Plan:
- this week: Power series
- next week: Differentiation
- last 2 weeks: Integration & related topics.

Power series "$\underbrace{\sum_{n=0}^{\infty} a_n x^n}$" $a_n \in \mathbb{R}$

$(s_0(x) = a_0,\ s_1(x) = a_0 + a_1 x,\ s_2(x) = a_0 + a_1 x + a_2 x^2 \cdots s_3(x) = \cdots)$

Q1: For what $x \in \mathbb{R}$ does $\sum_{n=0}^{\infty} a_n x^n$ converge?

Q2: Properties of the power series fcn?

Thm: $\beta := \limsup_{n\to\infty} |a_n|^{1/n}$, $R := 1/\beta$.

(note β would be 0 or $+\infty$, then R: $+\infty$ or 0)

Then $\sum a_n x^n$ conv. for $|x| < R$ ← radius of convergence of the power series $\sum a_n x^n$.

$\sum a_n x^n$ div. for $|x| > R$.

Pf: Root test $\Rightarrow$ if $\underbrace{\limsup_{n\to\infty} |a_n x^n|^{1/n} < 1}$ then $\sum a_n x^n$ conv.

$$\underbrace{\limsup_{n\to\infty} |a_n x^n|^{1/n}} = \underbrace{\underbrace{\limsup |a_n|^{1/n}}_{\beta} \cdot |x|}$$

$$\Updownarrow$$

$$|x| < 1/\beta = R$$

Similarly, $|x| > R \Leftrightarrow \limsup |a_n x^n|^{1/n} > 1 \xRightarrow{\text{root test.}} \sum a_n x^n$ div. $\square$

Rmk: $\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$

So, if $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists, then $\beta = \lim \left|\frac{a_{n+1}}{a_n}\right|$.

e.g. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ i.e. $a_n = 1 \ \forall n \ge 0$.

What's $R = ??$ 1

$\Rightarrow$ the power series conv. on $(-1, 1)$

div. on $(1, \infty)$, $(-\infty, -1)$

$x = 1$: $1 + 1 + 1 + \cdots \to$ div.

$x = -1$: $1 - 1 + 1 - 1 + \cdots \to$ div.

$\Rightarrow$ exact interval of conv. is $(-1, 1)$

e.g. $\boxed{\sum_{n=1}^{\infty} \frac{x^n}{n}}$ $a_n = \frac{1}{n} \to R = 1$

$x = 1$: $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ div.

$x = -1$: $-1 + \frac{1}{2} - \frac{1}{3} + \cdots$ conv.

$\Rightarrow$ exact interval of conv. is $[-1, 1)$

e.g. $\boxed{\sum_{n=1}^{\infty} \frac{x^n}{n^2}}$ $R = 1 \Rightarrow$ int. of conv. $[-1, 1]$

$x = 1$: $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ conv.

$x = -1$: $-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \cdots$ conv.

e.g.: $\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$ $\quad a_n = \frac{1}{n!}$

$$\rho = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim \frac{1}{n+1} = 0$$

$R = +\infty$

$\Rightarrow$ the power series conv $\forall x \in \mathbb{R}$.

e.g.: $\sum_{n=0}^{\infty} n!\, x^n$, $\quad a_n = n!$

$\rho = +\infty$, $R = 0$.

$\Rightarrow$ the power series conv. only at $x = 0$.

Rmk: $\forall x_0 \in \mathbb{R}$, $\boxed{\sum_{n=0}^{\infty} a_n (x - x_0)^n}$

↑ interval of conv. will be centered at x_0

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§ Properties of power series:

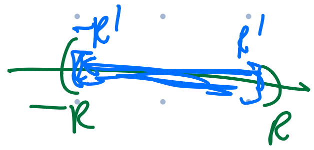
Thm: $\sum a_n x^n$ w/ radius of conv. $= R > 0$

Then $\sum a_n x^n$ conti. on $(-R, R)$

Lemma: $\sum a_n x^n$ w/ radius of conv. $= R > 0$.

Suppose $0 < R' < R$. then

$\sum a_n x^n$ conv. unif. on $[-R', R']$



Pf: $\forall x \in [-R', R']$

we have: $|a_n x^n| \le \boxed{|a_n| \cdot |R'|^n}$.

$f_n(x) = a_n x^n$

$\sum f_n$

M-test: doesn't depend on x

$|f_n(x)| \le M_n$

$\sum M_n < +\infty$

$\Rightarrow \sum f_n$ conv. unif.

$\sum |a_n| \cdot |R'|^n$ conv.??

$$\limsup \left(|a_n| \cdot |R'|^n\right)^{1/n}$$

$$= \limsup |a_n|^{1/n} \cdot R'$$

$$= \beta \cdot R'$$

$$< \beta \cdot R = 1.$$

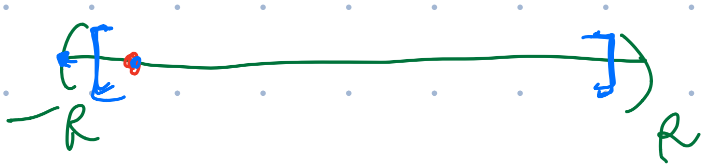
$\Rightarrow \sum a_n x^n$ conv. unif. on $[-R', R']$ by the Weierstrass M-test. □

pf of thm:

By lemma $\sum a_n x^n$ conti. on $[-R', R']$ $\forall\, 0 < R' < R$

(unif. limit of conti. fcns. is conti.)

$\Rightarrow \sum a_n x^n$ conti. on $(-R, R)$. □



$\underline{\text{Thm}}$: $f(x) = \sum a_n x^n$ has radius of conv. $R > 0$.

Then $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ also has radius of conv. $R > 0$

and $\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad \forall\, |x| < R$.

(i.e. " $\int$ " and " $\sum_{n=0}^{\infty}$ " can be exchanged in this case)

pf: ① $\limsup_{n\to\infty} \left|\frac{a_n}{n+1}\right|^{1/n} = \limsup |a_n|^{1/n} \cdot \boxed{\lim \left|\frac{1}{n+1}\right|^{1/n}}$

$\boxed{\lim \left|\frac{1}{n+1}\right|^{1/n}} = 1$

$= \limsup |a_n|^{1/n}$.

$(\lim_{n\to\infty} n^{1/n} = 1)$

② $\boxed{\underline{\text{Want}}: \int_0^{x_0} f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x_0^{n+1} \quad \forall\, |x_0| < R}$

• $\boxed{0 < x_0 < R}$: $\sum a_n x^n$ conv. unif. on $[0, x_0]$ (by the Lemma)

$$\Rightarrow \lim_{n\to\infty} \underbrace{\int_0^{x_0} \left(\sum_{k=0}^{n} a_k x^k\right) dx}_{= \lim_{n\to\infty}\left(\frac{a_0}{1} x_0 + \frac{a_1}{2} x_0^2 + \cdots + \frac{a_n}{n+1} x_0^{n+1}\right)} = \int_0^{x_0} \Big(\underbrace{\sum_{k=0}^{\infty} a_k x^k}_{= f(x)}\Big) dx$$

$$\lim_{n\to\infty}\left(\frac{a_0}{1} x_0 + \frac{a_1}{2} x_0^2 + \cdots + \frac{a_n}{n+1} x_0^{n+1}\right) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x_0^{n+1}. \quad \square$$

e.g. $\boxed{\sum_{n=0}^{\infty} x^n} = \frac{1}{1-x}$ for $|x|<1$.

Int. of conv. $(-1,1)$

$\Rightarrow \forall |x|<1$, $\boxed{\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}} = \int_0^x \frac{1}{1-t}\,dt$

$= -\log(1-x)$

int. of conv. $[-1,1)$

$\boxed{x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots}$

$\Rightarrow \underbrace{-\log(1-x)} = \underbrace{x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots}$ $\forall |x|<1$.

$x=-1$ well-defined for these two fns.

Q: Do we have the same equality at $x=-1$??

Thm (Abel) $f(x)=\sum a_n x^n$ has radius of conv. $R>0$

If f is convergent at $x=R$, then f is conti. at $x=R$

—————— $x=-R$, —————— $x=-R$

Abel thm $\Rightarrow$ Yes., i.e.

$$-\log 2 = -1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots$$

Prove it on Thursday

<u>Thm</u> $f(x) = \sum a_n x^n$ has r.o.c. $R > 0$

Then • $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has r.o.c. $R > 0$ ✓

and • f is differentiable on $(-R, R)$

• $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

<u>pf</u>: Define $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ (well-defined on $(-R, R)$)

On $|x| < R$, we have:

$$\int_0^x g(t)\,dt = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$$

by previous thm.

By Fundamental Thm of Calculus.

$$\Rightarrow \quad f'(x) = g(x). \quad \square$$