HOMEWORK 6
MATH 104, SECTION 6

Office Hours: Tuesday and Wednesday 9:30-11am at 735 Evans.

Nima's Office Hours: Monday, Tuesday and Thursday 9:30am-1pm at 1010 Evans.

Reading

There will be reading assigned for each lecture. You should come to the class having read the assigned sections of the textbook.

Due February 27: Ross, Section 15

Due March 3: Ross, Section 17

Problem set (9 problems; due February 27)

Submit your homework at the beginning of the lecture on Thursday. *Late homework will not be accepted under any circumstances.*

You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any.

Write clearly! Mastering mathematical writing is one of the goals of this course.

You have to staple your work if it is more than one page.

(1) Let (S, d) be a compact metric space. (Recall that a metric space is *compact* if every open cover has a finite subcover.) Let $E \subset S$ be a closed subset. Prove that E is compact. (Hint: Let $\{U_\alpha : \alpha \in I\}$ be an open cover of E. Then $\{U_\alpha : \alpha \in I\} \cup \{E^c\}$ is an open cover of S.)

(2) Determine each of the following series converges or not. Prove your answers.

$$(a)\ \sum \frac{(-1)^n(n-1)}{n};\ (b)\ \sum \frac{n^n}{(n+1)^{2n}};\ (c)\ \sum \frac{(-1)^n}{n^{1/12}};$$

$$(d)\ \sum \frac{1}{(2n-1)^2};\ (e)\ \sum \frac{1}{n \log n};\ (f)\ \sum n e^{-n^2}.$$

(3) Let $(a_n^{(1)})_{n=1}^\infty, (a_n^{(2)})_{n=1}^\infty, \ldots, (a_n^{(k)})_{n=1}^\infty$ denote k sequences of real numbers. (For instance, the first sequence is $(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}, \ldots)$.) Define another sequence $(b_n)_{n=1}^\infty$ where the n-th term is defined to be

$$b_n = a_n^{(1)} + a_n^{(2)} + \cdots a_n^{(k)}.$$

Suppose that the series $\sum_{n=1}^\infty a_n^{(i)}$ converges for each $i = 1, 2, \ldots, k$. Prove that

(a) the series $\sum_{n=1}^\infty b_n$ also converges; moreover,

(b)

$$\sum_{n=1}^{\infty} b_n = \sum_{i=1}^{k} \Big(\sum_{n=1}^{\infty} a_n^{(i)} \Big).$$

This is a discrete version of *Fubini's theorem.*

Also, find an example where $\sum b_n$ converges but $\sum a_n^{(i)}$ diverges.

(4) Let (a_n) and (b_n) be two sequences of real numbers satisfying:
 (a) The partial sums of (b_n) is bounded: there exists $L > 0$ such that $|s_k| = |b_1 + \cdots + b_k| < L$ for any k;
 (b) $\lim a_n = 0$;
 (c) $\sum |a_{n+1} - a_n|$ is convergent.

Prove that the series $\sum a_n b_n$ is convergent. This is known as *Abel's theorem.*

Hint: $\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} a_n(s_n - s_{n-1}) = \sum_{n=M}^{N-1}(a_n - a_{n+1})s_n + a_N s_N - a_M s_{M-1}$.

(5) Show that the series

$$\sum \frac{\cos(n\theta)}{n} \text{ and } \sum \frac{\sin(n\theta)}{n}$$

are convergent for any $0 < \theta < 2\pi$.

(Hint: use

$$\sum_{n=1}^{N} e^{in\theta} = e^{i\theta} \frac{1 - e^{iN\theta}}{1 - e^{i\theta}} = e^{i(N+1)\theta/2} \frac{\sin(N\theta/2)}{\sin(\theta/2)}$$

and the previous problem.)

(6) Let (a_n) be a decreasing sequence such that the series $\sum a_n$ converges. Prove that

$$\lim_{n\to\infty} n a_n = 0.$$

(7) Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and $s_k = a_1 + \cdots + a_k$ be the k-th partial sum.
 (a) Suppose that $\lim a_n = 0$, and there exists a $m \in \mathbb{N}$ such that the sequence $(s_{mk})_{k=1}^{\infty} = (s_m, s_{2m}, s_{3m}, \ldots)$ converges. Prove that $\sum a_n$ converges.
 (b) Find an example where $(s_{2k})_{k=1}^{\infty}$ converges and (a_n) doesn't converge to 0.
 (c) Find an example where $\lim a_n = 0$, and there is a subsequence (s_{k_n}) of (s_n) that converges, but $\sum a_n$ diverges.

(8) Prove the triangle inequality for series: if $\sum a_n$ converges absolutely, then

$$\Big| \sum_{n=1}^{\infty} a_n \Big| \leq \sum_{n=1}^{\infty} |a_n|.$$

(9) Show that the monotonicity assumption in the alternating series test is necessary: find a sequence of positive real numbers (a_n) with $\lim a_n = 0$, but $\sum (-1)^n a_n$ diverges.