HOMEWORK 11
MATH 104, SECTION 2

Some ground rules:

- You have to submit your homework via **Gradescope** to the corresponding assignment. The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You're allowed to use any result that is proved in the lecture; but if you'd like to use other results, you have to prove them before using them.

Problem set (5 problems; due April 27 at 11am PT)

(1) Define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } 1-2^{-2k} \le x \le 1-2^{-(2k+1)} \text{ for } k = 0,1,2,\dots \\ 0 & \text{if } 1-2^{-(2k+1)} < x < 1-2^{-(2k+2)} \text{ for } k = 0,1,2,\dots \\ 0 & \text{if } x = 1 \end{cases}$$

Prove that f is integrable on $[0,1]$, and compute $\int_0^1 f(x)dx$.

(2) Suppose that $f\colon [a,b] \to \mathbb{R}$ is integrable. Prove that the function $|f|\colon [a,b] \to \mathbb{R}$ which sends x to $|f(x)|$ is also integrable, and

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|dx.$$

(3) Let f be a positive and continuous function on $[0,1]$. Compute

$$\int_0^1 \frac{f(x)}{f(x)+f(1-x)} dx.$$

(4) Let $(\mathcal{C}[0,1], d_\infty)$ be the metric space of continuous functions on $[0,1]$, where the distance function is defined by

$$d_\infty(f,g) = \sup_{x\in[0,1]} |f(x)-g(x)|.$$

Consider the function $T\colon (\mathcal{C}[0,1], d_\infty) \to (\mathcal{C}[0,1], d_\infty)$ defined by

$$(Tf)(x) := \int_0^x f(t)dt.$$

Prove that:

(a) T is not a contraction, i.e. there does not exist $0 < K < 1$ such that

$$d_\infty(Tf, Tg) \le K \cdot d_\infty(f, g)$$

holds for any $f, g \in \mathcal{C}[0,1]$.

(b) T has a unique fixed point, i.e. there is a unique $f \in \mathcal{C}[0,1]$ satisfies $Tf = f$.

(c) T^2 is a contraction.

(5) Let f, g be integrable functions on $[a, b]$. Prove that

$$\Big(\int_a^b f(x)g(x)\Big)^2 \le \Big(\int_a^b f(x)^2 dx\Big)\Big(\int_a^b g(x)^2 dx\Big).$$

(Hint: Consider $\int_a^b (\int_a^b (f(x)g(y) - f(y)g(x))^2 dx) dy$.)