Just another algebraic geometry student trying to solidify their understanding by working through Hartshorne. Please email me if you find any mistakes!

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1 Chapter 1

1.1 Affine Varieties

1.1

1. Consider the ring map $k[t] \to k[x,y]/(x-y^2)$ given by $x \mapsto t$, which induces an isomorphism of schemes(varieties).

2. The multiplicative subgroup of units of $k[t]$ and $k[x,y]/(xy-1)$ are different.

3. Look at dimension of multiplicative subgroup of integers, which are just k-vector spaces. If $\dim = 1$ then the variety is isomorphic to case 1, if $\dim = 2$ then the variety is isomorphic to case 2. Since C has dimension 1, the dimension of subgroup of units cannot get any higher.

1.2

Generators of $I(Y)$ are $x^2 = y$ and $x^3 = z$ in $k[x, y, z]$. Since they are not multiples of each other, $k[x, y, z]/(I(Y)) = k[x, y, z]/(x^2 = y, x^3 = z)$ is irreducible. So by 1.8A $\dim(Y) = \dim k[x, y, z] - \mathrm{ht} I(Y) = 3 - 2 = 1$. The parametrization $A(Y) \to k[t]$ is given by $x \mapsto t, y \mapsto t^2, z \mapsto t^3$.

1.3

Y is the union of $V(x^2 - yz) \cup V(x) \cup V(z - x)$.

1.4

Consider $f(x, y) = xy - 1$. If the topology of $\mathbb{A}^2$ is the same as the product topology from $\mathbb{A}^1 \times \mathbb{A}^1$, then the projection of $V(f)$ to any of the $\mathbb{A}^1$ is closed. But an image of the projection is $\mathbb{A}^1 \backslash V(x)$ which is open. Contradiction.

1.5

($\Longleftarrow$) By 1.5 we can write $B = k[x_1, \cdots, x_m]/(f_1^{r_1} \cdots f_n^{r_n})$ for irreducible f_i's. Since each $V(f_i^{r_i})$ is irreducible all $r_n = 1$. By Chinese Reminder Theorem decompose B and so B is reduced. ($\Longrightarrow$) Write $B = k[x_1, \cdots, x_m]/(f_1^{r_1} \cdots f_n^{r_n})$ with f_i irreducible in $k[x_1, \cdots, x_n]$. Since B is reduced, all $r_i = 1$. So $V(f_1 \cdots, f_n) = V(f_1) \cup \cdots \cup V(f_n)$. Each $V(f_i)$ is irreducible since f_i is.

1.6

1. Any nonempty open subset U of an irreducible topological space X is dense and irreducible.

   Proof. If U is not dense, then $\bar{U} \cup X \backslash U$ shows that X is not irreducible. If $U = V_1 \cup V_2$ two proper closed subsets, then by subspace topology $V_i = U \cap W_i$ for some W_i closed in X. Note that $W_i \neq S$ since then $V_i = U$. Therefore W_1, W_2 are proper closed subsets of X. Let $W_3 = X \backslash U$ which is closed in U. Then $W_1, W_2 \cup W_3$ are two proper closed subsets of X such that $W_1 \cup (W_2 \cup W_3) = X$. So X is not irreducible as well. □

2. If Y is a subset of a topological space X which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

   Proof. (Math 631 Lecture 3) By taking complements of U, V, we have X is irreducible $\Longleftrightarrow$ for $U, V \subseteq X$ nonempty open, $U \cap V \neq \phi$.

Using this criterion, $Y \subseteq X$ is irreducible $\iff$ for any $U, V \subseteq X$ open that intersects Y it follows that $U \cap V \neq \phi$.
But notice that an open set U in X meets $Y \iff U$ meets $\overline{Y}$. □

1.12

Take Example 1.4.5. Note varieties like elliptic curves $y^2 = x^3 - x$ is reducible in Euclidean topology but not in Zariski topology.

1.2 Projective Varieties

2.1

(Homogeneous Nullstellensatz) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$ such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbb{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$.

Proof. As suggested in the hint, just interpret everything as in $\mathbb{A}^{n+1}$ and use the affine Nullstellensatz. Note that f still vanishes on every point in $V(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$. □

2.2

(i) $\iff$ (ii): ($\Longleftarrow$) is obvious by definition of $Z(-)$. By homogeneous Nullstellensatz we have $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ and $I(\phi) \supseteq S_+$.
(ii) $\iff$ (iii): ($\Longrightarrow$) is obvious by taking $d = 1$. To show ($\Longleftarrow$), suppose $\mathfrak{a} \supseteq S_d$ for some d. Then $x_0^d, \cdots, x_n^d \in S_d$, and so $\sqrt{\mathfrak{a}} \supseteq S_+$.

2 Chapter 2

2.1 Sheaves

1.2

(a) Use localization is exact to the exact sequence $0 \to \ker\phi \to \mathcal{F} \to \mathrm{Im}\phi \to 0$.
(b) ϕ is injective $\iff \ker\phi = 0 \iff (\ker\phi)_P = 0$ for all P. Then apply (a). Note that we can show epimorphism in the category of sheaves is the same as stalk-local surjectivity, so epimorphisms are surjective maps in the category. However, injectivity is not stalk-local, so they are different from monomorphism.
(c) It's a standard practice to break a long exact sequence into short exact sequences. Then apply part (b) to each of those short exact sequences.

1.7

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

1. Show that $\mathrm{Im}\phi \simeq \mathcal{F}/\ker\phi$.

   Proof. On stalk level we naturally have $(\mathrm{Im}\phi)_p \simeq \mathcal{F}_p/\ker\phi_p$. Since localization is exact $\mathcal{F}_p/\ker\phi_p \simeq (\mathcal{F}/\ker\phi)_p$. This holds for all $p \in X$ so $\mathrm{Im}\phi \simeq \mathcal{F}/\ker\phi$. □

2. Show that $\mathrm{coker}\phi \simeq \mathcal{G}/\mathrm{Im}\phi$.

   Proof. Similar to (a). $\square$

1.8.

(Left exactness of $\Gamma(U, -)$ for $U \subseteq X$ open)

For injectivity of first map, localize and use that injectivity is determined on the stalk level. For exactness in the middle, use the exact sequence $0 \to \mathcal{F}/\ker\phi \to \mathrm{Im}\phi \to 0$ and localize, where $\psi : \mathcal{F}' \to \mathcal{F}$ and $\phi : \mathcal{F} \to \mathcal{F}''$ are the given maps. Use the isomorphism is determined on the stalk level we are done.

Note that surjectivity on stalks (equivalently, on sheaves) is not equivalent to surjectivity on affine-open or open leve. See Stacks project 00WL.

1.14

(Support) Let $\mathcal{F}$ be a sheaf on X and let $s \in \mathcal{F}(U)$ be a section over an open set U. The support of s denoted $\mathrm{Supp}(s)$ is defined to be $\{P \in U : s_P \neq 0\}$. Show that $\mathrm{Supp}(s)$ is closed in U. Show that the support of a sheaf $\mathrm{Supp}\mathcal{F} = \{P \in X : \mathcal{F}_P \neq 0\}$ need not be closed.

Proof. Note that if we are working with schemes, then we can work affine locally and support of s is given by $\overline{D_s}$. But in a general setting, this follows from the definition of stalk. We prove that the complement of $\mathrm{Supp}(s)$ is open. If $s_P = 0$ for some P, then there is some small open neighborhood U of P where $s|U = 0$. So s vanishes on U.

Now consider the presheaf given by $\mathcal{G} := \underline{\mathbb{Z}}/i_P(\mathbb{Z})$, where $\underline{\mathbb{Z}}$ is the constant sheaf with values in $\mathbb{Z}$ and $i_P(\mathbb{Z})$ is the skyscraper sheaf supported at some point P. Since stalks don't change under sheafification, we can check that $\mathcal{G}_Q = 0$ iff $Q \notin \mathrm{Cl}(P)$, cf II Ex 1.17. So $\mathrm{Supp}(G)$ is open. $\square$

1.16

A sheaf $\mathcal{F}$ on a topological space is flasque if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

1. Show that a constant sheaf $\mathcal{F}$ on an irreducible topological space X is flasque.

   Proof. For any $\phi \neq V \subseteq U$, by Hartshorne Example 1.1.3 we know both V, U are irreducible in X. Since $\mathcal{F}(V)$ consist of locally constant functions on V, they are in fact constant function on V. Similarly for $\mathcal{F}(U)$. So $\mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism. $\square$

2. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ is flasque, then for any open set U, the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ of abelian groups is also exact.

   Proof. By Exercise 1.8 the section over U functor is left exact. Let $V \subseteq U$ and pick any $f \in \mathcal{F}''(V)$. Since localization is exact, we can cover V with $\{V_j\}$ where on each V_j, the restriction $f_j := f|_{V_j}$ can be lifted to some $g_j \in \mathcal{F}(V_j)$. We want to show that $\{g_j\}$ glue into an element in $\mathcal{F}(V)$. First consider g_1, g_2. Since $g_1 - g_2$ is mapped to 0, by left exactness of $\Gamma(-, V_1 \cap V_2)$ there is some $h_{12} \in \mathcal{F}''(V)$ that maps to $g_1 - g_2$. Modify g_2 to be $g_2 + \mathrm{Im}(h_{12})$, and merge V_1, V_2 in the case to $V_{1\cup 2} = V_1 \cup V_2$. Conclude by transfinite induction. □

3. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then $\mathcal{F}''$ is flasque.

   Proof. This follows from a general homological algebra results. Let $V \subseteq U$. Since the left two vertical arrows are surjective, so is the third.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}''(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}'(V) \arrow[r] & \mathcal{F}(V) \arrow[r] & \mathcal{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

□

4. Let $f : X \to Y$ be a continuous map, and if $\mathcal{F}$ is flasque on X, then $f_*\mathcal{F}$ is flasque on Y.

   Proof. Since f is continuous the preimage of open sets are open. Use the flasqueness on X. □

5. Let $\mathcal{F}$ be any sheaf on X and let $\mathcal{G}$ be the sheaf of discontinuous sections. Show that $\mathcal{G}$ is flasque, and that there is a natural injective morphism of $\mathcal{F} \hookrightarrow \mathcal{G}$.

   Proof. By definition of $\mathcal{G}$ it's flasque, and there is a natural morphism $i : \mathcal{F} \to \mathcal{G}$. To show i is an injection, check on localizations. □

1.20

(Subsheaf with supports) Let Z be a closed subset of X, and let $\mathcal{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathcal{F})$ to be the subgroup of $\Gamma(X, \mathcal{F})$ consisting of all sections whose support is contained in Z.

1. Show that the presheaf $V \mapsto \Gamma_{Z\cap V}(V, \mathcal{F}|_V)$ is a sheaf, called the subsheaf of $\mathcal{F}$ with supports in Z, and is denoted by $\mathcal{H}^0_Z(\mathcal{F})$. (proof skipped)

2. Let $U = X \backslash Z$ and let $j : U \to X$ be the inclusion. Show there is an exact sequence of sheaves on X,
$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U).$$
   Furthermore, if $\mathcal{F}$ is flasque, the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is surjective.

   Proof. Injectivity follows from the definition of $\mathcal{H}^0_Z(\mathcal{F})$. For exactness at $\mathcal{F}$, localize and use definitions. Since surjectivity can be check on the stalk level and U is open, using flasqueness we get surjectivity as well. □

1.22

Standard result and skipped.

2.2 Schemes

2.4

Standard and skipped.

2.5

Spec($\mathbb{Z}$) $= \{(p) : p$ prime$\} \cup \{\eta\}$. It is the final object for the category of schemes because by 2.4, mapping X to Spec($\mathbb{Z}$) is equivalent to a map of rings $\mathbb{Z} \to \Gamma(X, \mathcal{O}_X)$. Since $\mathbb{Z}$ is the initial object in the category of rings we are done.

2.7

Let K be any field. Show that to give a morphism Spec(K) $\to X$ is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

Proof. First show ($\to$). Let $x = \mathrm{Im}(\text{Spec}(K))$. We want to construct a map $k(x) \to K$, which is injective since $k(x)$ is a field. First construct a map $\mathcal{O}_x \to K$ by the following. Take affine neighborhoods $U_i = \text{Spec}(A_i)$ of x and the map Spec(K) $\to$ Spec(A_i) induces $A_i \to K$. Then take limits over A_i to get a map $\phi : \mathcal{O}_x \to K$. When taking limits, note that ϕ factors through $\mathfrak{m}_x$. Explicitly if $f \in \mathfrak{m}_x$ there is some Spec(A_i) where f vanishes. Then under $A_i \to K$, the image of f manifests as 0. So $\phi(f) = 0$. This gives a map $k(x) \to K$.

Then show ($\leftarrow$). On the space level map Spec(K) to x. To construct map of structural sheaves, use the following composition: $\mathcal{O}_X \to \mathcal{O}_x \to k(x) \to K$ where the last map is given. □

2.8

(fuzzier version of 2.7) Let X be a scheme. For any $x \in X$, we define the Zariski tangent space T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k and let $k[\epsilon]/\epsilon^2$ be the ring of dual numbers over k. Show that to give a k-morphism of Spec$k[\epsilon]/\epsilon^2$ to X is equivalent to giving a point $x \in X$ rational over k (i.e., $k(x) = k$) and an element of T_x.

Proof. We first show the forward direction. Let $f : \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2$, which is a local morphism. So $f(\mathfrak{m}) \subseteq (\epsilon)$. Therefore, the composed map $f_0 : \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2 \to k[\epsilon]/\epsilon^2/\epsilon = k$ has kernel containing $\mathfrak{m}$. Since kernel is prime, $\ker f_0 = \mathfrak{m}$. Since the residue field $\mathcal{O}_{X,x}/\mathfrak{m}$ is a field extension of k, it must equal to $\mathfrak{m}$. Hence the image of f is a rational point. Moreover, we show that f gives an element in T_x. For any $h \in \mathfrak{m}$, we have $f(h) \in (\epsilon)$. Define a k-module map $g : k[\epsilon]/\epsilon^2 \to k$ given by $1 \mapsto 0, \epsilon \mapsto 1$. Then $(g \circ f)(h) \in k$ and kills $\mathfrak{m}^2$. So this is an element in T_x.

Conversely, suppose we are given a rational point $x \in X$ and an element of T_x. This gives us a local map $\phi : \mathcal{O}_{X,x} \to k$ and an k-linear map $\psi : \mathfrak{m}/\mathfrak{m}^2 \to k$. Let $\gamma : \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2$ be $\phi + \psi\epsilon$. To construct a scheme map Spec($k[\epsilon]/\epsilon^2$) $\to X$, for each affine open U of x precompose with the localization map $\mathcal{O}(U) \to \mathcal{O}_{X,x}$, and for affine open U that does not contain x use the zero map.

Lastly check that those two constructions are inverse of each other. $\square$

2.9

If X is a topological space, and Z is an irreducible closed subset of X, a generic point for Z is a point ζ such that $Z = \overline{\zeta}$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

Proof. It suffices to consider $X = \text{Spec}(A)$. Let $Y \subseteq X$ be an irreducible closed subset. Then $Y = \overline{Y} = V(I(Y))$. Since Y is irreducible, $I(Y)$ is prime, and hence is the generic point of Y. $\square$

2.11

Let $k = \mathbb{F}_p$ be the finite field with p points. Describe Spec$(k[x])$. What are the residue fields of its points? How many points are there with a given residue field?

Proof. (cf. Vakil Example 6 in 3.2) Since $k[x]$ is an integral domain, there is a generic point. Since $k[x]$ is PID, all other prime ideals correspond to irreducible $f(x) \in k[x]$. Note that we have a complete characterization of these polynomials by Rabin's test for irreducibility, and there are infinitely many.

The residue fields of these points are finite dimensional subfields of $\mathbb{F}_p$, where $p = (f(x))$ and $[k_p : \mathbb{F}_p] = \dim f(x)$.

For any $\mathbb{F}_{p^n}$, there are $\sum_{d|n} \mu(n/d)p^n$ many points with that residue field. $\square$

2.16

(qcqs lemma, as Vakil calls it–super useful!) Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$ and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$.

1. If $U = \text{Spec}(B)$ is an open affine subscheme of X, and if $\bar{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

   Proof. ($\supseteq$) is clear. Pick some $x \in U \cap X_f$. Suppose BWOC $p \in V(\bar{f}) \subseteq \text{Spec}(B)$. That is, $\bar{f}_x \in \mathfrak{m}_x$. But $\bar{f}_x = f_x$. Contradiction to $x \in X_f$. So we have ($\subseteq$) too. $\square$

2. Assume that X is quasicompact. Let $A = \Gamma(X, \mathcal{O}_X)$ and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$.

   Proof. Use an affine cover $\{U_i\}_1^n$ of X. Let $a_i = a|_{U_i}$. By part (a) $U_i \cap X_f = D(\bar{f})$, and so $\bar{f}^{n_i}\bar{a} = 0$ on U_i. Let $n = \max\{n_i\}$ and we are done. $\square$

3. Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. (This hypothesis is satisfied, for example, if $sp(X)$ is noetherian.) Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of some element of A.

Proof. On each affine open U_i we can certainly find some n_i such that $\bar{f}^{n_i}\bar{b} = \bar{a}_i$ for some $a_i \in \Gamma(U_i, \mathcal{O}_{U_i})$ on U_i. Cover $U_i \cap U_j$ by affine opens $V_1, \cdots, V_k$. On each V_l, note that $\bar{a}_i - \bar{a}_j = 0$ on $D_{V_l}(\bar{f})$. So there is some m_{ij} such that $f^{m_{ij}}(\bar{a}_i - \bar{a}_j) = 0$. Take $M = \max_{i,j} m_{ij}$ and we can glue $\bar{a}_i$'s into a global section. □

4. With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) \simeq A_f$.

   Proof. By (c), we have a map $\phi : \Gamma(X_f, \mathcal{O}_{X_f}) \simeq A_f$, given by: starting with some $b \in \Gamma(X_f, \mathcal{O}_{X_f})$, take the element $a \in A$ such that $f^n b = a$ on X_f provided by part (c), and set $\phi(b) = a/f^n \in A_f$. This is clearly surjective, and we cliam is injective. Suppose $f^n b = 0$. Since if f is a zerodivisor then X_f is empty, we must have $b = 0$. □

2.19

Let A be a ring. Show that TFAE: i) Spec(A) is disconnected, ii) There is a pair of orthogonal idempotents; iii) A isomorphic to $A_1 \times A_2$, two nonzero rings.

Proof. $(i) \iff (iii)$ is obvious, by how Spec works. By taking $e_1 = (1, 0)$ and $e_2 = (0, 1)$ we have (iii) implies (ii). To show $(ii) \implies (i)$, note $D(e_1) \cup D(e_2) = D(e_1 e_2) = X$, and by de Morgen, $V(e_1) \cap V(e_2) = \phi$. Similarly since $D(e_1) \cap D(e_2) = D(e_1, e_2) = \phi$, we have $V(e_1) \cup V(e_2) = X$. So $X =$ Spec(A) is disconnected. □

2.3 First Properties of Schemes

3.6

Let X be an integral scheme. Show that the local ring $\mathcal{O}_\eta$ at the generic point η of X is a field, which is called the function field $K(X)$ of X. Show that if $U =$ Spec(A) is any affine open of X, then $K(X) \simeq \mathrm{Frac}(A)$.

Proof. Since X is integral there is only one generic point η. Since stalk can be computed locally, take any affine open $U =$ Spec(A) in X would contain η, in which η manifest as (0). Then $\mathcal{O}_\eta = A_{(0)} = \mathrm{Frac}(A)$ since A is integral. □

3.16

(Noetherian induction) Let X be a noetherian topological space. Let $\mathcal{P}$ be a property of closed subsets, with the feature that, if $\mathcal{P}$ holds for every proper subset of Y (where Y is any closed subset of X), then $\mathcal{P}$ holds for Y. Show that $\mathcal{P}$ holds for X as well.

Proof. By way of contradiction suppose $\mathcal{P}$ does not hold for X. Then there is some $Y_1 \subseteq X$ proper closed subset where $\mathcal{P}$ does not hold. Keep going and we have an infinite descending chain of proper closed subsets of X, contradicting the notherian hypothesis. □

2.4 Separated and Proper Morphisms

4.5.

Let X be an integral scheme of finite type over a field k, having function field K. We say that a valuation of K/k has center x if its valuation ring R dominates the local ring $\mathcal{O}_{X,x}$.

1. If X is separated over k, then the center of any valuation of K/k on X, if exists, is unique.

   Proof. Since v is a valuation of K/k, the inclusion $R \hookrightarrow K$ induces $\operatorname{Spec}(K) \to \operatorname{Spec}(R)$. Let the top map map $\operatorname{Spec}(K)$ to the generic point of X. Let the bottom map be induced by the inclusion $k \hookrightarrow R$, which exists because v is an K/k valuation. The right vertical map is the structural morphism. So we have the following commutative diagram:

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}(k) \end{tikzcd}$$

   where by valuative criterion of separatedness, there could be at most one diagonal dotted map. If there exists such a map, let $x \in X$ be the image of the closed point $\mathfrak{m}_R \in \operatorname{Spec}(R)$. We claim that this exactly means $\mathcal{O}_{X,x}$ is dominated by R.

   First, the map $\operatorname{Spec}(R) \to x$ induces a local ring map on the opposite direction, $j : \mathcal{O}_{X,x} \to R$. We claim that j is an inclusion. Suppose any $f \in \mathcal{O}_{X,x}$ is mapped to 0. Using the upper left triangle of the following diagram we see $f = 0$:

$$\begin{tikzcd} K & \mathcal{O}_{X,x} \arrow[l] \arrow[d] \arrow[dl] \\ R \arrow[u, hook] & k \arrow[l] \end{tikzcd}$$

   Since $\mathfrak{m}_R$ is mapped to x by construction, we have $f^*\mathfrak{m}_R = \mathfrak{m}_x$. So we indeed have R dominating $\mathcal{O}_{X,x}$. Therefore, there is at least a center of v if there is one diagonal map.

   We also need to show that the center if exists is unique. This is because any center of v will induce a lifting map. By uniqueness of lifting map from valuative criterion of separated morphisms, we are done. □

2. If X is proper over k then every valuation of K/k has a unique center on X.

   Proof. This is exactly like part (a), except that the valuative criterion for properness guarantees the existence of the unique diagonal map. □

3. Prove the converse of (a) and (b).

Proof. We prove the converse to (a) as the case for (b) is analogous. Suppose that the center of any valuation of K/k on X if exists is unique, and we want to show X is separated over k. By valuative criterion, we want to show that for any valuation ring R with $K = \text{Spec}(R)$, the commutative diagram has at most one diagonal map. (Not done.) $\square$

4. If X is proper over k, and if $k = \bar{k}$, show that $\Gamma(X, \mathcal{O}_X) = k$.

Proof. By way of contradiction pick $a \in \Gamma(X, \mathcal{O}_X) \backslash k$. Since $k = \bar{k}$, in the ring map $k \hookrightarrow \Gamma(X, \mathcal{O}_X)$ we have a is transcendental over k by the following lemma (from Math 614 2021 Fall, taught by Bhargav):

Lemma 2.1 (Integrality, an element is integral or a ring is nonzero)**.** Let $A \to B$ be any ring map, $b \in B$. Then either b is integral over A or $\frac{A[b^{-1}]}{(b^{-1})} \neq 0$, where $A[b^{-1}] \subseteq B_b$.

Proof. Assume $\frac{A[b^{-1}]}{(b^{-1})} = 0$. So $(b^{-1}) = (1)$, and there exists a polynomial $f(x) = \sum_{i=0}^n a_i x^i$ such that $1 = b^{-1} f(b^{-1})$ in $A[b^{-1}] \subseteq B_b$. Multiply by b^N for $N \gg 0$ to witness that b is integral over A. $\square$

In our situation, we just need to use that the local ring $k[\alpha^{-1}] \neq 0$ (as subring of $\Gamma(X, \mathcal{O}_X)[\alpha^{-1}]$). By existence of domination, we can find some $R \subseteq \Gamma(X, \mathcal{O}_X)[\alpha^{-1}]$ that dominates $k[\alpha^{-1}]_{(\alpha^{-1})}$. Let v be the valuation of R, and since $\mathfrak{m}_R \cap k[\alpha^{-1}]_{(\alpha^{-1})} = (\alpha^{-1})k[\alpha^{-1}]_{(\alpha^{-1})}$, we have $\alpha^{-1} \in R$. Moreover v has a unique center x on X by part (b) and properness assumption. So $v_x(\alpha^{-1}) > 0$, which implies $v_x(\alpha) < 0$. So α is not a global section on X. $\square$

2.5 Sheaves of modules

5.1

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the dual of $\mathcal{E}$, denoted $\mathcal{E}^\vee$, to be the sheaf $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

1. Show that $(\mathcal{E}^\vee)^\vee \simeq \mathcal{E}$.

Proof. There is a canonical isomorphism $\mathcal{O}_X^{\oplus n} \mapsto \mathrm{Hom}(\mathrm{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_X), \mathcal{O}_X)$ given by evaluation: $(a_1, \cdots, a_n) \mapsto ((f_1, \cdots, f_n) \mapsto \sum_i f_i(a_i))$. It has an inverse map, where for any $\phi : \mathrm{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_X)$, we get $(\phi(Id, 0, \cdots, 0), \cdots, \phi(0, \cdots, 0, Id)) \in \mathcal{O}_X^{\oplus n}$. $\square$

2. For any $\mathcal{O}_X$-module $\mathcal{F}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \simeq \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}$.

Proof. Using that $\mathcal{E}$ is locally free $\mathcal{O}_X$-module of finite rank, the functor $\mathrm{Hom}(\mathcal{E}, -)$ commutes with $\otimes$. That is, $\mathcal{E}^\vee \otimes \mathcal{F} \simeq \mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \otimes \mathcal{F} \simeq \mathrm{Hom}(\mathcal{E}, \mathcal{F})$. $\square$

3. For any $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, show $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}))$.

Proof. This follows from the canonical isomorphism given in the hom-tensor adjunction. $\square$

4. (Projection formula) We instead prove the version from Vakil 17.3.H: Suppose $\pi : X \to Y$ is quasicompact and quasiseparated, and $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves on X and Y respectively.
   (a) Describe a natural morphism $(\pi_*\mathcal{F}) \otimes G \to \pi_*(\mathcal{F} \otimes \pi^*\mathcal{G})$.
   (b) If $\mathcal{G}$ is locally free, show that this natural morphism is an isomorphism.
   (c) If π is affine, then show that this natural morphism is an isomorphism.

Proof. 1. We compute:

$$\begin{aligned}
&\mathrm{Hom}((\pi_*\mathcal{F}) \otimes \mathcal{G}, \pi_*(\mathcal{F} \otimes \pi^*\mathcal{G})) \\
=&\mathrm{Hom}(\pi^*\big((\pi_*\mathcal{F}) \otimes \mathcal{G}\big), \mathcal{F} \otimes \pi^*\mathcal{G}) && \text{by } \pi^*\text{-}\pi_* \text{ adjunction} \\
=&\mathrm{Hom}(\pi^*\pi_*\mathcal{F} \otimes \pi^*\mathcal{G}, \mathcal{F} \otimes \pi^*\mathcal{G}) && \text{since } \otimes \text{ distributes over pullback.}
\end{aligned}$$

By the π^*-π_* adjunction, there is a natural counit morphism $\epsilon : \mathrm{Hom}(\pi^*\pi_*\mathcal{F}, \mathcal{F})$. Then $\epsilon \otimes Id_{\pi^*\mathcal{G}} \in \mathrm{Hom}(\pi^*\pi_*\mathcal{F} \otimes \pi^*\mathcal{G}, \mathcal{F} \otimes \pi^*\mathcal{G})$ gives the wanted natural morphism.

2. Isomorphism is a local statement, so we can work on $U = \mathrm{Spec}(A) \subseteq Y$ where $\mathcal{G}$ is free. By Theorem 17.3.7 we have $(\pi_*\mathcal{F}) \otimes \mathcal{G} = (\pi_*\mathcal{F}) \otimes \mathcal{O}_X^{\oplus n} = (\pi_*\mathcal{F})^{\oplus n}$, and $\pi_*(\mathcal{F} \otimes \pi^*\mathcal{G}) = \pi_*(\mathcal{F} \otimes \pi^*(\mathcal{O}_X^{\oplus n})) = \pi_*(\mathcal{F} \otimes (\pi^*\mathcal{O}_X)^{\oplus n}) = \pi_*(\mathcal{F}^{\oplus n}) = (\pi_*\mathcal{F})^{\oplus n}$. So we have a natural $\mathcal{O}_X$-module morphism $Id : (\pi_*\mathcal{F})^{\oplus n} \xrightarrow{\sim} (\pi_*\mathcal{F})^{\oplus n}$.

3. Suppose π is affine. Let $U = \mathrm{Spec}(A) \subseteq Y$ be an affine open, and let $\pi^{-1}(U) = \mathrm{Spec}(B)$ Then since we don't need to sheafitify tensor products on affine opens, we have

$$(\pi_*\mathcal{F}) \otimes \mathcal{G}(U) = (\pi_*\mathcal{F})(U) \otimes \mathcal{G}(U) = \mathcal{F}(\pi^{-1}(U)) \otimes \mathcal{G}(U)$$

and

$$\begin{aligned}
&\pi_*(\mathcal{F} \otimes \pi^*\mathcal{G})(U) = (\mathcal{F} \otimes \pi^*\mathcal{G})(\pi^{-1}(U))\mathcal{F}(\pi^{-1}(U)) \otimes (\pi^*\mathcal{G})(\pi^{-1}(U)) \\
=&\mathcal{F}(\pi^{-1}(U)) \otimes \mathcal{G}(\pi(\pi^{-1}(U))) = \mathcal{F}(\pi^{-1}(U)) \otimes \mathcal{G}(U),
\end{aligned}$$

which are naturally isomorphic.

$\square$

5.2

Let R be a discrete valuation ring with $\mathrm{Frac}(R) = K$. Let $X = \mathrm{Spec}(R)$.

1. We have a correspondence

$$\left\{ \mathcal{O}_X\text{-modules} \right\} \longleftrightarrow \left\{ \begin{array}{l} R\text{-mod } M \\ K\text{-mod } L \\ \text{hom } \rho : M \otimes_R K \to L \end{array} \right\}$$

Proof. This is by definition of $\mathcal{O}_X$-modules, where ρ corresponds to the restriction map $X \to \{(0)\}$ (which is given by tensor product). $\square$

2. That $\mathcal{O}_X$-module is quasi-coherenet iff ρ is an isomorphism.

   Proof. This also follows from the definition of quasi-coherent $\mathcal{O}_X$-modules. $\square$

5.7

(Geometric Nakayama lemma) Let X be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf.

1. If the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_x$-module for some $x \in X$, then there is a neighborhood U of x such that $\mathcal{F}|_U$ is free.

   Proof. Note that we can start with fiber $\mathcal{F}|_x$ being free and use usual Nakayama lemma to get freeness of $\mathcal{F}_x$. Assume that $\mathcal{F}_x$ is freely generated by $t_1, \cdots, t_m$, germs of $\mathcal{F}$ at x. Since $\mathcal{F}$ is coherent, there is some affine neighborhood U of x where $\mathcal{F}(U)$ is finitely generated by $s_1, \cdots, s_n$. Then each $s_{j,x} = \sum a_i t_i$ for $a_i \in \mathcal{O}_{X,x}$. Write $a_i = f_i/g_i$, where f_i, g_i are local sections near x, where $g_i \notin x$. So we just need to invert finitely many elements g_i such that s_j can be written as linear combinations of t_i in a neighborhood of x. $\square$

2. $\mathcal{F}$ is locally free iff its stalks $\mathcal{F}_x$ are locally free for all $x \in X$.

   Proof. Suppose $\mathcal{F}$ is locally free. Then for any $x \in X$, there is an affine neighborhood U of x where $\mathcal{F}(U)$ is freely generated by $t_1, \cdots, t_n$ over $\mathcal{O}(U)$. That is, we have an isomorphism $\mathcal{O}(U)^n \simeq \mathcal{F}(U)$. Since localization is exact, $\mathcal{F}_x$ is also a free $\mathcal{O}_x$-module generated by $t_{1,x}, \cdots, t_{1,n}$.
   Now suppose that all stalks $\mathcal{F}_x$ are locally free. Then by part (a) we are done. $\square$

3. $\mathcal{F}$ is invertible (i.e., locally free of rank 1) iff there is a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \simeq \mathcal{O}_X$.

   Proof. Suppose $\mathcal{F}$ is invertible. Consider the sheaf $\mathcal{F}^\vee$ given by: on each affine open U in X, define $\mathcal{F}^\vee(U) := \left(\mathcal{F}(U)^\vee\right) = \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{F}(U), \mathcal{O}(U))$. Check this is a sheaf, and since locally $\mathcal{F}(U) \otimes \mathcal{F}^\vee(U) \simeq \mathcal{O}(U)$, we get $\mathcal{F} \otimes \mathcal{F}^\vee = \mathcal{O}_X$. $\square$

5.8

Let X be a noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. We consider the function $\phi(x) = \dim_{k(x)} \mathcal{F}_x \otimes k(x)$ where $k(x) = \mathcal{O}_x/\mathrm{m}_x$ is the residue field at point x. Use Nakayama's lemma to prove the following:

1. The function ϕ is upper semi-continuous, i.e., for any $n \in \mathbb{Z}$, the set $\{x \in X : \phi(x) \geq n\}$ is closed.

   Proof. The statement is the same as saying that, for any x, let $n_x = \phi(x)$, and we can find an open neighborhood U of x where for any $y \in U$ we have $\phi(y) \leq n_x$. This follow from 5.7(c). $\square$

2. If $\mathcal{F}$ is locally free, and X is connected, then ϕ is a constant function.

Proof. $\mathcal{F}$ being locally free implies that ϕ is locally constant, and X being connected implise ϕ is constant. ☐

3. Conversely, if X is reduced, and ϕ is constant, show that $\mathcal{F}$ is locally free.

   Proof. Since locally freeness is a local property, we may assume $X = \text{Spec}(A)$ where A is reduced. Moreover, we can replace X by some irreducible component, so that A is integral. By generic freeness, at η_X the local ring $\mathcal{F}_{\eta_X}$ is isomorphic to $\mathcal{O}_{\eta_X}^N$ for some N. Since ϕ is constant, by shrinking X we have $\mathcal{F}|_X \simeq \mathcal{O}|_X^N$. So $\mathcal{F}$ is locally free. ☐

5.11

(Generalization of Segre embedding) Let S and T be two graded rings with $S_0 = T_0 = A$. We define the cartesian product $S \times_A T$ to be the graded ring $\oplus_{d\geq 0} S_d \otimes_A T_d$. If $X = \mathrm{Proj} S$ and $Y = \mathrm{Proj} Y$, show that $\mathrm{Proj}(S \times_A T) \simeq X \times_A Y$, and show that the sheaf $\mathcal{O}(1)$ on $\mathrm{Proj}(S \times_A T)$ is isomorphic to the sheaf $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$ on $X \times Y$.

Proof. We first show that $\mathrm{Proj}(S \times_A T) \simeq X \times_A Y$. By the canonical inclusion $S \to S \times_A T$, we have an induced map $\mathrm{Proj}(S \times_A T) \to \mathrm{Proj} S$. By universal property of product, this gives a unique canonical morphism $\phi : \mathrm{Proj}(S \times_A T) \to X \times_A Y$. Using the tensor product construction it's easy to see that ϕ is an isomorphism on the space level. To show that it preserves structure sheaves, note that by general universal property of product, we have $\mathcal{O}_{X\times_A Y} \simeq \mathcal{O}_X \otimes_A \mathcal{O}_Y$. Moreover, an analog of Hartshorne 5.2(b) in projective case tells us $\mathcal{O}_{\mathrm{Proj}(S\times T)} \simeq \mathcal{O}_X \otimes_A \mathcal{O}_Y$. So ϕ induces an isomorphism of schemes. By the analog of Hartshorne 5.2(b) again the pulled backs sheaf on $X \times_A Y$ is $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$. Since in the diagram below, this corresponds to a closed embedding into $\mathbb{P}^{nm+n+m}$ on $X \times_A Y$, hence corresponds to $\mathcal{O}(1)$.

We can also directly show that $\mathcal{O}(1)$ is isomorphic to $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$ using global sections.

$$\begin{tikzcd}
 & & \mathrm{Proj}(S\times T) \arrow[d, dashed, "\phi"] \arrow[ddll, "\phi_1"'] \arrow[ddrr, "\phi_2"] & & \\
 & & X \times_A Y \arrow[dll, "p_1"] \arrow[drr, "p_2"'] & & \\
X \arrow[d, hook] & & & & Y \arrow[d, hook] \\
\mathbb{P}^n \arrow[rr] & & \mathbb{P}^{nm+n+m} & & \mathbb{P}^m \arrow[ll]
\end{tikzcd}$$

Note that everyone commutes in the diagram. Let $\mathcal{O}_X(1)$ be given by global sections $t_0, \cdots, t_n$, $\mathcal{O}_Y(1)$ by $s_0, \cdots, s_n$. Then by commutativity of the diagram, then induced map $\mathrm{Proj}(S \times T) \to \mathbb{P}^{nm+n+m}$ is induced by $\{t_i s_j\}_{i,j}$, which corresponds to the pullback of $\mathcal{O}_{\mathbb{P}^{nm+n+m}}(1)$ to $\mathrm{Proj}(S \times T)$. ☐

5.12

1. Let X be a scheme over a scheme Y and let $\mathcal{L}, \mathcal{M}$ be two very ample line bundles on X. Show that $\mathcal{L} \otimes \mathcal{M}$ is also very ample.

   Proof. We assume that a very ample line bundle is on a space X is a closed embedding (instead of locally closed embedding) $|\mathcal{L}| : X \to \mathbb{P}^N$ for some N, where $\mathcal{L}$ is the pullback of $\mathcal{O}(1)$. We can loosen the condition to assume that $\mathcal{L}$ is just basepoint free. This is Vakil 17.6.C. Denote by $|\mathcal{L}| : X \hookrightarrow \mathbb{P}^m$ the closed embedding induced by $\mathcal{L}$, and $|\mathcal{M}| : X \hookrightarrow \mathbb{P}^n$ the morphism. This induces a map $\Phi : X \to \mathbb{P}^n \times \mathbb{P}^m$, and a commutative diagram (and at the end, composed with the Segre embedding $\mathbb{P}^n \to \mathbb{P}^m \hookrightarrow \mathbb{P}^{mn+n+m}$:

$$\begin{tikzcd} X \arrow[rr, "\Phi"] \arrow[dr, hook, "|\mathcal{L}|"'] & & \mathbb{P}^n \times \mathbb{P}^m \arrow[dl, "\pi"] \\ & \mathbb{P}^m & \end{tikzcd}$$

   where π is the natural projection onto the second coordinate.

   By the Cancellation Theorem 11.2.1 for closed embeddings, to show that Φ is a closed embedding, it suffices to show that π is separated.

   By Corollary 11.3.15, any projective A-schemes are separated over A. So we consider the following commutative diagram:

$$\begin{tikzcd} \mathbb{P}^n \times \mathbb{P}^m \arrow[rr, "\pi"] \arrow[dr, "\alpha"'] & & \mathbb{P}^n \arrow[dl, "\beta"] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

   where α, β are the structure morphism.

   By the corollary, α and $\Delta\beta$ are separated. By the cancellation theorem for separated morphisms, π is separated as well. By the cancellation theorem for closed embeddings, Φ is a closed embedding. Moreover, since tensor product of base-point-free invertible sheaves are base-point-free, we conclude $\mathcal{L} \otimes \mathcal{M}$ is very ample. $\square$

2. Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes. Let $\mathcal{L}$ (resp. $\mathcal{M}$) be a very ample invertible sheaf on X relative to Y (resp. Y relative to Z). Show that $\mathcal{L} \otimes f^*\mathcal{M}$ is a very ample invertible sheaf on X relative to Z.

   Proof. Note that we have $\mathbb{P}^N_Y = \mathbb{P}^N_{\mathbb{Z}} \times Y$. For simplicity write $\mathbb{P}^N_{\mathbb{Z}}$ as $\mathbb{P}^N$. Consider the following composition of closed embeddings, and corresponding pullback of sheaves:

$$\begin{array}{ccccccc} f^*\mathcal{M} \otimes \mathcal{L} & & \mathcal{M} \otimes \mathcal{O}_{\mathbb{P}^n}(1) & & \mathcal{O}_{\mathbb{P}^m_Z}(1) \otimes \mathcal{O}_{\mathbb{P}^n}(1) & & \mathcal{O}_{\mathbb{P}^{mn+m+n}_Z}(1) \end{array}$$

$$X \xrightarrow{|\mathcal{L}|} Y \times \mathbb{P}^n \xrightarrow{|\mathcal{M}| \times Id_{\mathbb{P}^n}} (Z \times \mathbb{P}^m) \times \mathbb{P}^n \longrightarrow Z \times \mathbb{P}^{mn+m+n}.$$

   $\square$

5.13

(Vakil 7.4.D) Let S be a graded ring generated by S_1 as an S_0-algebra. For any $d > 0$, let $S^{(d)}$ be $\oplus_{d\geq 0} S_n^{(d)}$, where $S_n^{(d)} = S_{nd}$. Show that

1. $\mathrm{Proj} S^{(d)} \simeq X$,

   Proof. We have a natural inclusion $i : S^{(d)} \hookrightarrow S$, given by $S_n^{(d)} \hookrightarrow S_{dn}$. Pick $f \in S_+$ homogeneous of degree divisible by d. Then we want to identify $D_{\mathrm{Proj} S}(f)$ with $D_{\mathrm{Proj} S^{(d)}}(f)$. More explicitly, the maps given by $\phi : D_{\mathrm{Proj} S}(f) \to D_{\mathrm{Proj} S^{(d)}}(f)$, where $\mathfrak{p} \mapsto i^{-1}(\mathfrak{p}) = \mathfrak{p} \cap S^{(d)}$. This ϕ surjective: for any $\mathfrak{q} \in \mathrm{Proj}(S)$, let $\mathfrak{q}$ be finitely generated by $f_1, \cdots, f_n$ over $S_0 = S_0^{(d)}$. Then we need to show that $\mathfrak{q}' := \langle f_1, \cdots, f_n \rangle$ generated in S over S_0 does not contain S_+. For if it does, then $\mathfrak{q}$ is irrelevant as well (since both are generated over S_0).

   Now we show that ϕ is injective. Suppose $\mathfrak{p}_1, \mathfrak{p}_2 \in S$ restricts to the same ideal $\mathfrak{q} \in S^{(d)}$. Say $a \in \mathfrak{p}_2 \backslash \mathfrak{p}_1$. Then $a^d \in \mathfrak{q}$, and by primeness $a \in \mathfrak{q}$ as well. So $a \in \mathfrak{p}_1$. Contradiction. So $\mathfrak{p}_1 = \mathfrak{p}_2$. □

2. $\mathcal{O}(1)$ on $\mathrm{Proj} S^{(d)}$ corresponds to $\mathcal{O}_X(d)$.

   Proof. We want to show that the pullback of $\mathcal{O}_X(d)$ along ϕ constructed in part (a) is is $\mathcal{O}(1)$. We deal with the case $S = k[x, y]$, and it easily generalizes. The global sections of $\mathcal{O}_X(d)$ is S_d. Pulling back along ϕ, we have $\Gamma(\phi^* \mathcal{O}_X(d), \mathrm{Proj} S^{(d)}) = S_d = S_1^{(d)}$. So $\phi^* \mathcal{O}_X(d) \simeq \mathcal{O}(1)$ on $\mathrm{Proj} S^{(d)}$. □

5.15

(Extension of coherent sheaves) Let X be a noetherian scheme, let U be an open subset, and let $\mathcal{F}$ be a coherent sheaf on U. Then there is a coherent sheaf $\mathcal{F}'$ such that $\mathcal{F}'|_U \simeq \mathcal{F}$.

1. On an affine noetherian scheme, every quasi-coherent sheaf is the union of coherent subsheaves.

   Proof. Let $X = \mathrm{Spec}(A)$ where A is a notherian ring, and $\tilde{M}$ a quasicoherent sheaf on X, where M is a finitely generated module over A. Then we have a sequence

   $$K \hookrightarrow A^N \twoheadrightarrow M$$

   where K is not necessarily finitely generated. Let S be a generating set for K, and for each finite subset $S_0 \subseteq S$, let F_{S_0} denote the free module generated over elements in S_0. Note that if $S_0 \subseteq S_1$ then $A/S_0 \twoheadrightarrow A^N/S_1$, quotient given by further quotient. Using this direct system, form the direct limit $\lim_{S_0} A^N/S_0$, which surjects onto M. We claim that it also injects, since for any $f \in K$ there is some $S_0 \ni f$. □

2. Let X bean affine noetherian scheme, U an open set and $\mathcal{F}$ coherent on U. Then there exists a **coherent** sheaf $\mathcal{F}'$ on X with $\mathcal{F}'|_U \simeq \mathcal{F}$.

Proof. Let $i : U \hookrightarrow X$ be the natural inclusion and and by Proposition 5.8(c), $i_*\mathcal{F}$ is quasi-coherent as well. By (a) we can write $i_*\mathcal{F}$ as a union of coherent submodules $\mathcal{F}_j$. Let $\mathfrak{F} = \{\mathcal{F}_j\}$, and we enlarge $\mathfrak{F}$ to contain all coherent sheaves on X whose restriction to U are contained in $\mathcal{F}$.

Since X is noetherian, so is U. Let $\mathfrak{F}' := \{\mathcal{H}|_U : \mathcal{H} \in \mathfrak{F}\}$. Since all elements in $\mathfrak{F}'$ are contained in $\mathcal{F}$, there is a maximal element $\mathcal{F}^* = \mathcal{H}^*|_U$ where $\mathcal{H}^* \in \mathfrak{F}$, and by maximality we see $\mathcal{F}^*$ is just $\mathcal{F}$. So $\mathcal{H}^*$ is what we need. □

3. With $X, U, \mathcal{F}$ in (b) suppose furthermore we are given a quasi-coherent sheaf $\mathcal{G}$ on X such that $\mathcal{F} \subseteq \mathcal{G}|_U$. Show that we can find $\mathcal{F}'$ a coherent subsheaf of $\mathcal{G}$ with $\mathcal{F}'|_U \simeq \mathcal{F}$.

   Proof. In (b), let $\mathfrak{F}$ be the collection of all coherent subsheaf of $\mathcal{G}$ whose restriction to U is contained in $\mathcal{F}$. Since $\mathcal{G}$ is quasi-coherent (i.e., the corresponding module is noetherian), there is some maximal element $\mathcal{F}'$ in $\mathfrak{F}$ (that is, we don't need to use $\mathfrak{F}'$ any more). Run a similar argument in (b) and conclude $\mathcal{F}'|_U \simeq \mathcal{F}$. □

4. Now let U be any noetherian scheme, U an open subset, $\mathcal{F}$ a coherent sheaf on U, and $\mathcal{G}$ a quasi-coherent sheaf on X such that $\mathcal{F} \subseteq \mathcal{G}|_U$. Show that there is a coherent sheaf $\mathcal{F}' \subseteq \mathcal{G}$ on X with $\mathcal{F}'|_U \simeq \mathcal{F}$. Taking $\mathcal{G} = i_*\mathcal{F}$ proves the result announced in the beginning.

   Proof. Cover X with $V_1, V_2, \cdots, V_n$ noetherian affine opens. By (c) we can find some coherent sheaf $\mathcal{F}_1'$ on V_1 such that $\mathcal{F}_1' \subseteq \mathcal{G}|_{V_1}$ and $\mathcal{F}_1'|_U \simeq \mathcal{F}|_{V_1}$. Since X is noetherian, by Vakil 17.3.7, $\mathcal{F}_1'|_{V_1 \cap V_2}$ is still coherent. Applying (c) on V_2 with respect to $\mathcal{F}_1'|_{V_1\cap V_2}$ on $V_1 \cap V_2$ and we can extend $\mathcal{F}_1'$ one by one. Since X is quasicompact we are done. □

5. As an extra corollary, show that on a noetherian scheme, any quasi-coherent sheaf $\mathcal{F}$ is a the union of its coherent subsheaves.

   Proof. Let $\mathcal{F}$ be a quasi-coherent sheaf on a noetherian scheme X and U any affine open in X. For any section $s \in \mathcal{F}(U)$, let $\mathcal{H}$ be the coherent subsheaf of $\mathcal{F}|_U$ generated by s over U. By (d), there is some coherent subsheaf $\mathcal{H}'$ of $\mathcal{F}$ on X whose restriction to U is $\mathcal{H}$. Take the union (direct limit) of all these $\mathcal{H}$ and we claim that it equals to $\mathcal{F}$. But this follows from definition of union of subsheaves from (a). □

2.6 Projective morphisms

7.1

Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $f : \mathcal{L} \to \mathcal{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism.

Proof. Since isomoprhism and surjectivity are both local, it suffices to check that at each point $p \in X$ we have $\mathcal{L}_p \simeq \mathcal{M}_p$. Since $\mathcal{L}, \mathcal{M}$ are line bundles, we have $\mathcal{L}_p \simeq \mathcal{M}_p \simeq \mathcal{O}_p$. Consider the exact sequence $0 \to \ker(f) \to \mathcal{O}_p \to \mathcal{O}_p \to 0$. Since $k_p = \mathcal{O}_p/\mathfrak{m}_p$ is flat, tensoring with k_p preserves the exact sequence

$$0 \to \ker(f) \otimes k_p \to k \to k \to 0.$$

Since k is a field, we have $\ker(f) \otimes k_p = 0$. By property of tensor product, this is saying

$$\ker(f) \otimes \mathcal{O}_p/\mathfrak{m}_p \simeq \ker(f)/\mathfrak{m}_p \ker(f) = 0.$$

Since $\mathcal{O}_p$ is local, its Jacobson ideal is $\mathfrak{m}_p$, and by Nakayama lemma $\ker(f) = 0$. Hence f is injective. □

7.2

Let X be a scheme over a field k. Let $\mathcal{L}$ be an invertible sheaf on X and let $\{s_0, \cdots, s_n\}$ and $\{t_0, \cdots, t_m\}$ be two sets of sections of $\mathcal{L}$, which generate the same subspace $V \subseteq \Gamma(X, \mathcal{L})$, and which generate the sheaf $\mathcal{L}$ at every point. Suppose $n \leq m$. Show that the corresponding morphisms $\phi : X \to \mathbb{P}^n_k$ and ψ :

7.3

Let $\phi : \mathbb{P}^n_k \to \mathbb{P}^m_k$ be a morphism. Then:

1. (Vakil 17.4.L) either $\phi(\mathbb{P}^n)$ =pt or $m \geq n$ and $\dim \phi(\mathbb{P}^n) = n$.

   Proof. Since dimension is invariant under base change, we can assume that $k = \bar{k}$ and thus infinite. Recall that the dimension of image is no bigger than the dimension of the domain. Suppose that $\dim(\operatorname{Im}(\pi)) < \dim(\mathbb{P}^n) = n$. By way of contradiction also suppose that $\operatorname{Im}(\pi)$ is not a point, i.e., $\dim(\operatorname{Im}(\pi)) = m$ is such that $1 \leq m < n$. Let $i : X \hookrightarrow \mathbb{P}^N$, which corresponds to the data $(i^*\mathcal{O}_{\mathbb{P}^N}(1), i^*s_0 \cdots, i^*s_N)$. By Exercise 12.4.C, for each j the hypersurface $V(s_j) \cap X$ has nonempty intersections. We claim that there are at most m many $V(s_j)$ that intersects $\operatorname{Im}(\pi)$ properly (while the rest entirely contains $\operatorname{Im}(\pi)$).

   Lemma. Suppose $k = \bar{k}$. Suppose there are $m+1$ hypersurfaces with no common intersection in a closed subset Y of a projective space. Then $\dim Y \geq m$.
   Proof of lemma. Call the hypersurfaces $H_1, \cdots, H_{m+1}$. Let $Y = Y_0$, $Y_1 = H_1$. Note $\dim(Y_1) = \dim(Y_0) - 1$. Since the hypersurfaces have no common zero, up to reordering we can assume that $Y_2 := H_1 \cap H_2 \subsetneq Y_1$, and Y_2 is nonempty by Exercise 12.4.C(a). Note $\dim(Y_2) = \dim(Y_1) - 1$. Inductively define $Y_i := Y_{i-1} \cap H_i$ where $\dim(Y_{i-1}) \geq 1$ so that we can apply Exercise 12.4.C(a) in each step. At the end we have an nonempty chain of irreducible subsets in Y of length m. So $\dim(Y) \geq m$. This proves the lemma. □

   Take contrapositive of the lemma, we know that $\dim(\operatorname{Im}(\pi)) < n$ implies there are no n hypersurfaces in $\operatorname{Im}(\pi)$ with no common zeros. Therefore, at most $n-1$ many $V(i^*s_j)$ intersects $\operatorname{Im}(\pi)$ properly.
   If we further pullback those properly intersecting sections to $\mathbb{P}^n$, by Exercise 12.4.C(a), their vanishing locus have nonempty intersection. That is, those sections have common zeros, which is absurd (contradicting the correspondence in 17.4.1). Therefore, π contracts $\mathbb{P}^n$ into a point. □

2. in the second case, ϕ can be obtained as the composition of (1) a d-uple embedding $\mathbb{P}^n \to \mathbb{P}^N$ for a uniquely determined $d \geq 1$, (2) a linear projection $\mathbb{P}^N - L \to \mathbb{P}^m$, and (3) an automorphism of $\mathbb{P}^m$. Also, ϕ has finite fibers.

2.7 Differentials

3 Chapter 3

3.1 Derived Functors

No exercises for this subsection.

3.2 Cohomology of sheaves

2.1

1. Let $X = \mathbb{A}^1_k$ be the affine line over an infinite field k. Let P, Q be distinct closed points of X, and let $U = X - \{P, Q\}$. Show that $H^1(X, \mathbb{Z}_U) = 0$.

   Proof. Consider the exact sequence of sheaves $0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathcal{F} \to 0$, where $\mathcal{F}_P \simeq \mathcal{F}_Q \simeq \mathbb{Z}$. Write the long exact sequence of cohomology, and since $\mathbb{A}^1_k$ is affine we have a segment

   $$0 \qquad \mathbb{Z} \qquad \mathbb{Z} \oplus \mathbb{Z} \qquad 0$$

   where

   $$0 \longrightarrow H^0(\mathbb{Z}_U) \xrightarrow{0} H^0(\mathbb{Z}_X) \xrightarrow{(Id,Id)} H^0(\mathcal{F}) \xrightarrow{\phi} H^1(\mathbb{Z}_U) \longrightarrow H^1(\mathbb{Z}_X)$$

   ϕ is not the zero map since $\mathbb{Z}$ has rank 1 but $\mathbb{Z} \oplus \mathbb{Z}$ has rank 2. So $H^1(X, \mathbb{Z}_U) \neq 0$. □

2.2

Let $X = \mathbb{P}^1_k$ where $k = \overline{k}$, and show the exact sequence $0 \to \mathcal{O} \to \mathcal{K} \to \mathcal{K}/\mathcal{O} \to 0$ of (II. Ex 1.2.(d)) is a flasque resolution of $\mathcal{O}$. Conclude from (II. Ex. 1.21e) that $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

Proof. $\mathcal{K}$ is flasque by II Ex 1.16, and $\mathcal{K}/\mathcal{O}$ is flasque by its description by II. Ex. 1.21. So the short exact sequence is indeed a flasque resolution. By II. Ex. 1.21e the $\Gamma(X, -)$ functor preserves exactness. Taking its cohomology and we are done. □

2.3.

(Cohomology with supports) Let X be a topological space, let Y be a closed subset, and let $\mathcal{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X, \mathcal{F})$ denote the group of sections of $\mathcal{F}$ with supports in Y.

1. Show that $\Gamma_Y(X, \cdot)$ is an exact functor from $Ab(X)$ to Ab.

Proof. Note that for any $P \in Y$, we have $\Gamma_Y(X, \mathcal{G})_P \simeq \Gamma(X, \mathcal{G})_P$. Note there is a natural inclusion. For any $f \in \Gamma(X, \mathcal{G})_P$, let f be defined on open neighborhood U of P. Then $U \cap Y$ is an open neighborhood of P in Y, and $f|_{U \cap Y}$ defines a germ in $\Gamma_Y(X, \mathcal{G})_P$.

Recall that the global section functor $\Gamma(X, \cdot)$ is left exact, and exactness can be checked on the stalk level. Localize at each $P \in Y$ and we are done. □

2. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, with $\mathcal{F}'$ flasque, show that

$$0 \to \Gamma_Y(X, \mathcal{F}') \to \Gamma_Y(X, \mathcal{F}) \to \Gamma_Y(X, \mathcal{F}'') \to 0$$

is exact.

Proof. Follows from the observation about localization in part (a) and II Ex 1.16b. □

3. Show that if $\mathcal{F}$ is flasque then $H^i_Y(X, \mathcal{F}) = 0$ for all $i > 0$.

Proof. Use that exactness can be checked stalk-locally, observation in part (a) and Proposition 2.5. (So a subsheaf of flasque functor with support is also acyclic.) □

4. If $\mathcal{F}$ is flasque, show that the sequence

$$0 \to \Gamma_Y(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X - Y, \mathcal{F}) \to 0$$

is exact.

Proof. Use II Ex. 1.20 and take global sections. □

5. Take the long exact sequence of part (d).

6. (Excision) Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathcal{F}$,

$$H^i_Y(X, \mathcal{F}) \simeq H^i_Y(V, \mathcal{F}|_V).$$

Proof. This is saying we can excise $X - V$ and preserve the cohomology with supports. □

2.4.

(Mayer-Vietoris Sequence) Let Y_1, Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\cdots \to H^i_{Y_1 \cap Y_2}(X, \mathcal{F}) \to H^i_{Y_1}(X, \mathcal{F}) \oplus H^i_{Y_2}(X, \mathcal{F}) \to H^{i+1}_{Y_1 \cap Y_2}(X, \mathcal{F}) \to \cdots$$

Proof. Take an injective resolution of $F \to I^\bullet$ given by Proposition 2.2. Apply restricted global section functors, we have a short exact sequence of chain complexes:

$$0 \to I^\bullet_{Y_1 \cup Y_2}(X) \to I^\bullet_{Y_1}(X) \oplus I^\bullet_{Y_2}(X) \to I^\bullet_{Y_1 \cap Y_2}(X) \to 0,$$

where the third map is surjective by construction of $I^\bullet$ in Proposition 2.2. Lastly take the long exact sequence of cohomology. Note that by construction of $I^\bullet$, after restricting to some support Z, $I^\bullet_Z$ still flasque. By Proposition 1.2A the derived functors H^i can be computed via the above short exact sequence. □

3.3 Divisors

6.4.

We follow the hint. Since f is square free, the field K is a degree 2 Galois extension of $k(x_i)$. Hence any element $\alpha \in K$ can be written as $\alpha = g + hz$ with $g, h \in k(x_i)$, and has a minimal polynomial $X^2 - 2gX + (g^2 - h^2 f)$. We claim that α is integral over $k[x_i]$ iff $g, h \in k[x_1, \cdots, x_n]$. The ($\Longleftarrow$) direction is clear. If α is integral over $k[x_i]$, then there is a minimal polynomial over $k[x_i]$ of degree 2, say $X^2 + aX + b$. But by uniqueness of minimal polynomial we have $-2g = a$. So $g \in k[x_i]$, and $h^2 f \in k[x_i]$ as well. Note that $h^2 f = (hz)^2$ in K. Since $k[x_i]$ is integrally closed, we have $hz \in k[x_i]$. Suppose by way of contradiction that $h \notin k[x_i]$. By considering the most negative degrees, we have $z \in k[x_i]$, contradicting that f is square free. So ($\Longrightarrow$) also holds. The claim is exactly saying that A is the integral closure of $k[x_i]$ in K.

3.4 Cech Cohomology

4.1.

Let $f : X \to Y$ be an affine morphism of noetherian separated schemes. Show that for any quasi-coherent sheaf $\mathcal{F}$ on X, there are natural isomorphisms for all $i \geq 0$,

$$H^i(X, \mathcal{F}) \simeq H^i(Y, f_*\mathcal{F}).$$

Proof. The only thing to note is that pushforward of sheaves does not in general preserve quasicoherency. But by Theorem II.5.8(c) in our case $f_*\mathcal{F}$ is indeed quasicoherent. Use an affine open cover $\{U\}$ on Y and then $\{f^{-1}(U)\}$ is an affin open cover on X. Write down the Cech complex for both and they are the same. □

4.4.

On an arbitrary topological space X with an arbitrary abelian sheaf $\mathcal{F}$, Cech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1, there is an isomorphism if we take the limit over all coverings.

1. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of the topological space X. A refinement of $\mathcal{U}$ is a covering $\mathcal{V} = (V_j)_{j \in J}$ together with a map $\lambda : J \to I$ of the indices set, such that for

each $j \in J$, $V_j \subseteq U_{\lambda(j)}$. Show that if $\mathcal{V}$ is a refinement of $\mathcal{U}$ there is a natural induced map on Cech cohomology, for any abelian sheaf $\mathcal{F}$ and for each i:

$$\lambda^i : H^{\vee i}(\mathcal{U}, \mathcal{F}) \to H^{\vee i}(\mathcal{V}, \mathcal{F}).$$

Proof. We define a map on the chain complex level. Fix some U_i. For each $j \in \lambda^{-1}(i)$, use the restriction map $\mathcal{F}(U_i) \to \mathcal{F}(V_{\lambda(j)})$. Then induce the corresponding map on cohomology. □

2. For any

4.7.

Write $\mathbb{P}^2 = \mathrm{Proj} k[x, y, z]$, and let $u = x/y, v = z/y$. Since $(1, 0, 0) \notin X$, the open sets $U = D(y)$ and $V = D(z)$ cover X. Moreover, since $(1, 0, 0) \notin X$, without loss of generality, we can assume that $f(1, y, z)$ has a leading term y^d. So we can write $f(1, y, z) = u^d + vg(u, v) + c$. Using $(1, 0, 0) \notin X$ again we have $c \neq 0$. The Cech complex is

$$0 \xrightarrow{\delta_0} k[u, v]/f \oplus k[\frac{u}{v}, v^{-1}]/f \xrightarrow{\delta_1} k[u, v^{\pm}]/f.$$

Pick $(\alpha, \beta) \in k[u, v]/f \oplus k[\frac{u}{v}, v^{-1}]/f$, where $\alpha(u, v) = f_1(u) + vh_1(u, v) + c_1$, and $\beta(u/v, v^{-1}) = f_2(\frac{u}{v}) + \frac{1}{v}h_2(\frac{u}{v}, \frac{1}{v}) + c_2$. Suppose that $\delta_1(\alpha, \beta) = \gamma(u, v, v^{-1})(u^d + vg(u, v) + c)$, where $\gamma = f_3(u) + vf_4(u, v) + \frac{1}{v}h_3(u, v^{-1}) + c_3$. Collecting terms based on positive/negative/zero degrees of v we have $\alpha \equiv c_1$ and $\beta \equiv c_2$. So $\delta_1(\alpha, \beta) = 0$ means $c_1 - c_2 = c_3 c = 0$. Therefore, $H^0 = \ker(\delta_1) = k$.

Then we compute H^1, which can be seen to be generated by the basis $\{\frac{u^j}{v^i} : 0 < i < j < d, i \neq j\}$, which has $(d-1)(d-2)/2$ elements. So $H^1 \simeq k^{(d-1)(d-2)/2}$.

3.5 The Cohomology of Projective Space

5.1

Let X be a projective scheme over a field k and let $\mathcal{F}$ be a coherent sheaf. Show that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of coherent sheaves on X, then $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$.

Proof. By the snake lemma, get a long exact sequence of cohomology. By theorem 5.2(a), each term in the long exact sequence is a finite dimensional vector space. Alternating sum of ranks of vectors spaces in a long exact sequence is 0, so we get addivitity of χ. □

4 Chapter 4

4.1 Riemann-Roch Theorem

1.1

Let X be a curve and let $P \in X$ be a point. Then there is a nonconstant rational function $f \in K(X)$ which is regular everywhere except at P.

Proof. By Riemann-Roch, for any n we have $h^0(\mathcal{O}(nP)) - h^1(\mathcal{O}(nP)) = n - 1 + g$. By Serre Vanishing, $h^1(\mathcal{O}(nP)) = 0$ for $n \gg 0$. So $h^0(\mathcal{O}(nP)) = n - 1 + g \gg 0$ for $n \gg 0$, which proves the claim. $\square$

1.2

Let X be a curve and let $P_1, \cdots, P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles of some order at each of the P_i and regular elsewhere.

Proof. For each i let f_i be the function obtained from previous exercise, which has some poles at P_i and regular at all other points. Let $f = \prod_i f_i$. $\square$

1.3

Let C be an integral, separated, regular, one-dimensional scheme of finite type over k, which is not *proper* over k. Then C is affine.

Proof. Using the following commutative diagram, embed C into a projective integral separated regular curve $\overline{C}$:

$$\begin{tikzcd} C \arrow[r, hook, "\textbf{top. closure}"] \arrow[d, hook] & \overline{C} \arrow[d, hook] \\ \mathbb{A}^n \arrow[r, hook, "\textbf{compactification}"'] & \mathbb{P}^n \end{tikzcd}$$

The rest follows from Vakil 20.2.B, by modifying f in the proof to the f we obtained in Ex 1.2.

Lemma 4.1 (Vakil 20.2.B)**.** Suppose that k is algebraically closed. Show that $C \backslash \{p\}$ is affine.

Proof. Let $j > \max\{2g-3, g\}$. Since $\deg \mathcal{O}(jp) = j > 2g-3$, formula 20.2.5.1 applies, and we have $h^0(\mathcal{O}(jp)) = \deg \mathcal{O}(jp) - g + 1 = j - g + 1$ (using that $k = \overline{k}$). Similarly $h^0(\mathcal{O}((j-1)p)) = j - 1 - g + 1 = j - g$, which is one less than $h^0(\mathcal{O}(jp))$. So $\mathcal{O}(jp)$ is base point free by 20.2.7.

Moreover, $h^0(\mathcal{O}(jp)) = j - g + 1 > 1$, last inequality by assumption $j > g$. So we can pick two linearly independent global sections of $\mathcal{O}(jp)$, say 1 and s. Note $\operatorname{div}(1) = jp$. Consider the map $\phi : C \to \mathbb{P}^1$ given by $x \mapsto [1(x) : s(x)]$. Since $\mathcal{O}(jp)$ is base point free, $s(p) \neq 0$, so ϕ is well-defined. Moreover, since the section 1 only vanishes at p, ϕ is not constant. Because ϕ is a map of curves, it is dominant. By assumption k is algebraically closed, which implies ϕ is generically finite. Hence ϕ is a finite morphism.

Using $\mathcal{O}(jp)$ is base point free again, the set theoretic preimage of $[0:1]$ is just p. So $C \backslash \{p\}$ is the set theoretic preimage of $\mathbb{P}^1 \backslash [0:1] \simeq \mathbb{A}^1$. Recall that finite maps are affine. So $C \backslash \{p\} = \phi^{-1}(\mathbb{A}^1)$ is affine. $\square$

$\square$

1.5

For an effective divisor D on a curve X of genus g, show that $\dim |D| \leq \deg(D)$. Furthermore, equality holds iff $D = 0$ or $g = 0$.

Proof. By Riemann-Roch, $h^0(\mathcal{O}(D)) = h^0(\omega(-D)) + \deg(D) + 1 - g$. Since D is effective, $g \geq h^0(\omega(-D)) \geq g - \deg(D)$. So $\deg(D) + 1 \geq h^0(\mathcal{O}(D)) \geq 1$. By definition, $h^0(\mathcal{O}(D)) = \dim |D| + 1$. So $\deg(D) \geq \dim |D|$ as desired.

Now we show when equality holds. Suppose $D = 0$. Then $\mathcal{O}(D) \simeq \mathcal{O}$ and $\dim |D| = \deg(D) = 0$. Suppose $g = 0$ and $D \neq 0$. Then $\deg(D) \geq 1 \geq 2g - 2$. By Rimeann-Roch $h^0(\mathcal{O}(D)) = \deg(D) - g + 1 = \deg(D) + 1$. So equality holds.

Lastly suppose equality holds. By Riemann-Roch $h^0(\omega(-D)) = g = h^0(\omega)$. Note that we have $H^0(\omega(-D)) = H^0(\omega)$, which in terms induces $H^0(\mathcal{O}) = H^0(\mathcal{O}(D))$. But elements in $H^0(\mathcal{O})$ are constant functions. So $D = 0$. □

1.7

A curve X is called hyperelliptic if $g \geq 2$ and there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree 2.

1. If X is a curve of genus $g = 2$, show that the canonical divisor defines a complete linear system $|K|$ of degree 2 and dimension 1, without base points. Use (II. 7.8.1.) to conclude that X is hyperelliptic.

   Proof. We first prove a lemma:

   Lemma 4.2. Suppose $g(C) > 0$. Then ω_C is basepoint free.

   Proof. We want to show that for any closed point $p \in X$, we have $h^0(\omega_C(-p)) = h^0(\omega_C) - 1$. Using Riemann-Roch, $h^0(\omega_C(-p)) = g - 1$ or g. By way of contradiction suppose it's g for some p. Then by Riemann-Roch $h^0(\mathcal{O}(p)) = 2$. For any $s, q \in X$, we have $h^0(\mathcal{O}(p - s - q)) = 0$ by degree reason, so $\mathcal{O}(p)$ induces a closed embedding to $\mathbb{P}^1$. By (II. 7.8.1) the linear system $|\mathcal{O}(p)$ is an isomorphism of C and $\mathbb{P}^1$, contradicting $g > 0$. So ω_C is basepoint free. □

   So $\deg(\omega_C) = 2g - 2 = 2$, and $h^0(\omega_C) = 2$, so $\dim |\omega_C| = 2 - 1 = 1$. By (II 7.8.1) maps to $\mathbb{P}^1$ is either dominant or constant. It is nonconstant in our case (hence dominant), and because X is compact we are done. □

2. Show that the curves constructed in (1.1.1) all admit a morphism of degree 2 to $\mathbb{P}^1$. Thus there exist hyperelliptic curves of any genus $g \geq 2$.

   Proof. Recall that the curve from (1.1.1) looks like $f(t) = (x - \alpha_1) \cdots (x - \alpha_{2m})$. First consider the double cover of $\mathbb{A}^1$ given by Spec$(k[x, y]/y^2 - f(x) \to$ Spec$(k[x])$ Glue this to another double cover Spec$(k[u, v]/v^2 - u^{2m} f(1/u)) \to$ Spec$(k[u])$ via the maps $x \to 1/u, v \to y/x^m$. Hence we have a double cover (C, π) of $\mathbb{P}^1$. (This cover is branched at $\alpha_1, \cdots, \alpha_{2m}$: WLOG suppose $(\alpha_i, 0) \in \mathrm{Spec}(k[x,y]/y^2 - f(x)$ for all i. For any $x \in \mathbb{P}^1$, x has 1 preimage α via π iff $f(\alpha) = 0$ iff $\alpha = \alpha_1, \cdots, \alpha_{2m}$.)

Moreover, by computing the Cech complex of $\mathcal{O}_C$, we see $H^1(C, \mathcal{O}_C)$ is generated by $\{x^{-1}y, x^{-2}y, \cdots, x^{1-m}y\}$, which proves $g(C) = m - 1$. This works for all m, as desired. □

4.2 Embeddings in Projective Space

3.1.

If X is a curve of genus 2, show that a divisor D is very ample $\iff \deg(D) \geq 5$.

Proof. ($\Longleftarrow$) Follows from Corollary 3.2 since $5 = 2g + 1$.
($\Longrightarrow$) Suppose D is very ample. By Example 3.3.1 we have $\deg(D) > 0$. We divide the discussion into two cases. First suppose $\deg(D) \leq 2$. We claim that in this case $h^0(\omega(-D)) = 0$ and by Riemann Roch $h^0(\mathcal{O}(D)) = \deg(D) - 1$. If $\deg(D) < 2$, then since $\deg(\omega(-D)) < 0$ the claim holds. If $\deg(D) = 1$, suppose by way of contradiction $h^0(\omega(-D)) = 1$. Then $\omega(-D) = \mathcal{O}$, or $\omega = \mathcal{O}(D)$. So D is the canonical divisor and $|\mathcal{O}(D)|$ is the canonical embedding. By for genus 2 curves the canonical embedding is the hyperelliptic cover, which is certainly not a closed embedding. So $h^0(\omega(-D)) = 0$ as well. This checks the claim. Now $h^0(\mathcal{O}(D)) = \deg(D) - 1 \leq 2 - 1 = 1$. But by the h^0 criterion of very ampleness, $\mathcal{O}(D)$ cannot be very ample, as we cannot decrease $h^0(\mathcal{O}(D))$ by 2.

Now suppose $\deg(D) = 3, 4$. By Riemann Roch, $h^0(\mathcal{O}(D)) - h^0(\omega(-D)) + 1 = \deg(D) \leq 4$, or $h^0(\mathcal{O}(D)) \leq 3$.

By the same reason as before (h^0 criterion for very ampleness) $h^0(\mathcal{O}(D))$ cannot be 1.

If $h^0(\mathcal{O}(D))$ is 2, then we have a nonconstant map to $\mathbb{P}^1$, which must be dominant. But $g(X) = 2 \neq g(\mathbb{P}^1$.

If $h^0(\mathcal{O}(D)) = 3$, then we have a closed embedding $X \hookrightarrow \mathbb{P}^3$. By III Ex 4.7 $g = (\deg D - 1)(\deg D - 2)/2$. Plugging in $\deg D = 3, 4$ produces $g = 1, 3$, respectively. So $\deg D \neq 3, 4$. This finishes the proof. □