Math H54 Final Exam December 17, 2019

Name: ______________________________

- You have 170 minutes to complete the exam (3:10pm – 6:00pm).
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using. Exception: if you are asked to prove a theorem, you are not allowed to use it!
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total		

1. (2 points each) True/False questions. You don't need to justify your answers. No partial credit.

(a) Let A, B be two real 3×3 matrices. If $AB = 0$ and $\det(A) = 3$, then $B = 0$.

(b) There exists a real 2×2 matrix $A \neq I$ such that $A^5 = I$.

Hint: What's the geometric interpretation of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$?

(c) Let $T : V \to V$ be a linear transformation, and v_1, v_2, v_3 be three vectors in V. If $\{T(v_1), T(v_2), T(v_3)\}$ is a linearly dependent set, then $\{v_1, v_2, v_3\}$ is a linearly dependent set.

(d) If a real 2×2 matrix A satisfies $\langle \vec{v}, A\vec{v} \rangle = 0$ for any $v \in \mathbb{R}^2$. Then $A = 0$.

(e) Let A be a real $n \times n$ matrix. Let $\{\vec{x}_1(t), \ldots, \vec{x}_n(t)\}$ be a set of solutions of the homogeneous equation $\vec{x}\,'(t) = A\vec{x}(t)$. Suppose that $\{\vec{x}_1(1), \ldots, \vec{x}_n(1)\}$ is a linearly dependent set of vectors. Then $\{\vec{x}_1(t), \ldots, \vec{x}_n(t)\}$ is a linearly dependent set for any $t \in \mathbb{R}$.

2. (10 points; 3 parts) Let P_2 be the vector space consists of polynomials of degree less than or equal to 2. Define the linear transformation $T : P_2 \to \mathbb{R}^3$ by

$$T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$$

for $p \in P_2$.

(a) Show that T is a linear transformation.

(b) Is T injective? Surjective? Prove your answer.

(c) Find a polynomial p such that $T(p) = \begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix}$.

Hint: You should be solving a linear system with 3 equations and 3 unknowns.

3. Let A be a real symmetric $n \times n$ matrix. Suppose that v and w are eigenvectors of A that correspond to distinct eigenvalues. Show that v and w are orthogonal (with respect to the standard inner product on $\mathbb{R}^n$).

   Hint: Consider $\langle Av, w \rangle$.

4. (10 points) Let V be a finite-dimensional inner product space, and $W \subset V$ be a subspace. Let

$$W^{\perp} = \{v \in V : \langle v, w \rangle = 0 \text{ for any } w \in W\}$$

be the orthogonal complement of W. Prove that

$$\dim W + \dim W^{\perp} = \dim V.$$

Hint: You can prove it by showing that if $\{a_1, \ldots, a_n\}$ is a basis of W and $\{b_1, \ldots, b_m\}$ is a basis of $W^{\perp}$, then $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ is a basis of V.

Hint: You can also prove it via the rank-nullity theorem. (You don't have to prove the rank-nullity theorem.)

5. Let U be the set of 5×5 matrices with the property that the sum of each row is zero and the sum of each column is zero.

   (a) (3 points) Explain why U forms a vector space.

   (b) (7 points) Find the dimension of U and prove it.

   Hint: You may use the rank-nullity theorem, i.e. if $T : V \to W$ is a linear transformation, then $\dim V = \dim \text{Kernel}(T) + \dim \text{Image}(T)$.

   Hint: 15 is not the correct answer.

6. A square matrix A is called *unipotent* if $(A - I)^n = 0$ for some $n \in \mathbb{N}$.

   (a) (5 points) Prove that if λ is an eigenvalue of an unipotent matrix, then $\lambda = 1$.
   Hint: Suppose v is an eigenvector with eigenvalue λ. What's $(A - I)v$?

   (b) (5 points) Prove that if A is both unipotent and diagonalizable, then $A = I$.

7. (a) (5 points) Let A be a real $n \times n$ matrix, and let $\{\lambda_1, \ldots, \lambda_k\}$ be the set of distinct eigenvalues of A^2. Assume that λ_i is real and positive for any $1 \leq i \leq k$. Let S be the set of eigenvalues of A. Prove that:

   (i) S is a subset of $\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, \sqrt{\lambda_2}, -\sqrt{\lambda_2}, \ldots, \sqrt{\lambda_k}, -\sqrt{\lambda_k}\}$;

   (ii) For each $1 \leq i \leq k$, S contains at least one of $\pm\sqrt{\lambda_i}$.

   Hint: Consider $(A^2 - \lambda I) = (A - \sqrt{\lambda} I)(A + \sqrt{\lambda} I)$ for Part (ii).

(b) (5 points) Find a 3×3 matrix A such that

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 9 \end{bmatrix}$$

How many such matrices are there?

Hint: Show that A is diagonalizable and has the same eigenspaces as A^2.

(It's good enough to express A as something like BC^{-1} for two <u>explicit</u> 3×3 matrices B and C : you don't have to compute C^{-1} if it looks too complicated.)

8. (10 points) Find a function $y(t)$ satisfying

$$y''(t) - 2y'(t) + 2y(t) = 2t - 2; \quad y(0) = 4, \quad y'(0) = 2.$$

9. In this problem, you will find three functions $y_1(t), y_2(t), y_3(t)$ satisfying

$$y_1'(t) = 4y_2(t), \quad y_2'(t) = 4y_1(t) + 3y_3(t), \quad y_3'(t) = 3y_2(t)$$

and the initial conditions

$$y_1(0) = 0, \quad y_2(0) = -10, \quad y_3(0) = 0.$$

(a) (2 points) Write the differential equations as a matrix equation $\vec{y}\,'(t) = A\vec{y}(t)$ for some matrix A, and write the initial conditions as a vector equation $\vec{y}(0) = \vec{b}$ for some vector $\vec{b}$.

(b) (3 points) Find a diagonalization of A.

(c) (3 points) Find a fundamental matrix of $\vec{y}\,'(t) = A\vec{y}(t)$.

(d) (2 points) Finish the problem, i.e. find the functions $y_1(t), y_2(t), y_3(t)$.

10. (10 points; 2 parts)

   (a) Find the Fourier sine series of the function $f(x) = x(x - \pi)$ on $0 < x < \pi$.
   Hint: Integration by parts, with patience. (It's not too bad.)

(b) Find a function $u(x,t)$ satisfying

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$
$$u(0,t) = u(\pi,t) = 0, \quad t > 0,$$
$$u(x,0) = x(x-\pi), \quad 0 < x < \pi.$$