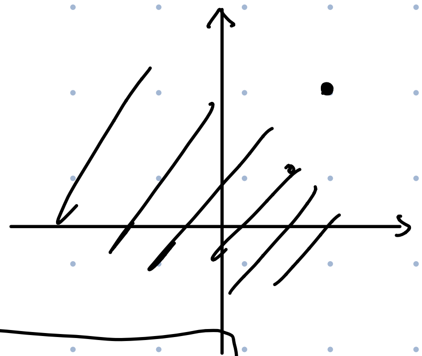
# Welcome to Math 185 !!

**Today:** Overview, basics of complex numbers.



$\mathbb{C} \cong \mathbb{R}^2$

**Holomorphic functions:** $f: \mathbb{C} \longrightarrow \mathbb{C}$

can take derivatives in the "complex sense"

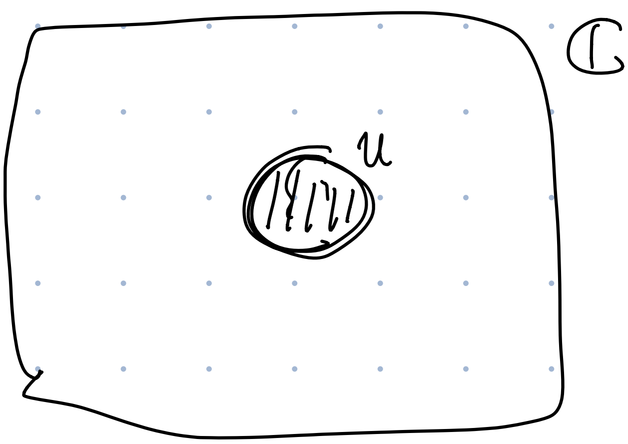
$$\frac{df}{dz}(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \quad \text{exists}$$

$z_0 \in \mathbb{C}$

Can be complex numbers.

**Amazing consequences of holomorphicity:**

1) "local determine global"




If $f$ and $g$ are holo. fcns that agreed on $\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{C}$

$$f(z) = g(z) \quad \forall z \in \mathcal{U},$$

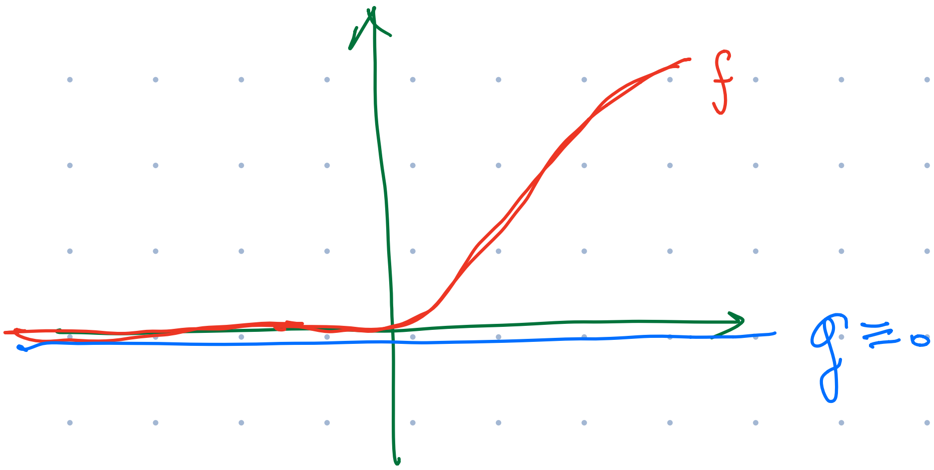
then

$$f \equiv g$$

Rmk: This is not true for smooth fcns in $\mathbb{R}$

e.g. (from 104)

$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x \le 0 \end{cases}$$




"local determine global" fails in $\mathbb{R}$.

2) "regularity": If $f$ is holo. (i.e. the 1st $f'$ derivative in the cpx sense exists), then $f''$, $f'''$, .... all exists

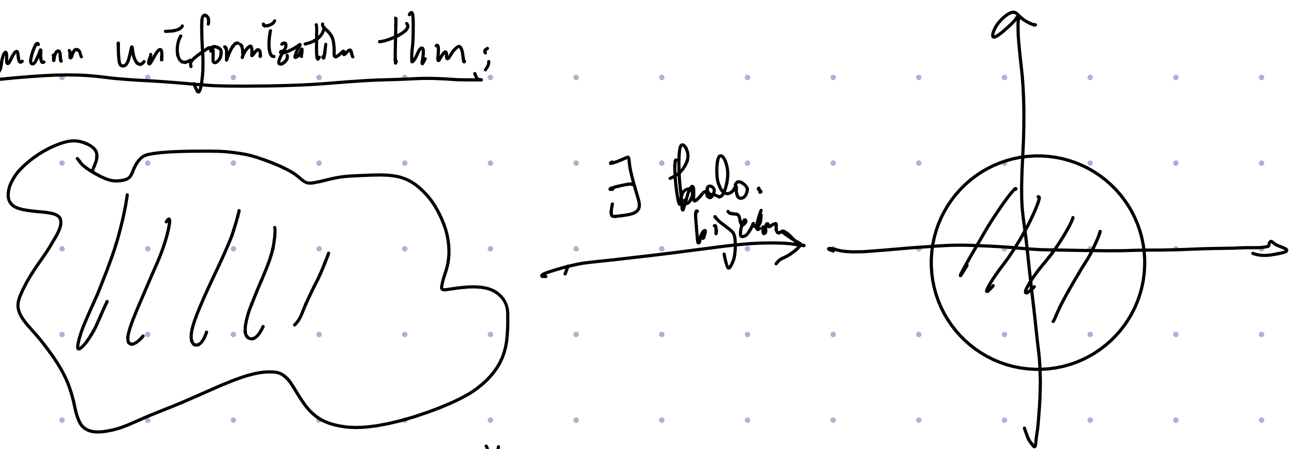
( Note: Again, NOT true over $\mathbb{R}$ )

3) "contour integral":


If $f$ is holo. in $\Omega$, then $\int_\gamma f(z)\,dz = 0$

**Conformal mapping**: "keep the shape"

**Riemann uniformization thm**:




any "simply connected" domain in $\mathbb{C}$

Another **Thm we'll prove: fundamental theorem of algebra**

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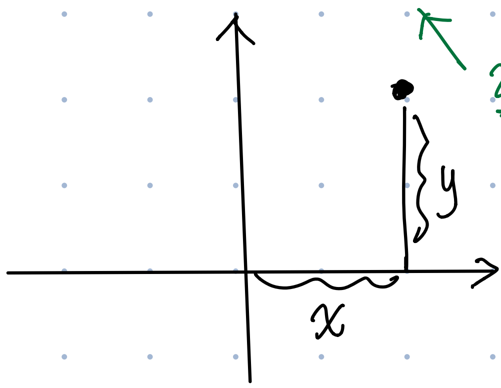
## § Complex numbers

As a set, $\mathbb{C} \cong \mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$

$x + iy$



$z \in \mathbb{C}$, write $z = x + iy$, where $x, y \in \mathbb{R}$

real part of $z$      Imaginary part of $z$

$x = \mathrm{Re}(z)$      $y = \mathrm{Im}(z)$

- Addition: $z = x + iy$, $w = u + iv$, $\in \mathbb{C}$

$x, y, u, v \in \mathbb{R}$

$$z + w := (x+u) + i(y+v) \in \mathbb{C}$$

multiplicat.

$$\boxed{i^2 = -1}$$

$$z \cdot w = (x+iy)(u+iv)$$

$$= xu + x(iv) + (iy)u + i^2 yv$$

$$= (xu - yv) + i(xv + yu)$$

division.

$$\frac{x+iy}{u+iv} = \frac{(x+iy)(u-iv)}{(u+iv)(u-iv)} = \frac{(xu+yv) + i(yu-xv)}{u^2+v^2}$$

$u, v$ not all $0$

$$= \frac{xu+yv}{u^2+v^2} + i \cdot \frac{yu-xv}{u^2+v^2}$$

Note: $(\mathbb{C}, +, \cdot)$ "division ring" "commutative" (= "field")
$zw = wz$

Another important example of division ring $\quad x, y, u, v \in \mathbb{R}$

Quaternion: $\mathbb{H} \cong \mathbb{R}^4 \quad q = x + \underline{i}y + \underline{j}u + \underline{k}v,$

where $i^2 = j^2 = k^2 = -1 \quad ij = k, \ldots$

$ij = -ji, \ldots$


quaternion plaque in Dublin

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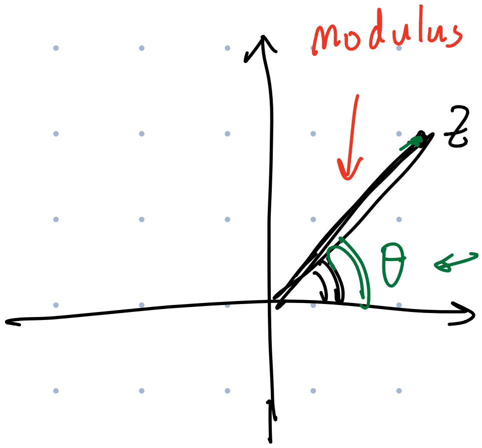
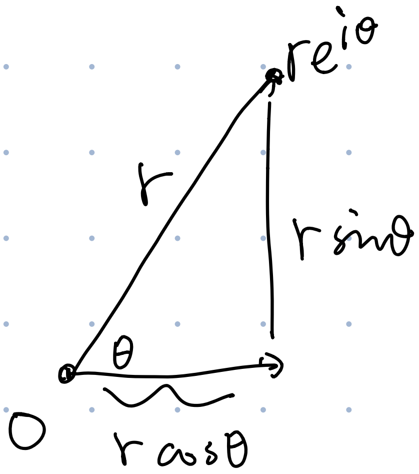
$z = x + iy$



modulus $r := |z| = \sqrt{x^2 + y^2}$ (absolute value of $z$)
$r \geq 0$

$z = x + iy$

$\theta$ ← argument of $z$
$\arg(z) \in \mathbb{R} \bmod 2\pi\mathbb{Z}$




$$z = r\cos\theta + i\, r\sin\theta$$
$$= r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

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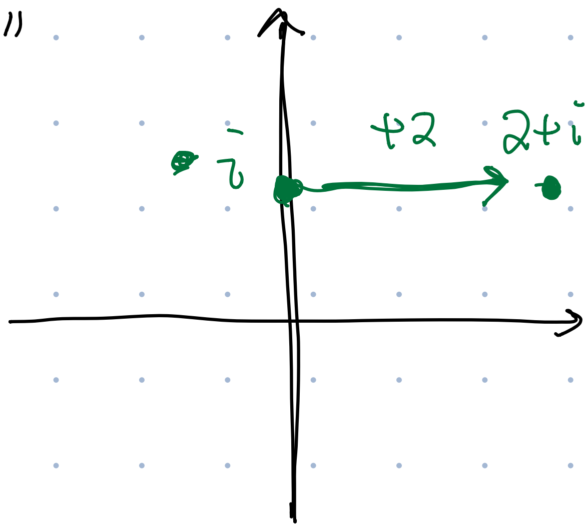
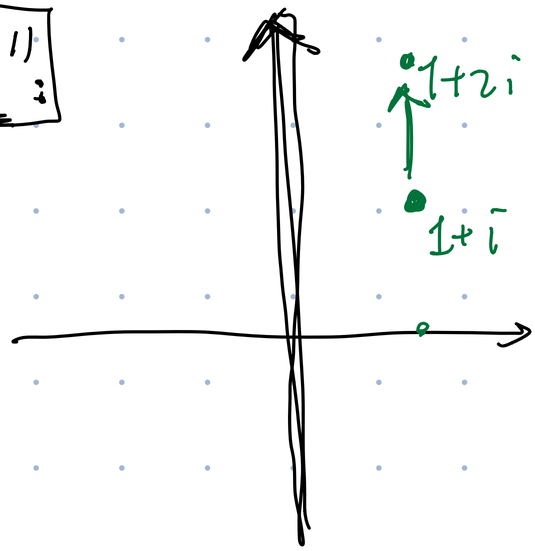
addition $\longleftrightarrow$ translation in $\mathbb{C}$

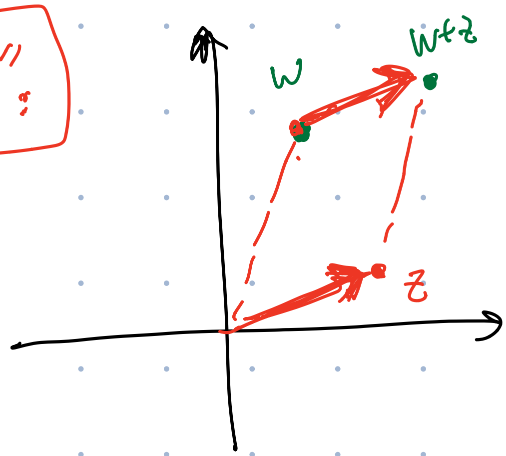
"+2"




"+i":

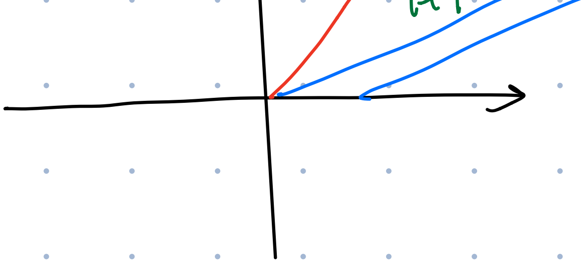



"+z":



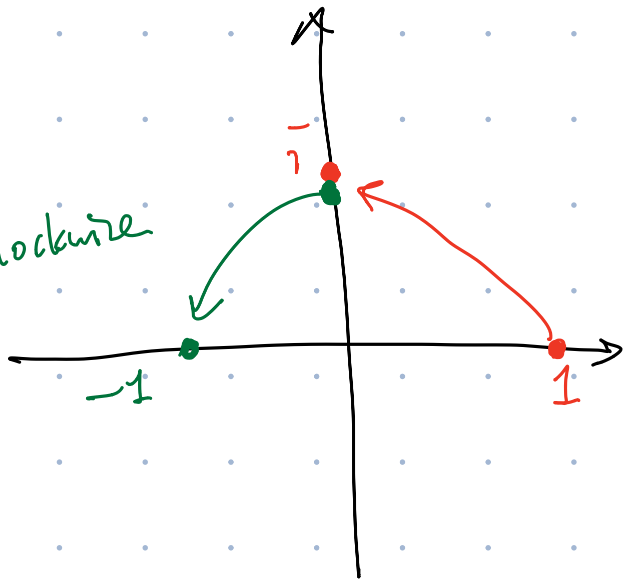

multiplication:

"×2":



$\boxed{\text{"} i \text{"}}$

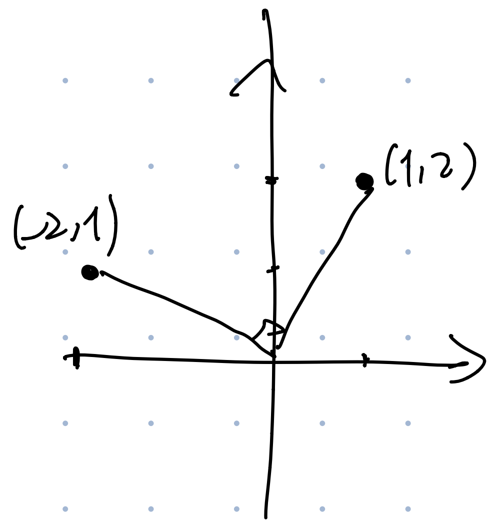
rotate $\frac{\pi}{2}$ counterclockwise




$\underline{x+iy}$

$(x+iy)i = -y+ix$

$(x,y) \longmapsto (-y,x)$



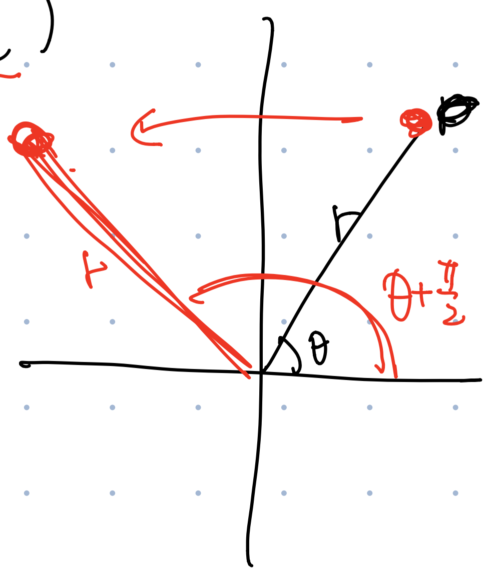

$\boxed{re^{i\theta}}$

$i = \boxed{e^{i\frac{\pi}{2}}}$

$\left(= \underbrace{\cos\frac{\pi}{2}}_{0} + i\underbrace{\sin\frac{\pi}{2}}_{1}\right)$

$re^{i\theta}\cdot i = re^{i\theta}\cdot e^{i\frac{\pi}{2}}$

$= re^{i(\theta+\frac{\pi}{2})}$

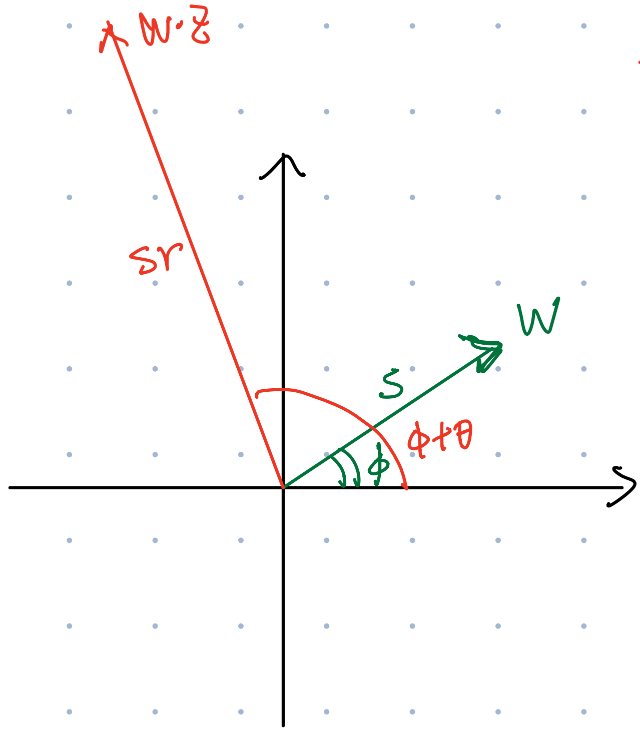


$$\boxed{\text{“} \times z \text{”}} : \quad \underset{\displaystyle re^{i\theta}}{\parallel}$$

$$W = se^{i\phi} \in \mathbb{C}$$

$$W \cdot z = (se^{i\phi})(re^{i\theta})$$

$$= (sr)\, e^{i(\phi+\theta)}$$




("$\times z$" = rotate counterclockwise by $\theta$ + scaling by $r$.)

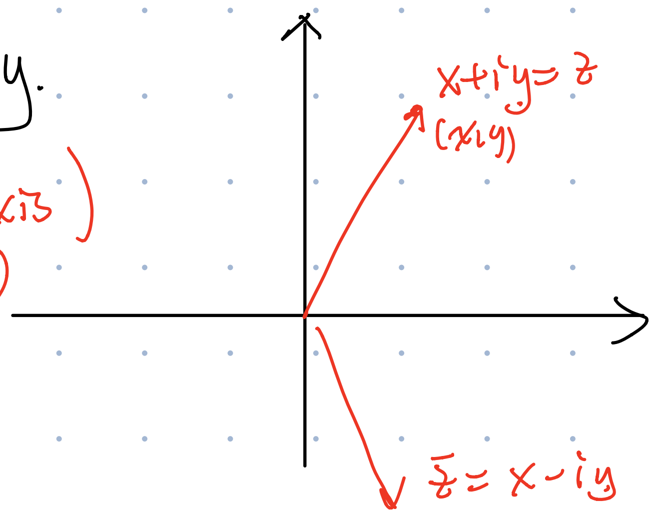
<u>Complex conjugation</u>: $z \longmapsto \bar{z}$

the complex conjugate of $z$

$z = x + iy$, $x, y \in \mathbb{R}$

Then $\bar{z} := x - iy$.

(reflection over real-axis (x-axis))



In terms of polar coordinates,

$$z = re^{i\theta}, \qquad \bar{z} = re^{-i\theta}$$

$$z = x + iy$$

$$\bar{z} = re^{-i\theta} = r\cos\theta - i\,r\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$x = r\cos\theta, \quad y = r\sin\theta \Rightarrow \bar{z} = \boxed{r\cos\theta - i\,r\sin\theta}$$

- $\forall\, z, w \in \mathbb{C}$, $\overline{z+w} = \bar{z} + \bar{w}$

$$\overline{zw} = \bar{z}\,\bar{w}$$

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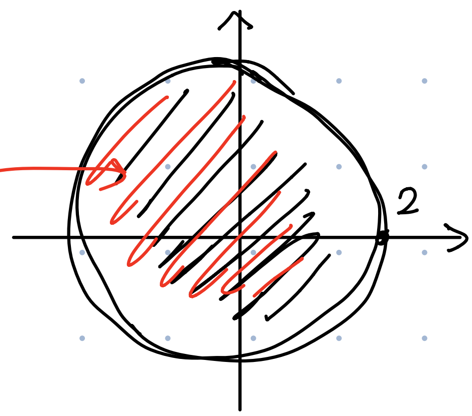
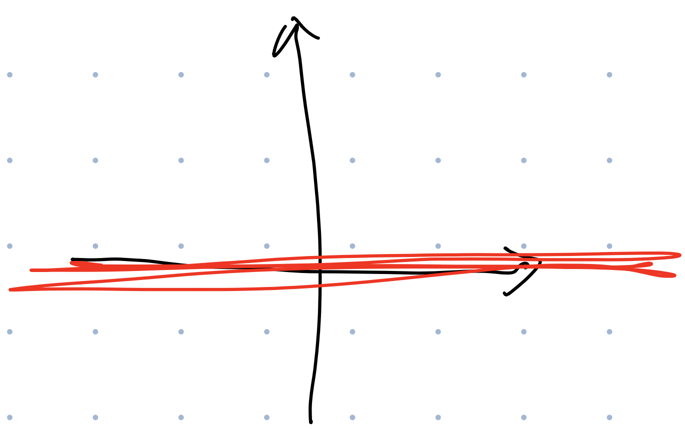
Basic shapes:

$$\boxed{\{z \in \mathbb{C} \mid |z| \leq 2\}}$$




$$\{z \in \mathbb{C} \mid \boxed{z = \bar{z}}\}$$

$$x + iy = x - iy$$
$$\Updownarrow$$
$$y = 0$$



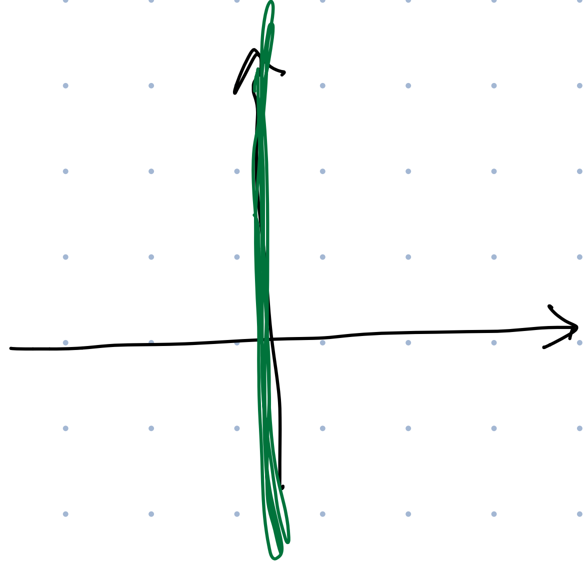
$\{z\in\mathbb{C} \mid z = -\bar{z}\}$

$z = x+iy$, $\bar{z} = x-iy$

$-\bar{z} = -x+iy$

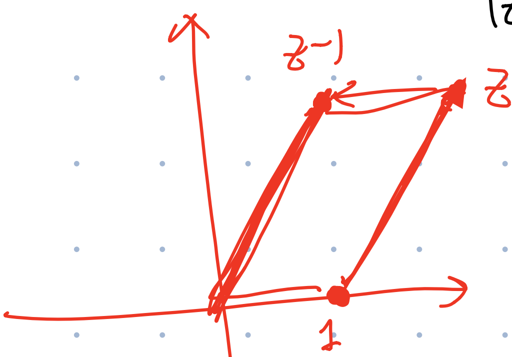
$\Updownarrow$

$x=0$

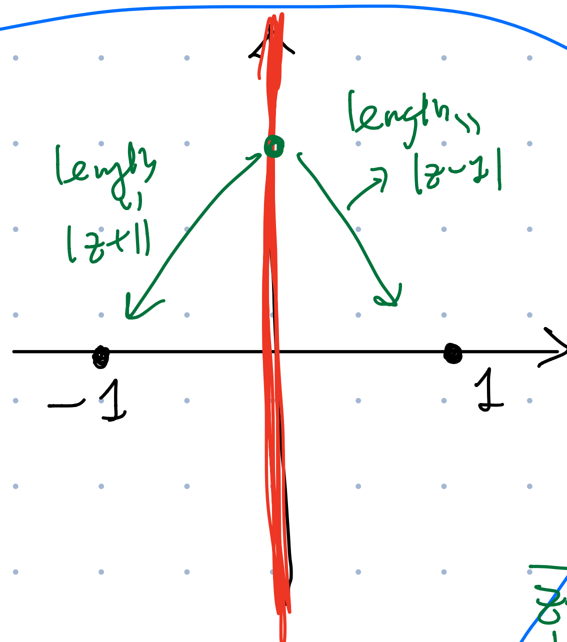


$\{z\in\mathbb{C} \mid |z-1| = |z+1|\}$

$|z+1| = |z-(-1)|$ = dist. b/w $z$ and $-1$




$|z-1|$ = distance between $z$ and $1$




$|z-w|$ = distance between $z$ and $w$.

$z, w\in\mathbb{C}$

$z = x+iy$, $w = u+iv$

$|z-w| = |(x-u)+i(y-v)|$

$= \sqrt{(x-u)^2+(y-v)^2}$

= distance b/w $(x,y)$ and $(u,v)$ in $\mathbb{R}^2$

$\overline{z-1} = \bar{z}-1$ b/c $\bar{1}=1$

Algebraically,

$|z-1|^2 = |z+1|^2$

$(z-1)(\bar{z}-1) = \cancel{z\bar{z}} - z - \bar{z} + \cancel{1}$

$(z+1)(\bar{z}+1) = \cancel{z\bar{z}} + z + \bar{z} + \cancel{1}$

$\Updownarrow$

$z+\bar{z}=0$

the previous example

Rmk: $|z|^2 = z \cdot \bar{z}$

$z = x + iy$ $\qquad$ $\bar{z} = x - iy$

$|z| = \sqrt{x^2 + y^2}$ $\qquad$ $z \cdot \bar{z} = (x+iy)(x-iy)$

$|z|^2 = x^2 + y^2$ $\Longrightarrow$ $= x^2 + y^2$