#1: Any $\vec{w} \in W$ can be written as $\vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$.

$$\Rightarrow \langle \vec{x}, \vec{w} \rangle = \langle \vec{x}, a_1\vec{v}_1 + \cdots + a_n\vec{v}_n \rangle$$
$$= a_1 \langle \vec{x}, \vec{v}_1 \rangle + \cdots + a_n \langle \vec{x}, \vec{v}_n \rangle = 0. \square$$

#2:

(a) If $\vec{x}_1, \vec{x}_2 \in W^\perp$,

then • $\vec{x}_1 + \vec{x}_2 \in W^\perp$ since $\langle \vec{x}_1 + \vec{x}_2, \vec{w} \rangle = \langle \vec{x}_1, \vec{w} \rangle + \langle \vec{x}_2, \vec{w} \rangle = 0$ $\forall \vec{w} \in W$.

• $c\vec{x}_1 \in W^\perp$ since $\langle c\vec{x}_1, \vec{w} \rangle = c\langle \vec{x}_1, \vec{w} \rangle = 0$ $\forall \vec{w} \in W$. $\square$

(b) If $\vec{x} \in W \cap W^\perp$,

then $\langle \underset{W}{\underset{\cap}{\vec{x}}}, \underset{W^\perp}{\underset{\cap}{\vec{x}}} \rangle = 0. \Rightarrow \vec{x} = \vec{0}. \square$

#3.

U orthogonal $\Leftrightarrow U^T U = I_n$

$\Leftrightarrow U$ is invertible and $U^{-1} = U^T$

$\Rightarrow (U^T)^T U^T = I_n$

$\Rightarrow U^T$ is orthogonal

$\Rightarrow U^T$ has orthonormal columns.

$\Rightarrow U$ has orthonormal rows. $\square$

#4: $U_1^T = U_1^{-1}$, $U_2^T = U_2^{-1}$

$$\Rightarrow (U_1 U_2)(\underbrace{U_2^T U_1^T}_{= (U_1 U_2)^T}) = I_n. \square$$

#5: Notice that $\langle \vec{v}_1, \vec{v}_2 \rangle_A = (A\vec{v}_1)^T (A\vec{v}_2) = \langle A\vec{v}_1, A\vec{v}_2 \rangle$

standard inner product on $\mathbb{R}^n$.

- $\langle \vec{v}, \vec{v} \rangle_A = \langle A\vec{v}, A\vec{v} \rangle = 0$ if and only if $A\vec{v} = \vec{0}$.

  $\Rightarrow \vec{v} = 0$ since A is invertible.

- rest of the axioms are easy to check. $\square$

#6: $\|\vec{v}_1 + \vec{v}_2\|^2 + \|\vec{v}_1 - \vec{v}_2\|^2 = \langle \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 \rangle + \langle \vec{v}_1 - \vec{v}_2, \vec{v}_1 - \vec{v}_2 \rangle$

$$= \left(\langle \vec{v}_1, \vec{v}_1 \rangle + 2\langle \vec{v}_1, \vec{v}_2 \rangle + \langle \vec{v}_2, \vec{v}_2 \rangle\right)$$

$$+ \left(\langle \vec{v}_1, \vec{v}_1 \rangle - 2\langle \vec{v}_1, \vec{v}_2 \rangle + \langle \vec{v}_2, \vec{v}_2 \rangle\right)$$

$$= 2\|\vec{v}_1\|^2 + 2\|\vec{v}_2\|^2. \quad \square$$

#7: $\|\vec{v}_1\| \geq \|\mathrm{proj}_{\mathrm{Span}\{\vec{v}_2\}}(\vec{v}_1)\|$, equality holds $\iff$ $\{\vec{v}_1, \vec{v}_2\}$ l.d.

$$\|\mathrm{proj}_{\mathrm{Span}\{\vec{v}_2\}}(\vec{v}_1)\| = \left\| \frac{\langle \vec{v}_1, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \right\| = \frac{|\langle \vec{v}_1, \vec{v}_2 \rangle|}{\|\vec{v}_2\|^2} \|\vec{v}_2\| = \frac{|\langle \vec{v}_1, \vec{v}_2 \rangle|}{\|\vec{v}_2\|}. \quad \square$$

#8: $\|\vec{v}_1 + \vec{v}_2\|^2 = \|\vec{v}_1\|^2 + 2\langle \vec{v}_1, \vec{v}_2 \rangle + \|\vec{v}_2\|^2$

By #7 $\longrightarrow$ $\leq \|\vec{v}_1\|^2 + 2\|\vec{v}_1\|\|\vec{v}_2\| + \|\vec{v}_2\|^2 = (\|\vec{v}_1\| + \|\vec{v}_2\|)^2. \quad \square$

#9.

$$\det \begin{bmatrix} -\lambda & 1 & & & \\ & -\lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}-\lambda \end{bmatrix}$$

cofactor exp.

$$= (-1)^{n+1}(-a_0)\cdot 1 + (-1)^{n+2}(-a_1)(-\lambda) + \cdots + (-1)^{n+n-1}(-a_{n-2})(-\lambda)^{n-2}$$
$$+ (-a_{n-1}-\lambda)(-\lambda)^{n-1}$$

$$= (-1)^n a_0 + (-1)^n a_1 \lambda + \cdots + (-1)^n a_{n-2}\lambda^{n-2} + (-1)^n a_{n-1}\lambda^{n-1} + (-1)^n \lambda^n.$$

Hence we can take:

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix}$$

$\square$

#10. (a) follows from the fact that the only continuous fn f s.t. $\int_{-\pi}^{\pi} f(x)^2\,dx = 0$ is the zero fn.

(b) follows from certain trigonometric identities.