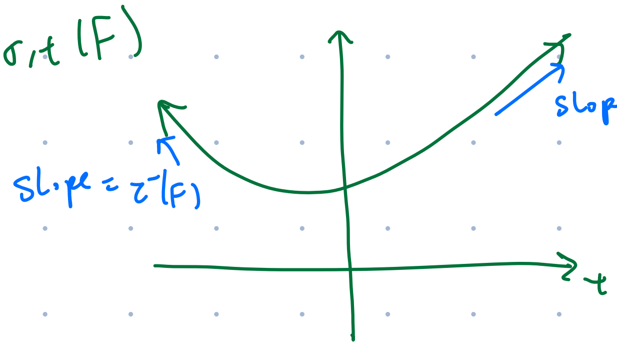
Properties of shifting #'s:

Recall: (shifting #) $F \in \mathrm{Aut}(\mathcal{D})$

1) $h_t(F)$ (via DHKK complexity function, or $\sum \dim \mathrm{Hom}$.)

$h_{\sigma,t}(F)$



2) $\sigma \in \mathrm{Stab}(\mathcal{D})$. G - split generator

$$\lim_{n \to \pm\infty} \frac{\phi_\sigma^{\pm}(F^n G)}{n} = \tau^{\pm}(F)$$

Poincaré Translation # $f \in \mathrm{Homeo}^{+}_{\mathbb{Z}}(\mathbb{R})$	Shifting #. $F \in \mathrm{Aut}(\mathcal{D})$
$\rho((+k)\circ f) = \rho(f) + k.$ ($k \in \mathbb{Z}$)	$\tau^{\pm}([k]\circ F) = \tau^{\pm}(F) + k.$
$\rho(f' f f'^{-1}) = \rho(f)$	$\tau^{\pm}(F' F F'^{-1}) = \tau^{\pm}(F).$
$\rho(f^n) = n\rho(f)$ $n \in \mathbb{Z}$	$\tau(F^n) = n\tau(F)$ $\boxed{\tau = \frac{\tau^{+}+\tau^{-}}{2}}$

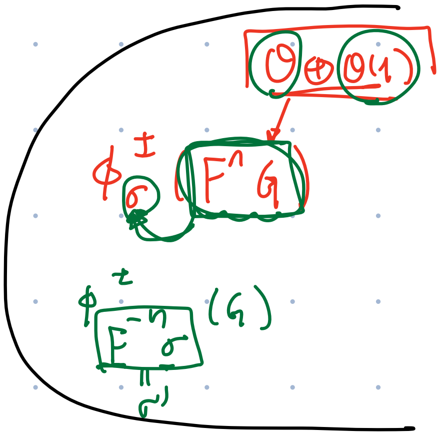
Ex:

$\rho : \mathrm{Homeo}^{+}_{\mathbb{Z}}(\mathbb{R}) \longrightarrow \mathbb{R}$ quasi-morphism.

i.e. $\exists\, C > 0$ s.t.

$$|\rho(f\circ g) - \rho(f) - \rho(g)| < C \quad \forall f, g \in \mathrm{Homeo}^{+}_{\mathbb{Z}}(\mathbb{R})$$

Question: Is $\tau: \mathrm{Aut}(\mathcal{D}) \longrightarrow \mathbb{R}$ a quasi-morphism?



e.g. $\mathcal{D} = D^b\mathrm{Coh}(X)$

e.g. $X =$ elliptic curve.

- Recall $\widetilde{GL^+_2(\mathbb{R})} \curvearrowright \mathrm{Stab}(\mathcal{D})$ acts freely & transitively.

- Fix any $\sigma_0 \in \mathrm{Stab}(\mathcal{D})$, define

$$\Phi_{\sigma_0}: \widetilde{GL^+_2\mathbb{R}} \xrightarrow{\sim} \mathrm{Stab}(\mathcal{D})$$
$$g \longmapsto \sigma_0 \cdot g.$$

Then we can define a group homomorphism:

$$s: \mathrm{Aut}(\mathcal{D}) \longrightarrow \widetilde{GL^+_2(\mathbb{R})}$$
$$F \longmapsto \Phi_{\sigma_0}^{-1}(F\cdot\sigma_0)$$

(this is a group homomorphism since $\mathrm{Aut}(\mathcal{D}) \curvearrowright \mathrm{Stab}(\mathcal{D})$ and $\widetilde{GL^+_2(\mathbb{R})} \curvearrowright \mathrm{Stab}(\mathcal{D})$ commute).

- Recall: $\widetilde{GL^+_2(\mathbb{R})} \ni (T, f)$, where $T \in GL^+_2(\mathbb{R})$, $f \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$, s.t. their actions on $\mathbb{R}^2\setminus\{0\}/\mathbb{R}_{>0} \cong S^1 \cong \mathbb{R}/2\mathbb{Z}$ compatible

Define $t: \widetilde{GL^+_2(\mathbb{R})} \longrightarrow \mathrm{Homeo}^+_{\mathbb{Z}}\mathbb{R}$ gp homom.

$$(T, f) \longmapsto f$$

We'll show that

$$\begin{tikzcd} \mathrm{Aut}(\mathcal{O}) \arrow[r, "s"] \arrow[rrr, bend right, "\tau = \tau^{t}"'] & \widetilde{GL_2^+\mathbb{R}} \arrow[r, "t"] & \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \arrow[r, "\xi"] & \mathbb{R} \end{tikzcd}$$

$(\Longrightarrow \quad \tau = \tau^{t}$ is a quasi-morphism$)$

- Take $G = \mathcal{O} \oplus \mathcal{O}(1)$ split generator

$$\tau^{t}(F) = \lim_{n\to\infty} \frac{\phi^{t}_{\sigma}(F^n G)}{n}$$

- $\mathcal{O}$, $\mathcal{O}(1)$ are stable w.r.t. $\underline{\text{any}}$ stability condition on $\mathcal{D}$.

- Denote: $\phi_0 = \phi_{\sigma_0}(\mathcal{O})$, $\quad \phi_1 = \phi_{\sigma_0}(\mathcal{O}(1))$

Then $\quad \phi_0 = \phi_{\sigma_0}(\mathcal{O}) = \phi_{F\sigma_0}(F\mathcal{O}) = \phi_{\sigma_0 \cdot \underbrace{s(F)}_{\in\, \widetilde{GL_2^+\mathbb{R}}}}(F\mathcal{O})$.

$\Rightarrow$ $F\mathcal{O}$ is a σ_0-stable object of phase $\underbrace{t(s(F))}_{\in\, \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})}(\phi_0)$.

- Similarly, $F^n\mathcal{O}$ is a σ_0-stable obj. of phase $(t(s(F)))^n(\phi_0)$
  $F^n\mathcal{O}(1)$ - - - - - - - - - - $(t(s(F))^n(\phi_1))$

$\Rightarrow$ $F^nG = F^n(\mathcal{O} \oplus \mathcal{O}(1))$ has 2 semistable factors w.r.t. σ_0, with phases given by $(t(s(F))^n(\phi_0)$ and $(t(s(F)))^n(\phi_1)$

$$\Rightarrow \quad \phi^{+}_{\sigma_0}(F^n G) = (t(s(F)))^n \max\{\phi_0, \phi_1\}. \quad x_1 \in \mathbb{R}$$

$$\phi^{-}_{\sigma_0}(F^n G) = (t(s(F)))^n \min\{\phi_0, \phi_1\}. \quad x_2 \in \mathbb{R}$$

$$\Rightarrow \quad \tau^{\pm}(F) = \lim_{n\to\infty} \frac{\phi^{+}_{\sigma_0}(F^n G)}{n}$$

$$= \lim_{n\to\infty} \frac{(t(s(F)))^n \, (\overset{x_2}{x_1})}{n}$$

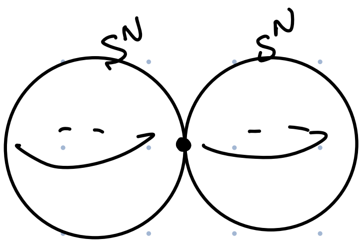
$$= g\big(t(s(F))\big). \quad \square$$

<u>Other known examples:</u>

- X = abelian surface
- $\pm K_X$ ample, $\left(\mathrm{Aut}(D) \xrightarrow{\text{shift}} \mathbb{Z} \hookrightarrow \mathbb{R} \right)$, $\tau = \tau^{\pm}$

  $Lf^*(-\otimes \mathcal{L})[k] \mapsto k$

- $D_{CY_N}(A_2) = D_N$, $N \geq 3$.



$\Big($ - $\mathrm{Hom}^\bullet(E,F) \simeq \mathrm{Hom}(F, E[N])^\vee$.

- generated by 2 N-spherical objects: S_1, S_2, $\quad \mathrm{Hom}^\bullet(S_i, S_i) = \mathbb{C} \oplus \mathbb{C}[-N]$

$\mathrm{Hom}^\bullet(S_1, S_2) = \mathbb{C}[-1]$ $\Big)$

$$\mathrm{Aut}_*(D_N) = \left\langle T_1, T_2, [1] \;\middle|\; \begin{array}{l} T_1T_2T_1 = T_2T_1T_2, \\ (T_1T_2)^3 = [4-3N], \quad T_i[1] = [1]T_i \\ \qquad\qquad\qquad\qquad\quad i=1,2 \end{array} \right\rangle,$$

$$\cap$$

$$\mathrm{Aut}(D_N)$$

$\mathrm{Aut}_*(D_N) \xrightarrow{\tau} \mathbb{R}$ is a quasi-morphism;

More precisely, $\tau = w - \frac{1}{6}(\phi \circ \alpha)$,

where:

- $W: \mathrm{Aut}_*(D_N) \longrightarrow \mathbb{R}$ is an explicit gp homom.

  $w(T_1) = w(T_2) = \dfrac{4-3N}{6}$, $w([1]) = 1$.

- $\alpha: \mathrm{Aut}_*(D_N) \longrightarrow PSL_2\mathbb{Z}$ gp homom.

  $T_1 \longmapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

  $T_2 \longmapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

  $[1] \longmapsto I_2$.

- $\phi: PSL_2\mathbb{Z} \longrightarrow \mathbb{R}$ is the (homogenization of the) Rademacher function (a classical quasi-morphism on $PSL_2\mathbb{Z}$).

- $PSL_2\mathbb{Z} = \langle s, u \mid s^2 = u^3 = 1 \rangle = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

  where $s = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $u = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = ST$

  $= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

- any element in $PSL_2\mathbb{Z}$ can be uniquely written as:

$$A = S^{\{0,1\}}\, u^{\overset{\varepsilon_1}{\widetilde{\pm 1}}}\, S\, u^{\overset{\varepsilon_2}{\widetilde{\pm 1}}}\, S \cdots\cdots S\, u^{\overset{\cdots}{\widetilde{\pm 1}}}\, S^{\{0,1\}}.$$

- Rademacher function: $PSL_2\mathbb{Z} \longrightarrow \mathbb{R}$

$$A \longmapsto \sum \varepsilon_i$$

- $S : \mathcal{C} \longrightarrow D$: an exact functor between two Δ cat's.

  Suppose S has left and right adjoint functors $L, R : D \longrightarrow \mathcal{C}$.

  i.e.

$$\mathrm{Hom}_D(SX, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, RY),$$

$$\mathrm{Hom}_{\mathcal{C}}(LX, Y) \cong \mathrm{Hom}_D(X, SY).$$

$\rightsquigarrow$ for the adjoint pair $S \dashv R$, we have:

- counit: $\varepsilon : SR \longrightarrow \mathrm{Id}_D$.
- unit: $\eta : \mathrm{Id}_{\mathcal{C}} \longrightarrow RS$.

Def (Anno-Logvinenko). $S : \mathcal{C} \longrightarrow D$ w/ L, R

Say S is a spherical functor if it satisfies:

1) the twist functor: $T_S := \mathrm{Cone}(SR \xrightarrow{\varepsilon} \mathrm{Id}_D)$

   is an autoequivalence of D.

2) the cotwist functor $C_S := \mathrm{Cone}(\mathrm{Id}_{\mathcal{C}} \xrightarrow{\eta} RS)[-1]$

   is an autoequivalence of $\mathcal{C}$.

3) $R \cong LT_S[-1]$.

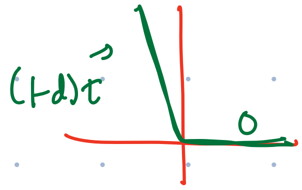
4) $R \cong C_S L[1]$.

Thm Any 2 conditions imply the other 2.

e.g. Let E be a d-spherical object of D, with sph. twist T_E. ($d \geq 2$)

Then $S := - \otimes_k E : D^b(k) \longrightarrow D$ is a spherical functor,

and $T_S = T_E$, $C_S = [-1-d]$



<u>Thm</u> (Kim). Let $S : \mathcal{C} \longrightarrow D$ spherical functor.

Assume the image of $R : D \to \mathcal{C}$ contains a split generator of $\mathcal{C}$.

Then $h_t(T_S) = h_t(C_S[2])$ for all t satisfying $h_t(C_S[2]) \geq 0$

① "<u>$h_t(T_S) \geq h_t(C_S[2])$</u>":

$$\begin{tikzcd} D \arrow[r, "R"] \arrow[d, "T_S"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C} \arrow[d, "C_S[2]"] \\ D \arrow[r, "R"] & \mathcal{C} \end{tikzcd}$$

$$\begin{aligned} C_S[2] R &\cong C_S[2] L T_S[-1] \\ &\cong C_S L T_S[1] \\ &\cong R T_S. \end{aligned}$$

- Let G be a split generator of D st. RG is a split gen. of $\mathcal{C}$.

Then

$$\begin{aligned} h_t(T_S) &= \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G, T_S^n G) \\ &\geq \lim_{n\to\infty} \frac{1}{n} \log \delta_t(RG, RT_S^n G) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \delta_t(RG, (C_S[2])^n RG) \\ &= h_t(C_S[2]). \quad \square \end{aligned}$$

② **Thm:** $S : \mathcal{C} \to \mathcal{D}$ sph. functor.

Assume:
- Image of $R : \mathcal{D} \to \mathcal{C}$ contains a split generator of $\mathcal{C}$.
- $h_t(C_S[2]) \geq 0$ for some t.

Then $h_t(T_S) \leq h_t(C_S[2])$.

---

pf: Let G, G' split generators of $\mathcal{D}$ s.t. RG, RG' split gen. of $\mathcal{C}$.

- By definition of T_S:

$$\begin{tikzcd} SRE \arrow[rr] & & E \arrow[dl] \\ & T_S E \arrow[ul, "+1"] & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[rr] & & T_S G \arrow[dl] \\ & SRG[1] \arrow[ul, "+1"] & \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd} T_S^{n-1} G \arrow[rr] & & T_S^n G \arrow[dl] \\ & T_S^{n-1} SRG[1] \arrow[ul, "+1"] & \end{tikzcd}$$

$$\Rightarrow \quad \delta_t(G', T_S^n G) \leq \delta_t(G', T_S^{n-1} G) + \delta_t(G', \underbrace{\boxed{T_S^{n-1} S}}_{S(C_S[2])^{n-1}} RG[1])$$

Lemma: $T_S^n S \cong S(C_S[2])^n$ (enough to show: $\boxed{T_S S = S C_S[2]}$)

pf: Recall: $T_S = \mathrm{Cone}(SR \xrightarrow{\varepsilon} \mathrm{Id}_{\mathcal{D}})$, $C_S = \mathrm{Cone}(\mathrm{Id} \xrightarrow{\eta} RS)[-1]$

$$\begin{tikzcd} S \arrow[r, "S\eta"] \arrow[dr, equal] & SRS \arrow[r] \arrow[d, "\varepsilon S"] & S C_S[1] \arrow[r] \arrow[d] & S[1] \\ & S \arrow[r] & 0 & \end{tikzcd}$$

octahedral axiom $\Rightarrow$ $T_S S \cong S C_S[2]$.

$$\begin{tikzcd} {} \arrow[d] \\ T_S S \arrow[d] \\ SRS[1] \end{tikzcd}$$

Lemma: $\Phi : \mathcal{C} \to D$ has right adjoint R. Then:

$$\delta_t(E, \Phi F) \leq (1 + \delta_t(E, T_\Phi E[-1]))\, \delta_t(RE, F).$$

where $T_\Phi = \mathrm{Cone}(\Phi R \to \mathrm{Id}_D)$.

pf <u>easy fact:</u>

$$\begin{tikzcd} E' \arrow[r] & E \arrow[dl] \\ E'' \arrow[u] & \end{tikzcd}, \quad \delta_t(G,F) \leq (\delta_t(G,E') + \delta_t(G,E''))\, \delta_t(E,F)$$

Consider:

$$\begin{tikzcd} T_\Phi E[-1] \arrow[r] & \Phi R E \arrow[dl] \\ E. \arrow[u] & \end{tikzcd}$$

$$\delta_t(E, \Phi F) \leq (\delta_t(E,E) + \delta_t(E, T_\Phi E[-1]))\, \delta_t(\underline{\Phi R E, \Phi F})$$

$$\leq (1 + \delta_t(E, T_\Phi E[-1]))\, \delta_t(RE, F). \quad \square$$

$$\delta_t(G', T_S^n G) \leq \delta_t(G', T_S^{n-1} G) + \delta_t(G', S(C_S[2])^{n-1} RG[1])$$

$$\leq \delta_t(G', T_S^{n-1} G) + (1 + \delta_t(G', T_S G'[-1]))\, \delta_t(RG', (C_S[2])^{n-1} RG[1])$$

$$\leq \dots$$

$$\leq \delta_t(G', G) + (1 + \delta_t(G', T_S G'[-1])) \sum_{i=0}^{n-1} \delta_t(RG', (C_S[2])^i RG[1])$$

$$\leq C(t)\left(1+\sum_{i=0}^{n-1} \boxed{\delta_t\left(RG', (C_s[2])^i\, RG[1]\right)}\right)$$

are split generators of $\mathcal{C}$.

Lemma: If $\{\log a_n\}$ is subadditive, then

$$\lim_{n\to\infty} \frac{1}{n}\log\left(1+\sum_{i=0}^{n} a_i\right) = \begin{cases} 0 & \text{if } \lim_{n\to\infty}\frac{1}{n}\log a_n \leq 0 \\ \lim_{n\to\infty}\frac{1}{n}\log a_n & \text{if } \lim_{n\to\infty}\frac{1}{n}\log a_n \geq 0 \end{cases}$$

$\Rightarrow$ If $h_t(C_s[2]) \geq 0$ for some t.

then $$\lim_{n\to\infty}\frac{1}{n}\log\left(1+\sum_{i=0}^{n-1}\delta_t\left(RG', (C_s[2])^i RG[1]\right)\right) = h_t(C_s[2]).$$

$\Rightarrow$ $h_t(T_s) \leq h_t(C_s[2])$ for any t s.t. $h_t(C_s[2]) \geq 0$.

$\square$

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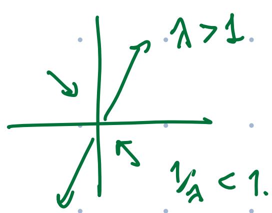
§ Pseudo-Anosov autoequivalences

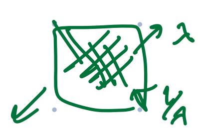
("what properties does a generic autoequivalence of a Δ cat. satisfy?")

Recap Pseudo-Anosov diffeomorphisms of Riemann surfaces.

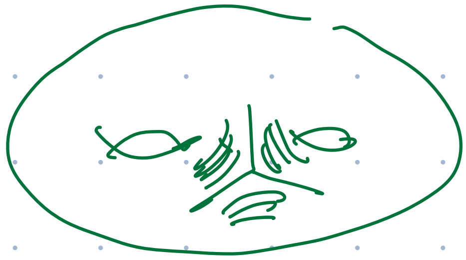
e.g. $SL(2,\mathbb{R}) \curvearrowright \mathbb{R}^2$



$SL(2,\mathbb{Z}) \circlearrowleft$ descends to $T^2 = \mathbb{R}^2/\mathbb{Z}^2$

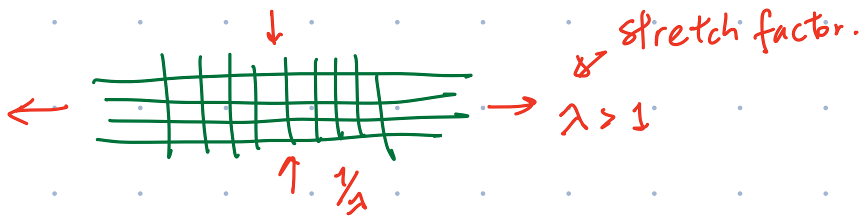


λ is a root of $x^2 + kx + 1 = 0$.



for $g \geq 2$, foliation would have singularities.

f is pseudo-Anosov if, at a generic point of the surface,



Thurston: $[f] \in MCG(S)$ is either
1) finite order
2) reducible
3) pseudo-Anosov. ← generic mapping class.

Three characterizations of pseudo-Anosov maps:

1) pair of transversed foliations with stretch factor $\lambda > 1$.

2) Thurston: $\forall\ [f] \in MCG(S)$, $\exists$ finite list of algebraic integers $1 \leq \lambda_1 < \lambda_2 < \dots < \lambda_k$

s.t. $\forall\ \alpha$ - simple closed curve, $\exists\ \lambda_i$

s.t. $\limsup_{n \to \infty} \text{length}_g (f^n \alpha)^{1/n} = \lambda_i$ $\forall$ Riemannian metric g on S.

Moreover, $[f]$ is pseudo-Anosov $\Leftrightarrow$ $k = 1$ and $\underset{\text{stretch factor.}}{\lambda_1 \overset{}{=}} > 1$.

3) <u>Def</u> (X, d) metric space $f \in \mathrm{Isom}(X)$.

Define the <u>translation length</u> of f: $\tau(f) := \inf_{x \in X} d(x, fx)$

Say f is <u>elliptic</u> if $\tau(f) = 0$ and the inf is achieved.

f is <u>parabolic</u> if the inf is not achieved,

f is <u>hyperbolic</u> if $\tau(f) > 0$ and the inf is achieved,

When $X = \mathrm{Teich}(S)$, $d =$ Teichmüller metric

pseudo-Anosov

---

surface S Teichmüller	vs	$\triangle$ cat. D Stab	(GMN, BS, HKK)
curve C		obj. E	
$C_1 \cap C_2$		$\mathrm{Hom}(E_1, E_2)$	
metric g		stability condition σ	
geodesics / straight lines / saddle connections		semistable obj.	
length		| central charge |	
$MCG(S)$		$\mathrm{Aut}(D)$	
shearing / rotating		$\widetilde{GL_2^+}(\mathbb{R})$	
$\mathrm{Teich}(S)$		$\mathrm{Stab}(D)$	

Def (DHKK pseudo-Anosov autoequivalence)

$F \in \mathrm{Aut}(D)$ is <u>DHKK pA</u> if.

$$\exists \ \sigma \in \mathrm{Stab}(D) \text{ and } g \in \widetilde{GL}^+(2,\mathbb{R}) \text{ s.t.} \qquad \pi \to GL^+(2,\mathbb{R})$$

1) $F\sigma = \sigma g$

2) $\pi(g) = \begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix} \in GL^+(2,\mathbb{R})$ for some $|r| > 1$.

λ → stretch factor of F. ($\lambda = |r|$)

<u>e.g.</u> $D = D^b\mathrm{Coh}(\text{elliptic curve})$.

$F \in \mathrm{Aut}(D)$ is DHKK pA $\iff$ $|\mathrm{tr}[F]| > 2$.

<u>e.g.</u> $D = D^b(\bullet \rightrightarrows \bullet)$, # arrows ≥ 3, Serre functor of D is DHKK pA.

<u>Main issue with this definition:</u> It's hard to find such F:

It's difficult to find F, σ that fixes the set of all σ-semistable objects, while acts on Z_σ in a nontrivial way.

In fact, no example of DHKK pA autoeq. is known for CY cat. of dim. > 1. (e.g. $D^b(K3)$, $D^b(CY3)$, ...)

<u>Alternative definition of pA autoeq.:</u> Say $F \in \mathrm{Aut}(D)$ is pA if

$$\limsup_{n\to\infty} \frac{1}{n} \log m_\sigma(F^n E) = \log \lambda > 0 \quad \forall E \neq 0.$$

Thm: F is DHKK pA $\Longrightarrow$ F is pA.

e.g. $\mathcal{D} = \mathcal{D}_{CY3}(A_2)$

some compositions of T_1 and T_2^{-1} are pA.

However, there is no DHKK pA auto. eq. on $\mathcal{D}$.

e.g. X - quintic CY3, $F = T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}(-1))$ is pA.