FINAL EXAM PRACTICE PROBLEMS
MATH H54, FALL 2021

(1) Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a linearly independent set of vectors in a real vector space V. Prove that

$$\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \dots, \vec{v}_{n-1} + \vec{v}_n, \vec{v}_n + \vec{v}_1\}$$

is linearly independent if and only if n is odd (not divisible by 2).

(2) Let A be a real $n \times n$ matrix. Prove that there exists a real $n \times n$ matrix B such that $BA = 0$ (the zero matrix) and $\operatorname{rank}(A) + \operatorname{rank}(B) = n$. (Hint: First show that there exists an invertible matrix P such that PA is the reduced echelon form of A.) (Hint: Then find a square matrix C such that $C(PA) = 0$ and $\operatorname{rank}(A) + \operatorname{rank}(C) = n$. Such C should not be hard to construct, using the fact that PA is of reduced echelon form.) (Hint: Finally, show that $B = CP$ has the desired properties.)

(3) Let V be a finite dimensional real inner product space, and let $W \subseteq V$ be a subspace.

(a) Define $T_W \colon V \to W$ to be the orthogonal projection onto W. Prove that for any $\vec{v}_1, \vec{v}_2 \in V$, one has $\langle \vec{v}_1, T_W(\vec{v}_2)\rangle = \langle T_W(\vec{v}_1), \vec{v}_2\rangle$.

(b) Conversely, suppose $T \colon V \to V$ is a linear transformation such that $T^2 = T$ and $\langle \vec{v}_1, T(\vec{v}_2)\rangle = \langle T(\vec{v}_1), \vec{v}_2\rangle$ holds for any $\vec{v}_1, \vec{v}_2 \in V$. Prove that T is the orthogonal projection onto its image $\operatorname{Im}(T)$. (Note: $T^2 = T \circ T$ denotes the composition of T with itself.) (Hint: Plug in $\vec{v}_1 = T(\vec{v})$ for any $\vec{v} \in V$, and use the condition $T^2 = T$.)

(4) Let W_1 and W_2 be two subspaces of an n-dimensional real vector space V, satisfying $\dim(W_1) + \dim(W_2) = n$. Prove that there exists a linear transformation $T \colon V \to V$ such that

$$\operatorname{Ker}(T) = W_1 \text{ and } \operatorname{Im}(T) = W_2.$$

(Hint: Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis of W_1. To construct the transformation T, you might want to use the fact that $\{\vec{v}_1, \dots, \vec{v}_k\}$ can be extended to a basis $\{\vec{v}_1, \dots, \vec{v}_k, \dots, \vec{v}_n\}$ of V.)

(5) Let A be an $n \times n$ matrix. Consider the linear transformation $T \colon \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ on the n^2-dimensional vector space $\mathrm{Mat}_{n\times n}(\mathbb{R})$ defined by $T(B) = AB$. Express $\det(T)$ in terms of $\det(A)$.

(6) Let A be a square matrix with columns given by unit vectors. Prove that $|\det(A)| \leq 1$. When does the equality hold?

(7) Let V be a finite-dimensional vector space, and let $T \colon V \to V$ be a diagonalizable linear transformation. Suppose $W \subseteq V$ is a subspace satisfying $T(W) \subseteq W$. Prove that the restriction $T|_W \colon W \to W$ also is diagonalizable.

(8) Consider a sequence of linear transformations between finite-dimensional vector spaces

$$\{0\} \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n \xrightarrow{T_n} \{0\}$$

Assume that $\mathrm{Im}(T_{i-1}) = \mathrm{Ker}(T_i)$ for all $1 \le i \le n$. What is the value of

$$\dim(V_1) - \dim(V_2) + \dim(V_3) - \cdots + (-1)^n \dim(V_n)?$$

(9) Let A be a real $n \times n$ matrix. Prove that the following two statements are equivalent:
(a) $A^2 = A$;
(b) $\mathrm{rank}(A) + \mathrm{rank}(\mathbb{I}_n - A) = n$.

(10) Let $\{\vec{v}_1, \ldots, \vec{v}_k\}$ be an orthonormal set in a finite-dimensional inner product space V. Suppose that for any $\vec{v} \in V$ we have

$$||\vec{v}||^2 = \langle \vec{v}_1, \vec{v} \rangle^2 + \cdots + \langle \vec{v}_k, \vec{v} \rangle^2 .$$

Prove that $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis of V.

(11) Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Suppose that $\mathbb{I}_m - AB$ is invertible. Prove that $\mathbb{I}_n - BA$ also is invertible.

(12) Let W_1 and W_2 be subspaces of a vectors space V. Consider the union

$$W_1 \cup W_2 := \{x \in V : x \in W_1 \text{ or } x \in W_2\}.$$

Prove that if $W_1 \cup W_2$ is a subspace of V, then we must have $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(13) Let $T : V \to V$ be a linear transformation on a (possibly infinite-dimensional) vector space V. Suppose that every subspace of V is invariant under V, i.e. $T(W) \subseteq W$ for any subspace $W \subseteq V$. Prove that T is a scalar multiple of the identity transformation.

(1) Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a linearly independent set of vectors in a real vector space V. Prove that

$$\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \ldots, \vec{v}_{n-1} + \vec{v}_n, \vec{v}_n + \vec{v}_1\}$$

is linearly independent if and only if n is odd (not divisible by 2).

- n is even: $n = 2m$. $m \in \mathbb{Z}$.

$$(\vec{v}_1 + \vec{v}_2) + (\vec{v}_3 + \vec{v}_4) + \cdots + (\vec{v}_{n-1} + \vec{v}_n)$$

$$= (\vec{v}_2 + \vec{v}_3) + (\vec{v}_4 + \vec{v}_5) + \cdots + (\vec{v}_n + \vec{v}_1)$$

$\Rightarrow$ the set is l.d.

- n is odd:

Suppose $\vec{0} = a_1(\vec{v}_1 + \vec{v}_2) + a_2(\vec{v}_2 + \vec{v}_3) + \cdots + a_n(\vec{v}_n + \vec{v}_1)$

$$= \underbrace{(a_1 + a_n)}_{=0}\vec{v}_1 + \underbrace{(a_1 + a_2)}_{=0}\vec{v}_2 + \underbrace{(a_2 + a_3)}_{=0}\vec{v}_3 + \cdots + \underbrace{(a_{n-1} + a_n)}_{=0}\vec{v}_n$$

$$(a_2 + a_3) + (a_4 + a_5) + \cdots + (a_{n-1} + a_n) = 0$$

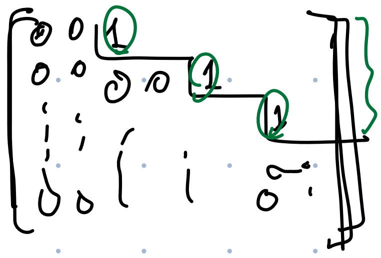
$$\Rightarrow \quad a_1 + a_2 + \cdots + a_n = a_1$$

$$(a_3 + a_4) + (a_5 + a_6) + \cdots + (a_{n-2} + a_{n-1}) + (a_n + a_1) = 0$$

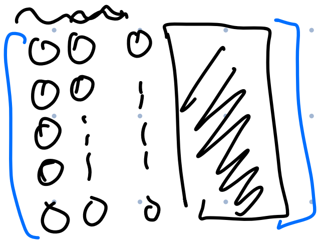
$$\Rightarrow \quad a_1 + \cdots + a_n = a_2$$

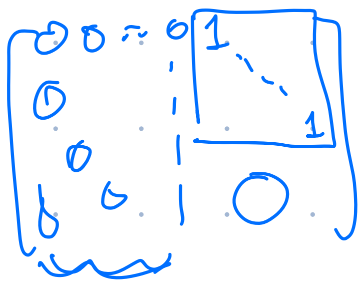
$$\Longrightarrow \quad a_1 = a_2 = \cdots = a_n = 0. \quad \square$$

(2) Let A be a real $n \times n$ matrix. Prove that there exists a real $n \times n$ matrix B such that $BA = 0$ (the zero matrix) and $\operatorname{rank}(A) + \operatorname{rank}(B) = n$. (Hint: First show that there exists an invertible matrix P such that PA is the reduced echelon form of A.) (Hint: Then find a square matrix C such that $C(PA) = 0$ and $\operatorname{rank}(A) + \operatorname{rank}(C) = n$. Such C should not be hard to construct, using the fact that PA is of reduced echelon form.) (Hint: Finally, show that $B = CP$ has the desired properties.)

$PA =$  # of nonzeros of the reduced echelon form $= \operatorname{rank}(PA) = \operatorname{rank}(A)$

rank(A)



$C =$ 

rk(A)

$rk(C) = n - rk(A)$

(3) Let V be a finite dimensional real inner product space, and let $W \subseteq V$ be a subspace.

(a) Define $T_W : V \to W$ to be the orthogonal projection onto W. Prove that for any $\vec{v}_1, \vec{v}_2 \in V$, one has $\langle \vec{v}_1, T_W(\vec{v}_2)\rangle = \langle T_W(\vec{v}_1), \vec{v}_2\rangle$.

(b) Conversely, suppose $T : V \to V$ is a linear transformation such that $T^2 = T$ and $\langle \vec{v}_1, T(\vec{v}_2)\rangle = \langle T(\vec{v}_1), \vec{v}_2\rangle$ holds for any $\vec{v}_1, \vec{v}_2 \in V$. Prove that T is the orthogonal projection onto its image $\operatorname{Im}(T)$. (Note: $T^2 = T \circ T$ denotes the composition of T with itself.) (Hint: Plug in $\vec{v}_1 = T(\vec{v})$ for any $\vec{v} \in V$, and use the condition $T^2 = T$.)

(a) $\langle \vec{v}_1, T_W(\vec{v}_2)\rangle = \langle \underset{\in W}{T_W(\vec{v}_1)} + \underset{\in W^\perp}{\vec{w}_1}, \underset{\in W}{T_W(\vec{v}_2)}\rangle$

$= \langle T_W(\vec{v}_1), T_W(\vec{v}_2)\rangle$

(b) Suppose $T^2 = T$, and $\langle \vec{v}_1, T(\vec{v}_2)\rangle = \langle T(\vec{v}_1), \vec{v}_2\rangle$

Want to show: $\vec{v} - T(\vec{v}) \in \operatorname{Im}(T)^\perp \quad \forall \vec{v} \in V.$

i.e. $\boxed{\langle \vec{v} - T(\vec{v}), T(\vec{w}) \rangle = 0 \quad \forall \vec{v}, \vec{w} \in V}$

$\langle T(\vec{v}), T(\vec{v_2}) \rangle = \langle T^2(\vec{v}), \vec{v_2} \rangle$

$= \langle T(\vec{v}), \vec{v_2} \rangle$

$\Rightarrow \langle T(\vec{v}), T(\vec{v_2}) - \vec{v_2} \rangle = 0 \quad \forall \vec{v}, \vec{v_2} \in V$

use the condition $T^2 = T$.)

(4) Let W_1 and W_2 be two subspaces of an n-dimensional real vector space V, satisfying $\dim(W_1) + \dim(W_2) = n$. Prove that there exists a linear transformation $T: V \to V$ such that

$$\operatorname{Ker}(T) = W_1 \text{ and } \operatorname{Im}(T) = W_2.$$

(Hint: Let $\{\vec{v}_1, \ldots, \vec{v}_k\}$ be a basis of W_1. To construct the transformation T, you might want to use the fact that $\{\vec{v}_1, \ldots, \vec{v}_k\}$ can be extended to a basis $\{\vec{v}_1, \ldots, \vec{v}_k, \ldots, \vec{v}_n\}$ of V.)

T: $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \longmapsto \vec{0}$

$\vec{v}_{k+1} \longmapsto \vec{w_1}$

$\vec{v}_{k+2} \longmapsto \vec{w_2}$

$\vdots$

$\vec{v_n} \longmapsto \vec{w}_{n-k}$.

$\{\vec{w}_1, \ldots, \vec{w}_{n-k}\}$ basis of W_2

Define T as follows:

$\forall \vec{v} \in V, \ \exists! \ c_1, \ldots, c_n$

s.t. $\vec{v} = c_1 \vec{v_1} + \cdots + c_k \vec{v_k} + \cdots + c_n \vec{v_n}$

Define $T(\vec{v}) := c_{k+1} \vec{w_1} + c_{k+2} \vec{w_2} + \cdots + c_n \vec{w}_{n-k}$.

Easily check that T is linear.

(8) Consider a sequence of linear transformations between finite-dimensional vector spaces

$$\{0\} \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n \xrightarrow{T_n} \{0\}$$

Assume that $\mathrm{Im}(T_{i-1}) = \mathrm{Ker}(T_i)$ for all $1 \le i \le n$. What is the value of

$$\dim(V_1) - \dim(V_2) + \dim(V_3) - \cdots + (-1)^n \dim(V_n)?$$

$\{0\} = \mathrm{Im}\, T_0 = \ker T_1$

rank-nullity thm: $\dim V_1 = \underbrace{\cancel{\dim \ker T_1}}_{} + \dim \mathrm{Im}\, T_1$ $\quad 0$

$$\dim V_2 = \underbrace{\dim \ker T_2}_{= \mathrm{Im}\, T_1} + \dim \mathrm{Im}\, T_2$$

$$\Rightarrow \dim V_1 - \dim V_2 = -\dim \mathrm{Im}\, T_2$$

$$\dim V_3 = \underbrace{\dim \ker T_3}_{= \mathrm{Im}\, T_2} + \dim \mathrm{Im}\, T_3$$

$$\Rightarrow \dim V_1 - \dim V_2 + \dim V_3 = \dim \mathrm{Im}\, T_3.$$

Continue this inductively $\Rightarrow$ the alternating sum of dim.

$= 0.$

(10) Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be an orthonormal set in a finite-dimensional inner product space V. Suppose that for any $\vec{v} \in V$ we have

$$||\vec{v}||^2 = \langle \vec{v}_1, \vec{v} \rangle^2 + \cdots + \langle \vec{v}_k, \vec{v} \rangle^2 .$$

Prove that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis of V.

$$\operatorname{proj}_{\operatorname{span}\{\vec{v}_1\}} \vec{v} = \frac{\langle \vec{v}_1, \vec{v}_1 \rangle}{\cancel{\langle \vec{v}_1, \vec{v}_1 \rangle}_{1}} \vec{v}_1$$

$$\|\operatorname{proj}_{\operatorname{span}\{\vec{v}_1\}} \vec{v}\|^2 = \langle \vec{v}, \vec{v}_1 \rangle^2$$

Suppose $W = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subsetneq V$

Choose any $0 \neq \vec{v} \in W^\perp$

Then $0 < \|\vec{v}\|^2 = \langle \vec{v}_1, \vec{v} \rangle^2 + \cdots + \langle \vec{v}_n, \vec{v} \rangle^2$

$= 0.$ $\square$