Compact $\Leftrightarrow$ Sequentially compact. $\Longrightarrow$ closed & bounded

We almost finish the proof in Lecture 8 & 12, except for proving the following lemma:

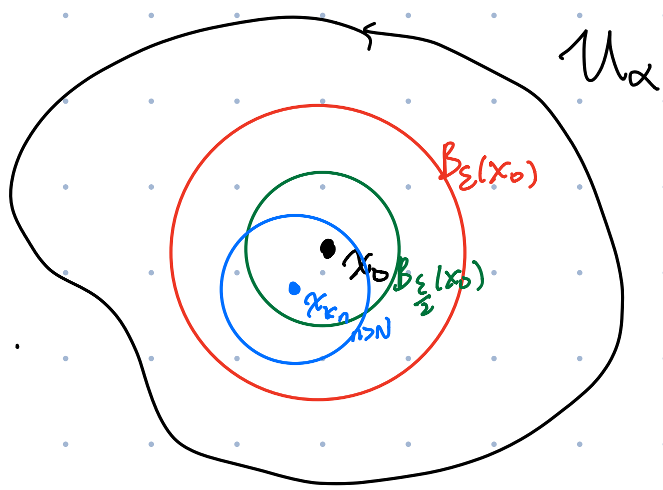
<u>Lemma 1</u>: If K is sequentially cpt, and if $\{U_\alpha\}_{\alpha\in I}$ is an open cover of K, then

$$\exists \delta>0 \text{ s.t. } \forall x\in K,\ \exists \alpha\in I \text{ s.t. } B_\delta(x)\subseteq U_\alpha.$$

<u>pf</u>: Prove by contradiction:

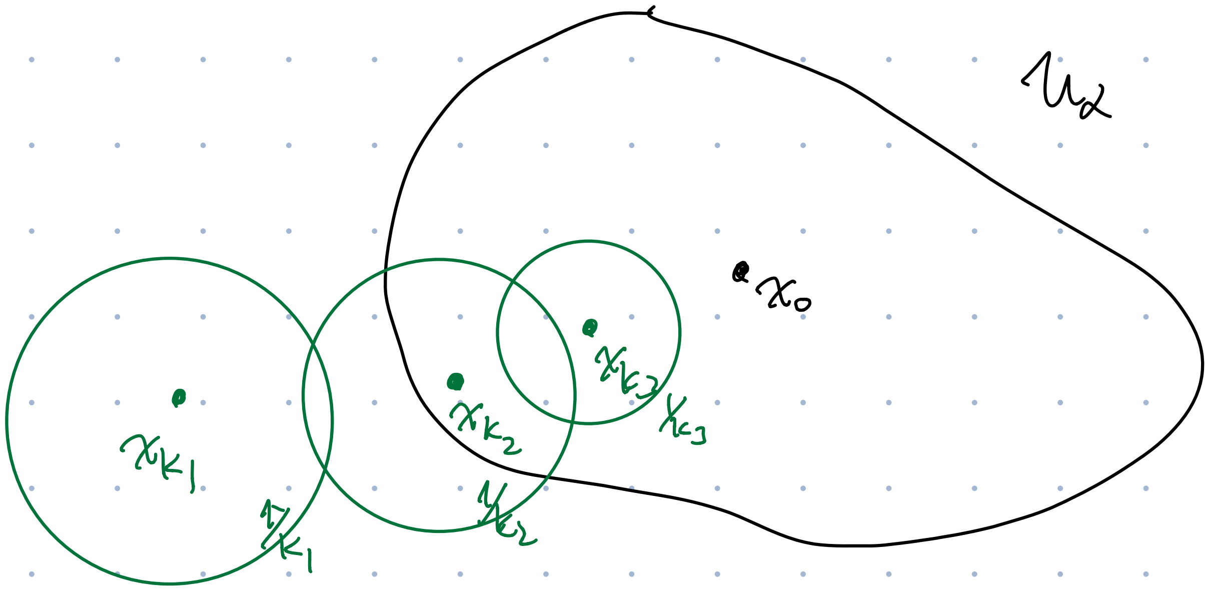
- Assume that $\forall \delta>0$, $\exists x\in K$ s.t. $B_\delta(x)\nsubseteq U_\alpha$ $\forall \alpha\in I$.
  Consider $\delta = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$
  we get a sequence $x_1, x_2, \dots, x_n, \dots$ in K
  s.t. $B_{1/n}(x_n) \nsubseteq U_\alpha$ $\forall \alpha\in I$.
- Since K is seq. cpt., so $\exists$ subseq. (x_{k_n}) s.t. $\lim_{n\to\infty} x_{k_n} = \boxed{x_0}\in K$.
- Since $\{U_\alpha\}_{\alpha\in I}$ is an open cover of K, so $\exists \alpha\in I$ st $x_0\in U_\alpha$.
- Since U_α is open, $\exists \varepsilon>0$ st $B_\varepsilon(x_0)\subseteq U_\alpha$.
- Since $\lim x_{k_n} = x_0$,
  $\exists N>0$ s.t. $d(x_{k_n}, x_0) < \varepsilon/2$ $\forall n>N$

$\Rightarrow$ $\underline{B_{\frac{\varepsilon}{2}}(x_{k_n})} \subseteq B_\varepsilon(x_0) \subseteq U_\alpha$ $\forall n>N$.



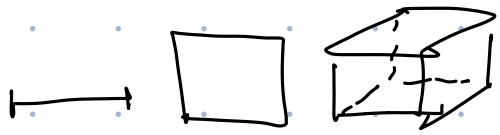
- Consider any $n > \max\{N, \frac{2}{\varepsilon}\}$

$B_{\frac{1}{k_n}}(x_{k_n}) \subseteq B_{1/n}(x_{k_n}) \subseteq B_{\frac{\varepsilon}{2}}(x_{k_n}) \subseteq U_\alpha \Rightarrow$ contradiction. $\square$



Thm (Heine-Borel) $E \subseteq \mathbb{R}^k$ compact $\Leftrightarrow$ E is closed and bounded.

Rmk: $(\Rightarrow)$ proved before



If $E \subseteq \mathbb{R}^k$ is closed and bounded,

$\Rightarrow \exists$ k-cell $I^k = [a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_k,b_k] \subseteq \mathbb{R}^k$

s.t. $E \subseteq I^k$

We'll show that $\boxed{I^k \text{ is compact.}}$

(this will imply any closed & bdd subset $E \subseteq \mathbb{R}^k$ is cpt:

(HW) $\underset{\text{closed}}{E \subseteq} X$, $\underset{\text{cpt}}{I \subseteq} X \Rightarrow \underset{\text{cpt.}}{E \cap I \subseteq} X$)

Bolzano-Weierstrass thm for $\mathbb{R}^k$: Any bounded seq. in $\mathbb{R}^k$ has a conv. subseq.

Idea: $(x_n) \subseteq \mathbb{R}^k$, $\quad x_n = (x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)}) \in \mathbb{R}^k$

$$x_1 = (x_1^{(1)}, \dots, x_1^{(k)})$$
$$x_2 = (x_2^{(1)}, \dots, x_2^{(k)})$$
$$\vdots$$

← bdd seq. in $\mathbb{R}$

By Bolzano-Weierstrass for $\mathbb{R}$ (we proved before),
$\exists\ k_1 < k_2 < k_3 < \dots$
s.t. $x_{k_1}^{(1)}, x_{k_2}^{(1)}, x_{k_3}^{(1)}, \dots$ conv. in $\mathbb{R}$

$$x_{k_1} = (x_{k_1}^{(1)}, x_{k_1}^{(2)}, \dots, x_{k_1}^{(k)})$$
$$x_{k_2} = (x_{k_2}^{(1)}, x_{k_2}^{(2)}, \dots, x_{k_2}^{(k)})$$
$$\vdots$$

conv. in $\mathbb{R}$

← bdd in $\mathbb{R}$.
$\Downarrow$ BW.
$\exists$ conv. subseq.

Repeat this process k-times, we get a subseq. (x_{ℓ_n}) of (x_n)
s.t.

$$x_{\ell_1} = (x_{\ell_1}^{(1)}, \dots, x_{\ell_1}^{(k)})$$
$$x_{\ell_2} = (x_{\ell_2}^{(1)}, \dots, x_{\ell_2}^{(k)})$$
$$x_{\ell_3} = \quad \vdots$$
$$\vdots$$

$\overset{\text{Ex:}}{\Longrightarrow}$ (x_{ℓ_n}) conv. in $\mathbb{R}^k$.

conv. seq. in $\mathbb{R}$

Prop: $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ "decreasing seq. of closed, bounded, nonempty subsets of $\mathbb{R}^k$"

$\Longrightarrow$ $F := \bigcap_{n=1}^{\infty} F_n$ is closed, bounded, nonempty

obvious.

Q: If we remove "closed" condition, is F always nonempty ??

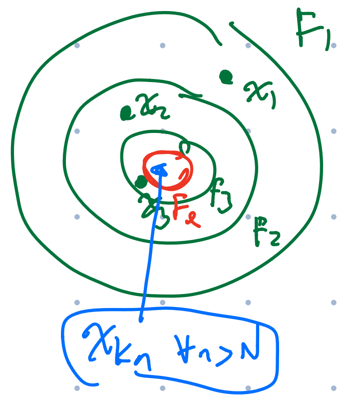
e.g. $F_n = (0, \frac{1}{n})$

Q: If we remove "bounded" condition, is F always nonempty ??

e.g. $F_n = [n, \infty)$

pf:

- Choose any pt $x_n \in F_n$ for each n.
- By Bolzano-Weierstrass, $\exists$ (x_{k_n}) subseq. of (x_n) s.t. $\lim x_{k_n} = x_0$
- Claim: $x_0 \in F$, i.e. $\boxed{x_0 \in F_\ell \quad \forall \ell}$



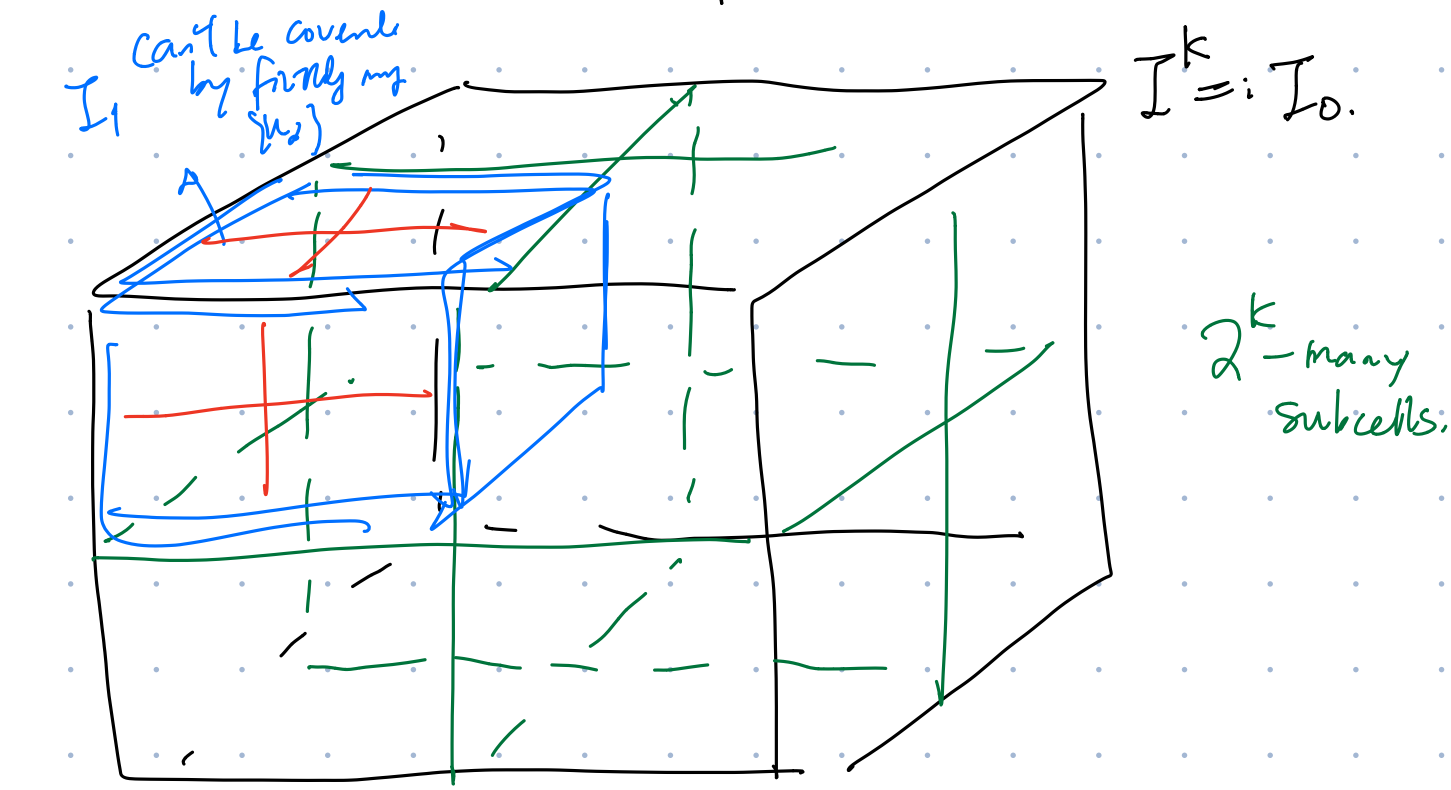
pf: $\forall \ell$, $\exists N > 0$ s.t. $k_n > \ell$ $\forall n > N$.

$\Rightarrow$ $x_{k_n} \in F_{k_n} \subseteq F_\ell$ $\forall n > N$.

$\Rightarrow$ $x_0 \in F_\ell$ since $\lim x_{k_n} = x_0$ and F_ℓ is closed.

$\square$

Proof of $I^k \subseteq \mathbb{R}^k$ is compact:



Prove by contradiction. Suppose $I^k \subseteq \mathbb{R}^k$ is not cpt.
So, $\exists$ $\{U_\alpha\}_{\alpha \in I}$ open cover of I^k which doesn't admit any finite subcover.

$I_1^{(1)}, I_1^{(2)}, \ldots, I_1^{(2^k)}$

There must exist some $I_1^{(*)}$ that can't be covered by finitely many open sets in $\{U_\alpha\}_{\alpha \in I}$.

We can get:

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \cdots I_K \supseteq \cdots$$

original k-cell diameter $= \delta$

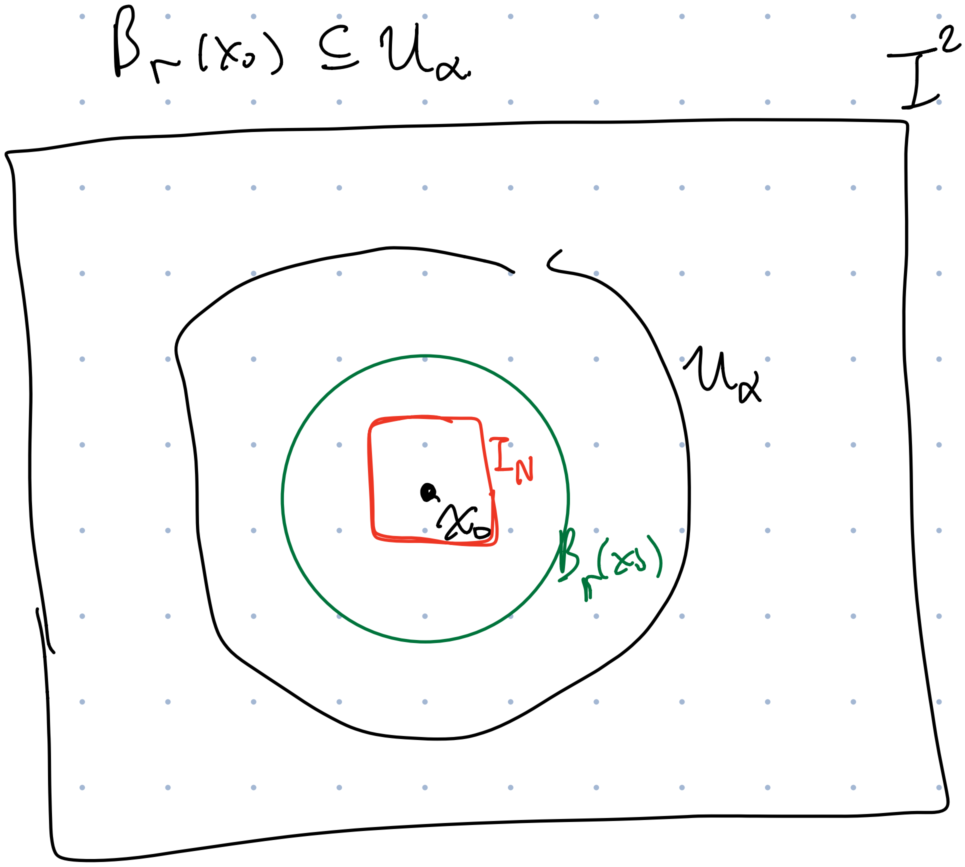
diam $= \dfrac{\delta}{2}$ diam $= \dfrac{\delta}{2^2}$ $\cdots$ diam $= \dfrac{\delta}{2^K}$

- each I_k can't be covered by finitely many open sets in $\{U_\alpha\}_{\alpha\in I}$.

- By previous Prop, $\exists\, x_0 \in \bigcap_{n=0}^{\infty} I_n$
- Since $\{U_\alpha\}_{\alpha\in I}$ is an open cover of I^k,

  so, $\exists\, \alpha\in I$ st. $x_0 \in U_\alpha$

  so, $\exists\, r>0$ st. $B_r(x_0) \subseteq U_\alpha$



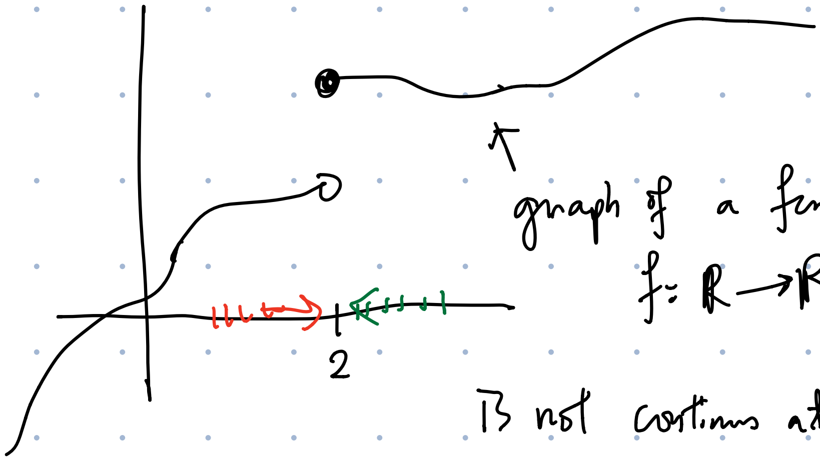
But $\exists\, N>0$ st. $\frac{\delta}{2^N} < r$

$\Downarrow$

$$I_N \subseteq B_r(x_0) \subseteq U_\alpha$$

Contradiction. $\square$

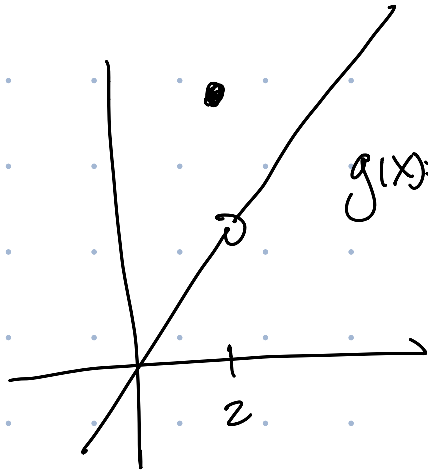
§ Continuous functions between metric spaces



graph of a fcn.

$f: \mathbb{R} \to \mathbb{R}$

is not continuous at 2



$g(x) = \begin{cases} x & x \neq 2 \\ 3 & x = 2 \end{cases}$

g is not conti. at 2.

We'll discuss 3 equivalent definitions of conti. fcns.

<u>Recall:</u> (X, d) metric, (x_n) seq. of pts in X.

Say $\lim x_n = x_0 \in X$ if

$\forall \varepsilon > 0, \ \exists N > 0$ st. $n > N \Rightarrow d(x_n, x_0) < \varepsilon$

<u>Def:</u> $f: (X, d_X) \to (Y, d_Y)$.

- Say f is <u>continuous at $x_0 \in X$</u> if:

  $\forall$ seq. (x_n) in X where $\lim x_n = x_0$,

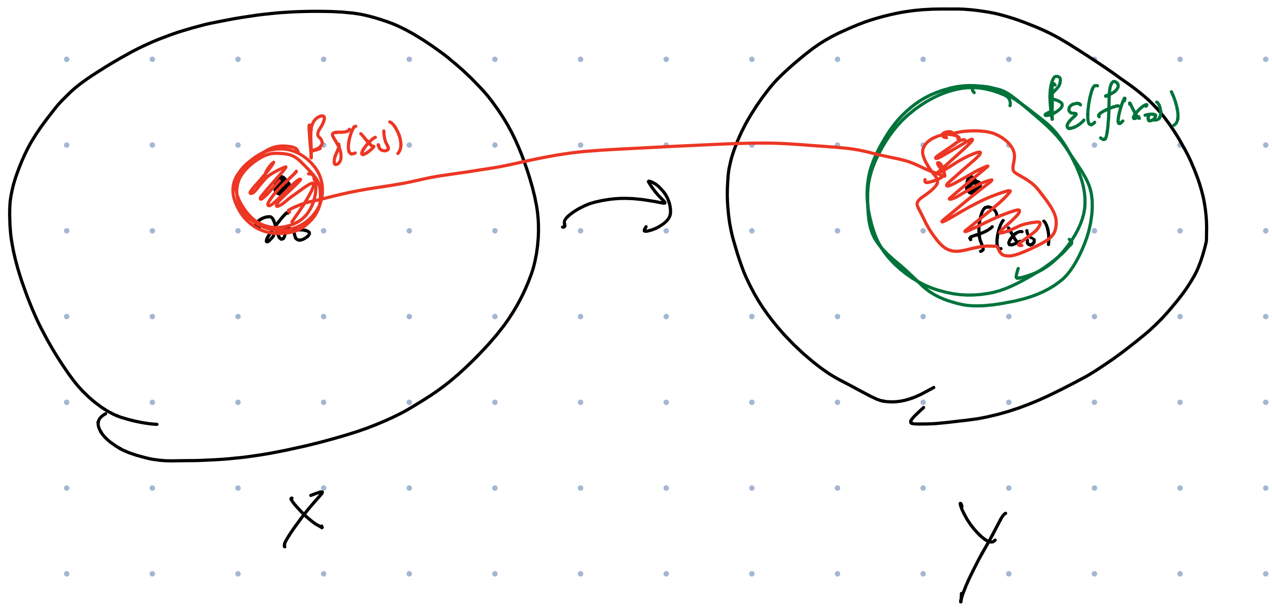
  we have: $\lim f(x_n) = f(x_0) \in Y$.

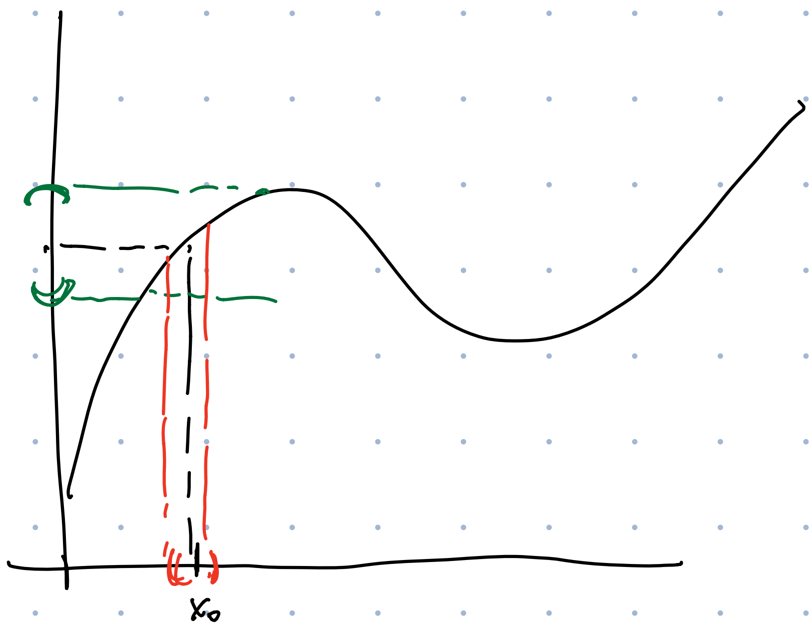
- Say f is <u>continuous</u> if it's continuous at all $x_0 \in X$.

Rmk: practically, it's difficult to use this definition to show a function is continuous.

Thm: $f: X \to Y$ continuous at $x_0 \in X$

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$





e.g. "$f(x)=x$ is conti. at $x_0 \in \mathbb{R}$."

"$\forall \varepsilon>0$, $\exists \delta>0$ st. $|x-x_0|<\delta \Rightarrow \underbrace{|f(x)-f(x_0)|}_{=|x-x_0|}<\varepsilon$"

We can just chose $\delta=\varepsilon$.

e.g. "$f(x)=x^2$ is conti. at $x_0 \in \mathbb{R}$"

"$\forall \varepsilon>0$, $\exists \delta>0$ st. $|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$"

depends on both ε, x_0.

$|f(x)-f(x_0)| = |x^2-x_0^2| = |x-x_0|\cdot|x+x_0|$

(If we choose $\delta<1$,

then $|x+x_0| \le |x-x_0| + 2|x_0| < 1+2|x_0|$)

Choose $\boxed{\delta := \min\left\{1, \dfrac{\varepsilon}{2|x_0|+1}\right\}} > 0$.

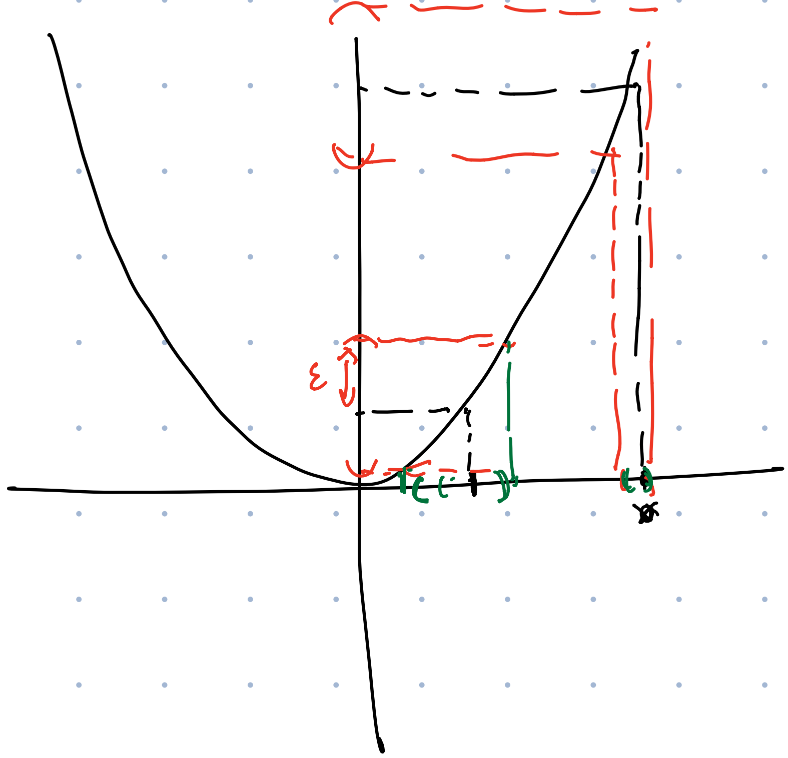
depends on both ε, x_0

Then: If $|x-x_0|<\delta \le 1$.

$\Rightarrow |x+x_0| < 1+2|x_0|$

$\Rightarrow |x^2-x_0^2| = |x-x_0|\,|x+x_0| < \delta\cdot(1+2|x_0|)$

$\le \dfrac{\varepsilon}{2|x_0|+1}\cdot(1+2|x_0|) = \varepsilon.$ $\square$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