#1

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} - \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}$$

$$= \begin{bmatrix} -3b+2c & -2a-3b+2d \\ 3a+3c-3d & 3b-2c \end{bmatrix}$$

(a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(T) \iff \begin{cases} -3b+2c=0 \\ -2a-3b+2d=0 \\ 3a+3c-3d=0 \\ 3b-2c=0 \end{cases}$

$$\iff \begin{bmatrix} 0 & -3 & 2 & 0 \\ -2 & -3 & 0 & 2 \\ 3 & 0 & 3 & -3 \\ 0 & 3 & -2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

row reduction

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ -2 & -3 & 0 & 2 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -3 & 2 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Null space = $\operatorname{Span}\left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\Rightarrow \left\{ \begin{bmatrix} -3 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of $\ker(T)$.

(1b) the pivot positions tell us that the first two columns form a basis of the column space of the 4×4 matrix.

Hence $\left\{ \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} -3 & -3 \\ 0 & 3 \end{bmatrix} \right\}$ is a basis of $Im(T)$.

$\square$

#2: Find C s.t. $C = B^{-1}A$.

$$\left[\underbrace{\begin{matrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{matrix}}_{B} \left| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{matrix}}_{A} \right.\right] \underset{\text{row reduction}}{\sim} \left[\begin{matrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} 1 & 0 & 0 \\ -1 & -2 & 0 \\ 2 & 5 & 2 \end{matrix} \right.\right]$$

$$\sim \left[\begin{matrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} 2 & 2 & 0 \\ -1 & -2 & 0 \\ 2 & 5 & 2 \end{matrix} \right.\right] \sim \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \left| \underbrace{\begin{matrix} -4 & -13 & -6 \\ 1 & 3 & 2 \\ 2 & 5 & 2 \end{matrix}}_{B^{-1}A \,=\, C} \right.\right]$$

$\square$

#3: Find a, b, c s.t. $|\det A| = 8 \left(= \frac{32\pi}{3} \Big/ \frac{4\pi}{3}\right)$

It's easy to get: $\det A = abc - a - c$.

$\Rightarrow |abc - a - c| = 8$. $\square$

#4:

(a) • T_A injective $\Leftrightarrow$ A has pivots in each column $\Rightarrow$ $m \geq n$.

• $\dim Nul(A) = 0$ since $Nul(A) = \{0\}$.

• $\dim Col(A) = n$ by rank-nullity thm.

(b) • T_A surjective $\Leftrightarrow$ A has pivots in each row $\Rightarrow$ $m \leq n$.

• $\dim Col(A) = m$ since $Col(A) = \mathbb{R}^m$.

• $\dim Nul(A) = n - m$ by rank-nullity thm $\square$

#5.

(a) Rank-nullity thm $\Rightarrow$ $n = \dim \ker(T) + \dim Im(T)$

Since $\ker(T) = Im(T)$, hence have the same dim.

$\Rightarrow$ n is even. $\square$

(b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$Ker(T_A) = Span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = Im(T_A)$. $\square$

#6.

(a) For any $\vec{v} \in V$, consider

$$\vec{v} = (\vec{v} - T(\vec{v})) + T(\vec{v}).$$

- $\vec{v} - T(\vec{v}) \in \ker(T)$ since

$$T(\vec{v} - T(\vec{v})) = T(\vec{v}) - T(T(\vec{v})) = \vec{0}.$$

(since $T^2 = T$)

- $T(\vec{v}) \in \operatorname{Im}(T)$.

$\Rightarrow \ker(T) + \operatorname{Im}(T) = V$.

(b) If $\vec{v} \in \ker(T) \cap \operatorname{Im}(T)$,

then $\exists\, \vec{w} \in V$ s.t. $T(\vec{w}) = \vec{v}$.

and $\vec{0} = T(\vec{v}) = T(T(\vec{w})) \underset{T^2 = T.}{=} T(\vec{w}) = \vec{v}$.

Hence $\ker(T) \cap \operatorname{Im}(T) = \{0\}$. $\square$