Last time:

- If A is real $n\times n$ matrix, then its complex (non-real) eigenvalues come in pairs:

$$\det(A-\lambda I) = \prod(\mu_j-\lambda)^{m_j}\prod(\lambda_i-\lambda)^{a_i}\prod(\bar{\lambda_i}-\lambda)^{a_i}$$

real poly. in λ. $\quad \mu_j \in \mathbb{R} \quad \lambda_i \in \mathbb{C}\setminus\mathbb{R}$

Today:

- 2×2 real matrix with complex eigenvalues.
- application: dynamical systems.
- Q & A / generalized eigenspaces (just for fun).

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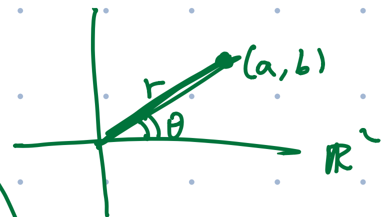
e.g. $\boxed{A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scale by } r}\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation by } \theta}$

$r = \sqrt{a^2+b^2}$

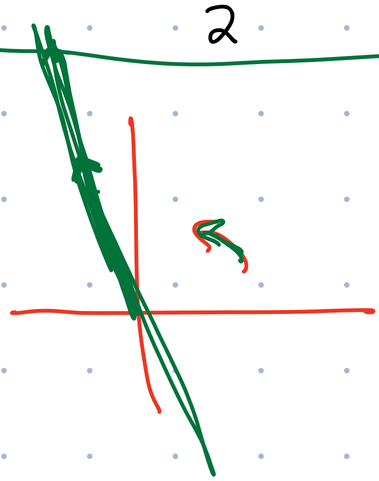


$$\det(A-\lambda I) = \det\begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix}$$

$$= (a-\lambda)^2 - (-b^2)$$

$$= \lambda^2 - 2a\lambda + (a^2+b^2)$$

$$\lambda = \frac{2a \pm \sqrt{4a^2-4(a^2+b^2)}}{2} = a \pm ib$$



Prop A 2×2 real, eigenvalues are not real

then $A = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix}P^{-1}$ for some $a, b \in \mathbb{R}$, $P \in M_{2\times 2}(\mathbb{R})$ invertible

pf $\lambda = a \pm ib$ $b \neq 0$

$a-ib$ $\quad A\vec{v} = (a-ib)\vec{v}$ $\quad \vec{v} \neq \vec{0}$ $\quad \vec{v} \in \mathbb{C}^2$

$\vec{v} = \mathrm{Re}\,\vec{v} + i\,\mathrm{Im}\,\vec{v}$. $\quad \mathrm{Re}\,\vec{v}, \mathrm{Im}\,\vec{v} \in \mathbb{R}^2$

$$A(\mathrm{Re}\,\vec{v} + i\,\mathrm{Im}\,\vec{v}) = (a-ib)(\mathrm{Re}\,\vec{v} + i\,\mathrm{Im}\,\vec{v})$$

$$A(\mathrm{Re}\,\vec{v}) + i\,A(\mathrm{Im}\,\vec{v}) \qquad = a\,\mathrm{Re}\,\vec{v} + i\,a\,\mathrm{Im}\,\vec{v} - ib\,\mathrm{Re}\,\vec{v} + b\,\mathrm{Im}\,\vec{v}$$

$$= (a\,\mathrm{Re}\,\vec{v} + b\,\mathrm{Im}\,\vec{v}) + i(a\,\mathrm{Im}\,\vec{v} - b\,\mathrm{Re}\,\vec{v})$$

$$\Rightarrow \quad A(\mathrm{Re}\,\vec{v}) = \boxed{a\,\mathrm{Re}\,\vec{v} + b\,\mathrm{Im}\,\vec{v}} \quad (a+bc)\mathrm{Re}\,\vec{v}$$

$$A(\mathrm{Im}\,\vec{v}) = a\,\mathrm{Im}\,\vec{v} - b\,\mathrm{Re}\,\vec{v}$$

Suppose $\mathrm{Re}\,\vec{v}, \mathrm{Im}\,\vec{v}$ are not l.i.

Suppose $\mathrm{Im}\,\vec{v} = c\,\mathrm{Re}\,\vec{v}$

$\Downarrow$

$A(\mathrm{Re}\,\vec{v}) = (a+bc)\mathrm{Re}\,\vec{v}$

$\Downarrow$

$a+bc$ is an eigvalue of A

$\therefore \mathrm{Re}\,\vec{v} = \vec{0}$

$$\Rightarrow \quad A\begin{bmatrix} | & | \\ \mathrm{Re}\,\vec{v} & \mathrm{Im}\,\vec{v} \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \mathrm{Re}\,\vec{v} & \mathrm{Im}\,\vec{v} \\ | & | \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$\Rightarrow$ invertible $= P$ $\Rightarrow$ $A = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix}P^{-1}$. $\square$

Rmk A real 2×2 matrix

- eigenvalues $\in \mathbb{R}$, $\cancel{\emptyset}$
- eigenvalues $\in \mathbb{C}\setminus\mathbb{R}$

$$\Rightarrow A = P\begin{bmatrix}\lambda & \\ & \bar{\lambda}\end{bmatrix}P^{-1} \quad P \in M_{2\times 2}(\mathbb{C})$$

$\lambda \in \mathbb{C}$

$$\to \quad A = P\underline{\begin{bmatrix}a & -b\\ b & a\end{bmatrix}}P^{-1} \quad P \in M_{2\times 2}(\mathbb{R})$$

A: $n\times n$ real matrix diagonalizable

$$A \sim \begin{pmatrix}\mu_1 & & & & & \\ & \ddots & & & & \\ & & \mu_k & & & \\ & & & \boxed{\begin{matrix}a_1 & -b_1\\ b_1 & a_1\end{matrix}} & & \\ & & & & \boxed{\begin{matrix}a_2 & -b_2\\ b_2 & a_2\end{matrix}} & \\ & & & & & \ddots\end{pmatrix}$$

$\in \mathbb{R}$

μ_j, $a_j \pm i b_j$ are eigenvalues of A

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e.g. $A = \begin{bmatrix}0.95 & 0.03\\ 0.05 & 0.97\end{bmatrix}$ $\vec{x}_0 = \begin{bmatrix}0.6\\ 0.4\end{bmatrix}$

$$\lim_{n\to\infty} A^n \vec{x}_0$$

eigenvalues of A: $1, \quad 0.92$

$$1 \to \mathrm{Nul}\left(\begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix}\right) = \mathrm{Span}\left\{\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right\}$$

$$0.92 \to \mathrm{Nul}\left(\begin{bmatrix} 0.03 & 0.03 \\ 0.05 & 0.05 \end{bmatrix}\right) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

$$A = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.92 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1}$$

$$A^n = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 1^n & \\ & 0.92^n \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1}$$

$$\to \boxed{\begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}}$$

$$= \lim A^n \vec{x}$$

$$= \frac{1}{8}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$$

A: Diagonalizable $\lambda_1, \dots, \lambda_k$

$$\mathbb{C}^n = \text{Nul}(A-\lambda_1 I) \oplus \cdots \oplus \text{Nul}(A-\lambda_k I)$$

Def (generalized eigenspace)

λ is an eigenvalue of A.

$$V_\lambda^g = \{\vec{v} \in \mathbb{C}^n \mid (A-\lambda I)^k \vec{v} = \vec{0} \text{ for some } k \geq 1\}$$

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e.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ 0 only eigenvalue

$$\text{Nul}(A - 0I) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

$$(A-0I)^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow V_{\lambda=0}^g = \mathbb{C}^2$$

$(A-\lambda I)$ $T_{A-\lambda I}$

$$0 \subseteq \ker(A-\lambda I) \subseteq \ker(A-\lambda I)^2 \subseteq \ker(A-\lambda I)^3 \subseteq \cdots$$

If $(A-\lambda I)v = 0 \Rightarrow (A-\lambda I)(A-\lambda I)v = 0$

$$\subsetneq \cdots \subseteq \underbrace{\ker(A-\lambda I)^k = \ker(A-\lambda I)^{k+1} = \cdots}$$

$\quad\quad \| $

stabilized kernel of $T_{A-\lambda I}$

$\quad\quad \|$

V^g_λ

Thm A: $n\times n$ $\{\lambda_1, \dots, \lambda_k\}$ distinct eigvalues

① $\mathbb{C}^n = V^g_{\lambda_1} \oplus \cdots \oplus V^g_{\lambda_k}$,

② $\underline{\dim V^g_{\lambda_i}} = \underline{\mathrm{mult}(\lambda_i)}$

$(1 \le \underline{\dim \mathrm{Nul}(A-\lambda_i I)} \le \underline{\mathrm{mult}(\lambda_i)})$

③ T_A preserves the decomposition:

$$T_A(V^g_{\lambda_i}) \subseteq V^g_{\lambda_i}.$$

$$AP = P\begin{bmatrix} \boxed{\begin{matrix}\lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1\end{matrix}} & & & \\ & \boxed{\begin{matrix}\lambda_2 & 1 \\ & \lambda_2\end{matrix}} & & \\ & & \lambda_3 & \\ & & & \ddots \end{bmatrix}$$

first 3 columns of P

$$A\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

v_1 is an eigenvector of λ_1

$$\begin{cases} \boxed{Av_1 = \lambda_1 v_1} \\ A\underline{v_2} = \underline{v_1} + \lambda_1 v_2 \\ A\underline{v_3} = v_2 + \underline{\lambda_1 v_3} \end{cases}$$

$$(A-\lambda I)v_2 = v_1$$

$$(A-\lambda I)^2 v_2 = (A-\lambda I)v_1 = 0$$

$$(A-\lambda I) v_3 = v_2$$

$$(A-\lambda I)^3 v_3 = (A-\lambda I)^2 v_2 = 0$$

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(3) $\boxed{T_A(V_\lambda^g) \subseteq V_\lambda^g}$:

$x \in V_\lambda^g$ $\quad \boxed{(A-\lambda I)^k x = 0 \quad k \geq 1}$

want to show: $Ax \in V_\lambda^g$ i.e.

$\exists\, l \geq 1$. $(A-\lambda I)^l (Ax) = 0$

Claim: $(A-\lambda I)^k(Ax) = 0$

$\parallel$

$A(A-\lambda I)^k x = 0$

$(A-\lambda I)^k A \overset{??}{=} A(A-\lambda I)^k$

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①, ③ $\Rightarrow$ $\dim V_\lambda^g = \text{mult}(\lambda)$

$\mathbb{C}^n = V_{\lambda_1}^g \oplus \cdots \oplus V_{\lambda_k}^g$

$\boxed{T_A|_{V_\lambda^g} : V_\lambda^g \to V_\lambda^g}$

$\{v_{\lambda_1}^{(1)}, \ldots, v_{\lambda_1}^{(a_1)}\} \cdots \{v_{\lambda_k}^{(1)}, \ldots, v_{\lambda_k}^{(a_k)}\}$

$\Rightarrow \{ \quad \cdots\cdots \quad \}$ basis of $\mathbb{C}^n$

$$P = \begin{bmatrix} \downarrow & \downarrow & & \downarrow & & \downarrow & \downarrow \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} \quad \end{bmatrix}$$

$\cancel{A\,P}$

$$AP = P\begin{bmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Big\} a_1$$

$$\boxed{A v_{\lambda_1}^{(1)}} \in \operatorname{Span}\{v_{\lambda_1}^{(1)}, \cdots, v_{\lambda_1}^{(a_1)}\} = V_{\lambda_1}^{g}$$

$$\parallel$$

$$* v_{\lambda_1}^{(1)} + \cdots + * v_{\lambda_1}^{(a_1)}$$

Claim: $\boxed{T_A|_{V_\lambda^g} : V_\lambda^g \to V_\lambda^g}$ has only 1 eigenvalue λ

$$x \neq 0 \quad \exists k \geq 1, \ (A - \lambda I)^k x = 0$$

$$\boxed{Ax = \mu x} \Rightarrow \mu = \lambda$$

$$A^l x = \mu^l x$$

$$0 = (A - \lambda I)^k x$$

$$= \left(A^k + \binom{k}{1}(-\lambda) A^{k-1} + \binom{k}{2}(-\lambda)^2 A^{k-2} + \cdots + (-\lambda)^k\right) x$$

$$= \left(\mu^k + \binom{k}{1}(-\lambda)\mu^{k-1} + \cdots + (-\lambda)^k\right) x$$

$$= (\mu - \lambda)^k \vec{x}$$