HW 8. - Yifei Wang

Question 3.

① Full Conditional for σ^2

We have $p(Y|\theta,\sigma^2) \propto_{\sigma^2} (\frac{1}{\sigma^2})^{\frac{n}{2}} \exp(-\frac{1}{2\sigma^2}\sum_{ijk}^{n}(y_{ijk}-\theta_{ij})^2)$.

We choose prior: $\sigma^2 \sim InvGamma(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2})$.

posterior. $p(\sigma^2|Y,\theta) \propto_{\sigma^2} p(\sigma^2)\cdot p(Y|\theta,\sigma^2)$

$$\propto (\frac{1}{\sigma^2})^{\frac{n+\nu_0}{2}+1} \times \exp(-\frac{1}{\sigma^2}(\frac{\nu_0+\sigma_0^2+\sum_k^n(y_{ijk}-\theta_{ij})^2}{2}))$$

thus, $\sigma^2|Y,\theta \sim InvGamma(\frac{\nu_0+n}{2}, \frac{\nu_0\sigma_0^2+\sum_k(y_{ijk}-\theta_{ij})^2}{2})$

② Full Conditional for σ_μ^2

$$P(\mu|\sigma_\mu^2) \propto (\frac{1}{\sigma_\mu^2})^{\frac{1}{2}} \exp(-\frac{\mu^2}{2\sigma_\mu^2})$$

We choose prior: $\sigma_\mu^2 \sim InvGamma(\frac{\nu_{\mu_0}}{2}, \frac{\nu_{\mu_0}\sigma_{\mu_0}^2}{2})$.

posterior: $p(\sigma_\mu^2|\mu, \nu_{\mu_0}, \sigma_{\mu_0}^2) \propto p(\sigma_\mu^2)\, p(\mu|\sigma_\mu^2)$

$$\propto (\frac{1}{\sigma_\mu^2})^{\frac{\nu_{\mu_0}+1}{2}+1} \times \exp(-\frac{1}{\sigma_\mu^2}(\frac{\nu_{\mu_0}\sigma_{\mu_0}^2+\mu^2}{2}))$$

thus, $\sigma_\mu^2|\mu,\nu_{\mu_0},\sigma_{\mu_0}^2 \sim InvGamma(\frac{\nu_{\mu_0}+1}{2}, \frac{\nu_{\mu_0}\sigma_{\mu_0}^2+\mu^2}{2})$

③

③ Full Conditional for σ_a^2 and σ_b^2 and σ_{ab}^2.

$$p(a_1 \cdots a_{m_1} | \sigma_a^2) \propto_{\sigma_a^2} \left(\frac{1}{\sigma_a^2}\right)^{\frac{m_1}{2}} \exp\left(-\frac{1}{2\sigma_a^2}\sum_i^{m_1} a_i^2\right)$$

We choose prior: $\sigma_a^2 \sim \text{InvGamma}\left(\frac{\nu_{a_0}}{2}, \frac{\nu_{a_0}\sigma_{a_0}^2}{2}\right)$

posterior: $p(\sigma_a^2 | a_1 \cdots a_m, \nu_{a_0}, \sigma_{a_0}^2) \propto p(\sigma_a^2) \cdot p(a_1 \cdots a_m | \sigma_a^2)$

$$\propto \left(\frac{1}{\sigma_a^2}\right)^{\frac{m_1+\nu_{a_0}}{2}+1} \times \exp\left(-\frac{1}{\sigma_a^2}\left(\frac{\nu_{a_0}\sigma_{a_0}^2 + \sum_i^{m_1} a_i^2}{2}\right)\right)$$

thus, $\sigma_a^2 | a_1 \cdots a_{m_1}, \nu_{a_0}, \sigma_{a_0}^2 \sim \text{InvGamma}\left(\frac{\nu_{a_0}+m_1}{2}, \frac{\nu_{a_0}\sigma_{a_0}^2 + \sum_i^{m_1} a_i^2}{2}\right)$

Similarly for σ_b^2, we have

$$\sigma_b^2 | b_1 \cdots b_{m_2}, \nu_{b_0}, \sigma_{b_0}^2 \sim \text{InvGamma}\left(\frac{\nu_{b_0}+m_2}{2}, \frac{\nu_{b_0}\sigma_{b_0}^2 + \sum_j^{m_2} b_j^2}{2}\right)$$

Similarly for σ_{ab}^2 (the only difference is the size), we have

$$\sigma_{ab}^2 | (ab)_1 \cdots (ab)_{m_1 m_2}, \nu_{ab_0}, \sigma_{ab_0} \sim \text{InvGamma}\left(\frac{\nu_{ab_0} + m_1 m_2}{2}, \frac{\nu_{ab_0}\sigma_{ab_0}^2 + \sum_j^{m_2}\sum_i^{m_1} (ab)_{ij}^2}{2}\right)$$

④ Full Conditional for μ.

$$p(Y | \theta, \sigma^2, \mu) \propto_\mu \exp\left(-\frac{1}{2\sigma^2}\sum_k^n (y_{ijk} - \theta_{ij})^2\right)$$

$$\propto_\mu \exp\left(-\frac{1}{2\sigma^2}\sum_{k=1}^n (y_{ijk} - \mu - a_i - b_j - (ab)_{ij})^2\right)$$

$$\boxed{\begin{aligned} \text{let } r_{ijk} &= \mu + \varepsilon_{ijk} \\ &= y_{ijk} - (a_i + b_j + (ab)_{ij}) \end{aligned}}$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\sum_{k=1}^n (r_{ijk} - \mu)^2\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\sum_{k=1}^n (\mu^2 - 2\mu r_{ijk})\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\left(n\mu^2 - 2\mu\sum_{k=1}^n r_{ijk}\right)\right)$$

~~We chose prior : $\mu \sim N($~~

posterior : $P(\mu | Y, a, b, (ab), \sigma^2) \propto P(\mu) \cdot P(Y | \theta, \sigma^2, \cdot)$

$$\propto \exp\left(-\frac{\mu^2}{2\sigma_\mu^2}\right) \times \exp\left(-\frac{1}{2\sigma^2}\left(n\mu^2 - 2\mu \sum_{k=1}^{n} r_{ijk}\right)\right)$$

thus, $\mu | Y, a, b, (ab), \sigma^2 \sim N(\mu_n, \sigma_{\mu_n}^2)$

where
$$\begin{cases} \sigma_{\mu_n}^2 = \left(\frac{1}{\sigma_\mu^2} + \frac{n}{\sigma^2}\right)^{-1} \\ \mu_n = \sigma_{\mu_n}^2 \cdot \left(\frac{\sum_{k=1}^{n} r_{ijk}}{\sigma^2}\right) = \sigma_{\mu_n}^2 \frac{\sum_{k=1}^{n}(y_{ijk} - a_i - b_j - (ab)_{ij})}{\sigma^2} \end{cases}$$

⑤ Full Conditional for a_i, b_j, and $(ab)_{ij}$.

~~Similarly to above μ, we have~~

Let $n_{i\cdot}$ be the number of observations in group a_i

$n_{\cdot j}$ be the number of observations in group b_j.

n_{ij} be the number of observations in group $(ab)_{ij}$.

$$P(Y_{i,\cdot,k} | \theta, \sigma^2) \propto \exp\left(-\frac{1}{2\sigma^2} \sum_{k}^{n_{i\cdot}} (y_{ij,k} - \theta_{ijk})^2\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2} \sum_{k}^{n_{i,\cdot}} (y_{ijk} - \mu - b_j - (ab)_{ij} - a_i)^2\right)$$

$$\boxed{\begin{aligned} \text{let } r_{ijk}' &= a_i + \varepsilon_{ijk} \\ &= y_{ijk} - (\mu + b_j + (ab)_{ij}) \end{aligned}} \rightarrow \propto \exp\left(-\frac{1}{2\sigma^2} \left(n_{i\cdot} a_i^2 - 2a_i \sum_{k}^{n_{i\cdot}} r_{ijk}'\right)\right)$$

posterior : $P(a_i | Y, b, (ab), \sigma^2) \propto P(a_i) \times P(Y_{i,\cdot,k} | \theta, \sigma^2)$

$$\propto \exp\left(-\frac{a_i^2}{2\sigma_a^2}\right) \times \exp\left(-\frac{1}{2\sigma^2}\left(n_{i\cdot} a_i^2 - 2a_i \sum_{k}^{n_{i\cdot}} r_{ijk}'\right)\right)$$

thus, $a_i | Y, \underset{\mu}{b}, (ab), \sigma^2 \sim N(\underset{\mu_{a_{in}}}{\cancel{\ \ }_n}, \sigma_{a_{in}}^2)$

where $\begin{cases} \sigma_{a_{in}}^2 = \left(\frac{1}{\sigma_a^2} + \frac{n_{i\cdot}}{\sigma^2}\right)^{-1} \\ \mu_{a_{in}} = \sigma_{a_{in}}^2 \cdot \frac{\sum_k^{n_{i\cdot}} r_{ijk}'}{\sigma^2} = \sigma_{a_{in}}^2 \cdot \frac{\sum_k^{n_{i\cdot}} (y_{ijk} - b_j - (ab)_{ij} - \mu)}{\sigma^2} \end{cases}$

Similarly, we have this for b_j.

$$b_j \mid Y, a, (ab), \mu, \sigma^2 \sim N(\mu_{b_{jn}}, \sigma_{b_{jn}}^2)$$

where $\begin{cases} \sigma_{b_{jn}}^2 = \left(\frac{1}{\sigma_b^2} + \frac{n_{\cdot j}}{\sigma^2}\right)^{-1} \\ \mu_{b_{jn}} = \sigma_{b_{jn}}^2 \cdot \frac{\sum_k^{n_{\cdot j}} (y_{ijk} - a_i - (ab)_{ij} - \mu)}{\sigma^2} \end{cases}$

Similarly, we have this for $(ab)_{ij}$.

$$(ab)_{ij} \mid Y, a, b, \mu, \sigma^2 \sim N(\mu_{(ab)_{ijn}}, \sigma_{(ab)_{ijn}}^2)$$

where $\begin{cases} \sigma_{(ab)_{ijn}}^2 = \left(\frac{1}{\sigma_{ab}^2} + \frac{n_{ij}}{\sigma^2}\right)^{-1} \\ \mu_{(ab)_{ijn}} = \sigma_{(ab)_{ijn}}^2 \cdot \frac{\sum_k^{n_{ij}} (y_{ijk} - a_i - b_j - \mu)}{\sigma^2} \end{cases}$