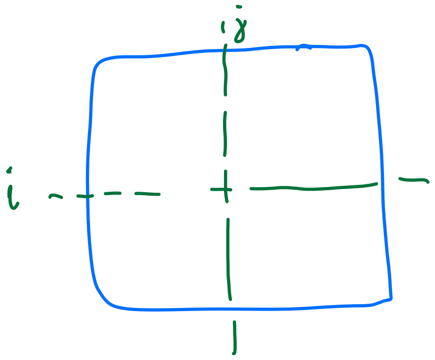
Today:

- Definition of determinant.
- $\det(A) \neq 0 \Leftrightarrow A$ is invertible
- $\det(AB) = \det(A)\det(B)$

Def: Let A: $n \times n$. Denote A_{ij} the $(n-1)\times(n-1)$-matrix w/ the i-th row & j-th column of A removed.



e.g.

$$A = \begin{array}{c} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \end{array}, \quad A_{21} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$$

Def: The determinants of square matrices are defined inductively as follows:
- 1×1 matrix: $\det([a_{11}]) = a_{11}$
- Suppose we've defined $\det(-)$ for all $(n-1)\times(n-1)$ matrices.

Now, suppose A: $n \times n$,

- Define the (i,j)-cofactor of A to be: $C_{ij} := (-1)^{i+j}\det(\underbrace{A_{ij}}_{(n-1)\times(n-1)})$
- Pick any row $a_{i1}, a_{i2}, \dots, a_{in}$

  or

  any column $a_{1j}, a_{2j}, \dots, a_{nj}$ of A.

Define

$$\det(A) := a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

or

$$\det(A) := a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}.$$

(cofactor expansion of the ith row or the jth column)

Thm det is well-defined, i.e. it's independent of the choice of column/row which we do the cofactor expansion.

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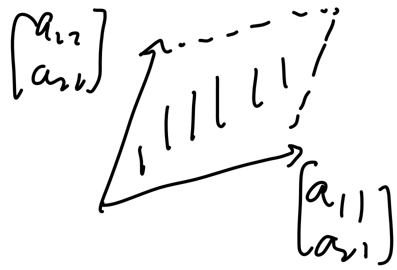
e.g. 2×2

$$\begin{bmatrix} a_{11} & a_{12} \\ \boxed{a_{21} \quad a_{22}} \end{bmatrix}$$

do cofactor exp. here.



$$\det = a_{21} \cdot \underbrace{C_{21}}_{= (-1)^{2+1} \det [a_{12}]} + a_{22} \cdot \underbrace{C_{22}}_{= (-1)^{2+2} \det [a_{11}]}$$

$$= -a_{21} a_{12} + a_{22} a_{11}$$

e.g. 3×3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

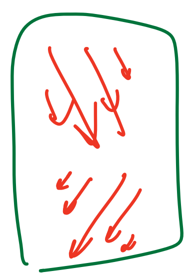
$$\det = a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31}$$

$$C_{11} = (-1)^{1+1} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = a_{22} a_{33} - a_{23} a_{32}$$

$$C_{21} = (-1)^{2+1} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} = -a_{12} a_{33} + a_{13} a_{32}$$

$$C_{31} = (-1)^{3+1} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} = a_{12} a_{23} - a_{13} a_{22}$$

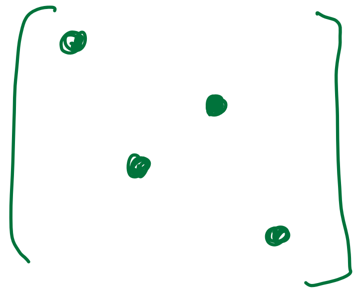
$$= a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{31} a_{12} a_{23}$$
$$- a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31}.$$



Rmk: not true for size > 3

Rmk: In general,

$$\det(A) = \sum_{\substack{\sigma:\{1,\dots,n\}\to\{1,\dots,n\}\\ \text{bijective.}}} (-1)^{\operatorname{sign}(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$



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e.g.

$$\det \begin{bmatrix} \boxed{a_{11} & & & 0} \\ * & a_{22} & & \\ * & * & \ddots & \\ * & * & * & a_{nn} \end{bmatrix} = a_{11} a_{22} \cdots a_{nn}$$

$$= a_{11} C_{11} + \cancel{a_{12}}^{\,0} C_{12} + \cdots + \cancel{a_{1n}}^{\,0} C_{1n}$$

$$C_{11} = (-1)^{1+1} \det \begin{bmatrix} a_{22} & & 0 \\ * & \ddots & \\ * & * & a_{nn} \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & & 0 \\ * & \ddots & \\ * & * & a_{nn} \end{bmatrix}$$

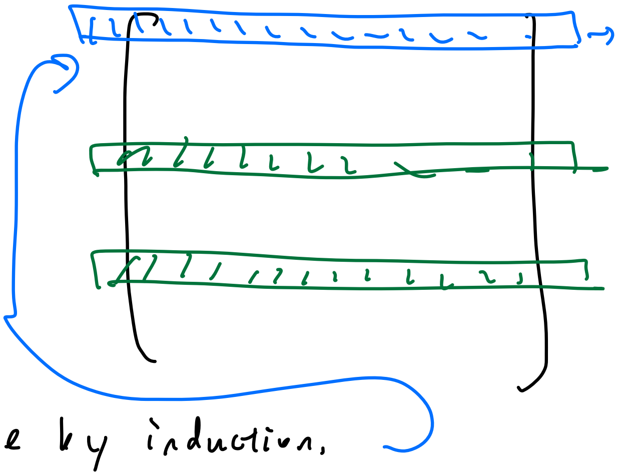
e.g.

$$\det \begin{bmatrix} \boxed{1 & & &} \\ & \ddots & & \\ & 1 & & \\ & & 0 & & 1 \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \\ & & 1 & & & & 0 \\ & & & & & & & 1 \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix} \to 1 \cdot C_{11} \qquad = -1$$

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$$

Lemma: If A has 2 identical rows or columns, then $\det(A) = 0$

pf:



$$a_{11}\underbrace{C_{11}}_{(-1)^{1+1}\det\left[\;\right]_{n-1\times n-1}\,=\,0} + a_{12}\underbrace{C_{12}}_{=0} + \cdots + a_{1n}\underbrace{C_{1n}}_{=0} = 0$$

$\square$

Prove by induction.

Statement is true for $n=2$: $\det\begin{bmatrix} a & b \\ a & b \end{bmatrix} = ab - ab = 0$

Lemma: det is linear in each row and column: i.e.

$$\det\begin{bmatrix} a_{11}+a_{11}' & a_{12}+a_{12}' & \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \\ a_{n1} & a_{n2} & \end{bmatrix} = \det\begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & \ddots & \\ \vdots & & \ddots \end{bmatrix} + \det\begin{bmatrix} a_{11}' & a_{12}' & \cdots \\ a_{21} & a_{22} & \\ & \vdots & \end{bmatrix}$$

$$\det\begin{bmatrix} ca_{11} & ca_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ & \vdots & \end{bmatrix} = c\det\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ & \vdots & \end{bmatrix}$$

Rmk:
$$\det\begin{bmatrix} a_{11}+a_{11}' & a_{12}+a_{12}' & \cdots \\ a_{21}+a_{21}' & a_{22}+a_{22}' & \cdots \\ a_{31} & a_{32} & \\ \vdots & \vdots & \end{bmatrix} \neq \det\begin{bmatrix} a_{11} & a_{12} & \\ & \vdots & \\ & \vdots & \end{bmatrix} + \det\begin{bmatrix} a_{11}' & a_{12}' & \cdots \\ a_{21}' & a_{22}' & \cdots \\ a_{31} & a_{32} & \\ & & \ddots \end{bmatrix}$$

pf:
$$\det = (a_{11}+a_{11}')\boxed{C_{11}} + (a_{12}+a_{12}')\boxed{C_{12}} + \cdots$$

$$C_{11} = (-1)^{1+1}\det\begin{bmatrix} a_{22} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$$

the same for these 3 matrices

$\square$

Prop: $\forall$ elementary matrix E, we have $\det(EA) = \det(E)\det(A)$

pf:

- $$\underbrace{\begin{bmatrix} 1 & c & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}}_{\det = 1} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots \\ & & \\ & & \end{bmatrix}}_{A} = \begin{bmatrix} a_{11}+ca_{21} & a_{12}+ca_{22} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \ddots \end{bmatrix}$$

$\det \begin{bmatrix} a_{11}+ca_{21} & a_{12}+ca_{22} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \ddots \end{bmatrix}$

$\parallel$ (linearity of det in each row/column)

$$\det \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ & & \ddots \end{bmatrix}}_{A} + c \underbrace{\det \begin{bmatrix} a_{21} & a_{22} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \ddots \end{bmatrix}}_{= 0}$$

- $$\underbrace{\begin{bmatrix} c & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}}_{\det = c} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots \\ & \vdots & \\ & & \ddots \end{bmatrix}}_{A} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \ddots \end{bmatrix}$$

$\det(\ \ ) = c\det(A)$

- $$\underbrace{\begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}_{\det = -1.} \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ a_{31} & a_{32} & \cdots \\ \vdots & & \ddots \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & \cdots \\ a_{11} & a_{12} & \cdots \\ a_{31} & a_{32} & \cdots \\ \vdots & & \end{bmatrix}$$

Prove by induction:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = bc - ad$$

$$a_{31}(-1)^{3+1} \det \begin{bmatrix} a_{12} & a_{13} & \cdots \\ a_{22} & a_{23} & \cdots \\ a_{42} & a_{43} & \cdots \\ \vdots & & \end{bmatrix} + \cdots$$

$$a_{31}(-1)^{3+1}\det\begin{bmatrix} a_{22} & a_{23} & \cdots \\ a_{12} & a_{13} & \cdots \\ a_{42} & a_{43} & \cdots \\ \vdots & & \end{bmatrix}$$

$\square$

Thm A is invertible $\iff$ $\det(A)\neq 0$

pf: ($\Rightarrow$) Since A invertible, $\exists$ elementary matrices $E_1,\dots,E_k$

s.t. $E_1E_2\cdots E_kA = I$.

$$\begin{aligned} 1 = \det(I) &= \det(E_1\cdots E_kA) \\ &= \det(E_1)\det(E_2\cdots E_kA) \\ &= \det(E_1)\det(E_2)\det(E_3\cdots E_kA) \\ &\ \ \vdots \\ &= \underbrace{\det(E_1)\cdots\det(E_k)}_{\neq 0}\cdot\det(A) \end{aligned}$$

by previous prop.

$\Rightarrow \det(A)\neq 0$

($\Leftarrow$) "$\det(A)\neq 0 \Rightarrow A$ invertible"

Suppose A is NOT invertible, (Goal: $\det(A)=0$)

then there exists some column/row has no pivot.

$$E_1\cdots E_kA = \begin{bmatrix} 1 & 0 & * & 0 & 0 & * \\ & 1 & * & 0 & 0 & * \\ & & & 1 & 0 & * \\ & & & & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\leftarrow$ reduced echelon form of A.

$\det = 0$

$$0 = \det(E_1 \cdots E_k A) = \underbrace{\det(E_1) \cdots \det(E_k)}_{\neq 0} \det(A)$$

$$\Rightarrow \det(A) = 0. \quad \square$$

Thm $\det(AB) = \det(A)\det(B)$

pf: <u>Case 1:</u> If $\det(A) \neq 0$.

Then A is invertible, so $\exists\, E_1, \ldots, E_k$ elementary matrices

st. $E_1 E_2 \cdots E_k A = I$

so, $A = \underbrace{E_k^{-1}}\, \underbrace{E_{k-1}^{-1}} \cdots \underbrace{E_1^{-1}}$

inverse of any elementary matrix is still an elementary matrix

$$\det(AB) = \det(E_k^{-1} \cdots E_1^{-1} B)$$
$$= \det(E_k^{-1}) \det(E_{k-1}^{-1} \cdots E_1^{-1} B)$$
$$\vdots$$
previous prop.
$$= \det(E_k^{-1}) \cdots \det(E_1^{-1}) \det(B)$$
$$\vdots$$
$$= \det(E_k^{-1} \cdots E_1^{-1}) \det(B)$$
$$= \det(A)\det(B).$$

<u>Case 2:</u> $\det(A) = 0 \overset{??}{\Longrightarrow} \det(AB) = 0.$

i.e. If A is not invertible, then we need to show that AB is not invertible.

(If AB is invertible, then $\exists\, C$ st. $(AB)C = I$,

$\Rightarrow \quad A(BC) = I \quad \Rightarrow \quad A$ is invertible)

proved last time.

$\square$