SECOND MIDTERM PRACTICE PROBLEMS
MATH 185, SECTION 3

(1) Compute the following integrals.

(a)
$$\int_{|z|=2} \frac{e^z}{z(z-1)} dz.$$

(b)
$$\int_{-\infty}^{\infty} \frac{1}{(x+i)(x+2i)} dx.$$

(c)
$$\int_{|z|=1} \frac{1}{\sin(1/z)} dz.$$

(2) For each of the following functions, classify the singularity at the indicated point z_0 as removable, pole, or essential. For poles, give the order of the pole.

(a)
$$f(z) = \frac{1-\cos(z)}{z^3(z-\pi)}, \ z_0 = 0.$$

(b)
$$f(z) = \frac{(z-3)(\sin(\pi z))^2}{z^2 \sin(\pi z)}, \ z_0 = 1.$$

(c)
$$f(z) = \frac{e^{2z} - 1 - 2z}{\sin(z) - z}, \ z_0 = 0.$$

(3) Determine the number of zeros (counting multiplicities) of
$$f(z) = 2(z-1)^3 - e^{-z}$$
inside the open disk $\mathbb{D}_1(1) = \{z \colon |z-1| < 1\}$.

(4) Let f be a holomorphic function on a neighborhood of $\overline{\mathbb{D}}$, such that $|f(z)| = 1$ for $|z| = 1$ and $f(z) \neq 0$ for $|z| < 1$. Prove that f is a constant function.

(5) Let f be an entire function satisfying $|f(2^{-n})| \leq 2^{-n^2}$ for all positive integer n. Prove that $f(z) = 0$ for all $z \in \mathbb{C}$.

(6) Let $f_1, \ldots, f_n$ be holomorphic functions on $\mathbb{D}$. Suppose that $|f_1(z)| + \cdots + |f_n(z)| = 1$ for all $z \in \mathbb{D}$. Prove that $f_1, \ldots, f_n$ are all constant functions.

(7) Let $\Omega \subseteq \mathbb{C}$ be an open subset (not necessarily simply connected), and let $f \colon \Omega \to \mathbb{C}\backslash\{0\}$ be a non-vanishing holomorphic function. Prove that if there exists a non-vanishing holomorphic function $g \colon \Omega \to \mathbb{C}\backslash\{0\}$ such that $f(z) = e^{g(z)}$ for all

$z \in \Omega$, then we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = 0$$

for any closed curve γ in Ω. (Note that Ω may not contain the interior of γ.)

(1) Compute the following integrals.

(a)

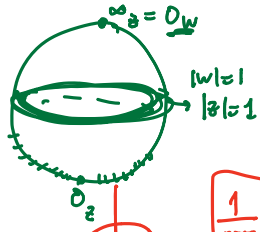
$$\int_{|z|=2} \frac{e^z}{z(z-1)}dz.$$

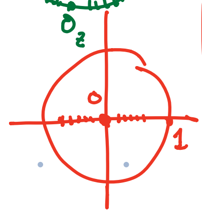
(b)

$$\int_{-\infty}^{\infty} \frac{1}{(x+i)(x+2i)}dx.$$

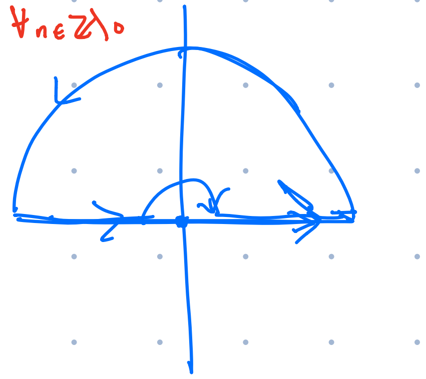
(c)

$$\boxed{\int_{|z|=1} \frac{1}{\sin(1/z)}dz.}$$

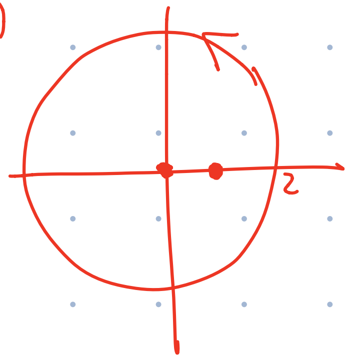




$\boxed{\frac{1}{n\pi}}$ sing pts $\forall n \in \mathbb{Z}\setminus 0$



(a)

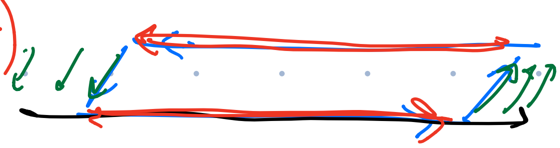


$$\int_{|z|=2} \frac{e^z}{z(z-1)} dz$$

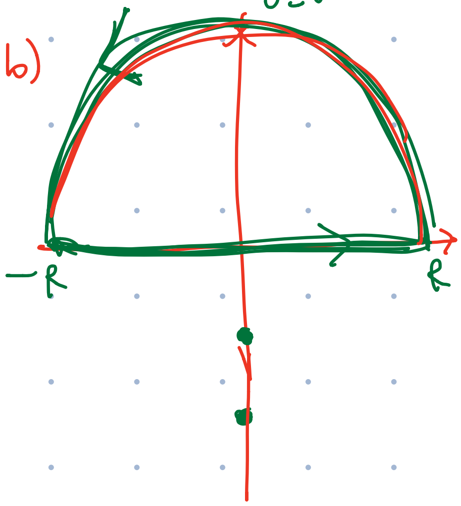
$$= 2\pi i \left(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=1} \right)$$

$$= 2\pi i \left(\frac{e^0}{0-1} + \frac{e^1}{1} \right)$$

$$= 2\pi i \left(-1 + e \right).$$



(b)



$$\left| \int_0^{\pi} \frac{iRe^{i\theta}}{(Re^{i\theta}+i)(Re^{i\theta}+2i)} d\theta \right|$$

$$\le \pi \cdot \sup_{0 \le \theta \le \pi} \left| \frac{iRe^{i\theta}}{(Re^{i\theta}+i)(Re^{i\theta}+2i)} \right|$$

$$\le \pi \cdot \frac{R}{(R-1)(R-2)} \to 0 \quad \text{as } R \to \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x+i)(x+2i)} = 0 \quad \text{by Cauchy's thm.}$$

(c) Change of var. $z = \frac{1}{w}$ $\quad dz = \frac{-1}{w^2}dw$

$$\int_{|z|=1} \frac{1}{\sin(\frac{1}{z})} dz = -\int_{|w|=1} \frac{1}{\sin(w)} \frac{-1}{w^2} dw$$

$$= \int_{|w|=1} \frac{dw}{w^2 \sin(w)} \to \text{sig at } w=0$$

$$= 2\pi i \cdot \operatorname{Res}_{w=0} \frac{1}{w^2 \sin w} = \frac{2\pi i}{6} \qquad \square$$

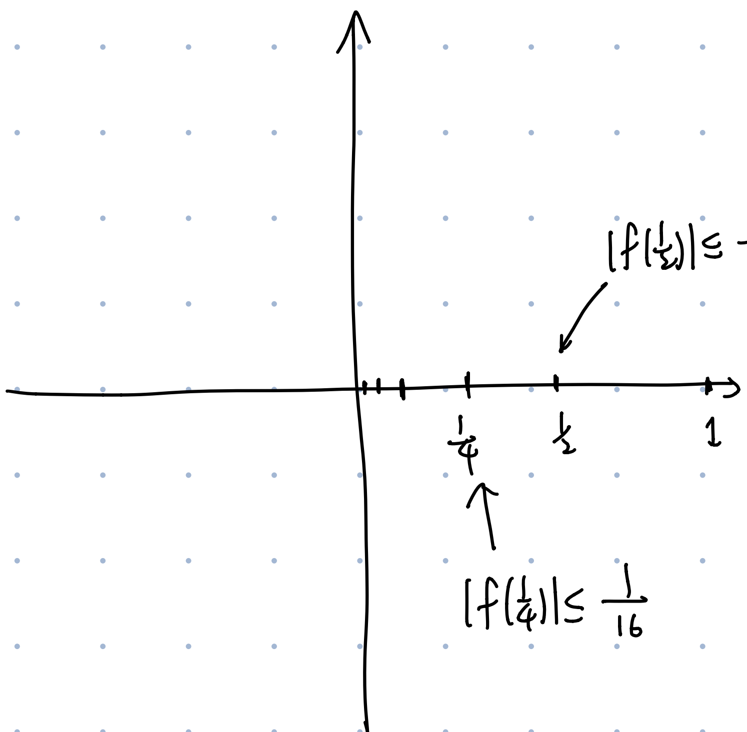
Laurent series exp. of $\frac{1}{w^2 \sin w}$ at $w=0$:

$$w^2 \sin w = w^3 - \frac{1}{3!} w^5 + \frac{1}{5!} w^7 - \cdots$$

$$= \underline{w^3} \left(1 - \frac{1}{3!} w^2 + \frac{1}{5!} w^4 - \cdots \right)$$

$$\frac{1}{w^2 \sin w} = w^{-3} \left(1 + \frac{1}{3!} w^2 + * \, w^4 + \cdots \right)$$

(5) Let f be an entire function satisfying $|f(2^{-n})| \le 2^{-n^2}$ for all positive integer n. Prove that $f(z) = 0$ for all $z \in \mathbb{C}$.



- $\lim 2^{-n} = 0$

$\Rightarrow \boxed{\lim f(2^{-n})} = f(0)$, $\lim f(2^{-n}) = 0$

$\Rightarrow \boxed{f(0) = 0}$

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases} \quad \text{holo entire}$$

$f'(0) = 0$

$\lim \frac{f(z)}{z} = f'(0)$

$\|$

$$|g(2^{-n})| = \left|\frac{f(2^{-n})}{2^{-n}}\right| \le \frac{2^{-n^2}}{2^{-n}} = 2^{-n^2+n}$$

$$\frac{f(z)-f(0)}{z-0}$$

$\Rightarrow$ $g(0)=0$ $\Rightarrow$ $h(z) = \begin{cases} \frac{g(z)}{z}, & z \neq 0 \\ \boxed{g'(0)}, & z = 0 \end{cases}$ entire

$g'(0) = f''(0) = 0$

$$|h(2^{-n})| \le 2^{-n^2+2n} \Rightarrow h(0)=0$$

By doing this inductively $\Rightarrow$ $f^{(k)}(0) = 0 \quad \forall k \ge 0$

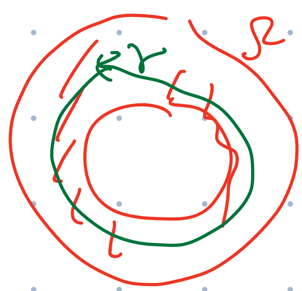
$\Rightarrow$ $f \equiv 0$

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \cdots \quad \forall z \in \mathbb{C}$$

(7) Let $\Omega \subseteq \mathbb{C}$ be an open subset (not necessarily simply connected), and let $f\colon \Omega \to \mathbb{C}\backslash\{0\}$ be a non-vanishing holomorphic function. Prove that if there exists a non-vanishing holomorphic function $g\colon \Omega \to \mathbb{C}\backslash\{0\}$ such that $f(z) = e^{g(z)}$ for all $z \in \Omega$, then we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = 0$$

for any closed curve γ in Ω. (Note that Ω may not contain the interior of γ.)



$$\frac{1}{2\pi i}\int_{\boxed{\gamma}} \frac{\boxed{g'(z)}\,\cancel{e^{g(z)}}}{\cancel{e^{g(z)}}}\,dz = 0$$

closed

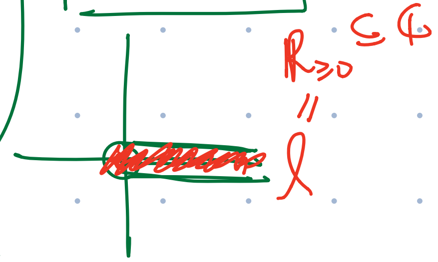
has a primitive $(g(z))$

$$D \backslash \left(\bigcup^n f_i(\ell)\right)$$

(6) Let $f_1, \dots, f_n$ be holomorphic functions on $\mathbb{D}$. Suppose that $|f_1(z)| + \cdots + |f_n(z)| = 1$ for all $z \in \mathbb{D}$. Prove that $f_1, \dots, f_n$ are all constant functions.

Claim: $\exists\, \mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{D}$, $\exists\, g_1, \dots, g_n$ holo. on $\mathcal{U}$, s.t. $f_i(z) = g_i(z)^2 \quad \forall z \in \mathcal{U}$.

pf of claim:



(1) If $\text{Image}(f_i) \not\subseteq \ell$, Consider $f_i^{-1}(\mathbb{C} \setminus \ell)$

(2) If $\text{Image}(f_i) \subseteq \ell$ $\Rightarrow \text{Im} f_i = 0 \Rightarrow f_i \equiv \text{const.}$

Assuming the claim first,

$\forall z \in \mathcal{U}$, $\quad 1 = \sum_{i=1}^{n} |g_i(z)|^2$

- $\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} 1 = 0$
- $\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \left(g_i(z) \overline{g_i(z)} \right) = \left| \frac{\partial g_i}{\partial z} \right|^2 = |g_i'(z)|^2$

$\Longrightarrow \quad 0 = \sum_{i=1}^{n} |g_i'(z)|^2$

$\Rightarrow \quad g_i'(z) \equiv 0$

$\Rightarrow \quad g_i$ are const. fns. on $\mathcal{U}$

$\Rightarrow \quad f_i$ are const. fns on $\mathcal{U}$

$\Rightarrow \quad f_i$ const. on $\mathbb{D}$.

(4) Let f be a holomorphic function on a neighborhood of $\overline{\mathbb{D}}$, such that $|f(z)| = 1$ for $|z| = 1$ and $f(z) \neq 0$ for $|z| < 1$. Prove that f is a constant function.

$\boxed{\text{If } f \text{ non-const}}$:

- Max principal $\to$ $\boxed{|f(z)| < 1}$ $\forall\, |z| < 1$.
- $g(z) = \dfrac{1}{f(z)}$ max. principle $\Rightarrow$ $|g(z)| < 1$ $\forall\, |z|<1$

$|g(z)| = 1$ for $|z| = 1$

$|g(z)| = \dfrac{1}{|f(z)|}$

$|g(z)| = \boxed{\dfrac{1}{|f(z)|}} > 1$

Contradiction. □

(2) For each of the following functions, classify the singularity at the indicated point z_0 as removable, pole, or essential. For poles, give the order of the pole.

(a) simple pole

$$f(z) = \frac{\boxed{1 - \cos(z)}}{z^3(z - \pi)}, \quad z_0 = 0.$$

$\cos z = 1 - \frac{1}{2!} z^2 + \dots$

(b) removable

$$f(z) = \frac{(z-3)\,(\sin(\pi z))^2}{z^2 \sin(\pi z)}, \quad z_0 = 1.$$

$= \sin(\pi z)$

(c) simple pole

$$f(z) = \frac{\boxed{e^{2z} - 1 - 2z}}{\sin(z) - z}, \quad z_0 = 0.$$

$\left(\cancel{1} + \cancel{2z} + \frac{1}{2!}\boxed{(2z)^2} + \dots\right) - \left(\cancel{1} + \cancel{2z}\right)$

$\left(\cancel{z} - \boxed{\frac{1}{3!} z^3} + \frac{1}{5!} z^5 - \dots\right) - \cancel{z}$

(3) Determine the number of zeros (counting multiplicities) of

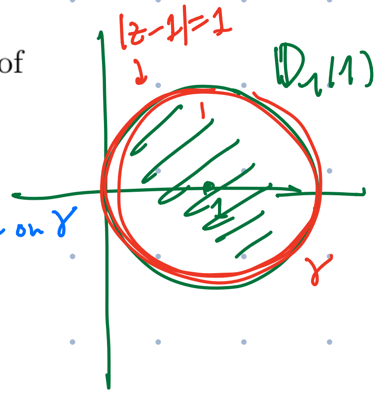
$$f(z) = \boxed{2(z-1)^3 - \boxed{e^{-z}}}$$

inside the open disk $\mathbb{D}_1(1) = \{z : |z-1| < 1\}$.

$\# = 3$

$|-| = 2$ on γ

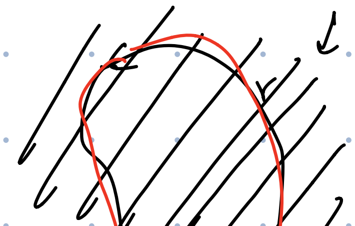
$|-| \le 1$ on γ



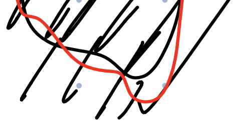
Rouché thm

f, g hol.

$\boxed{|f(z)| > |g(z)|}$ $\forall z \in \gamma$.

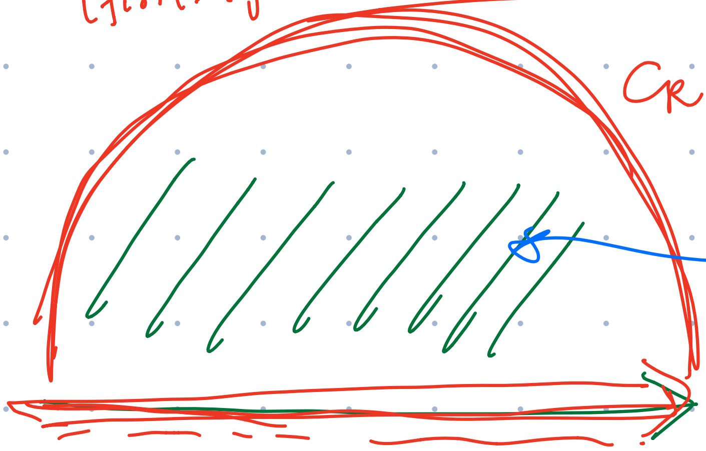


$\Rightarrow$ # zeros of f inside γ $=$ # zeros of $\boxed{f+g}$ inside γ.



$$|e^{-z}| = |e^{-(x+iy)}| = |e^{-x} e^{-iy}| = \boxed{|e^{-x}|} \le 1 \text{ on } \gamma$$

$|f(z)| > |g(z)|$ on C_R for $R > R_0$



\# of zeros of f = \# of zeros of $f+g$ inside half-disk $\forall R > R_0$

Suppose f, g

Suppose f has finitely many zeros in $\mathbb{H}$ (upper half plane)

$\Downarrow$

\# of zeros of f in $\mathbb{H}$

$=$

\# of zeros of $f+g$ in $\mathbb{H}$

$|f(z)| > |g(z)|$ on real-axis

$|f| > |g|$