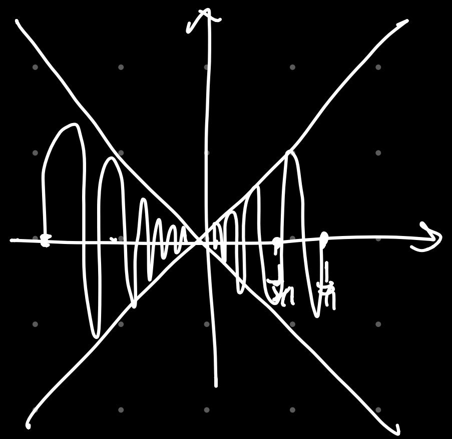
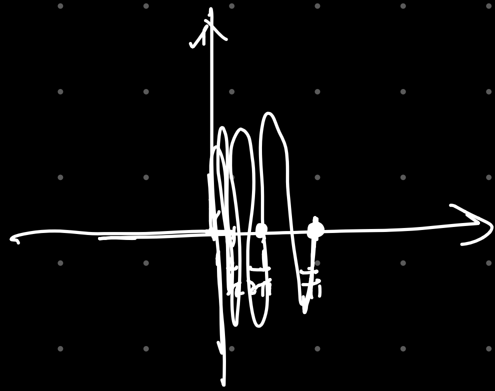
$f(x) = \begin{cases} x\sin(\frac{1}{x}), & x \neq 0 \\ 0 \end{cases}$ $\qquad$ $g(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ 0 \end{cases}$





#5: $[a,b] \subseteq \mathbb{R}$ is not of measure zero.

~~Suppose $\exists$~~

finite / countable

Def: Say $\{U_1, U_2, \dots\}$ open intervals is a "bad" covering of $[a,b]$ if its total length is less than $b-a$.

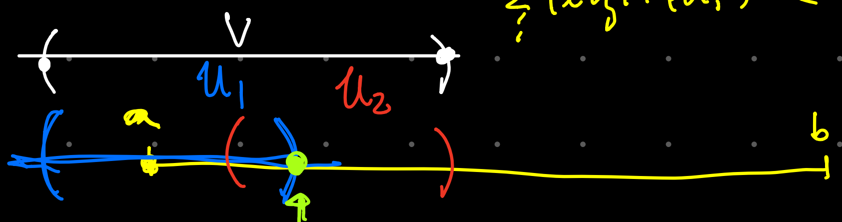
Suppose $\exists$ bad covering of $[a,b]$.

$\rightarrow$ $[a,b]$ cpt $\rightarrow$ $\exists$ finite subcover.

$\rightarrow$ still a bad covering.

$\rightarrow$ $\{U_1, \dots, U_n\}$ is a bad covering of $[a,b]$

open intervals $\quad \sum_i \text{length}(U_i) < b-a$.



$\text{Length}(V) < \text{Length}(U_1) + \text{Length}(U_2)$

Replace U_1, U_2 by V

$\{U_1, U_2, U_3, \cdots, U_n\} \rightsquigarrow \{V, U_3, U_4, \ldots, U_n\}$

n open intervals $\qquad$ $n-1$ open intervals.

#4. $B(X) = \{f : X \to \mathbb{R}, \text{ bdd.}\}$, cpt.

$$d_B(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$$

(a) $\boxed{\{f_n\} \subset B(X) \text{ Cauchy.}}$ $\qquad$ WTS: $\exists f \in B(X)$ s.t. $f_n \to f$ in $(B(X), d_B)$

$\forall \varepsilon > 0, \exists N > 0$

s.t. $\boxed{d_B(f_n, f_m) < \varepsilon}$ $\forall n, m > N$

$\sup_{x \in \mathbb{R}} |f_n(x) - f_m(x)|$

$\Leftrightarrow |f_n(x) - f_m(x)| < \varepsilon$ $\forall n, m > N$, $\forall x \in X$.

(unif. Cauchy)

$\exists f : X \to \mathbb{R}$ bdd

s.t. $\forall \varepsilon > 0, \exists N > 0$

s.t. $\boxed{d_B(f_n, f) < \varepsilon}$ $\forall n > N$

$|f_n(x) - f(x)| < \varepsilon$ $\forall n > N, \forall x \in X$.

$f_n \to f$ unif. (unif conv.)

(b) $\boxed{C(X)} = \{f : X \to \mathbb{R} \text{ conti.}\} \subseteq B(X)$, cpt. $\qquad$ closed subset.

i.e. $\{f_n\} \subseteq C(X)$, $\boxed{f_n \to f \text{ in } (B(X), d_B)}$

then $f \in C(X)$

(c) $A \subseteq B$

closed

complete metric space

(i.e. any Cauchy seq. conv. to some elt. in B)

$(a_n) \subseteq A$ Cauchy seq. WTS: $a_n \to a \in A$

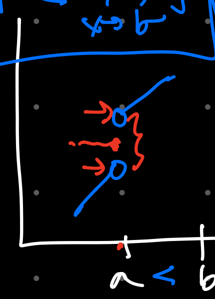
$\Downarrow$

$a_n \to a \in B$ $\Rightarrow$ $a \in A$.

b/c B is complete.

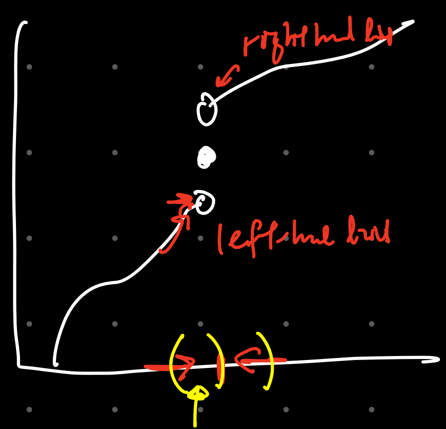
Ex: $f: [0,1] \to \mathbb{R}$ increasing

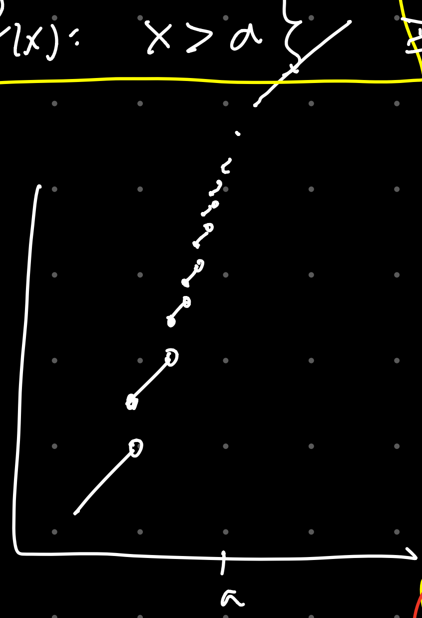
$\lim_{x \to a+} f \le \lim_{x \to b-} f$



(a) $\lim_{x \to a^-} f(x) = \sup \{ f(x) : x < a \} \le f(a)$

$\lim_{x \to a^+} f(x) = \inf \{ f(x) : x > a \} \ge f(a)$





(b)

$A = \{ x \in [0,1] \mid \lim_{x \to a^-} f < \lim_{x \to a^+} f \}$

$\forall x \in A$, Choose $g(x) \in \mathbb{Q}$ to be any rat'l btw $\lim_{x \to a^-}, \lim_{x \to a^+}$

$g: A \to \mathbb{Q}$ injective

general defn

Def (left-hand limit) Say the left hand limit of f at the point a exists if $\exists$ $L \in \mathbb{R}$

$\forall \varepsilon > 0$, $\exists \delta > 0$

st. $-\delta < x - a < 0 \Rightarrow |f(x) - L| < \varepsilon$.

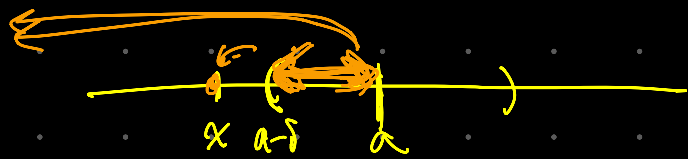
In this case, we'll denote "$\lim_{x \to a^-} f(x) = L$".

Fact: f conti. at a $\iff$ $\lim_{x\to a^-} f(x)$ & $\lim_{x\to a^+} f(x)$ exist and they both $= f(a)$.

Claim: $\lim_{x\to a^-} f(x) = \boxed{\sup\{f(x) : x < a\}} = L$

WTS: $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $-\delta < x - a < 0 \Rightarrow 0 \leq L - f(x) < \varepsilon$

$L = \sup\{\boxed{f(x) : x < a}\}$



$\Rightarrow \forall \varepsilon > 0,\ \exists x < a$ s.t.

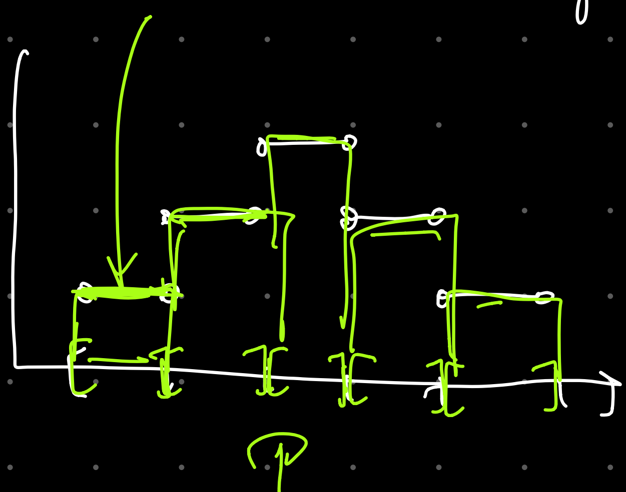
$$L - \varepsilon < f(x) \leq L$$

#2: $\forall \varepsilon > 0,\ \exists P$ partition of $[a,b]$

first $\rightarrow$ s.t. $0 \leq \int_a^b f(x)\,dx - L(f,P) < \varepsilon$

$L(f,P) = \sum (t_k - t_{k-1}) \inf\{f(x) : x \in [t_{k-1}, t_k]\}$

think of this as an integral of a step fcn



① Statement is true for step fns.

To prove it for step fns, it's enough to prove it for const. fns.

$$\boxed{\lim_{n\to\infty} \int_a^b \sin(nx)\, dx} = 0$$

$$-\frac{\cos(nx)}{n}\Big|_a^b = \frac{-\cos(nb)+\cos(na)}{n}$$

always bdd by 2.

② general case, f: integrable fn.

#2: $\forall \varepsilon>0$, $\exists P$ partition of $[a,b]$

first ⇒ s.t. $0 \le \int_a^b f(x)\,dx - \underbrace{L(f,P)}_{= \int_a^b s(x)\,dx} < \varepsilon$

certain step fn

$$\int_a^b f(x)\sin(nx)\,dx - \int_a^b s(x)\sin(nx)\,dx$$

→ 0 as $n\to\infty$

$$= \int_a^b (f(x)-s(x))\sin(nx)\,dx$$

$$\left|\int_a^b (f(x)-s(x))\sin(nx)\,dx\right| \le \int_a^b (f(x)-s(x))|\sin(nx)|\,dx < \varepsilon$$

$$\left| \int_a^b f(x) \sin(nx)\,dx - \int_a^b S(x) \sin(nx)\,dx \right| < \varepsilon \quad \forall n.$$

$\forall \varepsilon > 0$, $\exists$ step fcn $S(x)$ st. $0 \le \int_a^b f(x) - S(x)\,dx < \varepsilon$

$$\Rightarrow \left| \int_a^b f(x) \sin(nx)\,dx - \int_a^b S(x) \sin(nx)\,dx \right| < \varepsilon \quad \forall n.$$

$$\Rightarrow \int_a^b S(x) \sin(nx)\,dx - \varepsilon < \int_a^b f(x) \sin(nx)\,dx < \int_a^b S(x) \sin(nx)\,dx + \varepsilon$$

$$\Rightarrow \boxed{-\varepsilon \le \liminf_{n\to\infty} \int_a^b f(x) \sin(nx)\,dx \le \limsup_{n\to\infty} \int_a^b f(x) \sin(nx)\,dx \le \varepsilon}$$

$\forall \varepsilon > 0.$

$$\Rightarrow \liminf \int_a^b f(x) \sin(nx)\,dx = \limsup \int_a^b f(x) \sin(nx)\,dx = 0$$

$$\Rightarrow \lim \int_a^b f(x) \sin(nx)\,dx = 0.$$

$\bigcup U_i$ is open

↑

open

$\bigcap C_i$ is closed

$\Updownarrow$

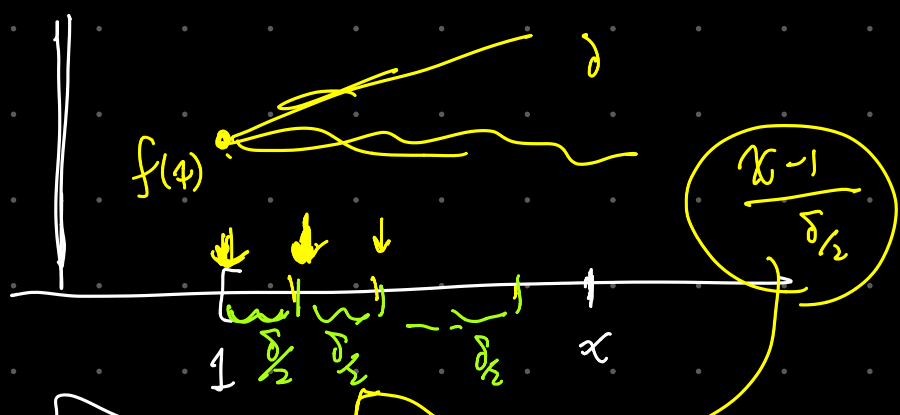
$\boxed{\left(\bigcap C_i\right)^c \text{ is open}}$

$\|$

$\bigcup \left(C_i^c\right)$

#6:

$|f(x)| \tilde{<} |f(1)| + \dfrac{x-1}{\delta/2}$



$\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $|x-y| < \delta \Rightarrow |f(x)-f(y)| < 1$

$\boxed{2, 4, 5, 6, 8}$

Continue on Sunday!!