Prof. Ryan Cotterell

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Question 1: Exploring the Kleene Star

a)

To prove:

$$a^* \stackrel{!}{=} 1 \oplus a \otimes a^*$$

$$\begin{aligned} a^* &= \bigoplus_{n=0}^{\infty} a^{\otimes n} && \text{(def.}*\text{)} && (1) \\ &= 1 \oplus \bigoplus_{n=1}^{\infty} a^{\otimes n} && \text{(def.}\oplus\text{)} && (2) \\ &= 1 \oplus a \otimes \bigoplus_{n=0}^{\infty} a^{\otimes n} && \text{(diss. of } \otimes \text{ over } \oplus\text{)} && (3) \\ & && && (4) \end{aligned}$$

b)

We want to find a Kleene Star for $\mathcal{W}_{log}$. Using the definition of Kleene Star, we have:

$$\begin{aligned} a^* &= \bigoplus_{n=0}^{\infty} a^{\otimes n} && \text{(def.}*\text{)} && (5) \\ &= 0 \oplus_{log} a \oplus_{log} a \otimes a \oplus_{log} \dots && && (6) \\ &= \ln(\exp(0) + \exp(a)) \oplus_{log} a + a \oplus_{log} \dots && \text{(def.}\oplus, \otimes\text{)} && (7) \\ &= \ln(\exp(\ln(\exp(0) + \exp(a))) + exp(2a)) \oplus_{log} \dots && && (8) \\ &= \ln(\exp(0) + \exp(a) + \exp(2a)) \oplus_{log} \dots && && (9) \\ &= \ln(\sum_{n=0}^{\infty} \exp(a \cdot n)) && && (10) \\ & && && (11) \end{aligned}$$

For $a \geq 0$ this series will diverge as $n \to \infty$, because $\lim_{n\to\inf} \exp(a \cdot n) \geq 1$ in that case.

Thus we assume that $a < 0$ for the rest of the proof. We can then rewrite the series as a geometric series as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \exp(a \cdot n) &= \sum_{n=0}^{\infty} g^n && \text{(where } g = \exp(a)) && (12)\\
&= \frac{1}{1-g} && \text{(limit of the geometric series)} && (13)\\
&= \frac{1}{1-\exp(a)} && (\text{def.}g) && (14)
\end{aligned}$$

For the Kleene Star expression we then get:

$$a^* = \ln\left(\frac{1}{1-\exp(a)}\right)$$

c)

We want to find a Kleene Star for the expectation semiring. Following the same pattern as in the previous part, the definition gives us:

$$\begin{aligned}
\text{let} a &= \langle x, y\rangle \in \mathcal{R} \times \mathcal{R} && (15)\\
a^* &= \bigoplus_{n=0}^{\infty} a^{\otimes n} && (16)\\
&= \bigoplus_{n=0}^{\infty} \langle x, y\rangle^{\otimes n} && (17)\\
&= \langle 1, 0\rangle \oplus \langle x, y\rangle \oplus \left\langle x^2, 2xy\right\rangle \oplus \left\langle x^3, 3x^2 y\right\rangle \oplus \dots && (18)\\
&= \bigoplus_{n=0}^{\infty} \left\langle x^n, nx^{n-1}y\right\rangle && (19)\\
&= \left\langle \sum_{n=0}^{\infty} x^n, \sum_{n=0}^{\infty} nx^{n-1}y \right\rangle && (20)
\end{aligned}$$

We can now explore the limit of both of the series individually. For the first one we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} && \text{(limit of the geometric series)} && (21)\\
&& && (22)
\end{aligned}$$

Assuming that $|x| < 1$, as otherwise the series would diverge. For the second one we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} nx^{n-1}y &= y \cdot (\sum_{n=0}^{\infty} nx^{n-1}) && (23)\\
&= y \cdot (\sum_{n=0}^{\infty} (n+1)x^{n}) && (24)\\
&= \frac{y}{(x-1)^2} \qquad \text{(limit of the power series)} && (25)
\end{aligned}$$

With the same assumption as before, we can now combine the two limits to get:

$$\langle x, y\rangle^* = \left\langle \frac{1}{1-x}, \frac{y}{(x-1)^2} \right\rangle$$

for $|x| < 1$.

d)

Now we want to find a Kleene Star for $\mathcal{W}_{lang}$. Using the definition of Kleene Star, we have:

$$\begin{aligned}
A* &= \bigoplus_{n=0}^{\infty} A^{\otimes n} \qquad \text{(def.}*\text{)} && (26)\\
&= \epsilon \cup A \cup A \otimes A \cup \ldots && (27)\\
&= \epsilon \cup A \cup A \circ A \cup \ldots && (28)\\
& && (29)
\end{aligned}$$

Question 2: Asterating Matrices in Idempotent Semirings

a)

For the tropical semiring, let $a \in \mathcal{R}_{\geq 0}$ be an element of the tropical semiring, we get:

$$\begin{aligned}
0 \oplus a &= \min(0, a) \qquad \text{(def.}\oplus\text{)} && (30)\\
&= 0 \qquad (a \in \mathcal{R}_{\geq 0}) && (31)\\
& && (32)
\end{aligned}$$

For the artic semiring, let $a \in \mathcal{R}_{\leq 0}$ be an element of the artic semiring, we get:

$$\begin{aligned}
0 \oplus a &= \max(0, a) \qquad \text{(def.}\oplus\text{)} && (33)\\
&= 0 \qquad (a \in \mathcal{R}_{\leq 0}) && (34)\\
& && (35)
\end{aligned}$$

Thus we can conclude that both the tropical and artic semirings are 0-closed.

b)

We prove that the M^n encodes the sum of paths of length n in the graph G using induction over n. For $n = 1$ we have: The matrix M encodes the transition matrix of the graph G. Thus it encodes all paths of length one.

Our induction hypothesis is that M^n encodes the sum of paths of length n in the graph G. We want to show that M^{n+1} encodes the sum of paths of length $n+1$. We have $M^{n+1} = M^n \otimes M$ as M^n encodes the sum of paths of length n and M encodes the sum of paths of length one. For some $i, j \in [1, ..., N]$ we have:

$$(M^{n+1})_{i,k} = \bigoplus_{k=1}^{N} (M^n)_{ik} \otimes M_{kj} \tag{36}$$

Thus as we can see $(M^{n+1})_{i,k}$ encodes the sum of all paths of length $n+1$ from i to k as it is the sum of all paths of length n from i to some k concatenated with a path of length one from k to j.

c)

Assume that there is a shortest path from i to j that traverses more than $N-1$ transitions/edges. It follows that at least one vertex has to be visited twice as there are only N vertices. We denote the path as follows:

$$i \overset{w_0}{\to} \underbrace{... \to}_{a_1} k \underbrace{\to ... \to}_{a_2} k \underbrace{\to ...}_{a_3} \overset{w_k}{\to} j \tag{37}$$

We can build a path of length $N-1$ from i to j by cutting out the path from k to k in the middle of the path.

Per definition of the shortest path, we get the following for the two paths:

$$\begin{aligned}
&w_0 \otimes a_1 \otimes a_2 \otimes a_3 \otimes w_k \oplus w_0 \otimes a_1 \otimes a_3 \otimes w_k && (38)\\
=&(w_0 \otimes a_1) \otimes (a_2 \otimes a_3 \otimes w_k \oplus a_3 \otimes w_k) && (39)\\
=&(w_0 \otimes a_1) \otimes (a_2 \oplus 1) \otimes (\otimes a_3 \otimes w_k) && (40)\\
=&(w_0 \otimes a_1) \otimes (1) \otimes (\otimes a_3 \otimes w_k) \quad \text{(0-closed)} && (41)\\
=&w_0 \otimes a_1 \otimes a_3 \otimes w_k && (42)
\end{aligned}$$

d)

$$\begin{aligned}
M^* &= \bigoplus_{n=0}^{\infty} M^n && \text{(def.}*\text{)} && (43)\\
&= \lim_{K\to\infty} \bigoplus_{n=0} KM^n && \text{(def.}\oplus\text{)} && (44)\\
& && && (45)
\end{aligned}$$

We we have shown in part b) that M^n encodes the sum of paths of length n in the graph G. Furthermore we have shown that the shortest path from i to j has uses at most $N-1$ transitions. Now using the definition of the shortest path:

$$Z(i,j) = \bigoplus_{\pi \in \prod(i,j)} w(\pi)$$

We get that $\prod(i,j)$ needs only to be over all path of length $N-1$ or less (see task c)). Thus we can rewrite the definition as follows:

$$Z(i,j) = \bigoplus_{n=1}^{N-1} w((M^n)_{ij})$$

As we made no further assuming on i, j this holds for all $i, j \in [1, ..., N]$. We can now rewrite the Kleene start in the same manner as the previous equality holds for all $i, j \in [1, ..., N]$:

$$\lim_{K\to\infty} \bigoplus_{n=0}^{K} M^n \tag{46}$$

$$= \lim_{K\to N-1} \bigoplus_{n=1}^{K-1} w((M^n)) \tag{47}$$

$$= \bigoplus_{n=1}^{N-1} w((M^n)) \tag{48}$$

$$\tag{49}$$

e)

Algorithm 1: Algorithm for computing M^*

Input: Adjacency matrix M
Output: M^*
$M' \leftarrow M$
$R \leftarrow M$
for $i = 1; i < N; i \leftarrow i + 1$ **do**
  $R \leftarrow R \oplus M'$
  $M' \leftarrow M' \otimes M$
end

The algorithm computes M^* using the equality we proved in part d). It simply aggregates the sum of the M^n for $n = 1, ..., N-1$ in the variable R. The variable M' is used to compute the M^n. The Runtime of the algorithm is $O(N^3)$ as we need to compute N matrix multiplications of size $N \times N$.

f)

We want to prove that every 0-closed semiring is also idempoent.

$$
\begin{aligned}
&a \oplus a && (50)\\
&\quad = a \otimes (1 \oplus 1) && (51)\\
&\quad = a \otimes 1 && (52)\\
&\quad = a && (53)
\end{aligned}
$$

Thus we have shown that every 0-closed semiring is idempoent.

g)