#1: • $\vec{x}(t)$ is a sol^{n} to $\vec{x}' = A\vec{x}$ since $X(t)\vec{c}$ is a sol^{n} for any fundamental matrix $X(t)$ and any constant vector $\vec{c}$.

• $\vec{x}(t_0) = \vec{x}_0$ is clear. $\square$

#2:

(a) $$A = \begin{bmatrix} 0 & -2 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & & \\ & -1 & \\ & & 5 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}^{-1}.$$

$$\Longrightarrow \begin{bmatrix} 0 & -2e^{-t} & e^{5t} \\ -e^{-3t} & e^{-t} & e^{5t} \\ e^{-3t} & e^{-t} & e^{5t} \end{bmatrix}$$ is a fundamental matrix.

(b) $$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 2 & & & \\ & -1 & & \\ & & 3 & \\ & & & 7 \end{bmatrix} \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}^{-1}$$

$$\Longrightarrow \begin{bmatrix} e^{2t} & -e^{-t} & e^{3t} & -e^{7t} \\ 0 & 3e^{-t} & 0 & e^{7t} \\ 0 & 0 & e^{3t} & 2e^{7t} \\ 0 & 0 & 0 & 8e^{7t} \end{bmatrix}$$ is a fundamental matrix.

(c) -2, -1 (multiplicity 2) are the eigenvalues of A.

eigenvector $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ eigenspace, $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ generalized eigenvector $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

Compute e^{tA} $\begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -1 \\ 4 & 1 & 0 \end{bmatrix}$:

- $$e^{tA}\begin{bmatrix}1\\-2\\4\end{bmatrix} = e^{-2t}e^{(A+2\mathbb{I})t}\begin{bmatrix}1\\-2\\4\end{bmatrix}$$
$$= e^{-2t}\left(\mathbb{I} + (A+2\mathbb{I})t + \cdots\right)\begin{bmatrix}1\\-2\\4\end{bmatrix}$$
$$= e^{-2t}\begin{bmatrix}1\\-2\\4\end{bmatrix}.$$

- Similarly, $e^{tA}\begin{bmatrix}1\\-1\\1\end{bmatrix} = e^{-t}\begin{bmatrix}1\\-1\\1\end{bmatrix}$.

- $$e^{tA}\begin{bmatrix}2\\-1\\0\end{bmatrix} = e^{-t}e^{(A+\mathbb{I})t}\begin{bmatrix}2\\-1\\0\end{bmatrix}$$
$$= e^{-t}\left(\mathbb{I} + t(A+\mathbb{I}) + \frac{t^2}{2}(A+\mathbb{I})^2 + \cdots\right)\begin{bmatrix}2\\-1\\0\end{bmatrix}$$
$$= e^{-t}\left(\begin{bmatrix}2\\-1\\0\end{bmatrix} + t\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)$$

$$\Longrightarrow \begin{bmatrix} e^{-2t} & e^{-t} & e^{-t}(2+t) \\ -2e^{-2t} & -e^{-t} & e^{-t}(-1-t) \\ 4e^{-2t} & e^{-t} & e^{-t}\cdot t \end{bmatrix}$$ is a fundamental matrix.

#3:

(a) $$A = \begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -2-i & 0 \\ 0 & -2+i \end{bmatrix}\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}^{-1}$$

$$e^{(-2-i)t}\begin{bmatrix}-1-i\\2\end{bmatrix} = e^{-2t}(\cos t - i\sin t)\begin{bmatrix}-1-i\\2\end{bmatrix}$$

$$= e^{-2t}\begin{bmatrix}(-\cos t-\sin t)+i(-\cos t+\sin t)\\2\cos t - 2i\sin t\end{bmatrix}$$

$\Rightarrow$ a fundamental matrix
$$X(t) = e^{-2t}\begin{bmatrix}-\cos t-\sin t & -\cos t+\sin t\\ 2\cos t & -2\sin t\end{bmatrix}$$

Find a constant vector $\vec{c}$ s.t. $X(0)\vec{c} = \begin{bmatrix}-1\\0\end{bmatrix}$.

$$X(0) = \begin{bmatrix}-1 & -1\\ 2 & 0\end{bmatrix} \Rightarrow \vec{c} = \begin{bmatrix}0\\1\end{bmatrix}$$

$\Rightarrow$ the sol^{n} is
$$\vec{x}(t) = e^{-2t}\begin{bmatrix}-\cos t-\sin t & -\cos t+\sin t\\ 2\cos t & -2\sin t\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$
$$= e^{-2t}\begin{bmatrix}-\cos t+\sin t\\ -2\sin t\end{bmatrix}.$$

(b) -2 is the unique eigenvalue.

eigenspace $= \text{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\4\\1\end{bmatrix}\right\}$,

and $(A+2I)\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0&0&0\\4&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\4\\1\end{bmatrix}$

Compute $e^{tA}\begin{bmatrix}0&0&1\\1&4&0\\0&1&0\end{bmatrix}$:

- $e^{tA}\begin{bmatrix}0&0\\1&4\\0&1\end{bmatrix} = e^{-2t}\begin{bmatrix}0&0\\1&4\\0&1\end{bmatrix}$

- $e^{tA}\begin{bmatrix}1\\0\\0\end{bmatrix} = e^{-2t}e^{(A+2I)t}\begin{bmatrix}1\\0\\0\end{bmatrix}$

$$= e^{-2t}\left(I + (A+2I)t + (A+2I)^2\frac{t^2}{2} + \cdots\right)\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$= e^{-2t}\left(\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\4\\1\end{bmatrix}\right)$$

$\Rightarrow X(t) = e^{-2t}\begin{bmatrix}0&0&1\\1&4&4t\\0&1&t\end{bmatrix}$ is a fund. matrix.

$$X(0) = \begin{bmatrix}0&0&1\\1&4&0\\0&1&0\end{bmatrix}$$

$$X(0)^{-1}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}5\\-1\\1\end{bmatrix}$$

$\Rightarrow$ the sol^{n} is $\vec{x}(t) = e^{-2t}\begin{bmatrix}0&0&1\\1&4&4t\\0&1&t\end{bmatrix}\begin{bmatrix}5\\-1\\1\end{bmatrix}$. $\square$

#4: Use variation of parameters:

- <u>Find a fundamental matrix of $\vec{x}' = \begin{bmatrix}0&1\\-1&0\end{bmatrix}\vec{x}$:</u>

i is an eigenvalue w/ an eigenvector $\begin{bmatrix}-i\\1\end{bmatrix}$.

$$e^{it}\begin{bmatrix}-i\\1\end{bmatrix} = (\cos t + i\sin t)\begin{bmatrix}-i\\1\end{bmatrix} = \begin{bmatrix}\sin t - i\cos t\\ \cos t + i\sin t\end{bmatrix}$$

$$\Rightarrow X(t) = \begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix} \text{ is a fundamental matrix.}$$

- $$\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}^{-1}\begin{bmatrix}8\sin t\\ 0\end{bmatrix} = \begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}8\sin t\\ 0\end{bmatrix}$$

$$= \begin{bmatrix}8\sin t\cos t\\ 8\sin^2 t\end{bmatrix}$$

- $$v_1(t) = \int^t 8\sin s\cos s\, ds = -4\cos^2 t \ (+\text{ const.})$$

$$v_2(t) = \int^t 8\sin^2 s\, ds = 4t - 4\sin t\cos t \ (+\text{ const.})$$

$$\Rightarrow \begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\begin{bmatrix}-4\cos^2 t\\ 4t - 4\sin t\cos t\end{bmatrix} = \begin{bmatrix}4t\sin t - 4\cos t\\ 4t\cos t\end{bmatrix}$$

is a sol^{n}

General sol^{n}: $$\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\begin{bmatrix}C_1\\ C_2\end{bmatrix} + \begin{bmatrix}4t\sin t - 4\cos t\\ 4t\cos t\end{bmatrix}.$$

$\square$

#5:

(a) $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(b) Let $\vec{x}_2 = te^{2t}\vec{u}_1 + e^{2t}\vec{u}_2$.

$$\vec{x}_2{}' = e^{2t}\vec{u}_1 + 2te^{2t}\vec{u}_1 + 2e^{2t}\vec{u}_2$$

$$\overset{??}{\parallel}$$

$$A\vec{x}_2$$

$$\parallel$$

$$te^{2t}A\vec{u}_1 + e^{2t}A\vec{u}_2$$

$$\parallel$$

$$2te^{2t}\vec{u}_1 + e^{2t}A\vec{u}_2$$

$$\Leftrightarrow \quad (A - 2I)\vec{u}_2 \overset{??}{=} \vec{u}_1$$

$$A - 2I = \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Pick $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. $\square$

(c) By similar calculation, it suffices to find $\vec{u}_3$ s.t. $(A - 2I)\vec{u}_3 = \vec{u}_2$.. Pick $\vec{u}_3 = \begin{bmatrix} 0 \\ -6/5 \\ 1/5 \end{bmatrix}$. $\square$

#6: eigenvalues: -3, 3 (mult. 2).

(a) eigenspace $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}\right\}$ $\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$

gen. eigenvectors: $\begin{bmatrix}2/3\\1/3\\0\end{bmatrix}$

$$\begin{bmatrix}-2 & 2 & -4\\0 & 0 & 0\\4 & -5 & -8\end{bmatrix}\begin{bmatrix}2/3\\1/3\\0\end{bmatrix}=\begin{bmatrix}2\\0\\1\end{bmatrix}$$

Compute $e^{tA}\begin{bmatrix}1 & 2 & 2/3\\0 & 0 & 1/3\\2 & 1 & 0\end{bmatrix}$.

$$e^{tA}\begin{bmatrix}2/3\\1/3\\0\end{bmatrix}=e^{3t}e^{(A-3I)t}\begin{bmatrix}2/3\\1/3\\0\end{bmatrix}$$

$$=e^{3t}\left(\begin{bmatrix}2/3\\1/3\\0\end{bmatrix}+t\begin{bmatrix}2\\0\\1\end{bmatrix}\right)$$

$$\Rightarrow X(t)=\begin{bmatrix}e^{-3t} & 2e^{3t} & e^{3t}\left(\frac{2}{3}+2t\right)\\0 & 0 & e^{3t}\left(\frac{1}{3}\right)\\2e^{-3t} & e^{3t} & e^{3t}(t)\end{bmatrix}$$ is a fund. matrix $\square$

(b) general sol$^{\text{ns}}$ are:

$$\vec{x}(t)=c_1\begin{bmatrix}e^{-3t}\\0\\2e^{-3t}\end{bmatrix}+c_2\begin{bmatrix}2e^{3t}\\0\\e^{3t}\end{bmatrix}+c_3\begin{bmatrix}e^{3t}\left(\frac{2}{3}+2t\right)\\\frac{1}{3}e^{3t}\\te^{3t}\end{bmatrix}.$$

Observe that if $c_2\neq0$ or $c_3\neq0$,

then $\vec{x}(t)$ is not bounded for all $t\geq0$.

Hence the initial conditions $\vec{x}(0)=\vec{x}_0$ are s.t.

$$\vec{x}(t)=c\begin{bmatrix}e^{-3t}\\0\\2e^{-3t}\end{bmatrix}$$ is the unique sol$^{\text{n}}$.

||

$$\vec{x}_0 = c\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ for some } c \in \mathbb{R}. \quad \square$$