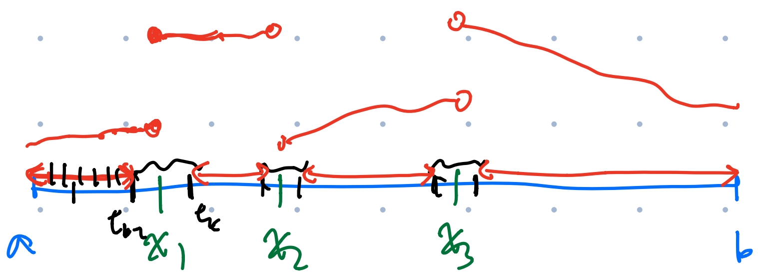
Thm (Riemann-Lebesgue): $f : [a,b] \longrightarrow \mathbb{R}$ bounded fcn.
Then f is integrable
$\iff$ $\{x \in [a,b] \mid f : \text{disconti. at } x\}$ has measure zero

---

e.g. Suppose $f : [a,b] \longrightarrow \mathbb{R}$ bdd fcn. $\nearrow$ $m \le |f(x)| < M \quad \forall x \in [a,b]$
and suppose f has finitely many disconti. pts



$\{a = t_0 < t_1 < \cdots < t_n = b\}$
$\parallel$
Want: f is integrable.
i.e. $\forall \varepsilon > 0$, we need to find a partition P
s.t. $\boxed{U(f,P) - L(f,P)} < \varepsilon$.

$$\| $$

$$\sum_{k=1}^{n} (t_k - t_{k-1}) \left(\sup_{x \in [t_{k-1}, t_k]} f(x) - \inf_{x \in [t_{k-1}, t_k]} f(x) \right)$$

- Suppose $[t_{k-1}, t_k]$ contains a disconti. pt.,
  then we don't have a good control on
  $$\sup_{x \in [t_{k-1}, t_k]} f(x) - \inf_{x \in [t_{k-1}, t_k]} f(x) \quad < M - m$$

But, we can choose t_{k-1}, t_k to be very close to x_i
to make $t_k - t_{k-1}$ very small.
$$< \frac{\varepsilon}{2(M-m)} \cdot \frac{1}{3}$$

• ~~For $[t_{k-1}, t_k]$~~ where f is conti. $\Rightarrow$ f is unif conti.

$\forall \varepsilon > 0 \quad \exists \delta > 0$

s.t. $|x-y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \dfrac{\varepsilon}{2(b-a)}$

$x, y \in [t_{k-1}, t_k]$

So, on the conti. part, if we choose partition so that each of the subinterval has length $< \delta$

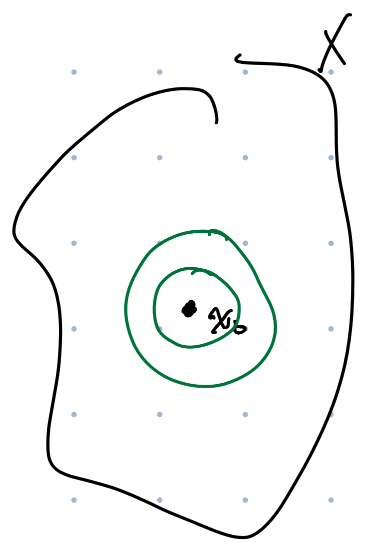
Then, on each of these subintervals, we have:

$$\sup f - \inf f < \frac{\varepsilon}{2(b-a)}$$

<u>Def</u>: X: metric space. $\quad f: X \longrightarrow \mathbb{R}$. bdd

$\cup$

$\mathcal{U}$

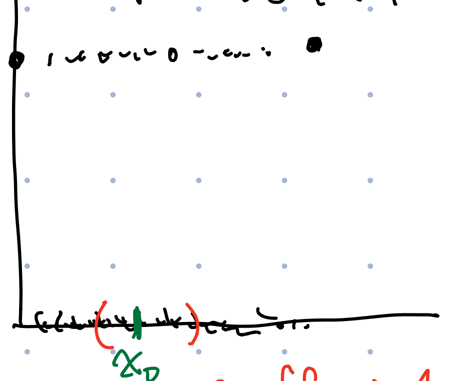


• $\operatorname{osc}(f, \mathcal{U}) := \sup\limits_{x \in \mathcal{U}} f(x) - \inf\limits_{x \in \mathcal{U}} f(x)$

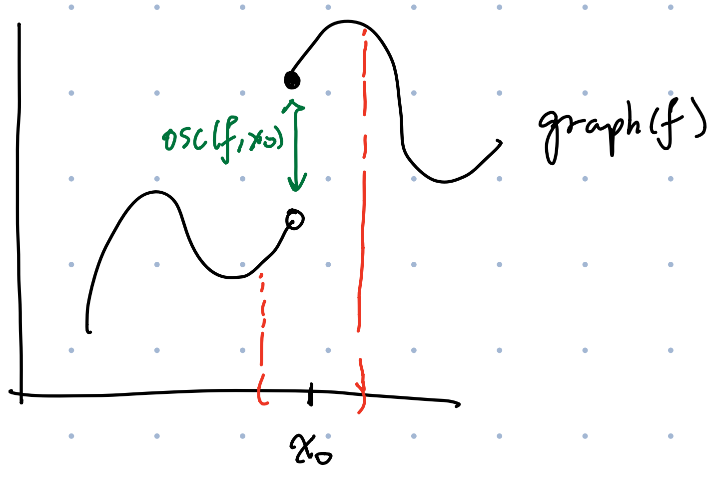
• $\operatorname{osc}(f, x_0) := \lim\limits_{\delta \to 0} \operatorname{osc}(f, B_\delta(x_0))$

$x_0 \in X$

$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$



$\operatorname{osc}(f, x_0) = 1.$



Ex: $\lim_{\delta \to 0} osc(f, B_\delta(x_0))$ exists.

Ex: f conti. at x_0 $\iff$ $osc(f, B_\delta(x_0)) = 0$.

Rmk: $osc(f, x_0)$ "measures" how disconti. f is at ~~a~~ the point x_0.

Thm(A): $f : [a,b] \longrightarrow \mathbb{R}$ bdd, integrable.
Then $A := \{x \in [a,b] \mid osc(f,x) > 0\}$ has measure zero.
$\left(= \{x \in [a,b] \mid f \text{ is disconti. at } x\} \right)$

pf: $A = \bigcup_{k=1}^{\infty} A_k$ where
$$A_k = \{x \in [a,b] \mid osc(f,x) \geq \tfrac{1}{k}\}.$$

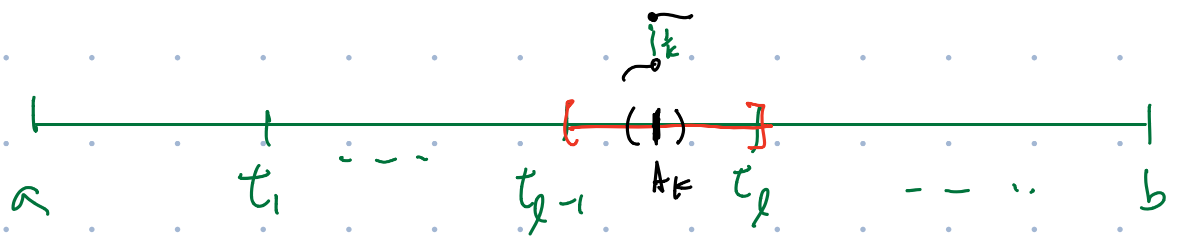
It's enough to show: A_k has measure zero $\forall k$.

(i.e. $\forall \varepsilon > 0$, we need find a (countably many) open intervals that covers A_k, with total length $< \varepsilon$)

f is integrable, $\forall \varepsilon > 0$, $\exists P$ st. $U(f,P) - L(f,P) < \varepsilon / k$

$$U(f,P) - L(f,P) = \sum (t_k - t_{k-1}) \left(\sup_{x \in [t_{k-1}, t_k]} f - \inf_{x \in [t_{k-1}, t_k]} f \right)$$

(Let's assume for now that each $t_\ell \notin A_k$)



$a = t_0$ $\quad$ $b = t_n$

Suppose $[t_{\ell-1}, t_\ell]$ contains some pts in A_k.

$$\Rightarrow \sup_{x \in [t_{\ell-1}, t_\ell]} f - \inf_{x \in [t_{\ell-1}, t_\ell]} f \geq \frac{1}{k}$$

$$\frac{\varepsilon}{k} > U(f,P) - L(f,P)$$

$$= \sum (t_\ell - t_{\ell-1}) \Big(\sup_{x\in(t_{\ell-1},t_\ell]} f(x) - \inf_{x\in[\cdots]} f(x) \Big)$$

$$\geq \sum_{\underline{[t_{\ell-1},t_\ell]}\cap A_k \neq \phi} (t_\ell - t_{\ell-1}) \underbrace{\Big(\sup_{x\in[\cdots} f(x) - \inf_{x\in[\cdots]} f(x) \Big)}_{\geq \frac{1}{k}}$$

$$\geq \frac{1}{k} \sum_{[t_{\ell-1},t_\ell]\cap A_k \neq \phi} (t_\ell - t_{\ell-1})$$

$$\Rightarrow \quad \varepsilon > \sum_{[t_{\ell-1},t_\ell]\cap A_k \neq \phi} (t_\ell - t_{\ell-1}). \qquad \square$$

Thm(B): $f : [a,b] \to \mathbb{R}$ bdd.

Suppose $A = \{x\in[a,b] \mid \operatorname{osc}(f,x) > 0\}$ has measure zero.

Then f is integrable.

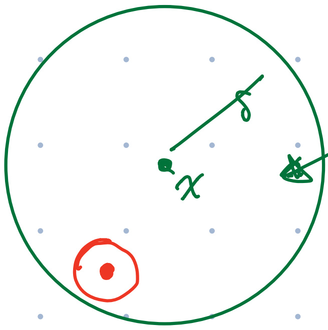
Lemma: $A_s := \{x\in[a,b] \mid \operatorname{osc}(f,x) \geq s\}$ is compact. $s\in\mathbb{R}$.

Pf: Need to show: $(A_s)^c$ is open.

$$(A_s)^c = \{x \mid \operatorname{osc}(f,x) < s\}.$$

$x\in(A_s)^c, \quad \operatorname{osc}(f,x) = t < s$

$\exists\, \delta > 0$ s.t. $\operatorname{osc}(f, B_\delta(x)) < \frac{t+s}{2} < s$



$\sup - \inf < \frac{t+s}{2}$

Claim: $\bullet \in B_\delta(x) \subseteq (A_s)^c$

---

pf of Thm (B): $A = \{x \in [a,b] \mid \operatorname{osc}(f,x) > 0\}$ measure zero

$\cup$

$\boxed{\forall \varepsilon > 0}$

Goal: Find P s.t. $U(f,P) - L(f,P) < \varepsilon$

$A_{\frac{\varepsilon}{2(b-a)}} = \left\{x \in [a,b] \mid \operatorname{osc}(f,x) \geq \frac{\varepsilon}{2(b-a)}\right\}$.

has measure zero, $\boxed{\text{compact set}}$

$\downarrow$

- $\exists$ open intervals $\{I_1, I_2, \cdots I_N\}$

s.t. $A_{\frac{\varepsilon}{2(b-a)}} \subseteq \bigcup_{n=1}^{\infty} I_n$

and $\sum \text{length}(I_n) < \frac{\varepsilon}{2(M-m)}$

---

compact

For $x \in \boxed{[a,b] \setminus \bigcup_{i=1}^{N} I_i}$, we have $\operatorname{osc}(f,x) < \frac{\varepsilon}{2(b-a)}$

$\Rightarrow \exists\, \delta_x > 0$ s.t. $\operatorname{osc}(f, B_{\delta_x}(x)) < \frac{\varepsilon}{2(b-a)}$

$\|$

$$= (x-\delta_x,\ x+\delta_x)$$

- $\{(x-\delta_x,\ x+\delta_x)\}_{x\in[a,b]\setminus \cup I_i}$ forms an open covering of the cpt set $[a,b]\setminus \cup I_i$

$\Rightarrow$ finite subcover:

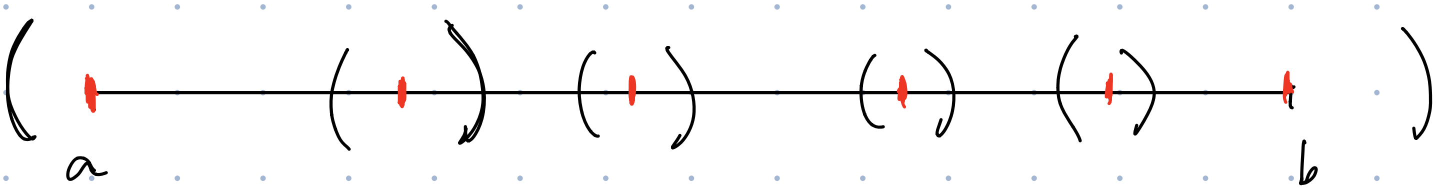
$$(x_1-\delta_1,\ x_1+\delta_1), \dots, (x_k-\delta_k,\ x_k+\delta_k)$$

$\mathcal{F} = \{$ • $I_1, \dots, I_N \}$ → ① $\sum \text{length}(I_i) < \frac{\varepsilon}{2(M-m)}$

$\mathcal{G} = \{$ • $(x_1-\delta_1,\ x_1+\delta_1), \dots, (x_k-\delta_k,\ x_k+\delta_k) \}$ → $\operatorname{osc}(f, (x_i-\delta_i,\ x_i+\delta_i)) < \frac{\varepsilon}{2(b-a)}$

covers $[a,b]$

Choose a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ so that each $[t_{i-1}, t_i]$ is completely in one of these open intervals



$$U(f,P) - L(f,P) = \sum (t_i - t_{i-1}) \left(\sup_{[t_{i-1},t_i]} f - \inf_{[t_{i-1},t_i]} f\right)$$

$$= \sum_{[t_{i-1},t_i]\subseteq \mathcal{F}} (t_i - t_{i-1}) \underbrace{\boxed{\left(\sup_{[t_{i-1},t_i]} f - \inf_{[t_{i-1},t_i]} f\right)}}_{\leq M-m}$$

$$+ \sum_{[t_{i-1},t_i]\subseteq \mathcal{G}} (t_i - t_{i-1}) \underbrace{\boxed{\left(\sup_{[t_{i-1},t_i]} f - \inf_{[t_{i-1},t_i]} f\right)}}_{< \frac{\varepsilon}{2(b-a)}}$$

$$< (M-m) \cdot \underbrace{\boxed{\sum_{[t_{i-1},t_i]\subseteq \mathcal{F}} (t_i - t_{i-1})}}_{< \frac{\varepsilon}{2(M-m)}}$$

$$+ \frac{\varepsilon}{2(b-a)} \underbrace{\boxed{\sum_{[t_{i-1},t_i]\subseteq \mathcal{G}} (t_i - t_{i-1})}}_{< b-a}$$

$$< \varepsilon. \quad \square$$