#1: Consider the quadratic form

$$Q(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2.$$

(1) Find a symmetric matrix A s.t. $Q(\vec{x}) = \vec{x}^T A \vec{x}$.

(2) Find an orthogonal diagonalization of A.

(3) Determine whether Q is positive (semi) definite, negative (semi) definite, or indefinite.

Sol^n:

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

positive definite. $\square$

#2: The unique orthogonal, symmetric matrix w/ only positive eigenvalues is $\mathbb{I}$.

pf:

- $A^T = A^{-1}$
- $A^T = A$

$\Rightarrow A^2 = \mathbb{I}$.

$$\underbrace{A^T = A}$$

$\Downarrow$

A is (orthogonally) diagonalizable.

$$A = PDP^T, \quad P^T = P^{-1}$$

$$\Rightarrow \mathbb{I} = A^2 = PD^2P^T. \Rightarrow D^2 = \mathbb{I}.$$

$\Rightarrow$ eigenvalues of A are ± 1.

By assumption, 1 is the only eigenvalue of A.

$$\Rightarrow D = \mathbb{I}. \Rightarrow A = PDP^T = \mathbb{I}. \square$$