HOMEWORK 12
MATH 104, SECTION 6

(1) Let X be a compact metric space, and let $\mathcal{B}(X)$ be the set of real-valued bounded functions on X. We define

$$d_{\mathcal{B}}(f,g) \coloneqq \sup_{x\in X} |f(x) - g(x)|.$$

Also, let $\mathcal{C}(X)$ be the set of real-valued continuous functions on X.

(a) Prove that $(\mathcal{B}(X), d_{\mathcal{B}})$ is a metric space.

(b) Moreover, prove that $\mathcal{B}(X)$ is a complete metric space, i.e. every Cauchy sequence in $\mathcal{B}(X)$ converges to some element in $\mathcal{B}(X)$.

(c) Prove that $\mathcal{C}(X)$ is a closed subset of $\mathcal{B}(X)$.

(d) Prove that a closed subset of a complete metric space is also complete, therefore concludes that $\mathcal{C}(X)$ is a complete metric space.

(2) Let (a_n) be a sequence of real numbers satisfying

$$0 \le a_{n+m} \le a_n + a_m \quad \text{for any} \quad n, m \in \mathbb{N}.$$

Define $b_n \coloneqq \frac{a_n}{n}$ for each n. Prove that the sequence (b_n) is convergent. (Hint: First prove that (b_n) is bounded. Let $z \coloneqq \limsup b_n$. There exists a subsequence (b_{k_n}) such that $\lim b_{k_n} = z$. For any $m \in \mathbb{N}$, you can write $k_n = \ell_n m + r_n$ where $0 \le r_n < m$. Then try to show that $z \le b_m$ by taking $n \to \infty$ for certain inequality obtained from the assumption.)

(3) Let S be the set of nonempty compact subsets of $\mathbb{R}^2$. For any $r > 0$ and $K \in S$, we define the *r-neighborhood* of K to be

$$B_r(K) \coloneqq \{x \in \mathbb{R}^2 \colon d(x,a) < r \text{ for some } a \in K\} = \bigcup_{a\in K} B_r(a).$$

For $K_1, K_2 \in S$, we define

$$d(K_1, K_2) \coloneqq \inf\{r > 0 \colon K_1 \subset B_r(K_2) \text{ and } K_2 \subset B_r(K_1)\}.$$

(a) Prove that (S, d) is a metric space, i.e. d is a distance function on S.

(b) Let F be the set of finite subsets of $\mathbb{R}^2$. Prove that F is dense in S.

(4) Let $f \colon [0,1] \to \mathbb{R}$ be an increasing function.

(a) Prove that for any $a \in (0,1)$, the *left hand limit* $\lim_{x\to a^-} f(x)$ and the *right hand limit* $\lim_{x\to a^+} f(x)$ of f at a both exists. (Recall Ross, §20 for the definition.)

(b) Define $A \coloneqq \{x \in [0,1] \colon f \text{ is not continuous at } x\}$. Prove that the set A is either finite or countable. (Hint: Define an injection from A to $\mathbb{Q}$ using (a).)

(5) An *open cube* in $\mathbb{R}^n$ is a product of open intervals

$$U = (a_1, b_1) \times \cdots \times (a_n, b_n).$$

Its *volume* is defined to be

$$\mathrm{vol}(U) = (b_1 - a_1) \cdots (b_n - a_n).$$

We say a subset $E \subset \mathbb{R}^n$ has *measure zero* if for any $\epsilon > 0$, there exists finite or countably many open cubes $U_1, U_2, \ldots$ such that

$$E \subset \bigcup_i U_i \quad \text{and} \quad \sum_i \mathrm{vol}(U_i) < \epsilon.$$

Let $f \colon [a,b] \to \mathbb{R}$ be a continuous map. Prove that the graph

$$\Gamma_f = \{(x, f(x)) \in \mathbb{R}^2 \colon x \in [a,b]\}$$

has measure zero in $\mathbb{R}^2$.

(6) Let $X = (\mathbb{R}^n, d_{\mathrm{std}})$ be the Euclidean space with the standard distance function

$$d_{\mathrm{std}}(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Prove that any linear map $T \colon X \to X$ is continuous.

(7) (a) Find the domain $E \subset \mathbb{R}$ of pointwise convergence of the series

$$\sum_{n=1}^{\infty} e^{-nx} \cos(nx),$$

i.e. find all possible $x \in \mathbb{R}$ such that the above series converges.

(b) Prove or disprove: the series converges uniformly on E.

(8) Let $a_1, a_2, \cdots, a_n$ be real numbers. Suppose that

$$|a_1 \sin x + a_2 \sin(2x) + \cdots + a_n \sin(nx)| \leq |\sin x| \text{ for any } x \in \mathbb{R}.$$

Prove that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$. (Hint: Let $f(x) = a_1 \sin x + a_2 \sin(2x) + \cdots + a_n \sin(nx)$ and consider $f'(0)$.)

(9) Suppose that the derivative and the second derivative of a function $f \colon (a,b) \to \mathbb{R}$ both exist on (a,b). Moreover, suppose that there exists $M > 0$ such that $|f''(x)| < M$ for any $x \in (a,b)$. Prove that f is uniformly continuous on (a,b).

(10) Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ satisfies $f(f(x)) = -x$ for any $x \in \mathbb{R}$. Prove that f is not a continuous function on $\mathbb{R}$.