Today:

- More examples of conti. fcns.
- revisit connectedness, IVT, fixed point thm.
- Q & A. (conti. fcns).

Reminder: Lecture this Thursday will be asynchronous.

$f: (X, d_X) \longrightarrow (Y, d_Y)$ is continuous if:

equivalent definitions

Ross, §21.

$$\boxed{\forall x_0 \in X,\ \forall (x_n) \subseteq X \text{ with } \lim x_n = x_0. \text{ we have: } \lim f(x_n) = f(x_0).}$$

$$\boxed{\forall x_0 \in X,\ \forall \varepsilon > 0,\ \exists \delta > 0. \text{ s.t. } d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon.}$$

$$\boxed{\forall U \subseteq Y \text{ open, we have } f^{-1}(U) \subseteq X \text{ open.}}$$

$\Downarrow$

Ross, 21.4.

$$\boxed{E \subseteq X \text{ cpt} \Rightarrow f(E) \subseteq E \text{ cpt.}}$$

$\Downarrow$

Ross, 21.5

Ross, 18.1

$$\boxed{\text{extreme value theorem. for compact sets. (e.g. } [a, b] \subseteq \mathbb{R})}$$

$\Downarrow$

Ross, 22.2.

$$\boxed{E \subseteq X \text{ connected} \Downarrow f(E) \subseteq Y \text{ connected.}}$$

$\Downarrow$

$$\boxed{\text{Intermediate value theorem.}}$$

Ross, 18.2.

Def $E \subseteq (X, d)$ is <u>disconnected</u> if $\exists\, U_1, U_2 \underset{\text{open}}{\subseteq} X$, that "separates E", <u>i.e.</u>

1) $E \cap U_1 \neq \phi$, $E \cap U_2 \neq \phi$
2) $E \subseteq U_1 \cup U_2$
3) $E \cap U_1 \cap U_2 = \phi$.

Otherwise, $E \subseteq X$ is called <u>connected</u>

<u>Thm</u> $f : X \longrightarrow Y$ conti.

If $E \subseteq X$ connected, then $f(E) \subseteq Y$ is connected

pf. Assume $f(E) \subseteq Y$ disconnected, i.e. $\exists\, U_1, U_2 \underset{\text{open.}}{\subseteq} Y$ that "separates $f(E)$".

- Since f is conti, $f^{-1}(U_1), f^{-1}(U_2) \subseteq X$ are open
- It's easy to check that: $f^{-1}(U_1), f^{-1}(U_2)$ "separates E". (ex).

(Ex: If $U_1, U_2 \underset{\text{open}}{\subseteq} Y$ separates $f(E)$, (i.e. they satisfy the condition 1) – 3),

then $f^{-1}(U_1), f^{-1}(U_2) \subseteq X$ separates E.)

$\Rightarrow$ E is disconnected. $\rightarrow\!\!\!\!\times$

$\square$

e.g. $f(x) = x^3 + x + 1 : \mathbb{R} \to \mathbb{R}$

Claim $\exists\, x_0 \in \mathbb{R}$ s.t. $f(x_0) = 0$

- If we can find $x_1, x_2 \in \mathbb{R}$ s.t. $f(x_1) < 0$, $f(x_2) > 0$,

  then by IVT, $\exists\, y$ between x_1, x_2

  s.t. $f(y) = 0$

$f(-1) = -1 < 0$

$f(1) = 3 > 0$

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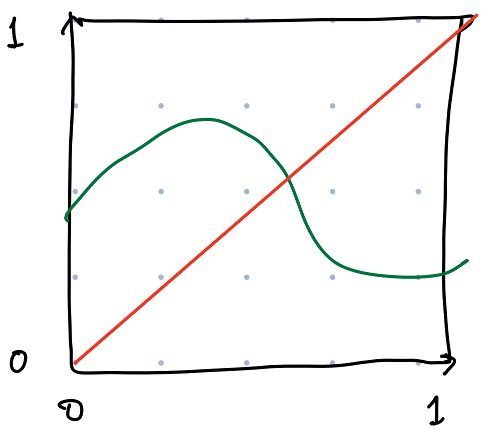
e.g. Any conti. fcn. from $[0,1]$ to itself has a fixed pt.

$f : [0,1] \to [0,1]$ conti.

Then $\exists\, x_0 \in [0,1]$ s.t. $f(x_0) = x_0$.



pf Consider $g(x) = f(x) - x$ conti.

Prove by contradiction.

Suppose $\forall x_0 \in [0,1]$, $f(x_0) \neq x_0$

Then $g(0) > 0$, $g(1) < 0$.

IVT $\Rightarrow \exists\, x_0 \in (0,1)$ s.t. $g(x_0) = 0$

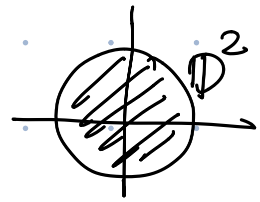
$\Updownarrow$

$f(x_0) = x_0$. ※.

$\square$

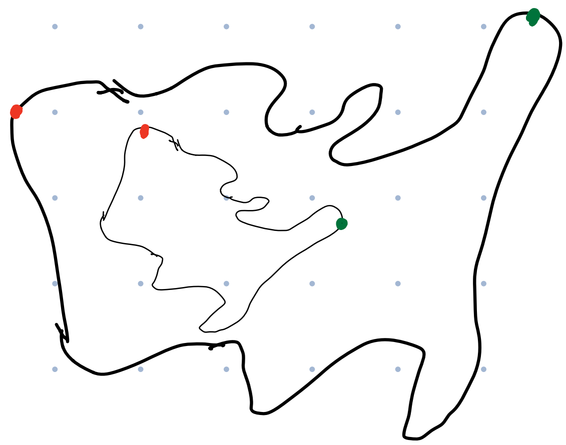
Brouwer fixed pt thm:

$$\mathbb{D}^n = \{x \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 \leq 1\} \subseteq \mathbb{R}^n$$



$\forall f : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ conti.,

$\exists\, x_0 \in \mathbb{D}^n$ s.t. $f(x_0) = x_0$.



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e.g. $f : (X, d_X) \longrightarrow (Y, d_Y)$, $y_0 \in Y$, the constant map that sends any pt in X to $y_0 \in Y$.

$$x \longmapsto y_0$$

Claim: f is conti.

Using 1st def^n: clear!

Using 2nd def^n: clear!

$\forall x_0 \in X, \forall \varepsilon > 0$, we need to find $\delta > 0$ (choose any)

s.t. if $d_X(x, x_0) < \delta$ then $\underbrace{d_Y(f(x), f(x_0))}_{=0} < \varepsilon$. ($f(x) = y_0$, $f(x_0) = y_0$)

Using 3rd def^n:

$\forall U \subseteq Y$ open, need to show: $f^{-1}(U) \subseteq X$ open.

(Reminder: $\forall$ metric space (X, d_X),
$\emptyset$ is open.
X is open.)

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e.g. $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

$f: \mathbb{R} \to \mathbb{R}$

Claim: f is discontinuous at any point $x_0 \in \mathbb{R}$.

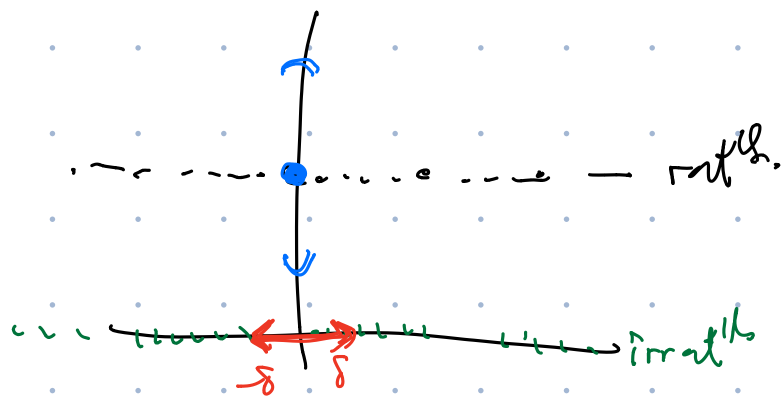
pf (using 1st def^n):

If $x_0 \in \mathbb{Q}$, (so $f(x_0) = 1$)

$\exists$ (x_n) irrat'ls s.t. $\lim x_n = x_0$

but $\lim \underbrace{f(x_n)}_{=0} = 0 \neq f(x_0) = 1$

Similarly, $x_0 \notin \mathbb{Q}$.



pf (using 2nd def^n)

If $x_0 \in \mathbb{Q}$,

$\boxed{\text{If } f \text{ is conti at } x_0: \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.}$

$\boxed{\text{If } f \text{ is not conti at } x_0: \quad \exists \varepsilon > 0, \text{ s.t. } \forall \delta > 0, \exists x \text{ where } |x - x_0| < \delta, |f(x) - f(x_0)| \geq \varepsilon}$

1/2 ''

pf (using 3rd defn)

$f^{-1}(\frac{1}{2}, \frac{3}{2}) = \mathbb{Q} \subseteq \mathbb{R}$

open interval

not open.

$f^{-1}(-\frac{1}{2}, \frac{1}{2}) = \mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$

open interval

not open.

$f(x) = x^2 : \mathbb{R} \to \mathbb{R}$

fix $x_0 \in \mathbb{R}$

$\forall \varepsilon > 0$,

we need find a $\delta > 0$

st. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$

$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0| |x + x_0|$

$\delta \leq 1$

$$|x + x_0| \leq |x - x_0| + 2|x_0| < \delta + 2|x_0| \leq 1 + 2|x_0|$$

$$|x^2 - x_0^2| = |x - x_0| |x + x_0| < \delta (1 + 2|x_0|) < \varepsilon$$

$$\delta < \frac{\varepsilon}{1 + 2|x_0|}$$