#1. Express the determinant of:

$$\begin{bmatrix} x & 1 & & & & \\ & x & 1 & & & \\ & & x & 1 & & \\ & & & x & 1 & \\ & & & & x & 1 \\ -2 & -3 & -4 & -5 & -6 & x-7 \end{bmatrix}$$

as a poly. in x.

Sol^n. Cofactor expression along the last row:

$$(-1)^{6+1}(-2)\cdot 1 + (-1)^{6+2}(-3)x + \cdots + (-1)^{6+5}(-6)x^4 + (-1)^{6+6}(x-7)x^5$$

$$= x^6 - 7x^5 + 6x^4 - 5x^3 + 4x^2 - 3x + 2.$$

#2. A: 4×4, $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^4$. Suppose the first three rows of A are given by $\vec{v}_1, \vec{v}_2, \vec{v}_1+\vec{v}_2$. Can one determine $\det(A) = ?$

Sol^n. $\underline{\det(A) = 0.}$

<u>Claim</u>: A is not invertible.

The rows of A are l.d. by assumption.

$\Rightarrow$ the columns of A^T are l.d.

$\Rightarrow$ A^T is NOT invertible.

We proved that A invertible $\Leftrightarrow$ A^T invertible.

hence A^T NOT invertible $\Leftrightarrow$ A NOT invertible. $\square$

#3. Consider

$$T: \mathrm{Poly}_{\leq 3} \longrightarrow \mathbb{R}^2$$
$$p \longmapsto \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$$

(1) Find a set of polynomial(s) in $\ker(T)$ that are l.i. and span $\ker(T)$.

(2) Find a set of vector(s) in $\mathrm{Im}(T)$ that are l.i. and span $\mathrm{Im}(T)$.

Sol^n: (1) $p \in \ker(T) \Leftrightarrow p(0) = p(1) = 0.$

$\Leftrightarrow p(x) = L(x) \cdot x(x-1)$ for some linear fcn $L(x) = Ax + B$ (since $p \in \mathrm{Poly}_{\leq 3}$).

Hence $\ker(T) = \mathrm{Span}\{x(x-1), x^2(x-1)\}$. $\square$

(2) Since $-x+1 \longmapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x \longmapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

So $\mathrm{Im}(T) = \mathbb{R}^2 = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. $\square$