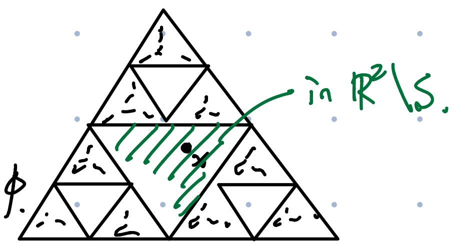
(1) Explain why the Sierpiński triangle (call it S) in $\mathbb{R}^2$ is compact. (You may want to read more about the construction of the Sierpiński triangle on Wikipedia.)

- By Heine-Borel thm, it suffices to show that S is closed & bounded.
- It's clear that S is bounded.
- $\forall x \in \mathbb{R}^2 \setminus S$, it locally looks like:



It's clear that $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap S = \phi$.

Hence S^c is open, so S is closed. $\square$

(2) Prove that any polynomial function of odd degree has at least one real root.

Let $f(x) = x^{2n+1} + a_{2n}x^{2n} + \cdots + a_0$.

We discussed in class that f is conti. on $\mathbb{R}$.

- Choose any $R > \max\{|a_{2n}|, |a_{2n-1}|, \cdots, |a_0|, 1\} \cdot 4n$.
- Then
$$f(R) = R^{2n+1} + a_{2n}R^{2n} + \cdots + a_0$$
$$> R^{2n+1} - \left(\frac{R}{4n}R^{2n} + \frac{R}{4n}\cdot R^{2n-1} + \cdots + \frac{R}{4n}\right)$$
$$> R^{2n+1} - \frac{2n+1}{4n}R^{2n+1} > 0.$$

- Similarly, $f(-R) < -R^{2n+1} + \frac{2n+1}{4n}R^{2n+1} < 0$.
- By Intermediate Value Thm, f has at least one real root between $-R$ and R. $\square$

(3) (a) Let $f\colon (X, d_X) \to (Y, d_Y)$ be a uniformly continuous function (on the whole domain X). Suppose that (x_n) is a Cauchy sequence in X. Prove that $(f(x_n))$ is a Cauchy sequence in Y. (See Ross, Definition 13.2 for the definition of Cauchy sequences in metric spaces.)

(b) Find an example of a continuous function $f\colon (X, d_X) \to (Y, d_Y)$ and a Cauchy sequence (x_n) in X such that $(f(x_n))$ is not Cauchy in Y.

(a) • Since f is unif. conti., $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.
If $z_1, z_2 \in X$ and $d_X(z_1, z_2) < \delta$, then $d_Y(f(z_1), f(z_2)) < \varepsilon$.

• Since $(x_n) \subseteq X$ Cauchy, $\exists N > 0$ s.t.
If $n, m > N$ then $d_X(x_n, x_m) < \delta$.
$\Rightarrow d_Y(f(x_n), f(x_m)) < \varepsilon$.

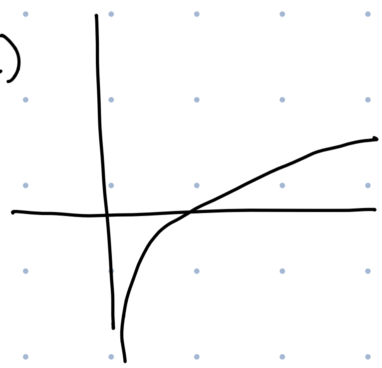
Hence $(f(x_n))$ is Cauchy in Y. $\square$

(b). $f\colon (0,1) \longrightarrow \mathbb{R}$ conti.
$x \longmapsto 1/x$

$(x_n = \frac{1}{n}) \subseteq (0,1)$ Cauchy.
But $(f(x_n) = n)$ not Cauchy. $\square$

(4) Determine whether the following functions are uniformly continuous, and give proofs:
(a) $A(x) = \log x$ on $(0,1)$.
(b) $B(x) = \frac{1}{x^2+1}$ on $\mathbb{R}$.
(c) $C(x) = \sin(\frac{1}{x})$ on $(0, \infty)$.
(Hint: Problem 3(a) could be useful for proving non-uniform continuity.)

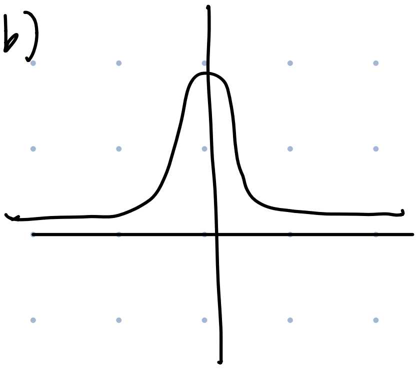
(a)



No.

$(x_n = e^{-n}) \subseteq (0,1)$ Cauchy.
But $(A(x_n) = -n)$ not Cauchy.
#3(a) $\Rightarrow$ NOT unif. conti.

(b)



Yes

Claim: $|B(x)-B(y)| \le |x-y| \quad \forall x,y \in \mathbb{R}.$

$$\Updownarrow$$

$$\left|\frac{1}{x^2+1}-\frac{1}{y^2+1}\right| \le |x-y|$$

$$\frac{|x+y||x-y|}{(x^2+1)(y^2+1)}$$

$$\Updownarrow$$

$$|x+y| \le (x^2+1)(y^2+1)$$

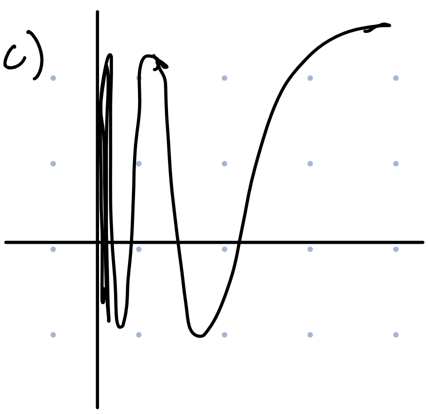
$$\Updownarrow$$

$$x^2+y^2+2xy = (x+y)^2 \le (x^2+1)^2(y^2+1)^2 = (x^4+2x^2+1)(y^4+2y^2+1)$$

Since $x^2+y^2+2xy \le 2(x^2+y^2) \le (x^4+2x^2+1)(y^4+2y^2+1)$,

the claim follows. $\square$

(c)



No.

$\left(x_n=\frac{2}{(2n-1)\pi}\right) \subseteq [0,\infty)$ Cauchy.

But $\left(C(x_n)=\sin\left(\frac{(2n-1)\pi}{2}\right)\right.$

$$=\left.\begin{cases} 1, & n\text{: odd} \\ -1, & n\text{: even}\end{cases}\right)$$ not Cauchy.

$\square$

(5) Consider the function $f\colon [0,\infty)\to\mathbb{R}$ defined by $f(x)=\sqrt{x}$.

(a) Prove that f is not Lipschitz continuous on $[0,\infty)$, i.e. there does not exist $K>0$ such that

$$|f(x)-f(y)| \le K|x-y| \text{ holds for any } x,y\ge 0$$

(b) Prove that f is uniformly continuous on $[0,\infty)$.

(a) Suppose such $K>0$ exists, then $\forall n$, we have:

$$\left|\frac{1}{n}-\frac{1}{n+1}\right| \leq K\left|\frac{1}{n^2}-\frac{1}{(n+1)^2}\right|$$

$$\Rightarrow \quad \frac{1}{n(n+1)} \leq K \cdot \frac{2n+1}{n^2(n+1)^2} \quad \forall n \in \mathbb{N}.$$

$$\Rightarrow \quad \frac{n(n+1)}{2n+1} \leq K \quad \forall n \in \mathbb{N}.$$ which is not possible. $\square$

(b) $\forall \varepsilon>0$, Let $\delta=\varepsilon^2>0$, then $\forall\ |x-y|<\delta=\varepsilon^2$,
we have:

$$|\sqrt{x}-\sqrt{y}| \leq \sqrt{|x-y|} < \sqrt{\delta} = \varepsilon. \quad \square$$

Claim: $|\sqrt{x}-\sqrt{y}| \leq \sqrt{|x-y|} \quad \forall x, y>0.$

pf: Without loss of generality, assume $x \geq y > 0$.

Then claim $\Leftrightarrow$ $x+y-2\sqrt{xy} \leq x-y$.

$\Leftrightarrow$ $y \leq \sqrt{xy}$, which is true. $\square$