HW7 sol'n

#1. $E \subset \mathbb{R}$ compact $\Rightarrow$ E is closed and bounded $\Rightarrow$ $\sup E, \inf E$ exists.

Claim: $z := \sup E \in E$ (the proof for $\inf E$ is similar).

pf. $\forall \varepsilon > 0$, $\exists\, x \in E$ s.t. $z - \varepsilon \le x \le z$.

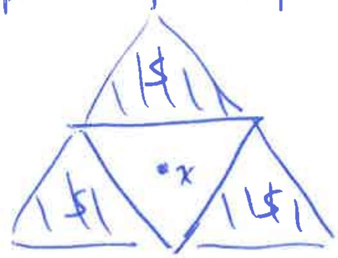
If $z \notin E$, then $z - \varepsilon < x < z$. $\Rightarrow$ $x \in B_\varepsilon(z)$

$\Rightarrow$ z is a limit of E.

Since E is closed, we have $E = \overline{E}$, hence $z \in E$. $\square$

#2. (a) It's clear that it's bounded.

For any point x in the complement of Sierpiński triangle S, it locally looks like:



It's clear that $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset S^c$.

Hence S^c is open, so S is closed.

In $\mathbb{R}^2$, closed and bounded $\Rightarrow$ compact. $\square$

(b) Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \in X$. Write $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^n$ the column vectors of A.

Then $\langle \vec{a}_i, \vec{a}_i \rangle = 1 \ \forall i$, $\langle \vec{a}_i, \vec{a}_j \rangle = 0 \ \forall i \ne j$.

$\Rightarrow$ $\sum_{j=1}^{n} |a_{ij}|^2 = 1 \ \forall i$, $\Rightarrow$ $|a_{ij}| \le 1 \ \forall i,j$.

$\Rightarrow$ X is bounded.

If $B \in M_n(\mathbb{R})$ and $B \notin X$, then either

- $\langle \vec{a}_i, \vec{a}_i \rangle \neq 1$ for some i, or
- $\langle \vec{a}_i, \vec{a}_j \rangle \neq 0$ for some i, j.

In either case, one can find $\varepsilon > 0$ s.t. $B_\varepsilon(B) \subset X^c$.

Hence X^c is open, so X is closed.

In $\mathbb{R}^{n^2}$, closed and bounded $\Rightarrow$ compact. $\square$

<u>#3</u>. <u>Claim</u>. E' is closed.

pf Let $y \in (E')^c$, i.e. y is not a limit pt of E.

Then $\exists\, r > 0$ s.t. $B_r(y) \cap E = \phi$ or $\{y\}$.

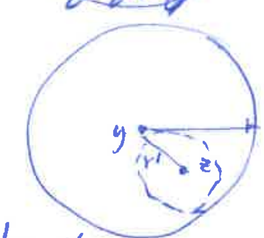
We claim that $B_r(y) \subset (E')^c$. (this implies $(E')^c$ is open),

i.e. $\forall z \in B_r(y)$, z is not a limit pt of E.

This is clear since

$$B_{\min\{r', r-r'\}}(z) \cap E = \phi,$$

where $r' = d(y,z) < r$. $\square$



<u>Claim</u>. If E has a limit pt, then E has infinitely many elements.

pf Let x be a limit pt of E, Then $\forall r > 0$, $B_r(x) \cap E$ contains at least one element other than x.

Say we start with $r_1 = 1$, $(B_1(x) \cap E) \overset{\exists}{\ni} x_1 \neq x$.

Let $r_2 = d(x, x_1)$, $(B_{r_2}(x) \cap E) \overset{\exists}{\ni} x_2 \neq x$, $x_2 \neq x_1$ since $d(x_2, x) < r_2$

Let $r_3 = d(x, x_2)$, $(B_{r_3}(x) \cap E) \overset{\exists}{\ni} x_3 \neq x$, $x_3 \neq x_1, x_2$ since $d(x_3, x) < r_3$.

Do this inductively. $\square$

#4 Claim $f(x)$ is continuous at $x=0$.

pf $\forall \varepsilon>0$, take $\delta=\varepsilon$, Then

$|f(x)| = |x \sin \frac{1}{x}| \leq |x| < \varepsilon$ for any $0<|x|<\delta=\varepsilon$. $\square$

Claim: $g(x)$ is not continuous at $x=0$

pf Consider $x_n = \dfrac{1}{2n\pi + \frac{\pi}{2}}$.

$(x_n) \longrightarrow 0$ but $(g(x_n))$ doesn't conv. to $g(0)=0$. $\square$

#5 (a) Take $\delta = \min\{\frac{1}{2}, \frac{1}{3}\varepsilon\} > 0$

Then $\forall\ |x-x_0| = |x-1| < \delta$, we have

$$|f(x)-f(x_0)| = \left|\frac{1}{x} - 1\right| = \frac{|x-1|}{|x|} \leq \frac{\delta}{\frac{1}{2}} < \varepsilon \qquad \square$$

since $|x-1| < \delta \leq \frac{1}{2}$, so $|x| \geq \frac{1}{2}$

(b) Take $\delta = \varepsilon^2 > 0$.

Then $\forall\ |x-x_0| = |x| < \delta$, we have

$$|f(x)-f(x_0)| = \sqrt{|x|} < \sqrt{\delta} = \varepsilon. \qquad \square$$

(c) Take $\delta = \min\{\frac{1}{2}, \varepsilon\} > 0$

Then $\forall\ |x-x_0| = |x-1| < \delta$, we have

$$|f(x)-f(1)| = |\sqrt{x} - 1| = \frac{|x-1|}{\sqrt{x}+1} \leq \frac{\delta}{\sqrt{\frac{1}{2}}+1} < \varepsilon. \qquad \square$$

#6 $f, g: X \longrightarrow \mathbb{R}$ continuous, WTS: $f+g: X \longrightarrow \mathbb{R}$ also continuous.

$\forall x_0 \in X$, $\forall (x_n) \subset X$ converging to x_0, WTS: $(f+g)(x_n)$ conv. to $(f+g)(x_0)$.

Since f is conti., we have $\lim_{n\to\infty} f(x_n) = f(x_0)$

— g — — — — — $\lim_{n\to\infty} g(x_n) = g(x_0)$.

By limit thm, we have $\lim_{n\to\infty} \left(f(x_n) + g(x_n)\right) = f(x_0) + g(x_0)$. □

#7. $\forall (x_n) \subset X$ converging to x_0,

Since f is conti. at x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Since g is conti. at $f(x_0)$, we have $\lim_{n\to\infty} g(f(x_n)) = g(f(x_0))$. □

#8 By #6, and the fact that $f(x) = x$ is conti., one can show that any polynomial fcn is continuous.

Let $f(x) = x^{2n+1} + a_{2n} x^{2n} + \cdots + a_0$ be an odd degree poly.

(without loss of generality, we can assume the leading coefficient is 1)

Let $R > \max\{|a_{2n}|, |a_{2n-1}|, \dots, |a_0|, 1\} \cdot (4n)$

Then
$$
\begin{aligned}
f(R) &= R^{2n+1} + a_{2n} R^{2n} + \cdots + a_0 \\
&> R^{2n+1} - \left(\frac{R}{4n} R^{2n} + \frac{R}{4n} R^{2n-1} + \cdots + \frac{R}{4n}\right) \\
&> R^{2n+1} - \frac{2n+1}{4n} R^{2n+1} > 0.
\end{aligned}
$$

Similarly $f(-R) < 0$.

By Intermediate Value thm, f has at least 1 real root. □

#9 Define $h(x) = f(x) - g(x)$. continuous fcn on $[a,b]$,

$h(a) < 0$, $h(b) > 0$.

Intermediate Value Thm $\Rightarrow$ $\exists\, c \in (a,b)$ s.t. $h(c) = 0$

$\Downarrow$

$f(c) = g(c)$. $\square$

#10 (a) $\forall x \in \mathbb{R}$, $\exists\, (x_n) \subset \mathbb{Q}$ s.t. $\lim_{n\to\infty} x_n = x$.

$\Rightarrow$ $\underbrace{\lim_{n\to\infty} f(x_n)}_{= 0} = f(x)$ since f is conti. on $\mathbb{R}$.

$\Rightarrow$ $f(x) = 0$ $\forall x \in \mathbb{R}$.

(b) Plug in $x = y = 0$ $\Rightarrow$ $f(0) = 2f(0) \Rightarrow f(0) = 0$.

Define $g(x) = f(x) - f(1)x$. continuous on $\mathbb{R}$

Then $g(1) = 0$.

nontrivial steps involved.

- Plug in x, y integers, you can show that $g(x) = 0$ $\forall x \in \mathbb{Z}$.
- Plug in x, y rational numbers, you can show that $g(x) = 0$ $\forall x \in \mathbb{Q}$

By Part (a), $g(x) = 0$ $\forall x \in \mathbb{R}$

$\Rightarrow$ $f(x) = f(1) \cdot x$ $\forall x \in \mathbb{R}$. $\square$