Last time: $y'' + by' + cy = 0$. —— ($\star$)

Consider the auxiliary eq^n $r^2 + br + c = 0$. —— (*)

- If (*) has two real roots $r_1 \neq r_2$, then general sol^n of ($\star$):
  $$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad \text{where } c_1, c_2 \in \mathbb{R}.$$
- If (*) has a unique root r, then general sol^n of ($\star$):
  $$y(t) = c_1 e^{rt} + c_2 t e^{rt}$$
- If (*) has two complex roots $\alpha \pm i\beta$, ($\beta \neq 0$), general sol^n:
  $$y(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t).$$

Today: non-homogeneous eq^n $y'' + by' + cy = f$. —— ($\star\star$)

---

e.g. $\boxed{y'' + by' + \overset{\overset{0}{\#}}{c}y = 1.}$ $\quad$ $y_0 = \frac{1}{c}$ is a sol^n.

- If y_1 and y_2 are l.i. sol^n to $\boxed{y'' + by' + cy = 0}$

  then $$y = c_1 y_1 + c_2 y_2 + y_0$$

  is a sol^n to $y'' + by' + cy = 1$.

Strategy: $y'' + by' + cy = f$ finding general sol^n to

① find a $\underline{\text{particular sol}^n}$ (y_0) to $y'' + by' + cy = f$

② find $\underline{\text{general sol}^n}$ ($c_1 y_1 + c_2 y_2$) to $y'' + by' + cy = 0$

③ Then general sol^n to $y'' + by' + cy = f$ is:

$$c_1 y_1 + c_2 y_2 + y_0$$

e.g. $y'' + by' + cy = t$

$y(t) = A_1 t + A_0$

$y'(t) = A_1$

$y''(t) = 0$

$$y'' + by' + cy = b(A_1) + c(A_1 t + A_0)$$
$$= \underbrace{cA_1}_{=1} t + \underbrace{(bA_1 + cA_0)}_{=0}$$

$A_1 = \frac{1}{c}$, $A_0 = \frac{-b}{c^2}$

e.g. $\boxed{y'' + by' + cy =} \; a_k t^k + a_{k-1} t^{k-1} + \cdots + a_0$

Guess: $y(t) = \boxed{B_k t^k + \cdots + B_0}$

Try to find $B_k, \ldots, B_0$ s.t. $y(t)$ satisfies

$$y'(t) = kB_k t^{k-1} + \cdots + 2B_2 t + B_1$$
$$y''(t) = k(k-1) B_k t^{k-2} + (k-1)(k-2) B_{k-1} t^{k-3} + \cdots + 2B_2.$$

$$\begin{aligned} y'' + by' + cy = \; & cB_k t^k + cB_{k-1} t^{k-1} + cB_{k-2} t^{k-2} + \cdots + cB_0 \\ & + bkB_k t^{k-1} + \cdots + bB_1 \\ & + k(k-1) B_k t^{k-2} + \cdots + 2B_2 \end{aligned}$$

$$a_k t^k + \cdots + a_0$$

$$cB_k = a_k \qquad cB_{k-1} + bkB_k = a_{k-1}$$

$$cB_{k-2} + b(k-1)B_{k-1} + k(k-1)B_k = a_{k-2}$$

$$\begin{bmatrix} c & 0 & 0 & \cdots & 0 \\ bk & c & 0 & \cdots & 0 \\ k(k-1) & b(k-1) & c & 0 \cdots & 0 \\ & \ddots & \ddots & c \ddots & 0 \\ 0 & & \ddots & \ddots & c \end{bmatrix} \begin{bmatrix} B_k \\ B_{k-1} \\ \vdots \\ \vdots \\ B_0 \end{bmatrix} = \begin{bmatrix} a_k \\ a_{k-1} \\ \vdots \\ \vdots \\ a_0 \end{bmatrix}$$

e.g. $\underline{y'' + by' + cy = e^{rt}}$ for some $r \in \mathbb{R}$.

Guess $y(t) = Ae^{rt}$

$y'(t) = rAe^{rt}$

$y''(t) = r^2 Ae^{rt}$

$$y'' + by' + cy = \boxed{Ae^{rt}(r^2 + br + c)} = e^{rt}$$

$$\boxed{A = \frac{1}{r^2 + br + c}}$$

(only works when $r^2 + br + c \neq 0$

i.e. r is not a sol^n to the auxiliary eq'n $r^2 + br + c$)

When r is a sol^n to the auxiliary eq^n

i.e. $\boxed{r^2 + br + c = 0}$ we need to modify this

$y'' + by' + cy = e^{rt}$.

Guess: $y(t) = A\, t e^{rt}$.

$$y' = A e^{rt} + A t \cdot r e^{rt}$$

$$= A e^{rt} (1 + rt)$$

$$y'' = A e^{rt} \cdot r(1+rt) + A e^{rt} (r)$$

$$= A e^{rt} (2r + r^2 t)$$

$$y'' + by' + cy = A e^{rt} \left(2r + r^2 t + b + brt + ct \right)$$

Since $r^2+br+c=0$ $\Rightarrow$ $= A e^{rt} (2r + b) \overset{??}{=} e^{rt}$

$$\boxed{A = \frac{1}{2r+b}}.$$

only works when $2r+b \neq 0$

$y'' + by' + cy = e^{r_0 t}$

- $r_0^2 + br_0 + c \neq 0 \Rightarrow \dfrac{1}{r_0^2 + br_0 + c} e^{r_0 t}$ is a sol$^{\underline{n}}$
- $r_0^2 + br_0 + c = 0$, $2r_0 + b \neq 0 \Rightarrow \dfrac{1}{2r_0+b} t e^{r_0 t}$ is a sol$^{\underline{n}}$
- $\boxed{r_0^2 + br_0 + c = 0,\ 2r_0 + b = 0} \Rightarrow \dfrac{1}{2} t^2 e^{r_0 t}$ is a sol$^{\underline{n}}$

$\Updownarrow$

r_0 is the double root of $r^2+br+c=0$.

$$b = -2r_0$$

$$r_0^2 + (-2r_0)r_0 + c = 0 \Rightarrow c = r_0^2$$

$$\begin{cases} b = -2r_0 \\ c = r_0^2 \end{cases}$$

Auxiliry eq'n: $r^2 + br + c = 0$

$$\begin{aligned} &r^2 + br + c = 0 \\ &\quad\| \\ &r^2 - 2r_0 r + r_0^2 \\ &\quad\| \\ &(r - r_0)^2 \end{aligned}$$

$$\boxed{r^2 + br + c = 0,\ 2r + b = 0} \qquad y'' + by' + cy = e^{rt}.$$

Guess: $y(t) = At^2 e^{rt}$

$$y'(t) = A \cdot 2t \cdot e^{rt} + At^2 \cdot re^{rt}$$
$$= Ae^{rt}(2t + t^2 r)$$
$$y''(t) = Are^{rt}(2t + t^2 r) + Ae^{rt}(2 + 2rt)$$

$$y'' + by' + cy = Ae^{rt}\big(2rt + r^2t^2 + 2 + 2rt + 2bt + brt^2 + ct^2\big)$$

$$= 2Ae^{rt} \cancel{=} e^{rt} \Rightarrow A = \frac{1}{2}$$

Method of undetermined coeff.: (textbook)

$$y'' + by' + cy = f$$

- $f = t^m e^{r_0 t}$

$\Rightarrow$ there is a sol^n of the form

$$y(t) = t^s (A_m t^m + \cdots + A_0) e^{r_0 t}.$$

where

- $s=0$, If r_0 is not a sol^n of $r^2 + br + c = 0$
- $s=1$, If r_0 is a single root of ———.
- $s=2$, If r_0 is a double root ———.

- $f = t^m e^{\alpha t} \cos\beta t$ or $t^m e^{\alpha t} \sin\beta t$

$\Rightarrow$ there is a sol^n of the form

$$y(t) = t^s (A_m t^m + \cdots + A_0) e^{\alpha t} \cos\beta t + t^s (B_m t^m + \cdots + B_0) e^{\alpha t} \sin\beta t.$$

where

- $s=0$ If $\alpha \pm i\beta$ not a sol^n of $r^2 + br + c = 0$
- $s=1$ if $\alpha \pm i\beta$ are sol^n of $r^2 + br + c = 0$.

---

e.g. $y'' + 2y' + y = t^2 + te^{-t}$

$$\boxed{y(0) = 2, \; y'(0) = 1}$$

① $y''+2y'+y=t^2 \rightarrow y_1$

② $y''+2y'+y=te^{-t} \rightarrow y_2$

③ $y''+2y'+y=0 \rightarrow \{y_3, y_4\}$

$\Rightarrow \boxed{c_1 y_3 + c_2 y_4 + y_1 + y_2}$ general sol^n^

---

① $y(t) = A_2 t^2 + A_1 t + A_0$

$2y'(t) = \quad 4A_2 t + 2A_1$

$y'' = \quad 2A_2$

$1t^2 + 0t + 0$

$A_2 = 1$

$A_1 = -4$

$A_0 = 6$

$y_1(t) = t^2 - 4t + 6$

---

② $y''+2y'+y = te^{-t}$

$y(t) = t^2(A_1 t + A_0)e^{-t}$

find A_1, A_0 st

$y_2(t) = \frac{1}{6}t^3 e^{-t}$

---

③ $c_1 e^{-t} + c_2 t e^{-t}$

$c_1 e^{-t} + c_2 t e^{-t} + t^2 - 4t + 6 + \frac{1}{6}t^3 e^{-t} \Rightarrow y(t)$

Find c_1, c_2 s.t. $\begin{cases} y(0) = 2 = c_1 + 6 \\ y'(0) = 1 = -c_1 + c_2 - 4 \end{cases}$

$$y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} + 2t - 4 + \frac{1}{2} t^2 e^{-t} - \frac{1}{6} t^3 e^{-t}$$

$\Rightarrow c_1 = -4, \quad c_2 = 1$

$$y(t) = -4e^{-t} + te^{-t} + t^2 - 4t + 6 + \frac{1}{6} t^3 e^{-t}$$

More generally, "Variation of parameters"

$$y'' + by' + cy = f(t)$$

Say $\{y_1(t), y_2(t)\}$ l.i. sol^{s} to $y'' + by' + cy = 0$

$\Rightarrow$ general sol^n of $y'' + by' + cy = 0$ is:

$$c_1 y_1(t) + c_2 y_2(t)$$

Find fcns $c_1(t), c_2(t)$ s.t.

$\boxed{y(t) = c_1(t) y_1(t) + c_2(t) y_2(t)}$ solves $y'' + by' + cy = f(t)$

$$\underline{y' = c_1' y_1 + c_1 y_1' + c_2' y_2 + c_2 y_2'}$$

Impose an extra condition: $\boxed{c_1'(t)\, y_1(t) + c_2'(t)\, y_2(t) = 0}$

Then $y' = c_1 y_1' + c_2 y_2'$

$$y'' = c_1' y_1' + c_1 y_1'' + c_2' y_2' + c_2 y_2''$$

$$y'' + by' + cy = c_1' y_1' + \underline{c_1 y_1''} + c_2' y_2' + \underline{c_2 y_2''}$$
$$+ \underline{b c_1 y_1'} + \underline{b c_2 y_2'} + \underline{c c_1 y_1} + \underline{c c_2 y_2}$$

$$= c_1' y_1' + c_2' y_2' \cancel{\overset{??}{=}} f$$

If we can find $c_1(t), c_2(t)$ s.t.

$$\begin{cases} c_1'(t)\, y_1(t) + c_2'(t)\, y_2(t) = 0 \\ c_1'(t)\, y_1'(t) + c_2'(t)\, y_2'(t) = f(t) \end{cases}$$

Then $y(t) = c_1 y_1 + c_2 y_2$ is a sol$^{\underline{n}}$ to $y'' + by' + cy = f$

$$\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

Invertible since $\{y_1, y_2\}$ is l.i.

$$\begin{cases} c_1'(t) = \dfrac{-f(t)\,y_2(t)}{y_1(t)\,y_2'(t) - y_2(t)\,y_1'(t)} \\[2ex] c_2'(t) = \dfrac{f(t)\,y_1(t)}{y_1(t)\,y_2'(t) - y_2(t)\,y_1'(t)} \end{cases}$$

$\Rightarrow$ We can find c_1, c_2 by integrating

---

e.g. find a sol^n on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to

$$y'' + y = \tan t$$

- $y'' + y = 0$ assoc. eq: $r^2 + 1 = 0$, roots $\pm i$

  $\{\cos t, \sin t\}$ l.i. sol$^{\text{ns}}$

$$\boxed{y_1(t) = \cos t, \quad y_2(t) = \sin t, \quad f(t) = \tan t}$$

$$\begin{cases} c_1'(t) = \dfrac{-f(t)\,y_2(t)}{y_1(t)\,y_2'(t) - y_2(t)\,y_1'(t)} \\[2ex] c_2'(t) = \dfrac{f(t)\,y_1(t)}{y_1(t)\,y_2'(t) - y_2(t)\,y_1'(t)} \end{cases}$$

$$c_1'(t) = \frac{-\tan t \cdot \sin t}{1} = \frac{-\sin^2 t}{\cos t}$$

$$c_2'(t) = \frac{\tan t \cos t}{1} = \sin t$$

$$c_1(t) = \int_0^t \frac{-\sin^2 s}{\cos s}\, ds = -\int_0^t \frac{1-\cos^2 s}{\cos s}\, ds$$

$$= \sin t - \log|\sec t + \tan t| + \text{const.}$$

$$c_2(t) = \int_0^t \sin s\, ds = -\cos t + \text{cont.}$$

$$\Rightarrow\ y(t) = \left(\sin t - \log|\sec t + \tan t| + \text{const.}\right)\cos t$$

$$+ \left(-\cos t + \text{const}\right)\sin t$$

$$= -\left(\log|\sec t + \tan t|\right)\cdot\cos t$$

$$+ (\text{const.})\cos t + (\text{const.})\sin t$$