$\boxed{3/5/2020}$

Thm (Heine-Borel) $E \subset \mathbb{R}^k$ compact $\Leftrightarrow$ closed and bounded.

Note: ($\Leftarrow$) not true for general metric space.

pf: $E \subset \mathbb{R}^k$ bounded $\Rightarrow$ $\exists$ k-cell $I^k = [a_1, b_1] \times \cdots \times [a_k, b_k] \subset \mathbb{R}^k$ that contains E.

Claim: I^k is compact

(Then $E \subset I^k$ is a closed subset in a cpt set $\Rightarrow$ E is cpt.).

It remains to prove this really beautiful proof

Bolzano-Weierstrass Thm in $\mathbb{R}^k$ Any bounded seq. in $\mathbb{R}^k$ has a conv. subseq.

Idea. $(x_n) \subset \mathbb{R}^k$. each $x_n = (x_n^{(1)}, \ldots, x_n^{(k)}) \in \mathbb{R}^k$

Look at their first coordinates $(x_n^{(1)}) \subset \mathbb{R}$.

By BW thm in $\mathbb{R}$, $\exists$ conv. subseq. $(x_{n_k}^{(1)})$ of $(x_n^{(1)})$.

For ~~this~~ the subseq. $(x_{n_k}) \subset \mathbb{R}^k$ there is a further subseq. s.t.

the 2^{nd}-coord. conv. ... keep taking subseq. k times. (Ross, 13).

Prop. $F_1 \supset F_2 \supset \cdots$ decreasing seq. of closed bounded nonempty sets in $\mathbb{R}^k$.

Then $F = \bigcap_{n=1}^{\infty} F_n$ is also closed, bounded, nonempty.

Q Is it true if $\{F_n\}$ are open sets? $(0, \frac{1}{n})$ ϕ

pf closed (HW), bounded - obvious.

Choose any pt. $x_n \in F_n$ for each n.

By BW thm, $\exists$ subseq. $(x_{n_k}) \longrightarrow x_0 \in \mathbb{R}^k$

Claim: $x_0 \in F$, i.e. $x_0 \in F_N$ $\forall N$.

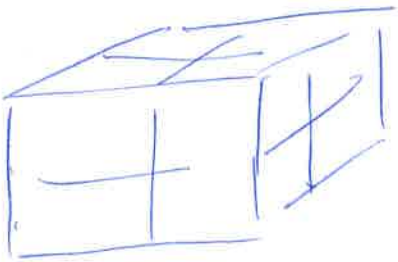
pf For any N, $\exists K > 0$ s.t. $n_k > N$ $\forall k > K$

$\Rightarrow$ $x_{n_k} \in F_{n_k} \subset F_N$ $\forall k > K$.

Since F_N is closed, $\lim x_{n_k} = x_0 \in F_N$. $\square$

<u>Proof of I^k is compact:</u> (Idea of subdivisions)

Suppose $\exists$ an open cover of I^k $\{U_\alpha\}$ that has <u>no</u> finite subcover.



divide I^k into 2^k smaller ~~cubes~~ cells.

At least one of them can't be covered by finitely many $\{U_\alpha\}$.

call it I_1.

$$\underset{\text{diameter } \delta}{I^k} \supset \underset{\text{diameter } \frac{\delta}{2}}{I_1} \supset \underset{\text{diameter } \frac{\delta}{2^2}}{I_2} \supset \cdots\cdots$$

each I_n: ~~no finite~~ can't be covered by finitely many $\{U_\alpha\}$.

By <u>Coro</u>, $\exists\, x_0 \in \bigcap_{n=1}^{\infty} I_n$.

$x_0 \in U_\alpha$ for some α. $\Rightarrow$ $\exists\, r>0$ s.t. $B_r(x_0) \subset U_\alpha$

But $\exists\, N>0$ s.t. $\delta \cdot 2^{-N} < r$. $\Rightarrow$ $I_N \subset B_r(x_0) \subset U_\alpha$.

$\times$ $\square$

<u>HW</u>: $E \subset \mathbb{R}$ compact $\Rightarrow$ $\sup E, \inf E \in E$.

(extreme value thm)

<u>Coro</u>: $f: (X,d) \longrightarrow \mathbb{R}$ continuous, ~~X cpt~~. $E \subset X$ cpt.

Then 1) f is bounded on E. ($\exists\, M>0$ s.t. $|f(x)| < M$ $\forall x \in E$)

2). f assumes its max. and min. on E.

($\exists\, x_1, x_2 \in E$ s.t. $f(x_1) \le f(x) \le f(x_2)$ $\forall x \in E$)

pf $f(E)$ is compact subset in $\mathbb{R}$. $\Rightarrow$ bounded. (1)

$\sup f(E), \inf f(E) \in f(E)$, $\exists\, x_1 \in E$ s.t. $f(x_1) = \inf f(E)$

$\exists\, x_2 \in E$ s.t. $f(x_2) = \sup f(E)$. $\square$

Uniform Continuity.

Recall: $f: X \to Y$ is continuous if $\forall x_0 \in X$, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, x_0) > 0$

s.t. $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$.

typicall depends on both ε and x_0.

Def. $f: (X, d_X) \to (Y, d_Y)$ is <u>uniformly continuous on $E \subset X$</u> if

$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t. $x, y \in E$

$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$.

e.g. Consider $f: \mathbb{R} \to \mathbb{R}$. $f(x) = x^2$.

1) Is it uniformly continuous on $\mathbb{R}$?

2) Is it uniformly continuous on $(0,1)$?

1) No. Let $\varepsilon = 1$.

Claim: $\forall \delta > 0$, $\exists x, y \in \mathbb{R}$ s.t. $|x-y| < \delta$ but $\underset{\overset{\|}{|x-y||x+y|}}{|x^2 - y^2|} \geq 1$.

Let's find x, y s.t. $|x-y| = \frac{\delta}{2}$, $|x+y| \geq \frac{2}{\delta}$.

e.g. Take $x = \frac{2}{\delta}$, $y = \frac{2}{\delta} + \frac{\delta}{2}$. $\square$

2) Yes. If $|x-y| < \delta$, then $|x^2 - y^2| = |x-y||x+y| < 2\delta$

Take $\delta = \frac{\varepsilon}{2}$. $\square$

Thm $f: (X, d_X) \to (Y, d_Y)$ conti. $E \subset X$ cpt.

Then f is uniformly conti. on E.

Rmk. Uniformly conti. can happen on non-cpt ~~space~~ sets. e.g. 2).

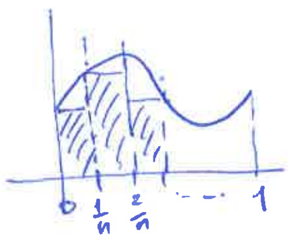
e.g. any subset of a cpt set E

Coro: $f: [a,b] \longrightarrow \mathbb{R}$ conti. $\Rightarrow$ uniformly conti.

pf. $[a,b]$ is compact..

Why uniformly conti. is important? - Preview of integration:

$f: [0,1] \longrightarrow \mathbb{R}$ conti. Want: Define its integral. $\int_0^1 f(x)\,dx$.



Goal. calculate Area below the graph.

$$L_n = \frac{1}{n} \cdot \sum_{k=1}^{n} \inf\left\{ f(x) : x \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right]\right\}.$$

$$U_n = \frac{1}{n} \sum_{k=1}^{n} \sup\left\{ f(x) : x \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right]\right\}.$$

$$\Rightarrow \quad L_n \leq \text{"} \int_0^1 f(x)\,dx \text{"} \leq U_n.$$

Hope: $\lim_{n\to\infty} (U_n - L_n) = 0$. then $\int_0^1 f(x)dx$ is well-defined.

$$\| \quad \frac{1}{n}\sum_{k=1}^{n}\left(\sup\left\{f(x) : x \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right]\right\} - \inf\left\{f(x) : x \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right]\right\}\right)$$

Since f is unif. conti, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Then for any $n > \frac{1}{\delta}$, we have:

$$\forall x, y \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right], \quad |x-y| \leq \frac{1}{n} < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

$$\Rightarrow 0 \leq U_n - L_n = \frac{1}{n}\left(\sum_{k=1}^{n}\left(\sup\left\{f(x) : x \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right]\right\} - \inf\left\{f(x) : x \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right]\right\}\right)\right)$$

$$\leq \frac{1}{n} \cdot n \cdot \varepsilon = \varepsilon.$$

$$\Rightarrow \lim_{n\to\infty} (U_n - L_n) = 0.$$