Yannick Wattenberg: Assignment 03

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Question 1: Exploring the Kleene Star

a)

To prove:

$$a^* \stackrel{!}{=} 1 \oplus a \otimes a^*$$

$$\begin{aligned}
a^* &= \bigoplus_{n=0}^{\infty} a^{\otimes n} && \text{(def.}*\text{)} && (1)\\
&= 1 \oplus \bigoplus_{n=1}^{\infty} a^{\otimes n} && \text{(def.}\oplus\text{)} && (2)\\
&= 1 \oplus a \otimes \bigoplus_{n=0}^{\infty} a^{\otimes n} && \text{(diss. of } \otimes \text{ over } \oplus\text{)} && (3)\\
& && && (4)
\end{aligned}$$

b)

We want to find a Kleene Star for $\mathcal{W}_{log}$. Using the definition of Kleene Star, we have:

$$\begin{aligned}
a^* &= \bigoplus_{n=0}^{\infty} a^{\otimes n} && \text{(def.}*\text{)} && (5)\\
&= 0 \oplus_{log} a \oplus_{log} a \otimes a \oplus_{log} \dots && && (6)\\
&= \ln(\exp(0) + \exp(a)) \oplus_{log} a + a \oplus_{log} \dots && \text{(def.}\oplus, \otimes\text{)} && (7)\\
&= \ln(\exp(\ln(\exp(0) + \exp(a))) + exp(2a)) \oplus_{log} \dots && && (8)\\
&= \ln(\exp(0) + \exp(a) + \exp(2a)) \oplus_{log} \dots && && (9)\\
&= \ln(\sum_{n=0}^{\infty} \exp(a \cdot n)) && && (10)\\
& && && (11)
\end{aligned}$$

For $a \geq 1$ this series will diverge as $n \to \infty$, because $\lim_{n \to \inf} \exp(a \cdot n) \geq 1$ in that case.

Thus we assume that $a < 1$ for the rest of the proof. We can then rewrite the series as a geometric series as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \exp(a \cdot n) &= \sum_{n=0}^{\infty} g^n && \text{(where } g = \exp(a)\text{)} && (12)\\
&= \frac{1}{1-g} && \text{(limit of the geometric series)} && (13)\\
&= \frac{1}{1-\exp(a)} && (\text{def.}g) && (14)
\end{aligned}$$

For the Kleene Star expression we then get:

$$a^* = \ln\left(\frac{1}{1-\exp(a)}\right)$$

c)

We want to find a Kleene Star for the expectation semiring. Following the same pattern as in the previous part, the definition gives us:

$$\begin{aligned}
\text{let} a &= \langle x, y \rangle \in \mathcal{R} \times \mathcal{R} && (15)\\
a^* &= \bigoplus_{n=0}^{\infty} a^{\otimes n} && (16)\\
&= \bigoplus_{n=0}^{\infty} \langle x, y \rangle^{\otimes n} && (17)\\
&= \langle 1, 0 \rangle \oplus \langle x, y \rangle \oplus \left\langle x^2, 2xy \right\rangle \oplus \left\langle x^3, 3x^2 y \right\rangle \oplus \dots && (18)\\
&= \bigoplus_{n=0}^{\infty} \left\langle x^n, n x^{n-1} y \right\rangle && (19)\\
&= \left\langle \sum_{n=0}^{\infty} x^n, \sum_{n=0}^{\infty} n x^{n-1} y \right\rangle && (20)
\end{aligned}$$

We can now explore the limit of both of the series individually. For the first one we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n &= \frac{1}{1-x} && \text{(limit of the geometric series)} && (21)\\
& && && (22)
\end{aligned}$$

Assuming that $|x| < 1$, as otherwise the series would diverge. For the second one we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} nx^{n-1}y &= y \cdot (\sum_{n=0}^{\infty} nx^{n-1}) && (23)\\
&= y \cdot (\sum_{n=0}^{\infty} (n+1)x^{n}) && (24)\\
&= \frac{y}{(x-1)^2} \qquad \text{(limit of the power series)} && (25)
\end{aligned}$$

With the same assumption as before, we can now combine the two limits to get:

$$\langle x, y\rangle^* = \left\langle \frac{1}{1-x}, \frac{y}{(x-1)^2} \right\rangle$$

for $|x| < 1$.

d)

Now we want to find a Kleene Star for $\mathcal{W}_{lang}$. Using the definition of Kleene Star, we have:

$$\begin{aligned}
A* &= \bigoplus_{n=0}^{\infty} A^{\otimes n} \qquad \text{(def.}*) && (26)\\
&= \epsilon \cup A \cup A \otimes A \cup \ldots && (27)\\
&= \epsilon \cup A \cup A \circ A \cup \ldots && (28)\\
& && (29)
\end{aligned}$$

Question 2: Asterating Matrices in Idempotent Semirings

a)

For the tropical semiring, let $a \in \mathcal{R}_{\geq 0}$ be an element of the tropical semiring, we get:

$$\begin{aligned}
0 \oplus a &= \min(0, a) \qquad (\text{def.}\oplus) && (30)\\
&= 0 \qquad (a \in \mathcal{R}_{\geq 0}) && (31)\\
& && (32)
\end{aligned}$$

For the artic semiring, let $a \in \mathcal{R}_{\leq 0}$ be an element of the artic semiring, we get:

$$\begin{aligned}
0 \oplus a &= \max(0, a) \qquad (\text{def.}\oplus) && (33)\\
&= 0 \qquad (a \in \mathcal{R}_{\leq 0}) && (34)\\
& && (35)
\end{aligned}$$

Thus we can conclude that both the tropical and artic semirings are 0-closed.

b)

We prove that the M^n encodes the sum of paths of length n in the graph G using induction over n. For $n = 1$ we have: The matrix M encodes the transition matrix of the graph G. Thus it encodes all paths of length one.

Our induction hypothesis is that M^n encodes the sum of paths of length n in the graph G. We want to show that M^{n+1} encodes the sum of paths of length $n + 1$. We have $M^{n+1} = M^n \otimes M$ as M^n encodes the sum of paths of length n and M encodes the sum of paths of length one. For some $i, j \in [1, ..., N]$ we have:

$$(M^{n+1})_{i,k} = \bigoplus_{k=1}^{N} (M^n)_{ik} \otimes M_{kj} \tag{36}$$

Thus as we can see $(M^{n+1})_{i,k}$ encodes the sum of all paths of length $n + 1$ from i to k as it is the sum of all paths of length n from i to some k concatenated with a path of length one from k to j.

c)

Assume that there is a shortest path from i to j that traverses more than $N - 1$ transitions/edges. It follows that at least one vertex has to be visited twice as there are only N vertices. We denote the path as follows:

$$i \overset{w_0}{\to} \underbrace{... \to}_{a_1} k \underbrace{\to ... \to}_{a_2} k \underbrace{\to ...}_{a_3} \overset{w_k}{\to} j \tag{37}$$

We can build a path of length $N - 1$ from i to j by cutting out the path from k to k in the middle of the path.

Per definition of the shortest path, we get the following for the two paths:

$$\begin{aligned}
& w_0 \otimes a_1 \otimes a_2 \otimes a_3 \otimes w_k \oplus w_0 \otimes a_1 \otimes a_3 \otimes w_k && (38)\\
=& (w_0 \otimes a_1) \otimes (a_2 \otimes a_3 \otimes w_k \oplus a_3 \otimes w_k) && (39)\\
=& (w_0 \otimes a_1) \otimes (a_2 \oplus 1) \otimes (\otimes a_3 \otimes w_k) && (40)\\
=& (w_0 \otimes a_1) \otimes (1) \otimes (\otimes a_3 \otimes w_k) \qquad \text{(0-closed)} && (41)\\
=& w_0 \otimes a_1 \otimes a_3 \otimes w_k && (42)
\end{aligned}$$

d)

$$\begin{aligned}
M^* &= \bigoplus_{n=0}^{\infty} M^n && \text{(def.}*\text{)} && (43)\\
&= \lim_{K \to \infty} \bigoplus_{n=0} KM^n && \text{(def.}\oplus\text{)} && (44)\\
& && && (45)
\end{aligned}$$

We we have shown in part b) that M^n encodes the sum of paths of length n in the graph G. Furthermore we have shown that the shortest path from i to j has uses at most $N-1$ transitions. Now using the definition of the shortest path:

$$Z(i,j) = \bigoplus_{\pi \in \prod(i,j)} w(\pi)$$

We get that $\prod(i,j)$ needs only to be over all path of length $N-1$ or less (see task c)). Thus we can rewrite the definition as follows:

$$Z(i,j) = \bigoplus_{n=1}^{N-1} w((M^n)_{ij})$$

As we made no further assuming on i,j this holds for all $i,j \in [1,...,N]$. We can now rewrite the Kleene start in the same manner as the previous equality holds for all $i,j \in [1,...,N]$:

$$\lim_{K\to\infty} \bigoplus_{n=0}^{K} M^n \tag{46}$$

$$= \lim_{K\to N-1} \bigoplus_{n=0}^{K-1} w((M^n)) \tag{47}$$

$$= \bigoplus_{n=0}^{N-1} w((M^n)) \tag{48}$$

$$\tag{49}$$

e)

Algorithm 1: Algorithm for computing M^*

Input: Adjacency matrix M

Output: M^*

$M' \leftarrow I$

$R \leftarrow M$

for $i = 1; i < N; i \leftarrow i+1$ **do**

  $R \leftarrow R \oplus M'$

  $M' \leftarrow M' \otimes M$

end

The algorithm computes M^* using the equality we proved in part d). It simply aggregates the sum of the M^n for $n = 1,...,N-1$ in the variable R. The variable M' is used to compute the M^n. The Runtime of the algorithm is $O(N^3)$ as we need to compute N matrix multiplications of size $N \times N$.

f)

We want to prove that every 0-closed semiring is also idempoent.

$$\begin{aligned}
&a \oplus a && (50)\\
&\quad =a \otimes (1 \oplus 1) && (51)\\
&\quad =a \otimes 1 && (52)\\
&\quad =a && (53)
\end{aligned}$$

Thus we have shown that every 0-closed semiring is idempoent.

g)

We want to prove the following equality:

$$\bigoplus_{n=0}^{K} M^n = (I \oplus M)^K$$

We do so by induction on K. The base case $K = 0$ is trivial as $M^0 = I$ thus:

$$I = (I \oplus M)^0 = M^0 = \bigoplus_{n=0}^{0} M^n \tag{54}$$

Now we assume that the equality holds for K and want to show that it also holds for $K+1$:

$$\begin{aligned}
(I \oplus M)^{K+1} &= (I \oplus M)^K \otimes (I \oplus M) && (55)\\
&= \bigoplus_{n=0}^{K} M^n \otimes (I \oplus M) && \text{(distributive)} \quad (56)\\
&= \bigoplus_{n=0}^{K} M^n \oplus M^{n+1} && (\text{def.}I) \quad (57)\\
&= \bigoplus_{n=0}^{K} M^n \oplus \bigoplus_{n=1}^{K+1} M^n && (58)\\
&= M^0 \oplus \left(\bigoplus_{n=1}^{K} M^n \oplus M^n \right) \oplus M^{K+1} && (59)\\
&= M^0 \oplus \left(\bigoplus_{n=1}^{K} M^n \right) \oplus M^{K+1} && \text{(Idempotent)} \quad (60)\\
&= \bigoplus_{n=0}^{K+1} M^n && (61)
\end{aligned}$$

h)

It is easy to see that the equation $M^n = \bigotimes_{k=0}^{\log_2 n} M^{\alpha_k 2^k}$ as it all natural numbers can be represented using the binary system. More concretely we can write $\bigotimes_{k=0}^{\log_2 n} M^{\alpha_k 2^k}$ as

$M^{\sum_{k=0}^{\log_2 n} \alpha_k 2^k}$ as $M^a \otimes M^b = M^{a+b}$. Then we choose $a_k = 1$ iff. the k-th bit of the binary representation of n is one and zero otherwise. In other words let a be the binary representation of n. Using this it becomes apparent that $\sum_{k=0}^{\log_2 n} \alpha_k 2^k = n$ as it is the way to convert the binary representation of n (a) to a decimal number.

A more efficient way to compute $M^* = \bigoplus_{n=0}^{N-1} M^n$ uses the following equality which holds as the semiring is 0-closed and thus idempoent (f):

$$\bigoplus_{n=0}^{N-1} M^n = (I \oplus M)^{N-1} \qquad \text{((g))} \tag{62}$$

$$(I \oplus M)^{N-1} = \bigotimes_{k=0}^{\log_2 N-1} (I \oplus M)^{a_k 2^k} \qquad \text{((h))} \tag{63}$$

$$\tag{64}$$

Where a_k is the k-th bit of the binary representation of $N-1$. The algorithm is as follows:

Algorithm 2: Faster Algorithm for computing M^*

Input: Adjacency matrix M
Output: M^*
$M' \leftarrow I$
$R \leftarrow (I \oplus M)$
for $i = 0; i < \log_2(N-1) + 1; i \leftarrow i + 1$ **do**
  if a_i *is* 1 **then**
    $R \leftarrow R \otimes M'$
  end
  $M' \leftarrow M' \otimes M'$
end

i)

In the following all norms will refer to the two norm. First we rewrite the definition of $\|M\|$:

$$\|M\| = sup_{x \neq 0} \frac{\|Mx\|}{\|x\|} \tag{65}$$

$$= sup_{x \neq 0; \|x\|=1} \|Mx\| \tag{66}$$

This holds as one can just normalize the vector x to have length 1.

Now let $U\Sigma V^T$ be the SVD of M. Then we have:

$$sup_{x \neq 0; \|x\|=1} \|U\Sigma V^T x\| \tag{67}$$

$$= sup_{x' \neq 0; \|x'\|=1} \|\Sigma x'\| \tag{68}$$

$$= \sigma_1 \tag{69}$$

The last equality follows from the min-max theorem for singular values. Thus we have shown that $\|M\| = \sigma_1$.

j)

First we reform the initial equation:

$$\|M^* - \sum_{n=0}^{K} A^n\| = \|\sum_{n=0}^{\infty} A^n - \sum_{n=0}^{\infty} M^n\| \tag{70}$$

$$= \| \sum_{n=K+1}^{\infty} M^n\| \tag{71}$$

$$sup_{x\neq 0;\|x\|=1}\| \sum_{n=K+1}^{\infty} M^n x\| \leq \sum_{n=K+1}^{\infty} sup_{x\neq 0;\|x\|=1}\|M^n x\| \tag{72}$$

$$\leq \sum_{n=K+1}^{\infty} \|M\|^n \qquad (\sigma = \sigma_{max}(A)) \tag{73}$$

$$= \sum_{n=K+1}^{\infty} \sigma^n \tag{74}$$

$$= \frac{\sigma^{K+1}}{1-\sigma} \tag{75}$$

The inequality in equation 72 holds as the one can choose the same x in the left and right. The inequality in equation 73 holds as the norm is submultiplicative and the equality in equation 75 holds as we assume $\sigma < 1$ which means the series is the geometric series. This is also the condition for the sum to converge to M^*. If $\sigma \geq 1$ then the sum diverges therefore there is no closed form solution for the error-term.

k)

$$\mathcal{O}\left(\frac{\sigma^{K+1}}{1-\sigma}\right) = \mathcal{O}\left(\sigma^{K+1}\right) \tag{76}$$

Where $\sigma < 1$. As this error term decayes exponentially fast we can say that the error term is acceptable and the truncation is a good approximation.