Today: More discussions on mirror symmetry.

§ Semi-flat mirror symmetry:

- V - finite dim'l $\mathbb{R}$-vector space.

  $V^* = \mathrm{Hom}(V, \mathbb{R})$

- $V \times V$ has a complex structure given by

  $$J(v_1, v_2) = -(v_2, v_1).$$

  $V \times V^*$ has a symplectic structure given by

  $$\omega\Big((v_1, w_1), (v_2, w_2)\Big) = \langle w_1, v_2 \rangle - \langle w_2, v_1 \rangle$$

- Write $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, where $\Lambda \cong \mathbb{Z}^n$.

  $$\Lambda^* := \{ w \in V^* \mid \langle w, v \rangle \in \mathbb{Z} \quad \forall v \in \Lambda \}.$$

$$\begin{tikzcd}
\rightsquigarrow \left(V \times V/\Lambda, J\right) \arrow[d, equal] & & \left(V \times V^*/\Lambda^*, \omega\right) \arrow[d, equal] \\
TV/\Lambda \arrow[dr] & & T^*V/\Lambda^* \arrow[dl] \\
& V &
\end{tikzcd}$$

dual torus fibration

More generally, let B be an integral affine manifold. (smooth manifold, with transition functions are in the affine group $\mathbb{R}^n \rtimes GL_n(\mathbb{Z})$), equivalently, B equips with a local system of integral lattices $\Lambda \subseteq TB$.,

Then $\forall b \in B$, there are lattices on T_b and T_b^* independent of the choice of charts,

$$\begin{tikzcd} (TB/\Lambda_B, J) \arrow[dr] & & (T^*B/\Lambda^*_B, \omega) \arrow[dl] \\ & B & \end{tikzcd}$$

coordinate on the fiber

$y_1, \dots, y_n$ $\qquad$ $\tilde{y}_1, \dots, \tilde{y}_n$

$x_1, \dots, x_n$ – coordinate on the base

- A Riemannian metric g on B is Hessian if g locally is given by $g_{ij} = \dfrac{\partial^2 K}{\partial x_i \partial x_j}$ for some potential function K.

$\rightsquigarrow$ Kähler structure on $(TB/\Lambda_B, J)$. where:

- canonical holomorphic coordinate: $z_i = x_i + \sqrt{-1} y_i$.
- symplectic form: $\omega = \frac{\sqrt{-1}}{2} g_{ij} \, z_i \wedge \bar{z}_j$

Kähler structure on $(T^*B/\Lambda_B^*, \omega)$, where:

- holo. coord: $\tilde{z}_i = \tilde{y}_i + \sqrt{-1} \frac{\partial K}{\partial x_i}$
- canonical symplectic form $\omega = d\tilde{y}_i \wedge dx_i$

Rmk: • $TB/\Lambda_B \to B$ and $T^*B/\Lambda^*_B \to B$ are both slag torus fibrations

• $(TB/\Lambda_B, J_{can}, \omega_K)$ is Ricci-flat

$\iff$ K satisfies the real Monge-Ampère equation $\det\left(\frac{\partial^2 K}{\partial x_i \partial x_j}\right) = \text{const.}$

$\iff$ $(T^*B/\Lambda_B, J_K, \omega_{can})$ is Ricci-flat.

<u>Strominger-Yau-Zaslow's strategy of constructing mirror:</u>

Given a Calabi-Yau manifold X, $B_0 \subset B$ — smooth fiber.

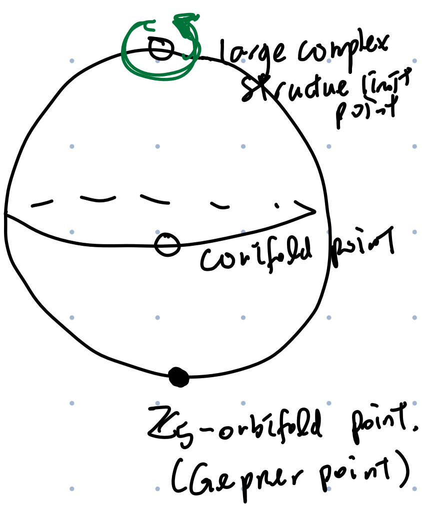
1) find a slag torus fib. $X \longrightarrow B$. (hard)

2) dualize the torus fib. over the smooth locus. $\rightsquigarrow$ $\check{X}_0 \to B_0$

3) compactify $\check{X}_0$, and get $\check{X} \longrightarrow B$. (need quantum corrections)

<u>e.g.</u> quintic Calabi-Yau 3-fold.

$\mathcal{M}_{cpx}$.



$$\mathcal{M}_{cpx}(X) \xhookrightarrow{\text{conjecture}} \mathrm{Aut}(D)\backslash \mathrm{Stab}(D^{\pi}\mathrm{Fuk}(X))/\mathbb{C}$$

$\cong$ HMS conj.

$$\mathrm{Aut}(D)\backslash \mathrm{Stab}(D^b\mathrm{Coh}(\check{X}))/\mathbb{C}$$

<u>Kontsevich, Horja:</u>

LCSL $\longleftrightarrow$ $\otimes \mathcal{O}(1)$

Conifold $\longleftrightarrow$ $T_{\mathcal{O}_{\check{X}}}$

Gepner $\longleftrightarrow$ $T_{\mathcal{O}_{\check{X}}} \circ (- \otimes \mathcal{O}(1))$

Ballard-Favero-Katzarkov:

$$\boxed{\left(T_{\mathcal{O}_{\check{X}}} \circ (- \otimes \mathcal{O}(1))\right)^5 = [2].}$$

Rmk: Gepner point $\rightsquigarrow$ stability condition with certain $\mathbb{Z}_5$-symmetry.
(cf. Toda's Gepner stability condition).

Rmk: It's easier to see this identity by considering
<u>matrix factorization category</u>:

- $\overset{A}{\overset{\parallel}{W \in \mathbb{C}[x_1, \cdots, x_n]}}$, homogeneous poly. of degree d.
  s.t. $(W=0)$ has isolated singularity at 0.

- For a graded A-module P, $(\deg x_i = 1)$
  denote P_i to be the degree i part of P,
  define $P(k)$ to be the shifted A-module where $P(k)_i = P_{i+k}$

$\rightsquigarrow$ Graded matrix factorization category $HMF^{gr}(W)$

- <u>Objects</u>: $P^0 \xrightarrow{f} P^1 \xrightarrow{g} P^0(d)$

  where $g \circ f = f(d) \circ g = W$

  (e.g. $W = x^5 + y^5 \in \mathbb{C}[x,y]$, $P^0 = P^1 = \mathbb{C}[x,y]^{\oplus 3}$,

  $f = \begin{pmatrix} x & y & 0 \\ 0 & x^3 & y \\ y^3 & 0 & x \end{pmatrix}$, $g = \begin{pmatrix} x^4 & -xy & y^2 \\ y^4 & x^2 & -xy \\ -x^3y^4 & y^4 & x^4 \end{pmatrix}$)

- $[1]$: sends $P^0 \xrightarrow{f} P^1 \xrightarrow{g} P^0(d)$
  to $P^1 \xrightarrow{-g} P^0(d) \xrightarrow{-f(d)} P^1(d)$

Rmk: Another autoequivalence $\tau : P \longmapsto P(1)$

$$\tau^d \simeq [2]$$

Rmk:

$$\begin{tikzcd}
\underset{\text{CYd.}}{D^b\mathrm{Coh}(X)} \arrow[r, "\text{Orlov}", "\cong"'] \arrow[d, "T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}(1))"', "\cong"] \arrow[dr, phantom, "\text{Ballard-Favero-Katzarkov}\ \circlearrowright"] & HMF^{gr}(W) \arrow[d, "\cong"', "\tau"] \\
D^b\mathrm{Coh}(X) \arrow[r, "\text{Orlov}", "\cong"'] & HMF^{gr}(W)
\end{tikzcd}$$

$X \subset \mathbb{P}^{d+1}$

W: defining poly of X. deg $d+2$, in $\mathbb{C}[x_1, \ldots, x_{d+2}]$

Fano-LG mirror symmetry:

$$\underset{\text{Fano}}{X} \longleftrightarrow \underset{\text{LG potential.}}{\check{X} \xrightarrow{W} \mathbb{C}}$$

$\check{X}$: noncompact Kähler manifold.

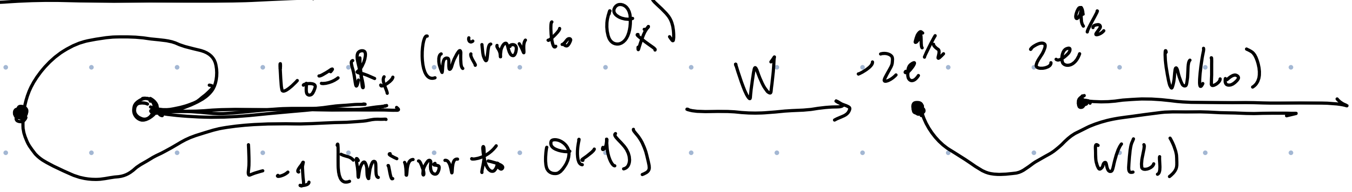
$$\begin{array}{ccc}
D^b\mathrm{Coh}(X) & & HMF^{gr}(W) \\
& \times & \\
D^{\pi}\mathrm{Fuk}(X) & & D^{\pi}\mathrm{Fuk}(\check{X}, W)
\end{array}$$

"Fukaya-Seidel category"

e.g. $X = \mathbb{P}^1$, $W = z + \frac{q}{z}$, where $q \in \mathbb{C}^*$.

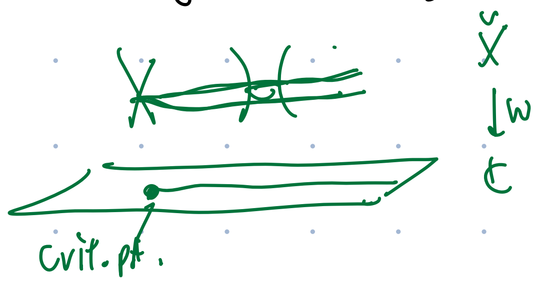
$W : \mathbb{C}^* \to \mathbb{C}$.

Objects in $D^{\pi}\mathrm{Fuk}(\check{X}, W)$:



$D^\pi \mathrm{Fuk}(\check{X}, W)$ is generated by L_0 and L_{-1}.

<u>Seidel</u>: If the critical points of W are isolated and nondegenerate, then $D^\pi \mathrm{Fuk}(\check{X}, W)$ is generated by a collection of Lefschetz thimbles:



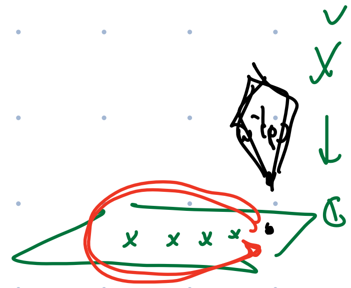
(Katzarkov-Kontsevich-Pantev)

<u>Rmk</u>: $X \xleftrightarrow{\text{MS}} W : \check{X} \to \mathbb{C}$

Fano $\qquad$ LG.

$Z \in |-K_X| \xleftrightarrow{\text{MS}} W^{-1}(p)$ for generic p.

smooth Calabi-Yau



$\otimes N_{Z/X}$ on $D^b\mathrm{Coh}(Z)$ $\longleftrightarrow$ autoequivalence on $D^\pi \mathrm{Fuk}(W^{-1}(p))$ induced by the monodromy along ∞

$\mathcal{X} \to 0 \in \Delta$ is a <u>Tyurin degeneration</u> of Calabi-Yau manifolds if generic fibers are smooth Calabi-Yaus, and

$$X_0 = X_1 \cup_Z X_2, \quad \text{where } X_1, X_2 \text{ are Fano},$$
$$Z \in |-K_{X_i}| \text{ for } i = 1, 2.$$

Conversely, given Fanos X_1, X_2 and $Z \in |-K_{X_i}|$

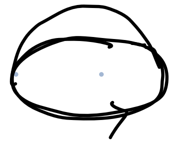
$X_1 \cup_Z X_2$ is smoothable $\iff N_{Z/X_1} \cong N_{Z/X_2}^{-1}$.

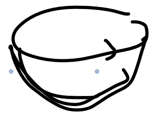
$$\begin{array}{c} X \\ \text{Tyurin deg}^{n} \rightsquigarrow \\ X_1 \cup_Z X_2. \end{array}$$

$Z \in |-K_{X_i}|$

$N_{Z/X_1} \otimes N_{Z/X_2} \cong \mathcal{O}_Z$

$$\begin{tikzcd} X_1 \arrow[r, leftrightarrow, "MS"] & Y_1 \arrow[d, "W_1"] \\ Z \arrow[u, phantom, "\cup"] \arrow[d, phantom, "\cap"] & \mathbb{C} \\ X_2 \arrow[r, leftrightarrow, "MS"] & Y_2 \arrow[d, "W_2"] \\ & \mathbb{C} \end{tikzcd}$$





$\Downarrow$

"monodromy on the mirror side matches with each other."

$\Downarrow$

expect: the two LG models glue, and get $Y \xrightarrow{W} \mathbb{P}^1$.

where: $X \xleftrightarrow{MS} Y$

$Z \xleftrightarrow{MS} W^{-1}(p)$ generic fiber.

(Doran-Harder-Thompson).

<u>Non-example:</u> $Y = \{f = g = 0\} \subseteq \mathbb{P}^5$, where f, g are cubic polynomials.

$$\begin{array}{c} Y \\ \updownarrow \text{MS} \\ X \rightsquigarrow X_1 \cup_Z X_2. \end{array}$$

where Z is a K3 surface w/ transcendental lattice $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

No $Y \to \mathbb{P}^1$ give the expected correspondence:

1) No such K3 fibration: $Y \to \mathbb{P}^1$; since $h^{1,1}(Y) = 1$

2) Z has no mirror K3; (in the sense of Dolgachev).

$$\overbrace{M}^{NS(Z)} \oplus \overbrace{U \oplus N}^{T(Z)} \quad \underbrace{M \oplus U}_{T(\check{Z})} \oplus \underbrace{N}_{NS(\check{Z})}$$

U: hyperbolic lattice sgn $(1,1)$

, since $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ doesn't contain the hyperbolic lattice U.

Conjecture There is a "noncommutative K3" mirror to Z, and a "noncommutative fibration $Y \to \mathbb{P}^1$" s.t. its fiber is

Consider $\mathcal{H}_L \subseteq \mathbb{P}^5 \times \mathbb{P}^1$. pencil of cubic fourfolds determined by f, g:

$$\begin{array}{ccc} \mathcal{H}_L & & \{sf + tg = 0\} \subseteq \mathbb{P}^5 \\ \downarrow \pi & & \uparrow \\ \mathbb{P}^1 & & \ni [s,t] \end{array}$$

base locus $= \underset{CY3}{Y} \subseteq \mathbb{P}^5$

- For a cubic 4 fold $W \subseteq \mathbb{P}^5$, there is a semiorthogonal decomposition.

$$D^b Coh(W) = \langle \mathcal{A}_W, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle.$$

where $\mathcal{A}_W$ is a K3 category, i.e. $S_{\mathcal{A}_W} = [2]$.

(Kuznetsov component).

- $D^b Coh(\mathcal{H}_L) = \langle \mathcal{A}_{\mathcal{H}_L}, \pi^* D^b(\mathbb{P}^1) \otimes \mathcal{O}(i,0), \quad i = 0,1,2 \rangle.$

|| by Homological projective duality.

$$D^b(Y)$$

- Take the "Fiber" of $D^b(\mathbb{P}^1) \xrightarrow{\pi^*} D^b(\mathcal{H}_2)$

$$\underbrace{D^b(\overset{t \in \mathbb{P}^1}{\overset{\shortparallel}{pt}}) \otimes_{D^b(\mathbb{P}^1)} D^b(\mathcal{H}_2)}_{\displaystyle \simeq D^b(\pi^{-1}(t))} \simeq \langle \text{"}\mathcal{A}_{\mathcal{H}_2}|_{\text{fiber}}\text{"}, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$$

$D^b(\pi^{-1}(t))$: $\pi^{-1}(t)$ cubic 4-fold.

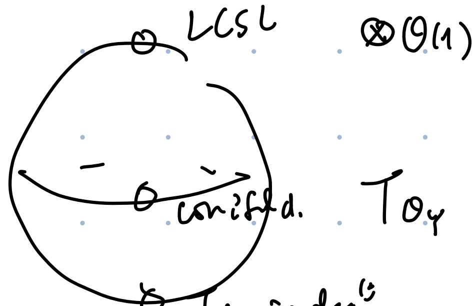
$\mathcal{A}_{\mathcal{H}_2}|_{\text{fiber}}$ $\downarrow$ Kuznetsov component of $D^b(\pi^{-1}(t))$

Conj (Doran-Harder-Thompson) W - generic cubic 4-fold,

$$D^b(W) = \langle \mathcal{A}_W, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle.$$

Then $\mathcal{A}_W \cong D^\pi \mathrm{Fuk}(Z)$,

Question: $\overset{CY3}{Y} \subseteq \mathbb{P}^5$



$\downarrow$

Express $T_{\mathcal{O}} \circ (- \otimes \mathcal{O}(1))$

in terms of the Tyurin degeneration?

Folklore conjecture: autoequivalence associated to $\overset{CY3}{X} \longrightarrow \mathbb{P}^1$

should be given by the twist/cotwist functor of the spherical functor $D^b(\mathbb{P}^1) \longrightarrow D^b(X)$.

In our case,

$$\begin{tikzcd} D^b(\mathbb{P}^1) \arrow[r, "\pi^*"] \arrow[rrr, bend right, "\text{spherical}^{\circledast}\text{ functor?}"'] & D^b(\mathcal{H}_2) \arrow[r, "j^L"] & \mathcal{A}_{\mathcal{H}_2} \arrow[r, phantom, "\cong"] & D^b(Y) \end{tikzcd}$$

Its twist/cotwist functor is $T_{\mathcal{O}} \circ (- \otimes \mathcal{O}(1))$?

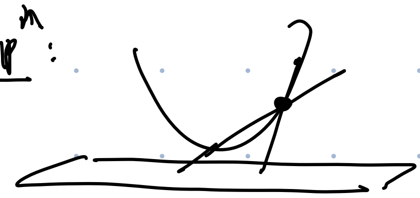
<u>Aside</u> rationality of cubic 4-fold.

($X \subseteq \mathbb{P}^5$ cubic hypersurface.

Question: Is X rational?

i.e. birational to $\mathbb{P}^4$)

<u>Rationality of hypersurfaces in projective space $\mathbb{P}^n$:</u>



<u>deg 1</u>: $\cong \mathbb{P}^{n-1}$ rational

<u>deg 2</u>: rational (projection from a point in X to a hyperplane)

<u>deg 3</u>: <u>dim 1</u>: elliptic curves $\to$ irrational.

<u>dim 2</u>: rational (Clebsch 1866).

<u>dim 3</u>: irrational (Clemens-Griffiths)

<u>dim 4</u>: ??

Questions: 1) Are cubic 4 folds rational?

2) How much birational data can we get from $D^b(\mathrm{Coh}(X))$?

- A cubic 4-fold is special if $\mathrm{rk}\, H^{2,2}(X,\mathbb{Z}) \geq 2$

  (generic cubic 4 fold $\to$ $H^{2,2}(X,\mathbb{Z}) \simeq \langle H^2 \rangle$, where H hyperplane class)

- $\mathcal{C}_d := \{$ special cubic 4-folds admitting a rank 2 saturated sublattice (labeling) $K \subseteq H^{2,2}(X,\mathbb{Z})$ st. $\mathrm{disc}(K) = d \}$.

- Hassett:
  - $\mathcal{C}_d \neq \phi \iff d \geq 8$ and $d \equiv 0, 2 \pmod 6$.

    $\downarrow$

    irred. divisor in moduli of cubic 4 folds.

  - Suppose $\langle H^2, T \rangle \subseteq H^{2,2}(X,\mathbb{Z})$ w/ $\mathrm{disc} = d$.

    $\exists$ polarized K3 surface S s.t.

    $$\langle H^2, T \rangle^{\perp} \cong H^2_{prim}(S,\mathbb{Z})(-1)$$

    $\iff$ d is not divisible by $4, 9$, or any odd prime $\equiv 2 \pmod 3$

These d's are called admissible.

Conj X (cubic 4 fold) is rational $\iff$ $X \in \mathcal{C}_d$ for some admissible d.

Addington-Thomas, Bayer-Lahoz-Macri-Nuer-Perry-Stellari.

Conj (Kuznetsov) $D^b(X) = \langle \mathcal{A}_X, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$

X is rational $\iff$ $\mathcal{A}_X \cong D^b(S)$ where S is a K3 surface

Conj (Huybrechts) $\mathcal{A}_X \cong \mathcal{A}_{X'} \implies X$ and X' are birational to each other.