Math 104 (Section 4) First Midterm October 1, 2019

Name: Solution

- You have 70 minutes to complete the exam.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	25	
2	20	
3	25	
4	30	
Total	100	

1. (a) (5 points) Write down the definition of a sequence (a_n) converging to a real number a.

$\forall \varepsilon > 0$, $\exists N > 0$ s.t.

$|a_n - a| < \varepsilon \quad \forall n > N$.

(b) (20 points) Prove the following statement based on the definition: Let (a_n) be a sequence converging to a and (b_n) be a sequence converging to b. Suppose that $a_n \leq b_n$ for every $n \in \mathbb{N}$. Then $a \leq b$. (Hint: Prove by contradiction.)

Suppose by contradiction that $b < a$.

Let $\varepsilon = \frac{a-b}{2}$.

$\exists N_1 > 0$ s.t. $|a_n - a| < \varepsilon \quad \forall n > N_1 \Rightarrow a_n > a - \varepsilon \ \forall n > N_1$

$\exists N_2 > 0$ s.t. $|b_n - b| < \varepsilon \quad \forall n > N_2. \Rightarrow b_n < b + \varepsilon \ \forall n > N_2$

Then for any $n > \max\{N_1, N_2\}$, we have

$a_n - b_n > (a-\varepsilon) - (b+\varepsilon) = (a-b) - 2\varepsilon = 0$

This contradicts with $a_n \leq b_n \ \forall n \in \mathbb{N}$.

Hence $a \leq b$. $\square$

2. (20 points) Let

$$a_n = \frac{n^2+1}{3n^2+5n}.$$

Prove that (a_n) converges to a real number based on the definition.

We prove that $\lim_{n\to\infty} a_n = \frac{1}{3}$.

$\forall \varepsilon > 0$. Choose $N = \frac{5}{9\varepsilon}$. Then $\forall n > N$, we have

$$\left| \frac{n^2+1}{3n^2+5n} - \frac{1}{3} \right| = \frac{5n-3}{3(3n^2+5n)} < \frac{5n}{9n^2} = \frac{5}{9n} < \frac{5}{9N} = \varepsilon. \quad \square$$

3. Let $a_1 = 3$ and $a_{n+1} = \sqrt{3a_n + 10}$ for $n \geq 1$.

(a) (20 points) Prove that (a_n) converges to a real number and find the limit.

Claim: $a_n < a_{n+1} < 5 \quad \forall n \geq 1$.

Prove by induction, For $n=1$, we have $a_1 = 3 < a_2 = \sqrt{19} < 5$.

Suppose $a_n < a_{n+1} < 5$. Then $a_{n+1} = \sqrt{3a_n+10} < \sqrt{3a_{n+1}+10} = a_{n+2}$,

and $a_{n+2} = \sqrt{3a_{n+1}+10} < \sqrt{3 \cdot 5 + 10} = 5$. This proves the claim.

Hence (a_n) is an increasing bounded sequence. Therefore $\lim a_n$ exists.

Let $A = \lim_{n\to\infty} a_n$.

We have $a_{n+1}^2 = 3a_n + 10 \quad \forall n$.

$\Rightarrow A^2 = (\lim_{n\to\infty} a_{n+1})^2 = \lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} (3a_n + 10) = 3A + 10$

$\Rightarrow A = 5$ or -2.

$\lim_{n\to\infty} a_n \neq -2$ since $a_n > 0 \quad \forall n$.

Hence the limit $\lim_{n\to\infty} a_n = \boxed{5}$.

(b) (5 points) Find $\liminf_{n\to\infty} a_n$. Give a brief reason for your answer.

$\liminf_{n\to\infty} a_n = \boxed{5}$

Since (a_n) converges, so $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

4. There are four statements below:

(I) Let (a_n) be a sequence such that $|a_{n+1} - a_n| < 2^{-n}$ for every $n \in \mathbb{N}$. Then (a_n) must be a convergent sequence. (Hint: Is (a_n) Cauchy?)

(II) Let (a_n) be a sequence that is bounded above. Then we have $\limsup_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.

(III) Let A and B be two real numbers. Suppose that we have $A < C$ for any rational number $C \in \mathbb{Q}$ satisfying $B < C$. Then $A \leq B$.

(IV) Let $M > 0$ and let (a_n) be any sequence satisfying $-M < a_n < M$ for every $n \in \mathbb{N}$. Then (a_n) admits a subsequence that converges to a real number $a \in \mathbb{R}$ such that $-M < a < M$.

(a) (15 points) Choose a statement that is true and prove it. *You are not allowed to choose more than one statement.*
My statement is (I) or (III).

(I): Claim: (a_n) is Cauchy ($\Rightarrow$ convergent).
$\forall \varepsilon > 0$. Choose $N = -\log_2 \varepsilon + 1$. (i.e. $2^{-N+1} = \varepsilon$).
Then for any $n > m > N$, we have

$$|a_n - a_m| \leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{m+1} - a_m|$$
$$< 2^{-(n-1)} + \cdots + 2^{-m}$$
$$< 2^{-(m-1)} < 2^{-N+1} = \varepsilon. \ \square$$

(III) Assume by contradiction that $A > B$.
By denseness of $\mathbb{Q}$, $\exists C \in \mathbb{Q}$ s.t. $A > C > B$.
So we find a rational number $C \in \mathbb{Q}$ satisfying $B < C$ but $A > C$.
Contradiction. $\square$

(b) (15 points) Choose a statement that is false. Give an explicit counterexample and justify it. *You are not allowed to choose more than one statement.*
My statement is (II) or (IV).

Counterexamples:

(II) $(a_n = \frac{1}{n})$, $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = 0$. But $\sup\{a_n : n \in \mathbb{N}\} = 1$.

(IV). $M = 1$, $(a_n = 1 - \frac{1}{n})$, $-1 < 1 - \frac{1}{n} < 1$, But $\lim_{n\to\infty} a_n = 1$.
So every subsequence of (a_n) converges to 1.