#1.

(a) If $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is l.d., i.e. $\exists\, c_1, \cdots, c_n \in \mathbb{R}$ not all 0, s.t.

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}.$$

Then $\vec{0} = T(\vec{0}) = T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n)$.

Hence $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is l.d. $\square$

(b) If $c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n) = \vec{0}$ for some $c_1, \cdots, c_n \in \mathbb{R}$,

then $\vec{0} = T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)$.

Since T is inj., we have $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$.

Hence $c_1 = \cdots = c_n = 0$. $\square$

#2.

Assume the contrary that Poly is a finite dim$^{\text{l}}$ v.s., then

$\exists\, \{p_1, \cdots, p_n\} \subseteq$ Poly gives a basis.

Choose $N > 0$ large enough s.t. $N > \max\{\deg p_1, \cdots, \deg p_n\}$.

Then it's not hard to see that $x^N \notin \operatorname{Span}\{p_1, \cdots, p_n\}$. $\square$

#3.

(a) $\operatorname{Col}(AB) \subseteq \operatorname{Col}(A) \Rightarrow \operatorname{rank}(AB) \leq \operatorname{rank}(A)$. $\square$

(b) $B\vec{x} = \vec{0} \Rightarrow AB\vec{x} = \vec{0}$, hence $\operatorname{Nul}(B) \subseteq \operatorname{Nul}(AB)$.

Therefore $\dim \operatorname{Nul}(B) \leq \dim \operatorname{Nul}(AB)$. $\square$

(c) $\operatorname{rank}(AB) = p - \dim \operatorname{Nul}(AB) \leq p - \dim \operatorname{Nul}(B) = \operatorname{rank}(B)$. $\square$

(d) If A invertible, then

$$\operatorname{rank}(B) \geq \operatorname{rank}(AB) \geq \operatorname{rank}(A^{-1}AB) = \operatorname{rank}(B).$$

Hence $\operatorname{rank}(B) = \operatorname{rank}(AB)$. The second statement can be proved similarly. $\square$

#4:

Let $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq H$ be a basis of H.

Then $T(H) = \text{Span}\{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$.

By the spanning set thm, there is a subset of $\{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ that gives a basis of $T(H)$. Hence

$$\dim T(H) \le n = \dim H. \quad \square$$

#5.

$\vec{x} = 3(-1+t) + (1-2t) = -2+t$.

Hence $[\vec{x}]_{\mathcal{C}}$ satisfies: $\begin{bmatrix} 13 & 5 \\ -5 & -2 \end{bmatrix} [\vec{x}]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

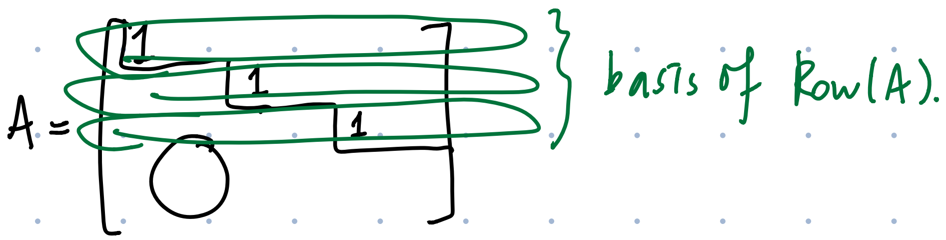
One can get: $[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. $\square$

#6.

$\text{rank}(A) = \dim \text{Col}(A) = \#$ pivots of A.

$\text{rank}(A^T) = \dim \text{Row}(A)$, where $\text{Row}(A) = \text{Span}\{\text{rows of } A\}$.

Note that row operations don't change the row space. Hence we can assume A is of reduced echelon form.



It's clear that $\dim \text{Row}(A) = \#$ pivots of A. $\square$

#7:

Claim: $\mathrm{Null}(A^TA) = \mathrm{Null}(A)$.

pf. By #3(b), we have $\mathrm{Nul}(A^TA) \supseteq \mathrm{Nul}(A)$.

It suffices to show $\mathrm{Nul}(A^TA) \subseteq \mathrm{Nul}(A)$.

$\vec{v} \in \mathbb{R}^n$, If $A^TA\vec{v} = \vec{0}$, then $\vec{0} = \vec{v}^TA^TA\vec{v} = (A\vec{v})^TA\vec{v}$.

Write $\vec{w} = A\vec{v}$, then $\vec{0} = \vec{w}^T\vec{w}$.

Write $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$.

Then $\vec{w}^T\vec{w} = [w_1 \cdots w_n] \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = w_1^2 + \cdots + w_n^2$.

Hence $0 = \vec{w}^T\vec{w} \Rightarrow w_1 = \cdots = w_n = 0$, i.e. $A\vec{v} = \vec{0}$. $\square$

By rank-nullity thm, it follows that

$$\mathrm{rank}(A^TA) = \mathrm{rank}(A). \ \square$$

#8:

Follow the hint. It suffices to show

$S := \{x_1, \cdots, x_n, y_1, \cdots, y_m, z_1, \cdots, z_k\}$ is a basis of $H_1 + H_2$.

① it spans $H_1 + H_2$:

$\forall v \in H_1 + H_2$, $\exists \underbrace{v_1 \in H_1}, \underbrace{v_2 \in H_2}$ s.t. $v = v_1 + v_2$.

$\Downarrow$ $\Downarrow$

$\exists a_1, \cdots, a_n, b_1, \cdots, b_m$ s.t.

$$v_1 = a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_my_m$$

$\exists c_1, \cdots, c_n, d_1, \cdots, d_k$ s.t.

$$v_2 = c_1x_1 + \cdots + c_nx_n + d_1z_1 + \cdots + d_kx_k.$$

Hence $v = (a_1+c_1)x_1 + \cdots + (a_n+c_n)x_n$
$+ b_1y_1 + \cdots + b_my_m$
$+ d_1z_1 + \cdots + d_kz_k$
$\in \mathrm{Span}\{S\}$. $\square$

② <u>S is a l.i. set.</u>

If $a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_my_m + c_1z_1 + \cdots + c_kz_k = 0$.
Consider

$$v := a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_my_m$$
$$= -c_1z_1 - \cdots - c_kz_k.$$

Then $v = a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_my_m \in H_1$

and $v = -c_1z_1 - \cdots - c_kz_k. \in H_2$.

Hence $v \in H_1 \cap H_2 = \mathrm{Span}\{x_1, \ldots, x_n\}$.

$\Rightarrow \exists\, d_1, \ldots, d_n$ s.t. $v = d_1x_1 + \cdots + d_nx_n$.

$\Rightarrow d_1x_1 + \cdots + d_nx_n = -c_1z_1 - \cdots - c_kz_k$.

Since $\{x_1, \ldots, x_n, z_1, \ldots, z_k\}$ is l.i.
hence $d_1 = \cdots = d_n = c_1 = \cdots = c_k = 0$.

$\Rightarrow v = 0$

$\Rightarrow a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_my_m = 0$

$\Rightarrow a_1 = \cdots = a_n = b_1 = \cdots = b_m = 0$. $\square$

#9.

Consider $T_A|_{Col(B)} : \underset{\text{}}{\overset{\operatorname{Im} T_B.}{\overset{\parallel}{Col(B)}}} \longrightarrow \mathbb{R}^m$.

$$\begin{aligned}
\Rightarrow \operatorname{rank}(B) &= \dim Col(B) \\
&= \dim \ker T_A|_{Col(B)} + \dim \operatorname{Im} T_A|_{Col(B)} \\
&= \dim \ker T_A|_{Col(B)} + \dim \operatorname{Im} T_{AB}. \\
&= \dim \ker T_A|_{Col(B)} + \operatorname{rank}(AB) \\
&\leq \operatorname{rank}(AB) + \dim \ker T_A \\
&= \operatorname{rank}(AB) + n - \operatorname{rank}(A). \quad \square
\end{aligned}$$