Next Tuesday: Review: Bring your questions!

OH: ~~Next Thursday~~ $\to$ Next Wed. 11:30 – 1pm PDT.

Next Thursday: 1st Midterm.

open: Wed. 1pm. close: Thu. 1pm 90 mins

- practice exam on the course website.
- the actual exam will be shorter (but perhaps slightly harder).

Last time:

- Basis: l.i. vectors that span V.
- given basis $\mathcal{B}$, we defined $V \xrightarrow{[\ ]_{\mathcal{B}}} \mathbb{R}^n$ (# of basis vectors.)
- find a basis of Nul(A).

Recap: $A \xrightarrow[\text{reduction}]{\text{row}} B = \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

* $\mathrm{Nul}(A) = \mathrm{Nul}(B)$.

* $\mathrm{Nul}(B) = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix} : x_2, x_4, x_5 \in \mathbb{R} \right\}$

they're l.i.
hence form a basis of Nul(B).

Find a basis of Col(A):

$A \xrightarrow[\text{reduction}]{\text{row}} B = \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

* $\mathrm{Col}(A) \neq \mathrm{Col}(B)$ in general.

e.g. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underset{\text{row reduce}}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\mathrm{Col} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \neq \mathrm{Col} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

* linearly dependence relations among the columns of A
= ___ ___ ___ ___ ___ ___ . of B.

$$\underset{A}{\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}} \underset{\text{row r.}}{\sim} \underset{B}{\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix}}$$

(HW1 Extra Credit)

$$\underline{x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{0}} \iff x_1\vec{b}_1 + x_2\vec{b}_2 + x_3\vec{b}_3 = \vec{0}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0} \iff B\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$$

$\{i_1, \cdots, i_k\} \subseteq \{1, \cdots, n\}$

$$\boxed{\begin{gathered} \{\vec{a}_{i_1}, \cdots, \vec{a}_{i_k}\} \text{ is a basis of } \mathrm{Col}(A) \\ \Updownarrow \\ \{\vec{b}_{i_1}, \cdots, \vec{b}_{i_k}\} \text{ is a basis of } \mathrm{Col}(B) \end{gathered}}$$

① $\{\vec{a}_{i_1}, \dots, \vec{a}_{i_k}\}$ l.i.

② $\vec{a}_j \in \operatorname{Span}\{\vec{a}_{i_1}, \dots, \vec{a}_{i_k}\}$ $\forall\, 1 \le j \le n$

These 2 conditions are preserved under row reductions

$\{\vec{a}_{i_1}, \dots, \vec{a}_{i_k}\}$ l.i. $\iff$ $\{\vec{b}_{i_1}, \dots, \vec{b}_{i_k}\}$ l.i.

b/c $\{\vec{a}_{i_1}, \dots, \vec{a}_{i_k}\}$ l.d. $\qquad \{\vec{b}_{i_1}, \dots, \vec{b}_{i_k}\}$ l.d.

$\Updownarrow \qquad\qquad \Updownarrow$

$\exists\, c_1, \dots, c_k$ not all 0 st. $\underline{c_1\vec{a}_{i_1} + \dots + c_k\vec{a}_{i_k} = \vec{0}}$ $\iff$ $\exists\, c_1, \dots, c_k$ not all 0 $c_1\vec{b}_{i_1} + \dots + c_k\vec{b}_{i_k} = \vec{0}$

$$B = \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Conclusion</u>: pivot columns give a basis of Col(A)

pivot columns give a basis of Col(B)

$\Downarrow$

A

$\{\vec{a}_1, \vec{a}_3\}$ gives a basis of Col(A)

Thm If a v.s. V admits a basis of n vectors, then *any* basis of V also consists of n vectors.

Def. • Such n is called the **dimension** of V, $\dim V$.

• If V can't be spanned by finitely many vectors, we call it infinite dimensional.

Lemma If $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of V, then any set of vectors in V with $> n$ vectors must be l.d.

pf.

$$V \xrightarrow[\cong]{[\ ]_B} \mathbb{R}^n$$

$$\left.\begin{matrix} \vec{w}_1 \\ \vdots \\ \vec{w}_{n+1} \end{matrix}\right\} \longmapsto \left.\begin{matrix} [\vec{w}_1]_B \\ \vdots \\ [\vec{w}_{n+1}]_B \end{matrix}\right\} \text{ they must be l.d.}$$

$$\begin{bmatrix} w_1 & \cdots & w_{n+1} \end{bmatrix}_{n \times (n+1)}$$

$\exists\, c_1, \dots, c_{n+1}$ not all 0

st. $c_1 [\vec{w}_1]_B + \cdots + c_{n+1} [\vec{w}_{n+1}]_B = \vec{0}$

$$\|$$

$[c_1 \vec{w}_1 + \cdots + c_{n+1} \vec{w}_{n+1}]_B$

$\Rightarrow c_1 \vec{w}_1 + \cdots + c_{n+1} \vec{w}_{n+1} = \vec{0}$ since $[\ ]_B$ bij.

$\Rightarrow \{\vec{w}_1, \dots, \vec{w}_{n+1}\}$ l.d. $\square$

pf of Thm. B basis of V of n vectors.

B' another basis

Lemma $\Rightarrow$ B' consists of $\leq n$ vectors

Let's say B' consists of n' vectors. $\underline{n' \leq n}$

Lemma $\Rightarrow$ $n \leq n'$

$\Rightarrow$ $n = n'$

$\square$

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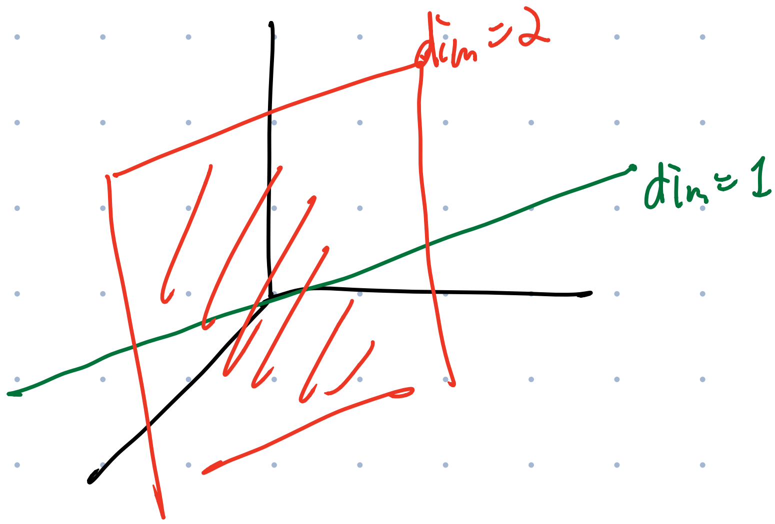
e.g. $\dim \mathbb{R}^n = n$, $\dim \mathrm{Poly}_{\leq n} = n+1$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots \right\}$ $\{1, x, x^2, \dots, x^n\}$



$\underline{\dim \{0\} := 0}$

e.g. $H = \{ \underline{x_1 + x_2 + \dots + x_n = 0} : x_1, \dots, x_n \in \mathbb{R} \} \subseteq \mathbb{R}^n$

(a hyperplane in $\mathbb{R}^n$)

$$\begin{bmatrix} 1 & \underbrace{1 \cdots 1}_{\text{free}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

$$H = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} \right\}$$

$\dim H = \underline{n-1}$.

$A: m \times n$

$\dim \underline{\mathrm{Nul}(A)} = \#$ free variables.

$= n - \#$ pivots.

$\boxed{\dim \underline{\mathrm{Col}(A)}} = \underline{\# \text{ pivots}}. \Longrightarrow : \boxed{\mathrm{rank}(A)}$

<u>Rank-Nullity Thm</u>:

$$\dim \mathrm{Nul}(A) + \mathrm{rank}(A) = \overset{\#\text{ of columns of } A}{n}$$

More generally, $T: V \longrightarrow W$, $\dim V < +\infty$.

$$\dim V = \dim \ker(T) + \dim \mathrm{Im}(T)$$

Thm $W \subseteq V$ subspace $\dim V < +\infty$.

Then $\dim W \le \dim V = n$

$V = \mathrm{Span}\{v_1, \dots, v_n\}$

$V = \mathbb{R}^2 \supset W = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Pf. Try to "build" a basis of W, show that # of vectors in the basis $\le n$.

1) $W = \{0\}$ ✓

2) $W \neq \{0\}$.

Pick any $0 \neq \vec{v}_1 \in W$.

Assume W has a basis, $\{w_1, \dots, \dots\}$ $\Rightarrow$ # of vectors in the basis $\le n$

- 1) $W = \mathrm{Span}\{\vec{v}_1\}$. $\Rightarrow$ $\{\vec{v}_1\}$ is a basis of W.

  2) $W \supsetneq \mathrm{Span}\{\vec{v}_1\}$

  Pick $\vec{v}_2 \in W \setminus \mathrm{Span}\{\vec{v}_1\}$.

  $\Rightarrow$ $\{\vec{v}_1, \vec{v}_2\}$ l.i.

  $$\{\vec{v}_1, \vec{v}_2\} \subseteq W$$

- $W = \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$ $\Rightarrow$ $\{\vec{v}_1, \vec{v}_2\}$ basis of W
- $W \supsetneq \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$.

  Pick $\vec{v}_3 \in W \setminus \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$

  $\Rightarrow$ $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ l.i.

$\boxed{\Rightarrow \text{ We can find } \vec{v}_1, \dots, \vec{v}_k \in W \text{ which gives a basis of } W}$

Change of basis $\quad V$

$$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}, \quad \mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$$

$$\begin{tikzcd}
\vec{x} \arrow[rr, bend left, mapsto] \arrow[dd, bend right, mapsto] & & {[\vec{x}]_{\mathcal{B}}} \\
& V \arrow[r, "{[\ ]_{\mathcal{B}}}", "\cong"'] \arrow[d, "{[\ ]_{\mathcal{C}}}"', "\cong"] & \mathbb{R}^n \arrow[dl, "\cong"] \\
{[\vec{x}]_{\mathcal{C}}} & \mathbb{R}^n &
\end{tikzcd}$$

$$?? \longleftrightarrow \vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vec{v} \longmapsto [\vec{v}]_{\mathcal{B}} = \vec{e}_i$$

$$\vec{v} = 0\vec{b}_1 + \cdots + 1\vec{b}_i + \cdots$$
$$= \vec{b}_i$$

$$\mathbb{R}^n \xrightarrow{[\ ]_{\mathcal{B}}^{-1}} V \xrightarrow{[\ ]_{\mathcal{C}}} \mathbb{R}^n$$

$$\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{e}_i \longmapsto \vec{b}_i \longmapsto [\vec{b}_i]_{\mathcal{C}}$$

Def:

$$\underset{C \leftarrow B}{P} := \begin{bmatrix} | & | & & | \\ [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \\ | & | & & | \end{bmatrix}_{n \times n}$$

change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$

$$\boxed{\underset{C \leftarrow B}{P} [\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}}$$

e.g.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "{[\ ]_B}"] \arrow[d, "{[\ ]_C}"'] & \mathbb{R}^n \\ \mathbb{R}^n & \end{tikzcd}$$

$B = \{\vec{b}_1, \ldots, \vec{b}_n\}$

$C = \{\vec{c}_1, \ldots, \vec{c}_n\}$

$$\boxed{\underset{C \leftarrow B}{P}} = \begin{bmatrix} [\vec{b}_1]_C & [\vec{b}_2]_C & \cdots \end{bmatrix}$$

$$[\vec{b}_i]_C = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\vec{b}_i = x_1 \vec{c}_1 + \cdots + x_n \vec{c}_n$$

$$= \begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix} [\vec{b}_i]_C \qquad P_C^{-1} P_B$$

$$\underbrace{\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix}}_{P_B} = \underbrace{\begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix}}_{P_C} \begin{bmatrix} [\vec{b}_1]_C & \cdots & [\vec{b}_n]_C \end{bmatrix}$$

$$\left[\, \boxed{C} \;\middle|\; B \,\right]_{n\times 2n} \xrightarrow[\text{reduce}]{\text{row}} \left[\, I \;\middle|\; \underline{C^{-1}B} \,\right]$$

$$\underline{C^{-1}B}$$