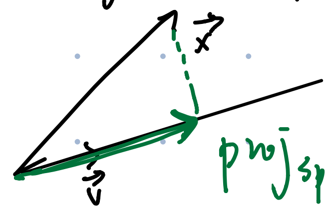
Last time:

- inner product space. $V, \langle -,- \rangle$.
- length. $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$, orthogonal $\langle \vec{v}, \vec{w} \rangle = 0$.
- orthogonal complement. $W \subseteq V \rightsquigarrow W^{\perp} \subseteq V$
- 

  $\text{proj}_{\text{span}\{\vec{v}\}}(\vec{x}) = \dfrac{\langle \vec{x}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}$.

- If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal basis of V, then $\forall \vec{x} \in V$,

$$\vec{x} = \frac{\langle \vec{x}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 + \cdots + \frac{\langle \vec{x}, \vec{v}_n \rangle}{\|\vec{v}_n\|^2} \vec{v}_n$$

$$= \text{proj}_{\text{span}\{\vec{v}_1\}} \vec{x} + \cdots + \text{proj}_{\text{span}\{\vec{v}_n\}} \vec{x}.$$

(Quiz 6 #1 is a counterexample of this statement when $\{\vec{v}_1, \dots, \vec{v}_n\}$ is NOT orthogonal)

Orthogonal decomposition thm: $W \subseteq V \leftarrow$ finite dim'l inner product space.

Every vector $\vec{v} \in V$ can be uniquely written as

$$\vec{v} = \vec{w} + \vec{z}, \qquad \boxed{\vec{w} = \text{proj}_W \vec{v}}$$

where $\vec{w} \in W$ and $\vec{z} \in W^{\perp}$

More concretely. If $\{\vec{u}_1, \dots, \vec{u}_n\}$ is an orthogonal basis of W, then

$$\vec{w} = \frac{\langle \vec{v}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 + \cdots + \frac{\langle \vec{v}, \vec{u}_n \rangle}{\|\vec{u}_n\|^2} \vec{u}_n$$

$$= \text{proj}_{\text{span}\{\vec{u}_1\}} \vec{w} + \cdots + \text{proj}_{\text{span}\{\vec{u}_n\}} \vec{w}$$

and $\vec{z} = \vec{v} - \vec{w}$.

Pf <u>Existence of decomp:</u>

Define $\vec{w}, \vec{z}$ in this way

$\boxed{\vec{w} \in W}$? ✓

$\boxed{\vec{z} \in W^{\perp}} = \{\vec{v} \in V \mid \langle \vec{v}, \vec{w} \rangle = 0 \quad \forall \vec{w} \in W\}$

$= \{\vec{v} \in V \mid \langle \vec{v}, \vec{u}_i \rangle = 0 \quad \forall i\}$

$\langle \vec{z}, \vec{u}_i \rangle$

$= \langle \vec{v}, \vec{u}_i \rangle - \langle \vec{w}, \vec{u}_i \rangle$

$= \langle \vec{v}, \vec{u}_i \rangle - \frac{\langle \vec{v}, \vec{u}_i \rangle}{\cancel{\|\vec{u}_i\|^2}} \cancel{\langle \vec{u}_i, \vec{u}_i \rangle}$

$= 0$ □

<u>Uniqueness of decomp:</u>

$\vec{v} = \vec{w}_1 + \vec{z}_1$ $\quad \vec{w}_1, \vec{w}_2 \in W$

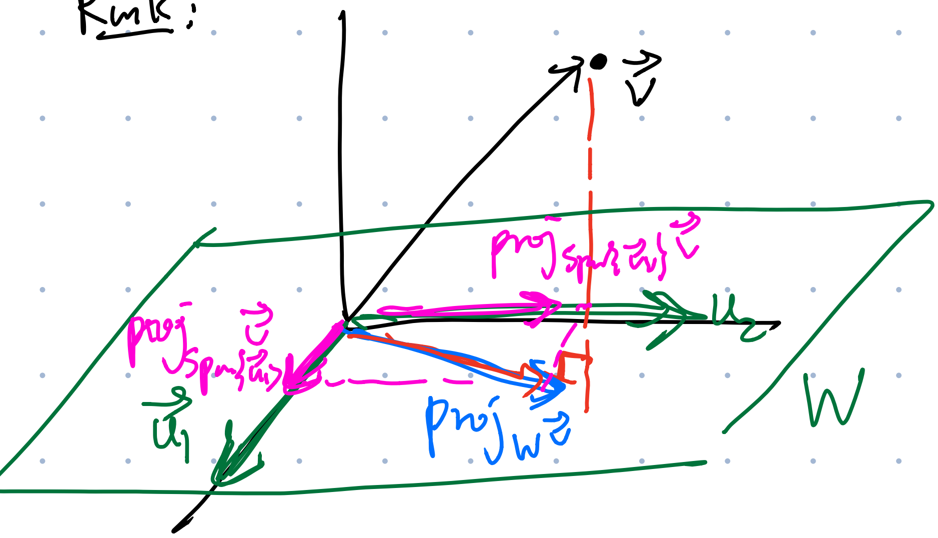
$= \vec{w}_2 + \vec{z}_2$ $\quad \vec{z}_1, \vec{z}_2 \in W^{\perp}$

<u>Want to show:</u> $\vec{w}_1 = \vec{w}_2$, $\vec{z}_1 = \vec{z}_2$.

$\underbrace{\vec{w}_1 - \vec{w}_2}_{\in W} = \underbrace{\vec{z}_2 - \vec{z}_1}_{\in W^{\perp}} = \vec{0}$ □

$W \cap W^{\perp} = \{\vec{0}\}$.

Rmk:



Rmk: $\vec v \in W$, $\boxed{\mathrm{proj}_W \vec v} = \vec v$

Rmk: If $\{\vec u_1, \dots, \vec u_n\}$ orthonormal basis of W

$$\mathrm{proj}_W \vec v = \frac{\langle \vec v, \vec u_1 \rangle}{\|\vec u_1\|^2} \vec u_1 + \cdots + \frac{\langle \vec v, \vec u_n \rangle}{\|\vec u_n\|^2} \vec u_n$$

$$= \boxed{\langle \vec v, \vec u_1 \rangle \vec u_1 + \cdots + \langle \vec v, \vec u_n \rangle \vec u_n}$$

$\| ?$

Let $U = \begin{bmatrix} | & & | \\ \vec u_1 & \cdots & \vec u_n \\ | & & | \end{bmatrix}$

$U^T U = I_n$

$UU^T \vec v.$

$$= \begin{bmatrix} \vec u_1 & \cdots & \vec u_n \end{bmatrix} \begin{bmatrix} \text{---} \vec u_1 \text{---} \\ \vdots \\ \text{---} \vec u_n \text{---} \end{bmatrix} \begin{bmatrix} | \\ \vec v \\ | \end{bmatrix}$$

$$\begin{bmatrix} \langle \vec u_1, \vec v \rangle \\ \vdots \\ \langle \vec u_n, \vec v \rangle \end{bmatrix}$$

Thm $(V, \langle , \rangle)$ finite dim'l inner product space

$W \subseteq V$.

$$\dim W + \dim W^{\perp} = \dim V$$

pf $\to W$ $\quad \{a_1, \dots, a_n\}$ basis of W

$\to W^{\perp}$ $\quad \{b_1, \dots, b_m\}$ basis of $W^{\perp}$

Claim: $\underline{\{\vec{a}_1, \dots, \vec{a}_n, \vec{b}_1, \dots, \vec{b}_m\}}$ is a basis of V.

① They span V:

orthogonal decomp. thm:

$$\underset{\in \operatorname{Span}\{a_1, \dots, a_n\}}{\vec{v} = \vec{w}} + \underset{\in \operatorname{Span}\{b_1, \dots, b_m\}}{\vec{z}} \in \operatorname{Span}\{a_1, \dots, a_n, b_1, \dots, b_m\}$$

② They're l.i.:

$$\underbrace{x_1\vec{a}_1 + \dots + x_n\vec{a}_n}_{W} + \underbrace{y_1\vec{b}_1 + \dots + y_m\vec{b}_m}_{W^{\perp}} = \vec{0}$$

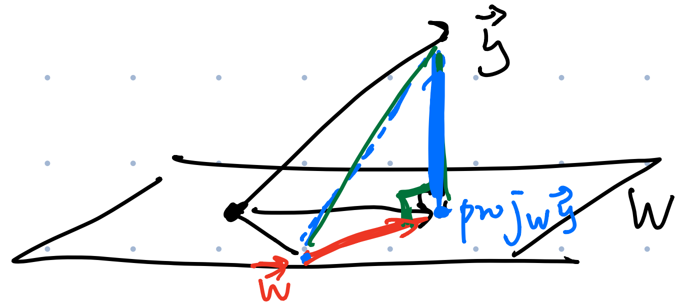
$$\boxed{x_1\vec{a}_1 + \dots + x_n\vec{a}_n} = -\boxed{(y_1\vec{b}_1 + \dots + y_m\vec{b}_m)} = \vec{0}$$

$$x_1 = \dots = x_n = 0 \qquad y_1 = \dots = y_m = 0. \qquad \square$$

Best approximation thm

$W \subseteq V$

$\vec{y} \in V$



$\mathrm{proj}_W \vec{y}$ is the closest point in W to $\vec{y}$

i.e. $\|\vec{y} - \mathrm{proj}_W \vec{y}\| < \|\vec{y} - \vec{w}\|$

for any $\vec{w} \in W \setminus \{\mathrm{proj}_W \vec{y}\}$

pf $\forall \vec{w} \in W \setminus \{\mathrm{proj}_W \vec{y}\}$,

Consider $\boxed{\vec{w} - \mathrm{proj}_W \vec{y}}$

$$\langle \underbrace{\vec{w} - \mathrm{proj}_W \vec{y}}_{\in W}, \underbrace{\vec{y} - \boxed{\mathrm{proj}_W \vec{y}}}_{\in W^\perp} \rangle = 0$$

$$\vec{y} - \vec{w} = (y - \mathrm{proj}_W \vec{y}) + (\mathrm{proj}_W \vec{y} - \vec{w})$$

Pythagorean thm $\Rightarrow$

$$\|\vec{y} - \vec{w}\|^2 = \|\vec{y} - \mathrm{proj}_W \vec{y}\|^2 + \|\underbrace{\mathrm{proj}_W \vec{y} - \vec{w}}_{\neq \vec{0}}\|^2$$

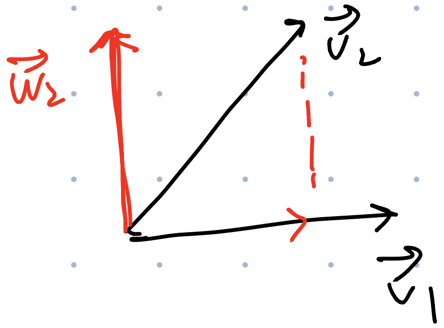
$$> \|\vec{y} - \mathrm{proj}_W \vec{y}\|^2$$

Gram-Schmidt process:

$\{\vec{v}_1, \dots, \vec{v}_n\}$ basis of V

① $\vec{v}_1$

② $\{\vec{v}_1, \vec{v}_2\}$



Idea replace $\vec{v}_2$ by $\vec{w}_2$

s.t.

- ✓ $\langle \vec{v}_1, \vec{w}_2 \rangle = 0$
- ✓ $\mathrm{Span}\{\vec{v}_1, \vec{v}_2\} = \mathrm{Span}\{\vec{v}_1, \vec{w}_2\}$

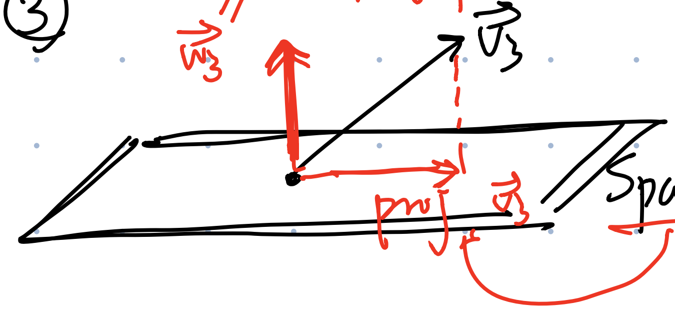
Define

$$\vec{w}_2 = \vec{v}_2 - \mathrm{proj}_{\mathrm{span}\{\vec{v}_1\}}(\vec{v}_2) = \vec{v}_2 - \frac{\langle \vec{v}_1, \vec{v}_2 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

- $\langle \vec{v}_1, \vec{w}_2 \rangle = 0$?

$$= \langle \vec{v}_1, \vec{v}_2 \rangle - \frac{\langle \vec{v}_1, \vec{v}_2 \rangle}{\cancel{\|\vec{v}_1\|^2}} \cancel{\langle \vec{v}_1, \vec{v}_1 \rangle} = 0$$

③ $\vec{w}_3 = \vec{v}_3 - \mathrm{proj}_{\mathrm{span}\{\vec{v}_1, \vec{v}_2\}} \vec{v}_3$



$\mathrm{Span}\{\vec{v}_1, \vec{v}_2\} = \mathrm{Span}\{\vec{v}_1, \vec{w}_2\}$

Claim:
- $\{\vec{v}_1, \vec{w}_2, \vec{w}_3\}$ is an orthogonal set.
- $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathrm{Span}\{\vec{v}_1, \vec{w}_2, \vec{w}_3\}$.

Continue this inductively

$\rightsquigarrow$ an orthogonal basis $\{\vec{v}_1, \vec{w}_2, \dots, \vec{w}_n\}$ of V.

Rmk $\left\{ \dfrac{\vec{v}_1}{\|\vec{v}_1\|}, \dfrac{\vec{w}_2}{\|\vec{w}_2\|}, \dots, \dfrac{\vec{w}_n}{\|\vec{w}_n\|} \right\}$ orthonormal basis of V

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e.g. $\boxed{\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix},}$ in $\mathbb{R}^3$

$$\vec{w}_2 = \vec{v}_2 - \operatorname{proj}_{\operatorname{span}\{\vec{v}_1\}} \vec{v}_2$$

$$= \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{\left\langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\rangle}{\left\| \begin{bmatrix}1\\0\\0\end{bmatrix} \right\|^2} \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

$$\vec{w}_3 = \underset{\begin{bmatrix}1\\1\\1\end{bmatrix}}{\underset{\parallel}{\vec{v}_3}} - \operatorname{proj}_{\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}} \vec{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\0\end{bmatrix}$$

Thm Suppose $\overset{m\times n}{A}$ has l.i. columns. then $\exists$ $\overset{m\times n}{Q}, \overset{n\times n}{R}$ s.t.

- $A = QR$
- Q has orthonormal columns.
- R upper triangular matrix where the diagonal entries are positive $\begin{bmatrix} \circledast & * & * \\ & \circledast & \\ 0 & & \circledast \end{bmatrix}$

(HW you'll show the decomp is unique)

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pf: follows from the Gram-Schmidt process,

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} | & | & \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} & * \\ & r_{22} & \\ & & r_{33} \\ 0 & & \ddots \\ & & & r_{nn} \end{bmatrix}}_{R}$$

$\vec{v}_1 \cdots \vec{v}_n$ l.i.

$\vec{u}_1 \cdots \vec{u}_n$ orthonormal

want to find

Q (orthonormal basis of $\mathrm{Col}(A)$)

$$R = \begin{bmatrix} \|\vec{v}_1\| & \langle \vec{v}_2, \vec{u}_1\rangle & \\ 0 & \|\vec{w}_2\| & \\ \vdots & 0 & \\ & \vdots & \\ 0 & 0 & \end{bmatrix}$$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} \qquad r_{11} = \|\vec{v}_1\|$$

$$\vec{w}_2 = \vec{v}_2 - \mathrm{proj}_{\mathrm{span}\{\vec{u}_1\}} \vec{v}_2$$

$$= \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1\rangle}{\|\vec{u}_1\|^2} \vec{u}_1$$

$$\vec{v}_2 = \vec{w}_2 + \langle \vec{v}_2, \vec{u}_1\rangle \vec{u}_1$$

$$= \underbrace{\|\vec{w}_2\|}_{r_{22}} \vec{u}_2 + \underbrace{\langle \vec{v}_2, \vec{u}_1\rangle}_{r_{12}} \vec{u}_1$$

$$\vec{u}_2 := \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{1}{\|\vec{w}_2\|}\vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1\rangle}{\|\vec{w}_2\|}\vec{u}_1$$

$$r_{12}\vec{u}_1 + r_{22}\vec{u}_2 = \vec{v}_2$$

Rmk: $A = QR \Rightarrow Q^T A = \underbrace{Q^T Q}_{I} R = R.$

suppose we know A, Q.