Math 104 (Section 6) First Midterm February 18, 2020

Name: Solution

- You have 75 minutes to complete the exam (8:10am - 9:25am).
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using. Exception: if you are asked to prove a theorem, you are not allowed to use it!
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- Do not detach pages from this exam packet or unstaple it.

Good Luck!

Question	Points
1	25
2	15
3	30
4	30
Total	100

1. (a) (5 points; no partial credits) State the precise definition of a sequence of real numbers (a_n) converging to a real number a.

$\forall \varepsilon > 0$, $\exists N > 0$ s.t.

$$|a_n - a| < \varepsilon \quad \forall n > N.$$

(b) (5 points; no partial credits) State the precise definition of a sequence of real numbers (a_n) being a Cauchy sequence.

$\forall \varepsilon > 0$, $\exists N > 0$ s.t.

$$|a_n - a_m| < \varepsilon \quad \forall n, m > N.$$

(c) (15 points) Let (a_n) be a convergent sequence. Prove that (a_n) is Cauchy. (You are not allowed to use any theorem for this problem.)

$\forall \varepsilon > 0$, $\exists N > 0$ s.t.

$$|a_n - \underset{\displaystyle\lim_{n\to\infty} a_n.}{\underset{\|}{a}}| < \frac{\varepsilon}{2} \quad \forall n > N,$$

Then $\forall n, m > N$, we have:

$$|a_n - a_m| \underset{\text{triangle ineq.}}{\leq} |a_n - a| + |a_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

2. (15 points) Let (a_n) be a sequence of real numbers where $a_1 = 1$ and

$$a_{n+1} = \frac{n}{n+3} a_n^2, \quad \text{for } n \geq 1.$$

Prove that (a_n) is convergent and find the limit.
(If you want to use theorems, you must state the precise statements you are using.)

Claim: $0 < a_{n+1} \leq a_n \leq 1 \quad \forall n \geq 1.$

Prove by induction. Clearly, we have $0 < a_2 \leq a_1 \leq 1$.

Suppose $0 < a_{n+1} \leq a_n \leq 1$, then

$$0 < a_{n+2} = \frac{n+1}{n+4} a_{n+1}^2 \underset{\text{since } a_{n+1} \leq 1}{\leq} \frac{n+1}{n+4} a_{n+1} < a_{n+1} \leq 1. \quad \square$$

Hence (a_n) is decreasing and bounded. $\underset{\text{Thm we proved in class}}{\Longrightarrow}$ (a_n) converges.

Let $\lim_{n\to\infty} a_n = a$.

Then

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{n}{n+3} a_n^2\right) \underset{\text{limit theorems.}}{=} \lim_{n\to\infty} \frac{n}{n+3} \cdot \left(\lim_{n\to\infty} a_n\right)^2$$

$$= 1 \cdot a^2 = a^2.$$

$\Rightarrow \quad a = 1$ or 0.

Observe that $a_n \leq a_2 = \frac{1}{4} \quad \forall n \geq 2$. Hence a can't be 1.

$\Rightarrow \quad \lim_{n\to\infty} a_n = a = 0. \quad \square$

3. (a) (5 points; no partial credits) Let (a_n) be a bounded sequence of real numbers. State the precise definition of

$$\limsup_{n\to\infty} a_n.$$

$$\| $$

$$\lim_{N\to\infty} \sup\{a_n : n > N\}.$$

(b) (10 points) Let (a_n) and (b_n) be two bounded sequences of real numbers. Prove that for any $N > 0$,

$$\sup\{a_n + b_n : n > N\} \leq \sup\{a_n : n > N\} + \sup\{b_n : n > N\}.$$

(Hint: For any $\epsilon > 0$, $\sup\{a_n + b_n : n > N\} - \epsilon$ is not an upper bound of the set $\{a_n + b_n : n > N\}$.)

$\forall \varepsilon > 0$, $\sup\{a_n + b_n : n > N\} - \varepsilon$ is not an upper bound of $\{a_n + b_n : n > N\}$.

$\Rightarrow$ $\exists\, n > N$ s.t. $a_n + b_n > \sup\{a_n + b_n : n > N\} - \varepsilon$

$\Rightarrow$ $\sup\{a_n : n > N\} + \sup\{b_n : n > N\} > \sup\{a_n + b_n : n > N\} - \varepsilon$.

This inequality holds for all $\varepsilon > 0$, therefore

$$\sup\{a_n : n > N\} + \sup\{b_n : n > N\} \geq \sup\{a_n + b_n : n > N\}. \quad \square$$

(c) (10 points) Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

for any bounded sequences (a_n) and (b_n).
(If you want to use theorems, you must state the precise statements you are using.)

By (b), $\forall N > 0$, we have:

$$\underbrace{\sup\{a_n : n > N\} + \sup\{b_n : n > N\} - \sup\{a_n + b_n : n > N\}}_{\downarrow} \ge 0.$$

$$\Rightarrow \lim_{N\to\infty}\Big(\qquad \Big) \ge 0$$

if the limit exists.

By the definition of limsup and limit theorem, we have

$$\lim_{N\to\infty}\left(\sup\{a_n : n > N\} + \sup\{b_n : n > N\} - \sup\{a_n + b_n : n > N\}\right) \ge 0$$

$$= \limsup a_n + \limsup b_n - \limsup(a_n + b_n). \qquad \square$$

(d) (5 points) Find an example of two bounded sequences (a_n) and (b_n) satisfying

$$\limsup_{n\to\infty}(a_n + b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

$(a_n) = (0, 1, 0, 1, \ldots)$ $\quad \limsup a_n = 1$

$(b_n) = (1, 0, 1, 0, \ldots)$ $\quad \limsup b_n = 1$

$(a_n + b_n) = (1, 1, 1, 1, \ldots)$ $\quad \limsup(a_n + b_n) = 1.$

4. There are four statements below:

(I) Consider the metric space $\mathbb{R}$ with the usual distance function $d(x,y) = |x-y|$. Let $E = \mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$. Then the closure $\overline{E} = \mathbb{R}$.

(II) Let (a_n) be a sequence of real numbers satisfying

$$\lim_{n\to\infty} |a_{n+1} - a_n| = 0.$$

Then (a_n) is convergent.

(III) Let (a_n) and (b_n) be two bounded sequences of real numbers. If $a_n \leq b_n$ for any $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n.$$

(Warning: this may be harder than you think.)

(IV) Let $a_n = (n!)^{1/n}$. Then the sequence (a_n) is convergent. (Recall $n! = 1 \cdot 2 \cdots n$.)

(a) (15 points) Choose a statement that is true and prove it. *You are not allowed to choose more than one statement.*

My statement is (I) or (III).

(I). $\forall x \in \mathbb{R}$ and $\forall r > 0$, $\exists\, y \in B_r(x) \cap \mathbb{Q}$ by the denseness of $\mathbb{Q}$.

$\Rightarrow \overline{E} = \mathbb{R}$.

(III). Claim: $\sup\{a_n : n > N\} \leq \sup\{b_n : n > N\}$ $\forall N$.

pf $\forall \varepsilon > 0$, $\sup\{a_n : n > N\} - \varepsilon$ is not an upper bound of $\{a_n : n > N\}$,

$\Rightarrow \exists\, n > N$ s.t. $a_n > \sup\{a_n : n > N\} - \varepsilon$

$\Rightarrow \sup\{b_n : n > N\} \geq b_n \geq a_n > \sup\{a_n : n > N\} - \varepsilon$ $\forall \varepsilon > 0$.

$\Rightarrow \sup\{b_n : n > N\} \geq \sup\{a_n : n > N\}$.

From the claim, take limit $N \to \infty$, we get

$\limsup\limits_{n\to\infty} a_n \leq \limsup\limits_{n\to\infty} b_n$. $\square$

(b) (15 points) Choose a statement that is false. Either prove the statement is false, or give an explicit counterexample and justify it. *You are not allowed to choose more than one statement.*

My statement is (II) or (IV).

(II). Counterexample: $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. (cf. HW 2)

(IV). For any even number n, we have

$$a_n = (n!)^{1/n} = (1 \cdot 2 \cdot 3 \cdot \cdots \cdot n)^{1/n}$$

$$> \left(\left(\tfrac{n}{2}+1\right)\left(\tfrac{n}{2}+2\right) \cdots n \right)^{1/n}$$

$$> \left(\left(\tfrac{n}{2}\right)^{\frac{n}{2}} \right)^{\frac{1}{n}} = \left(\tfrac{n}{2}\right)^{\frac{1}{2}}$$

Since $\lim_{n\to\infty} \left(\frac{n}{2}\right)^{\frac{1}{2}} = +\infty$, so (a_n) is not bounded, therefore is divergent. $\square$