Prof. Ryan Cotterell

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Question 1: Exploring the Kleene Star

a)

To prove:

$$a^* \overset{!}{=} 1 \oplus a \otimes a^*$$

$$\begin{aligned}
a^* &= \bigoplus_{n=0}^{\infty} a^{\otimes n} && \text{(def.}*\text{)} && (1)\\
&= 1 \oplus \bigoplus_{n=1}^{\infty} a^{\otimes n} && \text{(def.}\oplus\text{)} && (2)\\
&= 1 \oplus a \otimes \bigoplus_{n=0}^{\infty} a^{\otimes n} && \text{(diss. of } \otimes \text{ over } \oplus\text{)} && (3)\\
& && && (4)
\end{aligned}$$

b)

We want to find a Kleene Star for $\mathcal{W}_{log}$. Using the definition of Kleene Star, we have:

$$\begin{aligned}
a^* &= \bigoplus_{n=0}^{\infty} a^{\otimes n} && \text{(def.}*\text{)} && (5)\\
&= 0 \oplus_{log} a \oplus_{log} a \otimes a \oplus_{log} \dots && && (6)\\
&= \ln(\exp(0) + \exp(a)) \oplus_{log} a + a \oplus_{log} \dots && \text{(def.}\oplus, \otimes\text{)} && (7)\\
&= \ln(\exp(\ln(\exp(0) + \exp(a))) + exp(2a)) \oplus_{log} \dots && && (8)\\
&= \ln(\exp(0) + \exp(a) + \exp(2a)) \oplus_{log} \dots && && (9)\\
&= \ln(\sum_{n=0}^{\infty} \exp(a \cdot n)) && && (10)\\
& && && (11)
\end{aligned}$$

For $a \geq 0$ this series will diverge as $n \to \infty$, because $\lim_{n\to\inf} \exp(a \cdot n) \geq 1$ in that case.

Thus we assume that $a < 0$ for the rest of the proof. We can then rewrite the series as a geometric series as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \exp(a \cdot n) &= \sum_{n=0}^{\infty} g^n && \text{(where } g = \exp(a)\text{)} && (12)\\
&= \frac{1}{1-g} && \text{(limit of the geometric series)} && (13)\\
&= \frac{1}{1-\exp(a)} && (\text{def.}g) && (14)
\end{aligned}$$

For the Kleene Star expression we then get:

$$a^* = \ln\left(\frac{1}{1-\exp(a)}\right)$$

c)

We want to find a Kleene Star for the expectation semiring. Following the same pattern as in the previous part, the definition gives us:

$$\begin{aligned}
\text{let} a &= \langle x, y\rangle \in \mathcal{R} \times \mathcal{R} && (15)\\
a^* &= \bigoplus_{n=0}^{\infty} a^{\otimes n} && (16)\\
&= \bigoplus_{n=0}^{\infty} \langle x, y\rangle^{\otimes n} && (17)\\
&= \langle 1, 0\rangle \oplus \langle x, y\rangle \oplus \left\langle x^2, 2xy\right\rangle \oplus \left\langle x^3, 3x^2y\right\rangle \oplus \dots && (18)\\
&= \bigoplus_{n=0}^{\infty} \left\langle x^n, nx^{n-1}y\right\rangle && (19)\\
&= \left\langle \sum_{n=0}^{\infty} x^n, \sum_{n=0}^{\infty} nx^{n-1}y \right\rangle && (20)
\end{aligned}$$

We can now explore the limit of both of the series individually. For the first one we have:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{(limit of the geometric series)} \qquad (21)$$

$$(22)$$

Assuming that $|x| < 1$, as otherwise the series would diverge. For the second one we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} nx^{n-1}y &= y \cdot (\sum_{n=0}^{\infty} nx^{n-1}) && (23)\\
&= y \cdot (\sum_{n=0}^{\infty} (n+1)x^{n}) && (24)\\
&= \frac{y}{(x-1)^2} \qquad \text{(limit of the power series)} && (25)
\end{aligned}$$

With the same assumption as before, we can now combine the two limits to get:

$$\langle x, y\rangle^* = \left\langle \frac{1}{1-x}, \frac{y}{(x-1)^2} \right\rangle$$

for $|x| < 1$.

d)

Now we want to find a Kleene Star for $\mathcal{W}_{lang}$. Using the definition of Kleene Star, we have:

$$\begin{aligned}
A* &= \bigoplus_{n=0}^{\infty} A^{\otimes n} \qquad \text{(def.}*\text{)} && (26)\\
&= \epsilon \cup A \cup A \otimes A \cup \dots && (27)\\
&= \epsilon \cup A \cup A \circ A \cup \dots && (28)\\
& && (29)
\end{aligned}$$