

# GEOMETRY AND SYMMETRY, FALL 2022

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*Course Description.* Group is a mathematical concept describing the symmetry of objects, and has a wide range of applications in modern science. This course introduces basic concepts and methods of groups through examples and geometric phenomena, and aims to gain an intuitive understanding of how to use the methods of groups to study symmetries. The course starts from the symmetries of common geometric objects in plane and solid geometry, introduces the definition and basic principles of groups, and applies them to descriptions such as symmetry groups of regular polyhedra, plane symmetry groups (wallpaper classification), Möbius transformation, and other related content and method.

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