu / dx)
    return pu, pm, pd


sig = np.arange(0.05, 0.61, 0.05)
pu, pm, pd = prob(T, r, sig, q, N, dx)
plt.figure(1)
plt.xlabel("sigma")
plt.ylabel("probs")
plt.title("probs of implicit finite difference method")
plt.plot(sig, pu, label = 'pu')
plt.plot(sig, pm, label = 'pm')
plt.plot(sig, pd, label = 'pd')
plt.legend()
plt.show()
```

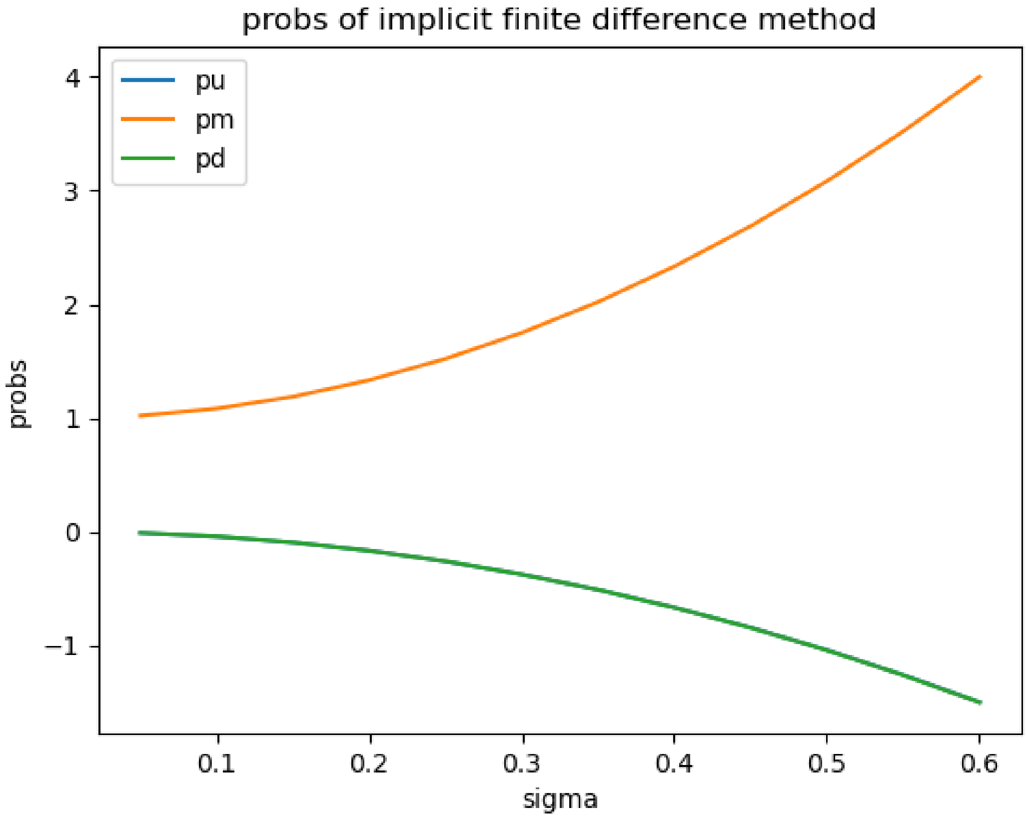

Figure 2: plot of part g

Note here p_u and p_d are very close to each other, so it looks like that p_u doesn't show in the plot. We can only plot p_u and p_m to find that:

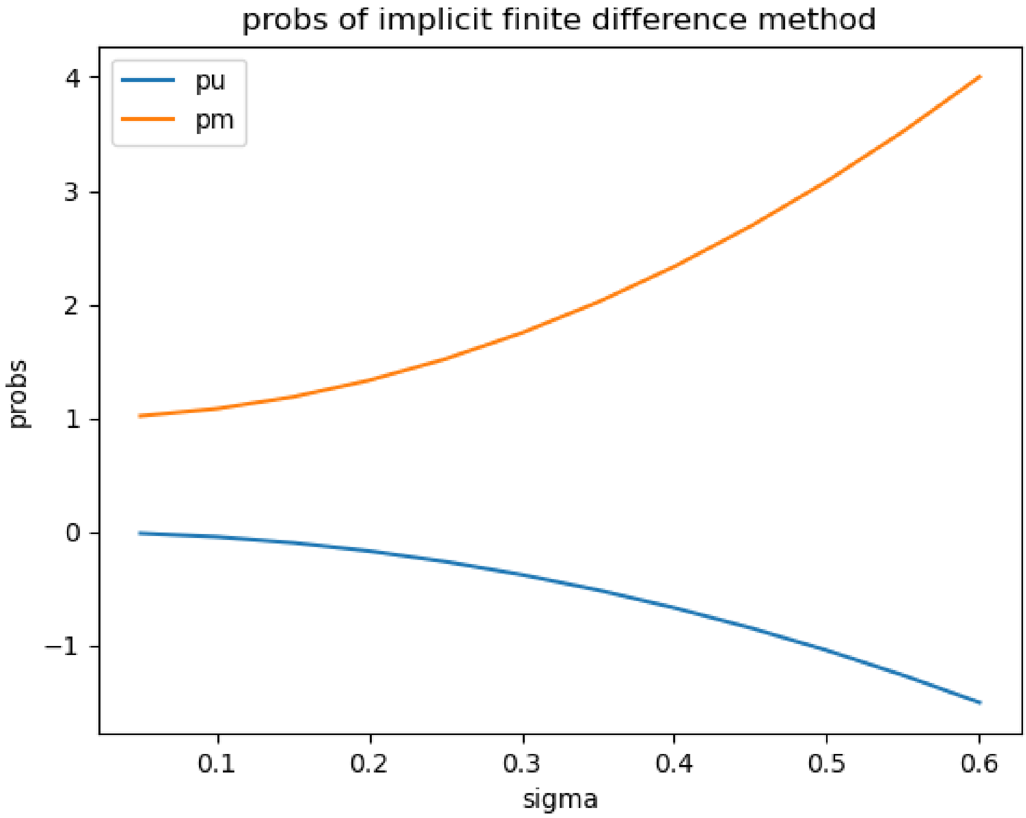

Figure 3: plot of pu and pm

Comment:

Because here we use the same parameters in part e, so we know Δx is very small. And by taking a look at p_u, p_m, p_d:

$$p_{\mathrm{u}} = -\frac{1}{2}\Delta t\left(\frac{\sigma^2}{\Delta x^2} + \frac{\nu}{\Delta x}\right)$$

$$p_{\mathrm{m}} = 1 + \Delta t\frac{\sigma^2}{\Delta x^2} + r\Delta t$$

$$p_d = -\frac{1}{2}\Delta t\left(\frac{\sigma^2}{\Delta x^2} - \frac{\nu}{\Delta x}\right)$$

- Obviously when Δx is very small, $\frac{\sigma^2}{\Delta x^2}$ is much more larger than $\frac{\nu}{\Delta x}$. As a result, p_u and p_d are approximately equal to $-\frac{1}{2}\Delta t \cdot \frac{\sigma^2}{\Delta x^2}$.

- When σ is larger, p_u and p_d becomes less, and p_m becomes larger.

(h) The following is the table of results calculated by 3 methods: (It has been calculated in part e)

type	EFD	IFD	CNFD
call	9.7285	9.7283	9.7284
put	5.8850	5.8848	8.8849

We can find that the result of Crank-Nicolson is just the average of explicit and implicit methods' results.

(i) For delta, gamma and theta, we can use the girds to calculate them directly:

```
def delta_gamma_theta(S, K, T, r, sigma, q, N, Nj, dx, op_type):
    # precompute
    dt = T / N
    nu = r - q - sigma ** 2 / 2
    pu = 0.5 * dt * ((sigma / dx) ** 2 + nu / dx)
    pm = 1 - dt * (sigma / dx) ** 2 - r * dt
    pd = 0.5 * dt * ((sigma / dx) ** 2 - nu / dx)

    # stock price and payoff at maturity
    st = np.arange(Nj, -Nj - 1, -1)
    st = np.exp(st * dx) * S
    p = payoff(op_type, st, K)

    def backward(p):
        temp1 = np.roll(p, -1)
        temp2 = np.roll(p, -2)
        temp3 = p * pu + temp1 * pm + temp2 * pd
        p[1:-1] = temp3[0:-2]
        if op_type == 'c':
            p[0] = p[1] + (st[0] - st[1])
            p[-1] = p[-2]
        elif op_type == 'p':
            p[0] = p[1]
            p[-1] = p[-2] + (st[-2] - st[-1])

    for i in range(N):
```

```
        if i == N-1:
            p1 = p[Nj]
        backward(p)

    delta = (p[Nj+1] - p[Nj-1]) / (st[Nj+1] - st[Nj-1])
    delta1 = (p[Nj+1] - p[Nj]) / (st[Nj+1] - st[Nj])
    delta2 = (p[Nj] - p[Nj-1]) / (st[Nj] - st[Nj-1])
    gamma = (delta1 - delta2) / (0.5 * (st[Nj+1] - st[Nj-1]))
    theta = (p1 - p[Nj])/dt
    return delta, gamma, theta
```

For vega, we need to apply small changes on σ, then use the difference of prices divided by small changes:

```
def vega(S, K, T, r, sigma, q, N, Nj, dx, op_type):
    p1 = e_fdm(S, K, T, r, sigma, q, N, Nj, dx, op_type, 'e')
    p2 = e_fdm(S, K, T, r, sigma+0.05, q, N, Nj, dx, op_type, 'e')
    return (p2-p1)/0.05
```

Use the same parameter in part e to calculate Greeks of call option:

```
# part i
delta, gamma, theta = delta_gamma_theta(
    S, K, T, r, sigma, q, N, Nj, dx, 'c')
vega = vega(S, K, T, r, sigma, q, N, Nj, dx, 'c')

print("delta: ", delta)
print("gamma: ", gamma)
print("vega: ", vega)
print("theta: ", theta)
```

Result:

delta	gamma	vega	theta
0.606	0.0019	37.501	-5.578

Problem 2

(a)

1. download data, save the data to a binary file.

```
import pandas as pd
import numpy as np
import yfinance as yf
from implied_vol import *
from finite_diff_methods import *

# download
spy = yf.Ticker("SPY")
equity_data = yf.download(tickers='SPY',period='1d')
expiry = ['2021-04-16', '2021-05-21', '2021-06-18']
call = []
put = []
for date in expiry:
    call.append(spy.option_chain(date)[0])
    put.append(spy.option_chain(date)[1])

# save
pd.to_pickle(call, "./datasets/call.pkl")
pd.to_pickle(put, "./datasets/put.pkl")
pd.to_pickle(equity_data, "./datasets/equity.pkl")
```

2. Read data and choose options with strike from 385 to 395.

```
call = pd.read_pickle("./datasets/call.pkl")
put = pd.read_pickle("./datasets/put.pkl")
equity_data = pd.read_pickle("./datasets/equity.pkl")
expiry = ['2021-04-16', '2021-05-21', '2021-06-18']

# clean
def clean(data):
    new_data = []
    for df,date in zip(data, expiry):
        df['expiry'] = pd.to_datetime(date)
        df['t2m'] = (df['expiry'] - pd.Timestamp('today')) / np.timedelta64(1, '
        df['s0'] = equity_data.iloc[0,3]
        df['market_price'] = df.bid/2 + df.ask/2
        # choose strike between 385 to 395
        new_df = df.loc[df.strike.isin(np.arange(385,395))].reset_index()
        del new_df['index']
        new_data.append(new_df)
    return new_data
```

```
call = clean(call)
put = clean(put)
```

3. Use bisection method to calculate implied volatility. (codes of `get_impliedVol` are pasted from Homework 1. I have put it in the appendix, you can find it here)

```
# get vol
r = 0.07/100
for df in call:
    df['vol'] = df.apply(lambda x: \
                        get_impliedVol('c', x.s0, x.strike, x.t2m, r, x.pri

for df in put:
    df['vol'] = df.apply(lambda x: \
                        get_impliedVol('c', x.s0, x.strike, x.t2m, r, x.pri
```

(b)

- Define a `fd_price` function, so that it can be applied on DataFrames in pandas:

```
def fd_price(x, epsilon, method):
dt = epsilon / (3 * x.vol ** 2 + 1)
dx = x.vol * np.sqrt(3 * dt)
N = int(np.ceil(x.t2m / dt))
Nj = int(np.ceil((2 * np.sqrt(3 * N) - 1) / 2))
return method(x.s0, x.strike, x.t2m, r, x.vol, 0, N, Nj, dx, x.type, 'e')
```

- Apply the function:

```
    for df1,df2 in zip(call, put):
df1['EFD'] = df1.apply(lambda x:
                       fd_price(x, epsilon, e_fdm),axis=1)
df1['IFD'] = df1.apply(lambda x:
                       fd_price(x, epsilon, i_fdm),axis=1)
df1['CNFD'] = df1.apply(lambda x:
                        fd_price(x, epsilon, cn_fdm),axis=1)
df2['EFD'] = df2.apply(lambda x:
                       fd_price(x, epsilon, e_fdm),axis=1)
df2['IFD'] = df2.apply(lambda x:
                       fd_price(x, epsilon, i_fdm),axis=1)
df2['CNFD'] = df2.apply(lambda x:
                        fd_price(x, epsilon, cn_fdm),axis=1)
```

Results are presented in the table of part d. You can see it here.

(c) The idea of part c is similar with part b:

```
def get_greeks(x):
    dt = epsilon / (3 * x.vol ** 2 + 1)
    dx = x.vol * np.sqrt(3 * dt)
    N = int(np.ceil(x.t2m / dt))
    Nj = int(np.ceil((2 * np.sqrt(3 * N) - 1) / 2))
    delta,gamma = delta_gamma(x.s0, x.strike, x.t2m, r, x.vol, 0, N, Nj, dx, x.type)
    vega_ = vega(x.s0, x.strike, x.t2m, r, x.vol, 0, N, Nj, dx, x.type)
    theta_ = theta(x.s0, x.strike, x.t2m, r, x.vol, 0, N, Nj, dx, x.type)
    return delta, gamma, vega_, theta_


for df1,df2 in zip(call, put):
    df1[['delta', 'gamma', 'vega', 'theta']] = \
    df1.apply(get_greeks,axis=1, result_type="expand")
    df2[['delta', 'gamma', 'vega', 'theta']] = \
    df2.apply(get_greeks,axis=1, result_type="expand")
```

Table of options' greeks maturing at 2021-04-16:

Call:

strike	market_price	vol	delta	gamma	theta	vega
385.0	10.985	0.1943	0.6281	0.0193	-56.3207	38.2343
386.0	10.18	0.189	0.6109	0.0202	-55.6071	38.7135
387.0	9.525	0.1871	0.5914	0.0207	-55.9917	39.2471
388.0	8.725	0.1811	0.572	0.0216	-54.5618	39.4786
389.0	8.165	0.1806	0.5503	0.0218	-54.8126	39.7086
390.0	7.495	0.1769	0.5286	0.0224	-54.0501	39.9371
391.0	6.825	0.1726	0.506	0.0231	-53.0269	40.1168
392.0	6.205	0.169	0.4824	0.0235	-51.6725	39.9814
393.0	5.695	0.1676	0.4586	0.0235	-50.9794	39.8435
394.0	5.17	0.1652	0.4339	0.0238	-49.979	39.7008

Put:

strike	market_price	vol	delta	gamma	theta	vega
394.0	8.07	0.1593	-0.5691	0.0246	-47.8872	39.6828
393.0	7.54	0.1604	-0.544	0.0246	-48.4817	39.8297
392.0	7.11	0.1633	-0.5188	0.0243	-49.6521	39.9765
391.0	6.735	0.1671	-0.4944	0.0238	-51.0491	40.1164

strike	market_price	vol	delta	gamma	theta	vega
390.0	6.365	0.1703	-0.471	0.0233	-51.7628	39.9324
389.0	5.985	0.1728	-0.4483	0.0228	-52.13	39.6961
388.0	5.68	0.1766	-0.4266	0.0221	-52.9115	39.4676
387.0	5.375	0.1799	-0.4058	0.0215	-53.4248	39.1789
386.0	5.175	0.1854	-0.3874	0.0206	-54.2052	38.6722
385.0	4.835	0.1869	-0.3677	0.02	-53.6883	38.1335

(d) First , save data to csv files:

```
# save to csv
for df1,df2 in zip(call, put):
    path1 = './p2_csv/' + df1.iloc[0,0][0:10] + '.csv'
    path2 = './p2_csv/' + df2.iloc[0,0][0:10] + '.csv'
    df1.to_csv(path1, index=False)
    df2.to_csv(path2, index=False)
```

Choose options maturing at 2021-05-21 to create the table: (other data can be found under *./codes/p2_csv/*)

```
info = ['t2m', 'strike', 'type', 'ask', 'bid',
       'market_price', 'vol', 'EFD', 'IFD', 'CNFD']
data1 = call[1][info].round(4)
data2 = put[1][info].round(4)
data1.to_csv('./p2_csv/d_res1.csv', index=False)
data2.to_csv('./p2_csv/d_res2.csv', index=False)
```

Tables Call option table:

t2m	strike	type	ask	bid	market_price	vol	EFD	IFD	CNFD
0.1706	395.0	c	9.12	9.08	9.1	0.1704	9.1024	9.0952	9.0988
0.1706	397.0	c	7.98	7.94	7.96	0.1654	7.9634	7.9567	7.9601
0.1706	435.0	c	0.4	0.38	0.39	0.1467	0.3908	0.3939	0.3923
0.1706	400.0	c	6.58	6.54	6.56	0.161	6.5584	6.5525	6.5554
0.1706	424.0	c	0.99	0.97	0.98	0.1449	0.9781	0.9804	0.9792
0.1706	390.0	c	12.0	11.96	11.98	0.1792	11.9824	11.9747	11.9786
0.1706	396.0	c	8.56	8.52	8.54	0.1682	8.5364	8.5294	8.5329
0.1706	430.0	c	0.6	0.58	0.59	0.1454	0.591	0.5939	0.5925
0.1706	405.0	c	4.56	4.52	4.54	0.1534	4.5395	4.5355	4.5375
0.1706	404.0	c	4.91	4.87	4.89	0.1546	4.8912	4.8868	4.889

Put option table:

t2m	strike	type	ask	bid	market_price	vol	EFD	IFD	CNFD
0.1706	371.0	p	5.98	5.94	5.96	0.2178	5.9633	5.9583	5.9608
0.1706	377.0	p	7.2	7.16	7.18	0.2049	7.1791	7.1728	7.1759
0.1706	385.0	p	9.2	9.17	9.185	0.1862	9.1846	9.1771	9.1808
0.1706	349.0	p	3.18	3.16	3.17	0.2658	3.1676	3.1679	3.1678
0.1706	390.0	p	10.89	10.85	10.87	0.1759	10.8726	10.865	10.8688
0.1706	380.0	p	7.88	7.86	7.87	0.198	7.8671	7.8603	7.8637
0.1706	384.0	p	8.87	8.84	8.855	0.1878	8.8587	8.8513	8.855
0.1706	350.0	p	3.27	3.24	3.255	0.2635	3.2527	3.2528	3.2528
0.1706	345.0	p	2.85	2.82	2.835	0.2742	2.8299	2.831	2.8304
0.1706	375.0	p	6.76	6.73	6.745	0.2093	6.7486	6.7427	6.7457

Choose call options maturing at 05-21 and 06-21 to plot:

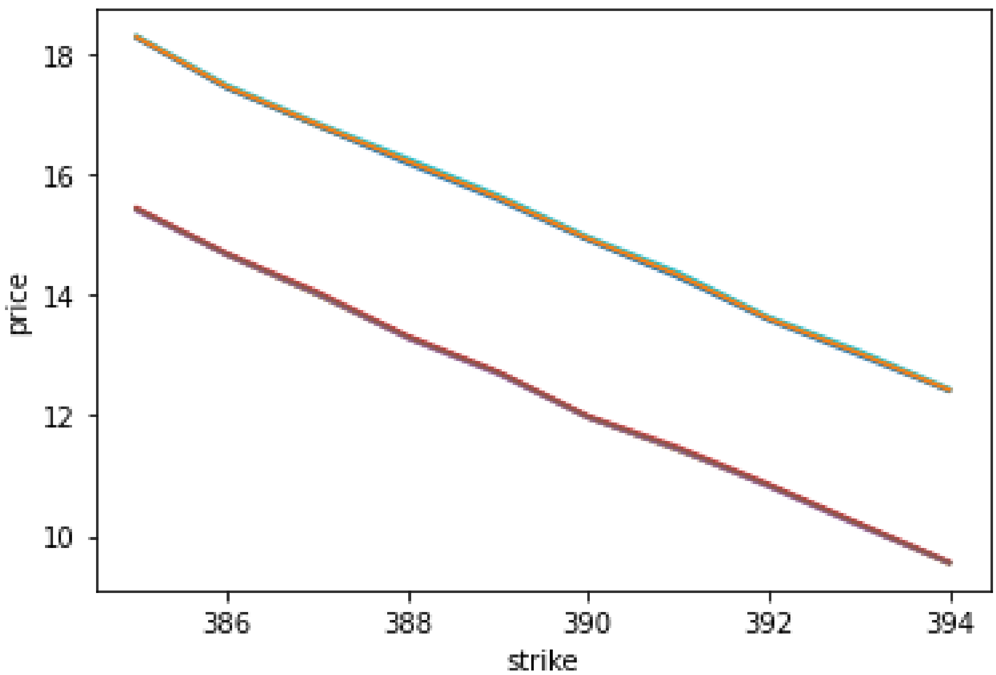

Figure 4: plot of d

From the table and plot, we find that A, B, C_M and prices calculated by finite difference are very close. And the upper lines are the prices of options maturing at 06-21 since their time to maturity is longer.

Problem 3

1 First we need to transform stock price process to return process: $x = \log S$ and choose $V(S,t) = V(e^x, t) = U(x,t)$. By applying chain rule, we have:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial x} = \frac{\partial V}{\partial S} \cdot \frac{\partial S}{\partial x} = \frac{\partial V}{\partial S} \cdot e^x = \frac{\partial V}{\partial S} \cdot S$$

and

$$\begin{aligned}\frac{\partial^2 U}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial S} \cdot \frac{\partial S}{\partial x}\right) \\ &= \frac{\partial^2 V}{\partial S^2} \cdot \frac{\partial S}{\partial x} \cdot \frac{\partial S}{\partial x} + \frac{\partial V}{\partial S} \cdot \frac{\partial^2 S}{\partial x^2}\end{aligned}$$

Thus

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 V}{\partial S^2} S^2 + \frac{\partial V}{\partial S} \cdot = \frac{\partial^2 V}{\partial S^2} S^2 + \frac{\partial V}{\partial x}$$

Therefore the PDE in this question becomes:

$$\frac{\partial V}{\partial t} + \frac{2\cos(S)}{S}\frac{\partial V}{\partial x} + 0.2S^{-\frac{1}{2}}\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}\right) - rV = 0$$

$$\frac{\partial V}{\partial t} + \left(\frac{2\cos(S)}{S} - 0.2S^{-\frac{1}{2}}\right)\frac{\partial V}{\partial x} + 0.2S^{-\frac{1}{2}}\frac{\partial^2 V}{\partial x^2} - rV = 0$$

Use forward difference to discretize the derivatives the equation:

$$\frac{V_{i+1,j} - V_{i,j}}{\Delta t} + a_{i,j}\frac{V_{i+1,j+1} - V_{i+1,j-1}}{2\Delta x} + b_{i,j}\frac{V_{i+1,j+1} - 2V_{i,j} + V_{i+1,j-1}}{\Delta x^2} - rV_{i+1,j} = 0$$

where $a_{i,j} = \frac{2\cos(S_{i,j})}{S_{i,j}} - 0.2S_{i,j}^{-\frac{1}{2}}$, $b_{i,j} = 0.2S_{i,j}^{-\frac{1}{2}}$.

Rearrange the function:

$$
\begin{aligned}
V_{i,j} &= p_u V_{i+1.j+1} + p_m V_{i+1,j} + p_d V_{i+1,j-1} \\
p_u &= \Delta t \left(\frac{b_{i,j}}{\Delta x^2} + \frac{a_{i,j}}{2\Delta x} \right) \\
p_m &= 1 - \Delta t \frac{2b_{i,j}}{\Delta x^2} - r\Delta t \\
p_d &= \Delta t \left(\frac{b_{i,j}}{\Delta x^2} - \frac{a_{i,j}}{2\Delta x} \right)
\end{aligned}
$$

2 For call option we need boundary conditions:

- When $S \to \infty$: $\frac{\partial V}{\partial S} = 1$
- When $S \to 0_+$: $\frac{\partial V}{\partial S} = 0$

3 From $-N_j$ to N_j we have different pu, pm, pd for different stock prices, but for girds at each time point, their pu, pm, pd are the same correspondingly. i.e. Prob of (i, j) are the same with $(i + 1, j)$ for any j.

Appendix Codes used to calculate implied vols:

```
import numpy as np
from scipy.stats import norm


def BS_formula(Type, S, K, T, sigma, r):
    d1 = (np.log(S / K) + (r + sigma ** 2 / 2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    if Type == 'c':
        return norm.cdf(d1) * S - norm.cdf(d2) * K * np.exp(-r * T)
    elif Type == 'p':
        return K * np.exp(-r * T) * norm.cdf(-d2) - norm.cdf(-d1) * S
    else:
        raise TypeError("Type must be 'c' for call, 'p' for put")


def vega(S, K, T, sigma, r):
    d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
    return np.sqrt(T) * S * norm.pdf(d1)
```

```python
def newton_method(f, f_prime, x0, tol=1e-6, N=100):
    for i in range(N):
        x1 = x0 - f(x0) / f_prime(x0)
        if abs(x1 - x0) < tol:
            break
        x0 = x1
    return x1


def bisection(f, a, b, tol=1e-6):
    if f(a) == 0:
        return a
    elif f(b) == 0:
        return b
    while abs(a - b) >= tol:
        c = (a + b) / 2
        if f(c) == 0:
            break
        if f(a) * f(c) < 0:
            b = c
        else:
            a = c
    return c


def get_impliedVol(Type, S, K, T, r, P):
    def price_diff(sigma):
        return BS_formula(Type, S, K, T, sigma, r) - P
    return bisection(price_diff, 0.001, 1)
```