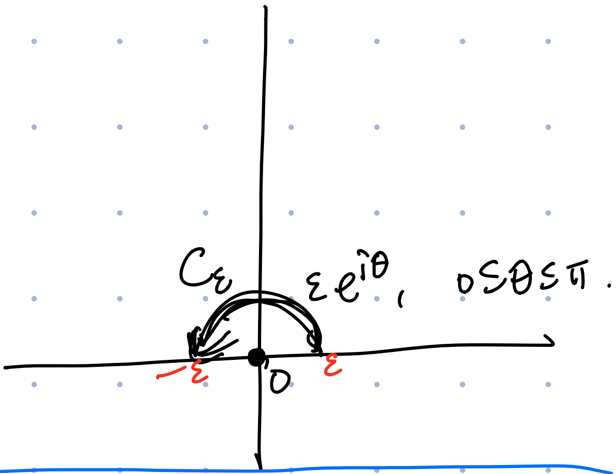
Correction:



f has $\boxed{\text{simple pole}}$ at 0. necessary!! (doesn't work for higher order pole, ess sg..)

$$\lim_{\varepsilon \to 0}\left(\int_{C_\varepsilon} f(z)\,dz\right) = 2\pi i \cdot \operatorname{Res}_{z=0} f \cdot \frac{1}{2}.$$

pf: Locally at 0, write

$$f(z) = \frac{\operatorname{Res} f}{z} + g \quad \leftarrow \text{holo. near } 0$$

$\uparrow$

$\exists$ primitive G near 0

Last time:

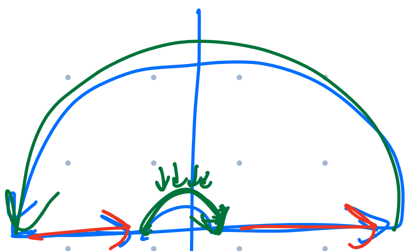
$$\int_{C_\varepsilon} \frac{\operatorname{Res} f}{z}\,dz = 2\pi i \cdot \operatorname{Res}_{z=0} f \cdot \frac{1}{2}.$$

$$\boxed{\int_{C_\varepsilon} g(z)\,dz} = G(-\varepsilon) - G(\varepsilon) \quad \forall \varepsilon > 0$$

Take $\varepsilon \to 0$: $G(-\varepsilon) \to G(0)$, $G(\varepsilon) \to G(0)$

So $\lim_{\varepsilon \to 0} \int_{C_\varepsilon} g(z)\,dz = 0$.

useful:



$$\boxed{\int_{-\infty}^{\infty} f} = \lim_{\substack{R\to\infty \\ \varepsilon \to 0}} \int_{\varepsilon}^{R} + \int_{-R}^{-\varepsilon}$$

Today: Argument principle, Rouché thm, Open mapping thm, Max. modulus principle.

Thm: (argument principle): $f : \Omega \to \hat{\mathbb{C}}$ meromorphic ($\exists$ isolated poles in Ω)

$\gamma \subseteq \Omega$ simple closed curve, positively oriented.

- Its interior is contained in Ω,

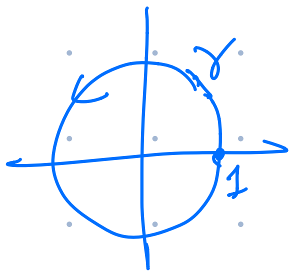
doesn't pass through any zeros or poles of f

Then

$$\underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz} = \begin{pmatrix}\#\text{ of zeros of } f \\ \text{inside } \gamma\end{pmatrix} - \begin{pmatrix}\#\text{ of poles of } f \\ \text{inside } \gamma\end{pmatrix}$$

counting multiplicities. (orders)

e.g. $f(z) = z^n$ (order n zero at $z=0$)



$$\int_\gamma \frac{f'}{f}\,dz = \int_\gamma \frac{n z^{n-1}}{z^n}\,dz$$

$$= n\int_\gamma \frac{1}{z}\,dz = 2\pi i\, n.$$

order of zero at 0.

e.g. $f(z) = \frac{1}{z^n}$ (order n pole at $z=0$)

$$\int_\gamma \frac{f'}{f}\,dz = \int_\gamma \frac{-n/z^{n+1}}{1/z^n}\,dz = -n\int \frac{1}{z}\,dz$$

$$= 2\pi i\,(-n)$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt$$

Chain rule

$$= \frac{1}{2\pi i}\int_a^b \frac{\frac{d}{dt}\left(f(\gamma(t))\right)}{f(\gamma(t))}\,dt$$

γ parametrized by

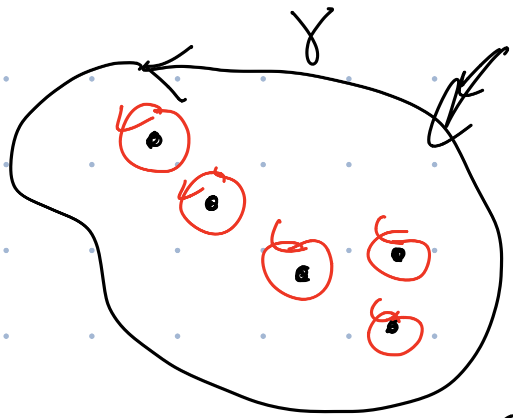
$\gamma(t): [a,b] \to \mathbb{C}$

$$= \frac{1}{2\pi i}\int_a^b \frac{\eta'(t)}{\eta(t)}\,dt = \boxed{\frac{1}{2\pi i}\int_\eta \frac{1}{z}\,dz}$$

Consider another curve η parametrized by

Winding # of η around 0

$$\begin{tikzcd}
{[a,b]} \arrow[r, "\gamma"] \arrow[rr, bend right, "\eta"'] & \Omega \arrow[r, "f"] & \mathbb{C}
\end{tikzcd}$$



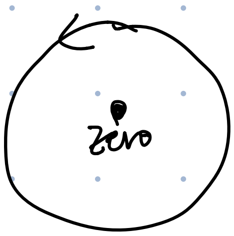
Some zeros & poles of f inside γ.

$$\boxed{\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz} = (\#\text{ zeros}) - (\#\text{ poles})$$

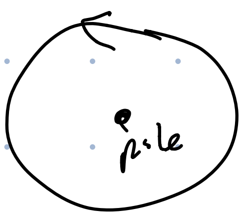
want to show.

$$= \frac{1}{2\pi i}\sum_{\substack{\text{circles}\\ \text{surrounding}\\ \text{zeros \& poles of } f.}} \int_{\odot} \frac{f'}{f}\,dz$$

<u>It suffices to show:</u>

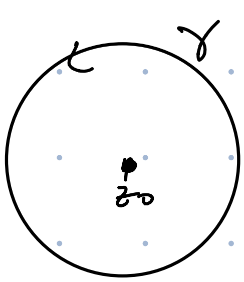
① 

$$\frac{1}{2\pi i}\int_{\circlearrowleft} \frac{f'}{f}\,dz = \text{order of zero}$$

② 

$$\frac{1}{2\pi i}\int_{\circlearrowleft} \frac{f'}{f}\,dz = -(\text{order of pole})$$

①



If f has order n zero near z_0,

Then $f(z) = (z-z_0)^n g(z)$, where

g is nonvanishing & holo. near z_0.

$$\frac{f'}{f} = \frac{n(z-z_0)^{n-1} g(z) + (z-z_0)^n g'(z)}{(z-z_0)^n g(z)}$$

$$= \frac{n}{z-z_0} + \boxed{\frac{g'(z)}{g(z)}} \rightarrow \text{holo. near } z_0.$$

$$\int_\gamma \frac{f'}{f}\,dz = \int_\gamma \frac{n}{z-z_0}\,dz = 2\pi i \cdot n. \quad \square$$

②

If f has order n pole at z_0,

then $f(z) = \dfrac{g(z)}{(z-z_0)^n}$, g: nonvanishing holo near z_0.

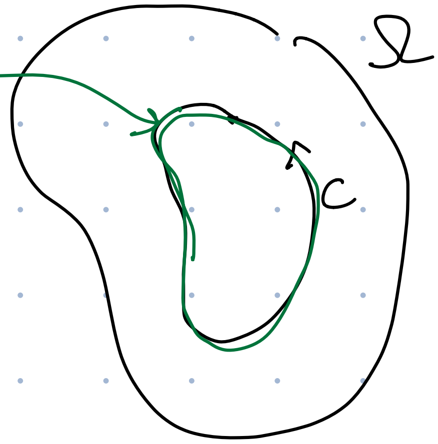
$$\frac{f'}{f} = \frac{\dfrac{g'(z)(z-z_0)^n - g(z)\cdot n(z-z_0)^{n-1}}{(z-z_0)^{2n}}}{g(z)/(z-z_0)^n}$$

holo near z_0

$$= \frac{g'(z)}{g(z)} + \frac{-n}{z-z_0}.$$

$$\Rightarrow \int_\gamma \frac{f'}{f}\,dz = \int_\gamma \frac{-n}{z-z_0}\,dz = 2\pi i(-n). \quad \square$$

Thm (Rouché) f, g holo. in Ω, $C \subseteq \Omega$ simple closed curve
its interior is contained in Ω.



If
- $\cancel{f(z) \neq 0 \ \forall z \in C}$
- $|f(z)| > |g(z)| \quad \forall z \in C$

Then # of zeros of f inside C

= # of zeros of $f+g$ inside C

pf.
- $(f+g)(z) = f(z) + g(z) \neq 0 \quad \forall z \in C$
- actually, $\boxed{(f+tg)(z) = f(z) + t\,g(z) \neq 0 \quad \forall\, 0 \leq t \leq 1,\ \forall z \in C}$

$f+tg$ holo. in Ω.

argument principle

$$\Rightarrow \frac{1}{2\pi i} \int_C \frac{(f+tg)'}{f+tg}\, dz = \text{\# of zeros of } f+tg \text{ inside } C. \in \mathbb{Z}$$

As t varies from 0 to 1,
the left hand side $\frac{1}{2\pi i} \int_C \frac{(f+tg)'}{f+tg}\, dz$ varies continuously

$$\Rightarrow \frac{1}{2\pi i} \int_C \frac{(f+tg)'}{f+tg}\, dz \equiv \text{const.} \quad \forall\, 0 \leq t \leq 1$$

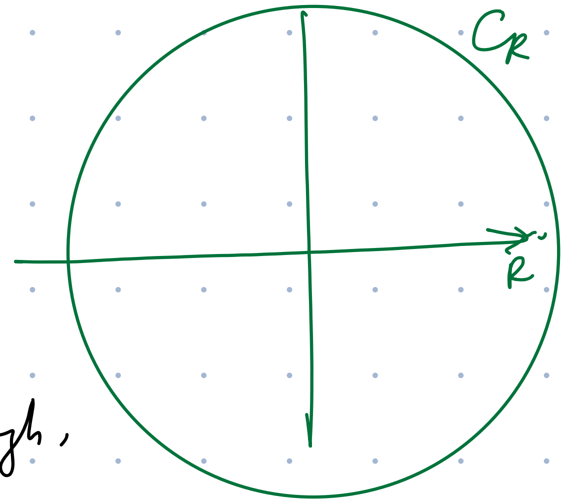
(b/c they're integral valued)

Take $t = 0, 1$. $\square$

e.g. (fundamental thm of alg)

$f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$,

$g(z) = -\frac{1}{2} z^n - a_{n-1} z^{n-1} - \cdots - a_0$



Claim: We can choose $R > 0$ large enough, s.t. $|f(z)| > |g(z)| \quad \forall z \in C_R$

($\underset{\text{Rouché}}{\Longrightarrow}$ # of zeros of f inside C_R

$=$ # of zeros of $\underbrace{f+g}_{\frac{1}{2}z^n}$ inside C_R $= n$)

Ex: We can choose $R > \max\{|a_{n-1}|, |a_{n-2}|, \ldots, 1\} \cdot 4n$

and show that $|f(z)| > |g(z)| \quad \forall z \in C_R$

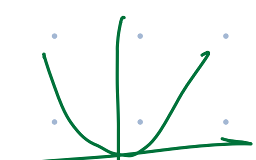
Thm (open mapping thm.) $f : \Omega \to \mathbb{C}$ nonconst. holo. fcn.

Then, for any $U \underset{\text{open}}{\subseteq} \Omega$, we have $f(U) \subseteq \mathbb{C}$ is open.

(recall: f is conti, if the preimage of any open set is open)

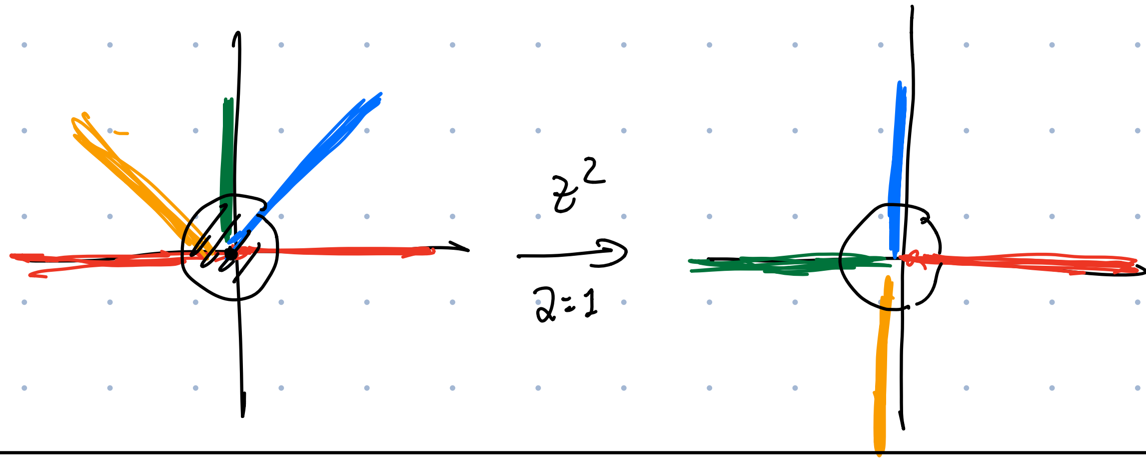
$f : X \to Y$, $U \underset{\text{open}}{\subseteq} Y \Rightarrow f^{-1}(U) \subseteq X$ is open

Rmks. not true / $\mathbb{R}$. $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$



$(-1,1) \underset{\text{open}}{\subseteq} \mathbb{R}$ but $f(-1,1) = [0,1)$ not open in $\mathbb{R}$

Why complex numbers save the day? $f(z)=z^2$



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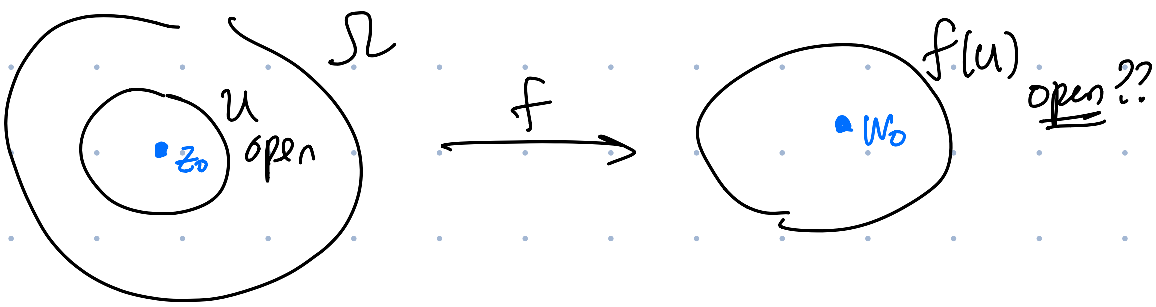
pf:



$\forall\, w_0 \in f(\mathcal{U})$, want to show $\exists\, \delta>0$ s.t. $\mathbb{D}_\delta(w_0) \subseteq f(\mathcal{U})$

- $\exists\, z_0 \in \mathcal{U}$ s.t. $f(z_0)=w_0$
- near z_0,

$$f(z) = f(z_0) + f'(z_0)\cdot(z-z_0) + \cdots$$

$$= w_0 + (z-z_0)^n\, h(z)$$

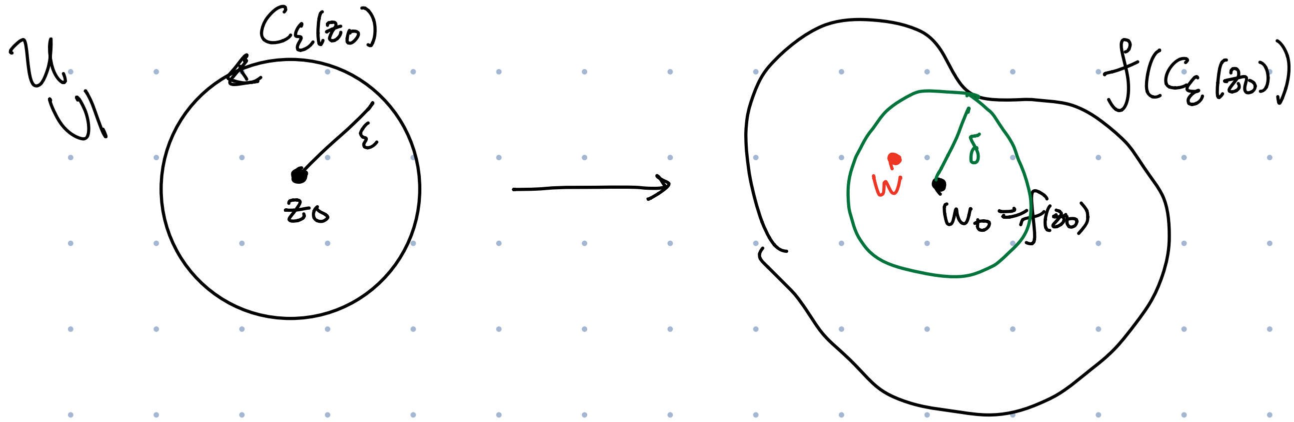
(where n is the smallest positive integer s.t. $f^{(n)}(z_0) \neq 0$.)

where $n \geq 1$,

$h(z)$ holo near z_0,

$h(z_0) \neq 0$

Choose $\varepsilon>0$ s.t. $h(z)\neq 0$ $\forall\, |z-z_0| \leq \varepsilon$.



Claim: $\delta := \inf_{z \in C_\varepsilon(z_0)} |f(z) - w_0| > 0$

$$\inf_{z \in C_\varepsilon(z_0)} |f(z) - w_0| = \inf_{z \in C_\varepsilon(z_0)} \Big| \underbrace{(z-z_0)^n}_{\varepsilon^n} \; \underbrace{h(z)}_{\text{nonvanishing } \forall |z-z_0| \le \varepsilon} \Big| > 0$$

Claim: $\forall w \in \mathbb{D}_\delta(w_0)$, $\exists z \in \mathbb{D}_\varepsilon(z_0)$ s.t. $f(z) = w$

- Consider $f(z) - w$ on $z \in \mathbb{D}_\varepsilon(z_0)$.

  (Want: $f(z) - w = 0$ for some $z \in \mathbb{D}_\varepsilon(z_0)$)

- $\underline{f(z) - w} = (\underline{f(z) - w_0}) - (w - w_0)$

- $|f(z) - w_0| \ge \delta \quad \forall z \in C_\varepsilon(z_0)$

$$\delta > |w - w_0|$$

- By Rouché thm, # of zeros of $f(z) - w_0$ in $\mathbb{D}_\varepsilon(z_0)$ $\overset{\ge 1 \text{ b/c } f(z_0) = w_0}{}$

  $=$ # of zeros of $f(z) - w$ in $\mathbb{D}_\varepsilon(z_0)$

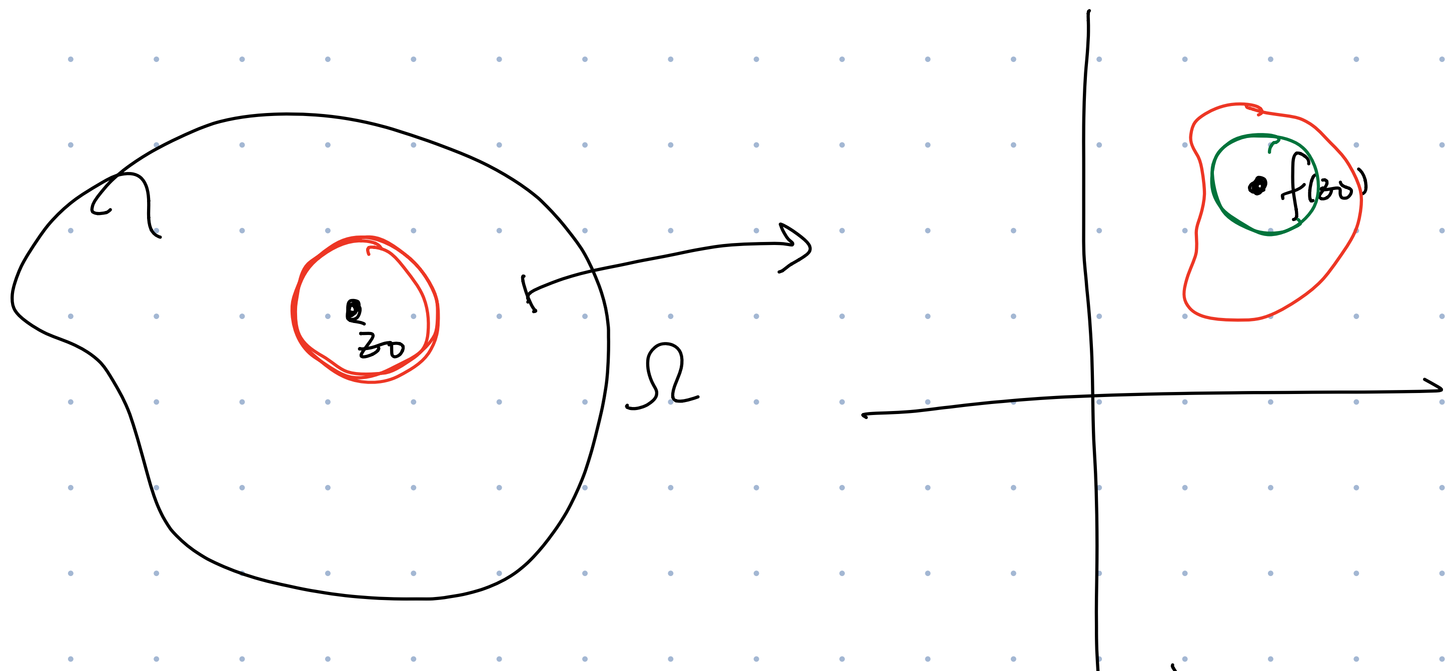
$\square$

Thm (max. modulus principle). Ω: open

$f: \Omega \to \mathbb{C}$ nonconst. holo. fcn

There is no $z_0 \in \Omega$ s.t. $|f(z_0)| = \sup_{z \in \Omega} |f(z)|$

pf: Assume there is such z_0



(direct consequence of open mapping thm.) $\square$