Recap: A: $n\times n$ matrix.

- <u>characteristic polynomial</u>: $\det(A-\lambda I) = \prod_{i=1}^{k} (\lambda_i - \lambda)^{\text{mult}(\lambda_i)}$,
  where $\{\lambda_1, \cdots, \lambda_k\}$ are distinct eigenvalues of A,
  and $\text{mult}(\lambda_i)$ is the <u>multiplicity</u> of the eigenvalue λ_i.
- for each eigenvalue λ_i,
  $$0 \subsetneq \text{Nul}(A-\lambda_i I) \text{ is the } \underline{\text{eigenspace}} \text{ of } \lambda_i.$$
  any nonzero vector in $\text{Nul}(A-\lambda_i I)$ is an <u>eigenvector</u> of λ_i
- A is <u>diagonalizable</u> $\overset{\text{definition}}{\Longleftrightarrow}$ A is similar to a diagonal matrix
  $\Longleftrightarrow$ $\exists$ an eigenbasis of A.
- if $\{\vec{v}_1, \cdots, \vec{v}_k\}$ are eigenvectors correspond to <u>distinct</u> eigenvalues then they form a linearly independent set.

<u>Thm</u>: A: $n\times n$. <u>If A is diagonalizable</u>, then any $\vec{v}\in\mathbb{C}^n$ can be uniquely written as
$$\vec{v} = \vec{v}_1 + \cdots + \vec{v}_k,$$
where each $\vec{v}_i$ is an eigenvector w.r.t. different eigenvalues

<u>pf</u>:
- A diagonalizable $\Rightarrow$ $\exists$ an eigenbasis of A.
  $\{\vec{w}_1, \cdots, \vec{w}_n\}$

$$\Big\{ \underbrace{\overset{w_1}{\underset{\shortparallel}{}}\vec{w}_{\lambda_1}^{(1)}, \cdots, \overset{w_{m_1}}{\underset{\shortparallel}{}}\vec{w}_{\lambda_1}^{(m_1)}}_{\text{eigenvects of } \lambda_1}, \underbrace{\vec{w}_{\lambda_2}^{(1)}, \cdots, \vec{w}_{\lambda_2}^{(m_2)}}_{\text{eigenvects of } \lambda_2}, \cdots \Big\}$$

- $\forall \vec{v}\in\mathbb{C}^n$, $\exists$ $c_1, \cdots, c_n \in \mathbb{C}$
  s.t. $\vec{v} = c_1\vec{w}_1 + \cdots + c_n\vec{w}_n$.

$$= (\underbrace{c_1 \vec{w}_1 + \cdots + c_{m_1} \vec{w}_{m_1}}_{\text{an eigenvector of } \lambda_1}) + (\cdots) + \cdots$$

- If
$$\vec{v} = \vec{v}_1 + \cdots + \vec{v}_k$$
$$= \vec{v}_1' + \cdots + \vec{v}_k'$$

eigenvectors of λ_1 ... eigenvectors of λ_k

$$\vec{0} = (\underbrace{\vec{v}_1 - \vec{v}_1'}_{\mathrm{Nul}(A-\lambda_1 I)}) + \cdots + (\underbrace{\vec{v}_k - \vec{v}_k'}_{\mathrm{Nul}(A-\lambda_k I)})$$

By the theorem we proved last time (eigenvectors corresp. to different eigenvalues must be l.i.)

we have $\vec{v}_1 - \vec{v}_1' = \vec{0}, \ \cdots, \ \vec{v}_k - \vec{v}_k' = \vec{0}$. $\square$

Rmk: $\mathbb{C}^n = \mathrm{Nul}(A - \lambda_1 I) \oplus \cdots \oplus \mathrm{Nul}(A - \lambda_k I)$

Rmk: The statement is NOT true if A is not diagonalizable:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \qquad \mathrm{Nul}(A - 2I) = \mathrm{Nul}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \subsetneq \mathbb{C}^2$$

$$\dim \mathrm{Nul}(A - 2I) = 1 < 2 = \mathrm{mult}(2).$$

Thm: (A: $n \times n$) $1 \le \underbrace{\dim \mathrm{Nul}(A - \mu I)}_{k} \le \mathrm{mult}(\mu) \quad \forall$ eigenvalue μ.

pf:
- $\mathrm{Nul}(A - \mu I)$ has a basis $\{\vec{v}_1, \cdots, \vec{v}_k\}$.
- we can choose $\vec{v}_{k+1}, \cdots, \vec{v}_n$ s.t. $\{\vec{v}_1, \cdots, \vec{v}_k, \vec{v}_{k+1}, \cdots, \vec{v}_n\}$ forms a basis of $\mathbb{C}^n$.

$$A \begin{bmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k & \cdots & \vec{v}_n \\ | & & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | & | & | & | \\ \mu\vec{v}_1 & \cdots & \mu\vec{v}_k & & & & \\ | & & | & | & | & | & | \end{bmatrix}$$

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}^{-1} A \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}^{-1} \cdot \begin{bmatrix} | & & | & | & | & | \\ \mu\vec{v}_1 & \cdots & \mu\vec{v}_k & | & | & | \\ | & & | & | & | & | \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} \mu & 0 & 0 & \\ 0 & \mu & & \\ \vdots & 0 & \ddots\ \mu & \\ \hline 0 & 0 & & \\ \vdots & \vdots & 0 & \\ 0 & 0 & & \end{array}\right] = B$$

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}^{-1} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} = I$$

- A is similar to B, so they have the same char. poly
- $\det(B - \lambda I) = \det \left[\begin{array}{ccc|c} \mu-\lambda & & 0 & \\ & \mu-\lambda & & \\ 0 & & \ddots\ \mu-\lambda & \\ \hline 0 & & & *-\lambda \\ 0 & 0 & & \ddots \\ 0 & & & *-\lambda \end{array}\right]$

$$= (\mu-\lambda)^k \det \underbrace{\left[\quad\right]}_{(n-k)\times(n-k)}$$

$\Rightarrow$ mult$(\mu) \geq k$. $\square$

Thm: A: diagonalizable $\iff$ $\dim \mathrm{Nul}(A - \lambda I) = \mathrm{mult}(\lambda)$ $\forall$ eigenvalue λ.

pf: ($\Rightarrow$) $A = PDP^{-1}$, where P invertible, D: diagonal.

If A is a diagonal matrix, then $\dim \mathrm{Nul}(A - \lambda I) = \mathrm{mult}(\lambda)$ $\forall$ eigvalue λ.

$$A = \begin{bmatrix} \overbrace{\lambda_1\ \lambda_1 \ddots \lambda_1}^{m_1} & & 0 \\ & \overbrace{\lambda_2 \ddots \lambda_2}^{m_2} & \\ 0 & & \ddots \end{bmatrix} \quad k$$

$$\det(A - \lambda I) = \prod_{i=1}^{k} (\lambda_i - \lambda)^{m_i}$$

$$\mathrm{Nul}(A - \lambda_1 I) = \mathrm{Nul}\left(\begin{bmatrix} \left.\begin{matrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{matrix}\right\}m_1 & \\ & \lambda_2-\lambda_1 \neq 0 & \\ & & \ddots \\ & & & \lambda_k-\lambda_1 \neq 0 \end{bmatrix}\right)$$

$$= \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} m_1$$

need to show: $\dim \mathrm{Nul}(\underbrace{A-\lambda I}_{}) = \dim \mathrm{Nul}(D-\lambda I)$. $\forall \lambda$.

$$A-\lambda I = PDP^{-1} - \lambda I = PDP^{-1} - \lambda PP^{-1} = P(D-\lambda I)P^{-1}$$

Ex: P invertible,
$\dim \mathrm{Nul}(PA) = \dim \mathrm{Nul}(A)$
$\dim \mathrm{Nul}(AP) = \dim \mathrm{Nul}(A)$

($\Leftarrow$): Fact: $\sum_{i=1}^{k} \mathrm{mult}(\lambda_i) = n$

$$\sum_{i=1}^{k} \mathrm{mult}(\lambda_i) = \sum_{i=1}^{k} \dim \mathrm{Nul}(A-\lambda_i I)$$

- For each eigenspace $\mathrm{Nul}(A-\lambda_i I)$, pick a basis $\{\vec{v}_{\lambda_i}^{(1)}, \dots, \vec{v}_{\lambda_i}^{(m_i)}\}$.

$\rightsquigarrow$ get n vectors $\{\vec{v}_{\lambda_1}^{(1)}, \dots, \vec{v}_{\lambda_1}^{(m_1)}, \dots\dots, \vec{v}_{\lambda_k}^{(1)}, \dots, \vec{v}_{\lambda_k}^{(m_k)}\}$

Claim: this is an eigenbasis for A.
It suffices to prove they're l.i.

$$\underbrace{a_{\lambda_1}^{(1)} \vec{v}_{\lambda_1}^{(1)} + \dots + a_{\lambda_1}^{(m_1)} \vec{v}_{\lambda_1}^{(m_1)}}_{\in\ \mathrm{Nul}(A-\lambda_1 I)} + \underbrace{\dots\dots}_{\in\ \mathrm{Nul}(A-\lambda_2 I)} \dots = \vec{0}$$

$$\Rightarrow a_{\lambda_i}^{(1)} \vec{v}_{\lambda_i}^{(1)} + \dots + a_{\lambda_i}^{(m_i)} \vec{v}_{\lambda_i}^{(m_i)} = \vec{0} \quad \text{for each } i.$$

basis of $\mathrm{Nul}(A-\lambda_i I)$.

$$\Rightarrow a_{\lambda_i}^{(*)} = 0. \quad \square$$

"Algorithm" to check diagonalizable? get a diagonalisation?

- char. poly. $\rightsquigarrow$ eigenvalues, multiplicities
- eigenspaces
- diagonalizable $\Leftrightarrow$ $\dim \mathrm{Nul}(A-\lambda I) = \mathrm{mult}(\lambda)$ $\forall \lambda$.
- Suppose it's diagonalizable,
  - pick a basis for each eigenspace
  - together, forms an eigenbasis of A say $\{\vec{v}_1, \dots, \vec{v}_n\}$.

Then
$$A\begin{bmatrix}\vec{v}_1 \cdots \vec{v}_n\end{bmatrix} = \begin{bmatrix}\vec{v}_1 \cdots \vec{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \ddots\end{bmatrix}$$

or
$$A = \begin{bmatrix}\vec{v}_1 \cdots \vec{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \ddots\end{bmatrix}\begin{bmatrix}\vec{v}_1 \cdots \vec{v}_n\end{bmatrix}^{-1}$$

Rmk: In general (A may not be diagonalizable), $\exists$ P invertible s.t.

$$P^{-1}AP = \begin{bmatrix} \begin{matrix}\lambda & 1 & \\ & \lambda & 1 \\ & & \lambda\end{matrix} & & & \\ & \boxed{\begin{matrix}\lambda & 1 \\ & \lambda\end{matrix}} & & \\ & & \lambda' & \\ & & & \ddots \end{bmatrix}$$

$A \in M_{n\times n}(\mathbb{R})$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$ is an eigenvalue

$A\vec{v} = \lambda\vec{v}$ $\quad \vec{v} \neq \vec{0}$

$\left(\vec{v} = \begin{bmatrix}v_1 \\ \vdots \\ v_n\end{bmatrix} \in \mathbb{C}^n, \quad \bar{\vec{v}} = \begin{bmatrix}\bar{v}_1 \\ \vdots \\ \bar{v}_n\end{bmatrix} \in \mathbb{C}^n.\right)$

$\Rightarrow$ $\bar{\lambda} \in \mathbb{C}$ is also an eigenvalue, and $\overline{\vec{v}}$ is an eigenvector for $\bar{\lambda}$.

$(A\bar{\vec{v}} = \overline{A\vec{v}} = \overline{\lambda\vec{v}} = \bar{\lambda}\,\bar{\vec{v}})$