MATHEMATICS FROM EXAMPLES, HOMEWORK 1: DRAFT
DUE XXX AT XXX

Some ground rules:

- Feel free to use English, Chinese, or both, in your solutions.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Make sure the writing in your submission is clear. Answers which are illegible won't be given credit.
- Feel free to use results that are proved in class. If you'd like to use other results, you have to prove them before using them.
- You're encouraged to work together on the assignments. In your solutions, you should acknowledge the students with whom you worked, and should **write solutions on your own**.

Problems:

(1) Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. Let $f\colon X \to \mathbb{R}$ be a measurable function that is T-invariant almost everywhere, i.e. $f \circ T(x) = f(x)$ holds for μ-almost every $x \in X$.

(a) Consider the set

$$E = \bigcup_{k \geq 0} T^{-k}(A), \qquad \text{where } A = \{x \in X \mid f(T(x)) \neq f(x)\}.$$

Prove that $\mu(E) = 0$ and that $T^{-1}(E) \subseteq E$.

(b) Define a new function $\tilde{f}\colon X \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \notin E, \\ 0 & \text{if } x \in E. \end{cases}$$

Prove that $f = \tilde{f}$ almost everywhere, and that $\tilde{f} \circ T = \tilde{f}$ *everywhere*.

(2) Given two measure-preserving systems $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$, one can define their product $(X_1 \times X_2, \mathcal{B}_1 \otimes \mathcal{B}_2, \mu_1 \times \mu_2, T_1 \times T_2)$, where:

- $\mathcal{B}_1 \otimes \mathcal{B}_2$ is the smallest σ-algebra over $X_1 \times X_2$ which contains $B_1 \times B_2$ for all $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$,
- $\mu_1 \times \mu_2$ is the unique measure on $(X_1 \times X_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$ such that $(\mu_1 \times \mu_2)(B_1 \times B_2) = \mu_1(B_1)\mu_2(B_2)$ for all $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$, and
- $(T_1 \times T_2)(x_1, x_2) = (T_1(x_1), T_2(x_2))$.

Here are the problems:

(a) Prove that the product system of two measure-preserving systems is measure-preserving.

(b) Consider a rotation $R_\alpha\colon S^1 \to S^1$ where α is irrational. We proved in lectures that it is ergodic (with respect to the standard Lebesgue measure). Let us now consider the product system $(S^1 \times S^1, R_\alpha \times R_\alpha)$. Prove the following statements:

(i) For any measurable sets $B_1, B_2 \subseteq S^1$, if $B_1 \times B_2$ is $R_\alpha \times R_\alpha$-invariant (i.e. $(R_\alpha \times R_\alpha)^{-1}(B_1 \times B_2) = B_1 \times B_2$), then $\mu(B_1 \times B_2)$ is either 0 or 1.

(ii) Prove that the product system $(S^1 \times S^1, R_\alpha \times R_\alpha)$ is *not ergodic*.

(Note: This problem shows that: (1) ergodicity is not preserved under taking products; (2) for checking ergodicity it is not enough to check the defining property ($\mu(B) \in \{0,1\}$ for T-invariant sets B) only on a generating set of the σ-algebra.)

(3) Consider the quadrupling map

$$T_4 \colon S^1 \to S^1; \qquad x \mapsto 4x \pmod 1.$$

Prove that it is measurably isomorphic to the product system $(S^1 \times S^1, T_2 \times T_2)$, where $T_2 \colon S^1 \to S^1$ is the doubling map.

(4) Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. Let $\mathcal{A} \subseteq \mathcal{B}$ be a subset such that for any $B \in \mathcal{B}$ and any $\epsilon > 0$, there exists $A \in \mathcal{A}$ with the property that $\mu(A \Delta B) < \epsilon$. Assume that

$$\lim_{n\to\infty} \mu(A \cap T^{-n}A') = \mu(A)\mu(A') \qquad \text{for all } A, A' \in \mathcal{A}.$$

Prove that

$$\lim_{n\to\infty} \mu(B \cap T^{-n}B') = \mu(B)\mu(B') \qquad \text{for all } B, B' \in \mathcal{B}.$$

(5) Prove that if a measure-preserving system $(X, \mathcal{B}, \mu, T)$ has the property that for any $A, B \in \mathcal{B}$ there exists $N > 0$ such that

$$\mu(A \cap T^{-n}B) = \mu(A)\mu(B) \qquad \text{for all } n \geq N,$$

then it is trivial in the sense that $\mu(A)$ is either 0 or 1 for every $A \in \mathcal{B}$.