Let $V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}$ be the eigenspace decomp. of the diagonalizable map $T : V \to V$.
(where $\lambda_1, \dots, \lambda_n$ are the distinct eigenvalues of T).

<u>Claim</u>: If $W \subseteq V$ and $T(W) \subseteq W$, then

$$W = (V_{\lambda_1} \cap W) \oplus \cdots \oplus (V_{\lambda_n} \cap W).$$

$(\Rightarrow T|_W : W \to W$ is diagonalizable$)$.

<u>Note</u>: It's clear that

$$(V_{\lambda_1} \cap W) \oplus \cdots \oplus (V_{\lambda_n} \cap W) \subseteq W.$$

But the converse inclusion is <u>NOT</u> true in general <u>without</u> the assumption $T(W) \subseteq W$.

<u>e.g.</u> $T = T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, where $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Then $\mathbb{R}^2 = \underbrace{\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}}_{V_2} \oplus \underbrace{\text{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}}_{V_3}$

is the eigenspace decomp. w.r.t. T.

Consider $W = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \subseteq \mathbb{R}^2$.

It is <u>NOT</u> invariant under T, i.e. $T(W) \not\subseteq W$,

and $W \neq \underbrace{(V_2 \cap W)}_{\{0\}} \oplus \underbrace{(V_3 \cap W)}_{\{0\}} = \{\vec{0}\}$.

So what we need to show is:

<u>Claim</u>: If $T(W) \subseteq W$, then

$$W \subseteq (V_{\lambda_1} \cap W) \oplus \cdots \oplus (V_{\lambda_n} \cap W).$$

For any $\vec{w} \in W$, $\exists!$ $\vec{v}_1, \ldots, \vec{v}_n \in V$

s.t. $\vec{w} = \vec{v}_1 + \cdots + \vec{v}_n$ and $\vec{v}_i \in V_{\lambda_i}$ $\forall i$

(the decomp. into eigenvectors)

If we can show that each $\vec{v}_i$ actually lies in W.
(using the fact that $T(W) \subseteq W$), then we'll have

$\vec{v}_i \in W \cap V_{\lambda_i}$ $\forall i$., which proves the claim.

This was proved in the solution.