On categorical dynamical systems

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Plan

- Topological entropy, examples
- Categorical dynamical systems
- Categorical entropy
  - Motivation from Teichmüller theory and stability conditions
  - Counterexamples of Gromov–Yomdin type results arising from mixing of holomorphic and symplectic dynamics
- Short break for questions
- New invariants – "Shifting numbers" (joint work with Simion Filip)
  - Complementary to categorical entropy
  - Motivation from Poincaré rotation numbers
- Further questions

Topological entropy

- (X, d) compact metric space.
- $f\colon X \to X$ continuous self-map. ("dynamical system")

For any $n \in \mathbb{N}$ and $\epsilon > 0$, define

$$N(n,\epsilon) := \max\left\{\ell\colon \exists x_1,\ldots,x_\ell \text{ s.t. } \max_{0\le k\le n}\{d(f^k(x_i), f^k(x_j))\} \ge \epsilon \ \forall x_i, x_j\right\}$$

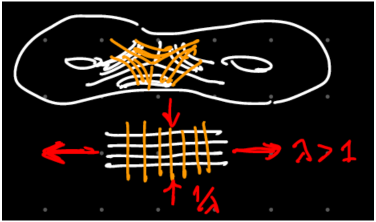
The **topological entropy** of f is defined to be

$$h_{\mathrm{top}}(f) := \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty}\frac{1}{n}\log N(n,\epsilon)\right) \in [0,\infty].$$

Basic properties:

- It's a topological invariant measuring the "complexity" of f.
- $f^n = \mathrm{id}_X \implies h_{\mathrm{top}}(f) = 0$.

Example: Pseudo-Anosov maps on Riemann surfaces



$$h_{\mathrm{top}}(f) = \log\lambda = \lim_{n\to\infty} \frac{1}{n}\log\iota(a, f^n(b)) = \lim_{n\to\infty}\frac{1}{n}\log\ell_g(f^n(c)).$$

- a, b, c: isotopy classes of simple closed curves
- ι: (geometric) intersection number
- ℓ_g: length with respect to a metric g

(They will lead to two categorical generalizations of entropy.)

Example: Holomorphic maps on compact Kähler manifolds

Theorem (Gromov, Yomdin)

If $f: X \to X$ is a surjective holomorphic map of a compact Kähler manifold, then

$$h_{\mathrm{top}}(f) = \log \rho(f^*)$$

where ρ is the spectral radius of $f^*: H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$.

Theorem (Cantat)

If a compact complex surface X admits an automorphism of positive topological entropy, then X is either a torus, a K3 surface, an Enriques surface, or a rational surface.

What are categorical dynamical systems?

A categorical dynamical system is a pair $(\mathcal{D}, F)$, where

- $\mathcal{D}$ is a triangulated category, and
- $F\colon \mathcal{D} \to \mathcal{D}$ is an endofunctor.

Recall that a triangulated category is an additive category with a translation functor [1] and a collection of exact triangles

$$A \to B \to C \to A[1].$$

Examples of triangulated categories:

- $\mathcal{D}^b\mathrm{Coh}(X)$, where X is a smooth complex projective variety (objects: (complex of) holomorphic vector bundles on X)
- $\mathcal{D}^\pi\mathrm{Fuk}(Y)$, where Y is a symplectic manifold (objects: Lagrangian submanifolds in Y, morphisms: $L_1 \cap L_2$)

Examples of categorical dynamical systems

- $\mathcal{D} = \mathcal{D}^b\mathrm{Coh}(X)$, where X is a smooth complex projective variety. Then a holomorphic self-map $f\colon X \to X$ induces an endofunctor

$$\mathbb{L}f^*\colon \mathcal{D} \to \mathcal{D}.$$

  Other standard examples include the shift functors $[n]$ and tensoring line bundles $-\otimes\mathcal{L}$.

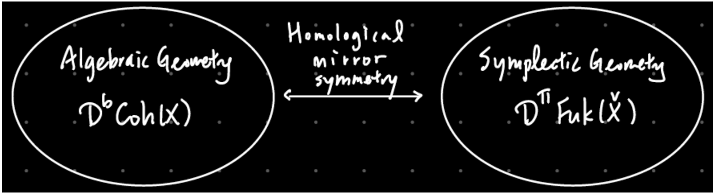
- $\mathcal{D} = \mathcal{D}^\pi\mathrm{Fuk}(Y)$, where Y is a symplectic manifold. Then a symplectomorphism $f\colon Y \to Y$ induces an autoequivalence

$$f_*\colon \mathcal{D} \to \mathcal{D}.$$

Why categorical dynamical systems?

- Rich connections with algebraic geometry, symplectic geometry, Teichmüller theory, rotation theory.

- There are examples of categorical dynamical systems that encompass both holomorphic and symplectic dynamical systems, which have interesting behaviors different from holomorphic dynamics alone. One class of such examples are provided by Calabi–Yau manifolds.



The rest of this talk will be centered around **invariants** that one can associate to categorical dynamical systems.

- Categorical entropy
  - Motivation from Teichmüller theory and stability conditions
  - Counterexamples of Gromov–Yomdin type results arising from mixing of holomorphic and symplectic dynamics
- Short break for questions
- New invariants – "Shifting numbers" (joint work with Simion Filip)
  - Complementary to categorical entropy
  - Motivation from Poincaré rotation numbers

Categorical entropy

$(\mathcal{D}, F)$ as before, $G, G' \in \mathcal{D}$ split generators. Dimitrov, Haiden, Katzarkov, and Kontsevich defined:

$$h_{\mathrm{cat}}(F) := \lim_{n\to\infty} \frac{1}{n} \log\Big(\sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}(G, F^n G'[k])\Big).$$

- cf. $h_{\mathrm{top}}(f) = \lim_{n\to\infty} \frac{1}{n} \log \iota(a, f^n(b))$ for pseudo-Anosov maps
- In Fukaya categories, $\dim \mathrm{Hom}$ is computed by intersection numbers

Basic properties:

- The limit exists, and is independent of the choice of G, G'.
- $F^n = [m] \implies h_{\mathrm{cat}}(F) = 0$.

Derived categories and Gromov–Yomdin type conjecture

$$h_{\mathrm{cat}}(F) := \lim_{n\to\infty} \frac{1}{n} \log \Big(\sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}(G, F^n G'[k]) \Big).$$

$\mathcal{D} = \mathcal{D}^b\mathrm{Coh}(X)$:

- Kikuta–Takahashi: $h_{\mathrm{cat}}(\mathbb{L}f^*) = h_{\mathrm{top}}(f)$.
- $h_{\mathrm{cat}}(- \otimes L) = 0$.

Conjecture (Kikuta–Takahashi)

Let $\mathcal{D} = \mathcal{D}^b\mathrm{Coh}(X)$ and $F\colon \mathcal{D} \to \mathcal{D}$ be an autoequivalence. Then

$$h_{\mathrm{cat}}(F) = \log \rho([F]_{H^*}).$$

The conjecture holds if X is a curve or $\pm K_X$ ample (K, K–T).

Counterexample to the Gromov–Yomdin type conjecture

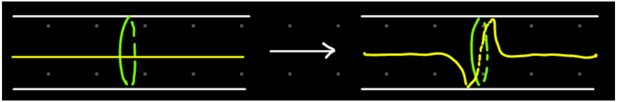
Counterexamples can be found for categorical dynamical systems that encompass both **holomorphic** and **symplectic** dynamics.

Theorem (F., 2018)

Let $d \geq 4$ even. Let $X \subseteq \mathbb{P}^{d+1}$ be a Calabi–Yau hypersurface of degree $d+2$, and let $F := T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}(-1))$. Then

$$h_{\mathrm{cat}}(F) > 0 = \log \rho([F]_{H^*}).$$

Here $T_{\mathcal{O}_X}$ is the *spherical twist* along $\mathcal{O}_X$. Spherical twists arise from *Dehn twists* along Lagrangian spheres of the mirror Calabi–Yau manifold.



Comments or questions?

Alternative definition of categorical entropy

Recall that in the pseudo-Anosov case we have

$$h_{\mathrm{top}}(f) = \lim_{n\to\infty} \frac{1}{n} \log \ell_g(f^n(c))$$

Parallel between Teichmüller theory and *Bridgeland stability conditions* on triangulated categories:

Riemann surfaces	Triangulated categories
curve C	object E
$C_1 \cap C_2$	$\mathrm{Hom}(E_1, E_2)$
metric g	Bridgeland stability condition σ
length $\ell_g(C)$	mass $m_\sigma(E)$

The parallel was established by recent work of Gaiotto–Moore–Neitzke, Bridgeland–Smith, Haiden–Katzarkov–Kontsevich.

Bridgeland stability conditions

A *Bridgeland stability condition* on $\mathcal{D}$ is a pair $\sigma = (Z, P)$:

- $Z \colon K_0(\mathcal{D}) \to \mathbb{C}$ group homomorphism (central charge)
- $P = \{P(\phi)\}_{\phi \in \mathbb{R}}$ additive subcategories (semistable of phase ϕ)

satisfying several axioms, including the Harder–Narasimhan property:

- for any $E \in \mathcal{D}$, there exists a unique sequence of exact triangles

$$\begin{tikzcd} 0 \arrow[rr] & & \star \arrow[rr] \arrow[dl] & & \star \arrow[r] \arrow[dl] & \dots \arrow[r] & \star \arrow[rr] & & E \arrow[dl] \\ & A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] \end{tikzcd}$$

where $A_i \in P(\phi_i)$ and $\phi_1 > \cdots > \phi_n$.

(Analogy in SG: $Z \leftrightarrow \int_L \Omega$, $P \leftrightarrow$ special Lagrangian submanifolds)
(Analogy in AG: $Z \leftrightarrow$ slope, $P \leftrightarrow$ slope semistable vector bundles)
(Analogy in flat surface: $Z \leftrightarrow$ slope, $P \leftrightarrow$ straight lines)

Bridgeland stability conditions (cont'd)

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr] & & \star \arrow[rr] \arrow[dl] & & \star \arrow[r] \arrow[dl] & \dots \arrow[r] & \star \arrow[rr] & & E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

where $A_i \in P(\phi_i)$ and $\phi_\sigma^+(E) \coloneqq \phi_1 > \cdots > \phi_n =: \phi_\sigma^-(E)$.

From $\sigma = (Z, P)$, one can extract the **mass** and the **phase** functions:

- $m_\sigma \colon \mathrm{Ob}(\mathcal{D}) \to \mathbb{R}_{\geq 0}$, defined as $E \mapsto m_\sigma(E) \coloneqq \sum_i |Z_\sigma(A_i)|$
- $\phi_\sigma^\pm \colon \mathrm{Ob}(\mathcal{D}) \to \mathbb{R}$, defined as $E \mapsto \phi_\sigma^\pm(E)$

(Analogy: mass $\leftrightarrow$ length of piecewise geodesic representative, $\phi_\sigma^\pm \leftrightarrow$ largest/smallest slopes in the geodesic)

Entropy via Bridgeland stability conditions

$(\mathcal{D}, F)$ as before. $G \in \mathcal{D}$ a split generator. $\sigma \in \mathrm{Stab}(\mathcal{D})$. Define

$$h_\sigma(F) := \limsup_{n\to\infty} \frac{1}{n} \log m_\sigma(F^n G)$$

- cf. $h_{\mathrm{top}}(f) = \lim_{n\to\infty} \frac{1}{n} \log \ell_g(f^n(c))$ for pseudo-Anosov maps
- In Fukaya categories of surfaces, mass = length (HKK)

Basic properties (Ikeda):

- $h_\sigma(F)$ is independent of the choice of G.
- $0 \leq h_\sigma(F) \leq h_{\mathrm{cat}}(F)$.
- $h_{\sigma_1}(F) = h_{\sigma_2}(F)$ if σ_1, σ_2 lie in the same connected component.

New invariants – "Shifting numbers"

Motivation: Categorical entropy doesn't capture the "amount of shifts".

- $h_{\mathrm{cat}}([n]) = 0$.
- The definition of $h_\sigma(F)$ only uses the mass function, not the phase functions $\phi^\pm$.

We'd like to define an invariant that captures precisely the amount of shifts of endofunctors of triangulated categories.

Idea: Use the idea of Poincaré translation/rotation numbers to define the asymptotic amount of shifts.

Poincaré translation numbers

Consider the central extension

$$0 \to \mathbb{Z} \to \mathrm{Homeo}_{\mathbb{Z}}^{+}(\mathbb{R}) \to \mathrm{Homeo}^{+}(S^1) \to 1$$

of orientation-preserving homeomorphisms of the circle.

Each element $f \in \mathrm{Homeo}_{\mathbb{Z}}^{+}(\mathbb{R})$ is a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x) + 1$.

Poincaré translation number:

$$\rho(f) := \lim_{n \to \infty} \frac{f^{(n)}(x_0) - x_0}{n}.$$

Translation numbers vs Shifting numbers

In our setting, we have a central extension

$$0 \to \mathbb{Z} \to \mathrm{Aut}(\mathcal{D}) \to \mathrm{Aut}(\mathcal{D})/[1] \to 1.$$

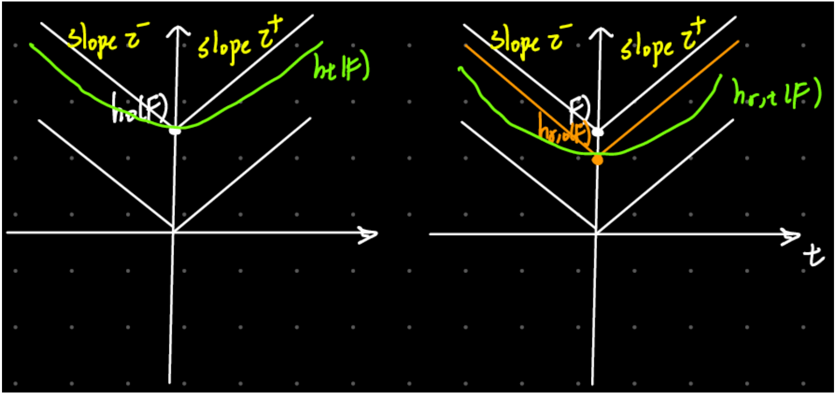
Translation numbers	Shifting numbers
$f \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$	$F \in \mathrm{Aut}(\mathcal{D})$
$x_0 \in \mathbb{R}$	$G \in \mathcal{D}$
amount of translation	phases $\phi^\pm_\sigma \colon \mathrm{Ob}(\mathcal{D}) \to \mathbb{R}$
$f^{(n)}(x_0) - x_0$	$\phi^\pm_\sigma(F^n G) - \phi^\pm_\sigma(G)$
translation number	upper/lower shifting numbers

Definition of shifting numbers

Theorem (F.–Filip, 2020)

- *The limit*
$$\lim_{n\to\infty} \frac{\phi_\sigma^\pm(F^nG) - \phi_\sigma^\pm(G)}{n}$$
always exists, and is independent of the choices of G and σ. They're defined to be the upper/lower shifting numbers, denoted by $\tau^\pm(F)$.
- *The function*
$$h_t(F) := \lim_{n\to\infty} \frac{1}{n} \log \Big(\sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}(G, F^nG[k]) e^{-kt} \Big),$$
is a convex function in t satisfying:
  - $t \cdot \tau^+(F) \leq h_t(F) \leq h_0(F) + t \cdot \tau^+(F)$ *for* $t \geq 0$, *and*
  - $t \cdot \tau^-(F) \leq h_t(F) \leq h_0(F) + t \cdot \tau^-(F)$ *for* $t \leq 0$.

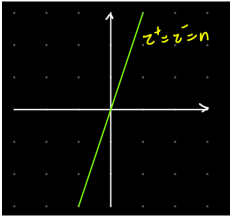
A similar statement holds for $h_{\sigma,t}(F)$.



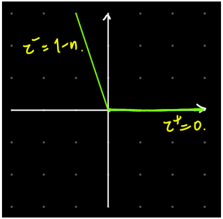
slope τ^-
slope τ^+
$h_t(F)$
$h_0(F)$
slope τ^-
slope τ^+
$h_{\sigma,t}(F)$
$h_{\sigma,0}(F)$
t

Examples

$F = [n]$:



F is a spherical twist in an n-dimensional variety (or Dehn twist along Lagrangian S^n):



Properties of shifting numbers

Translation numbers	Shifting numbers
$\rho((+k) \circ f) = \rho(f) + k$	$\tau^{\pm}([k] \circ F) = \tau^{\pm}(F) + k$
$\rho(f' f f'^{-1}) = \rho(f)$	$\tau^{\pm}(F' F F'^{-1}) = \tau^{\pm}(F)$
$\rho(f^n) = n\rho(f)$	$\tau(F^n) = n\tau(F)$

Here $\tau := (\tau^+ + \tau^-)/2$.

Question

$\rho\colon \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ is a quasimorphism. What about $\tau\colon \mathrm{Aut}(\mathcal{D}) \to \mathbb{R}$? i.e. does there exist $C > 0$ such that

$$|\tau(FF') - \tau(F) - \tau(F')| < C$$

hold for all $F, F' \in \mathrm{Aut}(\mathcal{D})$?

Quasimorphisms from shifting numbers

Theorem (F.–Filip, 2020)

Let $\mathcal{D} = \mathcal{D}^b\mathrm{Coh}(X)$. The shifting number $\tau\colon \mathrm{Aut}(\mathcal{D}) \to \mathbb{R}$ is a quasimorphism if X is an elliptic curve, abelian surface, or $\pm K_X$ ample.

- *When X is an elliptic curve, τ can be decomposed as*

  $$\mathrm{Aut}(\mathcal{D}) \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$$

  where the first map is a group homomorphism and the second map is the Poincaré translation number.
- *When $\pm K_X$ ample, $\mathrm{Aut}(\mathcal{D}) = (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)) \times \mathbb{Z}[1]$, and τ can be decomposed as*

  $$\mathrm{Aut}(\mathcal{D}) \to \mathbb{Z} \hookrightarrow \mathbb{R}$$

  where the first map is the projection to the $\mathbb{Z}[1]$-factor.
- *When X is an abelian surface, τ coincides with a standard quasimorphism on certain Lie group of Hermitian type.*

Quasimorphisms from shifting numbers (cont'd)

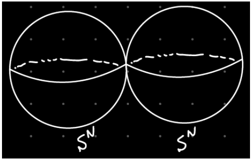
Theorem (F.–Filip, 2020)

Let $N \geq 3$ and $\mathcal{D}_N$ be the N-Calabi–Yau category of the A_2 quiver. Then

$$\tau\colon \mathrm{Aut}(\mathcal{D}_N) \to \mathbb{R}$$

is a quasimorphism related to the Rademacher function on $\mathrm{PSL}(2, \mathbb{Z})$.

This is the simplest example that admits spherical twists with non-trivial couplings.



Future directions

There are still a lot of problems to investigate!

- Algebraicity of entropy or shifting numbers?
- If $h_{\mathrm{cat}}(F) = 0$ and $h_{\mathrm{poly}}(F) > 0$, does F preserve some objects? Use this to study whether maps of polynomial degree growth preserve a fibration?
- Suppose X is an algebraic surface such that there exists $F \in \mathrm{Aut}(\mathcal{D}^b(X))$ with $h_{\mathrm{cat}}(F) > 0$, what can we say about the geometry of X?
- Categorical trichotomy: a notion of pseudo-Anosov autoequivalence was introduced in F.–Filip–Haiden–Katzarkov–Liu.
- Arithmetic categorical dynamics?

Thank you for your attention!

Reference:

- Dimitrov–Haiden–Katzarkov–Kontsevich. *Dynamical systems and categories*. arXiv:1307.8418.
- Fan. *Entropy of an autoequivalence on Calabi-Yau manifolds*. arXiv:1704.06957.
- Fan–Filip. *Asymptotic shifting numbers in triangulated categories*. arXiv:2008.06159.