Today: • classify isolated sing. from the perspective of Laurent series exp.
• winding number.

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$f$: isolated sing. at $z_0$ ($f$ is holo. in $\mathbb{D}_r^{\times}(z_0)$ for some $r>0$)

$\mathbb{D}_r^{\times}(z_0) = \{z \in \mathbb{C} \mid 0 < |z-z_0| < r\}$.

• removable: $\exists\, r>0$, $F$: holo. in $\mathbb{D}_r(z_0)$

s.t. $F(z) = f(z) \quad \forall z \in \mathbb{D}_r^{\times}(z_0)$.

$$\Longrightarrow \quad f(z) = F(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall z \in \mathbb{D}_r^{\times}(z_0)$$

• pole (of order $n$): $\Longrightarrow \quad f(z) = \dfrac{h(z)}{(z-z_0)^n}$ ← holo., non-vanishing near $z_0$.

$$\Longrightarrow \quad f(z) = \frac{\overset{\overset{0}{\#}}{a_{-n}}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{\overset{\operatorname{res}_{z=z_0} f(z)}{a_{-1}}}{z-z_0} + a_0 + \cdots\cdots \quad \forall z \in \mathbb{D}_r^{\times}(z_0)$$

$$= \sum_{k=-n}^{\infty} a_k (z-z_0)^k \quad \text{Laurent series expansion}$$

Q: What about essential singularities?
How to define its residue?

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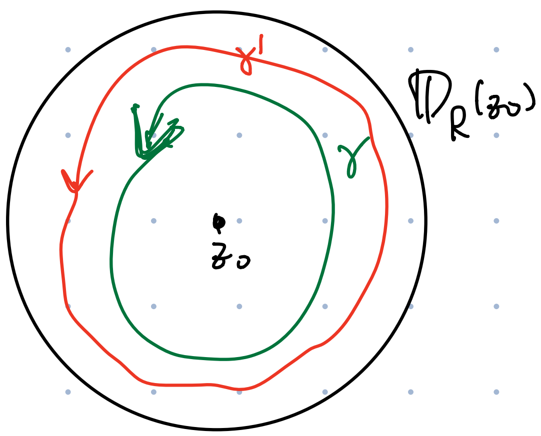
**Thm** $f: \mathbb{D}_R^\times(z_0) \longrightarrow \mathbb{C}$ holo.

Then $\exists! \, (a_n)_{n=-\infty}^{\infty}$ s.t. $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \forall z \in \mathbb{D}_R^\times(z_0)$.

In fact, $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}} \, dw$.




Rmk: the integral doesn't depend on the choice of the contour $\gamma$.

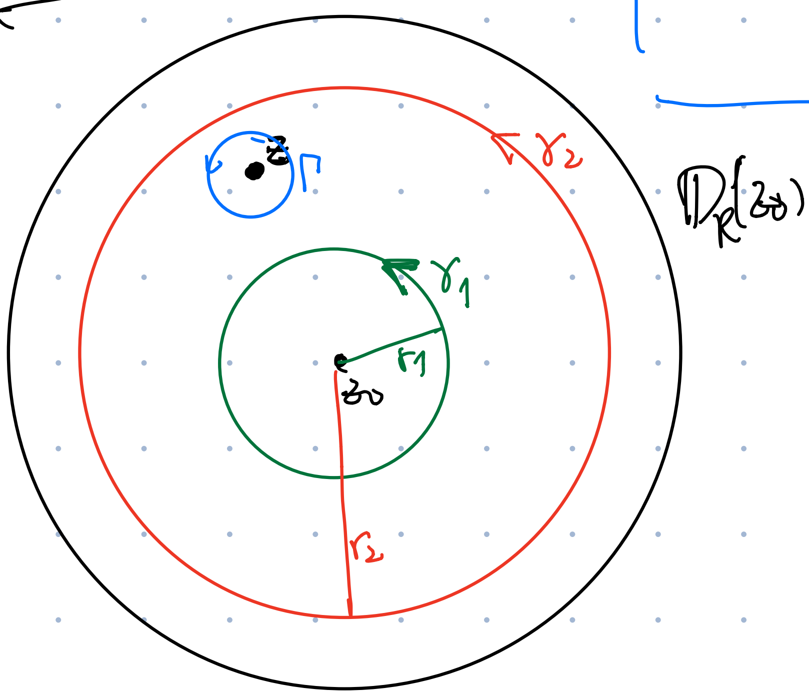
Rmk: if $f$ has removable sing at $z_0$ then if $n \le -1$,

$$\frac{f(w)}{(w-z_0)^{n+1}} = f(w)(w-z_0)^{-n-1}$$

$\Rightarrow a_n = 0 \quad \forall n \le -1.$

$\Rightarrow f = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

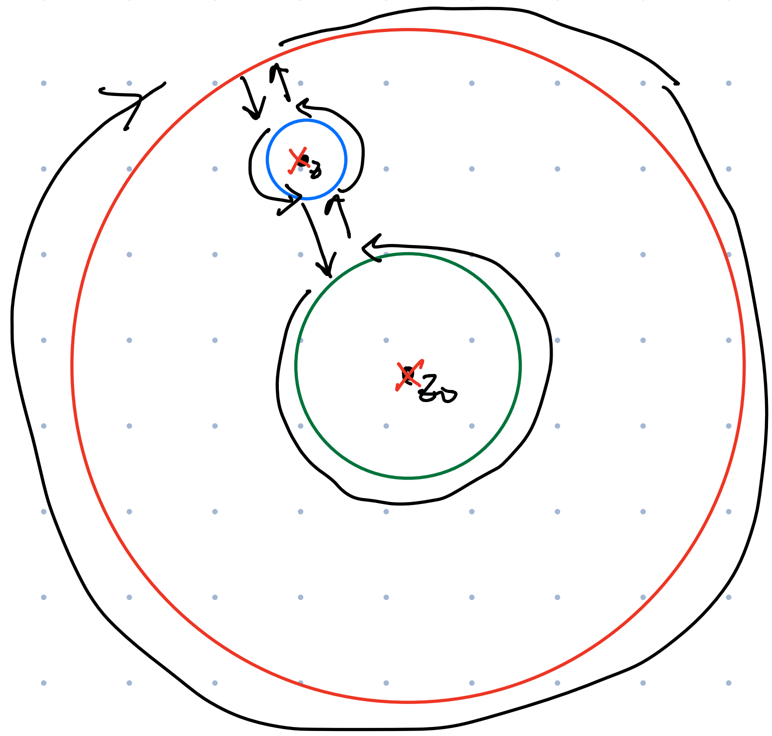
$\boxed{\text{Want to compute } f(z) \in \mathbb{D}_R^\times(z_0)}$

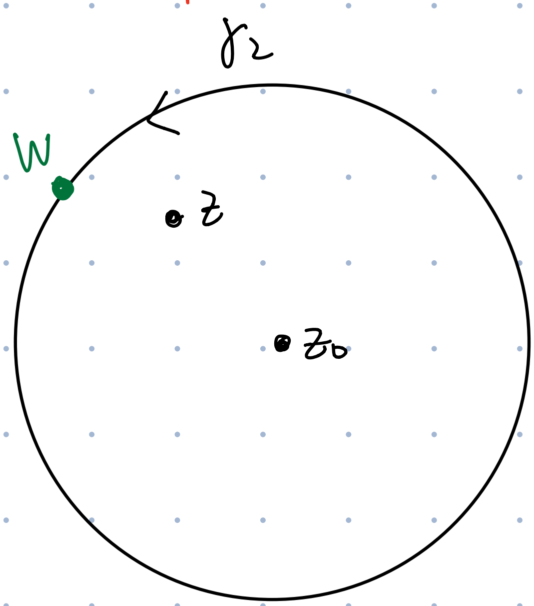



Choose $r_1, r_2$ s.t. $0 < r_1 < |z-z_0| < r_2 < R$

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(w)}{w-z}\,dw.$$ Cauchy integral formula

$$= \frac{1}{2\pi i}\left(\int_{\gamma_2}\frac{f(w)}{w-z}\,dw - \int_{\gamma_1}\frac{f(w)}{w-z}\,dw\right)$$




$|w-z_0| > |z-z_0|$

$$\int_{\gamma_2}\frac{f(w)}{w-z}\,dw$$

$$= \int_{\gamma_2}\frac{f(w)}{(w-z_0)-(z-z_0)}\,dw$$

$$= \int_{\gamma_2}\frac{f(w)}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}\,dw.$$

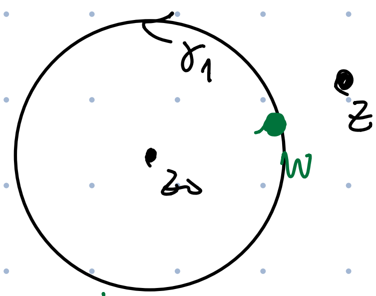
$|\cdot| < 1$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \forall |z|<1$$

$$= \int_{\gamma_2} \frac{f(w)}{w-z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n dw.$$

$$= \sum_{n=0}^{\infty} \int_{\gamma_2} \frac{f(w)}{(w-z_0)^{n+1}} (z-z_0)^n \, dw.$$

$$= \sum_{n=0}^{\infty} \underbrace{\left( \int_{\gamma_2} \frac{f(w)}{(w-z_0)^{n+1}} \, dw \right)}_{= 2\pi i \, a_n} \cdot (z-z_0)^n$$




$|z-z_0| > |w-z_0|$

$$\int_{\gamma_1} \frac{f(w)}{w-z} \, dw = \int_{\gamma_1} \frac{f(w)}{(w-z_0)-(z-z_0)} \, dw$$

$$= \int_{\gamma_1} \frac{f(w)}{-(z-z_0)} \cdot \frac{1}{1-\frac{w-z_0}{z-z_0}} \, dw$$

$|\cdot| < 1$

$$= -\int_{\gamma_1} \frac{f(w)}{z-z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{w-z_0}{z-z_0}\right)^n dw$$

$$= -\sum_{n=0}^{\infty} \int_{\gamma_1} \frac{f(w)}{(w-z_0)^{-n}} \, dw \cdot (z-z_0)^{-n-1}$$

$m = -n-1$

$$= -\sum_{m=-\infty}^{-1} \underbrace{\int_{\gamma_1} \frac{f(w)}{(w-z_0)^{m+1}}\, dw}_{= 2\pi i\, a_m} \cdot (z-z_0)^m$$

$\square$

**Rmk:** The Laurent series exp. is unique:

If $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$ for $z \in \mathbb{D}_r^\times(z_0)$,

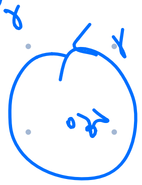

then

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\, dw$$

$(w-z_0)^{n-k-1}$

$$= \frac{1}{2\pi i} \int_\gamma \sum_{n=-\infty}^{\infty} \frac{b_n (w-z_0)^n}{(w-z_0)^{k+1}}\, dw$$

$\int_\gamma \frac{dw}{(w-z_0)^n} = 0$ unless $n=1$

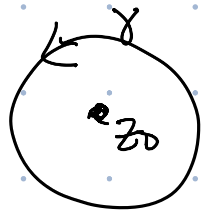



the integrand $= 0$ except when $\boxed{n-(k+1) = -1} \Rightarrow n = k$

$$= \frac{1}{2\pi i} \int_\gamma b_k \cdot \frac{1}{w-z_0}\, dw = b_k. \quad \square$$

**Def:** ~~Resid~~ $f: \mathbb{D}_R^\times(z_0) \to \mathbb{C}$ holo. $\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \forall z \in \mathbb{D}_R^\times(z_0)$

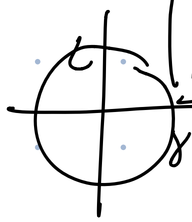
$$\boxed{\operatorname{Res}_{z=z_0} f = a_{-1}}$$




$\int_\gamma f(z)\,dz = 2\pi i \operatorname{Res}_{z=z_0} f$

residue formula also true

e.g. $e^{1/z}$ has ess. sing. at $z=0$



$$\boxed{\int_\gamma e^{1/z}\,dz} = 2\pi i \cdot 1 = 2\pi i$$

$\operatorname{Res}_{z=0} e^{1/z}$

$$e^{1/z} = 1 + \left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots \quad \forall z \neq 0$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

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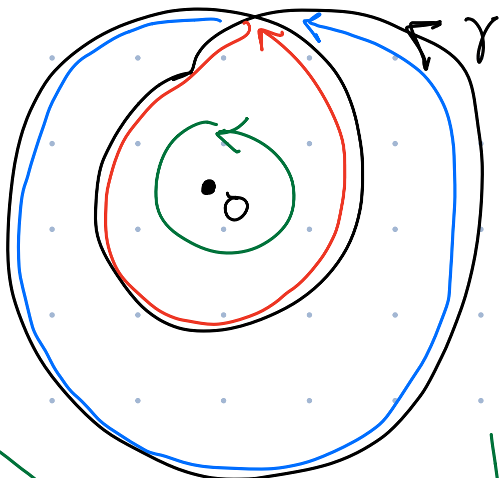
Q:



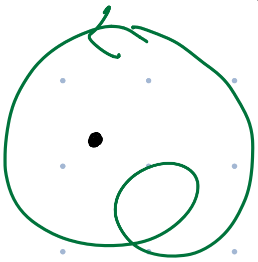

$$\int_\gamma \frac{1}{z}\,dz = ??$$

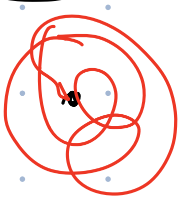
$4\pi i$

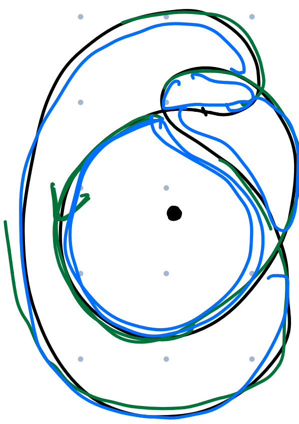
$\| $

$2\pi i \cdot 2$

winding # of $\gamma$ around 0






Thm: If a closed curve $\gamma$ doesn't pass through a point $z_0 \in \mathbb{C}$, then

$$\boxed{\int_\gamma \frac{1}{z - z_0}\,dz} \text{ is an } \underline{\text{integral}} \text{ multiple of } 2\pi i.$$

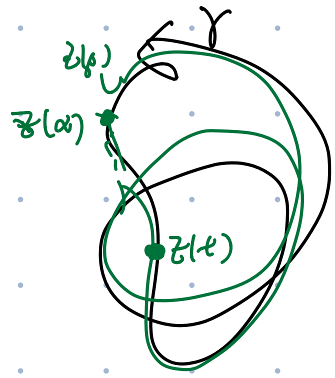
$= 2\pi i \cdot A$, where $A \in \mathbb{Z}$

(winding # around $z_0$)

pf: Let's parametrize $\gamma$: $\quad z(t): [\alpha, \beta] \longrightarrow \mathbb{C}$

$$\int_\gamma \frac{1}{z - z_0}\,dz = \int_\alpha^\beta \frac{1}{z(t) - z_0} \cdot z'(t)\,dt.$$

Consider $$h(t) := \int_\alpha^t \frac{z'(s)}{z(s)-z_0}\,ds$$




We want to prove: $h(\beta) \in 2\pi i \mathbb{Z}$

- $h(\alpha) = 0$
- $h'(t) = \dfrac{z'(t)}{z(t)-z_0}$.
- Define $H(t) := e^{-h(t)} \cdot (z(t) - z_0)$
- $$\begin{aligned} H'(t) &= -h'(t)\, e^{-h(t)} (z(t)-z_0) + e^{-h(t)} z'(t) \\ &= e^{-h(t)} \left( -h'(t)(z(t)-z_0) + z'(t) \right) \\ &= 0 \end{aligned}$$

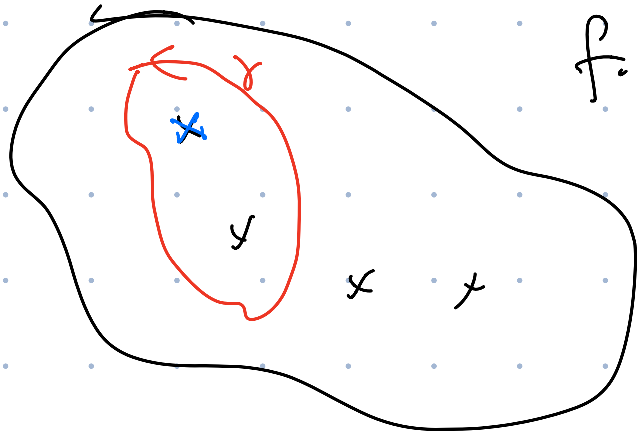
  i.e. $H(t) \equiv \text{const.}$

- $$H(\alpha) = e^{-h(\alpha)} \cdot (z(\alpha) - z_0)$$
  $$\|$$
  $$H(\beta) = e^{-h(\beta)} (z(\beta) - z_0)$$

  $\|$ $\gamma$ is closed

$$\Rightarrow \underbrace{e^{-h(\alpha)}}_{=1} = e^{-h(\beta)} \Rightarrow h(\beta) \in 2\pi i \mathbb{Z}. \quad \square$$

Rmk:




$f: \Omega \setminus \{z_1, \dots, z_k\} \to \mathbb{C}$

$\uparrow$ isolated sing.

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{k} \left(\operatorname*{Res}_{z=z_i} f\right) \cdot \underbrace{n(\gamma, z_i)}$$

$\uparrow$ winding # of $\gamma$ around $z_i$