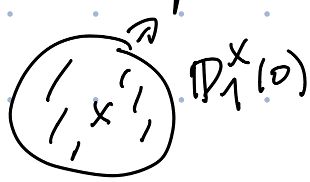
Today: classification of modular forms/functions.

Thm 1: Any modular form is a polynomial in $E_4(z)$ and $E_6(z)$.

Thm 2: Any mero. modular fcn. is a rational fcn in $j(z)$

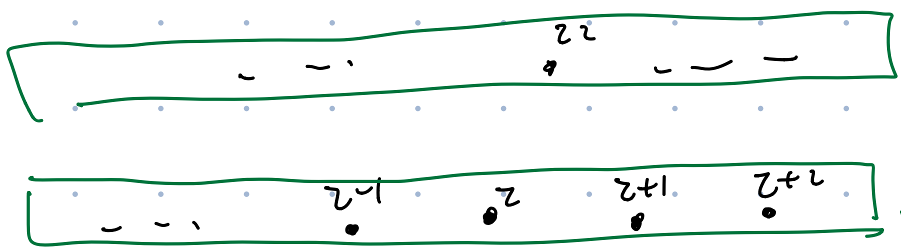
$E_{2k}(z)$ modular forms of wt $2k$, where $k \geq 2$.

$\underline{E_{2k}(q)}$ $\qquad q = e^{2\pi i z}$



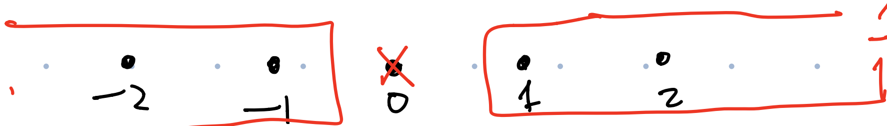
Power series expansion of E_{2k} at $q=0$:

(you can find in the textbook)



$\sum_{n\in\mathbb{Z}} \frac{1}{(2z+n)^{2k}}$

$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^{2k}}$



$\frac{1}{1^{2k}} + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \cdots = \zeta(2k)$

$\frac{1}{(-1)^{2k}} + \frac{1}{(-2)^{2k}} + \cdots$
$\quad \| $
$\zeta(2k)$

$$\boxed{\sum_{(m,n)\neq(0,0)} \frac{1}{(m+nz)^{2k}}}$$

For each $\sum_{n\in\mathbb{Z}} \frac{1}{(lz+n)^{2k}}$, we can use

$$\frac{\pi}{\tan(\pi l z)} = \frac{1}{lz} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{lz-n} + \frac{1}{n} \right).$$

$$E_{2k}(q) = 2\zeta(2k)\left[1 - \frac{4k}{B_{2k}} \sum_{r\geq 1} \sigma_{2k-1}(r)\, q^r\right]$$

where $$\frac{x}{e^x - 1} = \sum_{k\geq 0} \frac{B_k\, x^k}{k!},$$ Bernoulli #

k	0	1	2	3	4	5	6	7	8	9	10	11	12	…
B_k	1	$-\frac{1}{2}$	$1/6$	0	$-1/30$	0	$1/42$	0	$-1/30$	0	$5/66$	0	$\frac{691}{2730}$	…

$$\sigma_{2k-1}(r) = \sum_{d\mid r} d^{2k-1}$$

$$\tilde{E}_4(q) = 1 + 240 \sum_{r\geq 1} \sigma_3(r)\, q^r$$

$$\tilde{E}_6(q) = 1 - 504 \sum_{r\geq 1} \sigma_5(r)\, q^r$$

$$\tilde{E}_8(q) = 1 + 480 \sum_{r\geq 1} \sigma_7(r)\, q^r$$

$$\tilde{E}_{10}(q) = 1 - 264 \sum_{r\geq 1} \sigma_9(r)\, q^r$$

$$\tilde{E}_{12}(q) = 1 + \frac{65520}{691} \sum_{r\geq 1} \sigma_{11}(r)\, q^r.$$

Recall: If f_1, f_2 are modular forms of the same wt, then f_1/f_2 is a modular fcn.

$$\tilde{E}_4(q)^2 = \Big(1 + 240 \sum_{r\geq 1} \sigma_3(r)\, q^r\Big)^2$$

$$= \big(1 + 240(1^3)q + 240(1^3+2^3)q^2 + 240(1^3+3^3)q^3 + \cdots\big)^2$$

$$= (1 + 240q + 2160q^2 + 6720q^3 + \cdots)^2$$

$$= 1 + 480q + 61920q^2 + 1{,}050{,}240q^3 + \cdots$$

$$\tilde{E}_8(q) = 1 + 480 \sum_{r\geq 1} \sigma_7(r)\, q^r$$

$$= 1 + 480q + 480(1^7+2^7)q^2 + 480(1^7+3^7)q^3 + \cdots$$

$$= 1 + 480q + 61920q^2 + 1{,}050{,}240q^3 + \cdots$$

It turns out that $\boxed{\tilde{E}_4^2(q) = \tilde{E}_8(q).}$!!

$$\Rightarrow \tilde{E}_8(q) = 1 + 480 \sum \sigma_7(r)\, q^r$$

$$\tilde{E}_4(q)^2 = \Big(1 + 240 \sum \sigma_3(r) q^r\Big)\Big(1 + 240 \sum \sigma_3(s) q^s\Big)$$

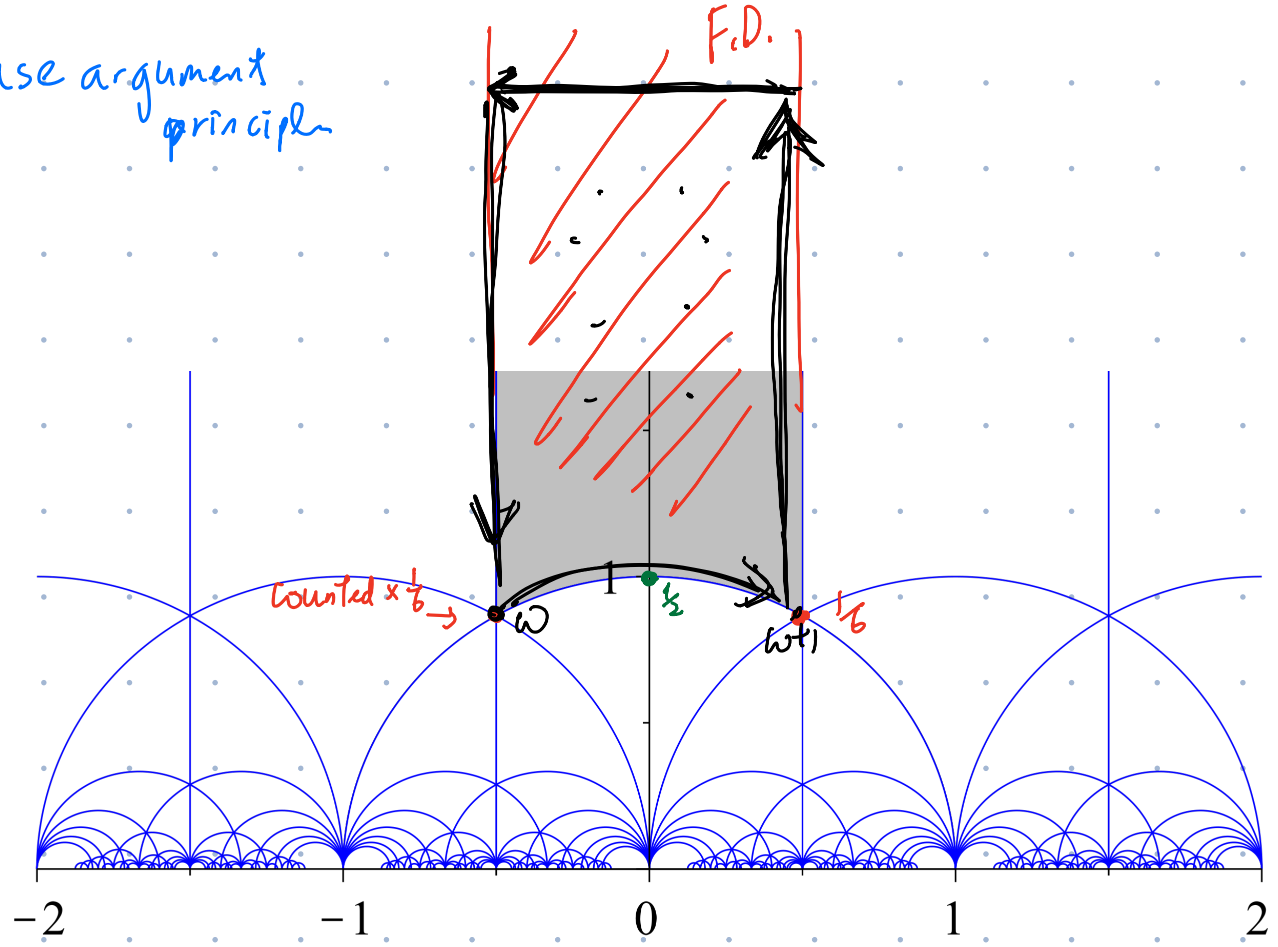
$$= 1 + 240 \sum_{n\geq 1} \Big(2\sigma_3(n) + 240 \sum_{\substack{p+q=n \\ p,q\geq 1}} \sigma_3(p)\sigma_3(q)\Big) q^n$$

$$\boxed{\sigma_7(r) = \sigma_3(r) + 120 \sum_{\substack{p+q=r \\ p,q\geq 1}} \sigma_3(p)\sigma_3(q)}$$

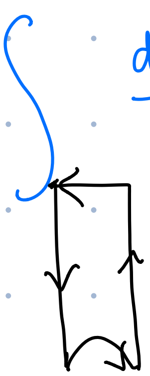
Key thm: f = modular form of wt k.

zeros of f in the F.D. (including $i\infty$) is $k/12$.
(of $SL_2\mathbb{Z} \curvearrowright \mathbb{H}$)

pf: use argument principle

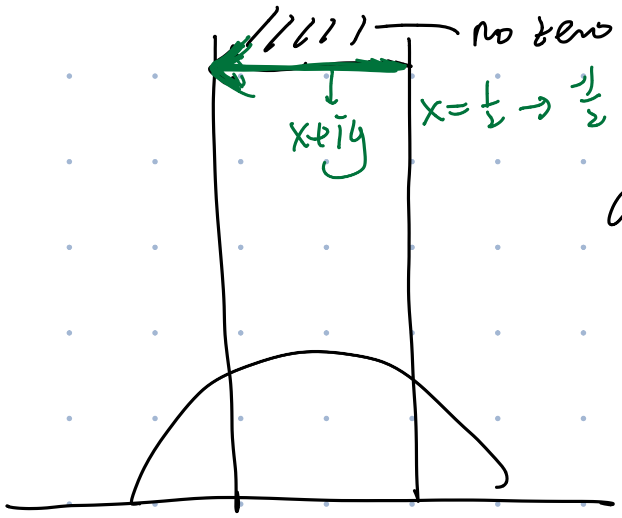


$\frac{1}{2\pi i}\int \frac{df}{f}$ + order of zero of f at $i\infty$ = # zero of f in the F.D.



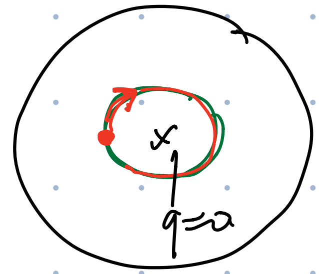
what we want to compute.

• $\int \frac{df}{f} + \int \frac{df}{f} = 0$ b/c $f(z) = f(z+1)$.

• 

$$e^{2\pi i(x+iy)} = e^{-2\pi y} e^{2\pi i x}$$

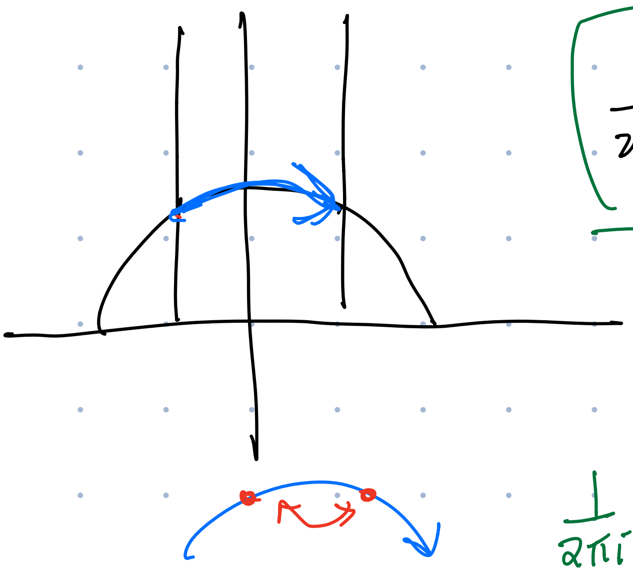
$$q = e^{2\pi i \tau} \longrightarrow$$



$$\frac{1}{2\pi i}\int \frac{df}{f} = \frac{1}{2\pi i}\int_{q=0} \frac{df}{f}$$

$=$ − order of zero of f at $q=0$ (or $\tau = i\infty$)

• So, what we need to show is:



$$\tau \mapsto -\frac{1}{\tau}$$

$$\boxed{f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)}$$

$$\boxed{\frac{1}{2\pi i}\int \frac{df}{f}} = \frac{k}{12}$$

If f is modular form of wt k.

$$\Downarrow$$

$$\frac{1}{2\pi i}\left(\int \frac{df}{f} - \int \frac{df}{f}\right) = \frac{k}{12}. \qquad \square$$

$$\oint \frac{df}{f} = \int \frac{df(-\frac{1}{z})}{f(-\frac{1}{z})} = \int \frac{d(z^k f(z))}{z^k f(z)}$$

$$= \int \frac{k z^{k-1} f(z) + z^k f'(z)}{z^k f(z)}\, dz$$

$$= \underbrace{\int \frac{k}{z}\, dz}_{= -\frac{2\pi i}{12}\cdot k} + \int \frac{f'}{f}\, dz$$

$\frac{1}{12} \times$ unit circle.

Ex: The only modular forms of wt 0 (holo. in $\mathbb{H}$, $i\infty$) are the constant fcns.

---

wt	# zeros in the FD.	modular forms	dimension.
0	0	const.	1.
2	1/6	NO	0
4	1/3 (simple zeros at $\omega, \omega+1$)	E_4.	1

f_1, f_2 both have simple zeros at $\omega, \omega+1$, and they don't have any other zeros in F.D.

Suppose f_1, f_2 mod. forms of wt 4.

f_1/f_2 holo. on $\mathbb{H} \cup i\infty$

6	$1/2$ (simple zero at i)	E_6	1
8	$2/3$ (double zeros at $\omega, \omega+1$)	$\tilde{E}_8 = \tilde{E}_4^2$	1
10	$5/6$ (simple zeros at $\omega, \omega+1, i$)	$\tilde{E}_{10} = \tilde{E}_4 \tilde{E}_6$	1
12	1	$E_4^3, E_6^2, E_{12}, \ldots$	?? 2

$\tilde{E}_4^3 - \tilde{E}_6^2$: q-expansion has const term $= 0$.

$\Rightarrow$ $\tilde{E}_4^3 - \tilde{E}_6^2$ has a zero at $i\infty$

$\Rightarrow$ it's nonvanishing on $\mathbb{H}$.

$$\Delta := \frac{\tilde{E}_4^3 - \tilde{E}_6^2}{1728}$$

modular form of wt 12, has zero at $i\infty$, nonvanishing on $\mathbb{H}$.

Claim: dim of (modular form of wt 12) = 2.

pf given any f modular form of wt 12,

$$f(q) = a_0 + a_1 q + \cdots$$

Consider $\boxed{f - a_0 \cdot \tilde{E}_4^3}$ → has zero at $i\infty$

$\Rightarrow$ $f - a_0 \tilde{E}_4^3 = (\text{const}) . \Delta$

$\Rightarrow$ $f \in \operatorname{Span}\{\tilde{E}_4^3, \Delta\}$. $\square$

Thm Any modular form is a polynomial in E_4, E_6.

pf: f modular form of wt $\underset{\text{even}}{k} > 12$.

- Find some $\underline{\tilde{E}_4^a \tilde{E}_6^b}$ s.t. it's a modular form of wt k.
- $f - * \tilde{E}_4^a \tilde{E}_6^b$ has a zero at $i\infty$

$\Rightarrow$ $\dfrac{f - * \tilde{E}_4^a \tilde{E}_6^b}{\Delta}$ modular form of wt $k-12$.

$\square$

Back to modular fcns:

Find all mero. modular fcns on $\mathbb{H}\cup\{i\infty\}$
$$\left(f(z) = f(z+1) = f\left(\tfrac{-1}{z}\right) \right)$$

Consider $\quad \tilde{j}(z) = \dfrac{\tilde{E}_4(z)^3}{\Delta(z)}$, $\quad \Delta(z) = \dfrac{\tilde{E}_4^3 - \tilde{E}_6^2}{1728}$

- holo. on $\mathbb{H}$, has a simple pole at $i\infty$
- $\tilde{j}(z)$ has zeros of order 3 at $\omega, \omega+1$
- $\tilde{j}(z) - 1728 = \dfrac{\tilde{E}_6^2}{\Delta(z)}$ has zero of order 2 at i

around $q=0$,
- $\tilde{j}(q) = q^{-1} + 744 + 196884\, q + \cdots\cdots$

Thm Any mero. modular fcn is a rational fcn of $\tilde{j}(z)$.
(i.e. $= \dfrac{p(\tilde{j}(z))}{q(\tilde{j}(z))}$ where p, q poly.)

Pf: f mero. modular fcn.

- get rid of poles of f in $\mathbb{H}$ by multiply $\tilde{j}(z) - \tilde{j}(z_0)$

$\Rightarrow$ reduce to the case where f is holo. in $\mathbb{H}$.

- around $q=0$, $\quad f(q) = a_{-n}\, q^{-n} + \cdots\cdots$

Consider $\boxed{a_{-n} j(q)^n} = a_{-n} q^{-n} + \cdots\cdots$

$$\boxed{f(q) - a_{-n} j(q)^n} = b_{-n+1} q^{-n+1} \cdots\cdots$$

modular fcn, holo. in $\mathbb{H}$

$\vdots$

$$\boxed{f(q) - a_{-n} j(q)^n - a_{-n+1} j(q)^{n-1} - \cdots - a_{-1} j(q)}$$

$\parallel$

holo. at $i\infty$

$\equiv$

const.