MATH 593 NOTES

Contents

1. Embeddings

Definition 1.1. *A topological embedding is a continuous map $F : M \to N$ that's injective and homeomorphic onto its image, where $F(M)$ has the subspace topology from N.*

An embedding is a smooth map $F : M \to N$ that is a topological embedding and an immersion.

Normalization of embeddings:

Theorem 1.2. *If $F : M \to N$ is an embedding, for any $p \in M$, there are charts (U, ϕ) of M with $p \in U$ and (V, ψ) of N with $f(p) \in V$, and $U \subseteq F^{-1}(V)$, such that in coordinates,*

$F(x^1, \cdots, x^n) = (x^1, \cdots, x^n, 0, \cdots, 0)$, *and*

$$F(U) = \{q \in V : \text{the last } m \text{ coordinates of } q \text{ are zeros}\} = V \cap F(M)$$

Note that the first equality is just the rank theorem. The point of the theorem is the second equality. Notice that $F(U) \subseteq V \cap F(M)$ automatically holds, and the theorem proves that if F is an embedding, then $F(M)$ cannot loop back and intersect V at somewhere else than $F(U)$. This says that locally, $S = F(M)$ looks like $\mathbb{R}^n \subseteq \mathbb{R}^m$. This implies that the image of any embedding is a submanifold. We have two equivalent definition of manifolds:

Definition 1.3. *1) (Tu, intrinsic): A subset $S \subseteq M$ is a (regular or embedded) submanifold of M $\iff$ S has the property that $\forall p \in S$, there exists $(U, \phi = (x^1, \cdots, x^n))$ with $p \in U$ such that $S \cap U = \{x^{d+1} = \cdots = x^n = 0\}$ and $d = \dim(S)$.*

2) (Lee, extrinsic): A subset $S \subseteq M$ is a submanifold $\iff$ S with the subspace topology has a C^∞ manifold structure, and the inclusion $i : S \hookrightarrow M$ is an embedding.

Notice that from Tu's definition, $F : U \to \mathbb{R}^{n-d}$ given by $F = (x^{d+1}, \cdots, x^n)$, the projection of last $n - d$ coordinates, then $S \cap U = F^{-1}(0)$ and 0 is a regular value of F since the chart U is a homeomorphism and projection is submersion. Such (F, U) is called a locally defined function for S.

Note that not all submanifolds have a globally defined functions (although the locally defined ones always exist by projection, as discussed above). For instance, consider the Mbius band. If $Mob = F^{-1}(0)$ for some F and 0 is a regular value, then dF would be surjective onto $\mathbb{R}$ and this means Mob has a globally defined normal vector field, which is untrue. (Combined with later lectures: existence of globally defined normal vector fields implies that the manifold is orientable. So any non-orientable manifold cannot be the regular level set of some F with domain $\mathbb{R}$.)

Some important examples of embeddings:

1) Any section $E \xrightarrow{s} M$ of a fiber bundle is an embedding, so $s(M) \subseteq E$ is a submanifold.

2) The zero section of TM, $s : TM \to M$ given by $p \mapsto (p, 0)$ is an embedding.

3) The regular value theorem extends to manifolds in a straightforward way:

Theorem 1.4. *Let $F : M \to N$ be smooth, $q \in N$ be such that for all $p \in F^{-1}(q)$, dF_q is surjective, then $F^{-1}(q)$ is a submanifold of M of codimension equals to $\dim(N)$.*

2. General results about embeddings, generalized regular value theorem

We first prove some general results about embeddings:

Proposition 2.1. *Let $F : M \to N$ be an injective, proper immersion. Then F is an embedding.*

As a corollary, if $F : M \to N$ is an injective immersion and M is compact, then F is an embedding.

It follows immediately that any immersion $F : M \to N$ is locally an embedding, i.e., for any $p \in M$, there exists some open neighborhood U of p open in M such that $F|_U : U \to N$ is an embedding.

The proof uses the general fact that a proper, continuous map into a Hausdorff, locally compact space is closed.

Tangent space to a submanifold $i : S \hookrightarrow M$: for any $p \in S$, the map $di_p : T_pS \in T_pM$ is injective. Moreover, if (U, F) is a locally defined function for $U \cap S$ and $p \in U$, then $T_pS = \ker dF_p$.

Toward generalized RVT:

Theorem 2.2. *If a map $F : M \to N$ has constant rank r, then every fiber $F^{-1}(q)$ of F is a submanifold of M with codimension r. (Intuition: information of dimension r is carried to N. If we fix a fiber, then information of dimension r is lost. Hence the codimension of $F^{-1}(q)$ is r.)*

Given any $F : M \to N$, a point $p \in N$ is a regular value of F if for any $p \in F^{-1}(q)$, $dF_p : T_pM \to T_qN$ is surjective. Otherwise q is called a critical value. Note that if $\dim(N) > \dim(M)$, then none points in $F(M)$ is regular. By Sard's Theorem $F(M)$ has measure 0.

Theorem 2.3. *If q is a regular value and $F^{-1}(q)$ is nonempty, then $F^{-1}(q)$ is a submanifold of codimension $= \dim(N)$.*

The proof reduces this theorem to last theorem by replacing M with the set $\{p \in M : dF_p \text{ is surjective}\}$, which can be shown to be open (using det is continuous).

3. Sard's Theorem

Theorem 3.1. *Let $F : M \to N$ be a smooth map. Then the set of regular values of F is dense in N; in fact, the set of critical values has measure 0.*

Since smooth maps are locally Lipschitz, we know that pushforward by smooth maps preserves measure 0. And for a manifold M, we say $S \subseteq M$ has measure 0 if and only if for any chart (U, ϕ) on M, $\phi(U \cap S)$ has measure 0. Using that countable collection of measure 0 set has measure 0, this definition is the same as saying that $S \subseteq M$ has measure 0 $\iff$ there is a countable cover $\{U_i, \phi_i\}$ of S by charts such that for each i, $\phi_i(U_i \cap S)$ has measure 0. Under these charts, we see that the complement of sets of measure 0 is dense.

4. Transversality

Definition 4.1. *Given $F : M \to N$ and $S \subseteq N$, we say that F is transverse to S if for any $p \in F^{-1}(S)$ we have*

$$T_{F(p)}N = T_{F(p)}S + Im\ dF(T_pM).$$

We will prove a transversality theorem, which really generalizes the regular value theorem:

Theorem 4.2. *If $F : M \to N, S \subseteq N$ is a submanifold transversal to F, then $F^{-1}S$ is a submanifold of M such that*

1) The codimension of $F^{-1}(S)$ in M equals to the codimension of S in N. (Codimension is preserved under pullback)

2) For any $p \in F^{-1}(S)$, we have $T_p(F^{-1})S \simeq dF_p^{-1}(T_{F(p)}S)$.

When S is a point, this is exactly the regular value theorem. A nice case is when $iLS_1 \hookrightarrow N$ is a submanifold, and $S \subseteq N$ is another submanifold. Then $i^{-1}(S) = S \cap S_1$. If S and S_1 intersect transversally, for any $q \in S \cap S_1$, $T_qN = T_qS + T_qS_1$. This means that (the codimension of $S \cap S_1$ in S) = (the codimension of S_1 in N), and (the codimension of $S \cap S_1$ in S_1) = (the codimension of S in N). This shows $S \cap S_1$ has codimension = codimS_1+codimS_2 in N.

5. Thom Transversality Theorem

We say that $\{F_s : M \to N\}$ is a smooth family of maps where $s \in \Sigma$, a manifold (also called parameter space) if $F : M \times \Sigma \to N$ given by $(p, s) \mapsto F_s(p)$ is smooth. Combining transversality and Sard's Theorem, we have

Theorem 5.1. *Thom Transversality Theorem*
Let $F : M \times \Sigma \to N$ be a smooth family of maps, $X \subseteq N$ a submanifold. Assume $F \pitchfork X$. Then for almost all $s \in \Sigma$, $F_s \pitchfork X$.

Corollary 5.2. *Let $S_1, S - 2 \subseteq \mathbb{R}^N$ be two submanifolds. For any $s \in \mathbb{R}^N$, let $S_2 + s := \{q + s : q \in S_2\}$. Then for almost all s, we have $S_1 \pitchfork (S_2 + s)$.*

The proof uses that $F : S_2 \times (\Sigma = \mathbb{R}^N) \to \mathbb{R}^N$ is a submersion and a submersion is transversal to all submanifolds.

6. Whitney Embedding Theorem

This is an application of Sard's Theorem.

Theorem 6.1. *Weak form. Any C^∞ manifold of* $\dim n$ *can be embedded in to $\mathbb{R}^{2n+1}$ and immersed in $\mathbb{R}^{2n}$.*
Strong form. Any C^∞ manifold of $\dim n$ *can be embedded in to $\mathbb{R}^{2n}$.*

Some important steps toward proving the first part of the theorem (embedding M into some $\mathbb{R}^N$ for $N >> 0$:

Definition 6.2. *The unit tangent bundle of $M \hookrightarrow \mathbb{R}^N$ is*

$$SM := \{(p, v) \in TM : ||v|| = 1\} \subseteq TM.$$

For instance, $S(S^2) \simeq SO(3)$, the isomorphism given by $(p, v) \mapsto (p, v, p \times v)$.

Proposition 6.3. *SM is an embedded submanifold of TM of dimension $2n - 1$.*

The proof uses the regular value theorem by showing that 1 is a regular value of the map $F : TM \to \mathbb{R}$ given by $(p, v) \mapsto ||v||^2$. We know need to show that $dF_{(p,v)}$ is not the zero map for any $(p, v) \in SM$. Considering the curve $\gamma(t) = (p, tv)$ in the radical direction and we see $dF_{(p,v)} = 2 \neq 0$. So SM has codimension 1.

7. Partition of unity

We first establish some preliminary results for paracompactness:

Proposition 7.1. *If M is second countable, then every cover of M has a countable subcover. This in particular implies that a manifold M can be covered by countably many charts or regular coordinate balls.*

Lemma 7.2. *Let M be a manifold. Then there exists a basis $\mathcal{B}$ of M consisting of pre-compact sets ($\forall B \in \mathcal{B}$, $\overline{B}$ is compact).*

Theorem 7.3. *Paracompactness of manifolds. Let $\mathcal{C} = \{C_\alpha\}_{\alpha \in A}$ be an open cover, $\mathcal{B}$ a basis of M. Then there exists a locally finite refinement $\mathcal{R}$ of $\mathcal{C}$ consisting of elements of $\mathcal{B}$. elements from the basis $\mathcal{B}$ that are fine enough to fit into $\mathcal{C}$ and not too many because of locally finiteness.*

Theorem 7.4. *Existence of partition of unity subordinate to any cover. Given M and a cover $\mathcal{C} = \{C_\alpha\}_{\alpha \in A}$, there exists $\{\psi_\alpha\}_{\alpha \in A}$ such that*
1) for any $\alpha \in A$, $\psi_\alpha \in C^\infty(M)$
2) $\{supp\psi_\alpha\}$ is locally finite
3) $supp\psi_\alpha \subseteq C_\alpha$
4) $\sum_{\alpha \in A} \psi_\alpha \equiv 1$ on M

8. Lie algebra and Lie bracket

For any $X, Y \in \mathcal{X}(M)$, we can check the commutator $[X, Y]$ is in $\mathcal{X}(M)$ as well (i.e., is $\mathbb{R}$-linear and satisfies the product rule). So $[-,-]$ is a map $\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$. In particular, it satisfies the Jacobi identity:
1) $[-,-]$ is $\mathbb{R}$-linear
2) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$
3) Product rule. $[X, fY] = f[X, Y] + (Xf) \cdot Y$

Definition 8.1. *A real Lie algebra is a real vector field $\mathfrak{g}$ together with a skew-symmetric bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity.*

As an example, $\mathbb{R}^3$ is a Lie algebra with $[u, v] = u \times v$, and $M_n(\mathbb{R})$ is a Lie algebra with $[A, B] = AB - BA$.

Pushforward of vector fields. For a general smooth map $F : M \to N$, we cannot pushforward a vector field X on M. However, if F is a diffeomorphism, then we can define $F_*(X) \in \mathcal{X}(M)$ by: for any $q \in N$, define $F_*(X)_q := dF_p(X_p)$ where $p = F^{-1}(q)$.

In general, fields don't have good functionality. But sometimes we have nice pair of vector fields:

Definition 8.2. *Given $F : M \to N$ smooth map, $X \in \mathcal{X}(M)$ anad $Y \in \mathcal{X}(N)$, we say they are F-related if for any $p \in M$, $dF_p(X_p) = Y_{F(p)}$.*

In terms of operators, this is the same as saying that for any $g \in C^\infty(N)$, we have

$$X(g \circ F) = (Yg) \circ F.$$

It should be noted that $Y_{F(p)}$, when applied to any $g \in C^\infty(N)$, the input are of the form $F(p)$ for $p \in M$ instead of directly from N. This follows from the definition, but is easy to ignore. Noticing this, the equivalence of the two definitions are clear.

9. Left invariant vector field

We first proved that the Lie bracket of two F-related vector fields are F-related as well. Then we define left-invariant vector fields for Lie groups G:

Definition 9.1. *A vector field $X \in \mathcal{X}(G)$ is left-invariant $\iff$ $\forall g \in G$, X is L_g-related to itself $\iff$ $\forall k, g \in G$,*
$$d(Lg)_k(X_k) = X_{gk}.$$

Note that a left-invariant vector field on G is determined by X_e, since for any $g \in G$, we have $X_g = dLg_e(X_e)$.

Definition 9.2. *Define $Lie(G) = \{X \in \mathcal{X}(G) : X$ left invariant vector field$\}$.*

Proposition 9.3. *We have an evaluation map $ev : Lie(G) \to \mathfrak{g}$ given by $X \mapsto X_e$. This is an isomorphism. So $Lie(G)$ is a vector space of $\dim G$.*

Using that F-relatedness is preserved under Lie bracket, we have the following proposition:

Proposition 9.4. *$Lie(G)$ is a Lie algebra (i.e., $[X, Y] \in Lie(G)$ for any $X, Y \in Lie(G)$).*

We can translate $[-, -]$ via the evaluation map: For any $A, B \in \mathfrak{g}$, define $[A, B] := [X^A, X^B]_e$, where $X^A \in Lie(G)$ is such that $ev_e(X^A) = A$. This makes $(\mathfrak{g}, [-, -]_e)$ the Lie algebra of G.

Using the product rule for Lie bracket, we can derive
$$[X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial y^j}] = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial y^j} - Y^j \frac{\partial X^i}{\partial y^j} \frac{\partial}{\partial x^i}.$$
Note that the coefficients can be computed in a neighborhood of e, so we don't need to extend $v \in \mathfrak{g}$ to the whole M to compute the Lie bracket; just a neighborhood suffices. Intuitively, this says Lie bracket only contains local information.

Note that the Lie algebra of the Lie group $(\mathbb{R}^n, +)$ is $\{X^i \frac{\partial}{\partial x^i} : X^j \text{ constant}\}$. So this Lie algebra is abelian, and the Lie bracket is constantly zero. An important non-abelian algebra if $\mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$ of $GL(n, \mathbb{R})$, where the Lie bracket is just the matrix commutator.

Passing a map of Lie groups to resp. Lie algebras. Let $F : G \to H$ be a morphism of Lie groups (i.e., F is smooth and a group morphism). Then $dF_e : \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie algebras, i.e., $\forall A, B \in \mathfrak{g}$, we have
$$[dF_e(A), dF_e(B)] = dF_e[A, B].$$

We can draw some important conclusions:

1) If G is a subgroup and embedded submanifold (i.e., Lie subgroup) of H, F is the inclusion map, then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{h}$.

2) In particular, for any matrix group G, it is a Lie subgroup of $GL(n, \mathbb{R})$, and so $\mathfrak{g} \hookrightarrow \mathfrak{gl}(n, \mathbb{R})$. Hence the Lie bracket of $\mathfrak{g}$ is the same as matrix commutator.

10. Dynamics: from flows to generators

A continuous time dynamical system on M is a C^∞ action of $(\mathbb{R}, +)$ where

1) $\theta : \mathbb{R} \times M \to M$ is a C^∞ map for any $t, s \in \mathbb{R}, p \in M$.

2) Group Law. $\theta(s, \theta(t, p)) = \theta(t + s, p)$.

For different purposes, we can write $\theta(t, p) = \theta_t(p)$, the time t map, or $\theta^p(t)$, an (integral) curve starts at p.

We can view flows as both operators and dynamics. Given θ as above, its infinitesimal generator is the vector field X on M given by, $\forall p \in M$,

$$X_p = \frac{d}{dt}\Big|_{t=0} \theta^p(t) \in T_pM.$$

In terms of the flow, we can rephrase the Group law as:

Proposition 10.1. *Let X be the infinitesimal generator of θ. Then for any $p \in M, t_0 \in \mathbb{R}$, we have*

$$\frac{d}{dt}\Big|_{t=t_0} \theta^p(t) = X_{\theta^p(t_0)}.$$

Moreover, for any $t \in \mathbb{R}$, X is θ_t-related to itself.

11. Dynamics: from generators to flows

In this section we reverse the process of last section. Given any $X \in \mathcal{X}(M)$, we can always find a local flow. If X is compact or a Lie group, then the global solution always exists. The main tool of studying this is the notion of an integral curve.

Definition 11.1. *We say $\gamma : (a,b) \to M$ is an integral curve of $X \in \mathcal{X}(M)$ if for any $t \in (a,b)$, $\frac{d\gamma}{dt}(t) = X_{\gamma(t)}$, and γ starts at $\gamma(0) = p$ (assuming $0 \in (a,b)$).*

Some properties:

Proposition 11.2.

1) Existence of local solution. For any $p \in M$, there exists some $\epsilon > 0$ and $\gamma : (-\epsilon, \epsilon) \to M$ that starts at p.

2) Uniqueness of solution. Two integral curves of X starting at the same point $p \in M$ agree on the intersection of their domain.

3) Translation lemma. If γ is an integral curve of X and $t_0 \in \mathbb{R}$, then $\tilde{\gamma}(t) := \gamma(t - t_0)$ with the corresponding domain is an integral curve of X as well.

Theorem 11.3. *Let $X \in \mathcal{X}(M)$.*

a) For any $p \in M$, there exists a unique $(\alpha(p), \beta(p)) \in \mathbb{R} \cup \{\pm\infty\}$ containing 0 such that there is an integral curve θ^p of X defined on $(\alpha(p), \beta(p))$ such that $\theta^p(0) = p$, and the domain of θ^p is maximal with respect to all such curves.

b) $\mathcal{D} = \cup_{p \in M}((\alpha(p), \beta(p)) \times \{p\}) \subseteq \mathbb{R} \times M$ is open in $\mathbb{R} \times M$.

c) $\theta : \mathcal{D} \to M$ given by $(t,p) \mapsto \theta^p(t)$ is C^∞, and satisfies the group law.

An example where the global solution does not exist is given by $M = \mathbb{R}$ and $X_x = x^2 \frac{d}{dx}$. By integrating both sides and we have $x(t) = \frac{1}{\frac{1}{x(0)} - t}$. If we assume that $x(0) \neq 0$ (so that the solutions are not constants), we see that when $t = \frac{1}{x(0)}$ the solution blows up, so doesn't exist.

Also observe that in $b)$ of the theorem, for any $p \in M$, there exists some $\epsilon > 0$ and $U \subseteq M$ open neighborhood of p such that θ is defined on $(-\epsilon, \epsilon) \times U \subseteq D$. We call

$$\theta : (-\epsilon, \epsilon) \times U \to M$$

a local flow of X.

Lemma 11.4. *Uniform time lemma. If $\epsilon > 0$ is such that $(-\epsilon, \epsilon) \times M$ is included in $\mathcal{D}$ (i.e., $(-\epsilon, \epsilon) \subseteq (\alpha(p), \beta(p))$), then X is complete.*

As a corollary, we find that for compactly supported X, using the existence of local flows and compact support, X satisfies the hypothesis in the Uniform time lemma, hence is complete.

12. Integral curves of F-related fields

We study the dynamical interpretation of F-relatedness:

Proposition 12.1. *Let $F : M \to N$ be C^∞, $X \in \mathcal{X}(M), Y \in \mathcal{X}(N)$. Then X and Y are F-related if and only if for any $\gamma : (a, b) \to M$ integral curve of X, $F \circ \gamma$ is an integral curve of Y.*

This point of view makes it easy to prove that left-invariant vector fields on Lie groups are complete.

Proposition 12.2. *Let G be a Lie group, $X \in Lie(G)$ a left-invariant vector field. Then X is complete.*

The proof uses that X is L_g-relate to itself $\forall g \in G$, and we can translate the local solution at e via L_g everywhere and get a local solution everywhere by the dynamical interpretation of L_g-relatedness. So the uniform time lemma is satisfied in this situation also, even if G could be non-compact (like $SL_2(\mathbb{R}), SL_2(\mathbb{C})$.)

This gives us a practical interpretation of vanishing of Lie bracket of two vector fields:

Theorem 12.3. *Given $X, Y \in \mathcal{X}(M)$, let θ be a flow of X, ϕ a flow of Y. TFAE:*
1) $[X, Y] = 0$
2) for any $t, s \in \mathbb{R}$, $\theta_t \circ \phi_s = \phi_s \circ \theta_t$
3) For any $t, s \in \mathbb{R}$, X is ϕ_s-related to itself, and Y is ψ_t-related to itself.

The equivalence (2) $\iff$ (3) follows from the dynamical interpretation of F-relatedness. To prove (1) $\iff$ (2), note that (2) is a condition on curve, so translate the action of vector on a function, $[X, Y]f$, into derivative on curves as usual and the result follows.

13. Definition of forms

We defined forms on vector spaces, and then defined forms on manifolds pointwise and smoothness condition. We also discussed an interesting example:

Let $M \subseteq \mathbb{R}^3$ be a surface, and B a smooth field on $\mathbb{R}^3$. Define a $2-$form β on M given by: $\forall p \in M, \beta_p(v_1, v_2) = \det(B_p, v_1, v_2)^T = B_p \cdot (v_1 \times v_2) = B_p \cdot \frac{v_1 \times v_2}{||v_1 \times v_2||} ||v_1 \times v_2||$. Note $\frac{v_1 \times v_2}{||v_1 \times v_2||}$ is the normal unit vector to M, and $||v_1 \times v_2||$ is the area of the parallelogram spanned by v_1 and v_2. So the whole expression is the flux of B_p across the parallelogram. So computing the flux is the same as computing the 2-form β.

14. Forms on manifolds

Definition 14.1. *A rough k-form α on M is an assignment*

$$M \ni p \mapsto \alpha_p \in \bigwedge^k (T_p^* M).$$

In local coorindates $(U, \phi = (x^1, \cdots, x^n))$ and any $p \in U$, we can write $\alpha_p = \sum_I' \alpha_I(p) dx_p^I$. We say that α is smooth if and only if α_I are smooth for all I.

Let V be a vector space. For any $k, l, \alpha \in \bigwedge^k V^*, \beta \in \bigwedge^l V^*$, we have anti-commutativity, i.e., $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$. This implies that even forms commute with all other forms, and $dx \wedge dx = -dx \wedge dx = 0$, i.e., wedge product of one-form with itself is 0. We also have the general statement that $\omega \wedge \omega = 0$ for an odd-degree form ω.

On manifolds, we formally define

$$\bigwedge^k T^*M = \bigcup_{p \in M} (\{p\} \times \bigwedge^k (T_p^* M)).$$

Theorem 14.2. *Let* $\dim M = n$. *There exists a unique* C^∞*-manifold structure on* $\bigwedge^k T^*M$ *that makes* $\pi : \bigwedge^k T^*M \to M$ *into a fiber bundle with fiber* $\bigwedge^k \mathbb{R}^n$. *Since fibers are vector spaces, this makes* π *into a vector bundle.*

So we can formally define $\Omega^k(M)$ to be the set of all C^∞-sections of $\bigwedge^k T^*M$. Any form $\alpha = \sum_I' \alpha_I dx^I$ corresponds to a section $(x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^n, a^I, \cdots)$.

15. Pullback forms

Properties of pullback of forms. Let $\Phi : V \to W$ be a map of vector spaces, and α, β two-forms on W. Then:

1) Φ^* is linear;
2) $\Phi^*(\alpha \wedge \beta) = \Phi^*(\alpha) \wedge \Phi^*(\beta)$;
3) Pullback commutes with composition of maps.

In the manifold case, let $f : M \to N$ be a smooth map. If $g \in \Omega^0(N) = C^\infty(N)$, then we have an additional property, commutation with d:

4) $F^*(dg) = d(F^*g)$.

A general formula for pulling back forms, which also proves that pullback of smooth form is smooth. Let $\beta \in \Omega^k(N)$, $(V, (y^1, \cdots, y^m))$ local coordinate on N, and $(U, (x^1, \cdots, x^n))$ local coordinate on M such that $U \subseteq F^{-1}(V)$. Write $\beta = \sum_I' \beta_I dy^I$ for $I = \{i_1 < \cdots < i_m\}$, $\beta_I \in C^\infty(V)$. Then we have

$$F^*(\sum_I{}' \beta_I dy^I) = (\beta_I \circ F) dF^{i_1} \cdots dF^{i_m}.$$

Special case: top-degree form. Let $M = \mathbb{R}^n$. Then $\bigwedge^n T_pM$ has dimension 1, generated by any top degree form $\omega = b dx^1 \wedge \cdots \wedge dx^n$. If $F : M \to M$, then $F^*\omega = f\omega$ for some $f \in C^\infty(\mathbb{R}^m)$. By evaluating on the standard basis $\{\partial x_i\}_{1 \leq i \leq n}$, we obtain that

$$F^*\omega = (b \circ F) \det(\frac{\partial F^i}{\partial x^j}) dx^1 \cdots dx^n.$$

Recall the change of variable formula for integration

$$\int_B b(y) dV = \int_A (b \circ F)(x) \left| \det(\frac{\partial F^i}{\partial x^j}) \right| dU,$$

this suggests that if $\det(\frac{\partial F^i}{\partial x^j}) > 0$, then $\int \omega = \int F^*\omega$.

16. The exterior differential

We first proved a theorem-definition: on any C^∞ manifold, there is a unique $d : \Omega^n(M) \to \Omega^{n+1}(M)$ such that

1) d is $\mathbb{R}$-linear
2) $d : \Omega^0 \to \Omega^1$ is the differential of functions
3) anti-derivation: for any $\alpha \in \Omega^k, \beta \in \Omega^l$, we have $d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$.
4) $d \circ d = 0$
5) If $F : M \to N$ is C^∞, then for any k, we have $d \circ F^* = F^* \circ d$ (This is a generalization of last section, where we have $F^*(dg) = d(F^*g)$.

Clairaut's Law. Suppose $\alpha = d\beta$ for some β. Locally write $\alpha = \sum_{j=1}^n \alpha_j dx^j$. Then $d\alpha = \sum_{i<j} \left(\frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j} \right) dx^i dx^j$. So $d\alpha = 0$ if and only if $\frac{\partial \alpha_j}{\partial x_i} = \frac{\partial \alpha_i}{\partial x_j}$. Since exact forms are closed, we have this equality. (If the equality holds, then α is closed, but not necessarily exact.)

17. Differential operator on general manioflds

We defined the de Rham complex and H^*_{dR}. Note that H^* is a contravariant functor, since we pullback forms. Moreover, $\wedge$ defined on $\Omega^*(M)$ naturally descends to $\wedge$ on $H^*(M)$. So $\bigoplus_{k=0}^n H^k(M)$ becomes a ring.

18. Orienting manifolds using atlases

[1]Given M, an atlas $\{U_\alpha, \phi_\alpha\}$ of M is coherently oriented if for all α, β such that $U_\alpha \cap U_\beta = \emptyset$, we have

$$\det(Jac(\phi_\alpha \circ \phi_\beta^{-1})) > 0.$$

We say that M is orientable if there exists such an atlas, and for any $\mu \in \Omega^n_c(M)$, where $n = \dim(M)$, we can define

$$\int_M \mu = \sum_\alpha \int (\phi_\alpha^{-1})^* (\overline{\chi_\alpha \mu})$$

where χ_α is a partition of unity subordinate to $\{U_\alpha\}$. We can show that this definition is independent of the choice of ϕ_α and atlases (as long as the the union of two atlas is coherently oriented).

19. Orienting manifolds using forms

On vector spaces: For any vector space V of dimension n, note $\bigwedge^n V$ is a vector space of dimension 1. Hence $\bigwedge^n V \backslash \{0\}$ has two connected components. There is a one to one correspondence between the choice of connected component and an orientation of V, given by $[f] \mapsto$ (μ's component), where $\mu \in \bigwedge^n V \backslash \{0\}$ is such that $\mu([f]) > 0$. If we are given an orientation on V, then basis forms in $\bigwedge^n V \backslash \{0\}$ that is equivalent to this given orientation is called positive.

[1]part of the material comes from lecture 33

On manifolds: An orientation of M is an assignment of an orientation to T_pM for each $p \in M$ that satisfies the smoothness condition in the following sense: for any $p \in M$, there exists a neighborhood U and $E_i \in \mathcal{X}(M)$ for $1 \le i \le n$ such that for any $q \in U$, $\{(E_1)_q, \cdots, (E_n)_q\}$ is a positive basis of T_qM. We call these E_i a moving frame.

Proposition 19.1. *The smoothness condition above is equivalent to the existence of a non-vanishing top-degree form on M.*

Next proposition shows that orientability defined using forms agree with orientability defined in last section, using coherent atlas:

Proposition 19.2. *M has a coherently oriented atlas $\iff$ one can orient T_pM for any $p \in M$ in a continuous fashion (i.e., satisfies the smoothness condition) $\iff$ there is a non-vanishing top-degree form on M.*

So the definitions using coherent atlas, smoothness condition, and non-vanishing forms are the same.

20. Stokes' Theorem for manifolds without boundary

Setup. Let M be an oriented manifold with a fixed orientation. Then for any $p \in M$, the vector space T_pM is oriented, and the function $p \mapsto$ orientation of T_pM is constant. With respect to this given orientation, we say a basis of T_pM positive if it is in the oriented class, and an atlas of M positive if for any $(x^1, \cdots, x^n)$ in the atlas, $\{\partial_{x^1}, \cdots, \partial_{x^n}\}$ is a positive basis of T_pM.

Definition 20.1. *Given M oriented, $\dim M = n$, define $\int_M \mu$ using a positive atlas as in 18. For any oriented submanifold $i : S \hookrightarrow M$ of dimension k, for any $\alpha \in \Omega_c^k(M)$, define $\int_S \alpha := \int_S i^*\alpha$.*

Note that submanifolds of an oriented manifold is not necessarily orientable: consider the Mbius band in $\mathbb{R}^3$. However, if $k = n-1$ and for any $p \in S$, we have a normal vector $n_p \in T_pM \backslash T_pS$ that varies continuously, then we can declare a basis $\{b_1, \cdots, b_{n-1}\}$ of T_pS to be continuous if and only if $\{n_p, b_1, \cdots, b_{n-1}\}$ is a positive basis of T_pM. So a submanifold is not orientable implies there is no continuous normal vector field.

We now prove Stokes' Theorem via a few steps:

Theorem 20.2. *If M is oriented, for any $\alpha \in \Omega_c^{n-1}(M)$, we have*

$$\int_M d\alpha = 0.$$

Proof. We first use Fubini's Theorem when $M = \mathbb{R}^n$. For general manifolds, use the partition of unity and product rule to break the integral into as difference, and use case 1 and $\sum_\alpha d(\chi_\alpha) = 0$ to show the integral is 0. ☐

Corollary 20.3. *Recall that if $n = \dim M$, we have*

$$H^n(M) = \frac{\ker(\Omega^n \to \Omega^{n+1})}{Im(\Omega^{n-1} \to \Omega^n)} = \frac{\Omega^n}{Im(\Omega^{n-1} \to \Omega^n)}.$$

If M is compact and oriented, then $\int_M$ induces a map $H^n(M) \to \mathbb{R}$ given by

$$[\mu] \mapsto \int_M \mu.$$

This is well-defined by Stokes' Theorem.

21. Stokes' Theorem for manifolds with boundary

Recall that $\mathbb{H}^n$ is defined to be the set $\{x^n \geq 0\} \subseteq \mathbb{R}^n$. Stokes' Theorem on $\mathbb{H}^n$:

Theorem 21.1. *For any $\alpha \in \Omega_c^{n-1}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{H}^n} d\alpha = \int_{\partial\mathbb{H}^n} \alpha.$$

First consider $\partial\mathbb{H}^n \simeq \mathbb{R}^{n-1}$ with the standard orientation, and we will see a sign problem to fix. Locally write $\alpha = \sum_i \alpha_i dx^1 \cdots \widehat{dx^i} \cdots dx^n$. Since α has compact support, by fundamental theorem of calculus and Fubini's theorem we have

$$\int_{\mathbb{H}^n} d\alpha = (-1)^n \int_{\mathbb{R}^{n-1}} \alpha_n(x^1, \cdots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}.$$

Now consider the right hand side, where

$$\int_{\partial\mathbb{H}^n} \alpha = \int_{\partial\mathbb{H}^n} i^*\alpha.$$

Write $i^*\alpha = g dx^1 \cdots dx^{n-1}$ for some $g \in C^\infty(\partial\mathbb{H}^n)$. By plugging in $\{\partial_{x^1}, \cdots, \partial_{x^{n-1}}\}$, we have $g = \alpha_n(x^1, \cdots, x^{n-1}, 0)$. So there is a sign difference of $(-1)^n$ between both sides.

To fix this, we need to orient $\partial\mathbb{H}^n$ in a way that will compensate the right hand side a factor of $(-1)^n$. Let e_n be the outward pointing normal vector of $\mathbb{H}^n$. Endow $\partial\mathbb{H}^n$ with the boundary orientation: $(e_1, \cdots, e_{n-1})$ is a positive basis of $\partial\mathbb{H}^n$ if and only if $(-e_n, e_1, \cdots, e_{n-1})$ is a positive basis of $\mathbb{R}^n$ (with respect to the standard basis of $\mathbb{R}^n$). Computing the determinant of the transition matrix we see that this orientation is positive if and only n is even. If n is odd, the boundary orientation is negative (w.r.to the standard orientation). We need to multiply RHS by -1 because we first computed RHS by regarding $\partial\mathbb{H}^n \simeq \mathbb{R}^{n-1}$.

Smooth structure on $\mathbb{H}^n$. We say that $F : V \to \mathbb{R}^m$ is smooth for some $V \subseteq \mathbb{H}^n$ open if there exists some $\widetilde{V} \subseteq \mathbb{R}^n$ open, $\widetilde{F} : \widetilde{V} \to \mathbb{R}^m$ such that

1) $V = \widetilde{V} \cap \mathbb{H}^n$
2) $\widetilde{F}|_V = F$
3) $\widetilde{F}$ is smooth.

Lemma 21.2. *For any V_1, V_2 open in $\mathbb{H}^n$ and a diffeomorphism $F : V_1 \to V_2$, F restricts to two diffeomorphisms on the interior and on the boundary, respectively. This in particular implies there is no diffeomorphism from an open set that does not intersect $\partial\mathbb{H}^n$ to another one that intersects $\partial\mathbb{H}^n$.*

22. Stokes' Theorem on manifolds with boundary

We defined a topological manifold with boundary in the last section. We say that M is a smooth manifold with boundary if M is a topological manifold with boundary together with a maximal atlas (ϕ_i, U_i) such that for any i, j, $\phi_i \circ \phi_j^{-1}$ is C^∞. From last time, this means that we can extend $\phi_i \circ \phi_j^{-1}$ to a smooth map between open sets of $\mathbb{R}^n$. For instance, if $\widetilde{M}$ is a manifold and $0 \in \mathbb{R}$ is a regular value of $f : \widetilde{M} \to \mathbb{R}$, then

$$M := \{p \in \widetilde{M} : f(p) \leq 0\}$$

is a manifold with boundary. This is because we can change coordinates on $\widetilde{M}$ to $(y^1, \cdots, y^{n-1}, f = y^n)$ and $M = \{y^n \leq 0\}$ in the new coordinates. Note that the manifold coordinates on M is $(y^1, \cdots, y^{n-1}, -y^n)$ since y^n is pointing toward the positive.

Since there is an invisible extension over ∂M, for any $p \in \partial M$, the tangent space T_pM is still n-dimensional. However, $T_p(\partial M)$ is a codimension 1 subspace of T_pM. So $T_pM \backslash T_p(\partial M)$ is the union of two open, disconnected subspaces, and this allows us to define inward and outward pointing vectors. There are two equivalent definitions:

Definition 22.1. *We say $v \in T_pM$ is inward pointing if for any $(x^1, \cdots, x^n)$ coordinate system near p, write $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$ and we have $v^n > 0$.*

This is independent of the choice of chart, since there is an equivalent intrinsic definition, namely there is a curve $\gamma : [0, \epsilon) \hookrightarrow M$ such that $\gamma(0) = p, \dot{\gamma}(0) \notin T_p(\partial M)$.

Even though many definitions and properties of manifolds without boundary naturally extend, it is not true that if M with $\partial M \neq \emptyset$ and is compact, all vector fields are complete. For instance, consider $X = -\frac{\partial}{\partial x^n}$ on $\mathbb{H}^2$. It has no integral solution on the boundary. But it is true that if M is compact, all vector fields X on ∂M is complete.

Lemma 22.2. *If M is orientable, then ∂M is orientable with the boundary orientation. (Orient using the outward pointing normal vector field, which exists from the definition.)*

Stokes' Theorem, complete form. If M is oriented and ∂M has the boundary orientation, then for any $\omega \in \Omega_c^{n-1}(M)$, we have

$$\int_M d\omega = \int_{\partial M} \omega,$$

where ∂M has the boundary orientation.

23. Lie derivative of a form

Take $X \in \mathcal{X}(M), f \in \Omega^0(M)$. We have a series of equalities, depending on how we view things:

$$X_p f = df_p(X_p) = \frac{d}{dt}\Big|_{t=0} f(\theta_t(p)) = \frac{d}{dt}\Big|_{t=0} (\theta_t^* f)(t)$$

where θ is an integral curve of X at p, and the second can be viewed as the pushforward of X_p by f, or the differential operator df acting on X_p. But the last definition does not only limit to 0-degree forms. For any n and $\omega \in \Omega^n(M)$, we define the Lie derivative

$$\mathcal{L}_X \omega = \frac{d}{dt}\Big|_{t=0} (\theta_t^* \omega).$$

This is a map from $\Omega^n(M) \to \Omega^n(M)$ for each n, and is $\mathbb{R}$-linear (since both pullback and differentiation are). The $\mathcal{L}_X$ has following properties:
1) If $k = 0$, then $\mathcal{L}_X f = Xf$.
2) Product Rule. $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta$.
3) Commutes with d. $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X \omega)$
4) Commutes with pullback (naturality). Given any $F : M \to M$ diffeomorphism, $\mathcal{L}_X(F^*\omega) = F^*(\mathcal{L}_{F_*X}\omega)$, where $F_*(X)_p = dF_{F^{-1}(p)}(X_{F^{-1}(p)})$.

We prove the naturality:

Corollary 23.1. *Fix $t_0 \in \mathbb{R}$. Take F as in naturality. Set $F = \theta_{t_0} : M \to M$ (assuming θ_{t_0} exists). Then we have*

$$\frac{d}{dt}(\theta_t^*\omega)_p\Big|_{t=t_0} = \mathcal{L}_X(\theta_{t_0}^*\omega) = \theta_{t_0}^*\mathcal{L}_X\omega,$$

where the first equality uses group law and the second is just naturality. This implies that

$$\theta_{t_0}^*\omega = \omega \text{ for any } t_0 \in \mathbb{R} \iff \mathcal{L}_X(\omega) = 0.$$

Contraction of a form. Given $X \in \mathcal{M}, \omega \in \Omega^k$, define $\iota_X\omega \in \Omega^{k-1}$ pointwise by putting in X_p as the first input. It has properties:
1) $\iota_X : \Omega^k \to \Omega^{k-1}$ is $C^\infty(M)$-linear (since the definition is pointwise)
2) Product rule. $\iota_X(\alpha \wedge \beta) = (\iota_X\alpha) \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge (\iota_X\beta)$.
Cartan's M-formula.

$$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d.$$

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^{k-1}(M) \arrow[r, "d"] \arrow[d, "\mathcal{L}_X"] & \Omega^k(M) \arrow[r, "d"] \arrow[d, "\mathcal{L}_X"] \arrow[dl, "\iota_X"] & \Omega^{k+1}(M) \arrow[r] \arrow[d, "\mathcal{L}_X"] \arrow[dl, "\iota_X"] & \cdots \\
\cdots \arrow[r] & \Omega^{k-1}(M) \arrow[r, "d"'] & \Omega^k(M) \arrow[r, "d"'] & \Omega^{k+1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

There are many applications. For instance, if $\omega \in \Omega^2(M)$ and is closed (such as sympletic forms), then $\mathcal{L}_X\omega = d \circ \iota_X(\omega)$. If, moreover, $\iota_X\omega$ is exact, then $\mathcal{L}_X\omega = 0$ since $d^2 = 0$.

24. de Rham cohomology

Define the k-th Betti number of M, β_k to be $\dim_{\mathbb{R}} H^k$. We will see that Betti numbers are homotopy invariants.

H^0: Since $\Omega^{-1} = 0$, we have $H^0(M) = \{f \in C^\infty(M) : df = 0\}$. Such f are locally constant, and thus are constant on each connected component. Hence $H^0 \simeq \mathbb{R}^{\beta_0}$, where β_0 is the number of connected components of M.

H^k for $k > n$: Since $\Omega^k = 0$ for such k, so is H^k.

$H^*(\mathbb{R})$: By the two discussion above, we have $H^0(\mathbb{R}) = \mathbb{R}$ and $H^n = 0$ for $n \geq 2$. So closed 1-forms $Z^1 = \Omega^1$. To compute B^1, note any $\omega \in \Omega^1$ can be locally written as fdx for some $f \in C^\infty$. Take $g = \int_{\mathbb{R}} fdx$. Then $dg = fdx$. This shows $B^1 = \Omega^1$, and so $H^1 = \Omega^1/\Omega^1 \simeq 0$.

$H_c^*(\mathbb{R})$: The compact case can be very different. Notice that $f \in C_c^\infty$ has a compactly supported anti-derivative if g and only if $\int_{\mathbb{R}} f dx = 0$ (by Fundamental Theorem of Calculus). So $B_c^1 = \ker(\int_{\mathbb{R}} : \Omega_c^1 \to \mathbb{R})$. The map $\int_{\mathbb{R}}$ is surjective, for example, by using scaler multiple of the bump function. Therefore, $H_c^1(\mathbb{R}) = \Omega_c^1/B_c^1 \simeq \mathbb{R}$.

$H^1(S^1)$: We need to find functions f on S^1 that has a 2π-periodic antiderivative. Using Fourier series, we know such antiderivative exists if and only if $\int_0^{2\pi} f dx = 0$. Hence we have a map $\int_0^{2\pi} : \Omega^1(S^1) \to \mathbb{R}$ as above, and $H^1(S^1) \simeq \mathbb{R}$.

Cohomology ring: We can define the cohomology ring $H^*(M) = \bigoplus_{k=0} H^k(M)$, with the cup product (the descent of wedge product on de Rham complex; we can show this is well-defined): for any $\omega \in Z^k$, write $[\omega] \in H^k$. For $\alpha \in Z^k, \beta \in Z^l$, we define $[\alpha] \cup [\beta] = [\alpha \wedge \beta]$. There are spaces with the same Betti numbers but different ring structures.

Maps between cohomology rings: Given a C^∞ map $F : M \to N$, for any k we can define $F^* : H^k(N) \to H^k(M)$ given by $F^*([\omega]) = [F^*\omega]$. This is a well-defined map between complexes because pullback commutes with d. This is a ring map since it can be shown to preserve cup product. It's easy to see that diffeomorphic manifolds have the same cohomology rings, but much more is true.

25. Homotopy invariance of cohomology rings:

We prove that two homotopy maps $f, g : M \to N$ induce the same map on the level of cohomology. First a lemma:

Lemma 25.1. *Let $X \in \mathcal{X}(M), \partial M = \emptyset$, and X complete with flow θ. Let $\theta_1 = \theta_t\Big|_{t=1}$, the time 1 map. Then the induced map $\theta_1^* : H^*(M) \to H^*(M)$ is the identity.*

The idea is to use Cartan's M-formula and FTC to show $\theta_1^*\omega - \theta_*^0\omega$ is exact, while θ_*^0 is just ω, so they represent the same class when descend to cohomology. Then we can prove the theorem:

Theorem 25.2. *Suppose $F, G : M \to N$ are homotopic. Then $F^* = G^*$ in de Rham cohomology.*

There are two important observations. First, we can regard a homotopy between F and G as a map $\mathbb{R} \times M \to N$ since $[0, 1]$ is a boundary with boundary and by definition, we can always extend it beyond its boundary a little bit. Second, θ_1 in the diagram below induces an automorphism of $H^*(\mathbb{R} \times M)$. Then use the diagram we can prove $G^*[\omega] = F^*[\omega]$ for any $\omega \in H^k(N)$; here $\mathcal{H}$ homotope G and F and $i_a(x) = (a, x)$:

$$\begin{tikzcd}
& \mathbb{R} \times M \arrow[r, "\mathcal{H}"] & N \\
M \arrow[ur, "i_1"] \arrow[urr, "G"] \arrow[dr, "i_0"] \arrow[drr, "F"] & & \\
& \mathbb{R} \times M \arrow[r, "\mathcal{H}"] & N
\end{tikzcd}$$

Corollary 25.3. *If M and N are homotopic equivalenet, then $H^*(M) \simeq H^*(N)$ as rings.*

Corollary 25.4. *If M is contractible, then $H^0(M) = \mathbb{R}$ and all higher cohomology groups vanish.*