# HW4 sol'n

**#1:** Claim: $a_n \le a_{n+1} \le 5 \quad \forall n.$

Pf: By induction, We have $a_1 = 3 < a_2 = \sqrt{19} < 5$.

Assume that $a_n \le a_{n+1} \le 5$. want to show: $a_{n+1} \le a_{n+2} \le 5$.

① $a_{n+1} \le a_{n+2} \iff a_{n+1} \le \sqrt{3a_{n+1}+10} \iff (a_{n+1}-5)(a_{n+1}+2) \le 0$

$\iff -2 \le a_{n+1} \le 5$ which is true by induction hypothesis

② $a_{n+2} \le 5 \iff \sqrt{3a_{n+1}+10} \le 5 \iff a_{n+1} \le 5$ which is true by induction hypothesis. $\square$

Hence $(a_n)$ is bounded & monotone, thus $\lim_{n\to\infty} a_n = a$ exists.

We have $a_{n+1}^2 = 3a_n + 10$.

$\Rightarrow a^2 = \lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty}(3a_n+10) = 3a+10$ by the limit thms.

$\Rightarrow a = 5$ or $-2$. $\quad a=-2$ isn't possible since $a_n > 0 \ \forall n$.

$\Rightarrow \lim_{n\to\infty} a_n = a = 5. \ \square$

**#2:**
$$\limsup_{n\to\infty}(-a_n) = \lim_{N\to\infty} \sup\{-a_n : n > N\}$$
$$= \lim_{N\to\infty} -\inf\{a_n : n > N\}$$
(see the proof of Ross, Corollary 4.5)
$$= -\lim_{N\to\infty} \inf\{a_n : n>N\}$$
$$= -\liminf_{n\to\infty} a_n. \ \square$$

**#3.** $\limsup_{n\to\infty} |a_n| = 0 \iff \lim_{N\to\infty} \sup\{|a_n| : n > N\} = 0.$

$\iff \forall \varepsilon > 0, \exists N > 0$ s.t. $|a_n| < \varepsilon \ \forall n > N.$

$\iff \lim_{n\to\infty} a_n = 0. \ \square$

**#4.** (a) Obvious, from triangle inequality.

(b) We prove $\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \le \liminf_{n\to\infty} (a_n + b_n)$:

Claim: $\forall N > 0,$

$$\underbrace{\inf\{a_n + b_n : n > N\}}_{I_N^{a,b}} \ge \underbrace{\inf\{a_n : n > N\}}_{I_N^a} + \underbrace{\inf\{b_n : n > N\}}_{I_N^b}.$$

pf. $\forall \varepsilon > 0$, $I_N^{a,b} + \varepsilon$ is NOT a lower bound of $\{a_n + b_n : n > N\}$.

$\Rightarrow \exists\, n > N$ s.t. $a_n + b_n < I_N^{a,b} + \varepsilon.$

$\Rightarrow I_N^{a,b} + \varepsilon > a_n + b_n \ge \inf\{a_n : n > N\} + \inf\{b_n : n > N\} = I_N^a + I_N^b.$

Since $I_N^{a,b} + \varepsilon > I_N^a + I_N^b$ holds $\forall \varepsilon > 0$

$\Rightarrow I_N^{a,b} \ge I_N^a + I_N^b. \ \square$

Hence $I_N^{a,b} - I_N^a - I_N^b \ge 0 \quad \forall N.$

$$\Rightarrow \liminf_{n\to\infty} (a_n + b_n) - \liminf_{n\to\infty} a_n - \liminf_{n\to\infty} b_n$$

$$= \lim_{N\to\infty} I_N^{a,b} - \lim_{N\to\infty} I_N^a - \lim_{N\to\infty} I_N^b$$

$$= \lim_{N\to\infty} \left( I_N^{a,b} - I_N^a - I_N^b \right) \ge 0. \ \square$$

(c) $(a_n) = (0, 1, 0, 1, 0, 1, \cdots)$

$(b_n) = (1, 0, 1, 0, 1, 0, \cdots)$

#5. (a) $\forall n<m$, we have

$$|a_n-a_m| \leq |a_n-a_{n+1}|+\cdots+|a_{m-1}-a_m| < c^n+c^{n+1}+\cdots+c^{m-1}<\frac{c^n}{1-c}$$

For any $\varepsilon>0$, there exists $N>0$ so that $\frac{c^N}{1-c}<\varepsilon$. (since $0<c<1$)

Hence $\forall n,m>N$, we have $|a_n-a_m|<\frac{c^{\min\{n,m\}}}{1-c}<\frac{c^N}{1-c}<\varepsilon$. $\square$

(b) No. e.g. $a_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ c.f. HW2 #9.

#6. (a) $L=\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{N\to\infty}\underbrace{\sup\left\{\left|\frac{a_{n+1}}{a_n}\right| : n>N\right\}}_{S_N}$

So $\forall L'>L$, $\exists N>0$ s.t. $S_N<L'$.

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right|<L' \quad \forall n>N$.

(b) $\forall n>N$, we have

$$|a_n| = \left|\frac{a_n}{a_{n-1}}\right|\left|\frac{a_{n-1}}{a_{n-2}}\right|\cdots\left|\frac{a_{N+1}}{a_N}\right||a_N| < (L')^{n-N}|a_N|.$$

(c) From (b), we have $|a_n|^{1/n} < B^{1/n}\cdot L' \quad \forall n>N$.

$$\Rightarrow \limsup_{n\to\infty}|a_n|^{1/n} \leq L'\cdot\limsup_{n\to\infty}B^{1/n} = L'\cdot\underbrace{\lim_{n\to\infty}B^{1/n}}_{\text{Converges to } 1\ \forall B>0} = L'.$$

(d) Since $\limsup_{n\to\infty}|a_n|^{1/n}\leq L' \quad \forall L'>L$.

$$\Rightarrow \limsup_{n\to\infty}|a_n|^{1/n}\leq L. \quad \square$$

#7. $\mathbb{Q} \subset \mathbb{R}$ not open:

$\forall x \in \mathbb{Q}$ and $\forall r > 0$, $B_r(x) = (x-r, x+r)$ contains irrational numbers.

$\mathbb{Q} \subset \mathbb{R}$ not closed: $\iff$ $\mathbb{Q}^c \subset \mathbb{R}$ not open:

$\forall x \in \mathbb{Q}^c$ (irrational) and $\forall r > 0$, $B_r(x)$ contains rational numbers. (denseness of $\mathbb{Q}$). $\square$

#8: (a) $x \in \left(\bigcup_\alpha S_\alpha\right)^c \iff x \notin \bigcup_\alpha S_\alpha \iff x \notin S_\alpha \ \forall \alpha$

$\iff x \in S_\alpha^c \ \forall \alpha \iff x \in \bigcap_\alpha S_\alpha^c$.

(b) $x \in \left(\bigcap_\alpha S_\alpha\right)^c \iff x \notin \bigcap_\alpha S_\alpha \iff \exists \alpha$ s.t. $x \notin S_\alpha$

$\iff \exists \alpha$ s.t. $x \in S_\alpha^c \iff x \in \bigcup_\alpha S_\alpha^c$. $\square$

#9. (a) $\{U_\alpha\}$ collection of open sets. Want: $\bigcup_\alpha U_\alpha$ is open.

$\forall x \in \bigcup_\alpha U_\alpha$, $\exists \alpha$ s.t. $x \in U_\alpha$.

Since $U_\alpha$ is open, $\exists r > 0$ s.t. $B_r(x) \subset U_\alpha \subset \bigcup_\alpha U_\alpha$. $\square$

(b) $U_1, \dots, U_n$ open sets. Want: $U_1 \cap \dots \cap U_n$ is open.

$\forall x \in U_1 \cap \dots \cap U_n$, $\exists r_1, \dots, r_n > 0$ s.t. $B_{r_k}(x) \subset U_k \ \forall 1 \le k \le n$.

Let $r := \min\{r_1, \dots, r_n\} > 0$. Then $B_r(x) \subset (U_1 \cap \dots \cap U_n)$. $\square$

(c) (d) Follows from #8 and #9(a)(b).

(e) • $\{[-1+\frac{1}{n}, 1-\frac{1}{n}]\}_{n\in\mathbb{N}}$ collection of closed sets. Their union $= (-1,1)$ not closed.

• $\{(-\frac{1}{n}, \frac{1}{n})\}_{n\in\mathbb{N}}$ collection of open sets. Their intersection $= \{0\}$ not open.