Last time:

- Basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ of V $\rightsquigarrow$ $V \xrightarrow{[\ ]_{\mathcal{B}}} \mathbb{R}^n$
- Find a basis of $\mathrm{Nul}(A)$.

Find a basis of Col(A):

$$A \underset{\text{row operations}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

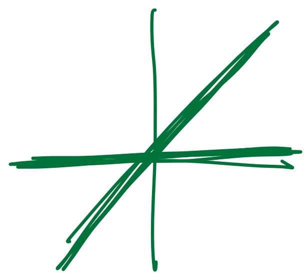
$\{\vec{b}_1, \vec{b}_3\}$ is a basis of $\mathrm{Col}(B)$

$\Rightarrow \{\vec{a}_1, \vec{a}_3\}$ is a basis of $\mathrm{Col}(A)$

$(\mathrm{Nul}(A) = \mathrm{Nul}(B))$

$\mathrm{Col}(A) \cancel{=} \mathrm{Col}(B)$??

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underset{\text{row operatn}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = B$$



$\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \neq \mathrm{Col}(B) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

Claim: Linear relations among the column vectors are preserved under row operations:

$$[\vec{a}_1 \cdots \vec{a}_n] = A \underset{\text{row operations}}{\rightsquigarrow} B = [\vec{b}_1 \cdots \vec{b}_n]$$

Claim: $c_1\vec{a}_1 + \cdots + c_n\vec{a}_n = \vec{0}$ $\Longleftrightarrow$ $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathrm{Nul}(A)$

$$A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\Longleftrightarrow c_1\vec{b}_1 + \cdots + c_n\vec{b}_n = \vec{0}$ $\Longleftrightarrow$ $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathrm{Nul}(B)$

$\|$ since A & B are related by row operations

Claim: $\{\vec{b}_{i_1}, \ldots, \vec{b}_{i_k}\}$ is a basis of $Col(B)$

$\Leftrightarrow$ $\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\}$ is a basis of $Col(A)$

pf: Assuming $\{\vec{b}_{i_1}, \ldots, \vec{b}_{i_k}\}$ is a basis of $Col(B)$, we want to show: $\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\}$ is a basis of $Col(A)$

- $\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\}$ l.i.:

$$c_1 \vec{a}_{i_1} + \cdots + c_k \vec{a}_{i_k} = \vec{0}$$

(by previous Claim)

$$\Rightarrow c_1 \vec{b}_{i_1} + \cdots + c_k \vec{b}_{i_k} = \vec{0}$$

$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0.$$

- $Span\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\} = Col(A)$:

$(\Leftrightarrow \vec{a}_j \in Span\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\} \ \forall j)$

$(\Leftrightarrow \forall j, \ \exists c_1, \ldots, c_k \in \mathbb{R}$ s.t. $\vec{a}_j = c_1 \vec{a}_{i_1} + \cdots + c_k \vec{a}_{i_k})$

Since $Span\{\vec{b}_{i_1}, \ldots, \vec{b}_{i_k}\} = Col(B)$;

so $\forall j, \ \exists c_1, \ldots, c_k \in \mathbb{R}$

s.t. $\vec{b}_j = c_1 \vec{b}_{i_1} + \cdots + c_k \vec{b}_{i_k}$.

(by previous Claim) $\Longrightarrow \vec{a}_j = c_1 \vec{a}_{i_1} + \cdots + c_k \vec{a}_{i_k}$

$$\Longrightarrow \vec{a}_j \in Span\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\} \ \forall j.$$

$$\Rightarrow Span\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\} = Col(A).$$

$\square$

Thm: Suppose a vector space V admits a basis of n vectors, then any basis of V also consists of n vectors.

Def: Such n is called the dimension of V, $\dim(V)$

- If V can't be spanned by finitely many vectors, then V is infinite dimensional, $\dim(V) = +\infty$
- The zero vector space $\{0\}$ has $\dim\{0\} = 0$.

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pf: Suppose $\mathcal{B}$ is a basis of V with n vectors.
Consider the coordinate mapping w.r.t. $\mathcal{B}$:

$$V \xrightarrow{[\ ]_{\mathcal{B}}} \mathbb{R}^n \quad \text{linear, bijective.}$$

- Suppose $\{\vec{w}_1, \cdots, \vec{w}_n, \cdots\}$ more than n vectors in V, consider $\{[\vec{w}_1]_{\mathcal{B}}, \cdots, [\vec{w}_n]_{\mathcal{B}}, \cdots\} \subseteq \mathbb{R}^n$

We proved last time that such set in $\mathbb{R}^n$ is l.d.

So $\exists\ c_1, \cdots, c_k$ s.t. $c_1[\vec{w}_1]_{\mathcal{B}} + \cdots + c_k[\vec{w}_k]_{\mathcal{B}} = \vec{0}$

$[c_1\vec{w}_1 + \cdots + c_k\vec{w}_k]_{\mathcal{B}}$ $=$ $[\ ]_{\mathcal{B}}$ is linear.

$\Rightarrow$ $c_1\vec{w}_1 + \cdots + c_k\vec{w}_k = \vec{0}$ since $[\ ]_{\mathcal{B}}$ is injective.

$\Rightarrow$ $\{\vec{w}_1, \cdots, \vec{w}_n, \cdots\}$ is l.d. $\square$

- Suppose $\{\vec{v}_1, \cdots, \vec{v}_k\}$ in V, $k < n$.

Consider $\{[\vec{v}_1]_B, \dots, [\vec{v}_k]_B\} \subseteq \mathbb{R}^n$

We proved last time that such set doesn't span $\mathbb{R}^n$,

$\exists \vec{x} \in \mathbb{R}^n$ s.t. $\vec{x} \notin \text{Span}\{[\vec{v}_1]_B, \dots, [\vec{v}_k]_B\}$.

Since $[\ ]_B$ is surjective, $\exists \vec{y} \in V$ s.t. $[\vec{y}]_B = \vec{x}$.

Claim: $\vec{y} \notin \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.

pf: Assume the contrary that $\vec{y} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$

then $\exists c_1, \dots, c_k$ s.t. $\vec{y} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$

$$\Rightarrow \underbrace{[\vec{y}]_B}_{\vec{x}} = [c_1\vec{v}_1 + \dots + c_k\vec{v}_k]_B$$

$$= c_1[\vec{v}_1]_B + \dots + c_k[\vec{v}_k]_B$$

$\Rightarrow \vec{x} \in \text{Span}\{[\vec{v}_1]_B, \dots, [\vec{v}_k]_B\}$. Contradiction. $\square$

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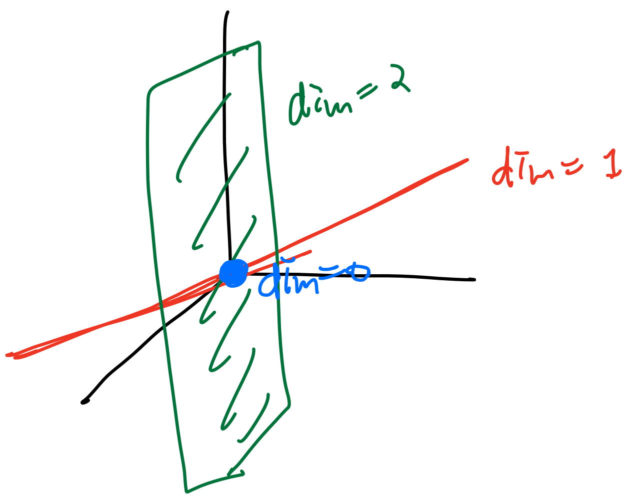
e.g. $\dim \mathbb{R}^n = n$

$\dim \text{Poly}_{\leq n} = n+1$ $\quad$ ($\{1, x, \dots, x^n\}$ is a basis)



$A: m \times n$ $\qquad \mathrm{Nul}(A) \subseteq \mathbb{R}^n, \quad \dim \mathrm{Nul}(A) = \#$ free variables $= n - \#$ pivots

$\mathrm{Col}(A) \subseteq \mathbb{R}^m.$

$\dim \mathrm{Col}(A) = \#$ pivots

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \| \quad$ $\mathrm{rank}(A)$

$T_A: \mathbb{R}^n \to \mathbb{R}^m$ $\qquad \mathrm{Nul}(A) = \ker(T_A)$

$\qquad \mathrm{Col}(A) = \mathrm{Im}(T_A)$

Rank-nullity theorem:

$$\dim \mathrm{Nul}(A) + \mathrm{rank}(A) = n \quad (\# \text{ of columns of } A).$$

More generally, $T: V \to W$ linear map btw v.s., $\dim V < +\infty$

Then

$$\dim V = \dim \ker(T) + \dim \mathrm{Im}(T)$$

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① Suppose $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$, $\quad \mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_n\} = V$

Claim: $\exists$ subset $\{\vec{v}_{i_1}, \ldots, \vec{v}_{i_k}\}$ that gives a basis of V.

("spanning set theorem")

(If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ l.d., we proved before that

$\exists i$ s.t. $\vec{v}_i \in \mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\}$,

and $\mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\} = \mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$)

Q: True for infinite spanning set?

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② $\dim V < +\infty$. $\qquad \{\vec{v}_1, \ldots, \vec{v}_n\}$ l.i.

Claim: $\exists\, \vec{v}_{n+1}, \ldots, \vec{v}_{n+k} \in V$ s.t. $\{\vec{v}_1, \ldots, \vec{v}_n, \ldots, \vec{v}_{n+k}\}$ is a basis of V.

(If $\{\vec{v}_1, \dots, \vec{v}_n\}$ l.i. and $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_n\} \neq V$,
then $\exists\ \vec{v}_{n+1} \in V$ s.t. $\vec{v}_{n+1} \notin \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$
$\Rightarrow \{\vec{v}_1, \dots, \vec{v}_{n+1}\}$ l.i.)

Thm: $W \subseteq V$ subspace of V. $\dim V < +\infty$.
Then $\dim W \leq \dim V = n$

pf: ① If $W = \{\vec{0}\}$, then $\dim W = 0 \leq n$

② If W contains some nonzero vectors, say $\vec{0} \neq \vec{v}_1 \in W$

$\operatorname{Span}\{\vec{v}_1\}$

use the argument in $\boxed{②}$ to get a basis of W,
$\{\vec{v}_1, \dots, \vec{v}_m\}$.

③ $\boxed{m \leq n:}$ If $m > n$, then $\vec{v}_1, \dots, \vec{v}_m \in V$
must be linearly dependent
since $\dim V = n$.

$$\begin{tikzcd}
V \arrow[r, "{[\ ]_B}", "\sim"'] \arrow[d, "{[\ ]_{B'}}"', "\wr"] & \mathbb{R}^n \arrow[dl, bend right] \\
\mathbb{R}^n &
\end{tikzcd}$$