Today: Automorphisms on $\mathbb{D}$ and $\mathbb{H}$

Def: A <u>fractional linear transformation</u> (or a <u>Möbius transformation</u>) is a map of the form
$$z \longmapsto \frac{az+b}{cz+d},$$
where $a,b,c,d \in \mathbb{R}$, and $ad-bc > 0$.
(rules out constant maps).

Rmk:

- the condition $ad-bc>0$ is equivalent to
$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0.$$

- the transformation is invariant under rescale a,b,c,d simultaneously. i.e. If we replace a,b,c,d by ra, rb, rc, rd for some $r \in \mathbb{R}\setminus 0$. then we get the same map. $\left(\frac{az+b}{cz+d} = \frac{(ra)z+(rb)}{(rc)z+(rd)}\right)$

  So: We can assume that $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$. $SL_2(\mathbb{R})$

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ give the same fractional linear transf.

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}$$
$a,b,c,d \in \mathbb{R}$

$$PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \pm 1 \xleftrightarrow{1-1} \text{fractional linear transf.}$$

Lemma: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{R})$

f_A: $z \mapsto \frac{az+b}{cz+d}$

$f_{A'}$: $z \mapsto \frac{a'z+b'}{c'z+d'}$

$f_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}(z) = \frac{1\cdot z+0}{0\cdot z+1} = z$

Then $f_A \circ f_{A'} = f_{AA'}$; $f_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = \mathrm{id}$

Ex:

$$z \xmapsto{f_{A'}} \frac{a'z+b'}{c'z+d'} \xmapsto{f_A} \frac{a\left(\frac{a'z+b'}{c'z+d'}\right)+b}{c\left(\frac{a'z+b'}{c'z+d'}\right)+d}$$

$$\| ??$$

$$f_{AA'}(z)$$

Prop: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$.

$z \mapsto \frac{az+b}{cz+d}$ is an automorphism of $\mathbb{H}$.

(i.e. it's a biholomorphism $\mathbb{H} \to \mathbb{H}$)

pf:

Need: • this is a bijection between $\mathbb{H}$ to $\mathbb{H}$

• holo. on $\mathbb{H}$.

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{(az+b)(c\bar{z}+d)}{(cz+d)(c\bar{z}+d)}\right) = \frac{1}{|cz+d|^2}\mathrm{Im}\left(ac|z|^2 + bd + adz + bc\bar{z}\right)$$

$$= \frac{1}{|cz+d|^2}\underbrace{(ad-bc)}_{=1}\underbrace{\mathrm{Im}(z)}_{>0}$$

so if $z \in \mathbb{H}$, then $\frac{az+b}{cz+d} \in \mathbb{H}$.

- f_A is a bijection $\mathbb{H} \to \mathbb{H}$:

Consider $\underset{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{\underset{\shortparallel}{A^{-1}}} \in SL_2(\mathbb{R})$, consider $f_{A^{-1}}$

Then $$f_A \circ f_{A^{-1}} \underset{*}{=} f_{AA^{-1}} = f_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = id.$$

$$f_{A^{-1}} \circ f_A = id.$$

$\Rightarrow$ f_A is bijective, and its inverse is $f_{A^{-1}}$.

Rmk:
- $$\begin{aligned} SL_2(\mathbb{R}) &\longrightarrow \mathrm{Aut}(\mathbb{H}) \quad \text{group homomorphism.} \\ A &\longmapsto f_A \end{aligned}$$
- kernel of this map is $\left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \subseteq SL_2(\mathbb{R})$
- $PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \hookrightarrow \mathrm{Aut}(\mathbb{H})$
- We'll show that any automorphism of $\mathbb{H}$ is a Möbius transformation, i.e. $PSL_2(\mathbb{R}) \cong \mathrm{Aut}(\mathbb{H})$

$\Rightarrow$ $\mathrm{Aut}(\mathbb{H})$ has "3 degree of freedoms".

§ Automorphisms on $\mathbb{D}$

e.g. $z \longmapsto e^{i\theta} z$ (rotation by angle θ)

is an automorphism on $\mathbb{D}$

Rmk:

$$\begin{array}{ccc} \mathbb{D} & \dashrightarrow & \mathbb{D} \\ {\scriptstyle i\frac{1-z}{1+z}}\Big\downarrow{\scriptstyle \text{biholo}} & & \Big\uparrow{\scriptstyle \frac{i-z}{i+z}}\ {\scriptstyle \text{biholo.}} \\ \mathbb{H} & \dashrightarrow & \mathbb{H} \end{array}$$

$\{\text{auto. on } \mathbb{H}\} \xleftrightarrow{1-1} \{\text{auto. on } \mathbb{D}\}$

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e.g. $\forall\, \alpha \in \mathbb{D}$, define $\psi_\alpha : \mathbb{D} \longrightarrow \mathbb{C}$ (cf. HW1 #7)

$$z \longmapsto \frac{\alpha - z}{1 - \bar{\alpha} z}$$

- $\psi_\alpha(0) = \alpha$, $\psi_\alpha(\alpha) = 0$
- $\psi_\alpha \circ \psi_\alpha = id_{\mathbb{D}}$
- ψ_α maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$

Lemma: $\psi_\alpha \in Aut(\mathbb{D})$

pf: • $\dfrac{\alpha - z}{1 - \bar{\alpha} z}$ holo. on $\mathbb{D}$:

- ψ_α extends continuously to $\overline{\mathbb{D}}$, and send $\partial\mathbb{D}$ to $\partial\mathbb{D}$

$\Rightarrow$ $|\psi_\alpha(z)| = 1 \quad \forall |z| = 1$

- By max. modulus principle, we have

$$|\psi_\alpha(z)| < 1 \quad \forall |z| < 1.$$

$\Rightarrow$ $\psi_\alpha : \mathbb{D} \longrightarrow \mathbb{D}$

- Since $\psi_\alpha \circ \psi_\alpha = id_{\mathbb{D}}$, so $\psi_\alpha = \psi_\alpha^{-1}$, and ψ_α is bijective. $\square$

Main thm: $\forall f \in Aut(\mathbb{D})$, $\exists\, \theta \in \mathbb{R}$, and $\alpha \in \mathbb{D}$ s.t. $f(z) = e^{i\theta}\psi_\alpha(z)$

Schwarz lemma: Suppose • $f: \mathbb{D} \longrightarrow \mathbb{D}$ holo. (not necessary bijective)
• $f(0) = 0$

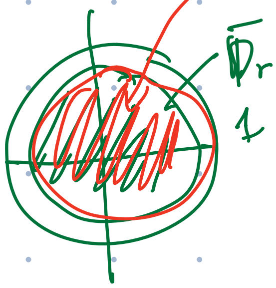
Then
1) $|f(z)| \leq |z| \quad \forall z \in \mathbb{D}$
2) If $\exists\, z_0 \in \mathbb{D}\setminus\{0\}$ s.t. $|f(z_0)| = |z_0|$ then f is a rotation. (i.e. $f(z) = e^{i\theta}z \ \forall z$ for some θ)
3) $|f'(0)| \leq 1$ and If $|f'(0)| = 1$ then f is a rotation.

pf: 1) $g(z) = \begin{cases} f(z)/z & \text{for } z \in \mathbb{D}\setminus\{0\} \\ f'(0) & \text{for } z = 0 \end{cases}$ holo. on $\mathbb{D}$

$\forall\, 0 < r < 1.$

$$\max_{\overline{\mathbb{D}}_r} |g(z)| = \max_{\partial \mathbb{D}_r} |g(z)| = \max_{\partial \mathbb{D}_r} \frac{|f(z)|}{r} < \frac{1}{r}$$

Since $f(z) \in \mathbb{D}$



Let $r \to 1$, then we have

$$\max_{z\in\mathbb{D}} |g(z)| \le 1.$$

$$\Rightarrow |g(z)| \le 1 \quad \forall z \in \mathbb{D}$$

$$\Rightarrow |f(z)| \le |z| \quad \forall z \in \mathbb{D}. \quad \square$$

2) If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}\setminus 0$

$\Rightarrow |g(z_0)| = 1$

But since $\max_{z\in\mathbb{D}} |g(z)| \le 1$,

By max. modulus principle $\Rightarrow g \equiv \text{const} = e^{i\theta}$ for some θ

$\Rightarrow f(z) = e^{i\theta} z \quad \forall z \in \mathbb{D}.$

i.e. f is a rotation. $\square$

3), $f'(0) = g(0)$

So $|f'(0)| \le 1$,

And, If $|f'(0)| = 1$, then by max. mod. prople
$$\underset{|g(0)|}{\shortparallel}$$

$\Downarrow$

g const. $= e^{i\theta}$

$\Downarrow$

f rotation. $\square$

<u>Coro</u>: $f \in \mathrm{Aut}(\mathbb{D})$ and $f(0)=0 \Longrightarrow f$ is rotation

Pf: $f, f^{-1} : \mathbb{D} \to \mathbb{D}$ holo. $f(0)=0$, $f^{-1}(0)=0$

Both satisfy the assumptn of the Schwarz lemma.

So
$$|f(z)| \le |z| \quad \forall z \in \mathbb{D}$$
$$|f^{-1}(z)| \le |z| \quad \forall z \in \mathbb{D}$$

$$\Rightarrow |f(z)| \le |z| = |f^{-1}(f(z))| \le |f(z)| \quad \forall z \in \mathbb{D}$$

$$\Rightarrow |f(z)| = |z| \quad \forall z \in \mathbb{D}$$

$\Rightarrow f$ is a rotation. $\square$

Pf of main thm.

Suppose $f \in \mathrm{Aut}(\mathbb{D})$,

- $\exists!\ \alpha \in \mathbb{D}$ st. $f(\alpha) = 0$
- Define $g = f \circ \psi_\alpha$
- Then $g(0) = f(\psi_\alpha(0)) = f(\alpha) = 0$
- By Coro; $g = e^{i\theta}$ is a rotation.
- $f = g \circ \psi_\alpha^{-1} = e^{i\theta} \circ \psi_\alpha$. $\square$