FE621: stability condition for trinomial tree:

$\Delta x \geq \sigma \cdot \sqrt{3\Delta t}$. Sec 3.5 of Clewlow

Midterm this weekend.

Monte Carlo method: (1) why it works? Law of large numbers:

$$\frac{1}{M}\sum_{i=1}^{M} X_i \longrightarrow \mu_X \quad \text{as } M \to \infty$$

(2) how good it works? Central Limit Theorem.

$$\text{as } M \to \infty, \quad \frac{1}{\sqrt{M}}\sum_{i=1}^{M} X_i \longrightarrow N\left(\mu_X, \frac{\sigma^2}{M}\right).$$

Black Scholes model: $\frac{dS_t}{S_t} = (r-\delta)\,dt + \sigma\,dW_t$. $S_0 = s_0$.

let $X_t = \ln S_t$, then $dX_t = \nu\,dt + \sigma\,dW_t$, where $\nu = r - \delta - \frac{1}{2}\sigma^2$.

2 steps: (1) simulate M number of paths.

(1a) for each path, use a time-discretization scheme to approximate X_T T is maturity.

(2) evaluate the payoff at the simulated asset price value, take sample average.

Euler scheme: $t \quad t+\Delta t$, $X_{t+\Delta t} - X_t = \nu \cdot \Delta t + \sigma \cdot \int_t^{t+\Delta t} dW_s$.

$$= \nu \cdot \Delta t + \sigma \cdot (W_{t+\Delta t} - W_t).$$

$$\begin{cases} S_{t+\Delta t} = \exp(X_{t+\Delta t}) = \exp(X_t + \nu\Delta t + \sigma(W_{t+\Delta t} - W_t)) \quad (1). \\ dS_t = (r-\delta)S_t\,dt + \sigma S_t\,dW_t . \\ S_{t+\Delta t} - S_t = (r-\delta)\int_t^{t+\Delta t} S_u\,du + \sigma\int_t^{t+\Delta t} S_u\,dW_u \\ \qquad \approx (r-\delta)S_t \cdot \Delta t + \sigma \cdot S_t \cdot (W_{t+\Delta t} - W_t). \quad (2). \end{cases}$$

$(1) \Leftrightarrow \quad S_{t+\Delta t} = S_t \cdot \exp(\nu\Delta t + \sigma(W_{t+\Delta t} - W_t)).$

$(2) \Leftrightarrow \quad S_{t+\Delta t} \approx S_t + (r-\delta)S_t\Delta t + \sigma S_t(W_{t+\Delta t} - W_t).$

$W_T - W_t \sim N(0, T-t). = \sqrt{T-t}\cdot Z,\ Z \sim N(0,1).$

In BS model, we have closed-form solution for S_t, so no need to discretize in time.

$$S_T = S_t \cdot \exp(\nu(T-t) + \sigma(W_T - W_t)). \quad (3)$$

CEV model: $\rightarrow dS_t = (r-\delta) S_t \, dt + \sigma \cdot S_t^{\beta} \, dW_t$, $\quad \beta \neq 1 \quad 0 < \beta < 2$

Euler discretization: $S_{t+\Delta t} = S_t + (r-\delta) S_t \cdot \Delta t + \sigma \cdot S_t^{\beta} \cdot (W_{t+\Delta t} - W_t)$.

assume we have n time steps;

$0 \quad t_1 \quad t_2 \cdots \quad t_{n-1} \quad T$ to t_n.

$\rightarrow t_i = i \cdot \frac{T}{n}$, $i = 0, 1, \ldots, n$. equidistant grid

Heston model:

$$dS_t = (r-\delta) S_t \, dt + S_t \cdot \sqrt{V_t} \, dW_t^{(1)}, \quad E[dW_t^{(1)} dW_t^{(2)}] = \rho \cdot dt$$

$$dV_t = \kappa(\theta - V_t) \, dt + \sigma_v \sqrt{V_t} \, dW_t^{(2)}, \quad 0 < \rho < 1.$$

steps:

(1) de-correlate the two BMs: Cholesky decomposition.

$$dW_t^{(1)} = \rho \cdot dW_t^{(2)} + \sqrt{1-\rho^2} \cdot dW_t^{(3)}, \quad \text{where } W_t^{(3)} \perp\!\!\!\perp W_t^{(2)}$$

independent.

$$E[dW_t^{(1)} dW_t^{(2)}] = \rho \, E[dW_t^{(2)} dW_t^{(2)}] + \sqrt{1-\rho^2} \, E[dW_t^{(3)} dW_t^{(2)}]$$

$$= \rho \, dt$$

Euler time discretization:

hidden Markov model

$$S_{t+\Delta t} - S_t = (r-\delta) S_t \Delta t + S_t \sqrt{V_t} \cdot \left(\rho \cdot (W^{(2)}_{t+\Delta t} - W^{(2)}_t) + \sqrt{1-\rho^2} (W^{(3)}_{t+\Delta t} - W^{(3)}_t) \right)$$

$$\cancel{A}\ V_{t+\Delta t} - V_t = \kappa(\theta - V_t)\Delta t + \sigma_v \cdot \sqrt{V_t} \cdot (W^{(2)}_{t+\Delta t} - W^{(2)}_t).$$

Goal: updating scheme from (S_t, V_t) to $(S_{t+\Delta t}, V_{t+\Delta t})$ $(S_{t+2\Delta t}, V_{t+2\Delta t})$

$$\begin{cases} V_{t+\Delta t} \text{ depends on } V_t \text{ and } W^{(2)}_{t+\Delta t}. \\ S_{t+\Delta t} \text{ depends on } S_t,\ V_t,\ W^{(2)}_{t+\Delta t}, \text{ and } W^{(3)}_{t+\Delta t}. \end{cases}$$

Pseudo-code for MC method in BS model:

initialize parameters $[K, T, S, sig, r, div, n, M]$.

n: # of Euler discretization

M: # of MC simulations.

precompute constants:

$dt = \frac{T}{n}$;

$nudt = (r - div - 0.5 * sig^2) * dt$;

$sigsdt = sig * sqrt(dt)$; % $\sigma\sqrt{\Delta t}$.

$lnS = \ln(S)$. initial stock price.

```
sum_CT = 0 ; % record the ~~accum~~ accumulated value.
sum_CT2 = 0 ; % record the accumulated squared value.
for j = 1 to M do
    lnSt = lnS ;
    for i = 1 to n, do
        ε = standard-normal-sample : % ε = randn(1);
        lnSt = lnSt + nudt + sigsdt * ε ; % update the log stock price.
    next i.
    end for.
    ST = exp(lnSt); % S_T
    CT = max(0, S_T - K); % evaluate the payoff.
    sum_CT = sum_CT + CT ;
    sum_CT2 = sum_CT2 + CT * CT ;
    ~~next j.~~
end for.
```

$$
\begin{cases}
\texttt{call\_price = sum\_CT/M * exp(-r*T).} \\
\texttt{SD = sqrt((sum\_CT2 - sum\_CT * sum\_CT/M) * exp(-2*r*T)/(M-1));} \\
\texttt{SE = SD / sqrt(M);}
\end{cases}
$$

Two asset case: consider a basket option: $E^Q\left[e^{-rT}\left(S_T^{(1)} + S_T^{(2)} - K\right)^+\right] = C(0)$.

$$
\begin{cases}
dS_t^{(1)} = r \cdot S_t^{(1)} dt + \sigma_1 S_t^{(1)}\, dW_t^{(1)}. \\
dS_t^{(2)} = r S_t^{(2)} dt + \sigma_2 S_t^{(2)}\, dW_t^{(2)},
\end{cases}
$$

here $E[dW_t^{(1)} \cdot dW_t^{(2)}] = \rho\, dt$.

$$dW_t^{(1)} = \rho \cdot dW_t^{(2)} + \sqrt{1-\rho^2}\, dW_t^{(3)}.$$

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after discretization,

$$S_{t+\Delta t}^{(1)} - S_t^{(1)} = r \cdot S_t^{(1)} \Delta t + \sigma_1 S_t^{(1)} \cdot \rho \cdot \sqrt{\Delta t} \cdot Z_1 + \sigma_1 S_t^{(1)} \sqrt{1-\rho^2} \sqrt{\Delta t}\, Z_2,$$

$$S_{t+\Delta t}^{(2)} - S_t^{(2)} = r \cdot S_t^{(2)} \Delta t + \sigma_2 S_t^{(2)} \sqrt{\Delta t} \cdot Z_1.$$

here $Z_1, Z_2 \sim N(0,1)$, $Z_1 \perp\!\!\!\perp Z_2$.

$$\left(S_t^{(1)}, S_t^{(2)}\right) \longrightarrow \left(S_{t+\Delta t}^{(1)}, S_{t+\Delta t}^{(2)}\right).$$

MC numerical integration; so-called MC integration method.

$$\int_a^b f(x)\,dx \longrightarrow \text{quadrature } (\checkmark)$$

$\to$ MC ?

Steps: (1). $[a,b]$. pick M randomly distributed points $x_1, \dots, x_M$ in $[a,b]$ (e.g. use uniform random variables.

(2). $\bar{f} = \frac{1}{M}\sum_{i=1}^{M} f(x_i)$,

(3) $\int_a^b f(x)\,dx = (b-a)\bar{f}$.

Compare MC and quadrature in terms of CPU time, given the same or very similar accuracy.

sample mean of MC sample is converging to the population mean.
how to compare MC estimators? Criteria is based on variace.
Thus it is important to carry out variace reduction. $\begin{cases} \text{antithetic variates} \\ \text{control variates.} \end{cases}$

(1) Antithetic variates: example, if $Z \sim N(0,1)$, then $-Z \sim N(0,1)$

and $\operatorname{cov}(Z,-Z) = -1$.

$\operatorname{Cov}(f(Z), g(-Z)) < 0$,

then $\operatorname{Var}(f(Z)+g(-Z)) = \operatorname{Var}(f(Z)) + \operatorname{Var}(g(-Z)) + \operatorname{Cov}(f(Z), g(-Z))$

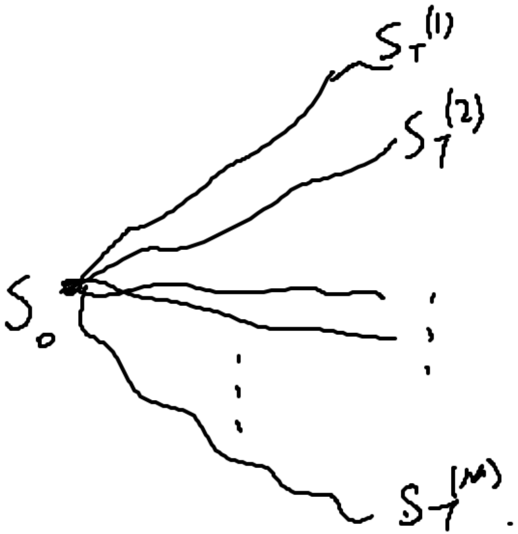
$< \operatorname{Var}(f(Z)) + \operatorname{Var}(g(-Z))$.

antithetic variates.

Copula theory, dependence of two random variables,

commonotonic : X, $2X$,

anti-commonotonic: Z, $-3Z$,



$(S_T^{(1)}, S_T^{(2)}, \dots, S_T^{(M)})$ want to design a dependence structure among them to result in the least possible variance of your MC estimator. $C^{MC} = \frac{1}{M}\sum_{i=1}^{M} G(S_T^{(i)})$.

optimization problem: $\min_{S_T^{(i)},\, i=1,\dots,M} \mathrm{Var}\left(\frac{1}{M}\sum_{i=1}^{M} G(S_T^{(i)})\right)$

open problem

subject to: ✓ $S_T^{(i)}$ have the same distribution;

(Rearrangement Algorithms)

✓ $S_T^{(1)} \overset{(d)}{=} S_T^{(2)} \overset{(d)}{=} S_T^{(3)} \cdots \overset{(d)}{=} S_T^{(M)}$.

equal in distribution.

original MC: $S_T^{(1)} \perp\!\!\!\perp S_T^{(2)} \perp\!\!\!\perp S_T^{(3)} \cdots \perp\!\!\!\perp S_T^{(M)}$ | $Z^{(1)} \perp\!\!\!\perp Z^{(2)} \perp\!\!\!\perp \cdots \perp\!\!\!\perp Z^{(M)}$.

Textbook antithetic variates: $S_T^{(i)} = f(Z^{(i)})$.

$\left((Z^{(1)}, -Z^{(1)}), (Z^{(2)}, -Z^{(2)}), \cdots (Z^{(M)}, -Z^{(M)})\right)$

$\downarrow$ $N(0,1)$

any other combinations??

Idea: keep the mean estimate the same, but try to blend in some dependence structure into the random variables to reduce the variance.

$$(\underbrace{z^{(1)}, z^{(2)}}, \cdots, z^{(M)}).$$

use $\left(\frac{z^{(1)}+z^{(2)}}{\sqrt{2}}, -\frac{z^{(1)}+z^{(2)}}{\sqrt{2}}\right)$ to replace $(z^{(1)}, z^{(2)})$.

use $\left(\frac{z^{(1)}+z^{(2)}+z^{(3)}}{\sqrt{3}}, -\frac{z^{(1)}+z^{(2)}+z^{(3)}}{\sqrt{3}}\right)$ to replace $(z^{(1)}, z^{(2)}, z^{(3)})$.

Pseudo code: initialize K, T, S, sig, r, div, n, M.

precompute: dt = T/N nudt = (r-div - 0.5*sig^2)*dt;

sigsdt = sig*sqrt(dt), lnS = ln(S);

sum_CT=0;

sum_CT2=0;

```
for j=1, to M, do:
    lnst1 = lnS;
    lnst2 = lnS;                ind1 = 0;
                                ind2 = 0;
  for i=1 to n, do:
          ε = randn(1);
      { lnst1 = lnst1 + nudt + sigsdt * ε
      { lnst2 = lnst2 + nudt + sigsdt * (-ε)
        put barrier check condition: { if lnst1 ≥ H
                                     { then ind1 = 1;
                                     { if lnst2 ≥ H
                                     { then ind2 = 1;
  end for
  { St1 = exp(lnst1);
  { St2 = exp(lnst2);
    CT = 0.5 * ( max(0, St1-K) (1-ind1) + max(0, St2-K) (1-ind2) )
    sum_CT = sum_CT + CT;
    sum_CT2 = sum_CT2 + CT * CT;
end for.
```

Arith1 = lnst1
Arith2 = lnst2

Asian option;
Arith1 = Arith1 + lnst1;
Arith2 = Arith2 + lnst2;

remaining steps same as before.

how to simulate $Z \sim N(0,1)$, pdf is $f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $x \in \mathbb{R}$.

To simulate random variables, a fundamental method is the inverse transform method, or inverse CDF method.

CDF → cumulative distribution function.

Let $F_X(x)$ denote the CDF of a random variable X, then we have;

$\boxed{F_X(X)} \sim Uni(0,1)$.

$$P(F_X(X) \leq u) = P(X \leq F_X^{-1}(u)) = F_X(F_X^{-1}(u)) = u.$$

monotonically increasing property of the CDF.

$P(Y \leq u) = u$. $\Rightarrow Y \sim Uni(0,1)$

$Y = F_X(X)$.

Algorithm:

(1) First, simulate $U \sim Uni[0,1]$

(2) Calculate $F_X^{-1}(x)$. ← challenge

(3) Assign $X = F_X^{-1}(U)$. then X has the desired distribution.

For standard normal, $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$

quantile function

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\,du .$$

then $F_X^{-1}(x)$ is not known and not in closed-form.

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step (3) equivalent to $F_X(X)=U$. solving this root-finding problem.

You can use bisection, Newton-Raphson and so on ...

New approach: based on projection idea in function space.

Idea: project $F_X^{-1}(x)$ onto orthonormal basis.

$$F_X^{-1}(x) = \sum_{k=1}^{\infty} c_k \cdot \varphi_k(x).$$

$$X \overset{(d)}{=} F_X^{-1}(U) = \sum_{k=1}^{\infty} c_k \cdot \varphi_k(U).$$

$\{\varphi_k(x)\}_{k=1,\dots,\infty}$ are orthogonal system

$$\int \varphi_k(x)\varphi_j(x)\,dx = 0 \quad \text{for } k \neq j.$$

here $C_k = \int_{\mathbb{R}} F_X^{-1}(x) \cdot \varphi_k(x)\, dx$.

$F_X(x) = P(X \le x) \in [0,1]$.

change of variable;
$u = F_X(x)$.

$$C_k = \int_0^1 F_X^{-1}(u)\, \varphi_k(u)\, du.$$

$$= \int_{-\infty}^{\infty} F_X^{-1}(F_X(x))\, \varphi_k(F_X(x))\, dF_X(x).$$

$$= \int_{-\infty}^{\infty} x \cdot \varphi_k(F_X(x))\, dF_X(x).$$

$$= \int_{-\infty}^{\infty} x\, \varphi_k(F_X(x))\, f_X(x)\, dx.$$

here $F_X(X) \sim \mathrm{Uni}[0,1]$.

$$= E[X \cdot \varphi_k(F_X(X))].$$

If we know $F_X(x)$ and/or $f_X(x)$ then we can compute C_k.

normal case $C_k = \int_{-\infty}^{\infty} x \cdot \varphi_k\left(\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$.

$$C_k = -\sqrt{2\pi} \int_{-\infty}^{\infty} \varphi_k\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}du\right) d\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right)$$

integration by parts:

$$C_k = -\sqrt{2\pi} \cdot \varphi_k(\cdots) \cdot \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right)\Big|_{-\infty}^{\infty}$$

$$+ \cancel{\sqrt{2\pi}} \int \frac{1}{\cancel{\sqrt{2\pi}}} e^{-\frac{1}{2}x^2} \, d\, \varphi_k\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}du\right).$$

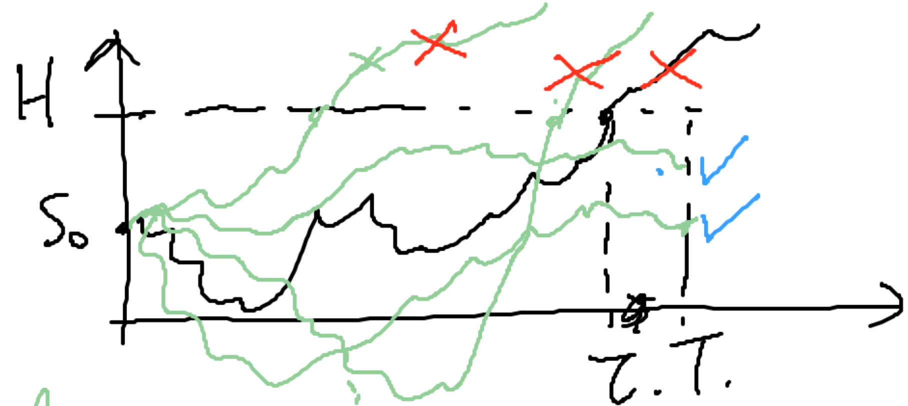
MC can be used for exotic path-dependent options.

- barrier options.
- Asian options (arithmetic average of asset prices).
- ⋮

American option: least squares Monte Carlo.

Barrier option: UOC:

$$(S_T-K)^+ \mathbb{1}_{\{\max_{0\le t\le T} S_t < H\}}.$$



Impose the barrier condition in your MC algorithm: general principle, discard those MC paths for which the barrier condition is breached.

Asian option: $\left(\frac{1}{m}\sum_{i=1}^{m} S_{t_i} - K\right)^+$.

Set $t_i, i=1,\dots,m$ as your time grid to carry out the Euler scheme!