SECOND MIDTERM SOLUTION
MATH H54, FALL 2021

Problem 1: (20 points) Let A be a real $n \times n$ matrix. Let $||\vec{v}||$ be the length of $\vec{v} \in \mathbb{R}^n$ with respect to the standard inner product on $\mathbb{R}^n$. Suppose that $||A\vec{v}|| = ||\vec{v}||$ for any $\vec{v} \in \mathbb{R}^n$. Prove that A is an orthogonal matrix.

Solution: For any $\vec{v} \in \mathbb{R}^n$, we have

$$\vec{v}^T A^T A \vec{v} = (A\vec{v})^T A\vec{v} = ||A\vec{v}||^2 = ||\vec{v}||^2 = \vec{v}^T \vec{v}.$$

(This does *not* directly imply that $A^T A = \mathbb{I}_n$; see the remark below.) Notice that $A^T A$ is a symmetric matrix, therefore, there exists an orthogonal matrix P and a diagonal matrix D such that $A^T A = PDP^T$. Note that the diagonal entries of D are the eigenvalues of $A^T A$. Suppose λ is an eigenvalue of $A^T A$ with an eigenvector $\vec{w} \neq \vec{0}$, then we have

$$||\vec{w}||^2 = \vec{w}^T \vec{w} = \vec{w}^T A^T A \vec{w} = \vec{w}^T (\lambda \vec{w}) = \lambda ||\vec{w}||^2,$$

hence $\lambda = 1$ (since $||\vec{w}|| > 0$). This proves that 1 is the only eigenvalue of $A^T A$, therefore $D = \mathbb{I}_n$. Hence $A^T A = PDP^T = P\mathbb{I}_n P^T = PP^T = \mathbb{I}_n$. Thus A is orthogonal.

Remark: Note that "$\vec{v}^T B \vec{v} = \vec{v}^T \vec{v}$ holds for any $v \in \mathbb{R}^n$" does *not* imply that B is the identity matrix. For instance, $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ also satisfies this property.

Problem 2: (20 points) Let B be a real symmetric positive definite $n \times n$ matrix. Recall that $\langle \vec{v}_1, \vec{v}_2 \rangle_B = \vec{v}_1^T B \vec{v}_2$ defines an inner product on $\mathbb{R}^n$. For $\vec{v} \in \mathbb{R}^n$, let $||\vec{v}||_B$ be the length of $\vec{v}$ with respect to the inner product $\langle -, - \rangle_B$, and let $||\vec{v}||$ be the length of $\vec{v}$ with respect to the standard inner product on $\mathbb{R}^n$.

Prove that there exists an eigenvalue λ of B such that $||\vec{v}||_B \geq \sqrt{\lambda}||\vec{v}||$ holds for any $\vec{v} \in \mathbb{R}^n$.

Solution: Since B is a real symmetric positive definite matrix, all of its eigenvalues are positive real numbers. We choose $\lambda > 0$ to be the *smallest* eigenvalue of B, and claim that $||\vec{v}||_B^2 \geq \lambda ||\vec{v}||^2$ holds for any $\vec{v} \in \mathbb{R}^n$.

Since B is symmetric, there exists an orthonormal eigenbasis of B, say $\{\vec{v}_1, \dots, \vec{v}_n\}$. For each $1 \leq i \leq n$, there is an eigenvalue λ_i of B such that $B\vec{v}_i = \lambda_i \vec{v}_i$. For any $\vec{v} \in \mathbb{R}^n$, there exists $c_1, \dots, c_n \in \mathbb{R}$ such that $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. Then

$$\begin{aligned}
||\vec{v}||_B^2 &= \vec{v}^T B \vec{v} \\
&= \vec{v}^T B(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \\
&= \vec{v}^T(\lambda_1 c_1 \vec{v}_1 + \cdots + \lambda_n c_n \vec{v}_n) \\
&= (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)^T(\lambda_1 c_1 \vec{v}_1 + \cdots + \lambda_n c_n \vec{v}_n) \\
&= \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2 \\
&\geq \lambda(c_1^2 + \cdots + c_n^2) \\
&= \lambda ||\vec{v}||^2.
\end{aligned}$$

Note that the last two equalities follow from the fact that $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal set.

Problem 3: (20 points) Let $T\colon V \to V$ be a linear transformation of an n-dimensional vector space V. Recall that for any basis $\mathcal{B}$ of V, the coordinate mapping $[-]_{\mathcal{B}}\colon V \to \mathbb{R}^n$ is a bijective linear map. One can then define a linear transformation $T_{\mathcal{B}}\colon \mathbb{R}^n \to \mathbb{R}^n$ by considering the composition $T_{\mathcal{B}} \coloneqq [-]_{\mathcal{B}} \circ T \circ [-]_{\mathcal{B}}^{-1}$:

$$T_{\mathcal{B}}\colon \mathbb{R}^n \xrightarrow{[-]_{\mathcal{B}}^{-1}} V \xrightarrow{T} V \xrightarrow{[-]_{\mathcal{B}}} \mathbb{R}^n.$$

Recall that there exists a unique $n \times n$ matrix, say denoted by $M_{T,\mathcal{B}}$, that represents the linear transformation $T_{\mathcal{B}}$ (i.e. $T_{\mathcal{B}} = T_{M_{T,\mathcal{B}}}$).

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be any two basis of V. Prove that the characteristic polynomial of $M_{T,\mathcal{B}_1}$ coincides with the characteristic polynomial of $M_{T,\mathcal{B}_2}$.

Solution: We claim that $M_{T,\mathcal{B}_1}$ and $M_{T,\mathcal{B}_2}$ are similar, therefore have the same characteristic polynomial. Define $S\colon \mathbb{R}^n \to \mathbb{R}^n$ to be the composition $S \coloneqq [-]_{\mathcal{B}_1} \circ [-]_{\mathcal{B}_2}^{-1}$, which is a bijective linear transformation. Then we have

$$\begin{aligned} T_{\mathcal{B}_2} &= [-]_{\mathcal{B}_2} \circ T \circ [-]_{\mathcal{B}_2}^{-1} \\ &= \left(S^{-1} \circ [-]_{\mathcal{B}_1}\right) \circ T \circ \left([-]_{\mathcal{B}_1}^{-1} \circ S\right) \\ &= S^{-1} \circ T_{\mathcal{B}_1} \circ S. \end{aligned}$$

Also, since $S\colon \mathbb{R}^n \to \mathbb{R}^n$ is a bijective linear map, it can be represented by an invertible matrix P (i.e. $S = T_P$). Hence we have $M_{T,\mathcal{B}_2} = P^{-1} M_{T,\mathcal{B}_1} P$.

Remark: Since $T\colon V \to V$ is a linear transformation of a general vector space (not necessarily $\mathbb{R}^n$), it doesn't make sense to "represent T by a matrix" without choosing a basis of V.

Problem 4: (20 points) Continue the notations in the previous problem. One defines the characteristic polynomial of T to be the characteristic polynomial of $M_{T,\mathcal{B}}$ for any basis $\mathcal{B}$ of V.

Let $V = M_{2\times 2}(\mathbb{R})$ be the vector space of all real 2×2 matrices. Consider the linear transformation $T\colon V \to V$ defined by $T(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ for $A \in V$. Find the characteristic polynomial of T.

Solution: Choose $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Then

$$M_{T,\mathcal{B}} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence the characteristic polynomial of T is $(\lambda - 1)^4$.

Problem 5: (20 points) Let $V = \mathcal{C}[-1,1]$ be the inner product space of real-valued continuous function defined on the interval $[-1,1]$, with inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} x^2 f(x)g(x)dx.$$

Find the orthogonal projection of $x^4 \in V$ onto the subspace $W = \mathrm{Span}\{1, x, x^2\} \subseteq V$.

Solution: First, we find an orthogonal basis of W by Gram–Schmidt. The set $\{1, x\}$ is orthogonal since x^3 is an odd function.

$$x^2 - \frac{\langle 1, x^2\rangle}{\langle 1, 1\rangle}1 - \frac{\langle x, x^2\rangle}{\langle x, x\rangle}x = x^2 - \frac{3}{5}.$$

Hence $\left\{1, x, x^2 - \frac{3}{5}\right\}$ is an orthogonal basis of W. The orthogonal projection of x^4 onto the subspace W is therefore

$$\frac{\langle 1, x^4\rangle}{\langle 1, 1\rangle}1 + \frac{\langle x, x^4\rangle}{\langle x, x\rangle}x + \frac{\langle x^2 - \frac{3}{5}, x^4\rangle}{\langle x^2 - \frac{3}{5}, x^2 - \frac{3}{5}\rangle}\left(x^2 - \frac{3}{5}\right) = \frac{3}{7} + \frac{10}{9}\left(x^2 - \frac{3}{5}\right) = \frac{10}{9}x^2 - \frac{5}{21}.$$

Remark: The projection formula

$$\mathrm{proj}_W \vec{v} = \frac{\langle \vec{v}, \vec{w}_1\rangle}{\langle \vec{w}_1, \vec{w}_1\rangle}\vec{w}_1 + \cdots + \frac{\langle \vec{v}, \vec{w}_n\rangle}{\langle \vec{w}_n, \vec{w}_n\rangle}\vec{w}_n$$

works only if $\{w_1, \ldots, w_n\}$ is an *orthogonal* basis of W. In this problem, $\{1, x, x^2\}$ is *not* an orthogonal set, so the projection formula doesn't apply to this basis.