Math 104 (Section 4) Final Exam (Practice Exam) December 18, 2019

Name: Solution

- You have 160 minutes to complete the exam (3:10pm – 5:50pm).
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using. Exception: if you are asked to prove a theorem, you are not allowed to use it!
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	30	
2	15	
3	20	
4	20	
5	20	
6	30	
7	30	
Total		

1. True/False questions. You don't need to justify your answers.

(a) (5 points) The sequence (s_n) defined by

$$s_1 = 1, \quad s_{n+1} = \frac{3s_n^2 + 1}{2s_n - 1} \quad (n \in \mathbb{N})$$

is a convergent sequence.

$\boxed{\text{False}}$.

$$S_{n+1} = \frac{3S_n^2+1}{2S_n-1} > \frac{3S_n^2}{2S_n} = \frac{3}{2}S_n.$$

Hence $\lim_{n\to\infty} S_n = +\infty$ doesn't converge.

(b) (5 points) Every Cauchy sequence is bounded.

$\boxed{\text{True}}$

(c) (5 points) The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & (x \in \mathbb{Q}) \\ x \sin x & (x \notin \mathbb{Q}) \end{cases}$$

is continuous at $x = 0$.

$\boxed{\text{True}}$

$|f(x)| \leq |x \sin x| \leq |x|$ for any $x \in \mathbb{R}$.

By squeeze lemma, we have $\lim_{x\to 0} f(x) = 0$.

Since $f(0) = 0$, so f is continuous at $x=0$.

(d) (5 points) The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(x^2)$ is uniformly continuous on $\mathbb{R}$.

$\boxed{\text{False}}$

The proof is similar to:

Practice Exam of Second Midterm. #4 (III).

(e) (5 points) Let (f_n) be a sequence of functions on $\mathbb{R}$. If (f_n) converges uniformly to a function f on $\mathbb{R}$, then f is also continuous on $\mathbb{R}$.

$\boxed{\text{True}}$

(f) (5 points) The following inequality holds:

$$\left| \int_{-\pi}^{\pi} x^2(2\cos(x^2) - \sin(e^{2x^2}))\,dx \right| \leq 2\pi^3.$$

$\boxed{\text{True}}$

$$\leq$$

$$\int_{-\pi}^{\pi} \left| x^2(2\cos(x^2) - \sin(e^{2x^2})) \right| dx$$

$$\leq$$

$$\int_{-\pi}^{\pi} 3x^2\,dx = 2\pi^3.$$

2. Find the following quantities. They are either real numbers or $\pm\infty$. You don't need to justify your answers.

(a) (5 points)

$$\lim_{x\to 0} \frac{1-\sin x}{x^2}.$$

$\boxed{+\infty}$

For $0<|x|<\frac{\pi}{6}$, we have $\frac{1-\sin x}{x^2} \geq \frac{1}{2x^2}$

Hence $\lim_{x\to 0} \frac{1-\sin x}{x^2} = +\infty$

(b) (5 points) The radius of convergence of the power series

$$\sum_{n=0}^{\infty} 3^{2n+(-1)^n} x^n.$$

$\boxed{1/9}$

$\limsup_{n\to\infty} \left|3^{2n+(-1)^n}\right|^{1/n} = \limsup_{n\to\infty} 9 \cdot 3^{(-1)^n/n} = 9.$

(c) (5 points)

$$\lim_{n\to\infty} n^{1/n}.$$

$\boxed{1}$

c.f. Theorem 9.7 in the textbook.

3. (20 points) Let $a < b$ be two real numbers, and let $f\colon (a,b) \to \mathbb{R}$ be a differentiable function. Assume that f attains its maximum at $c \in (a,b)$. Prove that $f'(c) = 0$.

- For any $x \in (c,b)$, we have $x - c > 0$ and $f(c) \geq f(x)$.

  Hence $\dfrac{f(x)-f(c)}{x-c} \leq 0$.

  $$\Rightarrow \quad f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \leq 0.$$

- For any $x \in (a,c)$, we have $x - c < 0$ and $f(c) \geq f(x)$.

  Hence $\dfrac{f(x)-f(c)}{x-c} \geq 0$.

  $$\Rightarrow \quad f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \geq 0.$$

Therefore $f'(c) = 0$. $\square$

4. (20 points) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Prove that f is integrable on $[0,1]$.

We'll use the following theorems: ① $f: [a,b] \longrightarrow \mathbb{R}$ is integrable if and only if $\forall \varepsilon > 0$, $\exists$ partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

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② Thm: A continuous function on a closed and bounded interval is uniformly continuous.

Proof: By ②, $f: [0,1] \longrightarrow \mathbb{R}$ is uniformly continuous.

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\underset{x,y \in [0,1]}{|x-y| < \delta} \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

Choose a partition $P = \{a = t_0 < \cdots < t_n = b\}$ s.t. $t_k - t_{k-1} < \delta$ $\forall k$.

Then
$$U(f,P) - L(f,P) = \sum_{k=1}^{n} (t_k - t_{k-1})\Big(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\Big)$$
$$< \sum_{k=1}^{n} (t_k - t_{k-1}) \cdot \frac{\varepsilon}{b-a} = \varepsilon.$$

By ①, f is integrable. $\square$

5. (20 points) Let (f_n) be a sequence of integrable functions on $[a, b]$, and suppose that $f_n \to f$ uniformly on $[a, b]$. Prove that f is integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

$f_n \to f$ unif. on $[a,b]$.

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $|f_n(x) - f(x)| < \dfrac{\varepsilon}{3(b-a)}$ $\quad \forall x \in [a,b]$, $n > N$.

- For any $n > N$, since f_n is integrable, $\exists$ Partition P of $[a,b]$ s.t. $U(f_n, P) - L(f_n, P) < \dfrac{\varepsilon}{3}$.

- For any partition subinterval $[t_{k-1}, t_k]$ of P, we have

$$M(f, [t_{k-1}, t_k]) \leq M(f - f_n, [t_{k-1}, t_k]) + M(f_n, [t_{k-1}, t_k])$$
$$< \frac{\varepsilon}{3(b-a)} + M(f_n, [t_{k-1}, t_k]).$$

$$m(f, [t_{k-1}, t_k]) \geq m(f - f_n, [t_{k-1}, t_k]) + m(f_n, [t_{k-1}, t_k])$$
$$> -\frac{\varepsilon}{3(b-a)} + m(f_n, [t_{k-1}, t_k]).$$

- Hence, $U(f,P) = \sum_{k} (t_k - t_{k-1}) \cdot M(f, [t_{k-1}, t_k]) < \dfrac{\varepsilon}{3} + U(f_n, P)$.

and $L(f,P) > L(f_n, P) - \dfrac{\varepsilon}{3}$.

$\Rightarrow$ $U(f,P) - L(f,P) < U(f_n,P) - L(f_n,P) + \dfrac{2\varepsilon}{3} < \varepsilon.$ $\Rightarrow$ f is integrable. $\square$

- $\int_a^b f_n - \frac{\varepsilon}{3} = \sup_P L(f_n,P) - \frac{\varepsilon}{3} \leq \sup_P L(f,P) = \int_a^b f = \inf_P U(f,P) \leq \frac{\varepsilon}{3} + \inf_P U(f_n,P) = \frac{\varepsilon}{3} + \int_a^b f_n$

$\Rightarrow$ $\left| \int_a^b f - \int_a^b f_n \right| \leq \dfrac{\varepsilon}{3}$. Hence $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$. $\square$

6. Consider the series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

(a) (5 points) Explain that the series converges. If you use a theorem, check that the assumptions of the theorem are all satisfied.

$\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a decreasing sequence that converges to 0.

By the alternating series theorem, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges.

(b) (10 points) Define the function $f\colon (-1,\infty) \to \mathbb{R}$ by
$$f(x) = \log_e(1+x).$$
Find the Taylor series for $f(x)$ at ~~x = 0~~. the point 0.

$$f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}$$

The Taylor series of $f(x)$ at 0 is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

(c) (15 points) Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log_e 2.$$

(Hint: Taylor's theorem.)

Consider the remainder:

$$R_n(1) := f(1) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} 1^k$$

We need to show that $\lim_{n\to\infty} R_n(1) = 0$.

By Taylor's thm, $\exists\ y_n \in (0,1)$ s.t.

$$R_n(1) = \frac{f^{(n)}(y_n)}{n!} 1^n = \frac{(-1)^{n-1}}{n(1+y_n)^n}$$

Since $y_n > 0$, we have $|R_n(1)| < \frac{1}{n}$.

Hence $\lim_{n\to\infty} R_n(1) = 0$.

Therefore

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot 1^k = f(1)$$

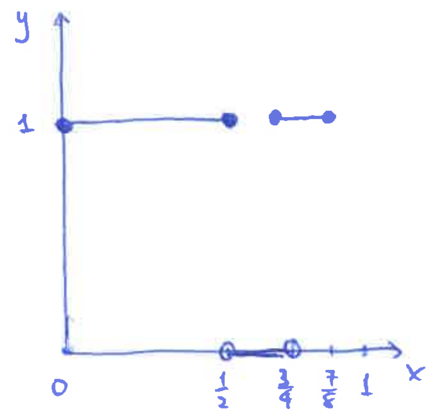
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \qquad \log_e 2$$

$\square$

7. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } 1 - 2^{-2k} \leq x \leq 1 - 2^{-(2k+1)} \text{ for } k = 0, 1, 2, \ldots \\ 0 & \text{if } 1 - 2^{-(2k+1)} < x < 1 - 2^{-(2k+2)} \text{ for } k = 0, 1, 2, \ldots \\ 0 & \text{if } x = 1 \end{cases}$$

(a) (5 points) Find the values $f(0), f(\frac{1}{2}), f(\frac{3}{4}), f(\frac{7}{8})$, and draw the graph of f on $[0, \frac{7}{8}]$.



(b) (5 points) Assume that f is integrable for now. Find $\int_0^1 f$ and express the value in the form $\frac{a}{b}$ $(a \in \mathbb{Z}, b \in \mathbb{N})$. You don't need to justify your answer.

$$\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots = \boxed{\frac{2}{3}}.$$

(c) (20 points) Prove that f is integrable.

$\forall \varepsilon > 0$, take N large s.t. $\frac{2N+1}{2^{2N}} < \varepsilon$.

Consider the partition: $P = \{0 = t_0 < \cdots < t_{2^{2N}} = 1\}$, $t_k = \frac{k}{2^{2N}}$.

Observe that:

$$M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k]) = \begin{cases} 1 & \text{if: } \begin{cases} \frac{k-1}{2^{2N}} = 1 - \frac{1}{2^{2l+1}} & \text{for some } l = 0,\dots,N-1 \\ \text{or} \\ \frac{k}{2^{2N}} = 1 - \frac{1}{2^{2l}} & \text{for some } l = 1,\dots,N \\ \text{or } k = 2^{2N} \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Hence $U(f,P) - L(f,P) = \frac{1}{2^{2N}} \cdot (2N+1) < \varepsilon$. $\square$