Existence & Uniqueness thm for systems of 1^{rd} diff'l eq$^{'n}$:

Given continuous functions $A(t): I \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{R})$

$\vec{f}(t): I \longrightarrow \mathbb{R}^n$

$\leftarrow$ open interval of $\mathbb{R}$. (e.g. $(0,1)$, $[0,+\infty)$, $\mathbb{R}$)

and any $\vec{x}_0 \in \mathbb{R}^n$ and $t_0 \in I$,

there exists a unique $\vec{x}(t): I \longrightarrow \mathbb{R}^n$

s.t.
$$\begin{cases} \vec{x}'(t) = A(t)\,\vec{x}(t) + \vec{f}(t) \\ \vec{x}(t_0) = \vec{x}_0. \end{cases}$$

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e.g. $\vec{x}'(t) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \vec{x}(t).$ $\left(\begin{cases} x_1'(t) = \lambda_1 x_1(t) \\ \quad\vdots \\ x_n'(t) = \lambda_n x_n(t) \end{cases} \right)$ $\begin{matrix} \rightarrow x_1(t) = c_1 e^{\lambda_1 t} \\ \\ \rightarrow x_n(t) = c_n e^{\lambda_n t}. \end{matrix}$

general sol^n:
$$c_1 \begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{\lambda_2 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_n t} \end{bmatrix}$$

(If we impose $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{bmatrix}$,,

then the sol^n to this initial value problem is:
$$\begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ e^{\lambda_2 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_n t} \end{bmatrix}$$
)

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e.g. $\vec{x}'(t) = A\,\vec{x}(t)$, where $A = PDP^{-1}$, $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$= PDP^{-1}\,\vec{x}(t)$

$$\Leftrightarrow \underbrace{P^{-1}\cdot\vec{x}'(t)}_{??\ \ \overset{\|}{\vec{y}'(t) = (P^{-1}\vec{x}(t))'}} = D\,\underbrace{P^{-1}\,\vec{x}(t)}_{\vec{y}(t)}$$

B - const. matrix , $\vec{x}(t)$.

$$(B\vec{x}(t))' = \left(\begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & & \cdots \\ \vdots & & \end{bmatrix}\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}\right)'$$

$$= \begin{bmatrix} b_{11}x_1(t)+b_{12}x_2(t)+\cdots b_{1n}x_n(t) \\ \vdots \\ \end{bmatrix}'$$

$$= \begin{bmatrix} (b_{11}x_1(t)+\cdots b_{1n}x_n(t))' \\ \vdots \\ \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}x_1'(t)+\cdots+b_{1n}x_n'(t) \\ \vdots \\ \end{bmatrix} = B\begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = B\vec{x}'(t)$$

$$\vec{y}'(t) = D\vec{y}(t)$$

We know general sol^n of $\vec{y}(t)$ is:

$$\vec{y}(t) = c_1\begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{\lambda_2 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n\begin{bmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_n t} \end{bmatrix}$$

Then $\begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}$, $A\vec{v}_i = \lambda_i\vec{v}_i$

$$\vec{x}(t) = P\vec{y}(t)$$

$$= c_1\boxed{P\begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix}} + c_2 P\begin{bmatrix} 0 \\ e^{\lambda_2 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots$$

$$= c_1\underline{e^{\lambda_1 t}\vec{v}_1} + c_2\underline{e^{\lambda_2 t}\vec{v}_2} + \cdots + c_n\underline{e^{\lambda_n t}\vec{v}_n}.$$

Def: $\{\vec{x}_1(t), \cdots, \vec{x}_n(t)\}$ l.d. If $\exists\ c_1, \cdots, c_n$ not all 0 s.t. $c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t) = \vec{0}$

Otherwise, they're called l.i.

Def: The Wronskian of $\{\vec{x}_1(t), \cdots, \vec{x}_n(t)\}$ is defined to be: the function:

$$W[\vec{x}_1, \cdots, \vec{x}_n](t) := \det\begin{pmatrix} \vec{x}_1(t) & \cdots & \vec{x}_n(t) \end{pmatrix}$$

Rmk: If $\{\vec{x}_1(t), \cdots, \vec{x}_n(t)\}$ l.d., then $W[\vec{x}_1, \cdots, \vec{x}_n](t) \equiv 0\ \forall t$.

Prop. If $\{\vec{x}_1(t), \cdots, \vec{x}_n(t)\}$ l.i., sol^{n}s of $\vec{x}'(t) = A\vec{x}(t)$, then $W[\vec{x}_1, \cdots, \vec{x}_n](t) \neq 0\ \forall t$.

pf: Suppose $W[\vec{x}_1, \cdots, \vec{x}_n](t_0) = 0$ for some t_0.

i.e. $\{\vec{x}_1(t_0), \cdots, \vec{x}_n(t_0)\}$ is a l.d. set.

$\Rightarrow \exists\ c_1, \cdots, c_n$ not all 0 s.t. $c_1\vec{x}_1(t_0) + \cdots + c_n\vec{x}_n(t_0) = \vec{0}$.

(Hope to show: $c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t) = \vec{0}\ \forall t$)

Consider the initial value problem:

$$\begin{cases} \vec{x}'(t) = A\vec{x}(t) \\ \vec{x}(t_0) = \vec{0} \end{cases}$$

$\vec{0}$ is obviously a sol^{n}.

$c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t)$ is also a sol^{n}

By uniqueness thm, we have

$c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t) = \vec{0}$.

$\Downarrow$

$\{\vec{x}_1(t), \cdots, \vec{x}_n(t)\}$ l.d. $\square$

Prop: If $\{\vec{x}_1(t), \dots, \vec{x}_n(t)\}$ l.i. sol^{ns} of $\vec{x}'(t) = A\vec{x}(t)$, then any sol^n can be written as $c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t)$.

pf: Let $\vec{x}_0(t)$ be any sol^n of $\vec{x}' = A\vec{x}$.

Pick any t_0,

$\underline{\{\vec{x}_1(t_0), \dots, \vec{x}_n(t_0)\}}$ l.i. $\Rightarrow$ basis of $\mathbb{R}^n$

$\exists$ $c_1, \dots, c_n$ st.

$$\vec{x}_0(t_0) = c_1\vec{x}_1(t_0) + \cdots + c_n\vec{x}_n(t_0).$$

Consider the initial value problem:

$$\begin{cases} \vec{x}'(t) = A\vec{x}(t). \\ \vec{x}(t_0) = c_1\vec{x}_1(t_0) + \cdots + c_n\vec{x}_n(t_0). \end{cases}$$

$\vec{x}_0(t)$ is a sol^n of this IVP.

$c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t)$ is also a sol^n of the IVP

By uniqueness, we have $\vec{x}_0(t) = c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t)$ $\forall t$.

$\square$

Notion: $\vec{x}'(t) = A\vec{x}(t)$. Suppose $\{\vec{x}_1(t), \dots, \vec{x}_n(t)\}$ is a l.i. set of sol^{ns}

then we say

$$X(t) = \begin{bmatrix} | & & | \\ \vec{x}_1(t) & \cdots & \vec{x}_n(t) \\ | & & | \end{bmatrix}$$

is a fundamental matrix of $\vec{x}'(t) = A\vec{x}(t)$

- By the Prop, any sol^n can be written as $X(t)\vec{c} = c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t)$
- If $X(t)$ and $\tilde{X}(t)$ are fundamental matrices of $\vec{x}'(t) = A\vec{x}(t)$, then $\exists$ Q: invertible st. $X(t) \cdot Q = \tilde{X}(t)$.

e.g. $\vec{x}'(t) = \underbrace{PDP^{-1}}_{A.}\vec{x}(t).$, $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$ $P = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$

We showed that $\{ e^{\lambda_1 t}\vec{v}_1, \dots, e^{\lambda_n t}\vec{v}_n \}$ l.i. sol^{ns}.

$$\begin{bmatrix} | & & | \\ e^{\lambda_1 t}\vec{v}_1 & \cdots & e^{\lambda_n t}\vec{v}_n \\ | & & | \end{bmatrix} = P \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

is a fundamental matrix of $\vec{x}'(t) = A\vec{x}(t)$

Matrix expotential:

($\vec{x}'(t) = A\vec{x}(t)$, if A is a 1×1 matrix, then we get $x'(t) = Ax(t)$

sol^n is $c\,e^{tA}$)

For $A \in M_{n\times n}(\mathbb{R})$, $t \in \mathbb{R}$

$$e^{tA} := \mathbb{I} + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

→ matrix-valued function in t.

(analogue: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$)

Fact: e^{tA} always converges. $\forall t$.

e.g. $A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}.$

$$e^{tA} = \mathbb{I} + \begin{bmatrix} ta_{11} & 0 \\ 0 & ta_{22} \end{bmatrix} + \begin{bmatrix} \frac{t^2}{2}a_{11}^2 & 0 \\ 0 & \frac{t^2}{2}a_{22}^2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + ta_{11} + \frac{t^2}{2}a_{11}^2 + \cdots & 0 \\ 0 & 1 + ta_{22} + \frac{t^2}{2}a_{22}^2 + \cdots \end{bmatrix} = \begin{bmatrix} e^{ta_{11}} & 0 \\ 0 & e^{ta_{22}} \end{bmatrix}$$

Fact:

- $e^{(t_1+t_2)A} = e^{t_1 A} \cdot e^{t_2 A}$
- e^{tA} is invertible ∀t, and $(e^{tA})^{-1} = e^{-tA}$.
- $\boxed{\frac{d}{dt} e^{tA} = A e^{tA}.}$

$\Rightarrow$ e^{tA} is a fundamental matrix of $\vec{x}'(t) = A\vec{x}(t)$.

e.g. $A = PDP^{-1}$, $P = \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

$$e^{tA} = I + tA + \frac{t^2}{2} A^2 + \cdots$$

$$= PP^{-1} + tPDP^{-1} + \frac{t^2}{2} PD^2P^{-1} + \frac{t^3}{3!} PD^3P^{-1} + \cdots$$

$$= P\left(I + tD + \frac{t^2}{2} D^2 + \cdots \right) P^{-1}$$

$$= P \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} P^{-1} \rightarrow \text{is a fundamental matrix}$$

Rmk: e^{tA} is a fundamental matrix of $\vec{x}' = A\vec{x}$, even when A is not diagonalizable

But when A is not diagonalizable, it's harder to compute e^{tA}.

Idea:

- ∃ generalized eigenbasis $\{\vec{v}_1, \dots, \vec{v}_n\}$ of A ∀ matrix A
- $e^{tA}\vec{v}$ is not hard to compute if $\vec{v}$ is a generalized eigenvector.

$\boxed{(A - \lambda I)^k \vec{v} = 0 \text{ for some } k \geq 1}$

$$\boxed{\begin{aligned} e^{tA}\vec{v} &= e^{t\lambda}\cdot e^{t(A-\lambda I)}\cdot\vec{v} \\ &= e^{t\lambda}\left(I + t(A-\lambda I) + \frac{t^2}{2}(A-\lambda I)^2 + \cdots\right)\vec{v} \\ &= e^{t\lambda}\left(I + t(A-\lambda I) + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{k-1}\right)\vec{v}. \end{aligned}}$$

$$e^{tA}\begin{bmatrix} \overbrace{\begin{matrix} | & & | \\ \vec{v}_1 & - & \vec{v}_n \\ | & & | \end{matrix}}^{\text{generalized eigenbasis}} \end{bmatrix} = \begin{bmatrix} | & & | \\ e^{tA}\vec{v}_1 & \cdots & e^{tA}\vec{v}_n \\ | & & | \end{bmatrix} \text{ is a fundamental matrix}$$

Rmk When $A\vec{v} = \lambda\vec{v}$, $e^{tA}\vec{v} = e^{t\lambda}\vec{v}$.