Today: Proof of limit thms; liminf & limsup.

Some limit theorems If $\lim a_n = a$, $\lim b_n = b$, then

1) $\forall r \in \mathbb{R}$, $\lim (r a_n) = ra$
2) $\lim (a_n + b_n) = a + b$.
3) $\lim (a_n b_n) = ab$
4) If $b_n \neq 0$ $\forall n$, $b \neq 0$, then $\lim \frac{a_n}{b_n} = \frac{a}{b}$

e.g. $a_n = \dfrac{4n^2 - 7n}{n^2 + 1}$.

$$= \frac{\overbrace{4 - \frac{7}{n}}^{=: b_n}}{\underbrace{1 + \frac{1}{n^2}}_{=: c_n}}$$

$$\lim b_n = \lim \left(4 - \frac{7}{n}\right) \underset{(2)}{=} \lim (4) - \lim\left(\frac{7}{n}\right) = 4$$

$$\lim c_n = 1.$$

By 4), $\lim a_n = \dfrac{\lim b_n}{\lim c_n} = \dfrac{4}{1} = 4.$ $\square$

Idea; ~~Pf~~ 3): " $\boxed{\lim a_n b_n = ab}$ "

What we have: $\lim a_n = a$, $\lim b_n = b$. $\boxed{|a_n - a|}$ $\boxed{|b_n - b|}$ small as n gets larger.

want to show

" $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $n > N \implies \boxed{|a_n b_n - ab|} < \varepsilon$ "

$$a_n b_n - ab = \underline{a_n b_n - a_n b} + \underline{a_n b - ab}$$

$$= a_n (b_n - b) + b(a_n - a)$$

$$|a_nb_n - ab| \le |a_n(b_n-b)| + |b(a_n-a)| \qquad (\Delta \text{ ineq.})$$

$$= |a_n| \cdot \underbrace{|b_n - b|} + \underbrace{|b|}_{\text{given}} \cdot \underbrace{|a_n - a|}$$

b/c (a_n) converges $\Rightarrow$ (a_n) bounded (last time) ; $\exists M>0$ s.t. $|a_n| < M$ $\forall n$.

can control these since we know $\lim a_n = a$, $\lim b_n = b$.

<u>pf of 3):</u> $\forall \varepsilon > 0$.

- Since $\lim a_n = a$,

  $\exists N_1 > 0$ s.t. $n > N_1 \Rightarrow |a_n - a| < \dfrac{\varepsilon}{2(|b|+1)}$

  Also, (a_n) is bdd, $\exists M > 0$ s.t. $|a_n| < M$ $\forall n \in \mathbb{N}$.

- Since $\lim b_n = b$,

  $\exists N_2 > 0$ s.t. $n > N_2 \Rightarrow |b_n - b| < \dfrac{\varepsilon}{2 \cdot M}$

Let $N = \max\{N_1, N_2\}$.
Then $\forall n > N$ ($\Rightarrow n > N_1$, $n > N_2$)
we have

$$|a_nb_n - ab| \le \underbrace{|a_n|}_{<M} \cdot \underbrace{|b_n - b|}_{<\frac{\varepsilon}{2M}\ (n>N_2)} + |b| \cdot \underbrace{|a_n - a|}_{<\frac{\varepsilon}{2(|b|+1)}\ (n>N_1)} < \varepsilon. \qquad \square$$

pf of 4): $\lim \frac{a_n}{b_n} = \frac{a}{b}$

Prop If $\lim b_n = b$, $b_n \neq 0\ \forall n$, $b \neq 0$,
then $\lim \frac{1}{b_n} = \frac{1}{b}$.

(Assuming the Prop. is true, then 4) is true.

$\lim a_n = a$, $\lim b_n = b$, $b_n \neq 0$, $b \neq 0$.

$\Downarrow$ Prop

use 3) $\lim \frac{1}{b_n} = \frac{1}{b}$

$$\lim \left(a_n \cdot \frac{1}{b_n}\right) = a \cdot \left(\frac{1}{b}\right)$$
$$\lim \frac{a_n}{b_n} \qquad\qquad \frac{a}{b}$$
)

Idea

we know $\lim b_n = b$ we can make $|b_n - b|$ as small as we want if n large.

Want to prove:

$\forall \varepsilon > 0$, $\exists N > 0$ s.t. $n > N \Longrightarrow \boxed{\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon}$

$$\left|\frac{b - b_n}{b_n \cdot b}\right| = \frac{\boxed{|b_n - b|}}{|b_n|\boxed{|b|}} < \frac{|b_n - b|}{M \cdot |b|}$$

We need to bound $|b_n|$ from below.

$|b_n| > ??$

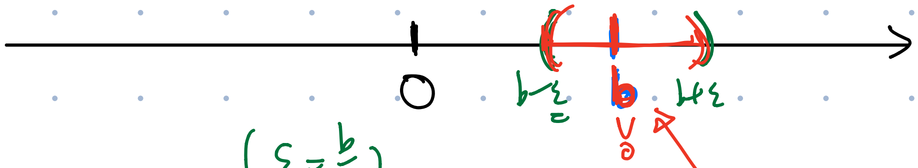
pf of Prop: $\forall \varepsilon > 0$

$\lim b_n = b$, $\exists N > 0$ (Is the M in the Lemma.)

s.t. $n > N \Rightarrow |b_n - b| < M \cdot |b| \cdot \varepsilon$

$$\Rightarrow \left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n| \cdot |b|} < \frac{M \cdot |b| \cdot \varepsilon}{M \cdot |b|} = \varepsilon. \quad \square$$

Lemma: If $b_n \neq 0\ \forall n$, $\lim b_n = b \neq 0$, then $\exists M > 0$ s.t. $|b_n| \geq M\ \forall n$.



$(\varepsilon = \frac{b}{2})$

$\exists N>0$ s.t. $\forall\, \boxed{n>N}$, b_n

In particular $b_n > b-\varepsilon = \boxed{\frac{b}{2}} > 0$

For $b_1, \dots, b_N$, $\neq 0$

$$M := \min\{|b_1|, \dots, |b_N|; \tfrac{b}{2}\}.$$

$$\Rightarrow |b_n| \geq M \quad \forall n$$

pf: Since $\lim b_n = b > 0$

$\exists N > 0$ s.t. for any $n > N$,

$$|b_n - b| < \frac{b}{2}$$

$$\Rightarrow |b_n| \geq |b| - |b - b_n| > b - \frac{b}{2} = \frac{b}{2} \quad \forall n > N$$

Define

$$M := \min\{\underset{\substack{\vee \\ 0}}{|b_1|}, \dots, \underset{\substack{\vee \\ 0}}{|b_N|}, \underset{\substack{\vee \\ 0}}{\tfrac{b}{2}}\} > 0$$

Then we have $|b_n| \geq M \quad \forall n \in \mathbb{N}$.

b/c if $n \leq N$, then $|b_1|, \dots, |b_N| \geq M$ by def^n of M,

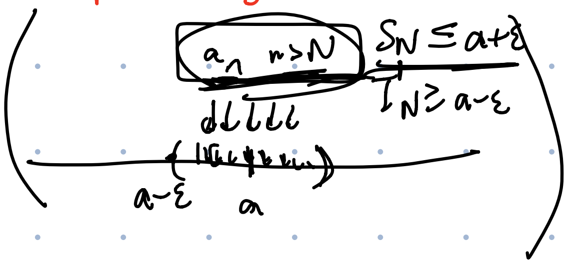
if $n > N$, then $|b_n| > \frac{b}{2} \geq M$. $\square$

§ "liminf", "limsup" of a bounded seq.

Q: When does a bounded seq. converge??

Motivation: The limiting behavior of a seq. (a_n) depends only on

$$\{a_n \mid n > N\} = \{a_{N+1}, a_{N+2}, \ldots\}.$$



Def: For a bounded seq. (a_n),

$$I_N := \inf\{a_n : n > N\} = \inf\{a_{N+1}, a_{N+2}, \ldots\}$$

$$S_N := \sup\{a_n : n > N\} = \sup\{a_{N+1}, a_{N+2}, \ldots\}.$$

e.g.

$$(a_n) = (\underset{a_1}{1}, \underset{a_2}{-1}, \underset{a_3}{\tfrac{1}{2}}, \tfrac{-1}{2}, \tfrac{1}{3}, \tfrac{-1}{3}, \ldots\ldots)$$

$\lim a_n = 0.$

$$I_1 = \inf\{\underset{-1}{a_2}, \underset{\frac{1}{2}}{a_3}, \underset{-\frac{1}{2}}{a_4}, \ldots\} = -1$$

$$I_2 = \inf\{\tfrac{1}{2}, \tfrac{-1}{2}, \tfrac{1}{3}, \ldots\} = \frac{-1}{2}$$

$$I_3 = -\frac{1}{2},$$

$$I_4 = \frac{-1}{3} = I_5, \ldots$$

$I_1 \le I_2 \le I_3 \le \ldots\ldots$

$\lim I_N = 0$

$$S_1 = \sup\{-1, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{3}, \ldots\} = \frac{1}{2}$$

$$S_2 = \frac{1}{2}$$

$$S_3 = \frac{1}{3}$$

$\vdots \qquad \vdots$

$S_1 \ge S_2 \ge S_3 \ge \ldots\ldots$

$\lim S_N = 0$

e.g. $(a_n) = (-1, 1, -1, 1, -1, 1, \dots)$ dIv.

$$I_1 = -1 \qquad S_1 = 1$$
$$I_2 = -1 \qquad S_2 = 1$$
$$I_3 = -1 \qquad S_3 = 1$$
$$\vdots \qquad\qquad \vdots$$

$\lim I_N$ exists $\qquad \lim S_N$ exists

$$\lim I_N = -1 \neq 1 = \lim S_N$$

Lemma: (a_n) bdd seq. , $\forall N > 0$

$$\underbrace{I_1}_{=\inf\{a_2, a_3, a_4, \dots\}} \leq \underbrace{I_2}_{=\inf\{a_3, a_4, a_5, \dots\}} \leq I_3 \leq \dots \leq \underbrace{I_N}_{=\inf\{a_n : n > N\}} \leq \underbrace{S_N}_{=\sup\{a_n : n > N\}} \leq S_{N+1} \leq \dots \underbrace{}_{=\sup\{a_3, a_4, \dots\}} \leq \underbrace{S_1}_{=\sup\{a_2, a_3, \dots\}}.$$

Ex: $S \subseteq S' \subseteq \mathbb{R}$

$\sup S \leq \sup S'$

$\inf S \geq \inf S'$

Coro: $(I_N), (S_N)$ converges (since they're bdd & monotone)

$$\lim I_N = \sup\{I_1, I_2, \dots\} =: \boxed{\liminf_{n\to\infty}} a_n$$

$$\lim S_N = \inf\{S_1, S_2, \dots\} =: \boxed{\limsup_{n\to\infty}} a_n$$

single notion

Thm (a_n) bdd.

$$\lim a_n \text{ exists} \iff \liminf a_n = \limsup a_n$$

In this case, we have $\lim a_n = \liminf a_n = \limsup a_n$.

Lemma: $\underline{I_1 \le I_2 \le \cdots \le I_N} \le \underset{n\to\infty}{\liminf}\, a_n \boxed{\le} \underset{n\to\infty}{\limsup}\, a_n \le \underline{S_N \le \cdots \le S_1}.$

$\liminf a_n = \lim I_N$, $\limsup a_n = \lim S_N$.

pf: $\underset{n\to\infty}{\liminf}\, a_n = \lim I_N = \sup\{I_1, I_2, \cdots\}$

so $\liminf a_n \ge I_N$ for any N.

Similarly, $\underset{n\to\infty}{\limsup}\, a_n \le S_N \quad \forall N.$

We need to show: " $\underset{n\to\infty}{\liminf}\, a_n \le \underset{n\to\infty}{\limsup}\, a_n$ "

$$\text{"}\ \boxed{\lim I_N \le \lim S_N}\ \text{"}$$

We already know $I_N \le S_M \quad \forall N, M > 0.$

Let's fix $N>0$ for now, $I_N \le S_M \quad \forall M > 0$

$$\Longrightarrow I_N \le \lim S_M = \underset{n\to\infty}{\limsup}\, a_n$$

So we have $I_N \le \underset{n\to\infty}{\limsup}\, a_n$ for any $N>0$.

$\Longrightarrow \liminf a_n = \lim I_N \le \limsup a_n \quad \square$