

# ALGEBRAIC COMBINATORICS II, SUMMER 2024

## 1. OVERVIEW OF THE COURSE

### Lecture 1

We will explore the *symmetries* of various *geometric spaces* in this course. The spaces that we will consider include: the Euclidean spaces  $\mathbb{R}^2$ ,  $\mathbb{R}^3$ , the spheres  $S^1$ ,  $S^2$ , the hyperbolic space  $\mathbb{H}^2$ , and some of their interesting subsets.

**Question 1.1.** Which of the following shapes is more “symmetric”?



**Question 1.2.** How to define “symmetries”?

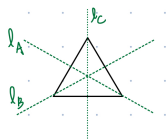
Each of the geometric spaces that we will consider ( $\mathbb{R}^2$ ,  $\mathbb{R}^3$ ,  $S^1$ ,  $S^2$ ,  $\mathbb{H}^2$ , etc.) has a natural metric (i.e. distance  $d(x, y)$  between any two points  $x, y$ ). The symmetries that we are interested in are the *isometries* (i.e. distance-preserving functions) of these spaces. For instance, an isometry of  $\mathbb{R}^2$  is a function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $d(f(x), f(y)) = d(x, y)$  for any  $x, y \in \mathbb{R}^2$ .

**Definition 1.3.** Let  $S \subseteq \mathbb{R}^2$  be a subset of  $\mathbb{R}^2$ . An isometry  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is called a *symmetry* of  $S$  if we have  $f(S) = S$ , i.e.

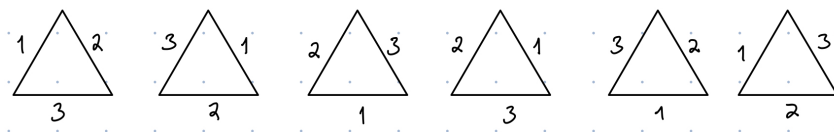
- for any  $p \in S$ , we have  $f(p) \in S$ ; and
- for any  $q \in S$ , there exists  $p \in S$  such that  $f(p) = q$ .

*Example.* Let us look at an easy example: an equilateral triangle. It has two kinds of symmetries:

- Rotational symmetries: one can rotate the triangle by  $\frac{2\pi}{3}$ ,  $\frac{4\pi}{3}$ , or  $2\pi$  without changing its appearance.
- Reflection symmetries: there are three “mirror lines” through which we can reflect the shape without changing its appearance.

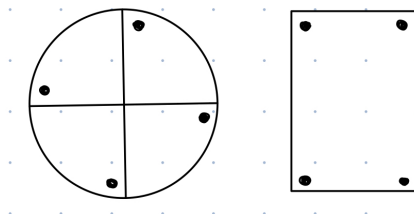


The easiest way to study the symmetries of a shape is by *counting*. In this example, it's easy to check that there are 6 symmetries. If we put labels on the edges of the triangle, then the effect of these symmetries look like:



However, counting alone is usually not good enough.

*Example.* Both of the following shapes have 4 symmetries. The shape on the



left has 4 rotational symmetries (by  $\frac{\pi}{2}$ ,  $\pi$ ,  $\frac{3\pi}{2}$ ,  $2\pi$ ), but no reflection symmetries. In contrast, the shape on the right has 2 rotational symmetries and 2 reflection symmetries. How can we distinguish them?

As we'll see later in this course, *group theory* provides rigorous tools to describe the symmetries of shapes. For any shape (or any geometric object), the set of its symmetries has a natural *group structure*. In the example above, although the sets of symmetries of both shapes have 4 elements, but their underlying group structures are different, and that's how we can tell them apart (e.g. consider the *orders* of elements in these two groups).

Another important tool that we will encounter is basic *linear algebra*, in particular *matrices* or *matrix groups*. The reason is that certain matrix groups ( $O(2, \mathbb{R})$ ,  $O(3, \mathbb{R})$ ,  $SL(2, \mathbb{R})$ ,  $SL(2, \mathbb{C})$ , etc.) act naturally as isometries on the spaces that we are interested in like  $\mathbb{R}^2$ ,  $\mathbb{R}^3$ ,  $S^1$ ,  $S^2$ ,  $\mathbb{H}^2$ . For instance,

you'll show in the homework that any isometry of the Euclidean space  $\mathbb{R}^n$  is a composition of a translation and a linear transformation.

## 2. A CRASH COURSE ON BASIC GROUP THEORY

**2.1. Binary operators.** Before discussing the actual definition of a *group*, let us first consider a more general notion of *binary operators*.

**Definition 2.1.** Let  $S$  be a set. A *binary operator* on  $S$  is a function

$$\circ: S \times S \rightarrow S.$$

*Example.* Addition on the set of positive integers (denoted by  $\mathbb{N}$ ), or the set of integers (denoted by  $\mathbb{Z}$ ), or the set of rational numbers (denoted by  $\mathbb{Q}$ ) or the set of real numbers (denoted by  $\mathbb{R}$ ), is a binary operator. Same for multiplication.

*Non-example.* Subtraction on the set of positive integers is *not* a binary operator. Division on the set of integers is *not* a binary operator.

**Definition 2.2.** Let  $(S, \circ)$  be a set with a binary operator. We say an element  $e \in S$  is an *identity element* if  $e \circ a = a \circ e = a$  for any  $a \in S$ .

*Example.* The element  $0 \in \mathbb{Z}$  is an identity element of  $(\mathbb{Z}, +)$ . The element  $1 \in \mathbb{Z}$  is an identity element of  $(\mathbb{Z}, \times)$ .

*Non-example.*  $(\mathbb{N}, +)$  has no identity element.

*Exercise.* Prove that any set with a binary operator  $(S, \circ)$  has at most one identity element.

**Definition 2.3.** Let  $(S, \circ, e)$  be a set with a binary operator and an identity element. We say an element  $a' \in S$  is an *inverse* of  $a \in S$  if  $a \circ a' = a' \circ a = e$ .

*Example.* For  $(\mathbb{Z}, +)$ , the inverse of  $a \in \mathbb{Z}$  is given by  $-a$ . For  $(\mathbb{R}, \times)$ , the inverse of  $a \in \mathbb{R}$  is given by  $1/a$ , provided that  $a \neq 0$ .

*Non-example.* For  $(\mathbb{Z}, \times)$ , any element  $a \in \mathbb{Z}$  has no inverse unless  $a = \pm 1$ .

**Definition 2.4.** Let  $(S, \circ)$  be a set with a binary operator. We say  $(S, \circ)$  is *associative* if  $(a \circ b) \circ c = a \circ (b \circ c)$  holds for any  $a, b, c \in S$ .

*Exercise.* Let  $(S, \circ, e)$  be a set with an associative binary operator and an identity element. Prove that any element in  $S$  has at most one inverse.

Most of the examples that we'll be discussing are associative. Here is a non-example (which we will not encounter in this course):

*Non-example.* The cross product  $\times$  on  $\mathbb{R}^3$  is *not* associative. Rather, it satisfies the *Jacobi identity*

$$\vec{v}_1 \times (\vec{v}_2 \times \vec{v}_3) + \vec{v}_2 \times (\vec{v}_3 \times \vec{v}_1) + \vec{v}_3 \times (\vec{v}_1 \times \vec{v}_2) = 0$$

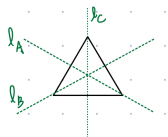
**Definition 2.5.** Let  $(S, \circ)$  be a set with a binary operator. We say  $(S, \circ)$  is *commutative* if  $a \circ b = b \circ a$  for any  $a, b \in S$ .

*Warning.* Many of the examples that we'll consider are *not* commutative.

*Non-example.* Consider the set of all six geometric transformations that give the symmetries of an equilateral triangle:

$$S = \left\{ \text{rotate } 0, \text{rotate } \frac{2\pi}{3}, \text{rotate } \frac{4\pi}{3}, \text{reflect along } \ell_A, \text{reflect along } \ell_B, \text{reflect along } \ell_C \right\}.$$

(note: rotations are typically assumed to be counterclockwise)

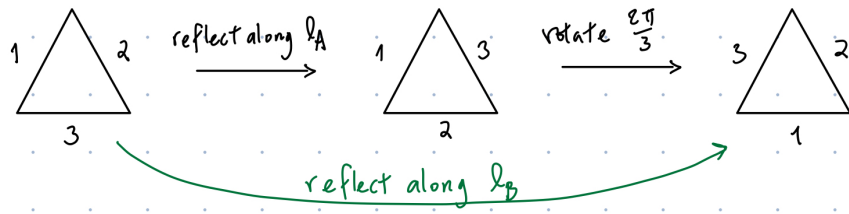


There is a binary operation on  $S$  given by *composing* these geometric transformations:

$$\circ: S \times S \rightarrow S,$$

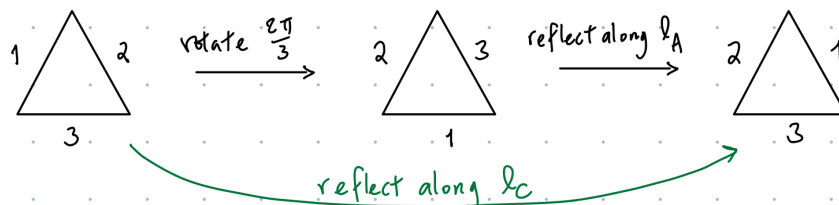
where  $a \circ b \in S$  is the transformation given by “do  $b$ , and then do  $a$ ”. For instance, we have

$$\left( \text{rotate } \frac{2\pi}{3} \right) \circ \left( \text{reflect along } \ell_A \right) = \text{reflect along } \ell_B.$$



On the other hand, by reversing the order one gets

$$\left(\text{reflect along } \ell_A\right) \circ \left(\text{rotate } \frac{2\pi}{3}\right) = \text{reflect along } \ell_C.$$



This shows that  $(S, \circ)$  is *not* commutative.

*Non-example.* Another important class of groups that we will discuss is the *matrix groups*. They are *not* commutative in most cases.

## 2.2. Groups.

**Definition 2.6.** Let  $(G, \circ)$  be a set with a binary operator. It is called a *group* if it satisfies the following conditions:

- (1) It is associative.
- (2) It has the identity element (which will usually be denoted by  $e$ ,  $e_G$ ,  $1$ , or  $1_G$ ).
- (3) Any element  $a \in G$  has an inverse (which will be denoted by  $a^{-1} \in G$ ).

*Remark 2.7.* Here are some notions that we will be using frequently:

- If a group  $(G, \circ)$  is commutative, then it is called an *abelian group*.
- We'll use  $|G|$  to denote the number of elements in the set  $G$ , and will call it the *order* of  $G$ . Note that the order of a group could be infinite in general.
- We quite often would omit “ $\circ$ ”, and simply denote  $a \circ b$  by  $ab$ , denote  $a \circ a$  by  $a^2$ , denote  $a \circ a \circ a$  by  $a^3$ , and so on.

*Example.* Consider the set of integers modulo  $n$

$$\mathbb{Z}/n\mathbb{Z} := \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}.$$

Addition and multiplication are well-defined on  $\mathbb{Z}/n\mathbb{Z}$ . It's not hard to show that  $(\mathbb{Z}/n\mathbb{Z}, +)$  is an abelian group of order  $n$ , with the identity given by  $\bar{0}$ .

*Example.* Consider the subset of  $\mathbb{Z}/n\mathbb{Z}$  consisting of elements that are coprime with  $n$ :

$$(\mathbb{Z}/n\mathbb{Z})^* := \left\{ \overline{m} \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1 \right\}.$$

It's not hard to show that  $((\mathbb{Z}/n\mathbb{Z})^*, \times)$  is an abelian group, with the identity given by  $\bar{1}$ .

*Example.* The set of all integers  $\mathbb{Z}$  under addition is an example of an abelian group with infinite order.

*Example.* The set  $\{0\}$  under addition is an example of a group with only one element (a trivial group).

*Example.* Let  $G_1$  and  $G_2$  be two groups. Consider the set

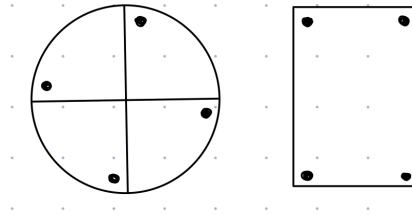
$$G_1 \times G_2 := \left\{ (g_1, g_2) : g_1 \in G_1 \text{ and } g_2 \in G_2 \right\}.$$

Define a binary operator on  $G_1 \times G_2$  as follows:

$$(g_1, g_2) \circ (g'_1, g'_2) := (g_1 \circ g'_1, g_2 \circ g'_2).$$

It's not hard to show that  $(G_1 \times G_2, \circ)$  is also a group. It's called the *direct product* of  $G_1$  and  $G_2$ .

*Example.* Let's come back to the following examples again. As discussed ear-



lier, the symmetries of a shape form a group, where the binary operation is given by composition. The symmetry group of the first shape is

$$G_1 := \left\{ \text{rotate } 0, \text{ rotate } \frac{\pi}{2}, \text{ rotate } \pi, \text{ rotate } \frac{3\pi}{2} \right\}.$$

One thing we might notice about this group is that all elements of the group can be obtained by taking one element of the set, and combining it different number of times. Let's denote rotate  $\frac{\pi}{2}$  by  $a$ . Then  $G_1$  can be rewritten as

$$G_1 = \{e, a, a^2, a^3\}.$$

Notice that  $a^4 = e$  since rotate  $2\pi$  is the same as rotate 0, i.e. the identity map. The same is true for  $\mathbb{Z}/4\mathbb{Z}$  (under addition) if one lets  $a = \bar{1}$  and note that  $a^4 = \bar{4} = \bar{0} = e$  in  $\mathbb{Z}/4\mathbb{Z}$ . In fact, we'll see that the symmetry group of the first shape and  $\mathbb{Z}/4\mathbb{Z}$  are *isomorphic*, which means that they are essentially the same group.

On the other hand, the symmetry group of the second shape is

$$G_2 := \left\{ \text{rotate } 0, \text{ rotate } \pi, \text{ reflect along } \ell_1, \text{ reflect along } \ell_2 \right\}.$$

It's not hard to see that there is no element  $a \in G_2$  such that  $G_2 = \{e, a, a^2, a^3\}$ . Therefore,  $G_2$  and  $G_1$  are not isomorphic. In fact, one can show that  $G_2$  is isomorphic to the direct product  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

**2.3. Homomorphisms.** For any mathematical structure (like groups), it is crucially important to understand how two structures of the same type (like two groups) are related in a meaningful way. Functions that bridge such two structures are called *homomorphisms*. (In the Ancient Greek language, “homo-” means “same”, and “morphe” means “form” or “shape”.) In general, a homomorphism is a function between two mathematical structures of the same type, that preserves the operations of the structures.

**Definition 2.8.** Let  $G$  and  $H$  be two groups. A function  $f: G \rightarrow H$  is called a *homomorphism* if for any  $g_1, g_2 \in G$  we have

$$f(g_1 g_2) = f(g_1) f(g_2)$$

Furthermore, a homomorphism that is both injective and surjective is called an *isomorphism*. In this case, we'll use the notation “ $G \cong H$ ”.

In other words, a homomorphism is a function that is compatible with the binary operations on the two groups.

*Exercise.* Let  $f: G \rightarrow H$  be a group homomorphism. Prove that

- It preserves the identity:  $f(e_G) = e_H$ .
- It preserves the inverses:  $f(g^{-1}) = f(g)^{-1}$  for any  $g \in G$ .

*Example.* We considered the symmetry group

$$G_1 := \left\{ \text{rotate } 0, \text{ rotate } \frac{\pi}{2}, \text{ rotate } \pi, \text{ rotate } \frac{3\pi}{2} \right\} = \{e, a, a^2, a^3\}$$

where  $a^4 = e$ . One can define a function

$$G_1 \rightarrow \mathbb{Z}/4\mathbb{Z}$$

by sending  $e \mapsto \bar{0}$ ,  $a \mapsto \bar{1}$ ,  $a^2 \mapsto \bar{2}$ , and  $a^3 \mapsto \bar{3}$ . It's an easy exercise to show that this function is an isomorphism.

## Lecture 2

*Remark 2.9.* A convenient way to present a group is by choosing elements that *generate* the group (which means that any element of the group can be written as a product of some of these generators and their inverses), and a set of *relations* among these generators. For instance,  $\mathbb{Z}/4\mathbb{Z}$  can be presented by

$$\mathbb{Z}/4\mathbb{Z} = \langle a : a^4 = e \rangle,$$

which means that one can find an element  $a \in \mathbb{Z}/4\mathbb{Z}$  such that any element in  $\mathbb{Z}/4\mathbb{Z}$  can be written as a power of  $a$ , and it satisfies  $a^4 = e$  (it's not hard to see that  $a$  can be chosen to be  $\bar{1}$  or  $\bar{3}$  in this case).

**Definition 2.10.** A group  $G$  that can be generated by a single element  $g$  is called a *cyclic* group (i.e. any element of  $G$  is of the form  $g^k$  for some  $k \in \mathbb{Z}$ ).

**Definition 2.11.** Let  $g$  be an element in a group  $G$ . If there exists a positive integer  $n$  such that  $g^n = e$ , then the smallest possible  $n$  satisfying  $g^n = e$  is called the *order* of  $g$ . If such  $n$  does not exist, then we say  $g$  is of infinite order.

*Exercise.* Let  $G$  be a cyclic group, and say it can be generated by an element  $g \in G$ .

- If  $g$  is of finite order, say  $\text{order}(g) = n$ . Prove that  $G \cong \mathbb{Z}/n\mathbb{Z}$ .
- If  $g$  is of infinite order, then prove that  $G \cong \mathbb{Z}$ .

Therefore, any cyclic group is isomorphic to either  $\mathbb{Z}$  or  $\mathbb{Z}/n\mathbb{Z}$  for some positive integer  $n$ .

*Exercise.* Prove that  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  is not a cyclic group.

*Example.* Let  $D_n$  be the symmetry group of a regular  $n$ -gon. It is not hard to show that  $D_n$  is generated by rotation by  $2\pi/n$  (which we'll denote by  $r$ ), and a reflection (which we'll denote by  $s$ ). The group  $D_n$  is of order  $2n$ , with elements given by

$$D_n = \{1, r, r^2, \dots, r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\}.$$



The generators  $r$  and  $s$  satisfy the relations  $r^n = s^2 = 1$  and  $s^{-1}rs = r^{-1}$ .

$$\begin{aligned} D_n &= \langle r, s \mid r^n = s^2 = 1, s^{-1}rs = r^{-1} \rangle \\ &= \langle r, s \mid r^n = s^2 = (rs)^2 = 1 \rangle. \end{aligned}$$

*Remark 2.12.* Since  $D_n$  is not commutative, it is not isomorphic to the direct product  $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . On the other hand, it is isomorphic to the *semi-direct product* of its order 2 subgroup  $\langle s \rangle$  and its order  $n$  normal subgroup  $\langle r \rangle$ :  $D_n \cong \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z}$ . We'll introduce these notations later on.

## 2.4. Subgroups.

**Definition 2.13.** Let  $G$  be a group. We say a subset  $H \subseteq G$  is a *subgroup* if:

- (1) it is closed under the binary operation of  $G$ : for any  $a, b \in H$ , we have  $ab \in H$ ;
- (2) it contains the identity element of  $G$ :  $e_G \in H$ ;
- (3) it is closed under taking inverse: for any  $a \in H$ , we have  $a^{-1} \in H$ .

*Exercise.* A subgroup  $H \subseteq G$  is itself a group, with the binary operator and the identity element inherit from  $G$ .

*Example.* For any group  $G$ , the subset  $\{e_G\} \subseteq G$  is always a subgroup, called the *trivial* subgroup of  $G$ . Also, the group  $G$  itself is a subgroup of  $G$ .

*Example.* For any positive integer  $n$ , the subset  $n\mathbb{Z} \subseteq (\mathbb{Z}, +)$  is a subgroup.

**Theorem 2.14.** Let  $G$  be a finite subgroup of  $O(2, \mathbb{R})$ . Then  $G$  is isomorphic to either a cyclic group or a dihedral group.

*Proof.* Any element of  $O(2, \mathbb{R})$  acts naturally on the unit circle  $S^1 \subseteq \mathbb{R}^2$ . Let  $g \in G$  be a non-identity element. It is not hard to show that  $g$  is either a rotation (when  $g$  does not fix any point of  $S^1$ ), or a reflection (when  $g$  fixes at least a point of  $S^1$ ).

First, suppose that all elements of  $G$  are rotations. Write  $r_\theta \in O(2, \mathbb{R})$  for the counterclockwise rotation by  $\theta$ , where  $0 \leq \theta < 2\pi$ . Choose  $r_\phi \in G$  with the smallest positive  $\phi$  (it is possible since  $G$  is finite). We claim that  $G$  is the cyclic group generated by  $r_\phi$ . Let  $r_\theta \in G$ , and write  $\theta = m\phi + \psi$  where  $m \in \mathbb{N}$  and  $0 \leq \psi < \phi$ . Then  $r_\psi = (r_\phi)^{-m}r_\theta \in G$ . Therefore  $\psi = 0$  by the minimality of  $\phi$ . Hence  $r_\theta = (r_\phi)^m$ .

Second, suppose  $G$  contains a reflection  $s$ . Let  $H \subseteq G$  be the subgroup consisting of rotations (including the identity). By the first case, we have  $H = \{1, r, \dots, r^{n-1}\}$  for some positive integer  $n$ . Consider any other reflection  $s' \in G$ . One can show that the composition of any two reflections is a rotation, hence  $ss' \in H$ . So  $s' = sr^k$  for some  $0 \leq k \leq n-1$ . This shows that

$$G = \{1, r, \dots, r^{n-1}, s, sr, \dots, sr^{n-1}\}.$$

It is easy to show that a rotation  $r$  and a reflection  $s$  satisfy the relation  $sr = r^{-1}s$ . Hence we get  $G \cong D_n$ .  $\square$

If  $H$  is a subgroup of  $G$ , then one can break  $G$  up into pieces, each of which looks like  $H$ . These pieces are called *cosets* of  $H$ , and they arise by “multiplying”  $H$  by elements of  $G$ .

**Definition 2.15.** Let  $G$  be a group and  $H \subseteq G$  be a subgroup. A *left coset* of  $H$  in  $G$  is a subset of the form

$$gH = \{gh \mid h \in H\} \text{ for some } g \in G.$$

The element  $g$  is called a *representative* of the coset  $gH$ . The collection of all left cosets is denoted by  $G/H$ . Its order  $|G/H|$  is called the *index* of  $H$  in  $G$ , and will sometimes be denoted by  $[G : H]$ .

Similarly, a *right coset* is a subset of the form

$$Hg = \{hg \mid h \in H\} \text{ for some } g \in G.$$

The collection of all right cosets is denoted by  $H \backslash G$ .

*Example.* Consider the subgroup  $n\mathbb{Z} \subseteq (\mathbb{Z}, +)$ . Since the group  $(\mathbb{Z}, +)$  is abelian, its left cosets and right cosets are identical. It is clear that the subgroup has exactly  $n$  cosets  $\bar{0}, \bar{1}, \dots, \overline{n-1}$ , where  $\bar{i} = i + n\mathbb{Z}$  consists of integers  $\equiv i$  modulo  $n$ . Hence  $n\mathbb{Z} \subseteq \mathbb{Z}$  is a subgroup of index  $n$ .

*Exercise.* The representative of a coset is *not* unique. In fact, show that a coset  $gH$  can be represented by any element of the form  $gh$  where  $h \in H$ .

**Proposition 2.16.** Let  $H \subseteq G$  be a subgroup. Prove that for any two cosets  $aH$  and  $bH$ , we have:

- either  $aH$  and  $bH$  are disjoint:  $aH \cap bH = \emptyset$ ,
- or  $aH$  and  $bH$  are exactly the same:  $aH = bH$ .

*Proof.* Suppose  $aH$  and  $bH$  are not disjoint. Then there exists  $h_1, h_2 \in H$  such that  $ah_1 = bh_2$ . For any  $h \in H$ , we have

$$ah = a(h_1h_1^{-1})h = b(h_2h_1^{-1}h) \in bH.$$

Hence  $aH \subseteq bH$ . Similarly, one can show that  $bH \subseteq aH$ . Therefore  $aH = bH$ .  $\square$

**Theorem 2.17** (Lagrange). *Let  $G$  be a finite group, and  $H \subseteq G$  be a subgroup. Then  $|G|$  is divisible by  $|H|$ . Moreover, we have  $|G| = |H|[G : H]$ .*

*Proof.* Since  $g \in gH$ , any element of  $G$  belongs to a left coset of  $H$ . Then the previous proposition shows that  $G$  is the disjoint union of the left cosets of  $H$ . Since each coset has exactly  $|H|$  elements, we can conclude that  $|G| = |H|[G : H]$ .  $\square$

### Lecture 3

*Exercise.* Consider the subgroup  $\mathbb{Z} \subseteq (\mathbb{R}, +)$ . The set of cosets  $\mathbb{R}/\mathbb{Z}$  can be identified with  $S^1$ , the unit circle in  $\mathbb{R}^2$ : Points of the circle are of the form  $e^{2\pi i\theta}$  where  $\theta \in \mathbb{R}$ . Show that the map  $t \mapsto e^{2\pi it}$  gives a bijection between  $\mathbb{R}/\mathbb{Z}$  and  $S^1$ .

*Exercise.* Let  $G$  be a finite group and  $g$  be an element of  $G$ . Prove that the order of  $g$  divides the order of  $G$ .

*Remark 2.18.* In the example  $n\mathbb{Z} \subseteq (\mathbb{Z}, +)$ , one can notice that the set of all cosets  $\{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$  also has a natural group structure inherits from the group structure on  $(\mathbb{Z}, +)$ : one defines  $\bar{i} + \bar{j}$  to be  $\overline{i+j}$ .

However, the set of all left cosets does *not* always admit a group structure! Let  $H \subseteq G$  be a subgroup and  $a, b \in G$  be two elements in  $G$ . It is tempting to define a group structure on  $G/H$  simply by declaring “ $aH \circ bH = (ab)H$ ”. In order for this definition to make sense, we need to show that, if  $a'$  is a representative of  $aH$  and  $b'$  is a representative of  $bH$ , then  $a'b'H = abH$ . This is equivalent to, for any  $a, b \in G$  and  $h_1, h_2 \in H$ , one needs  $ah_1bh_2H = abH$ , or equivalently,  $b^{-1}h_1b \in H$ . This is equivalent to the condition that for any  $g \in G$  one needs  $gH = Hg$ , i.e. the left and right cosets of  $H$  in  $G$  coincide, which is *not* true in general.

**Definition 2.19.** A subgroup  $H \subseteq G$  is called *normal* if  $gH = Hg$  for any  $g \in G$ .

By the previous remark, if  $H \subseteq G$  is a normal subgroup, then the set of (left) cosets  $G/H$  admits a group structure inherit from  $G$ : let  $aH$  and  $bH$  be two cosets, then  $aH \circ bH := (ab)H$  gives a well-defined group structure on  $G/H$ . The resulting group  $G/H$  is called the *quotient group*.

**Theorem 2.20** (First isomorphism theorem). *Let  $f: G \rightarrow H$  be a group homomorphism. Define*

$$\text{Ker}(f) := \{g \in G \mid f(g) = 1_H\} \subseteq G$$

and

$$\text{Im}(f) := \{h \in H \mid h = f(g) \text{ for some } g \in G\} \subseteq H.$$

Then

- (1)  $\text{Ker}(f)$  is a normal subgroup of  $G$ .
- (2)  $\text{Im}(f)$  is a subgroup of  $H$ .
- (3) There is an isomorphism between  $G/\text{Ker}(f)$  and  $\text{Im}(f)$ .

*Proof.* It is not hard to show that  $\text{Ker}(f) \subseteq G$  and  $\text{Im}(f) \subseteq H$  are subgroups (exercise). To show that  $\text{Ker}(f) \subseteq G$  is normal, one needs to show that for any  $g \in \text{Ker}(f)$  and  $g' \in G$ , we have  $g'gg'^{-1} \in \text{Ker}(f)$ . This is true because

$$f(g'gg'^{-1}) = f(g')f(g)f(g'^{-1}) = f(g')f(g)^{-1} = 1_H.$$

Now we define a map  $\bar{f}$  from  $G/\text{Ker}(f)$  to  $\text{Im}(f)$ : For any coset  $g\text{Ker}(f)$ , we define  $\bar{f}(g\text{Ker}(f)) := f(g)$ . This is a well-defined function on the set of cosets  $G/\text{Ker}(f)$ , because any representative of  $g\text{Ker}(f)$  is of the form  $gg'$  for some  $g' \in \text{Ker}(f)$ , and we have  $f(gg') = f(g)f(g') = f(g)$ . It is not hard to check that  $\bar{f}: G/\text{Ker}(f) \rightarrow \text{Im}(f)$  is a surjective group homomorphism. It is also injective: if  $\bar{f}(g_1\text{Ker}(f)) = \bar{f}(g_2\text{Ker}(f))$ , then we have  $f(g_1) = f(g_2)$ , or equivalently  $g_2^{-1}g_1 \in \text{Ker}(f)$ . Hence the cosets  $g_1\text{Ker}(f) = g_2\text{Ker}(f)$  coincide.  $\square$

*Example.* From Homework 1, we know that for any  $f \in \text{Isom}(\mathbb{R}^n)$ , there exists a unique pair of an orthogonal matrix  $A$  and a vector  $\vec{v}$  such that

$$f(\vec{x}) = A\vec{x} + \vec{v} \text{ for any } \vec{x} \in \mathbb{R}^n.$$

This gives a function

$$\pi: \text{Isom}(\mathbb{R}^n) \rightarrow \text{O}(n, \mathbb{R}), \quad f \mapsto A.$$

The function  $\pi$  is in fact a group homomorphism: suppose  $f_1(\vec{x}) = A_1\vec{x} + \vec{v}_1$  and  $f_2(\vec{x}) = A_2\vec{x} + \vec{v}_2$ , then

$$f_1(f_2(\vec{x})) = A_1(A_2\vec{x} + \vec{v}_2) + \vec{v}_1 = (A_1A_2)\vec{x} + (A_1\vec{v}_2 + \vec{v}_1).$$

Hence  $\pi(f_1f_2) = A_1A_2$ . The kernel of  $\pi$  is an isometry of the form  $f(\vec{x}) = \vec{x} + \vec{v}$ , which is simply the translation by  $\vec{v}$ . Hence  $\text{Ker}(\pi) = T(n, \mathbb{R}) \cong \mathbb{R}^n$ . This shows that the group of translations  $T(n, \mathbb{R})$  is normal in  $\text{Isom}(\mathbb{R}^n)$ . The homomorphism  $\pi$  is clearly surjective, so we have an isomorphism

$$\text{Isom}(\mathbb{R}^n)/T(n, \mathbb{R}) \cong O(n, \mathbb{R}).$$

Lecture 4

**2.5. Symmetry groups.** For any set  $X$ , a *permutation* of  $X$  is a bijective function  $f: X \rightarrow X$ . The *symmetric group*  $S_X$  *defined over*  $X$  is the set of all permutations of  $X$ , equipped with the group structure given by compositions. In particular, when  $X$  is a finite set of  $n$  elements  $\{1, 2, \dots, n\}$ , its symmetric group would be denoted by  $S_n$ . It is not hard to see that  $|S_n| = n!$ .

*Remark 2.21.* Symmetric groups arise naturally when we discuss the symmetry groups of Platonic solids. Let  $G$  be the symmetry group of a tetrahedron  $T$ . It is not hard to see that any symmetry of  $T$  sends a vertex of  $T$  to a vertex (not necessarily the same one); in other words, it gives rise to a permutation of the four vertices of  $T$ . This gives a group homomorphism  $\rho: \text{Aut}(T) \rightarrow S_4$ . Note that  $\rho$  is injective (why?), hence the symmetry group  $\text{Aut}(T)$  is isomorphic to a subgroup of the symmetric group  $S_4$ .

Any element of  $S_n$  can be represented by Cauchy's "two-line notation". Let  $\sigma \in S_n$  be a permutation of the set  $\{1, 2, \dots, n\}$ . Then we'll write

$$\sigma = \begin{bmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n). \end{bmatrix}$$

As usual, the composition  $\sigma_1\sigma_2 \in S_n$  is given by  $k \mapsto \sigma_1(\sigma_2(k))$ , i.e. first apply  $\sigma_2$  then apply  $\sigma_1$ . For instance, verify that

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Permutations are also often written in *cycle notation* ("decomposition into disjoint cycles"). To write down  $\sigma \in S_n$  in cycle notation, one proceeds as follows:

- Write an open bracket then select an arbitrary element  $x \in \{1, \dots, n\}$ , and write down:  $(x$
- Then trace the orbit of  $x$ : write down its value under successive applications of  $\sigma$ :  $(x \ \sigma(x) \ \sigma^2(x) \dots$
- Repeat until the value return to  $x$ , and write down a closing parenthesis rather than  $x$ :  $(x \ \sigma(x) \ \sigma^2(x) \dots)$
- Continue with any element  $y$  that is not yet written down, and proceed in the same way:  $(x \ \sigma(x) \ \sigma^2(x) \dots)(y \ \sigma(y) \dots)$
- Repeat until all elements of  $\{1, \dots, n\}$  are written in one of the cycles.

For instance,

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 6 & 2 & 3 & 5 \end{bmatrix} = (1)(24)(365) = (24)(365).$$

Here  $\sigma(1) = 1$  forms an 1-cycle, which is often omitted.

A 2-cycle is called a *transposition*. An important fact is that any element  $\sigma \in S_n$  can be written as a product of transpositions. To see this, it suffices to show that any cycle can be written as a product of transpositions, as any  $\sigma$  is a product of cycles. This can be easily verified:

$$(i_1 i_2 \dots i_k) = (i_1 i_k)(i_1 i_{k-1}) \dots (i_1 i_2).$$

It is not hard to see that there is no unique way to represent a permutation by a product of transpositions. For instance,  $(123) = (13)(12) = (12)(23) = (12)(23)(13)(13)$ . However, the *parity* (i.e. even or odd) of the numbers of transpositions of such representations is unique. (For instance,  $(123)$  can not be written as the product of odd number of transpositions.) This permits the *parity of a permutation* to be a well-defined notion.

The key idea of the proof is to define a group homomorphism

$$\text{sgn}: S_n \rightarrow \{+1, -1\} \text{ (under multiplication)}$$

so that all transpositions map to  $-1$ . Indeed, if we can find such a homomorphism, then for any representation  $\sigma = \tau_1 \dots \tau_k$  where  $\tau_i$ 's are transpositions, we have

$$\text{sgn}(\sigma) = \text{sgn}(\tau_1) \dots \text{sgn}(\tau_k) = (-1)^k.$$

This shows that the parity of  $k$  is independent of the choice of the decomposition.

Now, to define such group homomorphism  $\text{sgn}$ , we consider the Vandermonde polynomial

$$P(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

For  $\sigma \in S_n$ , define

$$\text{sgn}(\sigma) := \frac{P(x_{\sigma(1)}, \dots, x_{\sigma(n)})}{P(x_1, \dots, x_n)}.$$

Observe that the polynomials  $P(x_1, \dots, x_n)$  and  $P(x_{\sigma(1)}, \dots, x_{\sigma(n)})$  have the same factors except for the signs, therefore  $\text{sgn}(\sigma) = \pm 1$ .

It defines a group homomorphism  $\text{sgn}: S_n \rightarrow \{\pm 1\}$  since

$$\begin{aligned} \text{sgn}(\sigma_1 \sigma_2) &= \frac{P(x_{\sigma_1(\sigma_2(1))}, \dots, x_{\sigma_1(\sigma_2(n))})}{P(x_1, \dots, x_n)} \\ &= \frac{P(x_{\sigma_1(\sigma_2(1))}, \dots, x_{\sigma_1(\sigma_2(n))})}{P(x_{\sigma_2(1)}, \dots, x_{\sigma_2(n)})} \cdot \frac{P(x_{\sigma_2(1)}, \dots, x_{\sigma_2(n)})}{P(x_1, \dots, x_n)} \\ &= \text{sgn}(\sigma_1) \text{sgn}(\sigma_2). \end{aligned}$$

Also, it is easy to check that  $\text{sgn}$  sends any transposition to  $-1$ . This finishes the proof.

**Definition 2.22.** The subset of  $S_n$  consisting of all *even* permutations will be denoted by  $A_n$ . It is a *normal subgroup* of  $S_n$  since it is the kernel of the group homomorphism  $\text{sgn}$ . The group  $A_n$  is called the *alternating group* (of  $n$  elements).

*Exercise.* Show that  $A_n \subseteq S_n$  is a normal subgroup of index 2; it has two cosets, one of them consists of all even permutations, the other consists of all odd permutations.

**2.6. Group actions.** We will be interested in groups  $G$  that act as symmetries of a set  $X$  (for instance, the symmetry group of a tetrahedron acting on the set of its vertices). Let us introduce the formal definition of group actions.

**Definition 2.23.** We say that a group  $G$  *acts on a set*  $X$  if there is a map

$$G \times X \rightarrow X; \quad (g, x) \mapsto g \cdot x$$

satisfying:

- $e_G \cdot x = x$  for any  $x \in X$ ,

- $g \cdot (h \cdot x) = (gh) \cdot x$  for any  $g, h \in G$  and  $x \in X$ .

The dot “ $\cdot$ ” is sometimes omitted when the context is clear.

*Exercise.* Show that to give a group action of  $G$  on  $X$  is equivalent to give a group homomorphism  $\rho: G \rightarrow S_X$ . (Hint: Relate them by  $g \cdot x = \rho(g)(x)$ .)

*Example.* The symmetric group  $S_n$  acts on the set  $\{1, \dots, n\}$ .

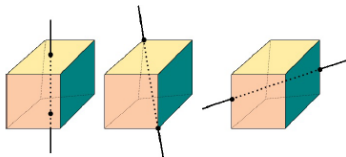
*Example.*  $\text{Isom}(\mathbb{R}^n)$  acts on  $\mathbb{R}^n$ .

*Example.*  $O(n, \mathbb{R})$  acts on the unit sphere  $S^{n-1} \subseteq \mathbb{R}^n$ , where

$$S^{n-1} = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| = 1\}.$$

*Example.* The dihedral group  $D_n$  acts on the set of vertices of a regular  $n$ -gon, which gives a group homomorphism  $D_n \rightarrow S_n$ . Similarly, the symmetry group of a Platonic solid  $P$  acts on the set of its vertices.

*Example.* Let  $C$  be a cube in  $\mathbb{R}^3$  centered at the origin. Denote  $\text{Aut}^+(C)$  the *rotational symmetric group* of  $C$ . Each element of  $\text{Aut}^+(C)$  is a rotation that fixes a line through the origin, and sends the cube  $C$  to itself. For instance:



- identity map;
- rotate  $\pi/2, \pi, 3\pi/2$  along the first (left-most) line: there are 3 such lines, so this gives in total 9 elements of  $\text{Aut}^+(C)$ ;
- rotate  $2\pi/3, 4\pi/3$  along the second line: there are 4 such lines, so this gives in total 8 elements of  $\text{Aut}^+(C)$ ;
- rotate  $\pi$  along the third line: there are 6 such lines, so this gives in total 6 elements of  $\text{Aut}^+(C)$ .

Hence  $|\text{Aut}^+(C)|$  is at least 24.

On the other hand, observe that  $\text{Aut}^+(C)$  gives an action on the set of the four main diagonals of  $C$ , therefore induces a group homomorphism

$$\rho: \text{Aut}^+(C) \rightarrow S_4.$$



One can show that  $\rho$  is injective (this is not a trivial observation: one needs to show that the antipodal map  $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3)$  is *not* a rotation). Now, combining with the fact that  $|\text{Aut}^+(C)| \geq 24$ , we can conclude that  $\rho$  is an isomorphism  $\text{Aut}^+(C) \cong S_4$ .

## Lecture 5

**Definition 2.24.** Let  $X$  be a set admitting a group action by  $G$ . For any  $x \in X$ , define its *orbit* to be

$$\text{orb}(x) := \{g \cdot x \mid g \in G\} \subseteq X.$$

It sometimes is also denoted by  $Gx$ .

The subset of  $G$  fixing  $x$

$$\text{Stab}(x) := \{g \in G \mid g \cdot x = x\}$$

is called the *stabilizer* of  $x$ , which is a subgroup of  $G$  (why?).

*Exercise.* Determine the orbits and stabilizers of the examples of group actions we mentioned above.

*Exercise.* Let  $X$  be a set admitting a group action by  $G$ . Let  $\text{orb}(x)$  and  $\text{orb}(y)$  be two orbits of the action. Prove that either  $\text{orb}(x) = \text{orb}(y)$  or  $\text{orb}(x) \cap \text{orb}(y) = \emptyset$ .

In other words, a group  $G$  acting on a set  $X$  decomposes  $X$  into disjoint union of the orbits of the action. The set of all orbits is denoted by  $X/G$ .

**Theorem 2.25.** Let  $X$  be a set admitting a group action by  $G$ . Let  $g \in G$  and  $x \in X$ .

- (1)  $\text{Stab}(gx) = g\text{Stab}(x)g^{-1}$ . In other words, the stabilizers of points on the same orbit are conjugate to each other.
- (2) (*Orbit-stabilizer theorem*) There is a bijective map between the orbit  $\text{orb}(x)$  and the set of left cosets  $G/\text{Stab}(x)$ . In particular, if  $|G|$  is finite then  $|G| = |\text{Stab}(x)||\text{orb}(x)|$ .

*Proof.* The first statement follows from

$$h \in \text{Stab}(gx) \Leftrightarrow hgx = gx \Leftrightarrow g^{-1}hgx = x \Leftrightarrow g^{-1}hg \in \text{Stab}(x).$$

To prove the second statement, consider the map

$$f: G \rightarrow \text{orb}(x); \quad g \mapsto gx.$$

The map is clearly surjective. For any two elements  $g_1, g_2 \in G$ ,

$$f(g_1) = f(g_2) \Leftrightarrow g_1x = g_2x \Leftrightarrow g_2^{-1}g_1x = x \Leftrightarrow g_2^{-1}g_1 \in \text{Stab}(x) \Leftrightarrow g_1 \in g_2\text{Stab}(x).$$

Hence  $f(g_1) = f(g_2)$  if and only if  $g_1$  and  $g_2$  lie in the same coset for the stabilizer subgroup  $\text{Stab}(x) \subseteq G$ . This proves the second statement.  $\square$

**Theorem 2.26** (Burnside's lemma). *Let  $X$  be a finite set admitting a group action by a finite group  $G$ . For any  $g \in G$ , denote  $X^g = \{x \in X \mid gx = x\}$  the collection of points fixed by  $g$ . Then the number of disjoint orbits satisfies*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

*Proof.* Consider the set of pairs

$$Z = \{(g, x) \in G \times X \mid gx = x\}.$$

On the one hand, for each  $g_0 \in G$  there exists  $|X^{g_0}|$  many elements in  $X$  such that  $(g_0, x) \in Z$ . Hence  $|Z| = \sum_{g \in G} |X^g|$ . On the other hand, for each  $x_0 \in X$ , there are  $|\text{Stab}(x_0)|$  many elements in  $G$  such that  $(g, x_0) \in Z$ . Hence

$$|Z| = \sum_{x \in X} |\text{Stab}(x)| = \sum_{x \in X} \frac{|G|}{|\text{orb}(x)|}.$$

Denote  $O_1, \dots, O_k$  the orbits of  $X$  under the  $G$ -action, where  $k = |X/G|$ . Then

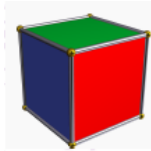
$$\sum_{x \in X} \frac{|G|}{|\text{orb}(x)|} = |G| \sum_{i=1}^k \sum_{x \in O_i} \frac{1}{|\text{orb}(x)|} = |G| \sum_{i=1}^k 1 = |G| |X/G|.$$

Therefore, we have

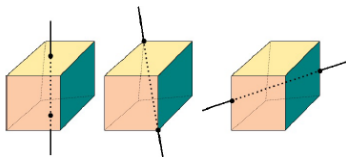
$$|G| |X/G| = |Z| = \sum_{g \in G} |X^g|.$$

$\square$

*Example.* How many different ways are there to color the faces of a cube with  $n$  colors (up to rotational symmetry)? Let  $X$  be the set of all possible colorings



of the cube, and let  $G = \text{Aut}^+(C)$ . The problem is equivalent to calculating the number of orbits  $|X/G|$ . By Burnside's lemma, it suffices to compute the size of the fixed point sets for each element of  $G$ .



- identity map: fixes all colorings, there are  $n^6$  of them;
- rotate  $\pi/2, 3\pi/2$  along lines of the first type (6 such rotations): each fixes  $n^3$  colorings;
- rotate  $\pi$  along lines of the first type (3 such rotations): each fixes  $n^4$  colorings;
- rotate  $2\pi/3, 4\pi/3$  along lines of the second type (8 such rotations): each fixes  $n^2$  colorings;
- rotate  $\pi$  along lines of the third type (6 such rotations): each fixes  $n^3$  colorings.

By Burnside's lemma, we have

$$|X/G| = \frac{1}{24} (1 \cdot n^6 + 6 \cdot n^3 + 3 \cdot n^4 + 8 \cdot n^2 + 6 \cdot n^3) = \frac{n^6 + 3n^4 + 12n^3 + 8n^2}{24}.$$

Lecture 6

### 3. A CRASH COURSE ON BASIC LINEAR ALGEBRA

**3.1. Matrix products, invertibility, determinants.** Elements of the vector space  $\mathbb{R}^n$  are of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

where  $x_1, \dots, x_n \in \mathbb{R}$ . To save space, we sometimes write down the *transpose* of  $\vec{x}$  instead:  $\vec{x} = [x_1 \cdots x_n]^T$ . There are two important operations on the vector space  $\mathbb{R}^n$ :

- *Addition*: Let  $\vec{x} = [x_1 \cdots x_n]^T$  and  $\vec{y} = [y_1 \cdots y_n]^T$  be two vectors in  $\mathbb{R}^n$ . Define  $\vec{x} + \vec{y} := [x_1 + y_1 \cdots x_n + y_n] \in \mathbb{R}^n$ .
- *Scalar multiplication*: Let  $\vec{x} = [x_1 \cdots x_n]^T$  and  $\lambda \in \mathbb{R}$ . Define  $\lambda \vec{x} := [\lambda x_1 \cdots \lambda x_n] \in \mathbb{R}^n$ .

**Definition 3.1.** Let  $\vec{v}_1, \dots, \vec{v}_k$  be vectors in  $\mathbb{R}^n$ . Then, for any  $c_1, \dots, c_k \in \mathbb{R}$ , the vector

$$c_1\vec{v}_1 + \dots + c_k\vec{v}_k \in \mathbb{R}^n$$

is called a *linear combination* of the set of vectors  $\vec{v}_1, \dots, \vec{v}_k$  (with weights  $c_1, \dots, c_k$ ). The *span* of the set of vectors  $\vec{v}_1, \dots, \vec{v}_k$  is defined to be the collection of all of their linear combinations:

$$\begin{aligned} \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} &= \{\text{linear combinations of } \vec{v}_1, \dots, \vec{v}_k\} \\ &= \{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}. \end{aligned}$$

*Remark 3.2.* The most fundamental question in linear algebra is to determine whether a linear system of equations has a solution:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

Consider the vectors  $\vec{v}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$  ( $1 \leq i \leq n$ ) and  $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ . Then the system

has a solution is equivalent to the statement that

$$\vec{b} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_n\}.$$

**Definition 3.3.** Let  $A = \begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_n \end{bmatrix}$  be an  $m \times n$  matrix with column vectors given by  $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^m$ . Let  $\vec{x} = [x_1 \dots x_n]^T \in \mathbb{R}^n$ . Define the *matrix-vector product* of  $A$  and  $\vec{x}$  to be the linear combination:

$$A\vec{x} := x_1\vec{a}_1 + \dots + x_n\vec{a}_n \in \mathbb{R}^m.$$

*Remark 3.4.* For any  $m \times n$  matrix  $A$ , the matrix-vector product gives rise to a function

$$T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m; \quad T_A(\vec{x}) := A\vec{x}.$$

The core of linear algebra is to study such a function. It is easy to check that the function  $T_A$  is *linear*, i.e. it is compatible with the additions and scalar multiplications on  $\mathbb{R}^n$  and  $\mathbb{R}^m$ :

- $T_A(\vec{v} + \vec{w}) = T_A(\vec{v}) + T_A(\vec{w})$ ,
- $T_A(\lambda\vec{v}) = \lambda T_A(\vec{v})$ .

*Exercise.* Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation (i.e. compatible with the additions and scalar multiplications on  $\mathbb{R}^n$  and  $\mathbb{R}^m$ ). Then there exists a unique  $m \times n$  matrix  $A$  such that  $T_A = T$ . In fact, the  $i$ -th column of  $A$  is given by  $T(\vec{e}_i)$ , where  $\vec{e}_i = [0 \cdots 0 1 0 \cdots 0]^T$  with the only nonzero entry at the  $i$ -th coordinate.

Therefore, there is a one-to-one correspondence between  $m \times n$  matrices and linear transformations  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ .

*Exercise.* Show that under the correspondence described above, the  $n \times n$  matrix corresponds to the identity transformation  $\text{id}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  ( $\text{id}(\vec{x}) = \vec{x}$  for all  $\vec{x}$ ) is

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & & \cdots & & 1 \end{bmatrix}$$

and is called the *identity matrix*.

**Definition 3.5.** Let  $A$  be an  $m \times n$  matrix and  $B$  be an  $n \times p$  matrix. We would like to define the *matrix product*  $AB$ , which will be an  $m \times p$  matrix.

The matrices  $A$  and  $B$  correspond to linear transformations  $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $T_B: \mathbb{R}^p \rightarrow \mathbb{R}^n$ . Consider the composition

$$T_A \circ T_B: \mathbb{R}^p \rightarrow \mathbb{R}^m; \quad \vec{x} \mapsto T_A(T_B(\vec{x})).$$

One can check the composition of linear maps is still linear, hence there exists a unique  $m \times p$  matrix, of which we define to be the matrix product  $AB$ , such that  $T_{AB} = T_A \circ T_B$ .

*Exercise.* Write down the entries of  $AB$  explicitly in terms of the entries of  $A$  and  $B$ .

*Remark 3.6.* The definition of matrix product we give here is more conceptual. It has many advantages: for instance, the *associativity* of matrix product

$A(BC) = (AB)C$  follows immediately from the associativity of compositions of functions.

*Exercise.* Let  $A$  be an  $m \times n$  matrix. Then  $A = AI_n = I_m A$ .

**Notation.** Let  $A$  and  $B$  be  $m \times n$  matrices. Let  $\lambda \in \mathbb{R}$ .

- (Addition)  $A + B$  is an  $m \times n$  matrix given by entry-wise addition.
- (Scalar multiplication)  $\lambda A$  is an  $m \times n$  matrix given by entry-wise scalar multiplication by  $\lambda$ .
- (Transpose)  $A^T$  is an  $n \times m$  matrix given by  $(A^T)_{ij} = A_{ji}$ .
- If  $A$  is a square matrix (i.e.  $m = n$ ), then  $A \cdot A$  makes sense and we denote  $A^2 = A \cdot A$ . Similarly,  $A^3 = A \cdot A \cdot A$ , and so on.

**Definition 3.7.** Let  $A$  be an  $n \times n$  matrix. We say  $A$  is *invertible* (or *non-singular*) if there exists  $n \times n$  matrices  $B$  and  $C$  such that

$$AB = I_n = CA.$$

In fact, such  $B$  and  $C$  must coincide since  $B = I_n B = (CA)B = C(AB) = CI_n = C$ . Moreover, one can easily show that such  $B$  is unique if it exists. When  $A$  is invertible, the matrix  $B$  such that  $AB = I_n = BA$  is called the *inverse* of  $A$ , and is denoted by  $A^{-1}$ .

*Exercise.* Prove the following statements.

- If  $A$  is invertible, then so is  $A^{-1}$ , and  $(A^{-1})^{-1} = A$ .
- If  $A, B$  are invertible matrices of the same size, then  $AB$  also is invertible, and  $(AB)^{-1} = B^{-1}A^{-1}$ .
- If  $A$  is invertible, then so is  $A^T$ , and  $(A^T)^{-1} = (A^{-1})^T$ .

A consequence of the first two statements is that, the set of all *invertible*  $n \times n$  matrices form a *group*, with operation given by matrix multiplication and identity element given by  $I_n$ . The group is called the *general linear group* and denoted by  $\text{GL}(n, \mathbb{R})$ .

*Remark 3.8.* Here is a basic fact on characterizing invertible matrices. Let  $A$  be an  $n \times n$  matrix. The following statements are equivalent:

- $A$  is invertible.
- $T_A$  is bijective.
- $T_A$  is injective.
- $T_A$  is surjective.

- There exists an  $n \times n$  matrix  $B$  such that  $AB = I_n$ .
- There exists an  $n \times n$  matrix  $C$  such that  $CA = I_n$ .

Note that these equivalences do *not* hold in general: they only hold for square matrices.

**Definition 3.9.** Let  $A$  be an  $n \times n$  matrix. Its *determinant* is defined to be

$$\det(A) := \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \in \mathbb{R}.$$

For instance, when  $n = 2$ , we have  $\det(A) = a_{11}a_{22} - a_{12}a_{21}$ .

**Theorem 3.10.** *Here are some important results of the determinants.*

- A square matrix  $A$  is invertible if and only if  $\det(A) \neq 0$ .
- For any two square matrices  $A$  and  $B$  of the same size, we have  $\det(AB) = \det(A) \det(B)$ .
- $\det(A) = \det(A^T)$ .
- Geometrically,  $|\det(A)|$  coincides with the volume of the ( $n$ -dimensional) parallelepiped spanned by the column (or row) vectors of  $A$ .

Denote  $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$  with the group structure given by multiplications. Then the determinants give a group homomorphism

$$\det: \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}^*$$

Its kernel (i.e. matrices with determinant one) is called the *special linear group* and denoted by  $\text{SL}(n, \mathbb{R})$ .

**3.2. Inner products, orthogonal matrices.** In Homework 1, we showed that any element  $T \in \text{O}(n, \mathbb{R})$  of origin-preserving isometries of  $\mathbb{R}^n$  is *linear*, and that  $T$  preserves the standard inner product on  $\mathbb{R}^n$ . There exists a unique  $n \times n$  matrix  $A$  such that  $T_A = T$ . Since  $T$  preserves the inner product, for any  $\vec{x}, \vec{y} \in \mathbb{R}^n$  we have

$$\langle \vec{x}, \vec{y} \rangle = \langle A\vec{x}, A\vec{y} \rangle, \text{ or equivalently } \vec{x}^T \vec{y} = \vec{x}^T A^T A \vec{y}.$$

It is an easy exercise to show that this would imply that  $A^T A = I_n$ . In fact, the converse is true, namely, if we have a matrix  $A$  satisfying  $A^T A = I_n$ , then  $T_A$  is an origin-preserving isometry of  $\mathbb{R}^n$ . Therefore, we can identify the origin-preserving isometries  $\text{O}(n, \mathbb{R})$  with the matrices satisfying  $A^T A = I_n$ .

**Definition 3.11.** An  $n \times n$  matrix  $A$  is called *orthogonal* if  $A^T A = I_n$ . We will also denote the group of orthogonal matrices by  $O(n, \mathbb{R})$ .

*Remark 3.12.* Let  $\{\vec{v}_1, \dots, \vec{v}_n\}$  be the columns of  $A$ . Then  $A$  is orthogonal if and only if  $\{\vec{v}_1, \dots, \vec{v}_n\}$  is an *orthonormal* set, i.e.  $\langle \vec{v}_i, \vec{v}_i \rangle = 1$  for all  $i$  and  $\langle \vec{v}_i, \vec{v}_j \rangle = 0$  for all  $i \neq j$ .

Observe that if  $A$  is orthogonal, then  $\det(A)^2 = \det(A^T A) = 1$ , hence  $\det(A) = \pm 1$ . The subgroup of  $O(n, \mathbb{R})$  with determinant one is called the *special orthogonal group*, and denoted by

$$SO(n, \mathbb{R}) = \{A \in O(n, \mathbb{R}) \mid \det(A) = 1\}.$$

There is a surjective group homomorphism  $\det: O(n, \mathbb{R}) \rightarrow \{\pm 1\}$  with kernel given by  $SO(n, \mathbb{R})$ , hence we have  $[O(n, \mathbb{R}) : SO(n, \mathbb{R})] = 2$ .

**Definition 3.13.** A *rotation* of  $\mathbb{R}^3$  about the origin is a map  $f \in O(3, \mathbb{R})$  such that

- $f$  fixes a line  $\ell$  through the origin (called the *axis of rotation*), and
- $f$  rotates the two-dimensional plane through the origin orthogonal to  $\ell$ .

It is a highly non-trivial fact that

$$SO(3, \mathbb{R}) = \{\text{rotations of } \mathbb{R}^3\}.$$

You will prove this in a homework assignment or a project.

#### 4. PLATONIC SOLIDS AND FINITE SUBGROUPS OF $SO(3, \mathbb{R})$

##### 4.1. Classification of the Platonic solids.

**Definition 4.1.** A *Platonic solid* is a convex polyhedron satisfying the following conditions:

- (1) all its faces are convex regular polygons, and are congruent (identical in shape and size);
- (2) none of its faces intersect except at their edges;
- (3) the same number of faces meet at each of its vertices.

Each Platonic solid is completely determined by two numbers  $p$  and  $q$ , where

- $p$  is the number of edges (or equivalently, vertices) of each face;



- $q$  is the number of faces (or equivalently, edges) that meet at each vertex.

**Fact 4.2.** *There are only five Platonic solids.*



<i>Polyhedron</i>	<i>Vertices <math>V</math></i>	<i>Edges <math>E</math></i>	<i>Faces <math>F</math></i>	$(p, q)$
<i>Tetrahedron</i>	4	6	4	$(3, 3)$
<i>Cube</i>	8	12	6	$(4, 3)$
<i>Octahedron</i>	6	12	8	$(3, 4)$
<i>Dodecahedron</i>	20	30	12	$(5, 3)$
<i>Icosahedron</i>	12	30	20	$(3, 5)$

*Proof.* We would like to show that there is no other possible  $(p, q)$  that can be used to form a Platonic solid. It is not hard to see that  $pF = 2E$  and  $qV = 2E$ . By the Euler's formula  $V - E + F = 2$ , one obtains

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E} > \frac{1}{2}.$$

Also, note that  $p$  and  $q$  must both be at least 3. One can then check that there are only 5 possibilities for  $(p, q)$ .  $\square$

**4.2. Symmetry groups of the Platonic solids.** In order to study the symmetry group  $\text{Aut}(P)$  of a Platonic solid  $P$ , one can move the solid so that its center is located at the origin  $\vec{0} = (0, 0, 0) \in \mathbb{R}^3$ . Then, any isometry of  $\mathbb{R}^3$  that fixes  $P$  must also fix the origin.

**Definition 4.3.** The *orthogonal group*  $O(3, \mathbb{R})$  is defined as:

$$O(3, \mathbb{R}) := \left\{ f \in \text{Isom}(\mathbb{R}^3) \mid f(\vec{0}) = \vec{0} \right\},$$

which consists of isometries of  $\mathbb{R}^3$  that fix the origin  $\vec{0} = (0, 0, 0)$  of  $\mathbb{R}^3$ .

*Exercise.* Prove that  $O(3, \mathbb{R})$  is a subgroup of  $\text{Isom}(\mathbb{R}^3)$ .

Given a Platonic solid  $P$  centered at the origin  $\vec{0} \in \mathbb{R}^3$ , we would like to study:

- the symmetry group  $\text{Aut}(P)$ , which is a subgroup of the orthogonal group  $\text{O}(3, \mathbb{R})$ ;
- the intersection  $\text{Aut}(P) \cap \text{SO}(3, \mathbb{R})$ , consisting of rotations that fix the solid  $P$ , will be called the *rotational symmetry group* of  $P$ , and will be denoted by  $\text{Aut}^+(P)$ .

We will prove that any finite subgroup of  $\text{SO}(3, \mathbb{R})$  is either cyclic, dihedral, or  $\text{Aut}^+(P)$  for some Platonic solid  $P$ . Therefore, the Platonic solids not only classify the regular polyhedrons in  $\mathbb{R}^3$ , but also provide a classification of finite subgroups of the rotation groups in dimension 3.

**The tetrahedron  $T$ :** We proved in class that

$$\text{Aut}(T) \cong S_4 \quad \text{and} \quad \text{Aut}^+(T) \cong A_4$$

by considering the action on the set of 4 vertices.

**The cube  $C$ :** We proved in class that  $\text{Aut}^+(C) \cong S_4$  by considering the action on the set of 4 main diagonals. Moreover, we proved that the group homomorphism

$$F: \text{Aut}(C) \rightarrow S_4$$

induced by the group action has kernel given by

$$\text{Ker}(F) = \{\text{id}, J\}$$

where  $J: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is the antipodal map that maps  $(x, y, z) \mapsto (-x, -y, -z)$ . Therefore, we obtain that

$$|\text{Aut}(C)| = |\text{Ker}(F)| \cdot |S_4| = 48.$$

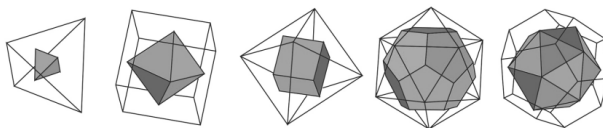
In the homework, you will show that there is an isomorphism

$$\text{Aut}(C) \cong \text{Aut}^+(C) \times (\mathbb{Z}/2\mathbb{Z}).$$

**The dodecahedron  $D$ :** Determining the group structures of  $\text{Aut}(D)$  and  $\text{Aut}^+(D)$  will be one of the projects.

*Remark 4.4.* Every polyhedron has a *dual polyhedron* with faces and vertices interchanged. One can construct the dual polyhedron by taking the vertices of the dual to be the centers of the faces of the original figure. Connecting the

centers of adjacent faces in the original forms the edges of the dual and thereby interchanges the number of faces and vertices while maintaining the number of edges. The dual of every Platonic solid is another Platonic solid, so we can arrange the five solids into dual pairs (where the tetrahedron is self-dual).



*Exercise.* The symmetry group of any polyhedron coincides with the symmetry group of its dual. (This is not hard to show by examining the construction of the dual polyhedron.) Therefore, there are only three symmetry groups associated with the Platonic solids rather than five.

#### 4.3. Finite subgroups of the rotation group $\text{SO}(3, \mathbb{R})$ .

**Theorem 4.5.** *Let  $G$  be a finite subgroup of  $\text{SO}(3, \mathbb{R})$ . Then  $G$  is isomorphic to precisely one of the following groups:*

- cyclic group  $\mathbb{Z}/n\mathbb{Z}$ ,
- dihedral group  $D_n$ ,
- $A_4$ : the rotational symmetry group of a tetrahedron,
- $S_4$ : the rotational symmetry group of a cube (or a octahedron),
- $A_5$ : the rotational symmetry group of a dodecahedron (or a icosahedron).