Today: Uniformly Cauchy, Series of fns, Weierstrass M-test.

HW6. #3(a): Show $f(x) = \frac{1}{x}$ is conti. at the pt $1 \in (0, \infty)$

Issue: Many of you write " $\left|\frac{x-1}{x}\right| \leq |x-1|$ ",

which is clearly not true unless $|x| \geq 1$.

Need to show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\boxed{|x-1| < \delta} \Rightarrow \boxed{\left|\frac{1}{x} - 1\right| < \varepsilon}$$

$$\left|\frac{1}{x}-1\right| = \left|\frac{x-1}{x}\right| = \frac{|x-1|}{|x|} < \boxed{\frac{\delta}{1/2} \leq \varepsilon}$$

want to bound $|x|$ fm below $|x| > ⊚$

If we choose δ s.t. $\delta \leq \frac{1}{2}$,

then $\forall |x-1| < \delta \Rightarrow |x| > \frac{1}{2}$

pf: $\forall \varepsilon > 0$, Choose $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\} > 0$

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e.g. $f_n(x) = \frac{1}{n}\sin(nx) : \mathbb{R} \longrightarrow \mathbb{R}$. $\underline{n \geq 1}$

Q: Does (f_n) converge pointwise to some fn $f: \mathbb{R} \to \mathbb{R}$??

i.e. Is $(f_n(x_0))$ a conv. seq. in $\mathbb{R}$ $\forall x_0 \in \mathbb{R}$?

$$\lim_{n\to\infty} \frac{1}{n}\sin(nx_0) = 0 \qquad \forall x_0 \in \mathbb{R}$$

$\Rightarrow$ (f_n) conv. to the zero fn $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto 0$ pointwisely

Q: Is $f_n \to f$ uniform?? $\underline{\text{Yes}}$

$\forall \varepsilon > 0, \ \exists N > 0$ s.t.

$$|\underbrace{f_n(x)}_{\frac{1}{n}\sin(nx)} - \underbrace{f(x)}_{0}| < \varepsilon \quad \forall n \geq N, \ \forall x \in \mathbb{R}.$$

$$\left|\frac{1}{n}\sin(nx)\right| \leq \frac{1}{n} < \varepsilon \quad \forall n \geq N, \ \forall x \in \mathbb{R}.$$

Choose N large enough s.t. $N > \frac{1}{\varepsilon}$

Def: X: a set. $f_n : X \to \mathbb{R}$ seq. of real-valued fns on X.

We say (f_n) is uniformly Cauchy if

$\forall \varepsilon > 0, \ \exists N > 0$

s.t. $|f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \geq N, \ \forall x \in X.$

HW: unif. conv. $\Rightarrow$ unif. Cauchy.

Thm (f_n) unif. Cauchy $\Rightarrow$ (f_n) conv. unif. to a fn $f : X \to \mathbb{R}$

pf: 1° "$\lim_{n\to\infty} f_n(x)$ exists $\forall x \in X$."

(f_n) unif. Cauchy $\Rightarrow$ $\forall x \in X$, $(f_n(x))$ is a Cauchy seq.

$\Rightarrow \lim_{n\to\infty} f_n(x)$ exists

So, we define $f : X \to \mathbb{R}$

by letting $f(x) := \lim_{n\to\infty} f_n(x)$.

2^o: "$f_n \to f$ uniformly" (need: $\forall \varepsilon>0$, $\exists N>0$ st. $|f_n(x)-f(x)|<\varepsilon$ $\forall n>N$ $\forall x\in X$)

$\forall \varepsilon>0$, $\exists N>0$

st. $|f_n(x)-f_m(x)| < \varepsilon/2$ $\quad \forall n,m>N$, $\forall x\in X$.

$$\Rightarrow \quad f_m(x)-\varepsilon/2 < f_n(x) < f_m(x)+\varepsilon/2 \quad \forall n,m>N \ \forall x\in X.$$

For any $m>N$, we have:

$$f_m(x)-\varepsilon/2 < \underbrace{f_n(x)}_{\substack{\downarrow \\ f(x) \\ \text{as } n\to\infty}} < f_m(x)+\varepsilon/2 \quad \forall n>N \ \forall x\in X$$

$$\boxed{\begin{gathered} A < a_n < B \\ \lim a_n = C \\ A \le C \le B \end{gathered}}$$

$$\Rightarrow \quad \boxed{f_m(x)-\varepsilon/2 \le f(x) \le f_m(x)+\varepsilon/2}$$

$$\Rightarrow \quad |f_m(x)-f(x)| \le \varepsilon/2 < \varepsilon. \quad \forall m>N, \ \forall x\in X$$

$\square$

Series of functions: Given a seq. of fns $f_n : \overset{\text{set}}{X} \to \mathbb{R}$

partial sums.
$$S_n : X \longrightarrow \mathbb{R}$$
$$x \longmapsto \sum_{k=1}^{n} f_k(x)$$

If S_n converges pointwisely to some fn on X,

then we define the series of fn: $\boxed{\sum_{n=1}^{\infty} f_n} : X \longrightarrow \mathbb{R}$.

where: $\left(\sum_{n=1}^{\infty} f_n\right)(x) := \lim_{n\to\infty} S_n(x)$

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e.g. $X = (-1,1) \subseteq \mathbb{R}$, $f_n(x) = x^n$. $\forall n \geq 0$.

$S_n(x) = 1 + x + \cdots + x^n$, $\lim_{n\to\infty} S_n(x) = \frac{1}{1-x}$ $\forall x \in (-1,1)$

" $\sum_{n=0}^{\infty} x^n$ " $= \frac{1}{1-x}$ $\forall x \in (-1,1)$.

We say the series $\sum_{n=1}^{\infty} f_n$ converges uniformly, if the seq. of fn: $(S_n(x))$ conv. uniformly.

Use uniform Cauchy criterion:

$\sum_{n=1}^{\infty} f_n$ conv. unif.

$\iff \forall \varepsilon > 0, \exists N > 0$

s.t. $|S_n(x) - S_m(x)| < \varepsilon$ $\forall n, m > N, \forall x \in X$

$$S_n(x) = \sum_{k=1}^{n} f_k(x), \quad S_m(x) = \sum_{k=1}^{m} f_k(x)$$

$\iff \forall \varepsilon > 0, \exists N > 0$ s.t. $\left|\sum_{k=m}^{n} f_k(x)\right| < \varepsilon$ $\forall x \in X$ $\forall n \geq m \geq N$.

(Recall: series (of real number) $\boxed{\sum a_n}$ conv.

$\iff \forall \varepsilon > 0, \exists N > 0$

s.t. $\left|\sum_{k=m}^{n} a_k\right| < \varepsilon$ $\forall n \geq m > N$)

Coro: If $\sum f_n$ conv. unif., then
$$\lim_{n\to\infty} \sup\{|f_n(x)| : x \in X\} = 0.$$

Recall: $\sum a_n$ conv. $\Rightarrow \lim a_n = 0$

pf: $\sum f_n$ conv. unif

$\iff \forall \varepsilon > 0, \exists N > 0$ st. $\left|\sum_{k=m}^{n} f_k(x)\right| < \varepsilon \quad \forall n \geq m > N, \ \forall x \in X.$

$\implies \forall \varepsilon > 0, \exists N > 0$ st. $\boxed{|f_n(x)| < \varepsilon} \quad \forall \underline{n > N}, \ x \in X.$

$\Rightarrow \underline{\sup\{|f_n(x)| : x \in X\}} \leq \varepsilon. \quad \forall \underline{n > N}.$

$\implies \lim_{n\to\infty} \sup\{|f_n(x)| : x \in X\} = 0.$ $\square$

Weierstrass M-test: Suppose (M_n) seq. of real numbers. $M_n \geq 0$, $\sum M_n < +\infty$.

Suppose (f_n): $f_n : X \longrightarrow \mathbb{R}$

- $|f_n(x)| \leq M_n \quad \forall x \in X, \quad \forall n.$

$\implies \sum f_n$ conv. unif.

pf: $\forall \varepsilon > 0, \exists N > 0$ st. $0 \leq \sum_{k=m}^{n} M_k < \varepsilon \quad \forall n \geq m > N.$

$\Rightarrow \left|\sum_{k=m}^{n} f_k(x)\right| \leq \sum_{k=m}^{n} |f_k(x)| \leq \sum_{k=m}^{n} M_k < \varepsilon.$ $\square$

e.g.: $\sum_{n=0}^{\infty} x^n$

We know this series is conv. $\iff x \in (-1,1)$

$\boxed{\text{Q: Does } \sum_{n=0}^{\infty} x^n \text{ conv. unif on } (-1,1)??}$

No:

$\sup\{|x^n| : x \in (-1,1)\} = 1.$

So, $\lim_{n\to\infty} \sup\{|x^n| : x \in (-1,1)\} \neq 0$

$\Rightarrow \sum_{n=0}^{\infty} x^n$ doesn't conv. unif. on $(-1,1)$.

Q: Does $\sum_{n=0}^{\infty} x^n$ conv. unif on $[-R,R]$ where $R<1$??

Yes: On $[-R,R]$, $\underbrace{|x^n|}_{f_n(x)} \le \underbrace{R^n}_{M_n}$

$\sum R^n < +\infty$

$\Rightarrow \sum x^n$ conv. unif by Weierstrass M-test. $\square$

Rmk: $\Rightarrow \sum_{n=0}^{\infty} x^n$ is conti. on $[-R,R]$ $\forall R<1$.

$\Rightarrow \sum_{n=0}^{\infty} x^n$ is conti. on $(-1,1)$.