Last time:

- $A: m\times n$, $B: n\times p$. $AB: m\times p$ is defined to be the matrix s.t. $T_{AB} = T_A \circ T_B$.
- $A: n\times n$ is <u>invertible</u> if $\exists\, A^{-1}: n\times n$ s.t. $A^{-1}A = AA^{-1} = I_n$.
  (what are the invertible matrices for $n=1$?)
- We proved that if A is invertible, then $A\vec{x}=\vec{b}$ has a unique sol'n $\forall\, \vec{b}\in\mathbb{R}^n$. i.e., T_A is <u>bijective</u> (both injective & surjective).

Today:

- Introduce "elementary matrices"
- Prove "T_A bijective" $\Rightarrow$ "A invertible".
- Introduction to determinants (if we have time)

Rmk: • $A: m\times n$, $m>n$
Then there doesn't exist $B: n\times m$ st. $AB = I_m$

• $A: m\times n$, $m<n$.
Then there doesn't exist $B: n\times m$ st. $BA = I_n$

Ex:
- If A is invertible, then so is A^{-1}, and $(A^{-1})^{-1} = A$
- If A, B invertible, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$
- If A is invertible, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

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e.g.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11} + 2\underline{a_{21}} & a_{12} + 2\underline{a_{22}} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

the effect of left-multiplying this matrix

= adding $2 \times (2^{nd}$ row$)$ to the 1^{st} row.

e.g.
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 2a_{11} & 2a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

left-multiplying this matrix $\Rightarrow$ replace 1^{st} row by $2\times(1^{st}$ row$)$

e.g.
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} \\ a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$$

Elementary matrices	effect of left-multplying the matrix
$\begin{array}{cc} & \begin{matrix} & j & & & \end{matrix} \\ \begin{matrix} \\ \\ \\ i \\ \\ \end{matrix} & \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & a & & \ddots & \\ & & & & 1 \end{bmatrix} \end{array}$	replace the i-th row by (i-th row) + a(j-th row)
$\begin{array}{cc} & \begin{matrix} & & & i & & & \end{matrix} \\ i & \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & a\neq 0 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \end{array}$	multiply the (i-th row) by $a \neq 0$.
$\begin{array}{cc} & \begin{matrix} & & & i & & & & j & & \end{matrix} \\ \begin{matrix} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \end{matrix} & \begin{bmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 0 & & & & 1 & & \\ & & & & 1 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 1 & & & \\ & & & 1 & & & & 0 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & \ddots \end{bmatrix} \end{array}$	Swapping (ith row) & (jth row)

Rmk: What happen if we multiply these matrices on the right?

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & 2a_{11}+a_{12} \\ a_{21} & 2a_{21}+a_{22} \end{bmatrix}$$

$\rightsquigarrow$ certain column operations.

Rmk: elementary matrices are all invertible:

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & a & \ddots & \\ & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & -a & \ddots & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & a & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1/a & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}^{2} = I.$$

Thm A: $n\times n$ invertible $\Longleftrightarrow$ T_A is bijective.

("$\Longrightarrow$" was proved last time)

$$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Note: T_A is bijective $\Longleftrightarrow$ A has pivots in each row and column

$\Updownarrow$

the reduced echelon form of A is I_n

pf: T_A bijective $\Leftrightarrow$ the reduced echelon form of A is I_n,

i.e. $\exists\ E_1, \ldots, E_k$ elementary matrices.

s.t. $E_1 E_2 \cdots E_k A = I_n$

$$\Rightarrow \underbrace{E_1^{-1} E_1}_{= I} E_2 \cdots E_k A = E_1^{-1} I = E_1^{-1}$$

$$\Rightarrow E_2 E_3 \cdots E_k A = E_1^{-1}$$

$$\Rightarrow E_3 \cdots E_k A = E_2^{-1} E_1^{-1}$$

$$\vdots$$

$$A = E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1}$$

Since $E_1, \ldots, E_k$ invertible,

so are $E_1^{-1}, \ldots, E_k^{-1}$,

so is $E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1} = A$.

$\Rightarrow$ A is invertible. $\square$

($A^{-1} = E_1 E_2 \cdots E_k$:

$$A^{-1} A = (E_1 E_2 \cdots E_k)(E_k^{-1} \cdots E_1^{-1}) = I$$

$$A A^{-1} = (E_k^{-1} \cdots E_1^{-1})(E_1 \cdots E_k) = I \quad)$$

Rmk: Suppose $A_{n\times n}$ invertible, how to compute A^{-1}?

$$E_1 \cdots E_k A = I$$
$$E_1 \cdots E_k I = A^{-1}$$

$$\left[\begin{array}{c|c} A & I_n \end{array}\right]_{n \times 2n}$$

$$\Big\downarrow \text{ elementary row operations } E_1 \cdots E_k$$

left-multiplying $E_1 \cdots E_k$

$$\left[\begin{array}{c|c} \overbrace{I_n}^{E_1 \cdots E_k A} & A^{-1} \end{array}\right]$$

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e.g. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right]$$ is the inverse of

e.g. Assume $a \neq 0$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{bc}{a} & -c/a & 1 \end{array}\right]$$

If the matrix is invertible, then $d-\frac{bc}{a} \neq 0$.

$(ad-bc \neq 0)$

$\|$

determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Suppose $ad-bc \neq 0$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

A: 2×2, Suppose $\det(A) = ad-bc \neq 0$, then A is invertible,

and $$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thm (criterions for invertibility): A: $n\times n$.

The following are equivalent:

1) A is invertible. (We proved A invertible $\Leftrightarrow$ T_A is bijective

2) T_A is surjective $\Leftrightarrow$ A has pivots in each row & column)

2') A has pivot in each row.

2'') $A\vec{x}=\vec{b}$ has a sol^n $\forall\ \vec{b}\in\mathbb{R}^n$.

3) T_A is injective

3') A has pivot in each column.

3'') $A\vec{x}=\vec{0}$ has only the trivial sol^n

4) $\exists$ B: $n\times n$ s.t. $BA=I_n$

5) $\exists$ C: $n\times n$ s.t. $AC=I_n$.

pf. 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) by what we proved earlier, (A invertible $\Leftrightarrow$ T_A bijective)

& by the fact that since A is a square matrix, "A has pivots in each row" $\Leftrightarrow$ "A has pivot in each column".

1) $\Rightarrow$ 4) and 5): by definition

Remaining: 4) or 5) $\Rightarrow$ 1)2)3).

$\boxed{4) \Rightarrow 3'')}$: Suppose $A\vec{x}=\vec{0}$ $\Rightarrow$ $\underbrace{BA}_{I}\vec{x}=B\vec{0}=\vec{0}$

$\vec{x} \;//$

$\boxed{5) \Rightarrow 2(i))}$: $AC = I \Rightarrow A\boxed{C\vec{b}} = \vec{b}$

$C\vec{b}$ is a sol^n to $A\vec{x} = \vec{b}$.

$\square$

Def: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is <u>invertible</u> if $\exists$ a function $S: \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$T(S(\vec{x})) = \vec{x} = S(T(\vec{x})) \quad \forall \vec{x} \in \mathbb{R}^n$$

i.e. $T \circ S = id_{\mathbb{R}^n} = S \circ T$

Ex: Suppose $T = T_A$ for some A: $n \times n$.

Then
- T is invertible $\Leftrightarrow$ A is invertible.
- In this case, $S(\vec{x}) = A^{-1}\vec{x}$ is the unique function s.t. $T \circ S = id_{\mathbb{R}^n} = S \circ T$.

Determinant (of square matrix)

- 1×1: $[a_{11}]$, $\det([a_{11}]) := a_{11}$
- 2×2: $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$\det(A) \neq 0 \Leftrightarrow A$ is invertible

$\det(AB) = \det(A)\det(B)$