(1) For each of the following power series, find the radius of convergence and determine the exact interval of convergence.

(a) $\sum \left(\frac{x}{n}\right)^n$; (b) $\sum \left(\frac{(-1)^n}{n^2 \cdot 4^n}\right) x^n$; (c) $\sum \left(\frac{(-1)^n}{n \cdot 4^n}\right) x^n$; (d) $\sum x^{n!}$.

(a) $\limsup_{n\to\infty} \left(\left|\left(\frac{1}{n}\right)^n\right|\right)^{1/n} = 0.$

$\Rightarrow$ $R = +\infty$, interval of conv. $= \mathbb{R}$.

(b) $\limsup_{n\to\infty} \left(\left|\frac{(-1)^n}{n^2 \cdot 4^n}\right|\right)^{1/n} = \frac{1}{4}$ $\Rightarrow$ $R = 4$.

At $x = 4$, get: $\sum \frac{(-1)^n}{n^2}$ conv.

At $x = -4$, get: $\sum \frac{1}{n^2}$ conv.

$\Rightarrow$ interval of conv. $= [-4, 4]$.

(c) Similarly, $R = 4$,

At $x = 4$, get: $\sum \frac{(-1)^n}{n}$ conv.

At $x = -4$, get: $\sum \frac{1}{n}$ div.

$\Rightarrow$ interval of conv. $= (-4, 4]$.

(d) coeff. $a_k = \begin{cases} 1, & k = n! \text{ for some } n. \\ 0 \end{cases}$

$\Rightarrow$ $\limsup_{k\to\infty} |a_k|^{1/k} = 1$. $\Rightarrow$ $R = \frac{1}{1} = 1$.

For $x = \pm 1$, the seq. $(a_k x^k)$ doesn't conv. to 0.

$\Rightarrow$ interval of conv. $= (-1, 1)$.

(2) Suppose $\sum a_n x^n$ has finite radius of convergence $R > 0$ and $a_n \geq 0$ for all n. Prove that if the series converges at R, then it also converges at $-R$.

- We have $\sum a_n R^n$ conv. and $a_n \geq 0$ $\forall n$.
- $|a_n (-R)^n| = a_n R^n$, hence $\sum a_n (-R)^n$ conv. by Comparison Test. $\square$

(3) Consider a power series $\sum a_n x^n$ with radius of convergence R. Prove that if $\limsup |a_n| > 0$, then $R \leq 1$.

- $\exists \varepsilon > 0$ and $N > 0$ s.t. $|a_n| > \varepsilon$ $\forall n > N$.
- Hence $\limsup |a_n|^{1/n} \geq \lim \varepsilon^{1/n} = 1$.

  $\Longrightarrow R \leq 1$. $\square$

(4) By mimicking what we discussed in class, prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

(Hint: First, we have $\sum (-1)^n x^{2n} = \frac{1}{1+x^2}$ for all $|x| < 1$.)

- By thm we proved in class,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \forall |x| < 1.$$

- the series is convergent at $x = 1$, By Abel's thm, we have

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \lim_{x \to 1} \tan^{-1} x$$

$$= \frac{\pi}{4}. \ \square$$

(5) Let (a_n) and (b_n) be two sequences of real numbers satisfying:

- The partial sums of (b_n) is bounded: there exists $L > 0$ such that $|b_1 + \cdots + b_k| < L$ for any k,
- $\lim a_n = 0$,
- $\sum |a_n - a_{n+1}|$ converges.

Prove that for any $k \in \mathbb{N}$, the series $\sum a_n^k b_n$ is convergent. (Hint: Same idea as the proof of Abel's theorem.)

- Denote $s_k := b_1 + \cdots + b_k$.
- $$\left|\sum_{n=M}^{N} a_n^k b_n\right| = \left|\sum_{n=M}^{N-1} (a_n^k - a_{n-1}^k) s_n + a_N^k s_N - a_M^k s_{M-1}\right|$$
$$\leq \sum_{n=M}^{N-1} |a_n^k - a_{n-1}^k||s_n| + |a_N^k||s_N| + |a_M^k||s_{M-1}|$$
$$< L\left(\sum_{n=M}^{N-1} |a_n - a_{n-1}|\,|\underbrace{a_n^{k-1} + \cdots + a_{n-1}^{k-1}}_{a_n^{k-1} + a_n^{k-2} a_{n-1} + \cdots + a_{n-1}^{k-1}}| + |a_N^k| + |a_M^k|\right)$$
- $\forall \varepsilon > 0$, Choose $\tilde{\varepsilon} > 0$ small enough s.t. $L(k+2)\tilde{\varepsilon}^k < \varepsilon$.
- $\exists N_1 > 0$ s.t. $|a_n| < \tilde{\varepsilon}$ $\forall n > N_1$,
- $\exists N_2 > 0$ s.t. $\sum_{n=M}^{N} |a_n - a_{n-1}| < \tilde{\varepsilon}$ $\forall N > M > N_2$.
- Then $\forall N > M > \max\{N_1, N_2\}$, we have:

$$\left|\sum_{n=M}^{N} a_n^k b_n\right| < L\left(\sum_{n=M}^{N-1} |a_n - a_{n-1}|\,|\underbrace{a_n^{k-1} + \cdots + a_{n-1}^{k-1}}_{\wedge \; k\tilde{\varepsilon}^{k-1}}| + |a_N^k| + |a_M^k|\right)$$
$$< L\left(k\tilde{\varepsilon}^{k-1} \sum_{n=M}^{N-1} |a_n - a_{n-1}| + 2\tilde{\varepsilon}^k\right)$$
$$< L(k+2)\tilde{\varepsilon}^k < \varepsilon.$$

Hence $\sum a_n^k b_n$ is convergent by Cauchy criterion. $\square$