(1) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that the derivative $f'(x)$ exists for any $x \in \mathbb{R}$, but $f'\colon \mathbb{R} \to \mathbb{R}$ is not a continuous function.

- for $x \neq 0$, $f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$
- for $x = 0$, $\lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{x^2\sin(\frac{1}{x})}{x} = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$

  since $|\sin(\frac{1}{x})| \leq 1$ bounded, Hence $f'(0) = 0$.
- $f'\colon \mathbb{R} \to \mathbb{R}$ is NOT continuous at 0:

  Consider $(x_n = \frac{1}{2n\pi})_{n\in\mathbb{N}}$, $\lim x_n = 0$, but: $\lim f'(x_n) = -1 \neq f'(0)$.

  (In fact, $\lim_{x\to 0} f'(x)$ doesn't exist.) $\square$

(2) We say a function $f\colon (a,b) \to \mathbb{R}$ is *strictly increasing* if $f(x) < f(y)$ for any $a < x < y < b$. Suppose f is differentiable on (a,b).

(a) Prove or disprove: If f is strictly increasing, then $f'(x) > 0$ for any $x \in (a,b)$.

(b) Prove or disprove: If $f'(x) > 0$ for any $x \in (a,b)$, then f is strictly increasing.

(Hint: Mean value theorem.)

(a) False. $f(x) = x^3$, $f'(0) = 0$.

(b) True: $\forall\, a < x < y < b$, by MVT, $\exists\, x < c < y$

s.t. $0 < f'(c) = \frac{f(y)-f(x)}{y-x}$

$\Rightarrow f(y) > f(x)$. $\square$

(3) Prove that the equation $e^x = 1 - x$ has a unique solution in $\mathbb{R}$.

$f(x) = e^x + x - 1$ has a zero at $x = 0$. Suppose $f(x) = 0$ for some $x \neq 0$.

By Rolle's thm, $\exists\, y \neq 0$ s.t. $f'(y) = 0$. $= e^y + 1 > 0$.

Contradiction. $\square$

(4) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying $|f(x) - f(y)| \le |x - y|^2$ for any $x, y \in \mathbb{R}$. Prove that f is a constant function.

<u>Claim:</u> $f'(x) = 0 \quad \forall x \in \mathbb{R}$.

Pf: $\forall x_0, x \in \mathbb{R}$, $\left| \dfrac{f(x) - f(x_0)}{x - x_0} \right| \le \dfrac{|x - x_0|^2}{|x - x_0|} = |x - x_0|$.

$$\Longrightarrow \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = 0. \quad \square$$

(5) Let $f\colon (a, b) \to \mathbb{R}$ be an unbounded differentiable function. Prove that the derivative $f'\colon (a, b) \to \mathbb{R}$ is also unbounded.

- Choose any point $c \in (a, b)$.
- Since f is unbounded on (a, b), $\forall M > 0$, $\exists\, d \in (a, b)$ s.t.

  $$|f(d)| > (b - a) M + |f(c)|$$

- By MVT, $\exists\, e$ between c & d s.t.

  $$|f'(e)| = \left| \frac{f(d) - f(c)}{d - c} \right| > \frac{|f(d)| - |f(c)|}{b - a} > M. \quad \square$$