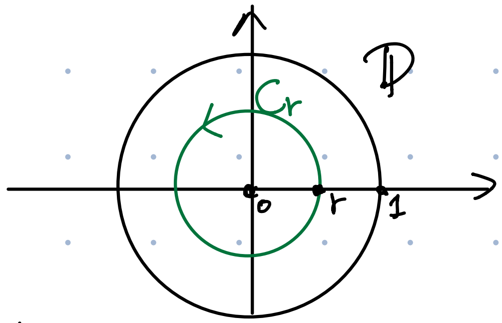
7. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,\, w \in \mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies

$$2|f'(0)| \leq d.$$

Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.



Let $0 < r < 1$, and consider:

$$f'(0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w^2}\, dw.$$

Substitute w by $-w$; we have:

$$f'(0) = \frac{1}{2\pi i} \int_{C_r} \frac{-f(-w)}{w^2}\, dw.$$

$$\Longrightarrow \quad 2f'(0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(w) - f(-w)}{w^2}\, dw.$$

$$\Longrightarrow \quad 2|f'(0)| \leq \frac{1}{2\pi} \sup_{w \in C_r} \frac{|f(w) - f(-w)|}{|w|^2} \cdot (2\pi r)$$

$$\leq \frac{1}{2\pi} \cdot \frac{d}{r^2} \cdot 2\pi r = \frac{d}{r}$$

Since this ineq., holds $\forall\, 0 < r < 1$, we have:

$$2|f'(0)| \leq \inf_{0<r<1} \frac{d}{r} = d. \quad \square$$

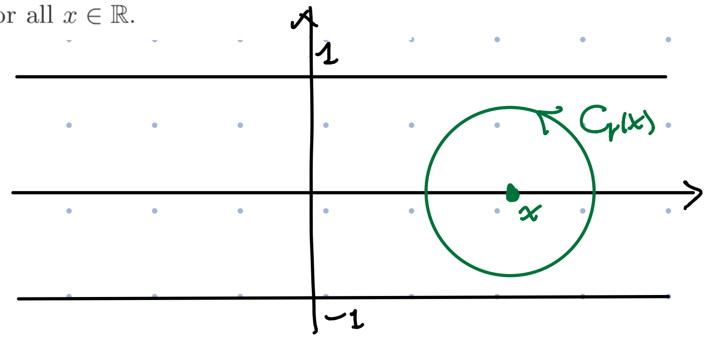
8. If f is a holomorphic function on the strip $-1 < y < 1$, $x \in \mathbb{R}$ with

$$|f(z)| \le A(1+|z|)^\eta, \qquad \eta \text{ a fixed real number}$$

for all z in that strip, show that for each integer $n \ge 0$ there exists $A_n \ge 0$ so that

$$|f^{(n)}(x)| \le A_n(1+|x|)^\eta, \qquad \text{for all } x \in \mathbb{R}.$$

Let $0<r<1$, consider



By Cauchy's ineq. $\Rightarrow$ $\displaystyle |f^{(n)}(x)| \le \frac{n!}{r^n} \sup_{z\in C_r(x)} |f(z)| \le \frac{n!}{r^n} A \cdot \sup_{z\in C_r(x)} (1+|z|)^\eta$

Claim: $\exists$ const. $C>0$ depends only on η,

(i.e. independent of $x\in\mathbb{R}$ and $0<r<1$)

s.t. $\displaystyle \sup_{z\in C_r(x)} (1+|z|)^\eta < C(1+|x|)^\eta$ holds for any $0<r<1$, $x\in\mathbb{R}$.

(Given the claim, we have:

$$|f^{(n)}(x)| \le \inf_{0<r<1}\left(\frac{n!}{r^n} A\, C \cdot (1+|x|)^\eta\right) = n!\, A \cdot C \cdot (1+|x|)^\eta,$$

which gives what we need to prove.)

pf of Claim: We'll show that one can take $C = 2^{|\eta|}$.

① If $\eta \ge 0$, then

$$\sup_{z\in C_r(x)} (1+|z|)^\eta \le (1+|x|+r)^\eta \le (2+|x|)^\eta$$

$$= 2^\eta\left(1+\frac{|x|}{2}\right)^\eta \le 2^\eta(1+|x|)^\eta. \quad \forall x\in\mathbb{R},\ 0<r<1$$

② If $\eta < 0$, then

$$\sup_{z \in C_r(x)} (1+|z|)^\eta \leq \begin{cases} 1, & \text{If } |x| < r. \\ (1+|x|-r)^\eta, & \text{If } |x| \geq r. \end{cases}$$

$$\leq \begin{cases} 1, & \text{If } |x| < 1 \\ |x|^\eta, & \text{If } |x| \geq 1. \end{cases} =: g(x)$$

For $|x| < 1$, we have:

$$\sup_{|x|<1} \frac{g(x)}{(1+|x|)^\eta} = \sup_{|x|<1} \frac{1}{(1+|x|)^\eta} = 2^{-\eta}.$$

For $|x| \geq 1$, we have:

$$\sup_{|x|\geq 1} \frac{g(x)}{(1+|x|)^\eta} = \sup_{|x|\geq 1} \frac{|x|^\eta}{(1+|x|)^\eta} = 2^{-\eta}.$$

$$\Longrightarrow \sup_{z \in C_r(x)} (1+|z|)^\eta \leq 2^{-\eta} (1+|x|)^\eta \quad \forall x \in \mathbb{R},\ 0<r<1. \quad \square$$

9. Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\varphi(z_0) = z_0 \quad \text{and} \quad \varphi'(z_0) = 1$$

then φ is linear.

- One can consider $\tilde{\Omega} := \{ z \in \mathbb{C} \mid z + z_0 \in \Omega \}$.
  and $\tilde{\varphi} : \tilde{\Omega} \longrightarrow \tilde{\Omega}$
  $$z \longmapsto \varphi(z+z_0) - z_0.$$
  Then $\tilde{\Omega} \subseteq \mathbb{C}$ is bdd, open, and $\tilde{\varphi} : \tilde{\Omega} \to \tilde{\Omega}$ holo.,
  $\tilde{\varphi}(0) = 0$, $\tilde{\varphi}'(0) = 1$,
  and "φ is linear" $\iff$ "$\tilde{\varphi}$ is linear".
  Therefore, we may assume $z_0 = 0$.

- Consider the power series exp. of φ in a nbhd of 0:
  $$\varphi(z) = \sum a_n z^n.$$
  By the assumption, we have $a_0 = 0$ and $a_1 = 1$.

- Suppose there exists some other nonzero a_n, and let m be the smallest integer s.t. $m \geqslant 2$ and $a_m \neq 0$. Then
  $$\varphi(z) = z + a_m z^m + O(z^{m+1}).$$

- Consider the k-th iteration $\varphi_k := \varphi \circ \varphi \circ \cdots \circ \varphi$ of φ.
  Then $\varphi_k : \Omega \to \Omega$ holo., and we still have

$\varphi_k(0)=0$ and $\varphi_k'(0)=1$. (by chain rule)

- Claim: The power series exp. of φ_k at 0 is of the form
$$\varphi_k(z) = z + k a_m z^m + O(z^{m+1}).$$

pf: Induction on k. $k=1$: ok. Consider $k+1$:

$$\begin{aligned}
\varphi_{k+1}(z) &= \varphi(\varphi_k(z)) \\
&= \varphi_k(z) + a_m(\varphi_k(z))^m + O(\varphi_k(z)^{m+1}) \\
&= (z + k a_m z^m + O(z^{m+1})) \\
&\qquad + a_m(z + k a_m z^m + O(z^{m+1}))^m + O(z^{m+1}) \\
&= z + (k+1) a_m z^m + O(z^{m+1}). \quad \square
\end{aligned}$$

- However, this contradicts with Cauchy's ineq.:
Choose $r>0$ s.t. $\overline{B_r(0)} \subseteq \Omega$,
and let $R := \sup_{z\in\Omega} |z| < +\infty$ (since Ω is bdd.)

Then
$$k|a_m| = \frac{|\varphi_k^{(m)}(0)|}{m!} \leq \frac{\frac{m!}{r^m} \sup_{z\in C_r(0)} |\varphi_k(z)|}{m!} \leq \frac{R}{r^m}. \quad \forall k.$$

Since $a_m \neq 0$, $\{k|a_m|\}_k$ is not bounded, contradiction.

$\square$

11. Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

(a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi.$$

(b) Show that

$$\mathrm{Re}\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}.$$

(a) If $z=0$, then the result follows from Cauchy integral formula.

$$f(0) = \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(w)}{w}\,dw.$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(Re^{i\varphi})}{Re^{i\varphi}}(iRe^{i\varphi})\,d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\,d\varphi.$$

Now suppose $z \neq 0$. Let $w = Re^{i\varphi}$. then

$$\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right) = \mathrm{Re}\left(\frac{w+z}{w-z}\right)$$

$$= \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{\bar{w}+\bar{z}}{\bar{w}-\bar{z}}\right)$$

$$= \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{R^2/w+\bar{z}}{R^2/w-\bar{z}}\right)$$

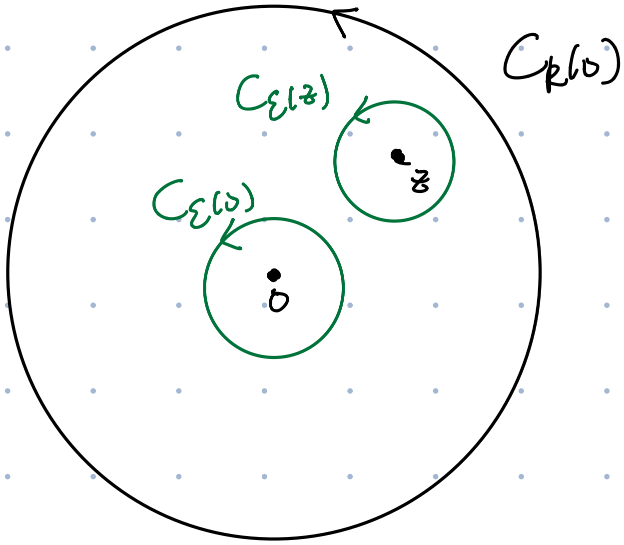
$$= \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{R^2/\bar{z}+w}{R^2/\bar{z}-w}\right)$$

Therefore, the integral becomes:

$$\frac{1}{2\pi}\int_{C_R(0)} f(w)\cdot\frac{1}{2}\left(\frac{w+z}{w-z} + \frac{R^2/\bar{z}+w}{R^2/\bar{z}-w}\right)\cdot\frac{1}{iw}\,dw$$

The integrand has 3 singularities: z, $R^2/\bar{z}$, and 0.

Note that $|R^2/\bar{z}| > R$, hence only 0 and z are in the disk $\mathbb{D}_R(0)$.



By the keyhole argument,

$$\int_{C_R(0)} \cdots = \int_{C_\varepsilon(0)} \cdots + \int_{C_\varepsilon(z)} \cdots$$

By Cauchy integral formula,

$$\frac{1}{2\pi} \int_{C_\varepsilon(0)} f(w) \cdot \frac{1}{2} \left(\frac{w+z}{w-z} + \frac{R^2/\bar{z} + w}{R^2/\bar{z} - w} \right) \cdot \frac{1}{iw}\, dw$$

$$= \left[f(w) \cdot \frac{1}{2} \left(\frac{w+z}{w-z} + \frac{R^2/\bar{z} + w}{R^2/\bar{z} - w} \right) \right] \Bigg|_{w=0} = 0.$$

and.

$$\frac{1}{2\pi} \int_{C_\varepsilon(z)} f(w) \cdot \frac{1}{2} \left(\frac{w+z}{w-z} + \frac{R^2/\bar{z} + w}{R^2/\bar{z} - w} \right) \cdot \frac{1}{iw}\, dw$$

$$= \left[f(w) \cdot \frac{1}{2} \cdot (w+z) \cdot \frac{1}{w} \right] \Bigg|_{w=z}.$$

$$= f(z) \cdot \frac{1}{2} \cdot 2z \cdot \frac{1}{z} = f(z). \quad \square$$

(b)
$$\mathrm{Re}\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \mathrm{Re}\left(\frac{(Re^{i\gamma}+r)(Re^{-i\gamma}-r)}{(Re^{i\gamma}-r)(Re^{-i\gamma}-r)}\right)$$
$$= \frac{\mathrm{Re}\left(R^2-r^2+rR\left(e^{-i\gamma}-e^{i\gamma}\right)\right)}{R^2+r^2-2Rr\cos\gamma} = \frac{R^2-r^2}{R^2+r^2-2Rr\cos\gamma}. \quad \square$$

12. Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\triangle u(x,y) = 0$$

for all $(x,y) \in \mathbb{D}$.

(a) Prove that there exists a holomorphic function f on the unit disc such that

$$\mathrm{Re}(f) = u.$$

Also show that the imaginary part of f is uniquely defined up to an additive (real) constant. [Hint: From the previous chapter we would have $f'(z) = 2\partial u/\partial z$. Therefore, let $g(z) = 2\partial u/\partial z$ and prove that g is holomorphic. Why can one find F with $F' = g$? Prove that $\mathrm{Re}(F)$ differs from u by a real constant.]

(b) Deduce from this result, and from Exercise 11, the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in the unit disc and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)u(\cancel{\varphi}^{\,e^{i\varphi}})\,d\varphi$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma) = \frac{1-r^2}{1-2r\cos\gamma+r^2}.$$

(a) Define $g(z) := u_x - iu_y$ $\left(= 2\frac{\partial u}{\partial z}\right)$, then g is continuously diff'able

$$\frac{\partial g}{\partial \bar z} = \frac{1}{2}\left[\frac{\partial}{\partial x}(u_x - iu_y) + i\frac{\partial}{\partial y}(u_x - iu_y)\right]$$

$$= \frac{1}{2}\left(u_{xx} - i\cancel{u_{xy}} + i\cancel{u_{xy}} + u_{yy}\right) = 0$$

since $\Delta u = 0$.

$\Rightarrow$ g is holo. on $\mathbb{D}$

$\Rightarrow$ g has a primitive F by thm 2.1.

Write $F = \tilde{u} + i\tilde{v}$,

Then $u_x - i u_y = g = \frac{\partial F}{\partial z} = \frac{1}{2}(F_x - iF_y)$

$$= \frac{1}{2}(\tilde{u}_x + i\tilde{v}_x - i\tilde{u}_y + \tilde{v}_y)$$

$$= \tilde{u}_x - i\tilde{u}_y \quad \text{by Cauchy-Riemann eq}^{n}.$$

$\Longrightarrow$ $u_x = \tilde{u}_x$ and $u_y = \tilde{u}_y$.

$\Longrightarrow$ $\mathrm{Re}(F) - u \equiv$ const. C.

Define $f = F - C$, then f is holo. and $\mathrm{Re}(f) = u$. $\square$

(b) By # 11(a)(b), $\forall$ $|z| < R < 1$, we have:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \cdot \mathrm{Re}\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right) d\varphi$$

Take the real part $\Longrightarrow$

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\varphi}) \cdot \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\varphi - \theta)} d\varphi$$

Let $R \longrightarrow 1$, we have:

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\varphi}) \cdot P_r(\theta - \varphi)\, d\varphi. \quad \square$$