Today: Aut($\mathbb{H}$), Preparation for Riemann mapping thm.

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Thm: Any $f \in \mathrm{Aut}(\mathbb{H})$ (i.e. biholomorphic map $f: \mathbb{H} \to \mathbb{H}$) is of the form

$$z \longmapsto \frac{az+b}{cz+d}$$

for some $a, b, c, d \in \mathbb{R}$ w/ $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$.

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pf: • $\mathbb{H} \longrightarrow \mathbb{D}$ biholo.

$$z \longmapsto \frac{i-z}{i+z}$$

$$i \longmapsto 0$$

$$\left( \begin{array}{l} A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \\ f_A : \mathbb{H} \to \mathbb{H} \\ \quad z \mapsto \frac{az+b}{cz+d} \end{array} \right)$$

Let $f$ be any automorphism of $\mathbb{H}$,

$\exists!\ z_0 \in \mathbb{H}$ s.t. $f(z_0) = i$.

① Find $A \in SL_2(\mathbb{R})$ s.t. $f_A(i) = z_0$.

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "f_A"] \arrow[d, "\frac{i-z}{i+z}"'] & \mathbb{H} \arrow[r, "f"] & \mathbb{H} \arrow[d, "\frac{i-z}{i+z}"] \\
\mathbb{D} & & \mathbb{D}
\end{tikzcd}$$

$$i \longmapsto z_0 \longmapsto i, \qquad i \mapsto 0, \quad i \mapsto 0$$

this composition is an automorphism of $\mathbb{D}$, that sends 0 to 0.

Last time, we showed that such automorphism of $\mathbb{D}$ must be a rotation (we used Schwarz lemma, max. mod. prin.)

② prove this composition is a Möbius transformtn. $= f_B$

$$\begin{tikzcd} \mathbb{H} \arrow[d, "\frac{i-z}{i+z}"'] & & \mathbb{H} \arrow[d, "\frac{i-z}{i+z}"] \\ \mathbb{D} \arrow[rr, "e^{i\theta}"] & & \mathbb{D} \end{tikzcd}$$

①, ② $\Rightarrow$ $f \circ f_A = f_B$

$\Rightarrow$ $f = f_B \circ f_A^{-1} = f_{\underbrace{BA^{-1}}_{\in SL_2(\mathbb{R})}}$

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①: $\boxed{\forall z_0 \in \mathbb{H}, \ \exists A \in SL_2(\mathbb{R}) \text{ s.t. } f_A(i) = z_0}$

$z_0 = x+iy,\ y>0$ $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad f_A(i) = \dfrac{ai+b}{ci+d}$ $\quad z_0 = x+iy$

$$\boxed{\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}} \qquad \frac{yi+x}{0i+1} = x+iy$$

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

Check: $\forall\, x+iy$, w/ $y>0$.

Take $A = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \in SL_2(\mathbb{R})$

$$f_A(i) = \frac{\sqrt{y}\cdot i + x/\sqrt{y}}{0\cdot i + 1/\sqrt{y}} = \frac{y\cdot i + x}{0\cdot i + 1} = x+yi. \quad \square$$

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② 

$$\begin{tikzcd} \mathbb{H} \arrow[d, "\frac{i-z}{i+z}"'] \arrow[r, phantom, "\text{is a Möbius transf.}"] & \mathbb{H} \arrow[d, "\frac{i-z}{i+z}", shift left] \\ \mathbb{D} \arrow[r, "e^{i\theta}"] & \mathbb{D} \arrow[u, bend right, "i\,\frac{1-z}{1+z}"'] \end{tikzcd}$$

$\frac{i-z}{i+z} = w$

$i-z = iw + zw$

$i - iw = z + zw$

$z = i\,\frac{1-w}{1+w}$

$$\boxed{\, i\,\frac{1 - e^{i\theta}\frac{i-z}{i+z}}{1 + e^{i\theta}\frac{i-z}{i+z}} \overset{\exists \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{R}) \text{ s.t. } \forall z \in \mathbb{H}}{=} \frac{az+b}{cz+d} \,}$$

$$\| $$

$$i\,\frac{(i+z) - e^{i\theta}(i-z)}{(i+z) + e^{i\theta}(i-z)} = \frac{z\left(i + ie^{i\theta}\right) + \left(-1 + e^{i\theta}\right)}{z\underbrace{\left(1 - e^{i\theta}\right)}_{-2i\sin\frac{\theta}{2} e^{\frac{\theta}{2}}} + \underbrace{i\left(1 + e^{i\theta}\right)}_{2i\cos\frac{\theta}{2} e^{\frac{\theta}{2}}}}$$

$$\begin{aligned} 1 - e^{i\theta} &= 1 - \cos\theta - i\sin\theta \\ &= 2\sin^2\tfrac{\theta}{2} - i\,2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} \\ &= -2i\left(\sin\tfrac{\theta}{2}\right)\left(\cos\tfrac{\theta}{2} + i\sin\tfrac{\theta}{2}\right) = -2i\sin\tfrac{\theta}{2}\, e^{\frac{\theta}{2}} \end{aligned}$$

$$1+e^{i\theta} = 1 + \cos\theta + i\sin\theta$$
$$= 2\cos^2\tfrac{\theta}{2} + i\, 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$

$$= \boxed{\frac{z\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{-z\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}}$$

$$\left(\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in SL_2\mathbb{R}\right). \quad \square$$

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Rmk: $\mathrm{Aut}(\mathbb{D}) \cong \mathrm{Aut}(\mathbb{H}) \cong PSL(2,\mathbb{R})$

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Riemann mapping thm: Any open, simply connected, connected of $\Omega \subsetneq \mathbb{C}$ is biholo. to $\mathbb{D}$.

i.e. $\exists\ f : \Omega \longrightarrow \mathbb{D}$ biholo.

This map will be constructed implicitly as a limit of certain sequence of holo. fns.

Recall: Several notions of convergence of holo fns: $(f_n) : \Omega \to \mathbb{C}$

① pointwise convergence: $\forall x \in \Omega,\ f_n(x) \to f(x)$ too weak: not much we can say about $f$.

② unif. convergence: too strong, not much examples

$\forall \varepsilon>0, \exists N>0$

s.t. $|f_n(x)-f(x)|<\varepsilon \quad \forall n>N, \forall x\in\Omega.$

③ unif. conv. on every cpt subsets of $\Omega$: $\to$ ✓

Recall: $f_n$: holo. on $\Omega$, $f_n \to f$ unif. on every cpt. subset of $\Omega$.

$\Rightarrow$ • $f$ holo. on $\Omega$. (cf. Lecture 9).

• $f_n' \to f'$ unif. on every cpt subset of $\Omega$.

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Def: A family $\mathcal{F}$ of holo. fcns. on $\Omega$, is called a normal family if $\forall \{f_n\} \subseteq \mathcal{F}$ any seq.

$\exists$ subseq. $\{f_{n_k}\}$ that unif. conv. over any $K \underset{\text{cpt.}}{\subseteq} \Omega$

Other notions:

- $\mathcal{F}$ is uniformly bounded if $\exists M>0$ s.t. $|f(x)|<M \quad \forall f\in\mathcal{F}, \forall x\in\Omega.$
- $\mathcal{F}$ is equicontinuous if $\forall\varepsilon>0, \exists\delta>0$ s.t. $|z_1-z_2|<\delta \Rightarrow |f(z_1)-f(z_2)|<\varepsilon. \quad \forall f\in\mathcal{F}.$ $z_1, z_2\in\Omega$

($\Rightarrow$ any $f\in\mathcal{F}$ is unif. conti.)

**Thm (Montel)** If $\mathcal{F}$ is family of holo. fns. on $\Omega$, suppose $\mathcal{F}$ is unif. bdd on every cpt. subset $K \subseteq \Omega$. Then

(a) $\mathcal{F}$ is equicontinuous on every cpt. subset $K \subseteq \Omega$.

(b) $\mathcal{F}$ is a normal family.

**Rmk:**

- In HW, you'll show: $\mathcal{F}$ is normal $\Rightarrow$ $\mathcal{F}$ is unif. bdd. on every cpt. subset $K \subseteq \Omega$.
  (so normal $\Leftrightarrow$ unif. bdd. on every cpt. subset.)
- Arzela-Ascoli thm: Any unif. bdd. equicontinuous sequence of fns defined over a cpt. set has a unif. conv. subseq.
  (essentially gives
  "unif. bdd. on every cpt subset + (a) $\Rightarrow$ (b)")
- part (a) is where we really need complex analysis

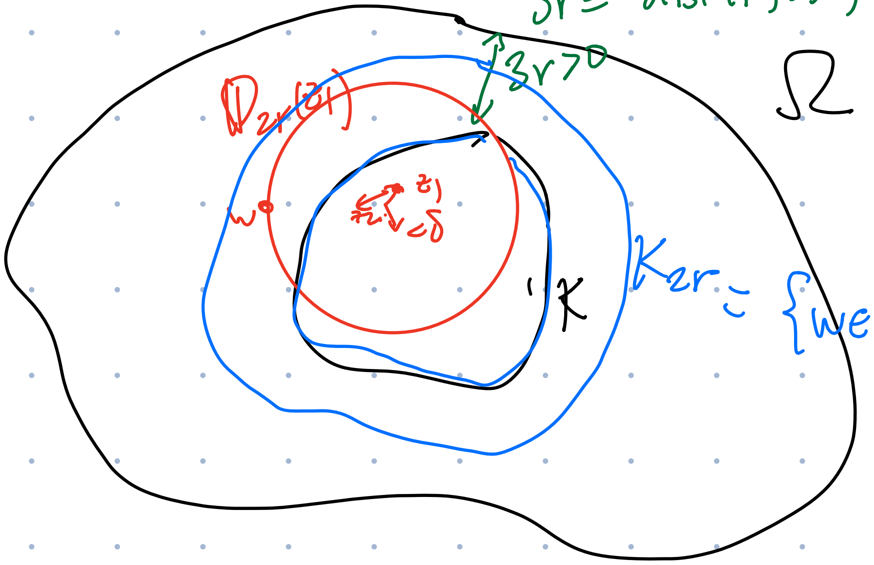
Pf of (a): $\mathcal{F}$ unif. bdd. on every cpt. subset $\Rightarrow$ equicontr. on every cpt subset.

$\forall K \subseteq \Omega$ $\forall \varepsilon > 0$,

we need to find $\delta > 0$

s.t. if $|z_1 - z_2| < \delta$, $z_1, z_2 \in K$ $\Rightarrow$ $|f(z_1) - f(z_2)| < \varepsilon$ $\forall f \in \mathcal{F}$

$3r = \operatorname{dist}(K, \partial\Omega)$




Choose $\delta < r$.

$\forall\, |z_1 - z_2| < \delta,\ z_1, z_2 \in K$

$K_{2r} = \{ w \in \Omega \mid \exists z \in K \text{ s.t. } |z - w| \le 2r \}$ cpt subset in $\Omega$.

$$f(z_1) - f(z_2) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_{2r}(z_1)} f(w) \left( \frac{1}{w - z_1} - \frac{1}{w - z_2} \right) dw$$

Cauchy integral formula

- $$\left| \frac{1}{w - z_1} - \frac{1}{w - z_2} \right| = \frac{|z_1 - z_2|}{\underbrace{|w - z_1|}_{= 2r}\ \underbrace{|w - z_2|}_{\ge r}} < \frac{|z_1 - z_2|}{2r^2}$$

- $$|f(z_1) - f(z_2)| \le \frac{1}{2\pi} \cdot 4\pi r \cdot \frac{|z_1 - z_2|}{2r^2} \cdot \sup_{w \in \partial \mathbb{D}_{2r}(z_1)} |f(w)|$$

$$\le \frac{1}{\cancel{2\pi}} \cdot \cancel{4\pi r} \cdot \frac{|z_1 - z_2|}{\cancel{2} r^{\cancel{2}}} \cdot \underbrace{\sup_{w \in K_{2r}} |f(w)|}_{\text{unif. bdd by assumption.}} < M$$

$$< \frac{|z_1 - z_2|}{r} M$$

depends only on $K$

unif. bdd of $\mathcal{F}$ on $K_{2r}$, i.e. $M$ depends only on $K$

So, choose $\delta = \frac{\varepsilon r}{M} > 0$

Then if $|z_1 - z_2| < \delta$, then

$$|f(z_1) - f(z_2)| < \frac{|z_1 - z_2|}{r} M < \frac{\delta M}{r} = \varepsilon. \quad \square$$

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<u>Thm</u> $\{f_n : \Omega \to \mathbb{C}\}$ holo. injective.

Suppose $f_n \to f$ unif. on every cpt subset $K \subseteq \Omega$.

(Then $f$ is holo.)

$\Longrightarrow$ $f$ is either injective or a constant fcn.