Math H54 First Midterm (Practice Exam) October 3, 2019

Name: ______________________________

- You have 80 minutes to complete the exam.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

| Question | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| Total | 100 | |

1. (4 points each) Determine if each statement is TRUE or FALSE, and give a short justification.

   (a) If $A$ and $B$ are both symmetric matrices, then $AB$ must be symmetric as well. ($A$ is called symmetric if $A = A^T$.)

   (b) If the determinant of a $4 \times 4$ matrix $A$ is 4, then $\text{rank}(A) = 4$.

   (c) If $A$ is an $n \times n$ matrix, then the determinant of $A$ is equals to the product of its diagonal entries.

   (d) The set of all polynomials $p$ satisfying $p(0) = 1$ forms a vector space.

   (e) If $A$ has pivot in each row, then its associated linear transformation $\vec{x} \to A\vec{x}$ must be injective.

2. (10 points each) Let

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

(a) Find $\det(A)$.

(b) Find $A^{-1}$.

3. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find a matrix $A$ such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

4. (10 points) Let $A$ be a $3 \times 3$ matrix with the property that the linear transformation $T$ defined by $\vec{x} \to A\vec{x}$ maps $\mathbb{R}^3$ *onto* $\mathbb{R}^3$. Explain why the transformation must be *injective.*

5. (10 points) Under what conditions on $a, b, c \in \mathbb{R}$ is the matrix $\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ invertible?

Be as explicit as possible.

6. (10 points) Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{pmatrix} -7 \\ 9 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \end{pmatrix} \right\}$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

7. (10 points) Let $A$ be an $m \times n$ matrix and $B$ be an $n \times p$ matrix. Show that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

8. (10 points) Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces, and $H \subset V$ be a subspace. We know that $T(H)$ is a subspace of $W$. Prove that $\dim T(H) \leq \dim H$.