Thm (Yomdin type lower bound) $F: D \to D$ endofunctor

$$h_{cat}(F) \geq \log \rho([F])$$

where: $[F]: N(D) \longrightarrow N(D)$ induced from F,

and $\rho =$ spectral radius of the linear map

$= \max\{|\text{eigenvalue}|\}$.

pf:

- Let λ be the eigenvalue of $[F]$ with largest $|\lambda|$, and with largest Jordan block.
- Let $\{v = v_0, v_1, \dots, v_s\}$ basis of max. Jordan block, and $v_k = (f - \lambda\, id)^k v$.
- Take objects $M_1, \dots, M_m$ of D s.t. $\{[M_1], \dots, [M_m]\}$ form a basis of $N(D)$. Define a norm on $N(D) \otimes \mathbb{C}$:

$$\|w\| := \sum_i |\chi_c([M_i], w)|.$$

- Write $v = v_0 = \sum a_i [M_i]$, $a_i \in \mathbb{C}$
  Choose $l \in \mathbb{Z}$ s.t. $l > |a_i|$ $\forall i$.
- Define $E := \bigoplus_i M_i^{\oplus l}$

$$\varepsilon\left(G \oplus \bigoplus_i M_i,\ F^n(G \oplus E)\right) \geq \varepsilon\left(\bigoplus_i M_i,\ F^n(E)\right)$$

$\varepsilon(A, B) := \sum_k \dim \operatorname{Hom}(A, B[k])$, $h_0(F) = \lim_{n\to\infty} \frac{1}{n} \log \varepsilon(G, F^n G')$ split generators

$$= \varepsilon\left(\bigoplus_i M_i,\ F^n\left(\bigoplus_j M_j^{\oplus l}\right)\right)$$

$$= \sum_i \sum_j l \cdot \varepsilon(M_i, F^n M_j)$$

$$\geq \sum_i \sum_j |a_j| \cdot \varepsilon(M_i, F^n M_j).$$

$$\geq \sum_i \sum_j |a_j| \cdot |\chi(M_i, F^n M_j)|$$

$$= \sum_j |a_j|\, \|F^n(M_j)\| = \|f^n v\|$$

$$= \left\| \lambda^n v_0 + \binom{n}{1} \lambda^{n-1} v_1 + \binom{n}{2} \lambda^{n-2} v_2 + \cdots + \binom{n}{s} \lambda^{n-s} v_s \right\|$$

$$h_{cat}(F) \geq \lim_{n\to\infty} \frac{1}{n} \log \left\| \lambda^n v_0 + \binom{n}{1} \lambda^{n-1} v_1 + \cdots \right\|$$

$$= \log |\lambda|$$

$$= \log \rho([F]). \quad \square$$

<u>Kikuta-Takahashi:</u> X - smooth proj. var. $/\mathbb{C}$, $\pm\omega_X$ is ample

$$h_{cat}(F) = \log \rho([F]) \quad \forall F \in \operatorname{Aut} D^b(X)$$

<u>pf:</u> • Bondal-Orlov: $\pm\omega_X$ ample $\Rightarrow$

$$\operatorname{Aut} D^b(X) \cong (\operatorname{Aut}(X) \ltimes \operatorname{Pic}(X)) \times \mathbb{Z}[1].$$

i.e. $\forall F = \boxed{f^*(- \otimes \mathcal{L}')[a] \text{ for some } f \in \operatorname{Aut}(X),\ \mathcal{L}' \in \operatorname{Pic}(X),\ a \in \mathbb{Z}.}$

- Choose $m \in \mathbb{Z}$ s.t. $\mathcal{L} := K_X^{\otimes m}$ is very ample
- Choose $G = \bigoplus_{i=1}^{d+1} \mathcal{L}^i$, $G^* = \bigoplus_{i=1}^{d+1} \mathcal{L}^{-i}$.
- Choose $l \in \mathbb{Z}$ st. $\mathcal{L}'' := \mathcal{L}' \otimes K_X^{l}$ is anti-ample
- Consider the autoequivalences $F_1 := f^*(- \otimes \mathcal{L}'')$
  $F_2 := (- \otimes K_X^{-l})[a]$.

Then • $F = F_1 \circ F_2$.
• $F_1 F_2 = F_2 F_1$.

$$\left(\Rightarrow h_0(F_1F_2) \leq h_0(F_1) + \underbrace{h_0(F_2)}_{=0} \right.$$

$$h_0(F_1) \leq h_0(F_1F_2) + \underbrace{h_0(F_2^{-1})}_{=0}$$

$$\left. \Rightarrow h_0(F_1) = h_0(F_1F_2) = h_0(F) \right)$$

$$(\text{Similarly, } \rho([F]) = \rho([F_1]))$$

$\Rightarrow$ reduce to the case where $F = f^*(- \otimes \underset{\substack{\uparrow \\ \text{anti-ample.}}}{\mathcal{L}''})$

$$\bullet\ \varepsilon(G, \overbrace{(f^*(- \otimes \mathcal{L}''))}^{F}{}^n G^*)$$

$$= \varepsilon(G, \underbrace{G^* \otimes \mathcal{L}'' \otimes f^*\mathcal{L}'' \otimes \cdots \otimes (f^*)^{n-1} \mathcal{L}''}_{\text{anti-ample.}})$$

$$\overset{\text{Kodaira vanishing}}{\Longrightarrow} |\chi(G, G^* \otimes \mathcal{L}'' \otimes \cdots \cdots \otimes (f^*)^{n-1} \mathcal{L}'')|$$

$$= \left| \chi\big(G, (\underbrace{f^*(- \otimes \mathcal{L}^{!})}_{F})^n G^*\big) \right|$$

$$\Rightarrow \quad h_{cat}(F) = \lim_{n\to\infty} \frac{1}{n} \log \left| \chi(G, F^n G^*) \right| = \log \rho(F) \qquad \square$$

<u>Spherical twists:</u> S: d-spherical object. $(d \geq 2)$

$$T_S(E) := \mathrm{Cone}(\mathrm{Hom}^\bullet(S,E) \otimes S \longrightarrow E)$$

- $$\begin{tikzcd} \mathrm{Hom}^\bullet(S,G)\otimes S \arrow[rr] & & G \arrow[dl] \\ & T_S(G) \arrow[ul, "+1"] & \end{tikzcd}$$

$T_S(S) = S[1-d]$

- $$\begin{tikzcd} \mathrm{Hom}^\bullet(S, T_S^{n-1}(G))\otimes S \arrow[rr] \arrow[d, equal] & & T_S^{n-1}(G) \arrow[dl] \\ \mathrm{Hom}^\bullet(T_S^{-(n-1)}(S), G)\otimes S \arrow[d, equal] & T_S^n(G) \arrow[ul, "+1"] & \\ \mathrm{Hom}^\bullet(S[(n-1)(d-1)], G)\otimes S & & \end{tikzcd}$$

- $$\varepsilon_t(G', T_S^n(G)) \leq \varepsilon_t(G', T_S^{n-1}(G)) + \varepsilon_t(G', \mathrm{Hom}^\bullet(S,G)\otimes S[(n-1)(1-d)+1])$$

$$= \varepsilon_t(G', T_S^{n-1}(G)) + \varepsilon_t(G', S)\,\varepsilon_t(S,G)\, e^{((n-1)(1-d)+1)t}.$$

$$\overset{\text{inductively}}{\leq} \cdots$$

$$\leq \varepsilon_t(G',G) + \varepsilon_t(G',S)\,\varepsilon_t(S,G) \sum_{k=1}^{n} e^{(k(1-d)+d)t}$$

$$\leq \varepsilon_t(G',G) + \varepsilon_t(G',S)\,\varepsilon_t(S,G)\cdot n \cdot \max\left\{1, e^{(n(1-d)+d)t}\right\}$$

1) If $t \le 0$, then $h_t(T_S) \le (1-d)t$.

On the otherhand,

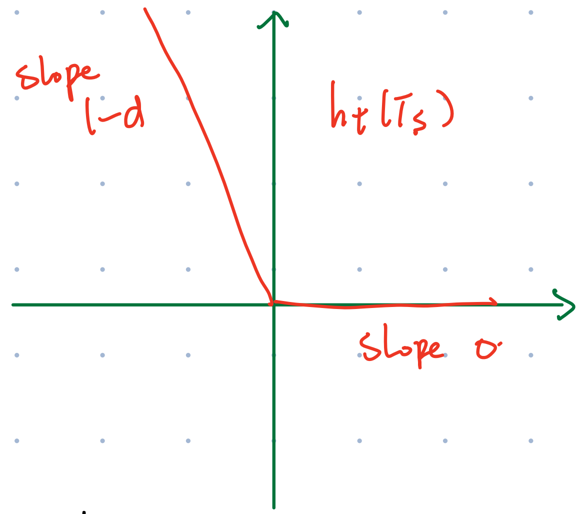
$$h_t(T_S) = \lim_{n\to\infty} \frac{1}{n} \log \varepsilon_t(G', T_S^n(G \oplus S))$$

$$\geq \lim_{n\to\infty} \frac{1}{n} \log \varepsilon_t(G', \underbrace{T_S^n(S)}_{\substack{\shortparallel \\ S[(1-d)n]}})$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \varepsilon_t(G', S)\, e^{n(1-d)t}$$

$$= (1-d)t.$$

$\Rightarrow \quad h_t(T_S) = (1-d)t.$



2) If $t \geq 0$, then $h_t(T_S) \le 0$

On the other hand, assume ${}^{\perp}S \neq \phi$,

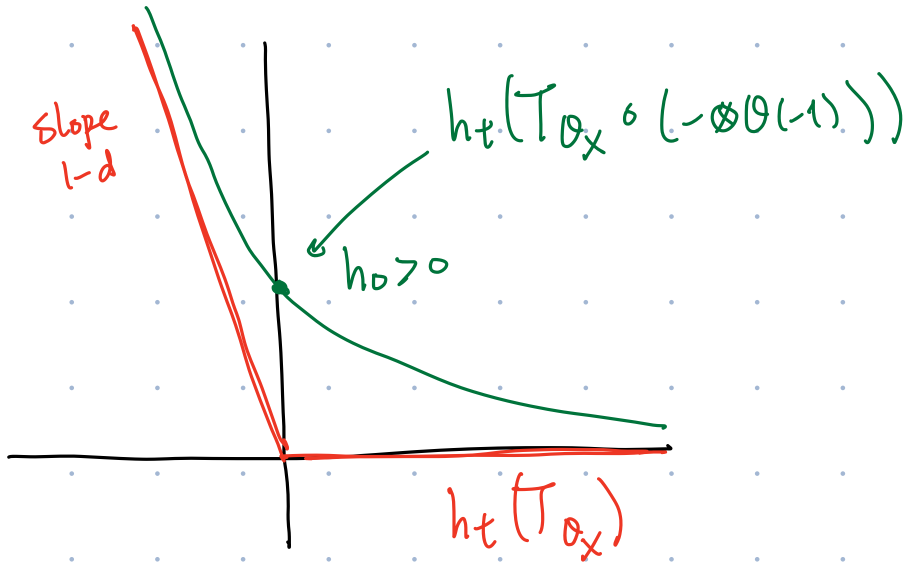
i.e. $\exists\, A \neq 0$ st. $\mathrm{Hom}(S, A) = 0$

$$\Rightarrow \quad T_S(A) = A$$

$$h_t(T_S) = \lim_{n\to\infty} \frac{1}{n} \log \varepsilon_t(G', T_S^n(G \oplus A))$$

$$\geq \lim_{n\to\infty} \frac{1}{n} \log \varepsilon_t(G', \underbrace{T_S^n(A)}_{\substack{\shortparallel \\ A}}) = 0$$

e.g. X - CY of dim $d \geq 3$.



- Consider $X^d \subseteq \mathbb{P}^{d+1}$ CY hypersurface with $2 | d$.
- $(T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}(1)))^{d+2} = [2]$.

$\Rightarrow \left([T_{\mathcal{O}_X}] \circ [- \otimes \mathcal{O}(1)]\right)^{d+2} = \mathrm{id}.$

- $2 | d \Rightarrow [T_{\mathcal{O}_X}]^2 = \mathrm{id}.$

$\Rightarrow \left([T_{\mathcal{O}_X}] \circ [- \otimes \mathcal{O}(-1)]\right)^{d+2} = \mathrm{id}.$

$\Rightarrow \log \rho\left([T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}(-1))]\right) = 0.$

$\rightsquigarrow$ Gives a counterexample to the Gromov-type equality:

$$h_{cat}(T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}(-1))) > \log \rho([T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}(-1))]).$$

Entropy via Bridgeland stability conditions:

$\sigma \in \mathrm{Stab}(D)$, $\forall\, E \neq 0$, $\exists !$

$$\begin{tikzcd}[column sep=small] 0 \arrow[rr] & & \cdots \arrow[r] & \cdots \arrow[rr] & & E \\ & A_1 \arrow[ul] & & & A_n \arrow[ul] & \end{tikzcd}$$

s.t. $A_i \in P(\phi_i)$, $\phi_1 > \cdots > \phi_n$.

$$m_{\sigma,t}(E) := \sum_i |Z_\sigma(A_i)|\, e^{\phi_i t}$$

Ikeda: $\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul] & \end{tikzcd}$, $\quad m_{\sigma,t}(B) \leq m_{\sigma,t}(A) + m_{\sigma,t}(C)$.

Corollary: $m_{\sigma,t}(B) \leq m_{\sigma,t}(A)\, \delta_t(A,B)$

pf:
- $B \cong 0$ ✓
- $B \notin \langle A \rangle$ ✓
- $B \in \langle A \rangle$, $\forall\, \varepsilon > 0$,

$$\exists\ \begin{tikzcd}[column sep=small] 0 \arrow[rr] & & \cdots \arrow[r] & \cdots \arrow[rr] & & B \oplus B' \\ & A[n_1] \arrow[ul] & & & A[n_k] \arrow[ul] & \end{tikzcd}$$

s.t. $\displaystyle\sum_i e^{n_i t} < \delta_t(A,B) + \varepsilon$.

Then $m_{\sigma,t}(B) \leq m_{\sigma,t}(B \oplus B') \overset{\text{Ikeda}}{\leq} \sum_i m_{\sigma,t}(A[n_i])$

$$= m_{\sigma,t}(A) \cdot \sum_i e^{n_i t}$$

$$< m_{\sigma,t}(A)\left(\delta_t(A,B) + \varepsilon\right). \quad \square$$

Thm 1) $\forall$ split generator G of D,

$$\limsup_{n\to\infty} \frac{1}{n} \log m_{\sigma,t}(F^n G) = \sup_{E \neq 0} \left\{ \limsup_{n\to\infty} \frac{1}{n} \log m_{\sigma,t}(F^n E) \right\}.$$

$\overset{..}{/\!/}$

$h_{\sigma,t}(F) \in [-\infty, \infty)$, later we'll show $h_{\sigma,t} \in \mathbb{R}$.

2) $h_{\sigma,t}(F) \leq h_t(F)$

pf: 1) $m_{\sigma,t}(F^n E) \leq m_{\sigma,t}(F^n G)\, \delta_t(F^n G, F^n E)$

$$\leq m_{\sigma,t}(F^n G)\, \underbrace{\delta_t(G, E)}_{\text{finite if } G \text{ is a split generator}}$$

$\Rightarrow \limsup_{n\to\infty} \frac{1}{n} \log m_{\sigma,t}(F^n E) \leq \limsup_{n\to\infty} \frac{1}{n} \log m_{\sigma,t}(F^n G). \quad \square$

2) $m_{\sigma,t}(F^n G) \leq m_{\sigma,t}(G)\, \delta_t(G, F^n G)$

$\Rightarrow h_{\sigma,t}(F) \leq h_t(F). \quad \square$

Rmk: $h_{\sigma,t}(F)$ is deformation invariant. (depends on the connected component of $\mathrm{Stab}(D)$)

(Bridgeland, Ikeda: $\forall \sigma \in \mathrm{Stab}(D)$, $\exists \varepsilon > 0$, $C_1, C_2 : \mathbb{R} \to \mathbb{R}_{>0}$
s.t. $\forall \tau \in B_\varepsilon(\sigma)$, $\forall\, 0 \neq E \in D$,

$$C_1(t)\, m_{\sigma,t}(E) < m_{\tau,t}(E) < C_2(t)\, m_{\sigma,t}(E) \Big)$$

Prop: (Yomdin type lower bound) $h_{\sigma,0}(F) \geq \log \rho([F])$.

pf: • Choose $M_1, \ldots, M_n \in D$ s.t. $\{[M_1], \ldots, [M_n]\}$ forms a basis of $N(D)$.

• Take an eigenvector $v = \sum \underset{\underset{\mathbb{C}}{\cap}}{a_i} [M_i]$ of λ w/ $|\lambda| = \rho$.

• May assume $Z_\sigma(v) \neq 0$ by deformation.

• Choose $l \in \mathbb{Z}$ s.t. $l > |a_i|$ $\forall i$.

$$\begin{aligned} h_{\sigma,0}(F) &= \limsup_{N\to\infty} \frac{1}{N} \log m_\sigma\Big(F^N\big(\underbrace{G \oplus \bigoplus_i M_i^{l}}\big)\Big) \\ &\geq \limsup_{N\to\infty} \frac{1}{N} \log m_\sigma\Big(F^N\big(\bigoplus_i M_i^{\oplus l}\big)\Big) \\ &= \limsup_{N\to\infty} \frac{1}{N} \log \Big(l \cdot \sum_i m_\sigma(F^N M_i)\Big) \\ &\geq \limsup_{N\to\infty} \frac{1}{N} \log \Big(l \cdot \sum_i \big|Z_\sigma(F^N M_i)\big|\Big). \end{aligned}$$

$$\geq \limsup_{N\to\infty} \frac{1}{N} \log \Big(\sum_i |a_i| |Z_\sigma(F^N(M_i))| \Big)$$

$$\geq \limsup_{N\to\infty} \frac{1}{N} \log |Z_\sigma([F]^N v)|$$

$$= \limsup_{N\to\infty} \frac{1}{N} \log \big(|\lambda|^N \underbrace{|Z_\sigma(v)|}_{\neq 0} \big)$$

$$= \log |\lambda|$$

$$= \log \rho([F]). \quad \square$$

Def • an abelian category is called <u>algebraic</u> if it's finite length with finitely many (isomorphic classes of) simple objects.

• $\sigma = (Z, P)$ is called <u>algebraic</u> if $\mathcal{H} = P[0,1]$ is algebraic

Rmk: • If $\mathcal{H} \subseteq D$ is an algebraic heart with simple objects $S_1, \dots, S_n$, then $K_0(D) = \bigoplus_i \mathbb{Z}[S_i]$

• $E \in \mathcal{H}$, $[E] = \sum_i d_i [S_i]$, where $d_i \geq 0$.

$\dim E := \sum d_i \geq 0$.

Lemma: $\mathcal{H} \subseteq D$ algebraic heart w/ simples $S_1, \dots, S_n$.

$G = \bigoplus S_i$

$\delta_t(G, E) \leq \dim E \quad \forall t, \ \forall E \in \mathcal{H}$.

pf $\forall E \in \mathcal{H}$, $\exists$ $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$

st. each $E_i / E_{i-1} \in \{S_1, \ldots, S_n\}$, and $n = \dim E$.

$\Rightarrow \exists$ $0 = E_0' \subseteq E_1' \subseteq E_2' \subseteq \cdots \subseteq E_n' = E \oplus \ast$

st. $E_i / E_{i-1} \cong G$.

$\Rightarrow \delta_t(G, E) \leq n = \dim E$. $\square$

- Let σ be an alg. stab. cond^n., $\mathcal{H} = \mathcal{P}_\sigma(0,1]$.
- Consider $\sigma_0 = (Z_0, \mathcal{H})$, where:

$$Z_0 : K_0(\mathcal{D}) \cong \bigoplus \mathbb{Z}[S_i] \longrightarrow \mathbb{C}$$
$$[S_j] \longmapsto i = \sqrt{-1} \in \mathbb{H}.$$

By deformation invar., $h_{\sigma_0,t} = h_{\sigma,t}$.

Prop: $\delta_t(G, \overset{\overset{\mathcal{D}}{\cup}}{E}) \leq e^{-\frac{1}{2}t} m_{\sigma_0,t}(E)$ $\to$ $|Z_{\sigma_0,t}(E)| e^{\phi(E)t}$

pf
- If $E \in \mathcal{H}$, then $\delta_t(G, E) \leq \dim E = e^{-\frac{1}{2}t} m_{\sigma_0,t}(E)$
- General $E \in \mathcal{D}$, let $H^k(E)$ be the k-th cohom. of E w.r.t. $\mathcal{H}$.

$$\begin{tikzcd} 0 \arrow[r] & \cdots \arrow[r] & E \end{tikzcd}$$
$$\cdots \ H^{-1}(E)[1] \quad H^0(E) \quad H^1(E)[-1] \cdots \qquad \in \mathcal{H}$$

- By Δ ineq. of δ_t,

$$\delta_t(G, E) \leq \sum_k \delta_t(G, H^k(E)) \cdot e^{-kt}$$
$$\leq \sum_k \dim H^k(E) \cdot e^{-kt}.$$

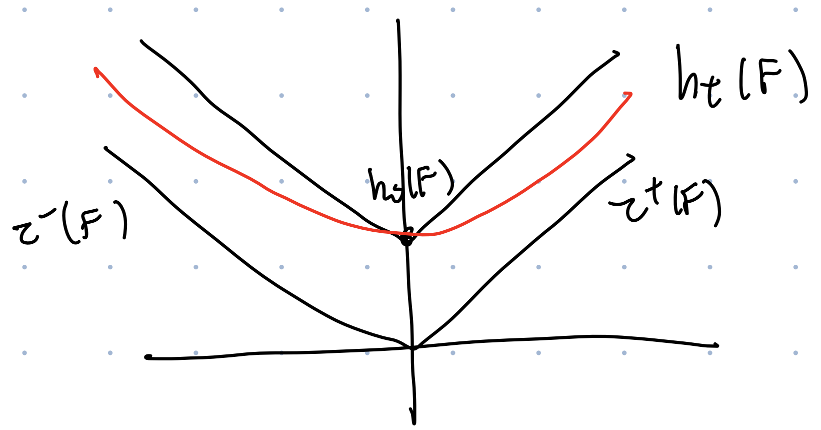
- $$m_{\sigma_0,t}(E) = \sum_k m_{\sigma_0,t}(H^k(E)) \cdot e^{-kt}$$
$$= \sum_k \dim H^k(E) \cdot e^{\frac{1}{2}t} e^{-kt} \qquad \square$$

$$\Rightarrow \quad e^{\frac{1}{2}t} \delta_t(G, F^n G) \leq m_{\sigma_0,t}(F^n G) \leq m_{\sigma_0,t}(G)\, \delta_t(G, F^n G)$$

$$\Rightarrow \quad h_{\sigma_0,t}(F) = h_t(F).$$

If a connected component $\mathrm{Stab}^\circ(D) \subseteq \mathrm{Stab}(D)$ contains an algebraic stability, then $\forall \sigma \in \mathrm{Stab}^\circ(D)$, we have

$$h_{\sigma,t}(F) = h_t(F). \quad \square$$



Motivation of shifting numbers $\tau^{\pm}(F)$:

(analogy with Poincaré translation numbers).

Consider the central extension

$$0 \to \mathbb{Z} \longrightarrow \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \longrightarrow \mathrm{Homeo}^+(S^1) \longrightarrow 1$$

$$\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \ni f: \mathbb{R} \to \mathbb{R}, \quad f(x+1) = f(x) + 1$$

Poincaré translation # of f:

$$\rho(f) := \lim_{n\to\infty} \frac{f^{(n)}(x_0) - x_0}{n}, \quad x_0 \in \mathbb{R}$$

In our Δ cat. setting, we also have a central extension

$$0 \to \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathcal{D}) \longrightarrow \mathrm{Aut}(\mathcal{D})/[1] \longrightarrow 1.$$

Translation #	Shifting # σ
$f \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$	$F \in \mathrm{Aut}(\mathcal{D})$.
$x_0 \in \mathbb{R}$	$G \in \mathcal{D}$.
amount of translation $f^{(n)}(x_0) - x_0$.	phases $\phi^{\pm}_{\sigma}: \mathrm{Ob}(\mathcal{D}) \to \mathbb{R}$ $\phi^{\pm}_{\sigma}(F^n G) - \phi^{\pm}_{\sigma}(G)$.
translation #	upper/lower shifting #.

$0 \to \cdots \to E$

$A_1 \qquad A_n$

$\phi(A_1) = \phi^+_\sigma(E) \qquad \phi(A_n) = \phi^-_\sigma(E)$

Thm (shifting # via stability conditions).

1) $\forall \sigma$, $\forall G$,

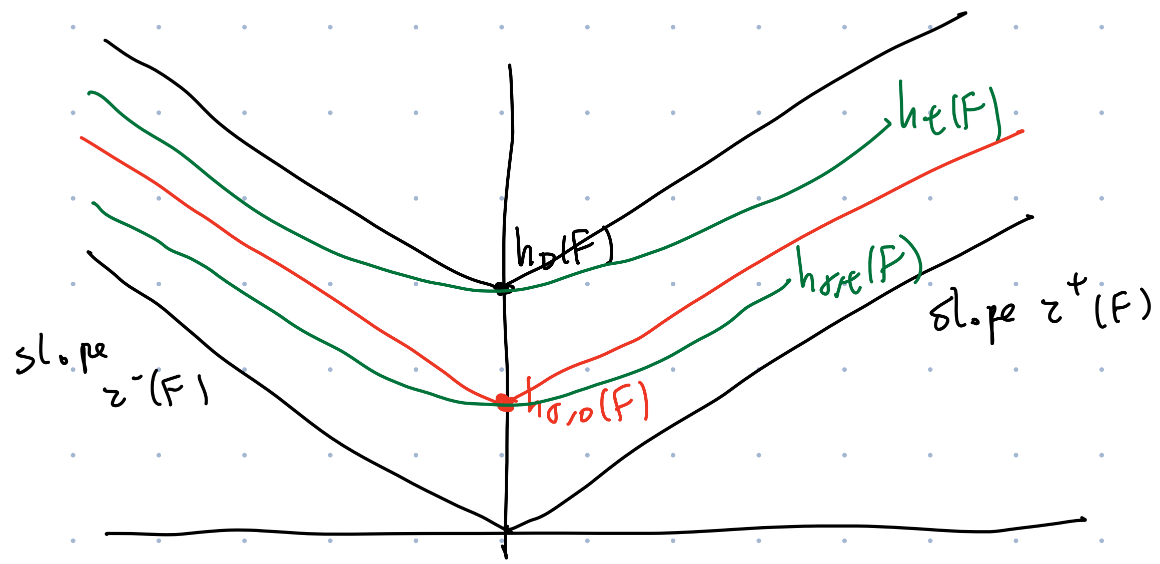
$$\lim_{n\to\infty} \frac{\phi_\sigma^+(F^n G) - \phi_\sigma^+(G)}{n} = \lim_{t\to+\infty} \frac{h_{\sigma,t}(F)}{t} = \tau^+(F)$$

limits exist.

2) If D admits a Serre functor, then

$$\lim_{n\to\infty} \frac{\phi_\sigma^-(F^n G) - \phi_\sigma^-(G)}{n} = \lim_{t\to-\infty} \frac{h_{\sigma,t}(F)}{t} = \tau^-(F).$$

3)
$$t\cdot\tau^+(F) \le h_{\sigma,t}(F) \le h_{\sigma,0}(F) + t\cdot\tau^+(F) \quad \forall t \ge 0$$
$$t\cdot\tau^-(F) \le h_{\sigma,t}(F) \le h_{\sigma,0}(F) + t\cdot\tau^-(F) \quad \forall t \le 0$$



pf: Fix any $\sigma \in \mathrm{Stab}(D)$.

- Support property: $\exists C > 0$ s.t. $|Z_\sigma(E)| > C$ $\forall E$: σ-semistable

- For $t > 0$,

t ... $F^n G$

A_1 ... A_n

$$\nearrow \sum |Z_\sigma(A_i)| e^{\cdots}$$

$$C \cdot e^{\phi_\sigma^+(F^n G)\cdot t} \leq m_{\sigma,t}(\boxed{F^n G}) \leq m_{\sigma,0}(F^n G) \cdot e^{\phi_\sigma^+(F^n G) t}$$

$$\Rightarrow \quad t \cdot \underbrace{\limsup_{n\to\infty} \frac{\phi_\sigma^+(F^n G)}{n}}_{\tau^+(F)} \leq h_{\sigma,t}(F)$$

$$\leq h_{\sigma,0}(F) + t \cdot \underbrace{\limsup_{n\to\infty} \frac{\phi_\sigma^+(F^n G)}{n}}_{\tau^+(F)}$$

$$\Rightarrow \quad \limsup_{n\to\infty} \frac{\phi_\sigma^+(F^n G)}{n} \leq \lim_{t\to+\infty} \frac{h_{\sigma,t}(F)}{t} \leq \limsup_{n\to\infty} \frac{\phi_\sigma^+(F^n G)}{n}$$

$$\Rightarrow \quad \lim_{t\to+\infty} \frac{h_{\sigma,t}(F)}{t} = \limsup_{n\to\infty} \frac{\phi_\sigma^+(F^n G)}{n}.$$

$$\varepsilon^+(G, F^n G) := \max\{k \in \mathbb{Z} \mid \mathrm{Hom}(G, F^n G[-k]) \neq 0\}$$

$$\Rightarrow \quad \phi_\sigma^+(F^n G) - \phi_\sigma^-(G) \geq \varepsilon^+(G, F^n G)$$

$$\Rightarrow \quad \liminf_{n\to\infty} \frac{\phi_\sigma^+(F^n G)}{n} \geq \liminf_{n\to\infty} \frac{\varepsilon^+(G, F^n G)}{n}$$

$$= \lim_{n\to\infty} \frac{\varepsilon^+(G, F^n G)}{n} = \tau^+(F)$$

$$= \lim_{t\to\infty} \frac{h_t(F)}{t}$$

$$\geq \lim_{t\to\infty} \frac{h_{\sigma,t}(F)}{t} = \limsup_{n\to\infty} \frac{\phi_\sigma^+(F^n G)}{n}.$$

$\square$