(1) Let $B \subseteq \mathbb{R}^n$ be the open ball of radius one centered at the origin, i.e.

$$B = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < 1\}.$$

Prove that any uniformly continuous function $f : B \to \mathbb{R}$ is bounded.

- Since $f: B \to \mathbb{R}$ is uniformly continuous, $\exists\, \delta > 0$ s.t.

$$\underset{x,y \in B}{d(x,y) < \delta} \implies |f(x) - f(y)| < 1.$$

denotes the origin of $\mathbb{R}^n$: $(0,0,\ldots,0)$

- For any point $x \in B$ we have $d(0,x) < 1$,
let N be any integer greater than $1/\delta$.
Then there exists N points $x_1, x_2, \ldots, x_N$ on the line segment connecting 0 and x, so that each of:

$$d(0,x_1),\ d(x_1,x_2),\ \ldots,\ d(x_{N-1},x_N),\ d(x_N,x)$$

is less than δ.
- Then:

$$|f(x)| \leq |f(0)| + |f(x_1) - f(0)| + |f(x_2) - f(x_1)| + \cdots + |f(x) - f(x_N)|$$

$$< |f(0)| + N + 1.$$

for any $x \in B$. $\square$

(2) Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational, or } x = 0, \text{ or } x = 1 \\ \frac{1}{q} & \text{if } x \in \mathbb{Q} \cap (0,1) \text{ and } x = \frac{p}{q} \text{ where } p, q \in \mathbb{N},\ \gcd(p,q) = 1. \end{cases}$$

Find all the points in $[0,1]$ at which f is continuous, and give a proof.
(Note: gcd denotes the greatest common divisor.)

f is continuous at x if $x \notin \mathbb{Q}$ or $x = 0, 1$;
f is discontinuous at x if $x \in \mathbb{Q} \setminus \{0,1\}$.

<u>Case 1</u>: $x \notin \mathbb{Q}$ or $x = 0, 1$

- Let $\varepsilon > 0$.
  There exists finitely many integers $q > 0$ s.t. $\frac{1}{q} \geq \varepsilon$.
  Hence there exists finitely many rational numbers $\frac{p}{q} \in [0,1]$ s.t. $f(\frac{p}{q}) \geq \varepsilon$.
- Choose $\delta > 0$ small enough s.t. $(x-\delta, x+\delta)$ doesn't contain any such rational numbers. Then $\forall y \in (x-\delta, x+\delta)$ we have:
  $$0 \leq f(y) < \varepsilon.$$
  Hence f is continuous at x. $\square$

<u>Case 2</u>: $x \in \mathbb{Q} \setminus \{0,1\}$

- In this case, $f(x) > 0$.
- By the same reasoning as above, $\exists\, \delta > 0$ s.t.
  if $y \in (x-\delta, x+\delta) \setminus \{x\}$, then $0 \leq f(y) < f(x)/2$.

Hence f is discontinuous at x. $\square$

(3) Consider the same function $f\colon [0,1] \to \mathbb{R}$ as in the previous problem. Using only the definition of upper integral, prove that ~~the~~ $U(f) = 0$. **You're not allowed to use any theorem in this problem.**

We'll show that $\forall \varepsilon > 0$, $\exists P$ s.t. $0 \leq U(f,P) < \varepsilon$.

this is clear for any P.

- There exists finitely many points, say $x_1 < \cdots < x_N$ in $[0,1]$, s.t. $f(x_i) \geq \frac{\varepsilon}{2}$.

- Choose a partition $P = \{ \underset{x_0^+}{\overset{\shortparallel}{0}} < x_1^- < x_1^+ < x_2^- < x_2^+ < \cdots < x_N^- < x_N^+ < \underset{x_{N+1}^-}{\overset{\shortparallel}{1}} \}$

  s.t.
  - $x_i^- < x_i < x_i^+$. $\forall\ 1 \leq i \leq N$.
  - $\displaystyle\sum_{i=1}^{N} (x_i^+ - x_i^-) < \frac{\varepsilon}{2}$.

- Then

$$U(f,P) = \sum_{i=1}^{N+1} (x_i^- - x_{i-1}^+) \cdot \boxed{\sup_{x \in [x_{i-1}^+, x_i^-]} f(x)} \leq \frac{\varepsilon}{2}$$

$$+ \sum_{i=1}^{N} (x_i^+ - x_i^-) \cdot \boxed{\sup_{x \in [x_i^-, x_i^+]} f(x)} \leq 1$$

$$\leq \frac{\varepsilon}{2} \cdot \sum_{i=1}^{N+1} (x_i^- - x_{i-1}^+) + 1 \cdot \sum_{i=1}^{N} (x_i^- - x_{i-1})$$

$$< \frac{\varepsilon}{2} \cdot 1 + 1 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

Rmk: The idea is essentially the same as the proof of the Riemann-Lebesgue theorem.

(4) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \left(\int_0^1 |f(x)|^n dx\right)^{1/n} = \sup\{|f(x)| \colon x \in [0,1]\}.$$

(Note: You need to show that the limit on the left hand side exists, and coincides with the right hand side.)
(Hint: For any $\epsilon > 0$, show that there exists some subinterval of $[0,1]$ such that the value of $|f|$ on this subinterval is at least $\sup\{|f(x)|\} - \epsilon$. Then use this to estimate the left hand side.)

- Since f is continuous on the compact set $[0,1]$, the supreme $M := \sup\{|f(x)|\}$ is achieved, say by $x_0 \in [0,1]$.
- $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

$$\Rightarrow |f(x)| > M - \varepsilon.$$

- Then: $$\left(\int_0^1 |f(x)|^n dx\right)^{1/n} > \left(\delta \cdot (M-\varepsilon)^n\right)^{1/n}$$

$$\Rightarrow \liminf_{n\to\infty} \left(\int_0^1 |f(x)|^n dx\right)^{1/n} \geq M - \varepsilon.$$

- Since the inequality holds for all $\varepsilon > 0$, we have:

$$\liminf_{n\to\infty} \left(\int_0^1 |f(x)|^n dx\right)^{1/n} \geq M.$$

- On the other hand,

$$\left(\int_0^1 |f(x)|^n dx\right)^{1/n} \leq \left(\int_0^1 M^n dx\right)^{1/n} = M.$$

$$\Rightarrow \limsup_{n\to\infty} \left(\int_0^1 |f(x)|^n dx\right)^{1/n} \leq M.$$

- Therefore: $M \leq \liminf_{n\to\infty} \left(\int_0^1 |f(x)|^n dx\right)^{1/n} \leq \limsup_{n\to\infty} \left(\int_0^1 |f(x)|^n dx\right)^{1/n} \leq M.$

$\Rightarrow \lim \left(\int_0^1 |f(x)|^n dx\right)^{1/n}$ exists and $= M$. $\square$

(5) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 x^n f(x)dx = 0 \text{ holds for any non-negative integer } n \geq 0.$$

Prove that f is the zero function.

(Hint: You can apply the Weierstrass Approximation Theorem.)

- Since f is continuous on the compact set $[0,1]$, f is bounded, say $|f(x)| < M$ $\forall x \in [0,1]$.
- By Weierstrass Approximation Theorem, $\forall \varepsilon > 0$, $\exists$ polynomial P s.t. $|P(x) - f(x)| < \varepsilon/M$ $\forall x \in [0,1]$.
- By the assumption, we have $\int_0^1 P(x) f(x)\, dx = 0$
- Then $$\int_0^1 f(x)^2 dx = \left| \int_0^1 f(x)\left(f(x) - P(x)\right) dx \right| < \varepsilon.$$
- Since the inequality holds for any $\varepsilon > 0$, we have: $$\int_0^1 f(x)^2 dx = 0.$$
- Assume the contrary that f is not the zero function, say $f(x_0) \neq 0$,

  By continuity of f, $\exists\, \delta > 0$ s.t. $f(y)^2 > \frac{f(x_0)^2}{2}$ $\forall |y - x_0| < \delta$.

  $\Longrightarrow \int_0^1 f(x)^2 dx > 0.$ Contradiction. $\square$

(6) An open cube in $\mathbb{R}^n$ is a product of open intervals $U = (a_1, b_1) \times \cdots \times (a_n, b_n)$, where its volume is defined to be $\mathrm{vol}(U) = (b_1 - a_1) \cdots (b_n - a_n)$. Recall that we say a subset $E \subseteq \mathbb{R}^n$ has measure zero if for any $\epsilon > 0$, there exists finitely or countably many open cubes $U_1, U_2, \ldots$ such that

$$E \subseteq \bigcup_i U_i \quad \text{and} \quad \sum_i \mathrm{vol}(U_i) < \epsilon.$$

Let $f \colon [a, b] \to \mathbb{R}$ be a continuous map. Prove that the graph

$$\Gamma_f = \{(x, f(x)) \in \mathbb{R}^2 \colon x \in [a, b]\}$$

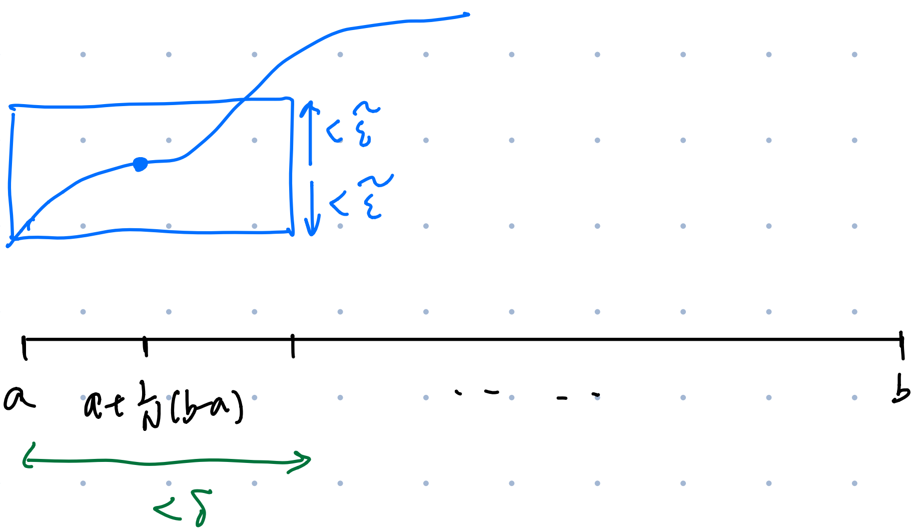
has measure zero in $\mathbb{R}^2$.

- f is uniformly continuous since $[a,b]$ is compact.

  Hence. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\underset{x,y \in [a,b]}{|x-y| < \delta} \Rightarrow |f(x) - f(y)| < \tilde{\varepsilon}$

  $\tilde{\varepsilon} = \dfrac{\varepsilon}{8(b-a)}$

- Choose an integer $N > 0$ large enough s.t. $\frac{2}{N}(b-a) < \delta$.

- 

The graph of f above $\left(a + \frac{i}{N}(b-a),\ a + \frac{i+2}{N}(b-a)\right)$ can be covered by an open cube of size $\frac{2}{N}(b-a) \times 2\tilde{\varepsilon}$.

- The graph of f over $[a,b]$ can be covered by $(N+1)$-many such open cubes, the total volume is:

$$(N+1) \cdot \frac{2}{N}(b-a) \cdot 2\tilde{\varepsilon} < 8(b-a)\tilde{\varepsilon} = \varepsilon. \qquad \square$$

(7) Let (a_n) be a decreasing sequence such that the series $\sum a_n$ converges. Prove that

$$\lim_{n\to\infty} na_n = 0.$$

- $\sum a_n$ converges $\Rightarrow$ $\lim a_n = 0$,
  and since (a_n) is decreasing, we have $a_n \geq 0$ $\forall n$
- By Cauchy criterion, $\forall \varepsilon > 0$, $\exists N > 0$
  s.t. $|a_m + a_{m+1} + \cdots + a_n| < \varepsilon/2$ $\forall n \geq m > N$.
  Since (a_n) is decreasing and $a_n \geq 0$ $\forall n$, we have,
  $\varepsilon/2 > a_m + a_{m+1} + \cdots + a_n \geq (n-m)a_n \geq 0.$ $\forall n \geq m > N$.
- Fix any $m > N$. Then:
  $$0 \leq (n-m)a_n < \varepsilon/2 \quad \forall n > m.$$
- Since $\lim a_n = 0$, and $a_n \geq 0$ $\forall n$,
  $\exists \tilde{N} > 0$ s.t. $0 \leq a_n < \varepsilon/2m$ $\forall n > \tilde{N}$.
- Then $\forall n > \max\{m, \tilde{N}\}$, we have:
  $$0 \leq na_n < \frac{\varepsilon}{2} + ma_n < \varepsilon.$$
  This proves $\lim na_n = 0$. $\square$