SECOND MIDTERM PRACTICE PROBLEMS
MATH 104, SECTION 2

(1) Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and $s_k = a_1 + \cdots + a_k$ be the k-th partial sum.
  (a) Suppose that $\lim a_n = 0$, and there exists a $m \in \mathbb{N}$ such that the sequence $(s_{mk})_{k=1}^{\infty} = (s_m, s_{2m}, s_{3m}, \ldots)$ converges. Prove that $\sum a_n$ converges.
  (b) Find an example where $(s_{2k})_{k=1}^{\infty}$ converges and (a_n) doesn't converge to 0.
  (c) Find an example where $\lim a_n = 0$, and there is a subsequence (s_{k_n}) of (s_n) that converges, but $\sum a_n$ diverges.

(2) (a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = 0$ for all $x \in \mathbb{Q}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.
  (b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then f is linear, i.e. there exists c so that $f(x) = cx$ for all x.

(3) Let $X = (\mathbb{R}^n, d_{\text{std}})$ be the Euclidean space with the standard distance function

$$d_{\text{std}}(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Prove that any linear map $T\colon X \to X$ is continuous.

(4) Let S be the set of nonempty compact subsets of $\mathbb{R}^2$. For any $r > 0$ and $K \in S$, we define the r-*neighborhood* of K to be

$$B_r(K) \coloneqq \{x \in \mathbb{R}^2 \colon d(x, a) < r \text{ for some } a \in K\} = \bigcup_{a \in K} B_r(a).$$

For $K_1, K_2 \in S$, we define

$$d(K_1, K_2) \coloneqq \inf\{r > 0 \colon K_1 \subset B_r(K_2) \text{ and } K_2 \subset B_r(K_1)\}.$$

  (a) Prove that (S, d) is a metric space, i.e. d is a distance function on S.
  (b) Let F be the set of finite subsets of $\mathbb{R}^2$. Prove that F is dense in S.

(5) Let (X, d) be a metric space and $E \subset X$ be a nonempty subset. Define a function $f\colon X \to [0, \infty)$ by:

$$f(x) \coloneqq \inf\{d(x, y) \colon y \in E\}.$$

Prove that f is uniformly continuous on X.

(6) Let (p_n) be a sequence of polynomials defined over real numbers, and let $f\colon \mathbb{R} \to \mathbb{R}$ be a real-valued function. Suppose that (p_n) converges uniformly to f on $\mathbb{R}$. Prove that f is also a polynomial.