(1) (15 points) Let $a_n = \frac{n^2+1}{3n^2+5n}$. Prove that $(a_n)$ is convergent based on the definition. (You're not allowed to use any theorem for this problem.)

Claim: $\lim a_n = \frac{1}{3}$.

pf: $\forall \varepsilon > 0$, let $N = \frac{5}{9\varepsilon} > 0$. Then $\forall n > N$, we have:

$$\left| \frac{n^2+1}{3n^2+5n} - \frac{1}{3} \right| = \frac{5n-3}{3(3n^2+5n)} < \frac{5n}{9n^2} = \frac{5}{9n} < \varepsilon. \quad \square$$

(2) (20 points) Let $(a_n)$ and $(b_n)$ be two sequences of real numbers. Suppose that $(a_n)$ is a bounded sequence, and suppose that $(b_n)$ converges to $0$. Prove that the sequence $(a_n b_n)$ converges to $0$.

- Since $(a_n)$ is bounded, $\exists M > 0$ s.t. $|a_n| < M$ $\forall n \in \mathbb{N}$.
- Since $\lim b_n = 0$, $\forall \varepsilon > 0$, $\exists N > 0$ s.t.
$$n > N \Rightarrow |b_n| < \frac{\varepsilon}{M}.$$
- Hence, $\forall n > N$, we have:
$$|a_n b_n - 0| = |a_n b_n| = |a_n||b_n| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

$\Rightarrow \lim a_n b_n = 0.$ $\square$

(3) (20 points) Let $(a_n)$ and $(b_n)$ be two bounded sequences of real numbers. Suppose $a_n \le b_n$ for any $n \in \mathbb{N}$. Prove that

$$\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n.$$

$\forall N > 0$, $\forall n > N$, we have

$$a_n \le b_n \le \sup\{b_n : n > N\} =: S_N^b$$

Since $a_n \le S_N^b$ for all $n > N$, therefore $S_N^a := \sup\{a_n : n > N\} \le S_N^b$

$$\Rightarrow \limsup_{n\to\infty} a_n = \lim_{N\to\infty} S_N^a \le \lim_{N\to\infty} S_N^b = \limsup_{n\to\infty} b_n. \quad \square$$

(4) (9 points each) Determine whether each of the following statements below is true or false. You have to prove the statement if you think the statement is true; otherwise, you have to provide an explicit counterexample and justify that it is indeed a counterexample. Answers without justification (or justification that does not make sense) will not be given credits. (Hint: There are more false statements than true statements!)

(a) Consider the metric space $\mathbb{R}$ with the usual distance function $d(x,y) = |x-y|$. The closure of $\mathbb{Q} \subseteq \mathbb{R}$ is $\overline{\mathbb{Q}} = \mathbb{R}$.

True. Claim: any $x \in \mathbb{R}$ is a limit point of $\mathbb{Q} \subseteq \mathbb{R}$.

pf: $\forall x \in \mathbb{R}$; $r > 0$. we need to show that there exists rational number in $(x-r, x+r) \setminus \{x\}$, which clearly follows from the denseness of $\mathbb{Q} \subseteq \mathbb{R}$.

(b) Let $(a_n)$ be a sequence of real numbers satisfying

$$\lim_{n\to\infty} |a_{n+1} - a_n| = 0.$$

Then $(a_n)$ is convergent.

False. $(a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n})$

(c) Let $(a_n)$ be a bounded sequence of real numbers. Then

$$\limsup_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

False: $(a_n = \frac{1}{n})$: $\limsup\limits_{n\to\infty} a_n = 0 \neq 1 = \sup\{a_n : n \in \mathbb{N}\}$.

(d) Let $a$ and $b$ be two real numbers. Suppose that $a < c$ for any rational number $c \in \mathbb{Q}$ satisfying $b < c$. Then $a \leq b$.

True: Assume the contrary that $b < a$. By denseness of $\mathbb{Q} \subseteq \mathbb{R}$, $\exists c \in \mathbb{Q}$ s.t. $b < c < a$. Contradiction. $\square$

(e) Let $M > 0$, and let $(a_n)$ be any sequence of real numbers satisfying $-M < a_n < M$ for any $n \in \mathbb{N}$. Then $(a_n)$ admits a subsequence that converges to a real number $a$ satisfying $-M < a < M$.

False. $M = 1$, $\{a_n = 1 - \frac{1}{n}\} \subseteq (-M, M)$

Any subseq. of $(a_n)$ converges to $1 \notin (-M, M)$.