HOMEWORK 5
MATH H54, FALL 2021

Part I (Due October 12, 11am)

Some ground rules:

- Please submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW5**.
- The submission should be a **single PDF file**.
- Late homework will not be accepted/graded under any circumstances.
- Make sure the writing in your submission is clear enough. Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own, and acknowledge the students with whom you worked.
- **For True/False questions**: You have to prove the statement if your answer is "True"; otherwise, you have to provide an explicit counterexample and justification.
- You are allowed to use any result that is proved in the lecture. But if you would like to use other results, you have to prove it first before using it.

Problems:

(1) Let A be a square matrix. Suppose that there exists a positive integer k such that $A^k = 0$. Prove that the only eigenvalue of A is 0.

(2) Prove that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

(3) Let A be an $n \times n$ matrix. Suppose that the sum of entries in each row of A equals to 1 (i.e. $a_{i1} + a_{i2} + \cdots + a_{in} = 1$ for any $1 \leq i \leq n$). Prove that 1 is an eigenvalue of A. (Hint: Find an eigenvector.) (Remark: These matrices naturally arise in the study of Markov chains.)

(4) Let A be an $n \times n$ matrix (not necessarily diagonalizable) with eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly with multiplicities).
  (a) Prove that $\det(A) = \lambda_1 \cdot \cdots \cdot \lambda_n$.
  (b) Prove that $\operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n$. (Recall that $\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.)
  (Hint: Consider the characteristic polynomial of A.)

(5) Prove that if two square matrices A and B are similar, then $\operatorname{rank}(A) = \operatorname{rank}(B)$.

(6) Suppose that A is a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly with multiplicities). Prove that

$$(\lambda_1 \mathbb{I} - A)(\lambda_2 \mathbb{I} - A) \cdots (\lambda_n \mathbb{I} - A) = 0 \text{ (the zero matrix)}.$$

(In fact, the statement holds true without the diagonalizable assumption; this is the Cayley–Hamilton theorem.)

(7) Suppose that A and B are two $n \times n$ matrices that commute, i.e. $AB = BA$. Also, suppose that B has n distinct eigenvalues.
   (a) Prove that if $B\vec{v} = \lambda\vec{v}$, then $BA\vec{v} = \lambda A\vec{v}$.
   (b) Prove that any eigenvector of B is also an eigenvector of A.
   (c) Prove that A, B, AB are all diagonalizable.

(8) (a) Let A be a real $n \times n$ matrix. Let $\{\lambda_1, \dots, \lambda_k\}$ be the set of distinct eigenvalues of A^2. Assume that λ_i is real and positive for each $1 \le i \le k$. Let $\mathcal{S}$ be the set of distinct eigenvalues of A.
      (i) Prove that $\mathcal{S}$ is a subset of $\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, \dots, \sqrt{\lambda_k}, -\sqrt{\lambda_k}\}$.
      (ii) Prove that for each $1 \le i \le k$, $\mathcal{S}$ contains at least one of $\pm\sqrt{\lambda_i}$. (Hint: Consider $(A^2 - \lambda\mathbb{I}) = (A - \sqrt{\lambda}\mathbb{I})(A + \sqrt{\lambda}\mathbb{I})$.)
   (b) Find a 3×3 matrix A such that
$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 9 \end{bmatrix}.$$
   (It's enough to write $A = PDP^{-1}$ with explicit P and D.)
   How many such matrices are there? Justify your answer.

(9) Let P be a square matrix such that $P^2 = P$.
   (a) Prove that $\mathrm{Col}(P) = \mathrm{Nul}(P - \mathbb{I})$.
   (b) Prove that P is diagonalizable. (Hint: rank-nullity theorem.)

(10) Let A be an $n \times n$ *real* symmetric matrix, i.e. $A^T = A$.
   (a) Let $\mathbf{x}$ be any vector in $\mathbb{C}^n$. Denote its conjugate by $\bar{\mathbf{x}} \in \mathbb{C}^n$. Prove that $\bar{\mathbf{x}}^T A\mathbf{x}$ is a *real* number. (Hint: Prove that $\bar{\mathbf{x}}^T A\mathbf{x} = \overline{\bar{\mathbf{x}}^T A\mathbf{x}}$.)
   (b) Prove that the eigenvalues of any real symmetric matrix are real, i.e. if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x} \in \mathbb{C}^n$, then $\lambda \in \mathbb{R}$. (Hint: Compute $\bar{\mathbf{x}}^T A\mathbf{x}$ and use Part (a).)
   (c) Same notations as is Part (b). Prove that the real and imaginary parts of $\mathbf{x}$ satisfy $A(\mathrm{Re}\mathbf{x}) = \lambda(\mathrm{Re}\mathbf{x})$ and $A(\mathrm{Im}\mathbf{x}) = \lambda(\mathrm{Im}\mathbf{x})$.

Part II (Extra credit problems, Due October 19, 11am)

The following problem worths up to 2 extra points in total (out of 10), so you can potentially get 12/10 for this homework.

You have to submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW5_extra_credit**, which is separated from Part I.

(1) Let $T\colon V \to V$ be a linear transformation of a finite dimensional vector space V. Consider the series of subspaces of V
$$\{0\} \subseteq \ker(T) \subseteq \ker(T^2) \subseteq \cdots \quad \text{and} \quad V \supseteq \mathrm{Im}(T) \supseteq \mathrm{Im}(T^2) \supseteq \cdots.$$
   (a) Prove that there exists $k > 0$ sufficiently large such that
$$\ker(T^k) = \ker(T^{k+1}) = \cdots \quad \text{and} \quad \mathrm{Im}(T^k) = \mathrm{Im}(T^{k+1}) = \cdots.$$

We'll call them the *stable kernel* and the *stable image* of T and denoted by

$$\ker^s(T) \coloneqq \ker(T^k) \quad \text{and} \quad \mathrm{Im}^s(T) \coloneqq \mathrm{Im}(T^k).$$

(b) Prove the following equivalent descriptions of stable kernel and stable image:

$$\ker^s(T) = \{\vec{v} \in V : T^\ell \vec{v} = \vec{0} \ \text{ for some } \ell\},$$

$$\mathrm{Im}^s(T) = \{\vec{v} \in V : \text{for any } \ell, \text{ there exists } \vec{w} \in V \text{ s.t. } \vec{v} = T^\ell(\vec{w})\}.$$

(c) Prove that $\ker^s(T) \cap \mathrm{Im}^s(T) = \{0\}$.

(d) Prove that $V = \ker^s(T) + \mathrm{Im}^s(T)$.

(e) The statement without stabilization "$V = \ker(T) + \mathrm{Im}(T)$" is not true in general. Find a counterexample.

(2) Same notations as previous problem.

(a) Prove that $T(\ker^s(T)) \subseteq \ker^s(T)$ and $T(\mathrm{Im}^s(T)) \subseteq \mathrm{Im}^s(T)$. (In other words, the linear transformation T respects the decomposition $V = \ker^s(T) \oplus \mathrm{Im}^s(T)$.) Hence one can define the restriction linear maps $T|_{\ker^s(T)} \colon \ker^s(T) \to \ker^s(T)$ and $T|_{\mathrm{Im}^s(T)} \colon \mathrm{Im}^s(T) \to \mathrm{Im}^s(T)$.

(b) Prove that there exists $L > 0$ such that $(T|_{\ker^s(T)})^L = 0$.

(c) Prove that $T|_{\mathrm{Im}^s(T)}$ is invertible.