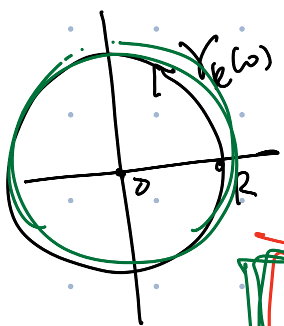
Today: end of §2, (start discussing §3).
Thursday: go over practice problems. & exam logistics
Next Tuesday: 1st midterm.

$$\frac{1}{2\pi}\int_0^{2\pi} f(w)\,\mathrm{Re}\left(\frac{w+z}{w-z}\right)d\varphi$$

$(w = \underline{Re^{i\varphi}})$



$$\parallel$$

$$\boxed{\frac{1}{2\pi}\int_0^{2\pi} f(w)\cdot\frac{1}{2}\left(\frac{w+z}{w-z}+\frac{R^2/\bar z + w}{R^2/\bar z - w}\right)d\varphi}$$

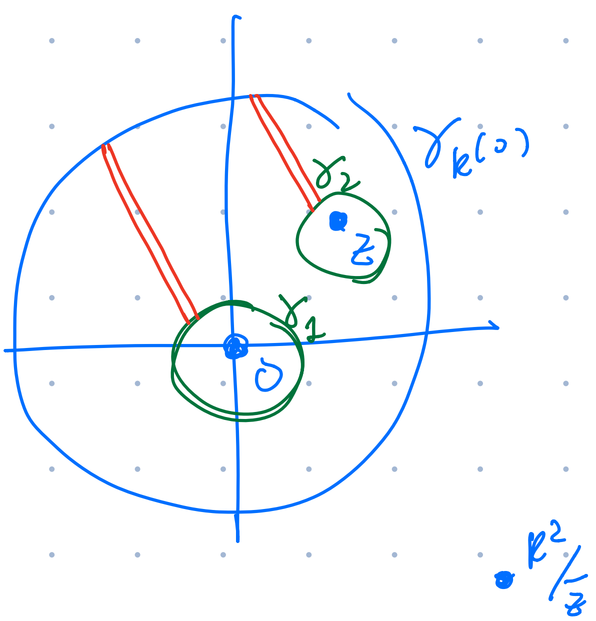
$$\parallel$$

$$\boxed{\frac{1}{2\pi}\int_{\gamma_R(0)} \underbrace{f(w)\cdot\frac{1}{2}\left(\frac{w+z}{w-z}+\frac{R^2/\bar z + w}{R^2/\bar z - w}\right)\frac{1}{iw}}_{G(w)}\,dw}$$

$$=\frac{1}{2\pi}\int_0^{2\pi} \Big[\qquad\qquad\Big]\boxed{\cancel{\frac{1}{iw}}}\cdot\left(\cancel{iRe^{i\varphi}}\right)d\varphi$$

$G(w)$

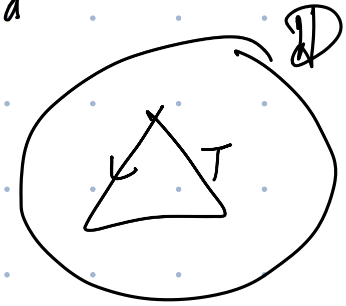
fails to be holo. at $w = 0,\ z,\ R^2/\bar z$



$$\int_{\gamma_R(0)} G(w)\,dw = \int_{\gamma_1} G(w)\,dw + \int_{\gamma_2} G(w)\,dw$$

Thm (Morera): If f is continuous on $\mathbb{D}$, and

$$\int_T f(z)\,dz = 0 \quad \forall\, T: \text{triangle in } \mathbb{D}$$



then f is holomorphic in $\mathbb{D}$.

pf: By the same proof we did last time, f has a primitive in $\mathbb{D}$.

i.e. $\exists\, F$: holo. in $\mathbb{D}$ s.t. $F' = f$.

By the regularity of holo fcn $\Rightarrow$ f is holo. $\square$

---

end of material of 1st midterm ↑

Thm
- $f_1, f_2, \cdots\cdots$ seq. of holo fcn. in $\Omega \overset{\text{open}}{\subseteq} \mathbb{C}$
- Suppose $\lim f_n = f$ uniformly in every cpt subset of Ω.

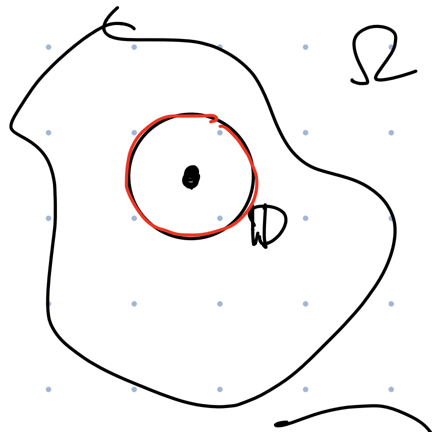
$$\left(\forall\, K \underset{\text{cpt}}{\subseteq} \Omega,\ \forall \varepsilon > 0,\ \exists\, N > 0 \text{ s.t. } |f_n(x) - f(x)| < \varepsilon \quad \forall n > N,\ \forall x \in K \right)$$

Then f is holo. in Ω.

(**Rmk**: NOT true / $\mathbb{R}$!!

Recall Weierstrass thm: $\forall$ conti. fcn f $[a,b]$,

$\exists\, P_n(x)$ poly. s.t. $P_n(x) \to f(x)$ uniformly) pf:



<u>want to show:</u>

f is holo. in $\mathbb{D}$.

By Morera's thm, It's enough to show:

- f is conti. in $\mathbb{D}$
- $\int_T f = 0$ $\forall\, T$ triangle in $\mathbb{D}$

$\overline{\mathbb{D}} \subseteq \Omega$ cpt. subset.

So $f_n \to f$ unif. in $\overline{\mathbb{D}}$.

$\Longrightarrow$

- f is conti. in $\overline{\mathbb{D}}$
- $\int_T f_n(z)\,dz \xrightarrow[n\to\infty]{} \int_T f(z)\,dz.$ $\forall\, T$ triangle in $\mathbb{D}$

$$\left(\left|\int_T f_n(z)\,dz - \int_T f(z)\,dz\right| = \left|\int_T (f_n(z) - f(z))\,dz\right| \le \boxed{\sup_{z\in T} |f_n(z) - f(z)|} \cdot \text{length}(T). \right)$$

- $\int_T f_n(z)\,dz = 0$ by Cauchy's thm. (f_n is holo.)

$\Rightarrow \int_T f(z)\,dz = 0.$

By Morera's thm, f is holo. in $\mathbb{D}$. $\square$

$\Rightarrow f$ is holo. in Ω.

Thm $\{f_n\}$ seq. of holo. fcn. in Ω

$f_n \to f$ conv. unif. on any cpt subset of Ω.

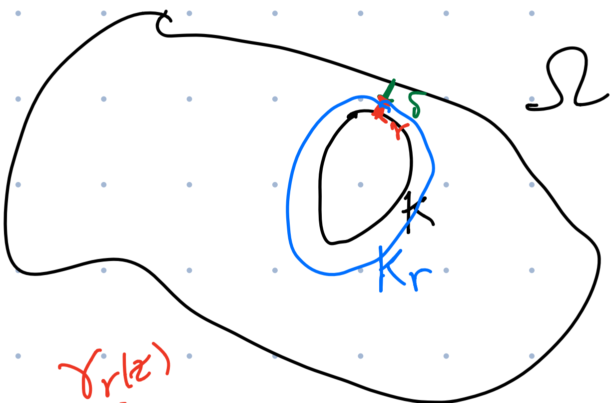
(by previous thm, f is holo.)

$\Longrightarrow$ $f_n' \to f'$ conv. unif. on any cpt subset of Ω.

(Rmk: This implies that $f_n^{(k)} \to f^{(k)}$ conv. unif. on any cpt subset of Ω)

pf: $K \subseteq \Omega$ cpt subset.

(WTS: $f_n' \to f'$ conv. unif on K).



$$\delta := \min_{z \in K} \operatorname{dist}(z, \partial\Omega) > 0$$

Let $r = \frac{1}{2}\delta > 0$ cpt

$K_r := \{z \in \Omega \mid \operatorname{dist}(z, K) \le r\} \subseteq \Omega$

$\overline{D_r(z)} \subseteq \Omega$

$$f'(z) = \frac{1}{2\pi i} \int_{\gamma_r(z)} \frac{f(w)}{(w-z)^2}\, dw.$$

$$|f_n'(z) - f'(z)| = \left| \frac{1}{2\pi i} \int_{\gamma_r(z)} \frac{f_n(w) - f(w)}{(w-z)^2}\, dw \right|$$

$$\le \frac{1}{2\pi} \sup_{w \in \gamma_r(z)} \frac{|f_n(w) - f(w)|}{\underbrace{|w-z|^2}_{= r^2}} \cdot 2\pi r.$$

$$= \frac{1}{r} \cdot \sup_{w \in \gamma_r(z)} |f_n(w) - f(w)|$$

$$\leq \frac{1}{r} \boxed{\sup_{w \in K_r} |f_n(w) - f(w)|}$$

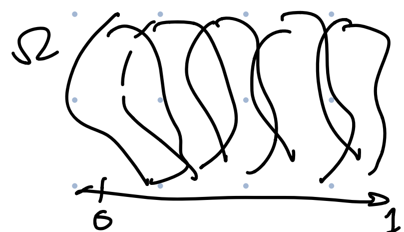
Since $f_n \to f$ unif. on K_r,

$\forall \varepsilon > 0$, $\exists N > 0$ s.t.

$$|f_n(z) - f(z)| < \varepsilon \cdot r \quad \forall n > N,\ z \in K_r.$$

$$\Rightarrow |f_n'(z) - f(z)| < \varepsilon \quad \forall z \in K. \quad \square$$

Thm: $F(z,s) : \Omega \times [0,1] \longrightarrow \mathbb{C}$



Suppose:

- $F(z,s)$ is holo. in z for any $s \in [0,1]$.
- $F(z,s)$ is continuous on $\Omega \times [0,1]$

Then $f(z) := \int_0^1 F(z,s)\,ds$ is holo. in Ω

pf: ~~Use~~ By Morera's thm, it's enough to show $\forall T \subseteq \mathbb{D} \subseteq \Omega$,

- f is conti. in $\mathbb{D}$. (clear)
- $\boxed{\int_T f(z)\,dz = 0.}$

$$\int_T \left(\int_0^1 F(z,s)\, ds \right) dz \overset{??}{=} 0$$

$$\int_T \left(\int_0^1 |F(z,s)|\, ds \right) dz \overset{??}{<} +\infty$$

$$\| ??$$

$$\int_0^1 \underbrace{\left(\int_T F(z,s)\, dz \right)}_{=0} ds$$

Since F is holo. $\forall s$. ($+$ Cauchy's thm).

$$\int_T \left(\int_0^1 |F(z,s)|\, ds \right) dz \le \boxed{\sup_{z \in T} \left(\int_0^1 |F(z,s)|\, ds \right)} \cdot \text{length}(T)$$

$$\int_0^1 |F(z,s)|\, ds \le \sup_{s \in [0,1]} |F(z,s)|$$

$$\sup_{\substack{z \in T \\ s \in [0,1]}} |F(z,s)|$$

- $T \times [0,1]$ is cpt subset of $\Omega \times [0,1]$
- F is conti.

$$\Rightarrow \sup_{\substack{z \in T \\ s \in [0,1]}} |F(z,s)| < +\infty$$

$F : \Omega \times [0,1] \to \mathbb{C}$ conti.

$\cup|$ cpt $\quad \cup$ cpt

$T \times [0,1] \quad \boxed{F(T \times [0,1])}$

any conti. fn. on any cpt set is bounded.

§ Zeros & poles

Def $f: \Omega \to \mathbb{C}$ holo.

say $z_0 \in \Omega$ is a <u>zero of f of order $n \geq 1$</u> if the power series expansion near z_0 is of the form:

$$f(z) = \underset{\neq 0}{a_n} (z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \cdots$$

Recall: $a_k = \dfrac{f^{(k)}(z_0)}{k!}$ $\forall k$.

$$f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0, \quad f^{(n)}(z_0) \neq 0$$

e.g. $n \geq 1$ $f(z) = z^n$, then f has zero of order n at $0 \in \mathbb{C}$.

$$f'(z) = n z^{n-1}, \quad f''(z) = n(n-1) z^{n-2}, \quad \cdots, \quad f^{(n-1)}(z) = n!\, z, \quad f^{(n)}(z) = n!$$

Thm f: holo in Ω, $f(z_0) = 0$, $z_0 \in \Omega$

Suppose $f \not\equiv 0$ in Ω

order of zero of z_0

$g(z) \neq 0 \; \forall z \in \mathcal{U}$

Then $\exists$ nbd $z_0 \in \mathcal{U} \subseteq \Omega$, $g: \mathcal{U} \to \mathbb{C}$ <u>nonvanishing</u>, holo. fcn. $n \geq 1$

s.t. $f(z) = (z-z_0)^n g(z)$ ~~for~~ $z \in \mathcal{U}$.

Pf. • Choose some nbd V of z_0 to do the power series exp.:

$$f(z) = \overset{\neq 0}{a_n}(z-z_0)^{n \geq 1} + a_{n+1}(z-z_0)^{n+1} + \cdots$$

$$= (z-z_0)^n \Big(\underbrace{a_n + a_{n+1}(z-z_0) + \cdots}_{g(z)} \Big)$$

holo. in V, and $g(z_0) \neq 0$

By continuity of g, $\exists$ nbd $z_0 \in U \subseteq V$,

s.t. $g(z) \neq 0 \quad \forall z \in U$. $\square$