Last time: A: $n\times n$ matrix.

- char. poly. $\det(A-\lambda I) = \prod_{i=1}^{k} (\lambda_i - \lambda)^{\mathrm{mult}(\lambda_i)}$, $\lambda_i \in \mathbb{C}$ in general

  where $\{\lambda_1, \dots, \lambda_k\}$ are distinct eigenvalues of A,

  and $\mathrm{mult}(\lambda_i)$ is the multiplicity of the eigenvalue λ_i.

- For each eigenvalue λ_i, $\{0\} \subsetneq \mathrm{Nul}(A-\lambda_i I)$ is the eigenspace of λ_i.

  Any nonzero vector in $\mathrm{Nul}(A-\lambda_i I)$ is an eigenvector of λ_i.

- A diagonalizable $\Leftrightarrow$ A is similar to a diagonal matrix $A = PDP^{-1}$.

  $\Leftrightarrow$ $\exists$ eigenbasis of A.

- A diagonalizable $\Rightarrow$ $\forall \vec{v} \in \cancel{\mathbb{R}^n}\ \mathbb{C}^n$, $\exists! \ \vec{v}_i \in \mathrm{Nul}(A-\lambda_i I)$

  $\{\lambda_1, \dots, \lambda_k\}$ distinct eigenvalues.

  s.t. $\vec{v} = \vec{v}_1 + \dots + \vec{v}_k$.

In other words: $\cancel{\mathbb{R}^n}\ \mathbb{C}^n = \mathrm{Nul}(A-\lambda_1 I) \oplus \dots \oplus \mathrm{Nul}(A-\lambda_k I)$.

Rmk: The direct sum decomposition does not hold if A is not diagonalizable, e.g. $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $\mathrm{Nul}(A-2I) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \neq \mathbb{R}^2$.

Today:

- $1 \leq \dim \mathrm{Nul}(A-\lambda I) \leq \mathrm{mult}(\lambda)$.
- diagonalizable $\Leftrightarrow$ $\dim \mathrm{Nul}(A-\lambda I) = \mathrm{mult}(\lambda)$ $\forall \lambda$.
- Complex eigenvalues of a real matrix.

Thm: A: $n\times n$

$1 \leq \dim \mathrm{Nul}(A-\mu I) \leq \mathrm{mult}(\mu)$ $\forall$ eigenvalue μ

pf

$\mathrm{Nul}(A-\mu I)$ has basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ $\dim \mathrm{Nul}(A-\mu I) = k$

$\mathbb{R}^n \supseteq$

We can pick $\vec{v}_{k+1}, \dots, \vec{v}_n$ s.t. $\boxed{\{\vec{v}_1, \dots, \vec{v}_k, \dots, \vec{v}_n\}}$

form a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$)

$$\begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}^{-1} A \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}$$

similar to A,
so it has the same char. poly. as A

b/c $\vec{v}_1, \dots, \vec{v}_k \in \mathrm{Null}(A - \mu I)$.

$$A \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix} = \begin{bmatrix} \mu\vec{v}_1 & \mu\vec{v}_2 & \cdots & \mu\vec{v}_k & | & | & | & | \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}^{-1} A \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}}_{\text{similar } A} = \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}^{-1} \begin{bmatrix} \mu\vec{v}_1 \cdots \mu\vec{v}_k & | & | & | \end{bmatrix}$$

??

$$= \begin{bmatrix} \mu & 0 & \cdots & 0 & ??? \\ 0 & \mu & & \vdots & \\ \vdots & 0 & \ddots & & \\ & \vdots & & \mu & \\ 0 & 0 & \cdots & 0 & \end{bmatrix} = B$$

size k

$$\begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}^{-1} \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ \vdots & & \ddots \\ 0 & & & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}^{-1} (\mu \vec{v}_1) = \mu \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$\det(B-\lambda I) = \det \begin{pmatrix} \begin{matrix} \mu-\lambda & & 0 \\ & \ddots & \\ 0 & & \mu-\lambda \end{matrix} & ??? \\ 0 & \begin{matrix} ??-\lambda & & ?? \\ & \ddots & \\ ?? & & ??-\lambda \end{matrix} \end{pmatrix}$$

(the first diagonal block is $k \times k$)

$$= (\mu-\lambda)^k \det\left(\qquad \right)$$

$$\Rightarrow \quad \text{mult}(\mu) \geq k := \dim \text{Nul}(A-\mu I). \quad \square$$

Thm diagonalizable $\iff$ $\dim \text{Nul}(A-\lambda I) = \text{mult}(\lambda)$ $\forall$ eigenvalue λ

pf $(\Rightarrow)$

$$A = PDP^{-1}$$

D diagonal

$$\dim \text{Nul}(A-\lambda I) = \dim \text{Nul}(D-\lambda I)$$

$$\text{Nul}(A-\lambda I) = \text{Nul}(PDP^{-1} - \lambda I)$$
$$= \text{Nul}(P(D-\lambda I)P^{-1})$$

$$D = \begin{bmatrix} \lambda & & & & & \\ & \lambda & & & & \\ & & \ddots & & & \\ & & & \mu & & \\ & & & & \ddots & \\ & & & & & \mu \end{bmatrix}$$

Fact. P invertible, ~~Null~~ $\dim \text{Nul}(PA) = \dim \text{Nul}(A)$
$= \dim \text{Nul}(AP)$

[A, P square $n \times n$]

$$D - \lambda I = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \neq 0 & & \\ & & & & \neq 0 & \\ & & & & & \ddots \end{bmatrix}$$

mult(λ)

$$\mathrm{Nul}(D - \lambda I) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

mult(λ)

$$\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \\ 0 \end{bmatrix}$$

"($\Rightarrow$)" diagonalizable $\Rightarrow$ $\exists$ eigenbasis

a basis of $\mathbb{R}^n$ where each vector is an eigenvector of A

$$v_{\lambda_1}^{(1)} \cdots v_{\lambda_1}^{(a_1)} \; v_{\lambda_2}^{(1)} \cdots v_{\lambda_2}^{(a_2)} \cdots$$

$\dim \mathrm{Nul}(A - \lambda_1 I) = a_1$

$\sum a_i = n$

Claim: $a_1 = \dim \mathrm{Nul}(A - \lambda_1 I)$

$\sum \mathrm{mult}(\lambda_i) = n$

$\Rightarrow \dim \mathrm{Nul}(A - \lambda_i I) = \mathrm{mult}(\lambda_i) \;\forall i$

pf $v_{\lambda_1}^{(1)}, \ldots, v_{\lambda_1}^{(a_1)}$ l.i.

$$\mathrm{Span}\{ v_{\lambda_1}^{(1)}, \ldots, v_{\lambda_1}^{(a_1)} \} \overset{??}{\subseteq} \mathrm{Nul}(A - \lambda_1 I)$$

If $\boxed{w \in \mathrm{Nul}(A - \lambda_1 I)}$.

$w = (\ \ ^{\lambda_1}\ \ ) + (\ \ ^{\lambda_2}\ \ ) + (\ldots) + \ldots$

0

$$w = \underbrace{c_{\lambda_1}^{(1)} v_{\lambda_1}^{(1)} + \cdots + c_{\lambda_1}^{(a_1)} v_{\lambda_1}^{(a_1)}}_{\in\, Nul(A-\lambda_1 I)} + \underbrace{c_{\lambda_2}^{(1)} v_{\lambda_2}^{(1)} + \cdots}_{\in\, Nul(A-\lambda_2 I)} + \cdots$$

$$\| $$

$$\underset{\in\, Nul(A-\lambda_1 I)}{w} + \underset{\in\, Nul(A-\lambda_2 I)}{0} + 0 + 0 \cdots + 0$$

$$\Rightarrow w = c_{\lambda_1}^{(1)} v_{\lambda_1}^{(1)} + \cdots + c_{\lambda_1}^{(a_1)} v_{\lambda_1}^{(a_1)}$$

$$\in \mathrm{Span}\{v_{\lambda_1}^{(1)}, \ldots, v_{\lambda_1}^{(a_1)}\}$$

$$\mathbb{R}^n = Nul(A-\lambda_1 I) \oplus \cdots \oplus Nul(A-\lambda_k I)$$

$(\Leftarrow)$ $\boxed{\dim \boxed{Nul(A-\lambda I)} = mult(\lambda)}$ $\forall \lambda$.

$\overset{??}{\Longrightarrow}$ diagonalizable $\Leftrightarrow$ $\exists$ eigenbasis

$Nul(A-\lambda_i I)$ pick a basis $\{v_{\lambda_i}^{(1)}, \ldots, v_{\lambda_i}^{(a_i)}\}$

$mult(\lambda_i) = a_i$

Claim $\{v_{\lambda_1}^{(1)}, \ldots, v_{\lambda_1}^{(a_1)}, v_{\lambda_2}^{(1)} \cdots v_{\lambda_2}^{(a_2)}, \ldots\}$ is a basis of $\mathbb{R}^n$

$$\sum a_i = \sum \text{mult}(\lambda_i) = n$$

$\boxed{\text{l.i.}}$:

$$0 = \underbrace{c_{\lambda_1}^{(1)} v_{\lambda_1}^{(1)} + \cdots + c_{\lambda_1}^{(a_1)} v_{\lambda_1}^{(a_1)}}_{\parallel \atop 0} + \underbrace{c_{\lambda_2}^{(1)} v_{\lambda_2}^{(1)} + \cdots\cdots}_{\parallel \atop 0} \; {\parallel \atop 0}$$

$\Rightarrow$

$\Rightarrow$ each $c_{\alpha}^{(i)} = 0$ $\square$

"Algorithm" to check diagonalizable? get diagonalization:

① char. poly

$\rightarrow$ eigenvalues, $\text{mult}(\lambda)$

② eigenspace $\text{Nul}(A - \lambda I)$

③ diagonalizable $\Leftrightarrow$ $\dim \text{Nul}(A - \lambda I) = \text{mult}(\lambda)$ $\forall \lambda$

④ Suppose diagonalizable,
Basis of $\text{Nul}(A - \lambda I)$ for each λ

$\rightsquigarrow$ a basis of $\mathbb{R}^n$

(eigenbasis)

$\rightsquigarrow$ $P = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ $\rightsquigarrow$ $\cancel{P^{-1}} AP = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$$A\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{bmatrix}$$

$$= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & 1 & & \\ & & \lambda_1 & & \\ & & & \lambda_2 & 1 \\ & & & & \lambda_2 \\ & & & & & \ddots \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$$

Jordan canonical form
(normal)

$\underline{A \in M_{n\times n}(\mathbb{R})}$ $\quad$ $\underline{\lambda \in \mathbb{C}\setminus\mathbb{R}}$

$\boxed{Av = \lambda v}$ $\quad v \neq 0$

$$\Downarrow$$

$$\overline{Av} = \overline{\lambda v}$$

$$\| \qquad\qquad \|$$

$$A\bar{v} \qquad \bar{\lambda}\,\bar{v}$$

$\bar{\lambda}$ eigenvalue

$\bar{v}$ eigenvector

$$A = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} \overline{a_{11}} & \cdots & \overline{a_{1n}} \\ \overline{a_{n1}} & \cdots & \overline{a_{nn}} \end{bmatrix}$$

$\boxed{\overline{AB} = \bar{A}\,\bar{B}}$

$\overline{c_1 c_2} = \bar{c_1}\,\bar{c_2}$

$c = x + iy \in \mathbb{C} \quad x, y \in \mathbb{R}$

$\bar{c} = x - iy$

$(x_1 - iy_1)(x_2 - iy_2)$

$\overline{(x_1 + iy_1)(x_2 + iy_2)}$

$= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}$

$= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1)$

A: real

$\det(A - \lambda I) = \prod_i (\mu_i - \lambda)^{a_i} \prod_i (\lambda_i - \lambda)^{b_i} \prod_i (\bar{\lambda}_i - \lambda)^{b_i}$

$\mu_i \in \mathbb{R}$ $\quad \lambda_i \in \mathbb{C} \setminus \mathbb{R}$

$\boxed{\text{real coeff. poly}}$