3/10/2020

§ Uniform continuity

Recall: $f : (X, d_X) \to (Y, d_Y)$ is unif. conti. on $E \subset X$

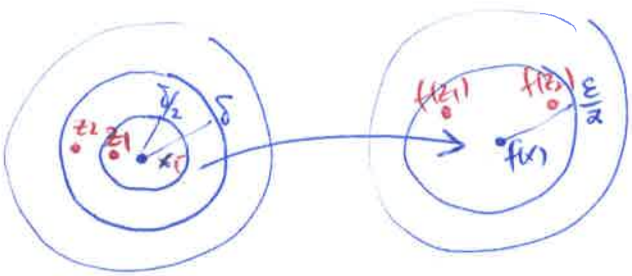
if $\forall \varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon) > 0$ s.t. $x, y \in E$, $d(x, y) < \delta$ $\Rightarrow$ $d(f(x), f(y)) < \varepsilon$.

Thm $f : (X, d_X) \to (Y, d_Y)$ conti., $E \subset X$ cpt.

$\Rightarrow$ f is unif. conti. on E.

pf: $\forall \varepsilon > 0$, $\forall x \in E$, $\exists\ \delta_x > 0$ s.t. $d(x', x) < \delta_x \Rightarrow d(f(x'), f(x)) < \frac{\varepsilon}{2}$.

X



Consider open cover $\{ B_{\frac{\delta_x}{2}}(x) \}_{x \in E}$ of E.

E cpt. $\Rightarrow$ $E \subset \left(B_{\frac{1}{2}\delta_{x_1}}(x_1) \cup \cdots \cup B_{\frac{1}{2}\delta_{x_n}}(x_n) \right)$ for some $x_1, \ldots, x_n \in E$.

Define $\delta := \min \{ \frac{1}{2}\delta_{x_1}, \ldots, \frac{1}{2}\delta_{x_n} \} > 0$.

Claim: $\forall z_1, z_2 \in E$ with $d(z_1, z_2) < \delta \rightarrow d(f(z_1), f(z_2)) < \varepsilon$.

pf $z_1 \in B_{\frac{1}{2}\delta_{x_i}}(x_i)$ for some i: $(1 \le i \le n)$.

Then $d(z_2, x_i) \le d(z_2, z_1) + d(z_1, x_i) < \delta + \frac{1}{2}\delta_{x_i} \le \delta_{x_i}$

$\Rightarrow$ Both $z_1, z_2 \in B_{\delta_{x_i}}(x_i)$.

$\Rightarrow$ $f(z_1), f(z_2) \in B_{\frac{\varepsilon}{2}}(f(x_i))$.

$\Rightarrow$ $d(f(z_1), f(z_2)) \le d(f(z_1), f(x_i)) + d(f(x_i), f(z_2)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$

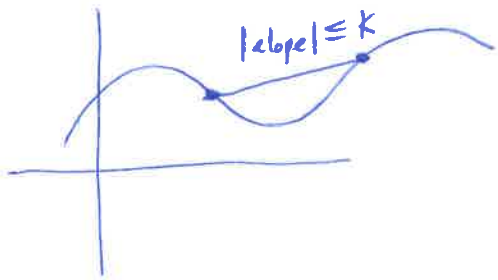
Def $f: (X, d_X) \longrightarrow (Y, d_Y)$ is Lipschitz conti. on $E \subset X$ if

$\exists\ K > 0$ s.t. $d_Y(f(x), f(y)) \leq K \cdot d_X(x, y)$ $\forall x, y \in E$.

Prop: $f: X \longrightarrow Y$ is Lip. conti on E $\Rightarrow$ unif. conti. on E.

pf Take $\delta = \frac{\varepsilon}{K}$. $\square$

e.g. $f: I \longrightarrow \mathbb{R}$ Lip. conti. : $|f(x) - f(y)| \leq K|x - y|$.



Rmk We'll show that if f is differentiable, then ~~with bounded derivatives,~~ ~~then~~ f is Lip. conti. (mean value thm.)

$\Updownarrow$

f has bounded derivatives.

However, $\exists\ f$ is differentiable w/ unbounded derivatives,

but f is unif. conti. (HW).

Convergence of seq. of fcns.

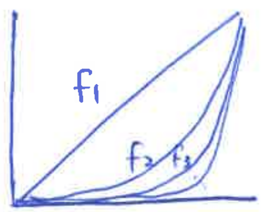
Weakest version: pointwise convergence.

Def: X: a set. (f_n): seq. of fcns. $f_n: X \longrightarrow \mathbb{R}$.

We say $(f_n) \longrightarrow f$ converges pointwisely to $f: X \longrightarrow \mathbb{R}$

if $\forall\, x_0 \in X$, we have $\lim f_n(x_0) = f(x_0)$.

e.g. $f_n: [0,1] \longrightarrow \mathbb{R}$
$x \longmapsto x^n$.



- For $x \in [0,1)$. $\lim f_n(x) = \lim x^n = 0$.
- For $x=1$, $\lim f_n(1) = \lim 1^n = 1$.

So $(f_n) \longrightarrow f$ pointwise, where $f(x) = \begin{cases} 0 & x \in [0,1). \\ 1 & x = 1. \end{cases}$

Bad News: pointwise limit of seq. of conti. fcns may not be conti.

More bad news:

e.g. $f_n(x) = \frac{1}{n}\sin(n^2 x): \mathbb{R} \longrightarrow \mathbb{R}$. Pointwise limit: $f(x) \equiv 0$.

$f_n'(x) = n\cos(n^2 x): \mathbb{R} \longrightarrow \mathbb{R}$.

For most $x \in \mathbb{R}$, $(f_n'(x))$ is unbounded,

so (f_n') doesn't conv. pointwise.

e.g. $f_n(x) = \dfrac{2n^2x}{(1+n^2x^2)^2}$ on $[0,1]$, Pointwise limit. $f \equiv 0$.

$$\int_0^1 f_n(x)\,dx = \int_0^1 \frac{2n^2x}{(\underbrace{1+n^2x^2}_{u})^2}\,dx = \int_1^{1+n^2} \frac{du}{u^2} = 1 - \frac{1}{1+n^2}$$

So $\lim \int_0^1 f_n(x)\,dx = 1, \neq \int_0^1 f(x)\,dx = 0.$

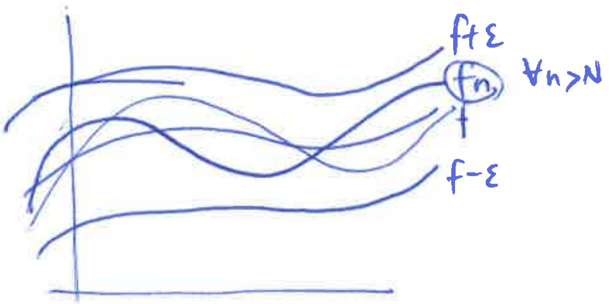
Need a stronger notion of convergence.

$(f_n) \to f$ pointwise $\Leftrightarrow$ $\forall \varepsilon > 0, \forall x_0 \in X, \exists N = N(\varepsilon, x_0) > 0$ s.t. $|f_n(x_0) - f(x_0)| < \varepsilon \quad \forall n > N.$

Def. X: a set. (f_n) seq. of fcns. $f_n : X \to \mathbb{R}$.

We say (f_n) converges uniformly to $f: X \to \mathbb{R}$

if $\forall \varepsilon > 0, \exists \underset{N(\varepsilon)}{\underset{\|}{N}} > 0$ s.t. $|f_n(x_0) - f(x_0)| < \varepsilon \quad \forall n > N \ \forall x_0 \in X.$



- unif. $\Rightarrow$ pointwise.

No bad news for unif. conv.:

Thm 1) (X, d) - metric space. $f_n : X \to \mathbb{R}$ conti.

$(f_n) \to f$ unif. conv.

Then f is also conti.

Thm 2) $f_n : [a,b] \longrightarrow \mathbb{R}$ differentiable, f_n' conti.

If $(f_n) \longrightarrow f$ unif. and $(f_n') \longrightarrow g$ unif.

Then f is differentiable and $f' = g$.

3) $f_n : [a,b] \longrightarrow \mathbb{R}$ integrable, $(f_n) \longrightarrow f$ unif.

Then f is integrable and $\lim \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

e.g. $f_n : [0,1] \longrightarrow \mathbb{R}$, $x \longmapsto x^n$. $\quad f_n \longrightarrow f = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

Uniformly??

$\boxed{\text{No}}$ Let $\varepsilon = \frac{1}{2}$. $\forall N > 0$,

Want to find $x \in [0,1]$, $n > N$ s.t. $|f_n(x) - f(x)| \geq \frac{1}{2}$.

Actually, $\forall n > 0$, we can find $x \in [0,1]$ s.t. $|f_n(x) - f(x)| \geq \frac{1}{2}$.

Since x^n is conti. $f_n(0) = 0$, $f_n(1) = 1$.

By IVT, $\exists\, x \in (0,1)$ s.t. $f_n(x) = \frac{1}{2}$.

$\Rightarrow |f_n(x) - \underbrace{f(x)}_{= 0}| = \frac{1}{2}$. $\square$

e.g. $f_n(x) = \frac{1}{n}\sin(nx)$. $: \mathbb{R} \longrightarrow \mathbb{R}$. $\quad (f_n) \to f \equiv 0$ unif.

$\forall \varepsilon > 0$, take $N = \frac{1}{\varepsilon} > 0$, Then

$\forall n > N$, $\forall x \in \mathbb{R}$, we have $|f_n(x) - f(x)| = \left|\frac{1}{n}\sin(nx)\right| \leq \frac{1}{n} < \varepsilon$. $\square$

Alternative interpretation of unif. conv.

X: a set.

Define a metric space $\mathcal{B}(X)$ as follows:

- $\mathcal{B}(X) = \{ \text{bounded fcns } f : X \longrightarrow \mathbb{R} \}$.
- $d(f_1, f_2) := \sup_{x \in X} |f_1(x) - f_2(x)|$.

Lemma (f_n) seq. of bounded fcns. $f_n : X \to \mathbb{R}$ converge unif. to f.

$\Leftrightarrow$ (f_n) conv. to f in $\mathcal{B}(X)$.

pf ① The unif. limit f of bounded fcns is also bounded (HW)

So $f \in \mathcal{B}(X)$.

② $(f_n) \to f$ in $\mathcal{B}(X)$ $\Leftrightarrow$ $\forall \varepsilon > 0, \exists N > 0$ s.t. $d_{\mathcal{B}(X)}(f_n, f) < \varepsilon \ \forall n > N$

$\Leftrightarrow$ $\forall \varepsilon > 0, \exists N > 0$ s.t. $|f_n(x) - f(x)| < \varepsilon \ \forall n > N, \forall x \in X$.

$\square$

pf of Thm(1) ($\frac{\varepsilon}{3}$- trick).

WTS: $\forall x_0 \in X, \forall \varepsilon > 0, \exists \delta > 0$ s.t. $d_X(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Trick: $|f(x) - f(x_0)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$.

Control by $(f_n) \to f$ unif.

Control by f_n conti.

- $\exists N > 0$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{3} \ \forall n > N, \forall x \in X$.
- Pick any $n > N$. Since f_n conti.

  $\exists \delta > 0$ s.t. $|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3} \ \forall d(x, x_0) < \delta$.

$\square$