SECOND MIDTERM PRACTICE
MATH H54

The difficulty of the second midterm will be similar to the first midterm. The problems below provide more practices for the proof-based problems (some of them are quite hard).

Some basic computations that you should be familiar with:

- Characteristic polynomials, eigenvalues, eigenspaces, diagonalization.
- Orthogonal projections, Gram–Schmidt process, QR decompositions.
- Orthogonal diagonalization of symmetric matrices.
- Relate symmetric matrices with quadratic forms.

(1) Let A be a 3×3 matrix satisfying $\operatorname{tr}(A) = 1$, $\operatorname{tr}(A^2) = 5$, and $\operatorname{tr}(A^3) = 7$. Prove that A is not invertible.

(2) Prove that the unique real 3×3 matrix A such that $A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(3) Let $\vec{x}, \vec{y}$ be vectors in an inner product space with lengths $||\vec{x}|| = 1$ and $||\vec{y}|| = 2$. What are the maximum and minimum of $||\vec{x} + \vec{y}||$? When do they occur?

(4) Let A be a real $n \times n$ symmetric matrix. Prove that there exists an eigenvalue λ of A such that $\langle A\vec{v}, \vec{v}\rangle \leq \lambda ||\vec{v}||^2$ for any $\vec{v} \in \mathbb{R}^n$.

(5) (a) Let A be a positive definite matrix. Prove that there exists an upper triangular matrix R with positive entries on its diagonal, such that $A = R^T R$. (Hint: HW8 Problem 4 and QR decomposition.)

(b) Let A be a positive definite matrix with diagonal elements $a_{11}, \dots, a_{nn}$. Prove that

$$\det(A) \leq \prod_{i=1}^{n} a_{ii} := a_{11} \cdots a_{nn}.$$

(6) Let A be a real symmetric matrix with $A^k = \mathbb{I}$ for some $k \geq 1$. Prove that $A^2 = \mathbb{I}$.

(7) Let A, B be two $n \times n$ matrices. Suppose that $AB - BA = A$. Prove that $A^k = 0$ for some $k \geq 1$.