FIRST MIDTERM PRACTICE
MATH H54

- You have 75 minutes to finish the exam and an additional 15 minutes to upload the solution to Gradescope. You can keep working on the exam after 75 minutes but you're responsible for uploading the solution within 90 minutes. No late submissions will be accepted.
- You'll need to upload two separate PDF files. The first one has solutions to the first three questions, the second one has solutions to the last three questions.
- You may use only the textbook, notes from the lectures, homework, quizzes and their solutions. In particular, you're NOT allowed to receive/give any assistance in any form during the exam.
- Make sure your argument is as clear as possible. In case you wish to use a theorem, you should write down the name of the theorem or state the precise result. You can use any statement that is proved in the lectures or appeared in homework assignments.
- Please write clearly. Answers that are not legible cannot be given credit.
- Good luck!

(Above are the instructions that you'll see in the actual exam on Gradescope.)

(1) Let $V = \mathrm{Mat}_{2\times 2}$ be the vector space of all real 2×2 matrices. Define the linear transformation $T\colon V \to V$ by

$$T(A) := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A - A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

(a) Find a basis of $\ker(T)$.
(b) Find a basis of $\mathrm{Im}(T)$.

(2) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}.$$

Find a matrix C such that $T_B(T_C(\vec{v})) = T_A(\vec{v})$ for any $\vec{v} \in \mathbb{R}^3$.

(3) Let

$$A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & 1 \\ 0 & 1 & c \end{bmatrix}.$$

Suppose that $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ sends the unit ball $\mathbb{B} = \{(x,y,z) \in \mathbb{R}^3 \colon x^2+y^2+z^2 \le 1\}$ to certain three dimensional region $T_A(\mathbb{B})$ with volume $\frac{32\pi}{3}$. What can you say about the real numbers a, b, c? (Recall that $\mathrm{Vol}(\mathbb{B}) = \frac{4\pi}{3}$.)

(4) (a) Let A be an $m \times n$ matrix, and $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ be the associated linear transformation. Suppose T_A is injective. What can you say about the relation between m and n? What can you say about the dimensions of the column space and the null space of A? Provide justifications for your answers.

(b) Same problems as is Part (a), but replace injective with surjective.

(5) Let V be an n-dimensional vector space and $T \colon V \to V$ a linear transformation such that $\ker(T) = \operatorname{Im}(T)$.

(a) Prove that n is even.

(b) Give an example of such a linear transformation T.

(6) Let V be a vector space (could be infinite dimensional) and $T \colon V \to V$ be a linear transformation. Suppose that $T^2 = T$, i.e.

$$T(T(\vec{v})) = T(\vec{v}) \quad \text{for any} \quad \vec{v} \in V.$$

Prove that

(a) $\ker(T) + \operatorname{Im}(T) = V$. (Hint: Consider $\vec{v} = (\vec{v} - T(\vec{v})) + T(\vec{v})$.)

(b) $\ker(T) \cap \operatorname{Im}(T) = \{0\}$.