Derivation of full conditionals.

We have $\beta | Z \sim N_p(0, D_Z R D_Z)$

where $D_Z = diag[a_1 z_1, \cdots, a_p z_p]$ and $R = I$ in our case.

$$a_i = \begin{cases} 1, & z_i = 0 \\ c_i, & z_i = 1 \end{cases}$$

sampling model: $Y | X, \beta, \sigma^2 \sim N_n(X\beta, \sigma^2 I_n)$

① $\beta | Y, X, Z, \cancel{\beta}, \sigma^2$:

$$p(\beta | Y, X, Z, \beta, \sigma^2) \propto P(Y|X, \beta, \sigma^2)\, P(\beta | Z)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}(\beta^T X^T X\beta - 2\beta^T X^T Y)\right) \times \exp\left(-\frac{1}{2}\beta^T (D_Z R D_Z)^{-1}\beta\right)$$

$$\propto \exp\left(-\frac{1}{2}\left[\beta^T\left(\frac{X^T X}{\sigma^2} + (D_Z R D_Z)^{-1}\right)\beta - 2\beta^T X^T Y/\sigma^2\right]\right)$$

Thus, $\beta | Y, X, Z, \beta, \sigma^2 \sim N(\mu_\beta, \Sigma_\beta)$

where $\Sigma_\beta = (X^T X/\sigma^2 + (D_Z R D_Z)^{-1})^{-1}$, $\mu_\beta = \Sigma_\beta \cdot \frac{X^T Y}{\sigma^2}$

② $\sigma^2 | Y, X, Z, \beta$

$$P(\sigma^2 | Y, X, Z, \beta) \propto P(Y|X, \beta, \sigma^2)\, P(\sigma^2)$$

We recognize this is an inverseGamma distribution. s.t.

$\sigma^2 | Y, X, Z, \beta \sim \text{inverseGamma}\left(\frac{\nu_0 + n}{2}, \frac{\nu_0 s_0^2 + SSR(\beta)}{2}\right)$, where $SSR(\beta) = (Y - X\beta)^T (Y - X\beta)$

③ $z_i | Y, X, \beta, \sigma^2, z_{-i}$

We know that $P(z_i = 1 | Y, \beta, \sigma^2, z_{-i}) = P(z_i = 1 | \beta, \sigma^2, z_{-i}) = \frac{a}{a+b}$

where
$$a = P(\beta | z_{-i}, z_i = 1) \times P(\sigma^2 | z_{-i}, z_i = 1) \times P(z_{-i}, z_i = 1)$$
$$b = P(\beta | z_{-i}, z_i = 0) \times P(\sigma^2 | z_{-i}, z_i = 0) \times P(z_{-i}, z_i = 0)$$

Since σ^2 is independant of z and $P(z) = 2^{-p}$, and $R = I$ (so z_i independent of z_{-i})

we have $a = P(\beta_i | z_i) \sim N(0, D_Z D_Z[i, i])$. Same for b.