Gist of Approximating Spread Option by using Bachelier's Model

Review of Bachelier's Model

When $S_1(t)$ and $S_2(t)$ are given by

$$
\begin{aligned}
dS_1(t) &= \mu S_1(t)dt + \sigma_1 dW_1(t) \\
dS_2(t) &= \mu S_2(t)dt + \sigma_2 dW_2(t)
\end{aligned}
$$

Set spread $S(t) = \alpha_2 S_2(t) - \alpha_1 S_1(t)$, by choosing $\sigma = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 - 2\rho\alpha_1\alpha_2\sigma_1\sigma_2}$ spread $S(t)$ can be written into the following SDE

$$dS(t) = \mu S(t)dt + \sigma dW(t)$$

which has closed-form solution

$$c_B(t,x;K,T,\sigma) = e^{-r(T-t)}\left(xe^{r(T-t)} - K\right)\Phi\left(\frac{xe^{r(T-t)} - K}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}}\right) \quad + e^{-r(T-t)}\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}\,\phi\left(\frac{x\cdots}{\sqrt{\frac{\sigma^2}{2r}\cdots}}\right.$$

Approximation option value under BS model using Bachelier's model

For BS Model

$$
\begin{aligned}
dS_1(t) &= S_1(t)\left[\mu_1 dt + \sigma_1 dW_1(t)\right] \\
dS_2(t) &= S_2(t)\left[\mu_2 dt + \sigma_2 dW_2(t)\right]
\end{aligned}
$$

Let u be the option value, define infinitesimal generator L to be

$$Lu = \frac{\partial u}{\partial t} + \frac{1}{2}\sigma_1^2 x_1^2 \frac{\partial^2 u}{\partial x_1^2} + \rho\sigma_1\sigma_2 x_1 x_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2 x_2^2 \frac{\partial^2 u}{\partial x_2^2} + \mu_1 x_1 \frac{\partial u}{\partial x_1} + \mu_2 x_2 \frac{\partial u}{\partial x_2}$$

u is the solution to

$$Lu = ru$$

Also, under Bachelier's model, we can define another infinitesimal generator $\bar{L}$

$$\bar{L}\bar{u} = \frac{\partial u}{\partial t} + \frac{1}{2}\sigma_1^2\frac{\partial^2 u}{\partial x_1^2} + \rho\sigma_1\sigma_2 x_1 x_2\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 u}{\partial x_2^2} + \mu x_1\frac{\partial u}{\partial x_1} + \mu x_2\frac{\partial u}{\partial x_2}$$

Even though we can use KM's method to do the following approximating

$$L\Delta u(t,x) + (L-\bar{L})\bar{u}(t,x) = r\Delta u(t,x)$$

$(L-\bar{L})\bar{u}(t,x)$ can be complex(Notice that under Bachelier's model, the coefficients for drift term is the same.)

solution(to be tested)

1. Using BS model as aux model, like approximating option price under Heston Model.
2. Still using Bachelier's model as aux model, and choose appropriate α_1, α_2 in spread $S(t) = \alpha_2 S_2(t) - \alpha_1 S_1(t)$, such that first-order partial derivative with respect to x_1 and x_2 can be canceled. This method may converge faster.

Transformation of Bachelier's model(solution 2)

Rewrite $S_1^*(t) = \alpha_1 S_1(t)$, $S_2^*(t) = \alpha_2 S_2(t)$, we have

$$dS_1^*(t) = \alpha_1 dS_1(t) = \mu\alpha_1 S_1(t)dt + \sigma_1\alpha_1 dW_1(t)$$
$$dS_2^*(t) = \alpha_2 dS_2(t) = \mu\alpha_2 S_2(t)dt + \sigma_2\alpha_2 dW_2(t)$$

Then spread $S(t) = S_2^*(t) - S_1^*(t)$ still follows the SED above and option value has the same closed-form solution.

Under our new setting, $(L-\bar{L})u$ can be more simple.

$\bar{L}$ is defined by

$$\bar{L}\bar{u} = \frac{\partial u}{\partial t} + \frac{1}{2}\alpha_1^2\sigma_1^2\frac{\partial^2 u}{\partial x_1^2} + \rho\alpha_1\alpha_2\sigma_1\sigma_2 x_1 x_2\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{1}{2}\alpha_2^2\sigma_2^2\frac{\partial^2 u}{\partial x_2^2} + \mu\alpha_1 x_1\frac{\partial u}{\partial x_1} + \mu\alpha_2 x_2\frac{\partial u}{\partial x_2}$$

Since α_1 and α_2 is flexible, we can set $\mu\alpha_1 = \mu_1$, $\mu\alpha_2 = \mu_2$, then $\alpha_1\sigma_1 = \sigma_1^*$, $\alpha_2\sigma_2 = \sigma_2^*$, partial derivatives to drift terms can be canceled.

$$dS_1^*(t) = \mu_1 S_1(t)dt + (\sigma_1^*)^2 dW_1(t)$$
$$dS_2^*(t) = \mu_2 S_2(t)dt + (\sigma_2^*)^2 dW_2(t)$$

$$(L-\bar{L})\bar{u}(t,x) = \frac{1}{2}(\sigma_1^2 x_1^2 - (\sigma_1^*)^2)\frac{\partial^2 u}{\partial x_1^2} + (\rho\sigma_1\sigma_2 x_1 x_2 - \rho\sigma_1^*\sigma_2^*)\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{1}{2}(\sigma_2^2 x_2^2 - (\sigma_2^*)^2)\frac{\partial^2 u}{\partial x_2^2}$$