STA 601/360 Homework 1

Yifei Wang

04 September, 2019

Exec 1. Hoff 2.3 Full conditionals: Let X, Y, Z be random variables with joint density (discrete or continuous) $p(x, y, z) \propto f(x, z)g(y, z)h(z)$. Show that

(a) $p(x|y, z) \propto f(x, z)$**, i.e.** $p(x|y, z)$ **is a function of** x **and** z**;**

Since

$$\begin{aligned}
p(x, y, z) &\propto f(x, z)g(y, z)h(z) \\
p(y, z) &= \int_X p(x, y, z)dx \\
&\propto \int_X f(x, z)g(y, z)h(z)dx \\
&\propto g(y, z)h(z) \int_X f(x, z)dx
\end{aligned}$$

Because $p(x|y, z)$ is a function of x, we have

$$\begin{aligned}
p(x|y, z) &= \frac{p(x, y, z)}{p(y, z)} \\
&= \frac{f(x, z)g(y, z)h(z)}{g(y, z)h(z) \int_X f(x, z)dx)} \\
&= \frac{f(x, z)}{\int_X f(x, z)dx} \\
&\propto f(x, z)
\end{aligned}$$

That is, $p(x|y, z)$ is a function of x and z.

(b) $p(y|x, z) \propto g(y, z)$**, i.e.** $p(y|x, z)$ **is a function of** y **and** z**;**

Since

$$\begin{aligned}
p(x, y, z) &\propto f(x, z)g(y, z)h(z) \\
p(x, z) &= \int_Y p(x, y, z)dy \\
&\propto \int_Y f(x, z)g(y, z)h(z)dy \\
&\propto f(x, z)h(z) \int_Y g(y, z)dy
\end{aligned}$$

Because $p(y|x, z)$ is a function of y, we have

$$
\begin{aligned}
p(y|x,z) &= \frac{p(x,y,z)}{p(x,z)} \\
&= \frac{f(x,z)g(y,z)h(z)}{f(x,z)h(z)\int_Y g(y,z)dy} \\
&= \frac{g(y,z)}{\int_Y g(y,z)dy} \\
&\propto g(y,z)
\end{aligned}
$$

That is, $p(y|x,z)$ is a function of y and z.

(c) X and Y are conditionally independent given Z.

First need to prove that $p(x \mid z) = p(x \mid y, z)$ and $p(y \mid z) = p(y \mid x, z)$

$$
\begin{aligned}
p(x \mid z) &= \frac{p(x,z)}{p(z)} \\
&= \frac{\int_Y p(x,y,z)dy}{\int_Y \int_X p(x,y,z)dydx} \\
&= \frac{\int_Y f(x,z)g(y,z)h(z)dy}{\int_Y \int_X f(x,z)g(y,z)h(z)dydx} \\
&= \frac{f(x,z)h(z)\int_Y g(y,z)\ dy}{h(z)\left(\int_Y g(y,z)dy\right)\left(\int_X f(x,z)dx\right)} \\
&= \frac{f(x,z)}{\int_X f(x,z)dx} \\
&= p(x \mid y,z)
\end{aligned}
$$

Similar to $p(x \mid z)$, we have

$$
\begin{aligned}
p(y \mid z) &= \frac{p(y,z)}{p(z)} \\
&= \frac{\int_X p(x,y,z)dx}{\int_Y \int_X p(x,y,z)dydx} \\
&= \frac{\int_X f(x,z)g(y,z)h(z)dx}{\int_Y \int_X f(x,z)g(y,z)h(z)dydx} \\
&= \frac{g(y,z)h(z)\int_X f(x,z)\ dx}{h(z)\left(\int_Y g(y,z)dy\right)\left(\int_X f(x,z)dx\right)} \\
&= \frac{g(y,z)}{\int_Y g(y,z)dx} \\
&= p(y \mid x,z)
\end{aligned}
$$

Therefore we get,

$$
\begin{aligned}
p(x,y \mid z) &= p(x \mid y,z)\,p(y \mid z) \\
&= p(x \mid z)\,p(y \mid z)
\end{aligned}
$$

This shows that X and Y are conditionally independentdent given Z.

Exec 2. Hoff 2.6 Conditional independence: Suppose events A and B are conditionally independent given C, which is written $A \perp B \mid C$. Show that this implies that $A^c \perp B \mid C$, $A \perp B^c \mid C$, and $A^c \perp B^c \mid C$, where A^c means "not A." Find an example where $A \perp B \mid C$ holds but $A \perp B \mid C^c$ does not hold.

Part (a)

If events A and B are conditionally independent given C, we have

$$p(A, B \mid C) = p(A \mid C)\, p(B \mid C)$$

For $A^c \perp B \mid C$ we have,

$$\begin{aligned} p(A^c, B \mid C) &= p(B \mid C) - p(A, B \mid C) \\ &= p(B \mid C) - p(A \mid C)\, p(B \mid C) \\ &= p(B \mid C)\,(1 - p(A \mid C)) \\ &= p(B \mid C)\, p(A^c \mid C) \end{aligned}$$

For $A \perp B^c \mid C$ we have,

$$\begin{aligned} p(A, B^c \mid C) &= p(A \mid C) - p(A, B \mid C) \\ &= p(A \mid C) - p(A \mid C)\, p(B \mid C) \\ &= p(A \mid C)\,(1 - p(B \mid C)) \\ &= p(A \mid C)\, p(B^c \mid C) \end{aligned}$$

For $A^c \perp B^c \mid C$ we have,

$$\begin{aligned} p(A^c, B^c \mid C) &= p(B^c \mid C) - p(A, B^c \mid C) \\ &= p(B^c \mid C) - p(B^c \mid C)\, p(A \mid C) \\ &= p(B^c \mid C)\,(1 - p(A \mid C)) \\ &= p(B^c \mid C)\, p(A^c \mid C) \end{aligned}$$

Part (b)

Suppose Chichi and Tina are playing a mind-reading game. Think about these 3 events:

- $A = \{$Chichi is thinking of the number 3$\}$
- $B = \{$Tina claims Chichi is thinking of the number 3$\}$
- $C = \{$Tina does not have extrasensory perception$\}$

In this case, when C is true, we know that both Chichi and Tina thinking of random number, so A and B are independent, which is $A \perp B \mid C$. However when C is not true, Tina knows exactly the number Chichi is thinking of. This shows that $A \perp B \mid C^c$ do not hold.

Exec 3. There are three coins in a bag; two fair coins (probability of heads = probability of tails) and one fake coin (probability of heads = 1).

Part a. You reach in and select one coin at random and throw it in the air. What is the probability that it lands on heads?

According to the rules of marginal probability,

$$\begin{aligned} P(\text{head}) &= P(\text{head} \mid \text{fake})P(\text{fake}) + P(\text{head} \mid \text{fair})P(\text{fair}) \\ &= 1 \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} \\ &= \frac{2}{3} \end{aligned}$$

Part b. You reach in and select one coin at random and throw it in the air and get heads. What is the probability that it is the fake coin?

According to Bayes rules,

$$\begin{aligned} P(\text{fake} \mid \text{head}) &= \frac{P(\text{head} \mid \text{fake})P(\text{fake})}{P(\text{head})} \\ &= \frac{1 \times \frac{1}{3}}{\frac{2}{3}} \\ &= \frac{1}{2} \end{aligned}$$

Exec 4. Monty Hall Problem! There are three doors on stage and a host hides a prize behind one of the doors and goats behind the other two doors. A contestant picks one of the three doors (say door 1) and then the game show host opens one of the remaining two doors (say door 3), making sure to reveal a goat. Should the contestant switch to door 2?

- Prior knowledge: The prize originally had 1/3 probability of being behind any of the three doors.
- The host's model: he knows which door has the prize. If he has to choose between a door with a goat and one with a prize, he chooses the one with the goat. If he has to choose between two doors with goats, he picks one of the doors with probability 1/2.

Supppose the prize is z and the host open door d. Then we have the prior of $p(z=1) = p(z=2) = p(z=3) = \frac{1}{3}$ and

$$P(y=3 \mid z=1) = \frac{1}{2} P(y=3 \mid z=2) = 1 P(y=3 \mid z=3) = 0$$

Then we have

$$\frac{P(z=2 \mid y=3)}{P(z=1 \mid y=3)} = \frac{P(y=3 \mid z=2)\, P(z=2)}{P(y=3 \mid z=1)\, P(z=1)} = \frac{1 \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3}} = 2$$

This means that in the above case, the probability of winning prize for switching to door 2 is twice of staying with door 1. Thus the contestant should switch to door 2 to achieve a higher winning chances.

Exec 5. Everyone: Read Why Isn't Everyone a Bayesian?