SOLUTION OF FINAL EXAM
MATH H54

Part I: Problems (1)-(9), each worth 14 points, maximum capped at 80 points.
Part II: Problems (10)-(12), each worth 10 points, maximum capped at 20 points.

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(1) (14 points) Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a linearly independent set of vectors in a real vector space V. Prove that

$$\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \dots, \vec{v}_{n-1} + \vec{v}_n, \vec{v}_n + \vec{v}_1\}$$

is linearly independent if and only if n is odd (not divisible by 2).
(Note: This is equivalent to show that the set of vectors is linearly independent if n is odd, and is linearly dependent if n is even.)

Solution. If n is even, then we have

$$(\vec{v}_1 + \vec{v}_2) + (\vec{v}_3 + \vec{v}_4) + \cdots + (\vec{v}_{n-1} + \vec{v}_n) = (\vec{v}_2 + \vec{v}_3) + (\vec{v}_4 + \vec{v}_5) + \cdots + (\vec{v}_n + \vec{v}_1),$$

which implies that the set $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \dots, \vec{v}_{n-1} + \vec{v}_n, \vec{v}_n + \vec{v}_1\}$ is linearly dependent.

If n is odd, we'd like to show that $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \dots, \vec{v}_{n-1} + \vec{v}_n, \vec{v}_n + \vec{v}_1\}$ is linearly independent. Suppose there exists $a_1, \dots, a_n \in \mathbb{R}$ such that

$$\begin{aligned} 0 &= a_1(\vec{v}_1 + \vec{v}_2) + a_2(\vec{v}_2 + \vec{v}_3) + \cdots + a_n(\vec{v}_n + \vec{v}_1) \\ &= (a_n + a_1)\vec{v}_1 + (a_1 + a_2)v_2 + \cdots + (a_{n-1} + a_n)\vec{v}_n, \end{aligned}$$

we'd like to show that $a_1 = \cdots = a_n = 0$. Since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent, the above relation implies that

$$a_n + a_1 = a_1 + a_2 = \cdots = a_{n-1} + a_n = 0.$$

Hence

$$(a_1 + a_2) + (a_3 + a_4) + \cdots + (a_{n-2} + a_{n-1}) = 0.$$

Therefore

$$a_1 + a_2 + \cdots + a_n = a_1.$$

Similarly, one can show that

$$a_1 + a_2 + \cdots + a_n = a_i \text{ for each } 1 \leq i \leq n.$$

Hence we have $a_1 = a_2 = \cdots = a_n = 0$.

(2) (14 points) Let A be a real symmetric positive-definite matrix, and let B be a real anti-symmetric matrix (i.e. $B = -B^T$). Prove that $A + B$ is invertible. (Hint: Show that $\vec{x}^T(A+B)\vec{x} = \vec{x}^T A\vec{x}$.) (Hint: Show that if $(A+B)\vec{x} = \vec{0}$ then $\vec{x} = \vec{0}$.)

Solution. It suffices to show that "$(A+B)\vec{x} = \vec{0}$ implies $\vec{x} = \vec{0}$". First, notice that $\vec{x}^T B\vec{x} = 0$ for any $\vec{x}$, since $\vec{x}^T B\vec{x} = -\vec{x}^T B^T\vec{x} = -(\vec{x}^T B^T \vec{x})^T = -\vec{x}^T B\vec{x}$. Now, suppose that $(A+B)\vec{x} = \vec{0}$ for some $\vec{x}$, then

$$\vec{x}^T A\vec{x} = \vec{x}^T(A+B)\vec{x} = 0.$$

Since A is positive-definite, this implies that $\vec{x} = \vec{0}$.

(3) Let A and B be two real square matrices.

(a) (7 points) Suppose that $\lambda \neq 0$ is an eigenvalue of AB. Prove that λ also is an eigenvalue of BA. (Note: One needs to show that there exists a NONZERO vector $\vec{v}$ such that $BA\vec{v} = \lambda\vec{v}$.) (Hint: Similar idea as in practice problem #13 might help.)

(b) (7 points) Does the same statement hold for $\lambda = 0$? Prove your answer. (Hint: What does it mean for a matrix to have 0 as an eigenvalue?)

Solution. (a) Let $\lambda \neq 0$ be an eigenvalue of AB and let $\vec{v} \neq \vec{0}$ be an eigenvector of λ, i.e. $AB\vec{v} = \lambda\vec{v}$. Then we have

$$BA(B\vec{v}) = BAB\vec{v} = B(\lambda\vec{v}) = \lambda(B\vec{v}).$$

Notice that $B\vec{v} \neq \vec{0}$: if $B\vec{v} = \vec{0}$, then $\vec{0} = A(B\vec{v}) = \lambda\vec{v}$; but we know that $\lambda\vec{v} \neq \vec{0}$ since $\lambda \neq 0$ and $\vec{v} \neq \vec{0}$. Therefore, we have λ is an eigenvalue of BA with an eigenvector given by $B\vec{v} \neq \vec{0}$.

(b) Yes. $\lambda = 0$ is an eigenvalue of AB if and only if AB is not invertible, which is equivalent to $\det(AB) = 0$; and we have $\det(AB) = \det(A)\det(B) = \det(BA)$.

(4) (14 points) Let A be a real $n \times n$ matrix. Prove that there exists a real $n \times n$ matrix B such that $BA = 0$ (the zero matrix) and $\operatorname{rank}(A) + \operatorname{rank}(B) = n$. (Hint: First show that there exists an invertible matrix P such that PA is the reduced echelon form of A.) (Hint: Then find a square matrix C such that $C(PA) = 0$ and $\operatorname{rank}(A) + \operatorname{rank}(C) = n$. Such C should not be hard to construct, using the fact that PA is of reduced echelon form.) (Hint: Finally, show that $B = CP$ has the desired properties.)

Solution. Recall that there exists an invertible matrix P such that PA is the reduced echelon form of A (the matrix P is a product of elementary matrices which correspond to elementary row operations). Let $r = \operatorname{rank}(A)$. Then the last $(n-r)$ rows of PA are all zeros. Define an $n \times n$ matrix Q as follows:

$$\begin{bmatrix} 0_{(n-r)\times r} & \mathbb{I}_{n-r} \\ 0_{r\times r} & 0_{r\times(n-r)} \end{bmatrix},$$

where the upper-right part of Q is an identity matrix of size $(n-r)$, and the remaining entries are all zeros. Then it's clear that $QPA = 0$. We define $B \coloneqq QP$. Then we have $BA = 0$ and $\operatorname{rank}(B) = \operatorname{rank}(QP) = \operatorname{rank}(Q) = n - r$ since P is invertible.

(5) (14 points) Let V_1, V_2, V_3 be real vector spaces, and $T\colon V_1 \to V_2$, $S\colon V_2 \to V_3$ be linear transformations. Prove that the following two statements are equivalent to each other:

(a) $\mathrm{Im}(S \circ T) = \mathrm{Im}(S)$;

(b) $\mathrm{Ker}(S) + \mathrm{Im}(T) = V_2$.

Recall that $S \circ T$ denotes the composition (first apply T, then apply S), and Ker (resp. Im) denotes the kernel (resp. image) of a linear transformation. (Hint: First show that $\mathrm{Im}(S \circ T) \subseteq \mathrm{Im}(S)$ always holds, therefore statement (a) is equivalent to $\mathrm{Im}(S) \subseteq \mathrm{Im}(S \circ T)$.) (Hint: Similarly, show that statement (b) is equivalent to $V_2 \subseteq \mathrm{Ker}(S) + \mathrm{Im}(T)$.) (Hint: Write down precisely the meaning of $\mathrm{Im}(S) \subseteq \mathrm{Im}(S \circ T)$. It shouldn't be hard to show that it is equivalent to $V_2 \subseteq \mathrm{Ker}(S) + \mathrm{Im}(T)$.)

Solution. Statement (a) is equivalent to: "for any $x \in V_2$, there exists $y \in V_1$ such that $S(x) = S(T(y))$", which is equivalent to: "for any $x \in V_2$, there exists $y \in V_1$ such that $x - T(y) \in \mathrm{Ker}(S)$", which is equivalent to: "for any $x \in V_2$, there exists $y \in V_1$ and $z \in \mathrm{Ker}(S)$ such that $x = T(y) + z$", which is equivalent to: "for any $x \in V_2$, there exists $w \in \mathrm{Im}(T)$ and $z \in \mathrm{Ker}(S)$ such that $x = w + z$", which is equivalent to Statement (b).

(6) Let V be a finite dimensional real inner product space, and let $W \subseteq V$ be a subspace.

(a) (7 points) Define $T_W \colon V \to W$ to be the orthogonal projection onto W. Prove that for any $\vec{v}_1, \vec{v}_2 \in V$, one has $\langle \vec{v}_1, T_W(\vec{v}_2) \rangle = \langle T_W(\vec{v}_1), \vec{v}_2 \rangle$.

(b) (7 points) Conversely, suppose $T \colon V \to V$ is a linear transformation such that $T^2 = T$ and $\langle \vec{v}_1, T(\vec{v}_2) \rangle = \langle T(\vec{v}_1), \vec{v}_2 \rangle$ holds for any $\vec{v}_1, \vec{v}_2 \in V$. Prove that T is the orthogonal projection onto its image $\mathrm{Im}(T)$. (Note: $T^2 = T \circ T$ denotes the composition of T with itself.) (Hint: Plug in $\vec{v}_1 = T(\vec{v})$ for any $\vec{v} \in V$, and use the condition $T^2 = T$.)

Solution. (a) Note that $\vec{v}_1 - T_W(\vec{v}_1) \in W^\perp$, hence $\langle \vec{v}_1, T_W(\vec{v}_2) \rangle = \langle T_W(\vec{v}_1), T_W(\vec{v}_2) \rangle$. Similarly, we also have $\langle T_W(\vec{v}_1), \vec{v}_2 \rangle = \langle T_W(\vec{v}_1), T_W(\vec{v}_2) \rangle$.

(b) Denote $W \coloneqq \mathrm{Im}(T)$. Our goal is to show that $T = T_W$. For any $\vec{v}_1, \vec{v}_2 \in V$, we have

$$\langle T(\vec{v}_1), T(\vec{v}_2) \rangle = \left\langle T^2(\vec{v}_1), \vec{v}_2 \right\rangle = \langle T(\vec{v}_1), \vec{v}_2 \rangle$$

since $T^2 = T$. Therefore, we have $\vec{v} - T(\vec{v}) \in W^\perp$ for any $\vec{v} \in V$. Hence

$$\vec{v} = T(\vec{v}) + (\vec{v} - T(\vec{v}))$$

is the unique orthogonal decomposition of $\vec{v}$ with $T(\vec{v}) \in W$ and $\vec{v} - T(\vec{v}) \in W^\perp$. This proves that $T(\vec{v}) = T_W(\vec{v})$ is the orthogonal projection of $\vec{v}$ onto W. Since this is true for any $\vec{v} \in V$, we conclude that $T = T_W$.

(7) (14 points) Let V be the set consisting of 5×5 real matrices with the property that the entries in each row and column sum to zero. More concretely, a 5×5 matrix $A = [a_{ij}]$ belongs to the set V if and only if

$$a_{i1} + a_{i2} + \cdots + a_{i5} = 0 \quad \text{and} \quad a_{1j} + a_{2j} + \cdots + a_{5j} = 0 \quad \text{for any} \quad 1 \le i, j \le 5.$$

It is not hard to see that V is a vector space. Find the dimension of V, and prove your answer. (Hint: Construct a linear transformation from the vector space of 5×5 matrices to another vector space, such that V is its kernel. Then apply the rank-nullity theorem to obtain the dimension of V.) (Note: 15 is not the correct dimension; watch out if there is any redundant equation.)

Solution. $\dim V = 16$. Consider the linear map

$$T \colon \mathrm{Mat}_{5\times 5} \to \mathbb{R}^9, \quad A \mapsto (c_1(A), c_2(A), \ldots, c_5(A), r_1(A), \ldots, r_4(A)),$$

where $c_i(A)$ denotes the sum of entries in the i-th column of A, and $r_i(A)$ denotes the sum of entries in the i-th row of A.

T is surjective since for any $(c_1, c_2, \ldots, c_5, r_1, \ldots, r_4) \in \mathbb{R}^9$ we have

$$T\left(\begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ r_2 & 0 & 0 & 0 & 0 \\ r_3 & 0 & 0 & 0 & 0 \\ r_4 & 0 & 0 & 0 & 0 \\ c_1 - r_1 - r_2 - r_3 - r_4 & c_2 & c_3 & c_4 & c_5 \end{bmatrix}\right) = (c_1, c_2, \ldots, c_5, r_1, \ldots, r_4).$$

The kernel of T is the vector space V, since $c_1(A) = c_2(A) = \cdots = c_5(A) = 0$ and $r_1(A) = \cdots = r_4(A) = 0$ imply that $r_5(A) = 0$. Hence the rank-nullity theorem gives

$$\dim V = \dim \mathrm{Mat}_{5\times 5} - \dim T(\mathrm{Mat}_{5\times 5}) = 25 - 9 = 16.$$

(8) (14 points) Let W_1 and W_2 be two subspaces of an n-dimensional real vector space V, satisfying $\dim(W_1) + \dim(W_2) = n$. Prove that there exists a linear transformation $T\colon V \to V$ such that

$$\operatorname{Ker}(T) = W_1 \text{ and } \operatorname{Im}(T) = W_2.$$

(Hint: Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis of W_1. To construct the transformation T, you might want to use the fact that $\{\vec{v}_1, \dots, \vec{v}_k\}$ can be extended to a basis $\{\vec{v}_1, \dots, \vec{v}_k, \dots, \vec{v}_n\}$ of V.)

Solution. Let $\{\vec{a}_1, \dots, \vec{a}_k\}$ be a basis of W_1 and let $\{\vec{b}_1, \dots, \vec{b}_{n-k}\}$ be a basis of W_2. Recall that one can find vectors $\{\vec{a}_{k+1}, \dots, \vec{a}_n\} \subseteq V$ so that $\mathcal{B} = \{\vec{a}_1, \dots, \vec{a}_k, \vec{a}_{k+1}, \dots, \vec{a}_n\}$ is a basis of V. This gives the coordinate mapping

$$[-]_{\mathcal{B}}\colon V \to \mathbb{R}^n,$$

where $[v]_{\mathcal{B}} = \vec{x} \in \mathbb{R}^n$ if $\vec{v} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$. We define another linear transformation $S\colon \mathbb{R}^n \to V$ by

$$S(\vec{x}) \coloneqq x_{k+1}\vec{b}_1 + x_{k+2}\vec{b}_2 + \cdots + x_n\vec{b}_{n-k}.$$

We claim the composition $T \coloneqq S \circ [-]_{\mathcal{B}}$ satisfies the desired properties.

First, $\vec{v} \in \operatorname{Ker}(T)$ if and only if $[\vec{v}]_{\mathcal{B}} \in \operatorname{Ker}(S)$ since the coordinate mapping is invertible. Since $\{\vec{b}_1, \dots, \vec{b}_{n-k}\}$ is linearly independent, the vector $\vec{x} = [\vec{v}]_{\mathcal{B}}$ lies in $\operatorname{Ker}(S)$ if and only if $x_{k+1} = \cdots = x_n = 0$, which is equivalent to $\vec{v} \in \operatorname{Span}\{a_1, \dots, a_k\} = W_1$. Therefore we have $\operatorname{Ker}(T) = W_1$.

Second, since the coordinate mapping is invertible, we have $\operatorname{Im}(T) = \operatorname{Im}(S)$. It's clear that $\operatorname{Im}(S) = \operatorname{Span}\{\vec{b}_1, \dots, \vec{b}_{n-k}\} = W_2$.

(9) (14 points) Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^n$. Then $\vec{u}\vec{v}^T$ is an $n \times n$ matrix. Prove that

$$\det(\mathbb{I}_n + \vec{u}\vec{v}^T) = 1 + \langle \vec{u}, \vec{v} \rangle .$$

Here $\langle -, - \rangle$ denotes the standard inner product on $\mathbb{R}^n$, and $\mathbb{I}_n$ denotes the $n \times n$ identity matrix. (Hint: Induction on n.)

Solution. Prove by induction. The statement is clearly true for $n = 1$. Now suppose the statement is true for sizes less than n.

$$\mathbb{I}_n + \vec{u}\vec{v}^T = \begin{bmatrix} 1 + u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & 1 + u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_1 & u_nv_2 & \cdots & 1 + u_nv_n \end{bmatrix}.$$

We compute its determinant by cofactor expansion along the first row. The first term is

$$(1 + u_1v_1) \det \begin{bmatrix} 1 + u_2v_2 & \cdots & u_2v_n \\ \vdots & \ddots & \vdots \\ u_nv_2 & \cdots & 1 + u_nv_n \end{bmatrix} = (1 + u_1v_1)(1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n)$$

by induction hypothesis. The second term in the cofactor expansion is

$$\begin{aligned}
&- u_1v_2 \det \begin{bmatrix} u_2v_1 & u_2v_3 & \cdots & u_2v_n \\ u_3v_1 & 1 + u_3v_3 & \cdots & u_3v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_1 & u_nv_3 & \cdots & 1 + u_nv_n \end{bmatrix} \\
&= -u_1v_2 \left(\det \begin{bmatrix} 1 + u_2v_1 & u_2v_3 & \cdots & u_2v_n \\ u_3v_1 & 1 + u_3v_3 & \cdots & u_3v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_1 & u_nv_3 & \cdots & 1 + u_nv_n \end{bmatrix} - \det \begin{bmatrix} 1 & u_2v_3 & \cdots & u_2v_n \\ 0 & 1 + u_3v_3 & \cdots & u_3v_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & u_nv_3 & \cdots & 1 + u_nv_n \end{bmatrix} \right) \\
&= -u_1v_2 \Big((1 + u_2v_1 + u_3v_3 + \cdots + u_nv_n) - (1 + u_3v_3 + \cdots + u_nv_n) \Big) \\
&= -u_1v_1u_2v_2
\end{aligned}$$

again by induction hypothesis. Similarly, one can show that the remaining cofactor expansions along the first row give $-u_1v_1u_3v_3, \ldots, -u_1v_1u_nv_n$. Hence

$$\begin{aligned}
\det(\mathbb{I}_n + \vec{u}\vec{v}^T) &= (1 + u_1v_1)(1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n) - u_1v_1(u_2v_2 + \cdots + u_nv_n) \\
&= 1 + u_1v_1 + \cdots + u_nv_n \\
&= 1 + \langle \vec{u}, \vec{v} \rangle .
\end{aligned}$$

(10) (10 points) Find the unique real-valued function $y(t)\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$y''(t) - 4y'(t) + 3y(t) = 3t - 7 \text{ for all } t \in \mathbb{R},$$

and the initial conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = 7.$$

Solution. $y(t) = 2e^{3t} + t - 1\,.$

(11) (10 points) Find the unique triple of real-valued functions $x_1(t), x_2(t), x_3(t)\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$x_1'(t) = x_2'(t) = x_3'(t) = x_1(t) + x_2(t) + x_3(t) \text{ for all } t \in \mathbb{R},$$

and the initial conditions

$$x_1(0) = 0, \;\; x_2(0) = 5, \;\; \text{and } x_3(0) = 4.$$

Solution. $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 3e^{3t} - 3 \\ 3e^{3t} + 2 \\ 3e^{3t} + 1 \end{bmatrix}.$

(12) (10 points) Find the unique real-valued function $u(x,t)\colon [0,\pi] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ on the domain $0 \leq x \leq \pi$ and $t \geq 0$, satisfying

$$\begin{cases} u_t(x,t) = 3u_{xx}(x,t) & \text{for all } 0 < x < \pi \text{ and } t > 0, \\ u(0,t) = u(\pi,t) = 0 & \text{for all } t > 0, \\ u(x,0) = x(\pi - x) & \text{for all } 0 < x < \pi. \end{cases}$$

Solution.

$$u(x,t) = \sum_{k=0}^{\infty} \frac{8}{(2k+1)^3\pi} \sin\Big((2k+1)x\Big) e^{-3(2k+1)^2 t}.$$