Math 104 (Section 6) First Midterm (Practice) February 18, 2020

Name: Solution

- You have 75 minutes to complete the exam.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using. Exception: if you are asked to prove a theorem, you are not allowed to use it!
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	25	
2	20	
3	25	
4	30	
Total	100	

1. (a) (5 points; no partial credits) Write down the precise definition of a sequence (a_n) converging to a real number a.

$\forall \varepsilon > 0$, $\exists N > 0$ s.t.

$n > N \Rightarrow |a_n - a| < \varepsilon$.

(b) (20 points) Prove the following statement based on the definition: Let (a_n) be a sequence converging to a and (b_n) be a sequence converging to b. Suppose that $a_n \leq b_n$ for every $n \in \mathbb{N}$. Then $a \leq b$. (Hint: Prove by contradiction.) (You are not allowed to use any theorem for this problem.)

Assume the contrary that $a > b$.

Let $\varepsilon = \frac{a-b}{2} > 0$.

$\exists N_1 > 0$ s.t. $|a_n - a| < \varepsilon \quad \forall n > N_1$

$\exists N_2 > 0$ s.t. $|b_n - b| < \varepsilon \quad \forall n > N_2$.

Then for any $n > \max\{N_1, N_2\}$, we have

$a_n - b_n > (a-\varepsilon) - (b+\varepsilon) = a - b - 2\varepsilon = 0$.

This contradicts with $a_n \leq b_n \quad \forall n$. $\square$

2. (20 points) Let

$$a_n = \frac{n^2+1}{3n^2+5n}.$$

Prove that (a_n) converges to a real number based on the definition.
(You are not allowed to use any theorem for this problem.)

We claim that $\lim_{n\to\infty} a_n = \frac{1}{3}$.

$\forall \varepsilon > 0$, Choose $N = \frac{5}{9\varepsilon}$, then $\forall n > N$, we have

$$\left|a_n - \frac{1}{3}\right| = \left|\frac{n^2+1}{3n^2+5n} - \frac{1}{3}\right| = \frac{5n-3}{3(3n^2+5n)} < \frac{5n}{9n^2} < \frac{5}{9N} = \varepsilon. \quad \square$$

3. (25 points) Let (a_n) be a bounded and decreasing sequence of real numbers. Prove that (a_n) is convergent.
(You are not allowed to use any theorem for this problem. You can (and should) use the completeness axiom of real numbers, which, by the way, is not a theorem.)

Since (a_n) is bounded, the set $\{a_n : n \in \mathbb{N}\}$ has the greatest lower bound. Let $z := \inf\{a_n : n \in \mathbb{N}\}$.

<u>Claim</u>: $\lim_{n \to \infty} a_n = z$.

• $\forall \varepsilon > 0$, $z + \varepsilon$ is not a lower bound of $\{a_n : n \in \mathbb{N}\}$.

$\Rightarrow$ $\exists N > 0$ s.t. $z \le a_N < z + \varepsilon$.

$\Rightarrow$ $\forall n > N$, $z \le a_n < z + \varepsilon$ since (a_n) is decreasing. (and z is a lower bound •).

$\Downarrow$

$|a_n - z| < \varepsilon$. $\square$

4. There are four statements below:

(I) Let (a_n) be a sequence such that $|a_{n+1} - a_n| < 2^{-n}$ for every $n \in \mathbb{N}$. Then (a_n) must be a convergent sequence. (Hint: Is (a_n) Cauchy?)

(II) Let (a_n) be a sequence that is bounded above. Then we have $\limsup_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.

(III) For all $a \in \mathbb{R}$, we have $\lim_{n\to\infty} \frac{a^n}{n!} = 0$.

(IV) For any metric space (S, d), the intersection of any infinitely many open subsets is open.

(a) (15 points) Choose a statement that is true and prove it. *You are not allowed to choose more than one statement.*
My statement is (I) or (III).

(I): Claim (a_n) is Cauchy ($\Rightarrow$ convergent).
$\forall \varepsilon > 0$, choose $N = -\log_2 \varepsilon + 1$.
Then $\forall\, n > m > N$, we have:

$$|a_n - a_m| \leq |a_n - a_{n-1}| + |a_{n-1} + a_{n-2}| + \cdots + |a_{m+1} - a_m|$$
$$< 2^{-(n-1)} + \cdots + 2^{-m}$$
$$< 2^{-(m-1)} < 2^{-(N-1)} = \varepsilon. \quad \square$$

(III) $\exists\, N \in \mathbb{N}$ s.t. $a < N$. Let $\delta = \frac{a}{N} < 1$.
Then $\forall\, n > N$, we have

$$\frac{a^n}{n!} = \underbrace{\boxed{\frac{\overbrace{a \cdots\cdots a}^{N}}{1\cdot 2 \cdots N}}}_{=A} \frac{a}{N+1}\,\frac{a}{N+2} \cdots \frac{a}{n}$$
$$= A\cdot\left(\frac{a}{N+1}\right)\left(\frac{a}{N+2}\right)\cdots\left(\frac{a}{n}\right) < A\cdot\delta^{n-N}.$$

$\Rightarrow \quad 0 < \frac{a^n}{n!} < A\cdot\delta^{n-N} \quad \forall n > N.$

Since $\lim_{n\to\infty} \delta^{n-N} = 0$ (since $\delta < 1$), by squeeze lemma, we have $\lim \frac{a^n}{n!} = 0. \quad \square$

(b) (15 points) Choose a statement that is false. Give an explicit counterexample and justify it. *You are not allowed to choose more than one statement.*
My statement is (II) or (IV).

(II) $a_n = \frac{1}{n}$.

$\limsup_{n\to\infty} a_n = 0$.

$\sup\{a_n : n \in \mathbb{N}\} = 1$.

(IV). $(S, d) = (\mathbb{R}, d_{standard})$

Consider open subsets.

$U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in \mathbb{N}$.

The intersection

$\bigcap_{n=1}^{\infty} U_n = \{0\}$ is not open.