STA 601/360 Homework 10

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1. Hoff problem 10.1: Reflecting random walks

It is often useful in MCMC to have a proposal distribution which is both symmetric and has support only on a certain region. For example, if we know $\theta > 0$, we would like our proposal distribution $J(\theta_1|\theta_0)$ to have support on positive θ values. Consider the following proposal algorithm:

- **sample $\tilde{\theta} \sim$ uniform$(\theta_0 - \delta, \theta_0 + \delta)$;**
- **if $\tilde{\theta} < 0$, set $\theta = -\tilde{\theta}$;**
- **if $\tilde{\theta} \geq 0$, set $\theta = \tilde{\theta}$.**

In other words, $\theta_1 = |\tilde{\theta}|$. Show that the above algorithm draws samples from a symmetric proposal distribution which has support on positive values of θ. It may be helpful to write out the associated proposal density $J(\theta_1|\theta_0)$ under the two conditions $\theta_0 \leq \delta$ and $\theta_0 > \delta$ separately.

Let's discuss two conditions $0 < \theta_0 + \theta_1 < \delta$ and $\theta_0 + \theta_1 \geq \delta$ separately. When $\theta_0 + \theta_1 \geq \delta$ we have $0 < |\theta_0 - \delta| \leq \theta_1 = \tilde{\theta} \leq \theta_0 + \delta$. Thus we have

$$J(\theta_1|\theta_0) = J(\tilde{\theta}|\theta_0) = \frac{1}{2\delta} = J(\theta_0|\tilde{\theta}) = J(\theta_0|\theta_1)$$

When $0 < \theta_0 + \theta_1 \leq \delta$, we have $0 < \theta_1 < |\theta_0 - \delta|$ and $\theta_0 - \delta \leq \tilde{\theta} \leq \theta_0 + \delta$. So for θ_1 it is either $0 < \theta_1 = \tilde{\theta}$ or $\tilde{\theta} < 0 < \theta_1 = -\tilde{\theta}$, and similarly for θ_0. Thus we have

$$\begin{aligned} J(\theta_1|\theta_0) &= J(\theta_1 = \tilde{\theta}|\theta_0) + J(\theta_1 = -\tilde{\theta}|\theta_0) \\ &= \frac{1}{2\delta} + \frac{1}{2\delta} \\ &= J(\theta_0 = \tilde{\theta}|\theta_1) + J(\theta_0 = -\tilde{\theta}|\theta_1) \\ &= J(\theta_0|\theta_1) \end{aligned}$$

The above two conditions cover all poosibilities of the choice of $theta_0$ and θ_1, and under either conditions we prove that $J(\theta_1|\theta_0) = J(\theta_0|\theta_1)$. Thus we could say that the proposal algorithm is a symmetric distribution.

2. Math Problem

Consider the following sampling model

$$y_1, \ldots, y_n|\theta \sim p(y|\theta_1, \theta_2),$$

and prior

$$\theta_1 \sim g(\theta_1), \ \theta_2 \sim h(\theta_2)$$

where

$$\theta_1, \theta_2 \in \mathbb{R}$$

Write down in the simplest form possible the acceptance probability for Metropolis-Hastings based on the following proposals:

- **Full conditional:**

$$J(\theta_1^\star | \theta_1^{(s)}, \theta_2^{(s)}) = p(\theta_1 | y_1, \ldots, y_n, \theta_2^{(s)})$$

- **Prior:**

$$J(\theta_1^\star | \theta_1^{(s)}, \theta_2^{(s)}) = g(\theta_1)$$

- **Random walk:**

$$J(\theta_1^\star | \theta_1^{(s)}, \theta_2^{(s)}) = Normal(\theta_1^{(s)}, \delta^2)$$

We know that the acceptance probabiliy of Metropolis-Hastings algorithm (when sampling a new θ_1) takes the form of

$$r = \frac{p(\theta_1^\star, \theta_2^{(s)} | y_1, \ldots, y_n)}{p(\theta_1^{(s)}, \theta_2^{(s)} | y_1, \ldots, y_n)} \times \frac{J(\theta_1^{(s)} | \theta_1^\star, \theta_2^{(s)})}{J(\theta_1^\star | \theta_1^{(s)}, \theta_2^{(s)})}$$

- **Full conditional:**

$$\begin{aligned}
r &= \frac{p(\theta_1^\star, \theta_2^{(s)} | y_1, \ldots, y_n)}{p(\theta_1^{(s)}, \theta_2^{(s)} | y_1, \ldots, y_n)} \times \frac{J(\theta_1^{(s)} | \theta_1^\star, \theta_2^{(s)})}{J(\theta_1^\star | \theta_1^{(s)}, \theta_2^{(s)})} \\
&= \frac{p(\theta_2^{(s)} | y_1, \ldots, y_n)\, p(\theta_1^\star | y_1, \ldots, y_n, \theta_2^{(s)})}{p(\theta_2^{(s)} | y_1, \ldots, y_n)\, p(\theta_1^{(s)} | y_1, \ldots, y_n, \theta_2^{(s)})} \times \frac{p(\theta_1^{(s)} | y_1, \ldots, y_n, \theta_2^{(s)})}{p(\theta_1^\star | y_1, \ldots, y_n, \theta_2^{(s)})} \\
&= 1
\end{aligned}$$

- **Prior:**

$$\begin{aligned}
r &= \frac{p(\theta_1^\star, \theta_2^{(s)} | y_1, \ldots, y_n)}{p(\theta_1^{(s)}, \theta_2^{(s)} | y_1, \ldots, y_n)} \times \frac{J(\theta_1^{(s)} | \theta_1^\star, \theta_2^{(s)})}{J(\theta_1^\star | \theta_1^{(s)}, \theta_2^{(s)})} \\
&= \frac{p(y_1, \ldots, y_n | \theta_1^\star, \theta_2^{(s)})\, g(\theta_1^\star)\, h(\theta_2^{(s)})}{p(y_1, \ldots, y_n | \theta_1^{(s)}, \theta_2^{(s)})\, g(\theta_1^{(s)})\, h(\theta_2^{(s)})} \times \frac{g(\theta_1^{(s)})}{g(\theta_1^\star)} \\
&= \frac{p(y_1, \ldots, y_n | \theta_1^\star, \theta_2^{(s)})}{p(y_1, \ldots, y_n | \theta_1^{(s)}, \theta_2^{(s)})}
\end{aligned}$$

- **Random walk:**

$$\begin{aligned}
r &= \frac{p(\theta_1^\star, \theta_2^{(s)} | y_1, \ldots, y_n)}{p(\theta_1^{(s)}, \theta_2^{(s)} | y_1, \ldots, y_n)} \times \frac{J(\theta_1^{(s)} | \theta_1^\star, \theta_2^{(s)})}{J(\theta_1^\star | \theta_1^{(s)}, \theta_2^{(s)})} \\
&= \frac{p(y_1, \ldots, y_n | \theta_1^\star, \theta_2^{(s)})\, g(\theta_1^\star)\, h(\theta_2^{(s)})}{p(y_1, \ldots, y_n | \theta_1^{(s)}, \theta_2^{(s)})\, g(\theta_1^{(s)})\, h(\theta_2^{(s)})} \times \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\delta^2}(\theta_1^{(s)} - \theta_1^\star)^2\right)}{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\delta^2}(\theta_1^\star - \theta_1^{(s)})^2\right)} \\
&= \frac{p(y_1, \ldots, y_n | \theta_1^\star, \theta_2^{(s)})\, g(\theta_1^\star)}{p(y_1, \ldots, y_n | \theta_1^{(s)}, \theta_2^{(s)})\, g(\theta_1^{(s)})}
\end{aligned}$$