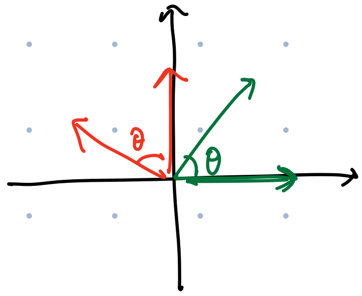
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Counterclockwise rotation by θ.

$$\det(A - \lambda I) = \det \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}$$

$$= \lambda^2 - 2\cos\theta\,\lambda + 1$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{-4\sin^2\theta}}{2} = \cos\theta \pm i\sin\theta$$

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad \text{eigenvalues} = a \pm ib$$

Let $r = \sqrt{a^2+b^2}$, so $\exists\, \theta$ s.t. $a = r\cos\theta$, $b = r\sin\theta$

$$\| $$

$$\begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = \begin{bmatrix} r & \\ & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Thm: $A \in M_{2\times 2}(\mathbb{R})$, eigenvalues of $A \in \mathbb{C} \setminus \mathbb{R}$

Then $A = P \begin{bmatrix} a & -b \\ b & a \end{bmatrix} P^{-1}$ for some $a, b \in \mathbb{R}$, $P \in M_{2\times 2}(\mathbb{R})$ invertible

Rmk: Such A is diagonalizable $/\mathbb{C}$: since if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is an eigenvalue of A, then so is $\bar{\lambda} \neq \lambda$.,

i.e. $A = Q \begin{bmatrix} \lambda & \\ & \bar{\lambda} \end{bmatrix} Q^{-1}$ for some $Q \in M_{2\times 2}(\mathbb{C})$

Pf: Say $a \pm ib$ are eigenvalues of A, where $a, b \in \mathbb{R}$, $b \neq 0$

$\exists\, \vec{v} \in \mathbb{C}^2$ s.t. $A\vec{v} = (a-ib)\vec{v}$.
$\vec{v} \neq \vec{0}$

$$A\vec{v} = A(\operatorname{Re}\vec{v} + i\operatorname{Im}\vec{v}) = A(\operatorname{Re}\vec{v}) + i\,A(\operatorname{Im}\vec{v})$$

$$(a-ib)\vec{v} = (a-ib)(\operatorname{Re}\vec{v} + i\operatorname{Im}\vec{v}) = (a\operatorname{Re}\vec{v} + b\operatorname{Im}\vec{v}) + i(a\operatorname{Im}\vec{v} - b\operatorname{Re}\vec{v})$$

$$\Rightarrow \begin{cases} A(\operatorname{Re}\vec{v}) = a\operatorname{Re}\vec{v} + b\operatorname{Im}\vec{v} \\ A(\operatorname{Im}\vec{v}) = a\operatorname{Im}\vec{v} - b\operatorname{Re}\vec{v} \end{cases}$$

$$\Rightarrow A\begin{bmatrix} | & | \\ \operatorname{Re}\vec{v} & \operatorname{Im}\vec{v} \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \operatorname{Re}\vec{v} & \operatorname{Im}\vec{v} \\ | & | \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

<u>Why $\begin{bmatrix} | & | \\ \operatorname{Re}\vec{v} & \operatorname{Im}\vec{v} \\ | & | \end{bmatrix}$ is invertible?</u>

- $\vec{v} = \operatorname{Re}\vec{v} + i\operatorname{Im}\vec{v}$ is an eigenvector of $a-ib$,
  $\overline{\vec{v}} = \operatorname{Re}\vec{v} - i\operatorname{Im}\vec{v}$ ———— $a+ib$.
- If $\{\operatorname{Re}\vec{v}, \operatorname{Im}\vec{v}\}$ were l.d., then
  $\operatorname{Span}_{\mathbb{C}}\{\operatorname{Re}\vec{v} + i\operatorname{Im}\vec{v}\} = \operatorname{Span}_{\mathbb{C}}\{\operatorname{Re}\vec{v} - i\operatorname{Im}\vec{v}\}$.

- Suppose $\{\operatorname{Re}\vec{v}, \operatorname{Im}\vec{v}\}$ is l.d., so $\exists\, r \in \mathbb{R}$ s.t. $\operatorname{Re}\vec{v} = r\operatorname{Im}\vec{v}$
  $\{\operatorname{Re}\vec{v}, \operatorname{Im}\vec{v}\} \subset \mathbb{R}^2$

$$\Rightarrow A(\operatorname{Im}\vec{v}) = a\operatorname{Im}\vec{v} - br\operatorname{Im}\vec{v} = \underbrace{(a-br)}_{\in\mathbb{R}}\operatorname{Im}\vec{v}$$

contradiction.

Rmk. more generally, if $A \in M_{n\times n}(\mathbb{R})$ and suppose A is diagonalizable $/\mathbb{C}$

Then $A \underset{\text{similar}/\mathbb{R}}{\sim} \begin{bmatrix} \mu_1 & & & & & \\ & \ddots & & & & \\ & & \mu_k & & & \\ & & & \boxed{\begin{matrix} a_1 & -b_1 \\ b_1 & a_1 \end{matrix}} & & \\ & & & & \boxed{\begin{matrix} a_2 & -b_2 \\ b_2 & a_2 \end{matrix}} & \\ & & & & & \ddots \end{bmatrix}$

$\underbrace{\mu_1, \dots, \mu_k}_{\in \mathbb{R}} \to$ real eigenvalues of A

$a_j \pm i b_j \in \mathbb{C}$ (complex) eigenvalues of A

Application: Dynamical system. $f: X \to X$

$$X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} X \to \cdots$$

"study long-term behavior of $f^{(n)}$ as $n \to \infty$".

e.g. $A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$. $\quad T_A : \mathbb{R}^2 \to \mathbb{R}^2$

$\vec{x}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$ $\quad$ Q: $\lim_{n\to\infty} T_A^n(\vec{x}_0) = ?$

$$= \lim_{n\to\infty} A^n \vec{x}_0$$

$(0.95-\lambda)(0.97-\lambda) - 0.03 \cdot 0.05 = \lambda^2 - 1.92\lambda + 0.92$

$= (\lambda - 1)(\lambda - 0.92)$

$\mathrm{Nul}(A - I) = \mathrm{Nul}\begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right\}$

$\mathrm{Nul}(A - 0.92 I) = \mathrm{Nul}\begin{bmatrix} 0.03 & 0.03 \\ 0.05 & 0.05 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$$\Rightarrow A = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0.92 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1}$$

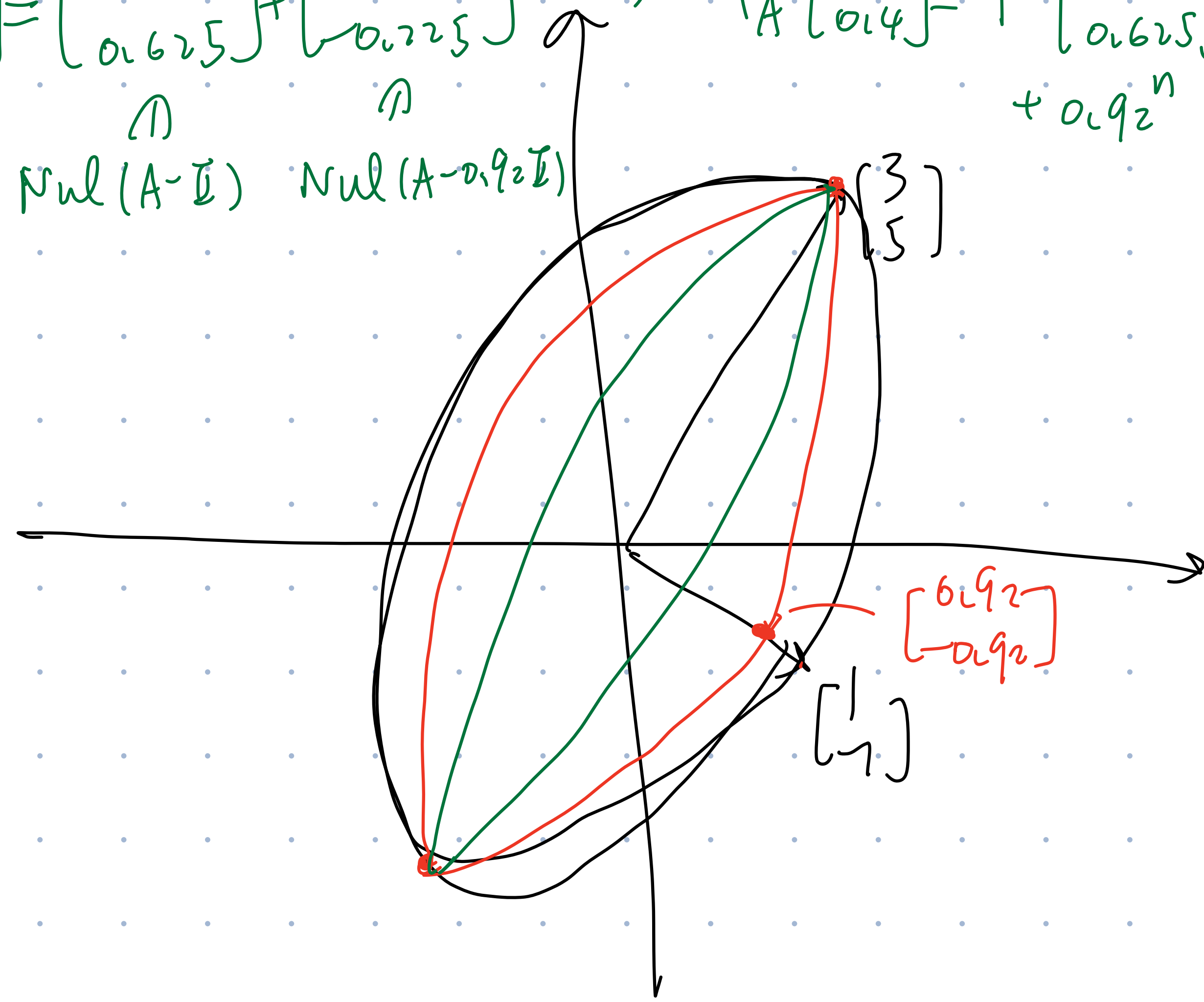
$$A^n = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0.92^n \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1}$$

$0.92^n \to 0$ as $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} A^n \vec{x}_0 = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$$

$$\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}}_{\in \, Nul(A - I)} + \underbrace{\begin{bmatrix} 0.225 \\ -0.225 \end{bmatrix}}_{\in \, Nul(A - 0.92 I)} \Rightarrow T_A^n \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = 1^n \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix} + 0.92^n \begin{bmatrix} 0.225 \\ -0.225 \end{bmatrix}$$



Recap A: $n \times n$ diagonalizable, $\{\lambda_1, \dots, \lambda_k\}$ distinct eigenvalues.

1) $\mathbb{C}^n = \mathrm{Nul}(A - \lambda_1 I) \oplus \dots \oplus \mathrm{Nul}(A - \lambda_k I)$.

2) $\dim \mathrm{Nul}(A - \lambda_i I) = \mathrm{mult}(\lambda_i) \quad \forall i$.

3) $T_A(\mathrm{Nul}(A - \lambda_i I)) \subseteq \mathrm{Nul}(A - \lambda_i I)$,

i.e. eigenspaces are invariant under T_A.

$$\left((A - \lambda_i I)\vec{v} = \vec{0} \Rightarrow A(A - \lambda_i I)\vec{v} = \vec{0} \right.$$
$$\left. A(A-\lambda_i I)\vec{v} = A^2\vec{v} - \lambda_i A\vec{v} = (A - \lambda_i I)A\vec{v} \right)$$

Def (generalized eigenspace of an eigenvalue λ)

$$V_\lambda^{gen} := \{ \vec{v} \in \mathbb{C}^n \mid (A - \lambda I)^k \vec{v} = \vec{0} \text{ for some } k \geq 1 \}.$$

Rmk.
- $0 \subseteq \mathrm{Nul}(A - \lambda I) \subseteq \mathrm{Nul}(A - \lambda I)^2 \subseteq \mathrm{Nul}(A - \lambda I)^3 \subseteq \dots$

A vector is in V_λ^{gen} if and only if it lies in one of the vector spaces in the chain above.

- $\exists\, l \geq 1$ s.t. $\mathrm{Nul}(A - \lambda I)^l = \mathrm{Nul}(A - \lambda I)^{l+1} = \mathrm{Nul}(A - \lambda I)^{l+2} = \dots$

e.g. $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ not diagonalizable, b/c 2 is the only eigenvalue

but $\mathrm{Nul}(A - 2I) = \mathrm{Nul}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \neq \mathbb{C}^2$

$\boxed{V_2^{gen}}$ $\quad (A - 2I)^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\mathrm{Nul}(A - 2I)^2 = \mathbb{C}^2$

Thm A: $n \times n$, $\{\lambda_1, \dots, \lambda_k\}$ distinct eigenvalues

1) $\mathbb{C}^n = V^{gen}_{\lambda_1} \oplus \dots \oplus V^{gen}_{\lambda_k}$

2) $\dim V^{gen}_{\lambda_i} = \text{mult}(\lambda_i) \quad \forall i$

3) $T_A\left(V^{gen}_{\lambda_i}\right) \subseteq V^{gen}_{\lambda_i} \quad \forall i.$

Suppose:

$$A = P \begin{bmatrix} \begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix} & & & \\ & \lambda_2 & & \\ & & \boxed{\begin{matrix} \lambda_3 & 1 \\ & \lambda_3 \end{matrix}} & \\ & & & \boxed{\ddots} \end{bmatrix} P^{-1}$$

$$\underbrace{\begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \lambda_1 & 1 \\ & & & \lambda_1 \end{bmatrix}}_{D}$$

(Recall: if $A = P \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} P^{-1}$, then the columns of P are eigenvectors.)

$$AP = P \begin{bmatrix} \begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix} & \\ & \ddots \end{bmatrix}$$

$$A \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \cdots \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \cdots \end{bmatrix} \begin{bmatrix} \begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix} & \\ & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | & \\ \lambda_1 \vec{v}_1 & \vec{v}_1 + \lambda_1 \vec{v}_2 & \vec{v}_2 + \lambda_1 \vec{v}_3 & \cdots \\ | & | & | & \end{bmatrix}$$

$$\Rightarrow \quad A\underset{\substack{\uparrow \\ \text{eigenvector of } \lambda_1}}{\vec{v}_1} = \lambda_1 \vec{v}_1, \quad A\underset{\uparrow}{\vec{v}_2} = \vec{v}_1 + \lambda_1 \vec{v}_2, \quad A\underset{\uparrow}{\vec{v}_3} = \vec{v}_2 + \lambda_1 \vec{v}_3.$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$(A - \lambda_1 I)\vec{v}_2 = \vec{v}_1 \qquad (A - \lambda_1 I)\vec{v}_3 = \vec{v}_2$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$(A - \lambda_1 I)^2 \vec{v}_2 = (A - \lambda_1 I)\vec{v}_1 = \vec{0} \qquad (A - \lambda_1 I)^3 \vec{v}_3 = \vec{0}.$$

$$\vec{v}_1, \vec{v}_2, \vec{v}_3 \in V^{gen}_{\lambda_1}.$$

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pf of 3) : $\underline{T_A(V^{gen}_\lambda) \subseteq V^{gen}_\lambda}$

i.e. $\forall\ \vec{v} \in V^{gen}_\lambda$, we want to show: $\underset{\substack{\parallel \\ A\vec{v}}}{T_A \vec{v}} \in V^{gen}_\lambda$

$$\Downarrow$$

$\exists\ k \geq 1$ s.t.

$$(A - \lambda I)^k \vec{v} = \vec{0} \Rightarrow A(A - \lambda I)^k \vec{v} = \vec{0}$$

$$\parallel$$

$$(A - \lambda I)^k A\vec{v}.$$
