(8) Let A be an $n \times n$ diagonalizable matrix with $n-1$ distinct eigenvalues. Prove ~~that~~ for any $\vec{v} \in \mathbb{R}^n$, the set $\{\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}\}$ is linearly dependent.

$\underline{PDP^{-1}}$

e.g.

$$A = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & 0 & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & 0 & & & \lambda_{n-1} \end{bmatrix}$$

not invertible

$$\begin{bmatrix} v_1 & \lambda_1 v_1 & \cdots & \lambda_1^{n-1} v_1 \\ \vdots & \vdots & & \\ v_{n-1} & \lambda_{n-1} v_{n-1} & \cdots & \lambda_{n-1}^{n-1} v_{n-1} \\ v_n & \lambda_{n-1} v_n & \cdots & \lambda_{n-1}^{n-1} v_n \end{bmatrix}$$

dependent

$$\{\vec{v}, \quad A\vec{v}, \quad \ldots, \quad A^{n-1}\vec{v}\}$$

$$c_0 \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{bmatrix} + c_1 \begin{bmatrix} \lambda_1 v_1 \\ \vdots \\ \lambda_{n-1} v_{n-1} \\ \lambda_{n-1} v_n \end{bmatrix} + \cdots + c_{n-1} \begin{bmatrix} \lambda_1^{n-1} v_1 \\ \vdots \\ \lambda_{n-1}^{n-1} v_{n-1} \\ \lambda_{n-1}^{n-1} v_n \end{bmatrix} = \vec{0}$$

$$\rightarrow \begin{cases} c_0 v_1 + c_1 \lambda_1 v_1 + \cdots + c_{n-1} \lambda_1^{n-1} v_1 = 0 \\ c_0 v_2 + c_1 \lambda_2 v_2 + \cdots + c_{n-1} \lambda_2^{n-1} v_2 = 0 \\ \vdots \\ c_0 v_{n-1} + c_1 \lambda_{n-1} v_{n-1} + \cdots + c_{n-1} \lambda_{n-1}^{n-1} v_{n-1} = 0 \\ c_0 v_n + c_1 \lambda_{n-1} v_n + \cdots + c_{n-1} \lambda_n^{n-1} v_n = 0 \end{cases}$$

We've proved that for a diagonal matrix D w/ $n-1$ distinct diagonal entries, the set $\{\vec{v}, D\vec{v}, \ldots, D^{n-1}\vec{v}\}$ is l.d.

$A = PDP^{-1}$, D has $n-1$ distinct diagonal entries

$$\boxed{\{\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}\} \text{ is l.d. ??}}$$

$$PP^{-1}\vec{v} \qquad PDP^{-1}\vec{v} \qquad PD^{n-1}P^{-1}\vec{v}$$

P is invertible.

$$\Longleftrightarrow \boxed{\{\underbrace{P^{-1}\vec{v}}_{\vec{w}}, D\underbrace{P^{-1}\vec{v}}_{\vec{w}}, \ldots, D^{n-1}\underbrace{P^{-1}\vec{v}}_{\vec{w}}\} \text{ is l.d.}}$$

Another proof:

$$\vec{v} = \vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_{n-1},$$

where $A\vec{v}_i = \lambda_i \vec{v}_i$, and $\{\lambda_1, \ldots, \lambda_{n-1}\}$ are the distinct eigenvalues of A.

$$\{\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}\} \text{ l.d.}$$

$\vec{v} = \vec{v}_1 + \cdots + \vec{v}_{n-1}$, $A\vec{v} = \lambda_1\vec{v}_1 + \cdots + \lambda_{n-1}\vec{v}_{n-1}$, $A^{n-1}\vec{v} = \lambda_1^{n-1}\vec{v}_1 + \cdots + \lambda_{n-1}^{n-1}\vec{v}_{n-1}$

not invertible.

$$\underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{n-1} \\ | & & | \end{bmatrix}}_{n \times (n-1)} \underbrace{\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_{n-1} & \cdots & \lambda_{n-1}^{n-1} \end{bmatrix}}_{(n-1) \times n} = \underbrace{\begin{bmatrix} | & | & & | \\ \vec{v} & A\vec{v} & \cdots & A^{n-1}\vec{v} \\ | & | & & | \end{bmatrix}}_{n \times n}$$

Yet another proof:

$$\{\vec{w}_1, \ldots, \vec{w}_n\} \text{ eigenbasis}$$

with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_{n-1}$.

$$\boxed{\vec{v} = a_1\vec{w}_1 + \cdots + a_n\vec{w}_n}$$

$$A\vec{v} = a_1\lambda_1\vec{w}_1 + \cdots + a_{n-1}\lambda_{n-1}\vec{w}_{n-1} + a_n\lambda_{n-1}\vec{w}_n$$

$$\vdots$$

$$A^{n-1}\vec{v} = a_1\lambda_1^{n-1}\vec{w}_1 + \cdots + a_{n-1}\lambda_{n-1}^{n-1}\vec{w}_{n-1} + a_n\lambda_{n-1}^{n-1}\vec{w}_n$$

$$\begin{bmatrix} | & & | \\ \vec{w}_1 & \cdots & \vec{w}_n \\ | & & | \end{bmatrix} \begin{bmatrix} a_1 & a_1\lambda_1 & \cdots & a_1\lambda_1^{n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-1} & a_{n-1}\lambda_{n-1} & \cdots & a_{n-1}\lambda_{n-1}^{n-1} \\ a_n & a_n\lambda_{n-1} & \cdots & a_n\lambda_{n-1}^{n-1} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{v} & A\vec{v} & \cdots & A^{n-1}\vec{v} \\ | & | & & | \end{bmatrix}$$

(7) Let $A_1, \ldots, A_k$ be $n \times n$ real symmetric matrices. Suppose that $A_1^2 + \cdots + A_k^2 = 0$ (the zero matrix). Prove that $A_1 = \cdots = A_k = 0$ (the zero matrix).

$$A_1 = \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$$ real symmetric

$$A_1^2 = A_1 A_1^T = \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\underline{\langle v_1, v_1 \rangle}} \geq 0 & \langle v_1, v_2 \rangle & \cdots \\ \langle v_2, v_1 \rangle & \underline{\underline{\langle v_2, v_2 \rangle}} \geq 0 & \\ \vdots & \vdots & \\ \langle v_n, v_1 \rangle & & \underline{\underline{}} \geq 0 \end{bmatrix}$$

$$A_1^2 + \cdots + A_k^2 = 0 \qquad \square$$

Another proof: $\quad 0 = \vec{x}^T (A_1^2 + \cdots + A_k^2) \vec{x}$

$$\| \quad \vec{x}^T A_1^2 \vec{x} + \cdots + \vec{x}^T A_k^2 \vec{x}$$

$$\| \quad \vec{x}^T A_1^T A_1 \vec{x} + \cdots + \vec{x}^T A_k^T A_k \vec{x} \qquad \forall \vec{x}$$

$$\| \quad \|A_1 \vec{x}\|^2 + \cdots + \|A_k \vec{x}\|^2$$

$$\Rightarrow \quad A_1 \vec{x} = \cdots = A_k \vec{x} = \vec{0} \quad \forall \vec{x}.$$

$$\Rightarrow \quad A_1 = \cdots = A_k = 0.$$

(6) Let $(V, \langle -, - \rangle)$ be an inner product space, and let $T\colon V \to V$ be a linear transformation. Suppose that $||T(\vec{x})|| = ||\vec{x}||$ for any $\vec{x} \in V$. Prove that

$$\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle \quad \text{for any } \vec{x}, \vec{y} \in V.$$

$$\|T(x+y)\|^2 = \|x+y\|^2$$

$$\|T(x+y)\|^2 = \langle T(x+y), T(x+y) \rangle = \cancel{\|T(x)\|^2} + \cancel{\|T(y)\|^2} + 2\langle T(x), T(y) \rangle$$

$$\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x,x \rangle + \langle y,y \rangle + 2\langle x,y \rangle = \cancel{\|x\|^2} + \cancel{\|y\|^2} + 2\langle x,y \rangle$$

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(5) Let A and B be two square matrices.

(a) Suppose that $\lambda \neq 0$ is an eigenvalue of AB. Prove that λ is also an eigenvalue of BA.

(b) Does the same statement hold for $\lambda = 0$?

(a)

$\exists\, \vec{v} \neq \vec{0}$ s.t. $AB\vec{v} = \lambda \vec{v}$.

$\Rightarrow \quad BA(B\vec{v}) = \lambda(B\vec{v})$

$\overset{??}{\Rightarrow} \quad \lambda$ is an eigenvalue of BA

We need to show $B\vec{v} \neq \vec{0}$

(b) "0 is an eigenvalue of AB" $\overset{??}{\Longleftrightarrow}$ "0 is an eigenvalue of BA".

0 is an eigenvalue of A $\Longleftrightarrow$ $\det(A - 0I) = 0$

$\Updownarrow$

$\det(A) = 0$.

$\det(AB) = \det(A)\det(B) = \det(BA)$

(4) Let W_1 and W_2 be two subspaces of a finite dimensional inner product space V.
(a) Prove that $W_1^\perp \cap W_2^\perp = (W_1 + W_2)^\perp$.
(b) Prove that $\dim(W_1) - \dim(W_1 \cap W_2) = \dim(W_2^\perp) - \dim(W_1^\perp \cap W_2^\perp)$.

(a) $\vec{v} \in W_1^\perp \cap W_2^\perp \iff \vec{v} \in W_1^\perp$ and $\vec{v} \in W_2^\perp$

$$\iff \boxed{\langle \vec{v}, \vec{w} \rangle = 0 \quad \forall\, \vec{w} \in W_1 \ \cancel{\text{and}} \text{ or } \vec{w} \in W_2.}$$

$$\vec{v} \in (W_1 + W_2)^\perp \iff \langle \vec{v}, \vec{y} \rangle = 0 \quad \forall\, \vec{y} \in W_1 + W_2$$

$$\iff \boxed{\langle \vec{v}, \vec{y}_1 + \vec{y}_2 \rangle = 0 \quad \forall\, \vec{y}_1 \in W_1,\ \vec{y}_2 \in W_2.}$$

(b)

$$\dim(W_1) + \dim(W_2) = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$$

$$\Rightarrow \quad \dim(W_1) - \dim(W_1 \cap W_2) = \dim(W_1 + W_2) - \dim(W_2)$$

$$= \left(\dim V - \dim(W_1 + W_2)^\perp\right) - \left(\dim(V) - \dim(W_2^\perp)\right)$$

$$= \dim W_2^\perp - \dim(W_1 + W_2)^\perp$$

$$\overset{(a)}{=} \dim W_2^\perp - \dim(W_1^\perp \cap W_2^\perp)$$

$M_{2\times 2}(\mathbb{R})$.

(3) Find all possible 5×5 real symmetric matrices A satisfying $A^3 - 2A = 4\mathbb{I}_5$.

$A = PDP^T$, P orthogonal, D diagonal real.
$P^T = P^{-1}$

$$PD^3P^T - 2PDP^T = 4\,\mathbb{I} = 4PP^T$$

$$\Rightarrow \boxed{P(D^3 - 2D - 4\mathbb{I})P^T = 0}$$

$\Rightarrow \quad D^3 - 2D - 4\mathbb{I} = 0$

$\Rightarrow$ $\forall$ eigenvalue λ of A, we have $\lambda^3 - 2\lambda - 4 = 0$

real

$$\lambda^3 - 2\lambda - 4 = (\lambda - 2)(\lambda^2 + 2\lambda + 2)$$

$\Rightarrow \quad \lambda = 2.$

$\Rightarrow$ 2 is the only eigenvalue of A.

$$\Rightarrow \quad A = P \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{bmatrix} P^T = 2\mathbb{I}. \quad \square$$

SECOND MIDTERM PRACTICE PROBLEMS
MATH H54, FALL 2021

(1) Consider the symmetric matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

eigenvalues $\rightarrow$ eigenspaces

Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

(2) Let $M_{2\times 2}(\mathbb{R})$ be the set of all real 2×2 matrices. It is naturally a vector space with the standard matrix addition and scalar multiplication. Consider the function $\langle -, - \rangle : M_{2\times 2}(\mathbb{R}) \times M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ given by

$$\langle A, B \rangle = \mathrm{tr}(AB^T).$$

(a) Show that $(M_{2\times 2}(\mathbb{R}), \langle -, - \rangle)$ is an inner product space.

(b) Construct an orthonormal basis (with respect to the inner product $\langle -, - \rangle$) of the subspace of $M_{2\times 2}(\mathbb{R})$ spanned by $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Gram-Schmidt.

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$\boxed{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \dfrac{\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}$

(c) Consider another function $\langle -, - \rangle_2 : M_{2\times 2}(\mathbb{R}) \times M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by $\langle A, B \rangle_2 = \mathrm{tr}(AB)$. Does $\langle -, - \rangle_2$ give an inner product on the vector space $M_{2\times 2}(\mathbb{R})$? No Find A s.t. $\mathrm{tr}(A^2) < 0$.

(3) Find all possible 5×5 real symmetric matrices A satisfying $A^3 - 2A = 4\mathbb{I}_5$.

(4) Let W_1 and W_2 be two subspaces of a finite dimensional inner product space V.

(a) Prove that $W_1^\perp \cap W_2^\perp = (W_1 + W_2)^\perp$.

(b) Prove that $\dim(W_1) - \dim(W_1 \cap W_2) = \dim(W_2^\perp) - \dim(W_1^\perp \cap W_2^\perp)$.

(5) Let A and B be two square matrices.

(a) Suppose that $\lambda \neq 0$ is an eigenvalue of AB. Prove that λ is also an eigenvalue of BA.

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$$\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle \quad \text{for any } \vec{x}, \vec{y} \in V.$$

(7) Let $A_1, \ldots, A_k$ be $n \times n$ real symmetric matrices. Suppose that $A_1^2 + \cdots + A_k^2 = 0$ (the zero matrix). Prove that $A_1 = \cdots = A_k = 0$ (the zero matrix).

(8) Let A be an $n \times n$ diagonalizable matrix with $n - 1$ distinct eigenvalues. Prove that for any $\vec{v} \in \mathbb{R}^n$, the set $\{\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}\}$ is linearly dependent.