

stat4500notes

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Preface

This is a Lecture note written for the course STAT 4500: Machine Learning offered at Auburn University at Montgomery. The course uses the textbook James et al. (2023).

This is a book wrtteen by Quarto book.

To learn more about Quarto books visit <https://quarto.org/docs/books>.

1 Setting up Python Computing Environment

1.1 on Your own computer

1. you can either `git clone` or download a zipped file containing the codes from the site: https://github.com/intro-stat-learning/ISLP_labs/tree/stable. If downloaded a zipped file of the codes, unzipped the file to a folder, for example, named `islp`. If `git clone` (preferred, you need to have Git installed on your computer, check this link for how to install Git <https://ywanglab.github.io/stat1010/git.html>), do `git clone https://github.com/intro-stat-learning/ISLP_labs.git`

2. Download and install the following software:

- **Anaconda:** Download anaconda and install using default installation options
- **Visual Studio Code (VSC):** Download VSC and install
- start VSC and install VSC extensions in VSC: Python, Jupyter, intellicode
- (optional) **Quarto** for authoring: Download Quarto and install

3. Create a virtual environment named `islp` for Python. Start an anaconda terminal.

```
conda create -n islp python==3.10
conda activate islp
conda install pip ipykernel
pip install -r https://raw.githubusercontent.com/intro-stat-learning/ISLP_labs/v2.1.2/
```

4. You are ready to run the codes using VSC or `jupyter lab`.

- Activate the venv: `conda activate islp`
- Start a Anaconda terminal, navigate to the folder using the command `cd path/to/islp`, where `path/to/islp` means the file path to the folder `islp`, such as `\Users\ywang2\islp`. Start VSC by typing `code .` in the anaconda terminal.
- open/create a `.ipynb` or `.py` file.
- Select the kernel `islp`
- Run a code cell by pressing **Shift+Enter** or click the triangular play button.

- Continue to run other cells.
- After finishing using VSC, close the VSC, and deactivate the virtual environment in a conda terminal: `conda deactivate`

1.2 Use Google Colab

All you need is a Google account. Sign in your Google account in a browser, and navigate to Google Colab. Google Colab supports both Python and R. Python is the default engine. Change the engine to R in **Connect->change runtime type**. Then you are all set. Your file will be saved to your Google Drive or you can choose to send it to your GitHub account (recommended).

1.2.1 How to run a project file from your Google Drive?

Many times, when you run a python file in Colab, it needs to access other files, such as data files in a subdirectory. In this case, it would be convenient to have the same file structure in the Google Colab user home directory. To do this, you can use Google Drive to store your project folder, and then mount the Google Drive in Colab.

Let's assume the project folder name, `islp/`. Here are the steps:

1. `git clone` the project folder (example: `git clone https://github.com/intro-stat-learning/ISLP_1`) to your local folder. This step is only needed when you want to clone some remote repo from GitHub.
2. **Upload** the folder (ex: `islp`) to Google Drive.
3. **Open the file using Colab**. In Google Drive, double click on the `ipynb` file, example, `ch06.ipynb` (or click on the three dots on the right end, and choose **open with**, then **Google Colaboratory**), the file will be opened by Google Colab.
4. **Mount the Google Drive**. In Google Colab, with the specific file (example, `ch06.ipynb`) being opened, move your cursor to the first code cell, and then click on the folder icon (this should be the fourth icon) on the upper left border in the Colab browser. This will open the file explorer pane. Typically you would see a folder named `sample_data` shown. On the top of the pane, click on the Google Drive icon to mount the Google Drive. Google Colab will insert the following code below the cursor in your opened `ipynb` file:

```
from google.colab import drive
drive.mount('/content/drive')
```

Run this code cell by pressing **SHIFT+ENTER**, and follow the prompts to complete the authentication. Wait for ~10 seconds, your Google Drive will be mounted in Colab, and it will be displayed as a folder named **drive** in the file explorer pane. You might need to click on the **Refresh** folder icon to see the folder **drive**.

5. Open a new code cell below the above code cell, and type the code

```
%cd /content/drive/MyDrive/islp/
```

This is to change the directory to the project directory on the Google Drive. Run this code cell, and you are ready to run the file **ch06.ipynb** from the folder **islp** on your personal Google Drive, just like it's on your local computer.

2 Chapter 2: Statistical Learning

2.1 What is statistical learning?

For the input variable $X \in \mathbb{R}^p$ and response variable $Y \in \mathbb{R}$, assume that

$$Y = f(X) + \epsilon,$$

where ϵ is a random variable representing **irreducible error**. We assume ϵ is *independent* of X and $E[\epsilon] = 0$. ϵ may include *unmeasured variables* or *unmeasurable variation*.

Statistical learning is to estimate f using various methods. Denote the estimate by \hat{f} .

- regression problem: when Y is a continuous (quantitative) variable. In this case $f(x) = E(Y|X = x)$ is the population *regression* function, that is, regression finds a conditional expectation of Y .
- classification problem: when Y only takes small number of discrete values, i.e., qualitative (categorical).

Logistic regression is a classification problem, but since it estimates class probability, it may be considered as a regression problem.

- supervised learning: training data $\mathcal{T}r = \{(x_i, y_i) : i \in \mathbb{Z}_n\}$: linear regression, logistic regression
- unsupervised learning: when only x_i are available. clustering analysis, PCA
- semi-supervised learning: some data with labels (y_i), some do not.
- reinforcement learning: learn a state-action policy function for an agent to interacting with an environment to maximize a reward function.

2.2 Why estimate f ?

We can use estimated \hat{f} to

- make predictions for a new X ,

$$\hat{Y} = \hat{f}(X).$$

The prediction error may be quantified as

$$E[(Y - \hat{Y})^2] = (f(X) - \hat{f})^2 + \text{Var}[\epsilon].$$

The first term of the error is *reducible* by trying to improve \hat{f} , where we assume f , \hat{f} and X are fixed.

- make inference, such as
 - Which predictors are associated with the response?
 - what is the relationship between the response and each predictor?
 - is the assumed relationship adequate? (linear or more complicated?)

2.3 How to estimate f

We use obtained observations called **training data** $\{(x_k, y_k) : k \in \mathbb{Z}_n\}$ to train an algorithm to obtain the estimate \hat{f} .

- Parametric methods: first assume there is a function form (shape) with some parameters. For example, a linear regression model with two parameters. Then use the *training data* to **train** or **fit** the model to determine the values of the parameters.

Advantages: simplify the problem of fit an arbitrary function to estimate a set of parameters.

Disadvantages: may not be flexible unless with large number of parameters and/or complex function shapes.

Example: linear regression,

- Non-parametric methods: Do not explicitly assume a function form of f . They seek to estimate f directly using data points, can be quite flexible and accurate.

****Disadvantage:** need large number of data points

Example: KNN (but breakdown for higher dimension. Typically only for $p \leq 4$), spline fit.

2.4 How to assess model accuracy

For regression problems, the most commonly used measure is the *mean squared error* (MSE), given by

$$MSE = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{f}(x_i))^2$$

For classification problems, typically the following **error rate** (classifications error) is calculated:

$$\frac{1}{n} \sum_{i=1}^n I(y_i \neq \hat{y}_i)$$

The accuracy on a training set can be arbitrarily increased by increasing the model flexibility. However, we are in general interested in the error on the test set rather on the training set, the model accuracy should be assessed on a test set.

Flexible models tend to overfit the data, which essentially means they follow the error or *noise* too closely in the training set, therefore cannot be generalized to *unseen cases* (test set).

2.5 Model Selection:

No free lunch theorem

There is no single best method for all data sets, which means some method works better than other methods for a particular dataset. Therefore, one needs to perform model selections. Here are some principles.

2.5.1 Trade-off between Model flexibility and Model Interpretability

More flexible models have higher *degree of freedom* and are less interpretable because it's difficult to interpret the relationship between a predictor and the response.

LASSO is less flexible than linear regression. GAM (generalized additive model) allows some non-linearity. Full non-linear models have higher flexibility, such as *bagging*, *boosting*, *SVM*, etc.

When *inference* is the goal, then there are advantages to using simple and less flexible models for interpretability.

When *prediction* is the main goal, more flexible model may be a choice. But sometimes, we obtain more accurate prediction using a simpler model because the underlying dataset has a simpler structure. Therefore, it is not necessarily true that a more flexible model has a higher prediction accuracy.

Occam's Razor: Among competing hypotheses that perform equally well, the one with the fewest assumptions should be selected.

2.5.2 Model Selection: the Bias-Variance Trade-off

As the model flexibility increases, the training MSE (or error rate for classification) will decrease, but the test MSE (error rate) in general will not and will show a characteristic **U-shape**. This is because when evaluated at a test point x_0 , the expected test MSE can be decomposed into

$$E[(y_0 - \hat{f}(x_0))^2] = \text{Var}[\hat{f}(x_0)] + (\text{Bias}(\hat{f}(x_0)))^2 + \text{Var}[\epsilon]$$

where the expectation is over different \hat{f} on a different training set or on a different training step if the training process is stochastic, and

$$\text{Bias}(\hat{f}(x_0)) = E[\hat{f}(x_0)] - f(x_0)$$

To obtain the least test MSE, one must trade off between variance and bias. Less flexible model tends to have higher bias, and more flexible models tend to have higher variance. An optimal flexibility for the least test MSE varies with different data sets. Non-linear data tends to require higher optimal flexibility.

2.6 Bayes Classifier

It can be shown that Bayes Classifier minimizes the classification test error

$$\text{Ave}(I(y_0 \neq \hat{y}_0)).$$

A Bayes Classifier assigns a test observation with predictor x_0 to the class for which

$$\Pr(Y = j | X = x_0)$$

is largest. Its error rate is given by

$$1 - E[\max_j \Pr(Y = j | X)]$$

where the expectation is over X . The Bayes error is analogous to the irreducible error ϵ .

Bayes Classifier is not attainable as we do not know $\Pr(Y|X)$. We only can estimate $\Pr(Y|X)$. One way to do this is by KNN. KNN estimate the conditional probability simply with a majority vote. The flexibility of KNN increases as $1/K$ increases with $K = 1$ being the most flexible KNN. The training error is 0 for $K = 1$. A suitable K should be chosen for an appropriate trade off between bias and variance. The KNN classifier will classify the test point x_0 based on the probability calculated from the k nearest points. KNN regression on the other hand will assign the test point x_0 the average value of the k nearest neighbors.

2.7 Homework (* indicates optional):

- Conceptual: 1,2,3,4*,5,6,7
- Applied: 8, 9*, 10*

2.8 Code Gist

2.8.1 OS

```
import os
os.chdir(path) # change dir
```

2.8.2 Python:

Concatenation using +

```
"hello" + " " + "world" # 'hello world'
[3,4,5] + [4,9,7] # [3,4,5, 4,9,7]
```

String formatting using `string.format()`

```
print('Total is: {0}'.format(total))
```

zip to loop over a sequence of tuples

```
for value, weight in zip([2,3,19],
                          [0.2,0.3,0.5]):
    total += weight * value
```

2.8.3 Numpy

2.8.3.1 Numpy functions:

`np.sum(x)`, `np.sqrt(x)` (entry wise). `x**2` (entry wise power), `np.corrcoef(x,y)` (find the correlation coefficient of array `x` and array `y`)

`np.mean(axis=None)`: axis could be `None` (all entries), 0(along row), 1(along column)

`np.var(x, ddof=0)`, `np.std(x, ddof=0)`, # Note both `np.var` and `np.std` accepts an argument `ddof`, the divisor is `N-ddof`.

`np.linspace(-np.pi, np.pi, 50)` # start, end, number of points 50

`np.multiply.outer(row,col)` # calculate the product over the mesh with vectors `row` and `col`.

`np.zeros(shape or int, dtype)` #eg: `np.zeros(5,bool)`

`np.ones(Boston.shape[0])`

`np.all(x)`, `np.any(x)`: check if all or any entry of `x` is true.

`np.unique(x)`: find unique values in `x`. `np.isnan(x)`: return a boolean array of `len(x)`.

`np.isnan(x).mean()`: find the percentage of `np.nan` values in `x`.

2.8.3.2 Array Slicing and indexing

`np.arange(start, stop, step)` # numpy version of `range`

`x[slice(3:6)]` # equivalent to `x[3:6]`

Indexing an array using `[row, col]` format. If `col` is missing, then index the entire rows. `len(row)` must be equal to `len(col)`. Otherwise use iterative indexing or use `np.ix_(x_idx, y_idx)` function, or use Boolean indexing, see below.

`A[1,2]`: index entry at row 1 and col 2 (recall Python index start from 0)

`A[[1,3]]` # row 1 and 3. Note the outer `[]` is considered as the operator, so only row indices

`A[:, [0,2]]` # cols 0 and 2

`A[[1,3], [0,2,3]]` # entry `A[1,0]` and `A[3,2]`

`A[1:4:2, 0:3:2]` # entries in rows 1 and 3, cols 0 and 2

`A[[1,3], [0,2,3]]` # syntax error

instead one can use the following two methods

`A[[1,3]][:,[0,2]]` # iterative subsetting

`A[np.ix_([1,3],[0,2,3])]` # use `.ix_` function to create an index mesh

`A[keep_rows, keep_cols]` # `keep_rows`, `keep_cols` are boolean arrays of the same length of rows

`A[np.ix_([1,3],keep_cols)]` # `np.ix_()` can be applied to mixture of integer array and boolean a

2.8.3.3 Random numbers and generators

```
np.random.normal(loc=0.0, scale=1.0,size=None) # size can be an integer or a tuple.
#
rng = np.random.default_rng(1303) # set random generator seed
rng.normal(loc=0, scale=5, size=2) #
rng.standard_normal(10) # standard normal distribution of size 10
rng.choice([0, np.nan], p=[0.8,0.2], size=A.shape)
```

2.8.3.4 Numpy array attributes

`.dtype`, `.ndim`, `.shape`

2.8.3.5 Numpy array methods

```
x.sum(axis=None) (equivalent to np.sum(x)), x.T (transpose),
x.reshape((2,3)) # x.reshape() is a reference to x.
x.min(), x.max()
```

2.8.4 Graphics

2.8.4.1 2-D figure

```
# Using the subplots + ax methods
fig, ax = subplots(nrows=2, ncols=3, figsize=(8, 8))
# explicitly name each axis in the grid
fig, ((ax1, ax2), (ax3, ax4)) = plt.subplots(nrows=2, ncols=2, figsize=(10,10))

ax[0,1].plot(x, y,marker='o', 'r--', linewidth=3); #line plot. `` suppresses the text output
ax.plot([min(fitted),max(fitted)],[0,0],color = 'k',linestyle = ':', alpha = .3)
ax.scatter(x, y, marker='o'); #scatter plot
ax.scatter(fitted, residuals, edgecolors = 'k', facecolors = 'none')
ax.set_xlabel("this is the x-axis")
ax.set_ylabel("this is the y-axis")
ax.set_title("Plot of X vs Y");
axes[0,1].set_xlim([-1,1]) # set x_lim. similarly `set_ylim()`

fig = ax.figure # get the figure object from an axes object
fig.set_size_inches(12,3) # access the fig object to change fig size (width, height)
fig # re-render the figure
```

```
fig.savefig("Figure.pdf", dpi=200); #save a figure into pdf. Other formats: .jpg, .png, etc
```

2.8.4.2 Contour and image

```
fig, ax = subplots(figsize=(8, 8))
x = np.linspace(-np.pi, np.pi, 50)
y = x
f = np.multiply.outer(np.cos(y), 1 / (1 + x**2))
ax.contour(x, y, f, levels=None); # nombre of levels. if None, automatically choose
ax.imshow(f); # heatmap colorcoded by f
```

2.8.5 Pandas

2.8.5.1 loading data

```
pd.read_csv('Auto.csv') # read csv
pd.read_csv('Auto.data',
            na_values=['?'], #specifying the na_values in the datafile.
            delim_whitespace=True) # read whitespaced text file
pd.read_csv('College.csv', index_col=0) # use column `0` as the row labels
```

2.8.5.2 Pandas Dataframe attributes and methods

```
Auto.shape
Auto.columns # gets the list of column names
Auto.index #return the index (labels) objects
Auto['horsepower'].to_numpy() # convert to numpy array
Auto['horsepower'].sum()

Auto.dropna() # drop the rows containing na values.
df.drop('B', axis=1, inplace=True) # drop a column 'B' inplace.
#equivalent to df.drop(columns=['B'], inplace=True)
df.drop(index=['Ohio', 'Colorado']) #equivalent to: df.drop(['Ohio', 'Colorado'], axis=0)
auto_df.drop(auto_df.index[10:86]) # drop rows with index[10:86] not including 86

Auto.set_index('name')# rename the index using the column 'name'.

pd.Series(Auto.cylinders, dtype='category') # convert the column `cylinders` to 'category` d
# the convertison can be done using `astype()` method
Auto.cylinders.astype('category')
Auto.describe() # statistics summary of all columns
```

```

Auto['mpg'].describe() # for selected columns

college.rename({'Unnamed: 0': 'College'}, axis=1): # change column name,
# alternativie way
college_df.rename(columns={college_df.columns[0] : "College"}, inplace=True) #

college['Elite'] = pd.cut(college['Top10perc'], # binning a column
                        [0,0.5,1], #bin edges
                        labels=['No', 'Yes'], # bin labels (names)
                        right=True, # True: right-inclusive (default) for each bin ( ]; False: left-inclusive ( [ );
                        )
college['Elite'].value_counts() # frequency counts
auto.columns.tolist() # equivalent to auto.columns.format() (rarely used)

```

2.8.5.3 Selecting rows and columns

Select Rows:

```

Auto[:3] # the first 3 rows.
Auto[Auto['year'] > 80] # select rows with boolean array
Auto_re.loc[['amc rebel sst', 'ford torino']] #label_based row selection
Auto_re.iloc[[3,4]] #integer-based row selection: rows 3 and 4 (index starting from 0)

```

Select Columns

```

Auto['horsepower'] # select the column 'horsepower', resulting a pd.Series.
Auto[['horsepower']] #obtain a dataframe of the column 'horsepower'.
Auto_re.iloc[:, [0,2,3]] # integer-based selection
auto_df.select_dtypes(include=['int16', 'int32']) # select columns by dtype

```

Select a subset

```

Auto_re.iloc[[3,4], [0,2,3]] # integer-based
Auto_re.loc['ford galaxie 500', ['mpg', 'origin']] #label-based
Auto_re.loc[Auto_re['year'] > 80, ['weight', 'origin']] # mix boolean indexing with labels

Auto_re.loc[lambda df: (df['year'] > 80) & (df['mpg'] > 30),
            ['weight', 'origin']]
] # using labmda function with loc[]

```

2.8.5.4 Pandas graphics

Without using subplots to get axes and figure objects

```
ax = Auto.plot.scatter('horsepower', 'mpg') #scatter plot of 'horsepower' vs 'mpg' from the c
ax.set_title('Horsepower vs. MPG');
fig = ax.figure
fig.savefig('horsepower_mpg.png');

plt.gcf().subplots_adjust(bottom=0.05, left=0.1, top=0.95, right=0.95) #in percentage of the
ax1.fig.suptitle('College Scatter Matrix', fontsize=35)
```

Using subplots

```
fig, axes = subplots( ncols=3, figsize=(15, 5))
Auto.plot.scatter('horsepower', 'mpg', ax=axes[1]);
Auto.hist('mpg', ax=ax);
Auto.hist('mpg', color='red', bins=12, ax=ax); # more customized
```

Boxplot using subplots

```
Auto.cylinders = pd.Series(Auto.cylinders, dtype='category') # needs to convert the `cylinder
fig, ax = subplots(figsize=(8, 8))
Auto.boxplot('mpg', by='cylinders', ax=ax);
```

Scatter matrix

```
pd.plotting.scatter_matrix(Auto); # all columns
pd.plotting.scatter_matrix(Auto[['mpg',
                                'displacement',
                                'weight']]); # selected columns
```

#Alternatively with sns.pairplot

Sns Graphic

```
# Scatter matrix
ax1 = sns.pairplot(college_df[college_df.columns[0:11]])

# Boxplot
sns.boxplot(ax=ax, x="Private", y="Outstate", data=college_df)
```


3 Chapter 3: Linear Regression

Linear regression is a simple supervised learning assuming a linear relation between Y and X . When there is only one predictor, it's a **simple linear regression**. When there are more than one predictors, it's called **multiple linear regression**. Note *multivariate regression* refer to the Y variable is a vector.

3.1 Simple Linear Regression

Assumes the *population regression line* model

$$Y = \beta_0 + \beta_1 X + \epsilon,$$

where, β_0 is the *expected* value of Y when $X = 0$, and β_1 is the *average* change in Y with a one-unit increase in X . ϵ is a “catch all” error term.

After training using the training data, we can obtain the parameter estimates $\hat{\beta}_0$ and $\hat{\beta}_1$. The we can obtain the prediction for x given by the *least square line*:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

The error at a data point x_i is given by $e_i = y_i - \hat{y}_i$, and the *residual sum of squares* (RSS) is

$$\text{RSS} = e_1^2 + \dots + e_n^2.$$

One can use the least square approach to minimize RSS to obtain

$$\hat{\beta}_1 = \frac{(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = r_{xy} \frac{\sigma_y}{\sigma_x}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

where, $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$ and $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$. If we assume the data matrix X is demeaned, then $\hat{\beta}_0 = \bar{y}$. and the correlation

$$r_{xy} = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} = \frac{(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}. \quad (3.1)$$

is the normalized covariance. Note $-1 \leq r_{xy} \leq 1$. When there is no intercept, that is $\beta_0 = 0$, then

$$\hat{y}_i = x_i \hat{\beta} = \sum_{i=1}^n a_i y_i$$

where,

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

That is, the fitted values are linear combinations of the response values when there is no intercept.

3.1.1 Assessing the accuracy of the coefficients

Let $\sigma^2 = \text{Var}(\epsilon)$, that is, σ^2 is the variance of Y , (estimated by $\sigma^2 \approx \text{RSE} = \text{RSS}/(n-p-1)$.) Assume each observation have *common variance* (homoscedasticity) and are *uncorrelated*, then the standard errors under repeated sampling

$$(\text{SE}[\hat{\beta}_1])^2 = \frac{1}{\sigma_x^2} \cdot \frac{\sigma^2}{n}$$

$$(\text{SE}[\hat{\beta}_0])^2 = \left[1 + \frac{\bar{x}^2}{\sigma_x^2} \right] \cdot \frac{\sigma^2}{n}$$

- when x_i are more spread out (with large σ_x^2), then $\text{SE}[\hat{\beta}_1]$ is small. This is because there are more *leverage* (of x values) to estimate the slope.
- when $\bar{x} = 0$, then $\text{SE}[\hat{\beta}_0] = \text{SE}[\bar{y}]$. In this case, $\hat{\beta}_0 = \bar{y}$.

Standard errors are used to construct CI and perform hypothesis test for the estimated $\hat{\beta}_0$ or $\hat{\beta}_1$. Under the assumption of **Gaussian error**, One can construct the CI of significance level α (e.g., $\alpha = 0.05$) as

$$\hat{\beta}_j = [\hat{\beta}_j - t_{1-\alpha/2, n-p-1} \cdot \text{SE}[\hat{\beta}_j], \hat{\beta}_j + t_{1-\alpha/2, n-p-1} \cdot \text{SE}[\hat{\beta}_j]]$$

Where $j = 0, 1$. Large interval including zero indicates β_j is not statistically significant from 0. When n is sufficient large, $t_{0.975, n-p-1} \approx 2$. With the standard errors of the coefficients, one can also perform **hypothesis test** on the coefficients. For $j = 0, 1$,

$$H_0 : \beta_j = 0$$

$$H_A : \beta_j \neq 0$$

The t -statistic of degree $n - p - 1$, given by

$$t = \frac{\hat{\beta}_j - 0}{\text{SE}[\hat{\beta}_j]}$$

shows how far away $\hat{\beta}_j$ is away from zero, normalized by its error $\text{SE}[\hat{\beta}_j]$. One can then compute the p -value corresponding to this t and test the hypothesis. Small p -value indicates **strong** relationship.

3.2 Multiple Linear Regression

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \epsilon.$$

The estimate of the coefficients $\hat{\beta}_j$, $j \in \mathbb{Z}_{p+1}$ are found by using the same least square method to minimize RSS. we interpret β_j as the *expected* (average) effect on Y with one unit increase in X_j , **holding all other predictors fixed**. This interpretation is based on the assumptions that *the predictors are uncorrelated*, so *each predictor can be estimated and tested separately*. When there are correlations among predictors, the variance of all coefficients tends to increase, sometimes dramatically, and the previous interpretation becomes hazardous because when X_j changes, everything else changes.

3.2.1 Model Assumption

- linearity: Y is linear in X . The change in Y associated with one unit of change in X_j is constant, regardless of the value of X_j . This can be examined visually by plotting the *residual plot* (e_i vs. x_i for $p = 1$ or e_i vs \hat{y}_i for multiple regression). If the linear assumption is true, then the residual plot should not exhibit obvious pattern. If there is a nonlinear relationship suggested by the residual plot, then a simple approach is to include transformed X , such as $\log X$, \sqrt{X} , or X^2 .
- additive: The association between X_j and Y is independent of other predictors.
- Errors ϵ_i are uncorrelated. This means ϵ_i provides no information for ϵ_{i+1} . Otherwise (for example, frequently observed in a time series, where error terms are positively correlated, and *tracking* is observed in the residuals, i.e., adjacent error terms take similar values), the estimated standard error will tend to be underestimated, hence leading less confidence in the estimated model.
- Homoscedasticity: $\text{Var}(\epsilon_i) = \sigma^2$. The error terms have constant variance. If not (heteroscedasticity), one may use transformed Y , such as \sqrt{Y} , or $\log(Y)$ to mitigate this; or use *weighted least squares* if it's known that for example $\sigma_i^2 = \sigma^2/n_i$.

- Non-colinearity: two variables are colinear if they are highly correlated with each other. Co-linearity causes a great deal of *uncertainty* in the coefficient estimates, that is, reducing the accuracy of the coefficient estimates, thus cause the standard error of β_j to grow, and hence smaller t -statistic. As a result, we may fail to reject $H_0 : \beta_j = 0$. This in turn means the power of Hypothesis test, the probability of correctly detecting a *non-zero* coefficient is reduced by colinearity. To detect colinearity,
 - use the correlation matrix of predictors. Large value of the matrix in absolute value indicates highly correlated variable pairs. But this approach cannot detect *multicollinearity*.
 - Use VIF (Variance inflation factor, $VIF \geq 1$) to detect multicollinearity. It is possible for colinearity exists between three or more variables even if no pair of variables has a particularly high correlation. This is the *multicollinearity* situation.

VIF is the ratio of the variance of $\hat{\beta}_j$ when fitting the full model divided by the variance of $\hat{\beta}_j$ if fit on its own. It can be calculated by

$$VIF(\hat{\beta}_j) = \frac{1}{1 - R_{X_j|X_{-j}}^2}$$

Where $R_{X_j|X_{-j}}^2$ is the R^2 from a regression of X_j onto all of the other predictors. A VIF value exceeds 5 or 10 (i.e., $R_{X_j|X_{-j}}^2$ close to 1) indicates colinearity.

To remedy a colinearity problem:

- drop a redundant variable (variables with colinearity should have similar VIF values.)
- Combine the colinear variables into a single predictor, e.g., taking the average of the standardized versions of those variables.

Claims of causality should be avoided for observational data.

3.2.2 Assessing existence of linear relationship

- test Hypothesis (test if there is a linear relationship between the response and predictors)

$$H_0 : \beta_1 = \beta_2 = \dots = \beta_p = 0$$

$$H_a : \text{at least one } \beta_j \text{ is non-zero.}$$

using F -statistic

$$F = \frac{SSB/df(B)}{SSW/df(W)} = \frac{(TSS - RSS)/p}{RSS/(n - p - 1)} \sim F_{p, n-p-1}$$

If H_0 is true, $F \approx 1$; if H_a is true, $F \gg 1$. F -statistic adjust with p . Note that one **cannot conclude** if an individual t -statistic is significant, then there is at least one predictor is related to the response, especially when p is large. This is related to *multiple testing*. The reason is that when p is large, there is α (eg 5%) chance that a predictor will have a small p -value by chance. When $p > n$, F -statistic cannot be used.

If the goal is to test that a particular subset of q of the coefficients are zero, that is, (for convenience, we put the q variables chosen at the end of the variable list)

$$H_0 : \beta_{p-q+1} = \beta_{p-q+2} = \dots = \beta_p = 0 \quad (3.2)$$

In this case, use

$$F = \frac{(\text{RSS}_0 - \text{RSS})/q}{\text{RSS}/(n-p-1)} \sim F_{q, n-p-1}$$

where, RSS_0 is the residual sum of squares of a second model that uses all variables *except* those last q variables. When $q = 1$, F -statistic in Equation ?? is the square of the t -statistic of that variable. The t -statistic reported in a regression model gives the *partial effect* of adding that variable, while holding other variables fixed.

3.2.3 Assess the accuracy of the future prediction

- confidence interval: Indicate how far away $\hat{Y} = \hat{f}(X)$ is from the population average $f(X)$ because the coefficients $\hat{\beta}_j$ are estimated, It quantifies *reducible error* around the predicted average response $\hat{f}(X)$, does-not include ϵ .
- prediction interval: Indicate how far away $\hat{Y} = \hat{f}(X)$ is from Y . predict an individual response $Y \approx \hat{f}(X) + \epsilon$. Prediction interval is always wider than the confidence interval, because it includes *irreducible error* ϵ .

3.2.4 Assessing the overall accuracy of the model

- RSE. To this end, first define the *lack of fit* measure **Residual Standard Error**

$$\text{RSE} = \sqrt{\frac{1}{n-p-1} \text{RSS}} = \sqrt{\frac{1}{n-p-1} \sum_{i=1}^n (y_i - \hat{y}_i)^2} \approx \sigma = \sqrt{\text{Var}(\epsilon)}$$

It is the *average amount* in \hat{Y} that a response deviates from the *true regression line* $(\beta_0 + \beta_1 X)$. Note, RSE can increase with more variables if the decrease of RSS doesnot offset the increase of p .

- Approach 2: Using *R-squared* (fraction of variance in Y explained by X), which is independent of the scale of Y , and $0 \leq R^2 \leq 1$:

$$R^2 = \frac{\text{TSS} - \text{RSS}}{\text{TSS}} = 1 - \frac{\text{RSS}}{\text{TSS}}$$

where, $\text{TSS} = \sum_{i=1}^n (y_i - \bar{y})^2$. When R^2 is near 0 indicates that 1) either the linear model is wrong 2) or the error variance σ^2 is high, or both. R^2 measures the linear relationship between X and Y . If computed on the training set, when adding more variables, the RSS always decrease, hence R^2 will always increase.

For simple linear regression, $R^2 = r_{xy}^2$, where the *sample correlation* measures the linear relationship between variables X and Y . See the formula r_{xy} above Equation ???. For multiple linear regression, $R^2 = (\text{Cor}(Y, \hat{Y}))^2$. The fitted linear model maximizes this correlation among all possible linear models.

3.3 Model Selection/Variable Selections: balance training errors with model size

- **All subsets (best subsets) regression:** compute the least square fit for all 2^p possible subsets and then choose among them based on certain criterion that balance training error and model size
- **Forward selection:** Start from the *null model* that only contains β_0 . Then find the best model containing one predictor that minimizing RSS. Denote the variable by β_1 . Then continue to find the best model with the lowest RSS by adding one variable from the remaining predictors, and so on. Continue until some stopping rule is met: e.g., when all remaining variables have a p -value greater than some threshold.
- **Backward selection:** start with all variables in the model. Remove the variable with the largest p -value (least statistically significant). The new $(p - 1)$ model is fit, and remove the variable with the largest p -value. Continue until a stopping rule is satisfied, e.g., all remaining variables have p -value less than some threshold.
- **Mixed selection:** Start with forward selection. Since the p -value for variables can become larger as new predictors are added, at any point if the p -value of a variable in the model rises above a certain threshold, then remove that variable. Continue to perform these forward and backward steps until all variables in the model have a sufficiently low p -value, and all variables outside the model would have a large p -value if added to the model.

Backward selection cannot be used if $p > n$. Forward selection can always be used, but might include variables early that later become redundant. Mixed selection can remedy this problem.

- **others** (Chapter 6): including Mallows's C_p , AIC (Akaike Information Criterion), BIC, adjusted R^2 , Cross-validation, test set performance.
- **not valid**: we could look at individual p -values, but when the number of variables p is large, we likely to make a false discoveries.

3.4 Handle categorical variables (factor variables)

For a categorical variable X_i with m levels, create one fewer dummy variables ($x_{ij}, 1 \leq j \leq m-1$). The level with no dummy variable is called the *baseline*. The coefficient corresponding to a dummy variable is the expected difference in change in Y when compared to the baseline, while holding other predictors fixed.

3.5 Adding non-linearity

3.5.1 Modeling interactions (synergy)

When two variables have interaction, then their product $X_i X_j$ can be added into the regression model, and the product maybe considered as a single variable for inference, for example, compute its SE, t -statistics, p -value, Hypothesis test, etc.

If we include an interaction in a model, then the **Hierarchy principle** should be followed: always include the main effects, even if the p -values associated with their coefficients are not significant. This is because without the main effects, the interactions are hard to interpret, as they would also contain the main effect.

3.5.2 Adding terms of transformed predictors

- 1) *Polynomial regression*: Add a term involving X_i^k for some $k > 1$.
- 2) other forms: Adding root or logarithm terms of the predictors.

3.6 Outliers (Unusual y_i that is far from \hat{y}_i)

It is typical for an outlier that does not have an unusual predictor value (with low leverage) to have little effect on the least squares fit, but it will increase RSE, hence deteriorate CI, p -value and R^2 , thus affecting interpreting the model.

An outlier can be identified by computing the

$$\text{studentized residual} = \frac{e_i}{\text{RSE}_i}$$

A studentized residual great than 3 may be considered as an outlier.

3.7 High leverage points (unusual x_i)

High leverage points tend to have sizeable impact on the regression line. To quantify the observation's leverage, one needs to compute the **leverage statistic**

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{j=1}^n (x_j - \bar{x})^2}.$$

$1/n \leq h_i \leq 1$ and $\text{Ave}(h_i) = (p+1)/n$. A large value of this statistic (for example, great than $(p+1)/n$) indicates an observation with high leverage. The leverage $1/n \leq h_i \leq 1$, reflects the amount an observation influences its own fit.

3.8 Compared to KNN Regression

KNN regression is a non-parametric method that makes prediction at x_0 by taking the average in a K -point neighborhood

$$\hat{f}(x_0) = \frac{1}{K} \sum_{x_i \in \mathcal{N}_{x_0}} y_i$$

A small value of K provides more flexible model with low bias but high variance while a larger value of K provides smoother fit with less variance. An optimal value of K depend on the *bias-variance tradeoff*. For non-linear data set, KNN may provides better fit than a linear regression model. However, in higher dimension (e.g., $p \geq 4$), even for nonlinear data set, KNN may perform much inferior to linear regression, because of the **curse of dimensionality**, as the K observations that are nearest to x_0 may in fact far away from x_0 .

3.9 Homework (* indicates optional):

- Conceptual: 1–6
- Applied: 8–15. at least one.

3.10 Code Gist

3.10.1 Python

```
dir() # provides a list of objects at the top level name space
dir(A) # display addtributes and methods for the object A
' + '.join(X.columns) # form a string by joining the list of column names by "+"
```

3.10.2 Numpy

```
np.argmax(x) # identify the location of the largest element
np.concatenate([x,y],axis=0) # concatenate two arrays x and y.
```

3.10.3 Pandas

```
X = pd.DataFrame(data=X, columns=['a','b'])

pd.DataFrame({'intercept': np.ones(Boston.shape[0]),
              'lstat': Boston['lstat']}) # make a dataframe using a dictionary
Boston.columns.drop('medv','age') # drop the elements 'medv' and 'age' from the list of columns

pd.DataFrame({'vif':vals},
              index=X.columns[1:]) # form a df by specifying index labels

X.values # Convert dataframe X to numpy array
X.to_numpy() # recommended to replace the above method
DataFrame.corr(numeric_only=True) # correlations between columns
x.sort_values(ascending=False)
pd.to_numeric(auto_df['horsepower'], errors='coerce') # if error, denote it by "NaN".
auto_df.dropna(subset= ['horsepower', 'mpg'], inplace=True) # looking for NaN in the columns

auto_df.drop('name', axis=1, inplace=True)

left2.join(right2, how="left") #join two databases by index.
left1.join(right1, on="key") # left-join by left1["key"] and the index of right1.
pd.concat([s1, s4], axis="columns", join="outer")
```

3.10.4 Graphics

```
xlim = ax.get_xlim() # get the x_limit values xlim[0], xlim[1]
ax.axline() # add a line to a plot
ax.axhline(0, c='k', ls='--'); # horizontal line
line, = ax.plot(x,y,label="line 1") # "line 1" is the legend
# alternatively the label can be set by
line.set_label("line 1")
ax.scatter(fitted, residuals, edgecolors = 'k', facecolors = 'none')
ax.plot([min(fitted),max(fitted)], [0,0], color = 'k', linestyle = ':', alpha = .3)
ax.legend(loc="upper left", fontsize=25) # adding legend
ax.annotate(i,xy=(fitted[i],residuals[i])) # annotate at the xy position with i.

plt.style.use('seaborn') # pretty matplotlib plots
plt.rcParams.update({'font.size': 16})
plt.rcParams["figure.figsize"] = (8,7)

plt.rc('font', size=10)
plt.rc('figure', titlesize=13)
plt.rc('axes', labelsiz=10)
plt.rc('axes', titlesize=13)
plt.rc('legend', fontsize=8) # adjust legend globally
```

3.10.5 Using Sns

```
sns.set(font_scale=1.25) # set font size 25% larger than default
sns.heatmap(corr, cbar=True, annot=True, square=True, fmt='.2f', annot_kws={'size': 10})
ax = sns.regplot(x=x, y=y)
```

3.10.6 Using Sklearn

```
from sklearn.linear_model import LinearRegression
## Set the target and predictors
X = auto_df['horsepower']

### To get polynomial features
poly = PolynomialFeatures(interaction_only=True, include_bias = False)
X = poly.fit_transform(X)
```

```

y = auto_df['mpg']

## Reshape the columns in the required dimensions for sklearn
length = X.values.shape[0]
X = X.values.reshape(length, 1) #both X and y needs to be 2-D
y = y.values.reshape(length, 1)

## Initiate the linear regressor and fit it to data using sklearn
regr = LinearRegression()
regr.fit(X, y)
regr.intercept_
regr.coef_

pred_y = regr.predict(X)

```

3.10.7 Using statsmodels and ISLP

```

from ISLP import load_data
from ISLP.models import (ModelSpec as MS,
                        summarize,
                        poly)

import statsmodels.api as sm
import statsmodels.formula.api as smf
from statsmodels.stats.outliers_influence \
    import variance_inflation_factor as VIF
from statsmodels.stats.anova import anova_lm

#Training
Boston = load_data("Boston")
#hand-craft the design matrix X
X = pd.DataFrame({'intercept': np.ones(Boston.shape[0]), #design matrix. intercept column
                  'lstat': Boston['lstat']})
#the following is the preferred method to create X
design = MS(['lstat']) # specifying the model variables. Automatically add an intercept, add.
design = design.fit(Boston) # do intial computation as specified in the model object design

X = design.transform(Boston) # apply the fitted transformation to the data to create X
#alternatiely,
X = design.fit_transform(Boston) # this combines the .fit() and .transform() two lines

y = Boston['medv']

```

```

model = sm.OLS(y, X) # setup the model
model = smf.ols('mpg ~ horsepower', data=auto_df) # alternatively use smf formula, y~x
smf.ols("y ~ x -1" , data=df).fit() # "-1" not including the intercept
results = model.fit() # results is a dictionary:.summary(), .params

results.summary()
results.params # coefficients
results.resid # residual array
results.rsquared # R^2
results.pvalues
np.sqrt(results.scale) # RSE
results.fittedvalues # fitted \hat{y}_i at x_i in the training set

summarize(results) # summarize() is from ISLP to show the essential results from model.fit()

# Making prediction
new_df = pd.DataFrame({'lstat':[5, 10, 15]}) # new test-set containing data where to make prediction
newX = design.transform(new_df) # apply the same transform to the test-set
new_predictions = results.get_prediction(newX);
new_predictions.predicted_mean #predicted values
new_predictions.conf_int(alpha=0.05) #for the predicted values

new_predictions.conf_int(obs=True, alpha=0.05) # prediction intervals by setting obs=True

# Including an interaction term
X = MS(['lstat',
        'age',
        ('lstat', 'age')]).fit_transform(Boston) #interaction term ('lstat', 'age')

# Adding a polynomial term of higher degree
X = MS([poly('lstat', degree=2), 'age']).fit_transform(Boston) # Note poly is from ISLP, # age
# Given a qualitative variable, `ModelSpec()` generates dummy
variables automatically, to avoid collinearity with an intercept, the first column is dropped

# Compare nested models using ANOVA
anova_lm(results1, results3) # result1 is the result of linear model, and result3 is the result of quadratic model

# Identify high leverage x
infl = results.get_influence()
# hat_matrix_diag calculate the leverage statistics
np.argmax(infl.hat_matrix_diag) # identify the location of the largest leverage

```

```
# Calculate VIF
vals = [VIF(X, i)
        for i in range(1, X.shape[1])] #excluding column 0 because it's all 1's in X.
vif = pd.DataFrame({'vif':vals},
                   index=X.columns[1:])
vif # VIF exceeds 5 or 10 indicates a problematic amount of colinearity
```

Useful Code Snippets

```
def abline(ax, b, m, *args, **kwargs):
    "Add a line with slope m and intercept b to ax"
    xlim = ax.get_xlim()
    ylim = [m * xlim[0] + b, m * xlim[1] + b]
    ax.plot(xlim, ylim, *args, **kwargs)
```

```
# Plot scatter plot with a regression line
ax = Boston.plot.scatter('lstat', 'medv')
abline(ax,
       results.params[0],
       results.params[1],
       'r--',
       linewidth=3)
```

```
# Plot residuals vs. fitted values (note, not vs x, therefore works for multiple regression)
ax = subplots(figsize=(8,8))[1]
ax.scatter(results.fittedvalues, results.resid)
ax.set_xlabel('Fitted value')
ax.set_ylabel('Residual')
ax.axhline(0, c='k', ls='--');
```

```
# Alternatively
sns.residplot(x=X, y=y, lowess=True, color="g", ax=ax)
```

```
# Plot the smoothed residuals-fitted by LOWESS
from statsmodels.nonparametric.smoothers_lowess import lowess
smoothed = lowess(residuals,fitted) # Note the order (y,x)
ax.plot(smoothed[:,0],smoothed[:,1],color = 'r')
```

```
# QQ plot for the residuas (obtain studentized residuals for identifying outliers)
import scipy.stats as stats
sorted_student_residuals = pd.Series(smf_model.get_influence().resid_studentized_internal)
```

```

sorted_student_residuals.index = smf_model.resid.index
sorted_student_residuals = sorted_student_residuals.sort_values(ascending = True)
df = pd.DataFrame(sorted_student_residuals)
df.columns = ['sorted_student_residuals']

#stats.probplot() #assess whether a dataset follows a specified distribution
df['theoretical_quantiles'] = stats.probplot(df['sorted_student_residuals'], dist = 'norm',

x = df['theoretical_quantiles']
y = df['sorted_student_residuals']
ax.scatter(x,y, edgecolor = 'k',facecolor = 'none')

# Plot leverage statistics
infl = results.get_influence()
ax = subplots(figsize=(8,8))[1]
ax.scatter(np.arange(X.shape[0]), infl.hat_matrix_diag)
ax.set_xlabel('Index')
ax.set_ylabel('Leverage')
np.argmax(infl.hat_matrix_diag) # identify the location of the largest leverage

```

4 Chapter 4: Classification

Given a feature vector X and a *qualitative* (categorical) response Y taking finite values in a set \mathcal{C} , the classification task is to build a classifier $C(X)$ that takes an input X and predicts its class $Y = C(X) \in \mathcal{C}$. This is often done by model $P(Y = k|X = x)$ for each $k \in \mathcal{C}$.

4.1 Linear regression and Classification

- For a *binary* classification, one can use linear regression and does a good job. In this case, the linear regression classifier is equivalent to LDA, because

$$P(Y = 1|X = x) = E[Y|X = x]$$

However, linear regression may not represent a probability as it may give a value outside the interval $[0, 1]$.

- When there are more than two classes, linear regression is not appropriate, because any chosen coding of the Y variable imposes an ordering and fixed differences among categories, which may not be implied by the data set. If the coding changes, a dramatic function will be fitted, which is not reasonable. One should turn to *multiclass logistic regression* or *Discriminant Analysis*.

4.2 Logistic Regression

Logistic regression is a *discriminative learning*, because it directly calculates the conditional probability $P(Y|X)$ to make classification.

4.2.1 Binary classification

with a single variable Logistic regression simply convert the linear regression to probability by

$$p(X) = Pr(Y = 1|X) = \frac{e^{\beta_0 + \beta_1 X}}{1 + e^{\beta_0 + \beta_1 X}}.$$

Note the *logit* or *log odds* is linear

$$\log \left(\frac{p(X)}{1-p(X)} \right) = \beta_0 + \beta_1 X.$$

Increasing X by one unit, changes the log odds by β_1 . Equivalently, it multiplied the odds by e^{β_1} . The rate of change of $p(X)$ is no longer a constant, but depends on the current value of X . Positive β_1 implies increasing $p(X)$, and vice versa.

The parameters are estimated by maximizing the *likelihood*

$$\ell(\beta_0, \beta_1) = \prod_{i:y_i=1} p(x_i) \prod_{i:y_i=0} (1-p(x_i))$$

With the estimated parameters $\hat{\beta}_j, j = 0, 1$, one can calculate the probability

$$p(X) = Pr(Y = 1|X) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 X}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 X}}$$

4.2.2 with multiple variables

In this case, simply let the logit be a linear function of p variables.

Note when there are multiple variables, it's possible to have variables confounding (especially when two variables are correlated): the coefficient of a variable may changes significantly or may change sign, this is because the coefficient represents the rate of change in Y of that variable when holding other variable constants. The coefficient reflects the effect when other variables are hold constant, how the variable affects Y , and this effect may be different than when only this variable is used in the model.

i Note

One can include a nonlinear term such as a quadratic term in the logit model, similar to a linear regression that includes a non-linear term.

4.2.3 Multi-class logistic regression (multinomial regression) with more than two classes

in this case, we use the *softmax* function to model

$$Pr(Y = k|X) = \frac{e^{\beta_{0k} + \beta_{1k} X_1 + \dots + \beta_{pk} X_p}}{\sum_{\ell=1}^K e^{\beta_{0\ell} + \beta_{1\ell} X_1 + \dots + \beta_{p\ell} X_p}} = a_k$$

for each class k . Note $\sum_k a_k = 1$ and the cross-entropy loss function is given by $-\log \ell(\beta) = -\sum_k \mathbb{1}_k \log a_k$, where β represents all the parameters.

The log odds between k th and k' th classes equals

$$\log\left(\frac{\Pr(Y = k|X = x)}{\Pr(Y = k'|X = x)}\right) = (\beta_{k0} - \beta_{k'0}) + (\beta_{k1} - \beta_{k'1}) + \dots + (\beta_{kp} - \beta_{k'p})$$

4.3 Discriminant Classifier: Approximating Optimal Bayes Classifier

Apply the Bayes Theorem, the model

$$\Pr(Y = k|X = x) = \frac{\Pr(X = x|Y = k) \cdot \Pr(Y = k)}{\Pr(X = x)} = \frac{\pi_k f_k(x)}{\sum_{\ell=1}^K \pi_\ell f_\ell(x)}$$

where $\pi_k = \Pr(Y=k)$ is the *marginal* or *prior* probability for class k , and $f_k(x) = \Pr(X = x|Y = k)$ is the *density* for X in class k . Note the denominator is a *normalizing constant*. So when making decisions, effectively we compare $\pi_k f_k(x)$, and assign x to a class k with the largest $\pi_k f_k(x)$.

Discriminant uses the full likelihood $P(X, Y)$ to calculate $P(Y|X)$ to make a classification, so it's known as *generative learning*.

- when f_k is chosen as a normal distribution with constant variance (σ^2) for $p = 1$ or correlation matrix Σ for $p > 1$, this leads to the LDA. For $p = 1$, the *discriminant score* is given by

$$\delta_k(x) = x \cdot \frac{\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} + \log(\pi_k)$$

when $K = 2$ and $\pi_1 = \pi_2 = 0.5$ then the *decision boundary* is given by

$$x = \frac{\mu_1 + \mu_2}{2}.$$

When $p \geq 2$, assume that $X = (X_1, X_2, \dots, X_p)$ is drawn from a multivariate Gaussian distribution $X \sim N(\mu_k, \Sigma)$, with a class-specific mean vector and a common variance matrix.

$$\delta_k(x) = x^T \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k + \log \pi_k = c_{k0} + c_{k1} x_1 + \dots + c_{kp} x_p.$$

The score function (posterior probability) is *linear* in x . With $\hat{\delta}_k(x)$ for each k , it can be converted to the class probability by the *softmax* function

$$\hat{\Pr}(Y = k|X = x) = \frac{e^{\hat{\delta}_k(x)}}{\sum_{\ell=1}^K e^{\hat{\delta}_\ell(x)}}$$

The π_k , μ_k and σ are estimate the follwing way:

$$\hat{\pi}_k = \frac{n_k}{n}$$

$$\hat{\mu}_k = \frac{1}{n_k} \sum_{i:y_i=k} x_i$$

$$\hat{\sigma}^2 = \frac{1}{n-K} \sum_{k=1}^K \sum_{i:y_i=k} (x_i - \hat{\mu}_k)^2 = \sum_{k=1}^K \frac{n_k-1}{n-K} \hat{\sigma}_k^2$$

where $\hat{\sigma}_k^2 = \frac{1}{n_k-1} \sum_{i:y_i=k} (x_i - \hat{\mu}_k)^2$ is the estimated variance for the k -th class.

i Note

One can include a nonlinear term such as a quadratic term in the LDA model, similar to a linear regression that includes a non-linear term.

- when each class chooses a different Σ_k , then it's QDA. It assumes an observation from the k -th class is $X \sim N(\mu_k, \Sigma_k)$. The score function has a *quadratic* term

$$\delta_k(x) = -\frac{1}{2}(x - \mu_k)^T \Sigma_k^{-1} (x - \mu_k) + \log \pi_k - \frac{1}{2} \log |\Sigma_k|$$

QDA has much more parameters $Kp(p+1)/2$ to estimate compared to LDA (Kp), hence has higher flexibility and may lead to higher variance. When there are few training examples, LDA tend to perform better and reducing variance is crucial. When there is a large traning set, QDA is recommended as variance is not a major concern. LDA is a special case of QDA.

- when the features are modeled independently, i.e., there is no association between the p predictors, $f_k(x) = \prod_{j=1}^p f_{jk}(x_j)$, the method is *naive Bayes*, and Σ_k are diagonal. Any classifier with a linear decision boundary is a special case of NB. So LDA is a special case of NB. To estimate f_{kj} , one can
 - assume that $X_j|Y = k \sim N(\mu_{jk}, \sigma_{jk}^2)$, that is, a class specific covariance but is diagonal. QDA's Σ_k is not diagonal. If we model $f_{kj}(x_j) \sim N(\mu_{kj} + \sigma_j^2)$ (Note σ_j^2 is shared among clases), In this case NB is a special case of LDA that has a diagonal Σ and
 - use a non-parametric estimate such as histogram (or a smooth kernel density estimator) for the observations of the j th Predictor within each class.
 - If X_j is qualitative, then one can simply count the proportion of training observations for the j th predictor corresponding to each class.

- Can applied to *mixed* feature vectors (qualitative and quantitative). NB does not assume normally distributed predictors.
- Despite strong assumptions, performs well, especially when n is not large enough relative to p , when estimating the joint distribution is difficult. It introduces some biases but reduces variance, leading to a classifier that works quite well as a result of bias-variance trade-off.
- Useful when p is very large.
- NB is a *generalized additive model*.
- Neither NB nor QDA is a special case of the other. Because QDA contains interaction term $x_i x_j$, while NB is purely additive, in the sense that a function of x_i is added to a function of x_j . Therefore, QDA potentially is a better fit when the interactions among predictors are important.

4.3.1 Why discriminant analysis

- When the classes are well-separated, the parameter estimation of logistic regression is unstable, while LDA does not suffer from this problem.
- if the data size n is small and the distribution of X is approximately normal in each of the classes, then LDA is more stable than logistic regression. Also used when $K > 2$.
- when there are more than two classes, LDA provides low-dimensional views of the data hence popular. Specifically, when there are K classes, LDA can be viewed exactly in $K - 1$ dimensional plot. This is because it essentially classifies to the closest centroid, and they span a $K - 1$ dimensional plane.
- For a two-class problem, the logit of $p(Y = 1|X = x)$ by LDA (generative learning) is a linear function in X , the same as a logistic regression (discriminative learning). The difference lies in how the parameters are estimated. But in practice, they are similar.
- LDA assumes the predictors follow a multivariable normal distribution with a shared Σ among classes. So when this assumption holds, we expect LDA performs better; and Logistic regress performs better when this assumption does not hold.

4.4 KNN

KNN is a non-parametric method and doesnot assume a shape for the decision boundary. KNN assign the class of popularity to $X = x$ in a K -neighborhood.

- KNN dominates LDA and Logistic Regression when the decision boundary is highly non-linear, provided n is large and p is small. As KNN breaks down when p is large.

- KNN requires large $n \gg p$, this is because KNN is non-parametric and tends to reduce bias but increase variance.
- When the decision boundary is non-linear but n is only modest and p is not very small, QDA may outperform KNN. This is because QDA provides a non-linear boundary while taking advantage of a parametric form, which means that it requires smaller size for accurate classification.
- Unlike logistic regression, KNN does not tell which predictors are more important: We don't get a table of coefficients.
- When the decision boundary is linear, LDA or logistic regression may perform better, when the boundary is moderately non-linear, QDA or NB may perform better; For a much more complicated decision boundary, KNN may perform better.

4.5 Poisson Regression

When Y is discrete and non-negative, a linear regression model is not satisfactory, even with the transformation of $\log(Y)$, because \log does not allow $Y = 0$.

- Poisson Regression: typically used to model counts,

$$\Pr(Y = k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, \dots,$$

where, $\lambda = E(Y) = \text{Var}(Y)$. This means that if Y follows a Poisson distribution, the larger the mean of Y , the larger its variance. Poisson regression can handle this when variance changes with mean, but linear regression cannot, because it assumes constant variance.

Assume

$$\log(\lambda(X_1, X_2, \dots, X_p)) = \beta_0 + \beta_1 X_1 + \dots + \beta_p X_p$$

Then one can use maximum likelihood

$$\ell(\beta_0, \beta_1, \dots, \beta_p) = \prod_{i=1}^n \frac{e^{-\lambda(x_i)} \lambda(x_i)^{y_i}}{y_i!}$$

to estimate the parameters.

- Interpretation: An increase in X_j by one unit is associated with a change in $E(Y) = \lambda$ by a *factor* (percentage) of $\exp(\beta_j)$.

4.6 Generalized Linear Models (GLM)

Perform a regression by modeling Y from a particular member of the *exponential family* (Gaussian, Bernoulli, Poisson, Gamma, negative binomial), and then transform the mean of Y to a linear function.

- Use predictors X_1, \dots, X_p to predict Y . Assume Y conditional on X follow some distribution: For linear regression, assume Y follows a normal distribution; for logistic regression, assume Y follows a Bernoulli (multinomial distribution for multi-class logistic regression) distribution; For poisson distribution, assume Y follows a poisson distribution.
- Each approach models the mean of Y as a function of X using a *linking function* η to transform $E[Y|X]$ to a linear function.

- for linear regression

$$E(Y|X) = \beta_0 + \beta_1 X_1 + \dots + \beta_p X_p$$

$$\eta(\mu) = \mu$$

- for logistic regression

$$E(Y|X) = P(Y = 1|X) = \frac{e^{\beta_0 + \beta_1 X_1 + \dots + \beta_p X_p}}{1 + e^{\beta_0 + \beta_1 X_1 + \dots + \beta_p X_p}}$$

$$\eta(\mu) = \log(\mu/(1 - \mu))$$

- for Poisson regression

$$E(Y|X) = \lambda(X) = e^{\beta_0 + \beta_1 X_1 + \dots + \beta_p X_p}$$

$$\eta(\mu) = \log(\mu).$$

- Gamma regression and negative binomial regression.

4.7 Assessment of a classifier

- Confusion matrix
- Overall error rate: equals to

$$\frac{FP + FN}{N + P}$$

- Class-specific performance: One can adjust the decision boundary (posterior probability threshold) to improve class specific performance at the expense of lowered overall performance.

- percentage of TP detected among all positives

$$\text{sensitivity (recall, power)} = TPR = \frac{TP}{TP + FN} = \frac{TP}{P} = 1 - \text{Type II error} = 1 - \beta$$

this is equal to $1 - FNR$, where, FNR is The fraction of positive examples that are classified as negatives

$$FNR = \frac{FN}{FN + TP} = \frac{FN}{P}$$

- percentage of TN detected among all negatives

$$\text{specificity} = TNR = \frac{TN}{TN + FP} = \frac{TN}{N}$$

This is equal to $1 - FPR$, where, False positive rate (FPR): the fraction of negative examples (N) that are classified as positive:

$$FPR = \frac{FP}{FP + TN} = \frac{FP}{N} = \text{Type I error}(\alpha)$$

- ROC (receiver operating characteristic curve): plot true positive rate (TPR=1-Type II error) ~ false positive rate (FPR= 1- specificity=Type I error) as a threshold for the posterior probability of positive class changes from 0 to 1. The point on the ROC curve closest to the point (0,1) corresponds to the best classifier.
- AUC (area under the ROC): Overall performance of a classifier summarized over all thresholds. AUC measures the probability a random positive example is ranked higher than a random negative example. A larger AUC indicates a better classifier.
- class-specific prediction performance

–

$$\text{precision} = \frac{TP}{TP + FP} = \frac{TP}{\text{predicted positives}} = 1 - \text{false discovery proportion}$$

4.8 Homework:

- Conceptual: 1,2,3,4, 5,6,7,8, 9, 10, 12
- Applied: 13, 14*,15*,16*

4.9 Code Gist

4.9.1 Python

4.9.2 Numpy

```
np.where(lda_prob[:,1] >= 0.5, 'Up','Down')
np.argmax(lda_prob, 1) #argmax along axis=1 (col)
np.asarray(feature_std) # convert to np array
np.allclose(M_lm.fittedvalues, M2_lm.fittedvalues)
#check if corresponding elts are equal within rtol=1e-5 and atol=-1e08
```

4.9.3 Pandas

```
Smarket.corr(numeric_only=True)
train = (Smarket.Year < 2005)
Smarket_train = Smarket.loc[train] # equivalent to Smarket[train]
Purchase.value_counts() # frequency table
feature_std.std() #calculate column std
S2.index.str.contains('mnth')
Bike['mnth'].dtype.categories # get the categories of the categorical data
obj2 = obj.reindex(["a", "b", "c", "d", "e"])# rearrange the entries in obj according to the
```

4.9.4 Graphics

```
ax_month.set_xticks(x_month) # set_xticks at the place given by x_month
ax_month.set_xticklabels([l[5] for l in coef_month.index], fontsize=20)
ax.axline([0,0], c='black', linewidth=3,
          linestyle='--', slope=1);#axline method draw a line passing a given point with a g
```

4.9.5 ISLP and Statsmodels

```
from ISLP import confusion_table
from ISLP.models import contrast

# Logistic Regression using sm.GLM() syntax similar to sm.OLS()
design = MS(allvars)
X = design.fit_transform(Smarket)
y = Smarket.Direction == 'Up'
glm = sm.GLM(y,
```

```

        X,
        family=sm.families.Binomial())
results = glm.fit()
summarize(results)
results.pvalues
probs = results.predict() #without data set, calculate predictions on the training set.
results.predict(exog=X_test) # on test set
# Prediction on a new dataset
newdata = pd.DataFrame({'Lag1':[1.2, 1.5],
                        'Lag2':[1.1, -0.8]});
newX = model.transform(newdata)
results.predict(newX)
confusion_table(labels, Smarket.Direction) #(predicted_labels, true_labels)
np.mean(labels == Smarket.Direction) # calculate the accuracy

hr_encode = contrast('hr', 'sum') #coding scheme for categorical data: the unreported coefficient

#Poisson Regression
M_pois = sm.GLM(Y, X2, family=sm.families.Poisson()).fit()
#`family=sm.families.Gamma()` fits a Gamma regression
model.

```

4.9.6 sklearn

```

from sklearn.discriminant_analysis import \
    (LinearDiscriminantAnalysis as LDA,
     QuadraticDiscriminantAnalysis as QDA)
from sklearn.naive_bayes import GaussianNB
from sklearn.neighbors import KNeighborsClassifier
from sklearn.preprocessing import StandardScaler
from sklearn.model_selection import train_test_split
from sklearn.linear_model import LogisticRegression

#LDA
lda = LDA(store_covariance=True) #store the covariance of each class
X_train, X_test = [M.drop(columns=['intercept']) # drop the intercept column
                  for M in [X_train, X_test]]
lda.fit(X_train, L_train) # LDA() model will automatically add a intercept term

lda.means_ # mu_k (n_classes, n_features)
lda.classes_
lda.priors_ # prior probability of each class

```



```

#Linear discriminant vectors
lda.scalings_ #Scaling of the features in the space spanned by the class centroids. Only available for LDA

lda_pred = lda.predict(X_test) #predict class labels
lda_prob = lda.predict_proba(X_test) #ndarray of shape (n_samples, n_classes)

#QDA
qda = QDA(store_covariance=True)
qda.fit(X_train, L_train)
qda.covariance_[0] #estimated covariance for the first class

# Naive Bayes
NB = GaussianNB()
NB.fit(X_train, L_train)
NB.class_prior_
NB.theta_ #means for (#classes, #features)
NB.var_ #variances (#classes, #features)
NB.predict_proba(X_test)[:5]

# KNN
knn1 = KNeighborsClassifier(n_neighbors=1)
X_train, X_test = [np.asarray(X) for X in [X_train, X_test]]
knn1.fit(X_train, L_train)
knn1_pred = knn1.predict(X_test)

# When using KNN one should standarize each variables
scaler = StandardScaler(with_mean=True,
                        with_std=True,
                        copy=True) # do calculaton on a copy of the dataset
scaler.fit(feature_df)

#train test split
X_std = scaler.transform(feature_df)
(X_train,
 X_test,
 y_train,
 y_test) = train_test_split(np.asarray(feature_std),
                            Purchase,
                            test_size=1000,
                            random_state=0)

# Logistic Regression
logit = LogisticRegression(C=1e10, solver='liblinear') #use solver='liblinear' to avoid warni

```

```
logit.fit(X_train, y_train)
logit_pred = logit.predict_proba(X_test)
```

4.9.7 Useful code snippet

```
# Tuning KNN
for K in range(1,6):
    knn = KNeighborsClassifier(n_neighbors=K)
    knn_pred = knn.fit(X_train, y_train).predict(X_test)
    C = confusion_table(knn_pred, y_test)
    templ = ('K={0:d}: # predicted to rent: {1:>2},' + # > for right alignment
            ' # who did rent {2:d}, accuracy {3:.1%}')
```

```
pred = C.loc['Yes'].sum()
did_rent = C.loc['Yes','Yes']
print(templ.format(
    K,
    pred,
    did_rent,
    did_rent / pred))
```

5 Chapter 5: Resampling Methods

Resampling methods are mainly used to estimate the test error by resampling the training set. Two methods: cross-validation and bootstrap. These methods refit a model to samples from the training set, in order to obtain additional information (eg. prediction error on the test set, standard deviation and bias of estimated parameters) about the fitted model.

Recall training error in general *dramatically underestimate* the test error, and in general, training error decreases as the model flexibility increases, but the test error shows a characteristic U-curve due to the bias-variance trade-off of the test error.

Model Assessment: evaluating a model's performance *Model selection*: selecting the proper level of flexibility.

5.1 how to estimate test error

- use a large designated test set, but often not available.
- make adjustment to the training error to estimate the test error, e.g., Cp statistic, AIC and BIC.
- validation set approach: estimate the test error by *holding out* a subset of the training set, also called a *validation set*.
 - the estimate of the test error can be highly variable, depending on the random train-validation split.
 - Only a subset of the training set is used to fit the model. Since statistical methods tend to perform worse when trained on a smaller data set, which suggests the validation error tends to *overestimate* the test error compared to the model that uses the entire training set.
- K-fold Cross-Validation: randomly divide the data into K equal-sized parts C_1, C_2, \dots, C_K . For each k , leave out part k , fit the model on the remaining $K - 1$ parts (combined, $(K - 1)/K$ of the original training set), and then evaluate the model on the part k . Then repeat this for each k , and weighted average of the errors is computed:

$$CV_{(K)} = \sum_{k=1}^K \frac{n_k}{n} \text{MSE}_k$$

where $\text{MSE}_k = \sum_{i \in C_k} (y_i - \hat{y}_i) / n_k$.

For classification problem, simply replace MSE_k with the misclassification rate $\text{Err}_k = \sum_{i \in C_k} I(y_i \neq \hat{y}_i)/n_k$.

The estimated standard error of CV_k can be calculated by

$$\hat{\text{SE}}(CV_k) = \sqrt{\frac{1}{K} \sum_{k=1}^K \frac{(\text{Err}_k - \bar{\text{Err}}_k)^2}{K-1}}$$

The estimated error tends bias upward because it uses only $(K-1)/K$ of the training set. This *bias is minimized with $K = n$ (LOOCV)*, but LOOCV estimate has *high variance* due to the high correlation between folds.

- LOOCV: it's a special case of K-fold CV with $K = n$. For least squares linear or polynomial regression, the LOOCV error can be computed by

$$CV_{(n)} = \frac{1}{n} \sum_{i=1}^n \left(\frac{y_i - \hat{y}_i}{1 - h_i} \right)^2$$

Where h_i is the leverage statistic of x_i . There is no randomness in the error. The leverage $1/n \leq h_i \leq 1$, reflects the amount an observation influences its own fit. The above formula doesn't hold in general, in which case the model has to refit n times to estimate the test error.

- for LOOCV, the estimate from each fold are highly correlated, hence their average can have high variance.
- better choice is $K = 5$ or $K = 10$ for bias-variance trade-off, because large k leads to low bias but high variance due to the increased correlation between models. Despite the estimated test error sometimes *underestimate* the true test error, they then to be close to identify the correct flexibility where the test error is minimum.
- Bootstrap: Primarily used to estimate the standard error, or a CI (called *bootstrap percentile*) of an estimate. Repeatedly sampling the training set with replacement and obtain a *bootstrap set* of the *the same size* as the original training set. One can fit a model and estimate a parameter with each bootstrap data set, and then estimate the *standard error* using the estimated parameters by the bootstrap model, assuming there are B bootstrap data sets:

$$SE_B(\hat{\alpha}) = \sqrt{\frac{1}{B-1} \sum_{r=1}^B (\hat{\alpha}^{*r} - \bar{\hat{\alpha}}^*)^2}$$

- Note sometimes sampling with replacement must take caution, for example, one can't simply sample a time series with replacement because the data are sequential.

- Estimate prediction error: Each bootstrap sample has significant overlap with the original data, in fact, about $2/3$ of the original data points appear in each bootstrap sample. If we use the original data set as the validation set, This will cause the bootstrap to seriously *underestimate* the true prediction error. To fix this, one can only use predictions on those samples that do not occur (by chance) in a bootstrap sample.
- Bootstrap vs. Permutation test: permutation methods sample from an estimated *null* distribution for the data, and use this to estimate p -values and *False Discovery Rates* for hypothesis tests.

The bootstrap can be used to test a null hypothesis in simple situation. Eg. If $H_0 : \theta = 0$, we can check whether the confidence interval for θ contains zero.

5.2 Homework

- Conceptual: 1,2,3,4
- Applied: 5–9, at least one.

5.3 Code Gist

5.3.1 Python

```
np.empty(1000) #create an array without initializing
quartiles = np.percentile(arr, [25, 50, 75])
```

5.3.2 Numpy

```
c = np.power.outer(row, col) # mesh of row[i]^col[j] power.
# random choice
rng = np.random.default_rng(0)
alpha_func(Portfolio,
            rng.choice(100, # random numbers are selected from arange(100)
                       100, #size
                       replace=True))
```

5.3.3 Pandas

```
np.cov(D[['X','Y']].loc[idx], rowvar=False) #cov compute corr of variables. rowvar=False: co
```

5.3.4 Graphics

5.3.5 ISLP and statsmodels

```
# function that evaluates MSE for training a model
def evalMSE(terms,      #predictor variables
            response, #response variable
            train,
            test):

    mm = MS(terms)
    X_train = mm.fit_transform(train)
    y_train = train[response]

    X_test = mm.transform(test)
    y_test = test[response]

    results = sm.OLS(y_train, X_train).fit()
    test_pred = results.predict(X_test)

    return np.mean((y_test - test_pred)**2)

# Compare polynomial models of different degrees
MSE = np.zeros(3)
for idx, degree in enumerate(range(1, 4)):
    MSE[idx] = evalMSE([poly('horsepower', degree)],
                      'mpg',
                      Auto_train,
                      Auto_valid)

MSE

# Estimating the accuracy of a LR model using bootstrap

# Compute the SE of the bootstrapped values computed by func
def boot_SE(func,
            D,
            n=None,
            B=1000,
```

```

        seed=0):
    rng = np.random.default_rng(seed)
    first_, second_ = 0, 0
    n = n or D.shape[0]
    for _ in range(B):
        idx = rng.choice(D.index,
                          n,
                          replace=True)
        value = func(D, idx)
        first_ += value
        second_ += value**2
    return np.sqrt(second_ / B - (first_ / B)**2) #compute var.
def boot_OLS(model_matrix, response, D, idx):
    D_ = D.loc[idx]
    Y_ = D_[response]
    X_ = clone(model_matrix).fit_transform(D_) #clone create a deep copy.
    return sm.OLS(Y_, X_).fit().params

quad_model = MS([poly('horsepower', 2, raw=True)]) #raw=True: not normalize the feature
quad_func = partial(boot_OLS,
                    quad_model,
                    'mpg')
boot_SE(quad_func, Auto, B=1000)

```

5.3.6 sklearn

```

from functools import partial
from sklearn.model_selection import \
    (cross_validate,
     KFold,
     ShuffleSplit)
from sklearn.base import clone
from ISLP.models import sklearn_sm #wrapper to feed a sm model to sklearn

#Cross Validation
hp_model = sklearn_sm(sm.OLS,
                      MS(['horsepower']))
X, Y = Auto.drop(columns=['mpg'], Auto['mpg']
cv_results = cross_validate(hp_model,
                             X,
                             Y,
                             cv=Auto.shape[0]) #cv=K.loocv. Can use cv=KFold()object

```

```

cv_err = np.mean(cv_results['test_score']) # test_score: MSE
cv_err

# Use KFold to partition instead of using an integer.
cv_error = np.zeros(5)
cv = KFold(n_splits=10,
           shuffle=True, #shuffle before splitting
           random_state=0) # use same splits for each degree
for i, d in enumerate(range(1,6)):
    X = np.power.outer(H, np.arange(d+1))
    M_CV = cross_validate(M,
                          X,
                          Y,
                          cv=cv)
    cv_error[i] = np.mean(M_CV['test_score'])
cv_error

# using ShuffleSplit() method
validation = ShuffleSplit(n_splits=10,
                          test_size=196,
                          random_state=0)
results = cross_validate(hp_model,
                        Auto.drop(['mpg'], axis=1),
                        Auto['mpg'],
                        cv=validation)
results['test_score'].mean(), results['test_score'].std()

#View sklearn fitting results using model.results_
hp_model.fit(Auto, Auto['mpg']) # hp_model is a sklearn model sk_model.fit(X, Y) for training
model_se = summarize(hp_model.results_['std err']) #summarize is an ISLP function
model_se

```

5.3.7 Useful code snippet

6 Chapter 6: Linear Model Selection and Regularization

Linear models are *interpretable* and often shows small variance. They are fitted by *OLS*. There are other methods that can either provide alternatives or improve linear regression models in terms of *prediction accuracy* (especially when $p > n$) and automatic *feature selection* for improved interpretability.

There are three classes of methods:

- subset selection: pick a subset of the p predictors that best explains the response.
- Shrinkage (regularization). With an added regularizing term, estimated parameters are shrunk to zero relative the OLS estimates. If L^2 regularization is used, all coefficients are shrunk toward zero by the same proportion; while if L^1 is used, then some coefficients will become zero, leading to actual *variable selection* or *sparse representation*.
- Dimension reduction. Project p -predictors to a M ($M < p$) dimensional subspace. Each new direction is a linear combination (or projection) of the p -variables. These M -projections can then be used to fit a linear regression model(**PCR** if the M -projections are obtained in an unsupervised way; or **PLR** if these M -projections are obtained in a supervised way))

6.1 Best Subset Selection

Algorithm

1. Fit the data with the *null model* \mathcal{M}_0 , which contains no predictors. This model simply set $Y = \text{mean}(y_i)$.
2. for $k = 1, 2, \dots, p$: fit $\binom{p}{k}$ models containing exactly k predictors. Pick the best one that having the smallest RSS or largest R^2 (or *deviance* for classification problem, i.e., $-2 \max \log(\text{likelihood})$ on the *training set*, called \mathcal{M}_k . Note for each categorical variable with L -level, there are $L - 1$ dummy variables.
3. Select the best one among $\mathcal{M}_0, \dots, \mathcal{M}_p$ using cross-validation or other measures such as $C_p(\text{AIC})$, BIC or adjusted R^2 . If cross-validation is used, then Step 2 is repeated on each training fold, and the validation errors are averaged to select the best k . The the model \mathcal{M}_k fit on the full training set is delivered for chosen k .

Best subset selection suffers - high computation: needs to compute 2^p models - overfitting due to the large search space of models

6.2 Stepwise selection

Both Forward and Backward selection are stepwise selection. They are used when p is large. They searches over $1 + p(p + 1)/2$ models and are *greedy* algorithm and is not guaranteed to find the best possible model out of all 2^p models.

- Backward selection requires $n > p$ (so that the full model can be fit);
- Forward selection can be used when $n < p$ but only fits up to models with n variables.
- One can combine forward and backward selection to a hybrid selection.

6.2.1 Forward Stepwise Selection

Adding one variable at a time that offers the greatest additional improvement.

Algorithm

1. Fit the data with the *null model* \mathcal{M}_0 , which contains no predictors. This model simply set $Y = \text{mean}(y_i)$.
2. for $k = 1, 2, \dots, p - 1$:
 - fit all $p - k$ models that augment the predictors in \mathcal{M}_k with one additional predictor.
 - Pick the best one that having the smallest RSS or largest R^2 on the training set, called \mathcal{M}_{k+1} .
3. Select the best one among $\mathcal{M}_0, \dots, \mathcal{M}_p$ using cross-validation or other measures such as $C_p(AIC)$, BIC or adjusted R^2 .

6.2.2 Backward Stepwise Selection

It begins with the full model with all variables, and iteratively removing one variable at a time.

Algorithm

1. Fit the data with the *full model* \mathcal{M}_p , which contains all predictors.
2. for $k = p, p - 1, \dots, 1$:
 - fit all k models that contains all but one of the predictors in \mathcal{M}_k .
 - Pick the best one that having the smallest RSS or largest R^2 on the training set, called \mathcal{M}_{k-1} .

3. Select the best one among $\mathcal{M}_0, \dots, \mathcal{M}_p$ using cross-validation or other measures such as $C_p(AIC)$, BIC or adjusted R^2 .

6.3 Model selection

Models with all predictors always have the smallest RSS or largest R^2 on the *training set*. Therefore they are not suitable to choose the best one among models with different number of predictors.

We ought to *estimate the test error* on a test set. This may be done - indirectly by adjusting the training error to account for the bias due to overfitting: C_p (equivalently, AIC in case of linear model with Gaussian errors), BIC and adjusted R^2 .

- Mallows's C_p :

$$C_p = \frac{1}{n}(RSS + 2d\hat{\sigma}^2)$$

where, d is the number of parameters and $\hat{\sigma}^2 \approx Var\epsilon$, typically estimated by using the *full model* containing all variables. C_p is an unbiased estimate of test MSE.

- AIC is defined for models fit by maximum likelihood.

$$AIC = -2\log L + 2d$$

where, L is the maximum likelihood function for the estimated model. For linear regression with Gaussian error, $AIC \propto C_p$.

- BIC

$$BIC = \frac{1}{n}(RSS + \log(n)d\hat{\sigma}^2)$$

Since $\log n > 1$ for $n > 7$, the BIC places a higher penalty on models with many variables, and hence select smaller models than C_p .

- Adjusted R^2 (larger value is better)

$$\text{Adjusted } R^2 = 1 - \frac{RSS/(n - d - 1)}{TSS/(n - 1)}$$

$RSS/(n - d - 1)$ may increase or decrease depends on d . Unlike R^2 , adjusted R^2 pays a price for the inclusion of unnecessary variables in a model.

C_p , AIC, BIC, adjusted R^2 , are not appropriate in high-dimensional setting, as the estimated $\hat{\sigma}^2 \approx 0$ (when $p \geq n$).

- directly by cross-validation (or validation). It does not require estimate σ^2 . It has a wide range of usage, as it may difficult to estimate d or σ^2 . One can choose the model that has the smallest test error or using the *one-standard-error rule* to select the model that has a smaller size:
 - calculate the SE of the estimated test MSE for each model size.
 - identify the lowest test MSE
 - choose the smallest model for which its test MSE is within one SE of the lowest point.

6.4 Shrinkage methods for Variable selection

The shrinkage offers an alternative to selecting variables by adjusting a hyperparameter that trades-off RSS and the model parameter magnitudes. Shrinkage methods will produce a different set of coefficients for a different λ . Cross-validation may be used to select the best λ . After the λ is selected, one can fit a final model using the entire training data set.

The reason shrinkage methods may perform better than OLS is rooted in bias-variance trade-off: as λ increases, the flexibility of the model decreases because of shrunk coefficients, leading to decreased variance but increased bias.

6.4.1 Ridge regression: minimize the following objective

$$RSS + \lambda \sum_{j=1}^p \beta_j^2$$

The ridge regression is equivalent to

$$\text{minimize } RSS \quad \text{subject to } \sum_{j=1}^p \beta_j^2 \leq s$$

for some $s \geq 0$.

- It encourages the model parameters to shrink toward zero and find a balance between RSS and model parameter magnitudes. Cross-validation is used to find the best tuning parameter λ . When λ is large, $\beta_j \rightarrow 0$. Note, Ridge shrinks all coefficients and include all p variables.
- The OLS coefficients estimates are *scale equivariant*: regardless of how X_j is scaled, $X_j \hat{\beta}_j$ remain the same: if X_j is multiplied by c , this will simply leads to $\hat{\beta}_j$ be scaled by a factor of $1/c$.

- In contrast, when multiplying X_j by a factor, this may significantly change the ridge coefficients. Ridge coefficients depends on λ and the scale of X_j , and may even on the scaling of other predictors. Therefore, it is best practice to *standardize the predictors* before fitting a ridge model:

$$\tilde{x}_{ij} = \frac{x_{ij}}{\frac{1}{n} \sum_{i=1}^n (x_{ij} - \bar{x}_j)}$$

- Ridge regression works best in situations where the OLS estimates have high variance, especially when p is large.
- Ridge will include all p variables in the final model.

6.4.2 The Lasso (Least Absolute Shrinkage and Selection Operator)

- The Lasso replaces the ℓ^2 error with ℓ^1 penalty. Lasso can force some coefficients to become exactly zero when λ is large enough. Thus it can actually performs *variable selection* hence better interpretation. Again, cross-validation is employed to select λ .
- The reason Lasso can perform variable selection is because the objective function is equivalent to

$$\text{minimizing RSS, subject to } \sum_{j=1}^p |\beta_j| \leq s$$

for some s . The contour of RSS in general only touch the ℓ_1 ball at its vertex, at which a minimum is obtained with some variables vanishes. In contrast, in the ridge situation, the ℓ_2 ball is round, and in general, the contour of the RSS function only touches the sphere at a surface point where a minimum is obtained with no variable vanishes.

- Neither ridge nor the lasso will universally dominate the other. When the response depends on a small number of predictors, one may expect lasso performs better; but in practice, this is never known in advance.
- Combining ridge and lasso leads to *elastic net* method.
- it is well known that ridge tends to give similar values coefficient values to correlated variables, while lasso may give quite different coefficient values to correlated variables.
- ridge regression shrinks all coefficients by the same proportion. While lasso perform **soft-thresholding**, shrink all coefficients by similar amount, and sufficient small coefficients are shrunk all the way to zero.

both ridge and lasso can be considered as computationally feasible approximation to the *best subset selection* which can be equivalently formulated as:

$$\text{minimize } RSS \quad \text{subject to } \sum_{j=1}^p I(\beta_j \neq 0) \leq s.$$

- **Bayesian formulation:** Both ridge and lasso can be interpreted as maximize the **posterior probability** (MAP)

$$p(\beta|X, Y) \propto f(Y|X, \beta)p(\beta|X) = f(Y|X, \beta)p(\beta)$$

where $p(\beta) = \prod_{j=1}^p g(\beta_j)$ with some density function g is the believed prior on β .

- if g is Gaussian with mean zero and standard deviation a function of λ , then it follows the solution β given by the ridge is the same as maximizing the posterior $p(\beta|X, Y)$, that is, β is the posterior mode. In fact, β is also the posterior mean. Since the Gaussian prior is flat at near zero, ridge assumes the coefficients are randomly distributed about zero.
- if g is double-exponential (Laplace) with mean zero and scale parameter a function of λ , then it follows the solution β given by the lasso is the same as maximizing the posterior $p(\beta|X, Y)$, that is, β is the posterior mode. In this case β is **not** the posterior mean. Since the Laplacian prior is steeply peaked at zero, lasso expects a priori that many coefficients are (exactly) zero.

6.5 Dimension reduction methods: transforming X_j .

There are two types of dimension reduction methods for regression: a) PCA regression, b) Partial list squares PLS. Both are designed to handle when the OLS breaks down due to that there are large number of correlated variables.

6.5.1 PCA regression: first use PCA to obtain M - PCA as linear combinations (directions) of the original p predictors:

$$Z_m = \sum_{j=1}^p \phi_{mj} X_j, \quad 1 \leq m \leq M, \quad (6.1)$$

where, the ϕ_{mj} are called **PCA loadings**, and subject to the norm $\sum_{j=1}^p \phi_{mj}^2 = 1$ for each m . Note Z_m is a vector of length equal to the length of X_j , which is the number of data points n . The component of Z_m : z_{im} , $1 \leq i \leq n$ are called **PCA scores**. z_{im} is a *single number summary* of the p predictors with the m -th PCA for the i -th observation. PCA is not a feature selection method.

The first PCA defines the direction that contains the largest variance in X , and minimize the sum of squared perpendicular distances to each point (the projection error on the PCA), that is it defines the line that is *as close as possible* to the data; (In fact, the first PCA is given by the eigenvector of the largest eigenvalue of the covariance matrix $\frac{1}{n-1}X^T X$). The second PCA is orthogonal to the first PCA and has the second largest variance and is uncorrelated with the first, and so on. These directions are obtained in an *unsupervised way*, as Y is not used to obtain these components. Consequently, there is no guarantee that the directions that best explain the predictors will also be the best directions to use for predicting the response.

PCA is typically conducted after standardizing the data X , as without scaling, the high variance variables will tend to have higher influence on the obtained PCAs.

We then use OLS to fit a linear regression model

$$y_i = \theta_0 + \sum_{m=1}^M \theta_m z_{im} + \epsilon_i, \quad i = 1, 2, \dots, n \quad (6.2)$$

After substitute Equation ?? into equation Equation ??, one can find that

$$\sum_{m=1}^M \theta_m z_{im} = \sum_{j=1}^p \beta_j x_{ij}$$

with

$$\beta_j = \sum_{m=1}^M \theta_m \phi_{mj}. \quad (6.3)$$

Eq. Equation ?? has the potential to bias the coefficient estimates, but selecting $M \ll p$ can significantly reduce the variance. So model Equation ?? is a special case of linear regression subject to the constants Equation ??.

PCR and ridge are closely related and one can think of ridge regression as a continuous version of PCR.

6.5.2 Partial Least Squares

Similar to PCAR, PLS also first identifies a new set of features Z_1, Z_2, \dots, Z_m , each of which is a linear combinations of the original features, and then fits a linear model via OLS with these new M features.

But PLS identifies these new features in a *supervised way*, that is, PLS uses Y in order to identify the new features that not only approximate the old features well, but also are *related to the response*, i.e., these new features explain both the response and the predictors.

First PLS standardizes the p predictors. PLS identifies the first component $Z_1 = \sum_{j=1}^p \phi_{1j} X_j$ by choosing $\phi_{1j} = \langle X_j, Y \rangle$, the coefficient from the simple linear regression of Y onto X_j .

Since this coefficient is equal to the correlation between Y and X_j , PLS places the highest weight on the variables that are most strongly related to Y . The PLS direction does not fit the predictors as closely as does PCA, but it does a better job explaining the response.

Next, PLS orthogonalizes each X_j with respect to Z_1 , that is, replace each X_j with the residual by regressing X_j on Z_1 , and then repeat the same process.

When p is large, especially $p > n$, the forward selection method, shrinkage methods (lasso or ridge), PCR, PLR fit a *less flexible* model, hence particularly useful in performing regression in high-dimensional settings.

6.6 Homework:

- Conceptual: 1–4
- Applied: At least one.

6.7 Code Snippet

6.7.1 Python

```
np.isnan(Hitters['Salary']).sum()
```

6.7.2 Numpy

```
np.linalg.norm(beta_hat) #L2 norm. ord=1: L1 ord='inf': max norm.
```

6.7.3 Pandas

```
Hitters.dropna();
soln_path = pd.DataFrame(soln_array.T,
                        columns=D.columns,
                        index=-np.log(lambdas))
soln_path.index.name = 'negative log(lambda)'
```


6.7.4 Graphics

```
ax.errorbar(np.arange(n_steps),
            cv_mse.mean(1), #mean of each row (model)
            cv_mse.std(1) / np.sqrt(K), #estimate standard error of the mean
            label='Cross-validated',
            c='r') # color red

ax.axvline(-np.log(tuned_ridge.alpha_), c='k', ls='--') # plot a vertical line
```

6.7.5 ISLP and statsmodels

```
#Estimate Var(epsilon)

design = MS(Hitters.columns.drop('Salary')).fit(Hitters)
design.terms # to see the variable names in the design matrix
Y = np.array(Hitters['Salary'])
X = design.transform(Hitters)
sigma2 = OLS(Y,X).fit().scale #.scale: RSE: residual standard error estimating

# Forward Selection using ISLP.models and a scoring function
from ISLP.models import \
    (Stepwise,
     sklearn_selected,
     sklearn_selection_path)
strategy = Stepwise.first_peak(design,
                               direction='forward',
                               max_terms=len(design.terms))
hitters_Cp = sklearn_selected(OLS,
                              strategy,
                              scoring=neg_Cp)
                              #default scoring MSE, will choose all variables
hitters_Cp.fit(Hitters, Y) # the same as hitters_Cp.fit(Hitters.drop('Salary', axis=1), Y)
hitters_Cp.selected_state_

#Forward selection using cross-validation
strategy = Stepwise.fixed_steps(design,
                                len(design.terms),
                                direction='forward')
full_path = sklearn_selection_path(OLS, strategy) #using default scoring MSE
full_path.fit(Hitters, Y) # there are , 19 variables, 20 models
Yhat_in = full_path.predict(Hitters)
```

```

#calculate in-sample mse

mse_fig, ax = subplots(figsize=(8,8))
insample_mse = ((Yhat_in - Y[:,None])**2).mean(0) #Y[:,None]: add a second axis, create a column of ones
# [yw] mean(0): calculate mean along row, i.e., for each col. mean(1)

#Cross-validation
K = 5
kfold = skm.KFold(K,
                  random_state=0,
                  shuffle=True)
Yhat_cv = skm.cross_val_predict(full_path,
                                Hitters,
                                Y,
                                cv=kfold)

# Cross-validation mse
cv_mse = []
for train_idx, test_idx in kfold.split(Y):
    errors = (Yhat_cv[test_idx] - Y[test_idx,None])**2
    cv_mse.append(errors.mean(0)) # column means
cv_mse = np.array(cv_mse).T

#validation approach using ShuffleSplit
validation = skm.ShuffleSplit(n_splits=1, # only split one time.
                             test_size=0.2,
                             random_state=0)
for train_idx, test_idx in validation.split(Y):
    full_path.fit(Hitters.iloc[train_idx], #note needing to use iloc
                  Y[train_idx])
    Yhat_val = full_path.predict(Hitters.iloc[test_idx])
    errors = (Yhat_val - Y[test_idx,None])**2
    validation_mse = errors.mean(0)

```

6.7.6 sklearn

```

from sklearn.pipeline import Pipeline
from sklearn.decomposition import PCA
from sklearn.cross_decomposition import PLSRegression

#Best subset selection using 10bnb

```

```

D = design.fit_transform(Hitters)
D = D.drop('intercept', axis=1) #needs to drop intercept
X = np.asarray(D)
path = fit_path(X,
                Y,
                max_nonzeros=X.shape[1]) #fit_path: a function from l0nb. use all variables
                # max_nonzeros: max nonzero coefficients in the fitted model.

# Ridge Regression
soln_array = skl.ElasticNet.path(Xs, # standardized, no intercept
                                Y,
                                l1_ratio=0., #ridge
                                alphas=lambdas)

# Using pipeline
ridge = skl.ElasticNet(alpha=lambdas[59], l1_ratio=0)
scaler = StandardScaler(with_mean=True, with_std=True)
pipe = Pipeline(steps=[('scaler', scaler), ('ridge', ridge)])
pipe.fit(X, Y)
ridge.coef_

# Validation

validation = skm.ShuffleSplit(n_splits=1,
                              test_size=0.5,
                              random_state=0) # validation is a generator

ridge.alpha = 0.01
results = skm.cross_validate(ridge,
                              X,
                              Y,
                              scoring='neg_mean_squared_error',
                              cv=validation) # using the strategy defined in validation

- results['test_score']

# GridSearchCV()
param_grid = {'ridge__alpha': lambdas}
grid = skm.GridSearchCV(pipe,
                        param_grid,
                        cv=validation, # or use cv=kfold (5-fold CV defined separately)
                        scoring='neg_mean_squared_error') #default scoring=R^2

grid.fit(X, Y)
grid.best_params_['ridge__alpha']
grid.best_estimator_
grid.cv_results_['mean_test_score']

```

```

grid.cv_results_['std_test_score']

#Plot CV MSE
ridge_fig, ax = subplots(figsize=(8,8))
ax.errorbar(-np.log(lambdas),
            -grid.cv_results_['mean_test_score'],
            yerr=grid.cv_results_['std_test_score'] / np.sqrt(K))
ax.set_ylim([50000,250000])
ax.set_xlabel('$-\log(\lambda)$', fontsize=20)
ax.set_ylabel('Cross-validated MSE', fontsize=20);

# Use ElasticNetCV()
ridgeCV = skl.ElasticNetCV(alphas=lambdas, # ElasticNetCV accepts a sequence of alphas
                           l1_ratio=0,
                           cv=kfold)
pipeCV = Pipeline(steps=[('scaler', scaler), # scaling is done once.
                          ('ridge', ridgeCV)])

pipeCV.fit(X, Y)
tuned_ridge = pipeCV.named_steps['ridge']
tuned_ridge.mse_path_
tuned_ridge.alpha_ # best alpha
tuned_ridge.coef_

# Evaluating test Error of Cross-validated Ridge

outer_valid = skm.ShuffleSplit(n_splits=1,
                               test_size=0.25,
                               random_state=1)
inner_cv = skm.KFold(n_splits=5,
                     shuffle=True,
                     random_state=2)
ridgeCV = skl.ElasticNetCV(alphas=lambdas, # a sequence of lambdas
                           l1_ratio=0,
                           cv=inner_cv) # K-fold validation
pipeCV = Pipeline(steps=[('scaler', scaler),
                          ('ridge', ridgeCV)]);

results = skm.cross_validate(pipeCV,
                              X,
                              Y,
                              cv=outer_valid,

```

```

                                scoring='neg_mean_squared_error')
- results['test_score']

# Lasso regression
lassoCV = skl.ElasticNetCV(n_alphas=100, #test 100 alpha values
                           l1_ratio=1,
                           cv=kfold)
pipeCV = Pipeline(steps=[('scaler', scaler),
                           ('lasso', lassoCV)])
pipeCV.fit(X, Y)
tuned_lasso = pipeCV.named_steps['lasso']
tuned_lasso.alpha_
tuned_lasso.coef_
np.min(tuned_lasso.mse_path_.mean(1)) # minimum avg mse

# to get the soln path
lambdas, soln_array = skl.Lasso.path(Xs, # standarsized, no -intercept
                                     Y,
                                     l1_ratio=1,
                                     n_alphas=100)[:2]

# PCA and PCR
pca = PCA(n_components=2)
linreg = skl.LinearRegression()
pipe = Pipeline([('scaler', scaler),
                  ('pca', pca),
                  ('linreg', linreg)])
pipe.fit(X, Y)
pipe.named_steps['linreg'].coef_
pipe.named_steps['pca'].explained_variance_ratio_

# perform Grid search
param_grid = {'pca__n_components': range(1, 20)} #PCA needs n_components >0
grid = skm.GridSearchCV(pipe,
                        param_grid,
                        cv=kfold,
                        scoring='neg_mean_squared_error')
grid.fit(X, Y)

# cross-validation a null model
cv_null = skm.cross_validate(linreg,
                              Xn,
                              Y,

```

```

                                cv=kfold,
                                scoring='neg_mean_squared_error')
-cv_null['test_score'].mean()

#PLS
pls = PLSRegression(n_components=2,
                    scale=True) # standardize the data
pls.fit(X, Y) # X has no-intercept

# Cross-validation
param_grid = {'n_components':range(1, 20)}
grid = skm.GridSearchCV(pls,
                        param_grid,
                        cv=kfold,
                        scoring='neg_mean_squared_error')

grid.fit(X, Y)

```