#1:

Let $P = A+B$. Then

$$A(A+B)^{-1}B = AP^{-1}(P-A)$$
$$= A - AP^{-1}A$$
$$= (P-A)P^{-1}A = B(A+B)^{-1}A. \quad \square$$

#2:

Consider the restriction of T_A to $\mathrm{Im}(T_B)$:

we have a surjective linear map: $T_A\big|_{\mathrm{Im}(T_B)} : \mathrm{Im}(T_B) \longrightarrow \mathrm{Im}(T_{AB})$

By rank-nullity thm,

$$\mathrm{rank}(B) - \mathrm{rank}(AB) = \dim \ker\left(T_A\big|_{\mathrm{Im}(T_B)}\right)$$
$$= \dim\left(\ker(T_A) \cap \mathrm{Im}(T_B)\right)$$

Similarly, consider the restriction of T_A to $\mathrm{Im}(T_{BC})$;

one gets:

$$\mathrm{rank}(BC) - \mathrm{rank}(ABC) = \dim \ker\left(T_A\big|_{\mathrm{Im}(T_{BC})}\right)$$
$$= \dim\left(\ker(T_A) \cap \mathrm{Im}(T_{BC})\right).$$

The desired inequality follows from:

$$\ker(T_A) \cap \mathrm{Im}(T_{BC}) \subseteq \ker(T_A) \cap \mathrm{Im}(T_B). \quad \square$$

#3: Choose a basis of $\mathrm{Mat}_{n\times n}(\mathbb{R})$:

$$\{e_{11}, e_{21}, \dots, e_{n1}, e_{12}, e_{22}, \dots, e_{n2}, \dots, e_{1n}, \dots, e_{nn}\},$$

where e_{ij} is the $n \times n$ matrix:

$$\begin{array}{cc} & j\text{th column} \\ i\text{th row} & \begin{bmatrix} 0 & 0 & 0 \\ 0 \cdots 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}.$$

Then

$$T(e_{ij}) = A\left(\begin{array}{cc} & j\text{th column} \\ i\text{th row} & \begin{bmatrix} 0 & 0 & 0 \\ 0 \cdots 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \right).$$

$$= \begin{array}{c} \begin{bmatrix} 0 & \cdots & 0 & a_{1i} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{ni} & 0 & \cdots & 0 \end{bmatrix} \\ \uparrow \\ j\text{-th column.} \end{array}$$

$$= a_{1i} e_{1j} + \cdots + a_{ni} e_{nj}.$$

With respect to the basis $\{e_{ij}\}$, T can be expressed as the $n^2 \times n^2$ matrix:

$$\begin{bmatrix} \boxed{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{matrix}} & 0 & 0 \\ 0 & \begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{matrix} & 0 \\ & & \ddots & \\ 0 & & & \begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} A & & & & O \\ & A & & & \\ & & A & & \\ O & & & \ddots & \\ & & & & A \end{bmatrix}$$

Hence $\det(T) = \det(A)^n$. $\square$

#4:

$$f(A) = tr(A^2) = tr \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & \end{pmatrix}$$

$$= (a_{11}^2 + a_{12}a_{21} + \cdots + a_{1n}a_{n1}) + (a_{21}a_{12} + a_{22}^2 + \cdots + a_{2n}a_{n2}) + \cdots$$

$$= (a_{11}^2 + \cdots + a_{nn}^2) + 2\sum_{1 \le i < j \le n} a_{ij}a_{ji}$$ is a quad. form.

The signature of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is $(1, 0, 1)$.

Hence the signature of f is

$$\left(n + \frac{n(n-1)}{2}, 0, \frac{n(n-1)}{2}\right)$$

$$= \left(\frac{n(n+1)}{2}, 0, \frac{n(n-1)}{2}\right). \square$$

#5:

$$A^TA = \begin{bmatrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{bmatrix}, \implies tr(A^TA) = n.$$

Since A^TA is symmetric, and positive-semidefinite since

$$\vec{x}^T A^T A \vec{x} = \langle A\vec{x}, A\vec{x} \rangle \geq 0 \quad \forall \vec{x},$$

hence its eigenvalues $\lambda_1, \dots, \lambda_n$ are real and ≥ 0.

$$\implies 1 = \frac{tr(A^TA)}{n} = \frac{\lambda_1 + \dots + \lambda_n}{n} \geq (\lambda_1 \cdots \lambda_n)^{1/n} \geq 0.$$

$$\implies 0 \leq \underbrace{\det(A^TA)}_{= \det(A)^2} \leq 1.$$

$$\implies |\det(A)| \leq 1. \quad \square$$

#6: Let $\vec{x} \in W$ be any vector in W.

Since $T: V \longrightarrow V$ is diagonalizable, $\exists! \, \vec{v}_1, \dots, \vec{v}_n \in V$

s.t. $\vec{x} = \vec{v}_1 + \dots + \vec{v}_n$ and each $\vec{v}_i$ is an eigenvector of T, corresponding to <u>distinct</u> eigenvalues $\lambda_1, \dots, \lambda_n$.

To prove the statement, it suffices to show that $\vec{v}_1, \dots, \vec{v}_n \in W$.

By the assumption $T(W) \subseteq W$, we have $T(\vec{x}) \in W$.

$$W \ni T(\vec{x}) = T(\vec{v}_1 + \dots + \vec{v}_n)$$

$$= \lambda_1 \vec{v}_1 + \dots + \lambda_n \vec{v}_n.$$

$$W \ni \lambda_1 \vec{x} = \lambda_1 \vec{v}_1 + \lambda_1 \vec{v}_2 + \dots + \lambda_1 \vec{v}_n.$$

Hence $(\underbrace{\lambda_2-\lambda_1}_{\neq 0})\vec{v}_2 + \cdots + (\underbrace{\lambda_n-\lambda_1}_{\neq 0})\vec{v}_n \in W$.

Proceed this argument inductively, one can show that $\vec{v}_i \in W \ \forall i$. $\square$

Note: This is similar to the argument we used to prove eigenvectors corresp. to distinct eigenvalues are l.i.

#7: The value is 0.

- $\ker(T_1) = \mathrm{Im}(T_0) = \{0\}$, i.e. T_1 is injective.

$$\Rightarrow \dim V_1 = \dim \mathrm{Im}(T_1)$$
$$= \dim \ker(T_2)$$
$$= \dim V_2 - \dim \mathrm{Im}(T_2)$$

$$\Rightarrow \dim V_2 - \dim V_1 = \dim \mathrm{Im}(T_2)$$
$$= \dim \ker(T_3)$$
$$= \dim V_3 - \dim \mathrm{Im}(T_3)$$

$\cdots$ proceed this inductively. $\square$

#8: Let λ be an eigenvalue of A and $A\vec{v} = \lambda\vec{v}$, $\vec{v} \neq \vec{0}$.

Then $\lambda \in \mathbb{R}$ since A is symmetric;

$$\vec{0} = (A^3 - 2A - 4I)\vec{v} = (\lambda^3 - 2\lambda - 4)\vec{v}.$$

Since $\vec{v} \neq \vec{0} \Rightarrow \lambda^3 - 2\lambda - 4 = 0. = (\lambda-2)(\lambda^2+2\lambda+2)$

The only real root is $\lambda = 2$,
Hence 2 is the unique eigenvalue of A,

$$\Rightarrow A = P \begin{bmatrix} 2 & & \\ & \ddots & \\ & & 2 \end{bmatrix} P^{-1} = 2 I_5. \quad \square$$

#9: eigenvalues are ± 1, and f is diagonalizable.

eigenbasis:

$$\left\{ \underbrace{\begin{bmatrix} 1 & & 0 \\ & 0 & \\ 0 & & \ddots & \\ & & & 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 & \\ & & & 1 \end{bmatrix}}_{}, \right.$$

n diagonal matrices (eigenvectors of 1)

$\frac{n(n-1)}{2}$ symmetric matrices. (eigenvectors of 1)

$$\underbrace{\begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & O \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & \\ 0 & 0 & 0 & O \\ 1 & 0 & 0 & \\ O & & & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix}}_{},$$

$\frac{n(n-1)}{2}$ anti-sym. matrices (eigenvectors of -1)

$$\left. \underbrace{\begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & \ddots & \\ & & & 0 \end{bmatrix}, \cdots\cdots, \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}}_{} \right\}$$

#10: Recall in HW5 #9 we showed that $A^2 = A \Rightarrow A$ is diagonalizable and $\{0,1\}$ are the only possible eigenvalues of A. Hence (a) $\Rightarrow$ (b).

Conversely, (b) $\Rightarrow$ A is diagonalizable & $\{0,1\}$ are the only eigenvalues of A,

Since
$$\begin{aligned} n &= rk(A) + rk(I_n - A) \\ &= (n - \dim Nul(A)) + (n - \dim Nul(I_n - A)) \end{aligned}$$

$\Rightarrow \dim \mathrm{Nul}(A) + \dim \mathrm{Nul}(A - I_n) = n.$

$A = PDP^{-1}$, where D is diagonal matrix w/ 0 or 1 on diagonal.

$\Rightarrow D^2 = D.$

$\Rightarrow A^2 = A. \quad \square$

#11: Suppose $\mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subsetneq V$.

Then $\langle \vec{v}_1, \dots, \vec{v}_k \rangle^\perp \neq \{0\}$.

Choose any $\vec{0} \neq \vec{w} \in \langle \vec{v}_1, \dots, \vec{v}_k \rangle^\perp \subseteq V$.

Then $0 < \|\vec{w}\|^2 = \langle \vec{v}_1, \vec{w} \rangle^2 + \cdots + \langle \vec{v}_k, \vec{w} \rangle^2$

$= 0.$ Contradiction. $\square$

#12:

(a) $\vec{x} \in (W_1 + W_2)^\perp \iff \langle \vec{x}, \vec{w}_1 + \vec{w}_2 \rangle = 0 \quad \forall \vec{w}_1 \in W_1, \vec{w}_2 \in W_2$

$\langle \vec{x}, \vec{w}_1 + \vec{w}_2 \rangle = \langle \vec{x}, \vec{w}_1 \rangle + \langle \vec{x}, \vec{w}_2 \rangle$

(why?) $\iff \langle \vec{x}, \vec{w}_1 \rangle = \langle \vec{x}, \vec{w}_2 \rangle = 0 \quad \forall \vec{w}_1 \in W_1, \vec{w}_2 \in W_2$

$\iff \vec{x} \in W_1^\perp \cap W_2^\perp. \quad \square$

(b) $\dim(W_1) - \dim(W_1 \cap W_2) = \dim(W_1 + W_2) - \dim(W_2)$

$= (\dim V - \dim(W_1+W_2)^\perp) - (\dim V - \dim(W_2)^\perp)$

$= \dim(W_2^\perp) - \dim(W_1^\perp \cap W_2^\perp). \quad \square$

#13: Suppose $(I - BA)\vec{x} = \vec{0}$.

Then $\vec{x} = BA\vec{x}$.

$\Rightarrow A\vec{x} = ABA\vec{x} \Rightarrow (I - AB)(A\vec{x}) = \vec{0}$.

$\Rightarrow A\vec{x} = 0$ since $I - AB$ is invertible.

$\Rightarrow \vec{x} = BA\vec{x} = \vec{0}$. $\square$

#14: Suppose $W_1 \cup W_2 \subseteq V$ is a subspace.

and assume that $\exists\, x_1 \in W_1 \setminus W_2$ and $x_2 \in W_2 \setminus W_1$.

Then

- $x_1 + x_2 \in W_1 \cup W_2$ since $x_1, x_2 \in W_1 \cup W_2$ and $W_1 \cup W_2$ is a subspace
- $x_1 + x_2 \notin W_1$: otherwise, if $x_1 + x_2 \in W_1$, then $x_2 = (x_1 + x_2) - x_1 \in W_1$, contradiction.
- Similarly, $x_1 + x_2 \notin W_2$

$\Rightarrow x_1 + x_2 \notin W_1 \cup W_2$. Contradiction. $\square$

#15: $A(B - I - A) = I \Rightarrow A$ is invertible

$\Rightarrow B = A^{-1}(A^2 + A + I) = A + I + A^{-1}$

$\Rightarrow AB = A(A + I + A^{-1}) = (A + I + A^{-1})A = BA$. $\square$

#16: By assumption, we have $\forall \vec{v} \in V$, $\exists c_{\vec{v}} \in \mathbb{R}$ s.t. $T(\vec{v}) = c_{\vec{v}} \vec{v}$.

We need to show that $c_{\vec{v}}$ is the same constant $\forall \vec{v} \in V$.

Suppose $T(\vec{v}_1) = c_1 \vec{v}_1$, $T(\vec{v}_2) = c_2 \vec{v}_2$. and $c_1 \neq c_2$.

Then $\{\vec{v}_1, \vec{v}_2\}$ is l.i. and

$$T(\vec{v}_1 + \vec{v}_2) = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

$$= c_3 (\vec{v}_1 + \vec{v}_2) \quad \text{for some } c_3 \in \mathbb{R}.$$

$$\Rightarrow (c_1 - c_3) \vec{v}_1 + (c_2 - c_3) \vec{v}_2 = \vec{0}$$

$$\Rightarrow c_1 = c_2 = c_3. \quad \text{Contradiction.} \square$$