HOMEWORK 9
MATH H54, FALL 2021

Due November 23, 11am

Some ground rules:

- Please submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW9**.
- The submission should be a **single PDF file**.
- Late homework will not be accepted/graded under any circumstances.
- Make sure the writing in your submission is clear enough. Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own, and acknowledge the students with whom you worked.
- **For True/False questions**: You have to prove the statement if your answer is "True"; otherwise, you have to provide an explicit counterexample and justification.
- You are allowed to use any result that is proved in the lecture. But if you would like to use other results, you have to prove it first before using it.

You have to write down your computations, not just the final answers.

(1) Find general solutions to the following differential equations:
  (a) $6y'' + y' - 2y = 0$.
  (b) $4y'' + 20y' + 25y = 0$.
  (c) $y'' + 4y' + 8y = 0$.
  (d) $y''(t) + 4y(t) = \sin t - \cos t$.
  (e) $y''(t) - 2y'(t) + y(t) = t^{-1}e^t$.
  (f) $y''(t) + 16y(t) = \sec(4t)$.

(2) Solve the following initial value problems:
  (a) $y'' - 4y' - 5y = 0\,;\ y(-1) = 3$ and $y'(-1) = 9$.
  (b) $y'' + 2y' + 2y = 0\,;\ y(0) = 2$ and $y'(0) = 1$.
  (c) $y''(t) + y'(t) - 12y(t) = e^t + e^{2t} - 1\,;\ y(0) = 1$ and $y'(0) = 3$.

(3) When the values of a solution to a differential equation are specified at *two different points*, these conditions are called *boundary conditions.* The purpose of this problem is to show that for boundary value problems there is no existence-uniqueness theorem.
  (a) Find general solutions to the differential equation:

$$y'' + y = 0. \tag{*}$$

(b) Show that there is a unique solution to (*) that satisfies the boundary conditions $y(0) = 2$ and $y(\pi/2) = 0$.
(c) Show that there is no solution to (*) that satisfies $y(0) = 2$ and $y(\pi) = 0$.
(d) Show that there are infinitely many solutions to (*) that satisfy $y(0) = 2$ and $y(\pi) = -2$.

(4) One way to define the *hyperbolic functions* is by means of differential equations. Consider the differential equation:

$$y'' - y = 0. \tag{**}$$

The *hyperbolic cosine*, denoted $\cosh t$, is defined as the solution of (**) subject to the initial values: $y(0) = 1$ and $y'(0) = 0$. The *hyperbolic sine*, denoted $\sinh t$, is defined as the solution of (**) subject to the initial values: $y(0) = 0$ and $y'(0) = 1$.
(a) Solve these two initial value problems to derive explicit formulas for $\cosh t$ and $\sinh t$. Also, show that $(\cosh t)' = \sinh t$ and $(\sinh t)' = \cosh t$.
(b) Prove that a general solution of (**) is given by $y(t) = c_1 \cosh t + c_2 \sinh t$.

(5) To see the effect of changing the coefficient b in the initial value problem

$$y'' + by' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 0,$$

solve the problem for $b = 5$, $b = 4$, and $b = 2$, and sketch the solutions.

(6) Prove the *sum of angles formula* for the sine function by following these steps. Fix $x \in \mathbb{R}$.
(a) Let $f(t) = \sin(x + t)$. Verify that $f''(t) + f(t) = 0$, $f(0) = \sin x$, and $f'(0) = \cos x$.
(b) Solve the initial value problem: $y'' + y = 0$; $y(0) = \sin x$ and $y'(0) = \cos x$.
(c) By uniqueness, the solution in Part (b) is the same as $f(t)$ from Part (a). Write this equality; this should be the standard sum of angles formula for $\sin(x + t)$.

(7) All that is known concerning a mysterious second-order constant-coefficient differential equation $y'' + by' + cy = f(t)$ is that $t^2 + 1 + e^t \cos t$, $t^2 + 1 + e^t \sin t$, and $t^2 + 1 + e^t \cos t + e^t \sin t$ are solutions.
(a) Determine two linearly independent solutions to the corresponding homogeneous equation $y'' + by' + cy = 0$.
(b) Find a suitable choice of $b, c \in \mathbb{R}$ and function $f(t)$ that enables these solutions.

(8) Use the method of variation of parameters to show that

$$y(t) = c_1 \cos t + c_2 \sin t + \int_0^t f(s) \sin(t - s) ds$$

is a general solution to the differential equation $y'' + y = f(t)$.

(9) Suppose the auxiliary equation $r^2 + br + c = 0$ of the differential equation $y'' + by' + cy = 0$ have two real roots. Prove that a nonzero solution to the differential equation can take the value 0 at most once, i.e. if $y(t)$ is a nonzero solution, then there is at most one point $t_0 \in \mathbb{R}$ such that $y(t_0) = 0$.