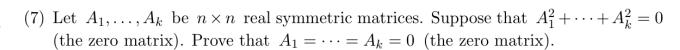


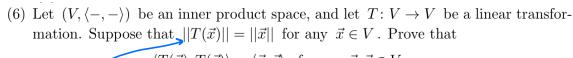
Another prof: where  $A \vec{J}_i = \lambda_i \vec{J}_i$ , and  $\{\lambda_1, -, \lambda_{n-1}\}$ once the distinct eigenles of A. {\$\frac{1}{2}, \frac{1}{2}, \fr  $\begin{bmatrix}
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Every enders ATT - a . 2 . 2 . - - a apra- varia an Iniva 



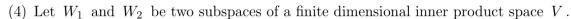
$$A_{i}^{2} = A_{i}A_{i}^{T} = \begin{bmatrix} -v_{i} \\ -v_{n} - \end{bmatrix} \begin{bmatrix} v_{i} \\ v_{i} = v_{n} \end{bmatrix}$$

Y Z



$$\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$$
 for any  $\vec{x}, \vec{y} \in V$ .

- (5) Let A and B be two square matrices.
  - (a) Suppose that  $\lambda \neq 0$  is an eigenvalue of AB. Prove that  $\lambda$  is also an eigenvalue of BA.
  - (b) Does the same statement hold for  $\lambda = 0$ ?



(a) Prove that  $W_1^{\perp} \cap W_2^{\perp} = (W_1 + W_2)^{\perp}$ .

(b) Prove that 
$$\dim(W_1) - \dim(W_1 \cap W_2) = \dim(W_2^{\perp}) - \dim(W_1^{\perp} \cap W_2^{\perp})$$
.

$$\frac{1}{2} \int_{-\infty}^{\infty} d\tau_{m}(w_{1}) - d\tau_{m}(w_{2}) d\tau_{m}(w_{1} + w_{2}) - d\tau_{m}(w_{2})$$

$$= \left( d\tau_{m} V - d\tau_{m}(w_{1} + w_{2})^{\perp} \right)$$

$$\frac{1}{2} = d\tau_{m} \cdot w_{s}^{\perp} - d\tau_{m} \cdot (w_{l} + w_{s})^{\perp}$$

(3) Find all possible 
$$5 \times 5$$
 real symmetric matrices A satisfying  $A^3 - 2A = 4\mathbb{I}_5$ .

PDPT, Porthigal, Dobuguel PTPT, veal,

$$P D^{3} P^{T} - 2 P D P' = 4 I = 4 P P^{T}$$

$$\Rightarrow P (D^{3} - 2D - 4I) P^{T} = 0$$

$$\Rightarrow 0^3 - 20 - 41 = 0$$

$$\Rightarrow \forall \text{ eigenvalue $\lambda$ of $\lambda$; we have $\lambda^3 - 2\lambda - 4 = 0$}$$

$$\text{Verl}$$

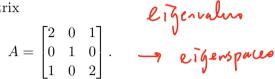
$$(\lambda - 2)(\lambda^2 + 2\lambda + 2)$$

$$\Rightarrow \lambda = 2.$$

$$\Rightarrow A = P \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} P^{T} = 2 I.0$$

## SECOND MIDTERM PRACTICE PROBLEMS MATH H54, FALL 2021

(1) Consider the symmetric matrix



Find an orthogonal matrix P and a diagonal matrix D such that  $A = PDP^T$ .

(2) Let  $M_{2\times 2}(\mathbb{R})$  be the set of all real  $2\times 2$  matrices. It is naturally a vector space with the standard matrix addition and scalar multiplication. Consider the function  $\langle -, - \rangle : M_{2 \times 2}(\mathbb{R}) \times M_{2 \times 2}(\mathbb{R}) \to \mathbb{R}$  given by

$$\langle A, B \rangle = \operatorname{tr}(AB^T).$$

- (a) Show that  $(M_{2\times 2}(\mathbb{R}), \langle -, \rangle)$  is an inner product space.
- (b) Construct an orthonormal basis (with respect to the inner product  $\langle -, \rangle$ ) of the subspace of  $M_{2\times 2}(\mathbb{R})$  spanned by  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .
- (c) Consider another function  $\langle -, \rangle_2 : M_{2\times 2}(\mathbb{R}) \times M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$  defined by  $\langle A, B \rangle_2 = \operatorname{tr}(AB)$ . Does  $\langle -, \rangle_2$  give an inner product on the vector space  $M_{2\times 2}(\mathbb{R})$ ?  $M_{2\times 2}(\mathbb$
- (4) Let  $W_1$  and  $W_2$  be two subspaces of a finite dimensional inner product space V.
  - (a) Prove that  $W_1^{\perp} \cap W_2^{\perp} = (W_1 + W_2)^{\perp}$ .
  - (b) Prove that  $\dim(W_1) \dim(W_1 \cap W_2) = \dim(W_2^{\perp}) \dim(W_1^{\perp} \cap W_2^{\perp})$ .
- (5) Let A and B be two square matrices.
  - (a) Suppose that  $\lambda \neq 0$  is an eigenvalue of AB. Prove that  $\lambda$  is also an eigenvalue of BA.
  - (b) Does the same statement hold for  $\lambda = 0$ ?
- (6) Let  $(V, \langle -, \rangle)$  be an inner product space, and let  $T: V \to V$  be a linear transformation. Suppose that  $||T(\vec{x})|| = ||\vec{x}||$  for any  $\vec{x} \in V$ . Prove that

$$\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle \quad \text{for any } \vec{x}, \vec{y} \in V.$$

- (7) Let  $A_1, \ldots, A_k$  be  $n \times n$  real symmetric matrices. Suppose that  $A_1^2 + \cdots + A_k^2 = 0$ (the zero matrix). Prove that  $A_1 = \cdots = A_k = 0$  (the zero matrix).
- (8) Let A be an  $n \times n$  diagonalizable matrix with n-1 distinct eigenvalues. Prove that for any  $\vec{v} \in \mathbb{R}^n$ , the set  $\{\vec{v}, A\vec{v}, \dots, A^{n-1}\vec{v}\}$  is linearly dependent.