(1) For each of the following power series, find the radius of convergence and determine the exact interval of convergence.

$$(a) \ \sum \left(\frac{x}{n}\right)^n; \quad (b) \ \sum \left(\frac{(-1)^n}{n^2 \cdot 4^n}\right) x^n; \quad (c) \ \sum \left(\frac{(-1)^n}{n \cdot 4^n}\right) x^n; \quad (d) \ \sum x^{n!}.$$

(a)
$$\lim_{n\to\infty} \left(\left(\frac{1}{n} \right)^n \right)^k = 0.$$

(16)
$$\lim_{n\to\infty} \left(\frac{(-1)^n}{n^2 \cdot 4^n} \right)^{\frac{1}{n}} = \frac{1}{4} \implies R=4.$$

At
$$x=4$$
, get: $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2}$ conv.

At
$$x = -4$$
, get: $\sum_{h^2} \frac{1}{h^2} conv$.

At
$$x=4$$
, get: $\sum_{n=1}^{\infty} (1)^n conv$.

At
$$x = -4$$
, get: $\sum_{h} \frac{1}{h} div$.

(d) coeff.
$$a_k = \begin{cases} 1 \\ 0 \end{cases}$$
 $k=n!$ for some n .

$$\Rightarrow$$
 limsup $|a_k|^k = 1$. $\Rightarrow R = \frac{1}{1} = 1$.

For
$$x=\pm 1$$
, the seq. $(a_{i}cx^{k})$ doesn't conv. to D .
 \Rightarrow interval if $conv. = (-1,1)$.

- (2) Suppose $\sum a_n x^n$ has finite radius of convergence R > 0 and $a_n \ge 0$ for all n. Prove that if the series converges at R, then it also converges at -R.
- · We have EarR conv. and an zo . Yn.
- · $|a_n(-R)^n| = a_n R^n$, hence $\sum a_n (-R)^n$ conv. by Comparison Test.
 - (3) Consider a power series $\sum a_n x^n$ with radius of convergence R. Prove that if $\limsup |a_n| > 0$, then $R \le 1$.
- · JE70 and N>0 st. |an| > E Yn> N.
- Hence $\limsup_{n \to \infty} |a_n|^n \ge \lim_{n \to \infty} \varepsilon^n = 1$.
 - (4) By mimicking what we discussed in class, prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

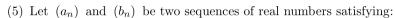
(Hint: First, we have $\sum (-1)^n x^{2n} = \frac{1}{1+x^2}$ for all |x| < 1.)

By thin we proved in class,

$$\tan^{1} x = x - \frac{x^{3}}{3} + \frac{x^{5}}{5} - \frac{x^{7}}{7} + \dots$$
 $\forall x | < 1$.

· the series is convergent at x=1, By Abel's thm,

We have
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = 1 + \frac{1}{5} + \frac{1}{5} = 1 + \frac{1}{5} +$$



- The partial sums of (b_n) is bounded: there exists L > 0 such that $|b_1 + \cdots + b_k| < L$ for any k,
- $\lim a_n = 0$,
- $\sum |a_n a_{n+1}|$ converges.

Prove that for any $k \in \mathbb{N}$, the series $\sum a_n^k b_n$ is convergent. (Hint: Same idea as the proof of Abel's theorem.)

$$\left| \sum_{n=m}^{N} a_{n}^{k} b_{n} \right| = \left| \sum_{n=m}^{N-1} (a_{n}^{k} - a_{n-1}^{k}) s_{n} + a_{n}^{k} s_{n} - a_{n}^{k} s_{n-1} \right|$$

$$\leq \sum_{n=m}^{N-1} |a_{n}^{k} - a_{n-1}^{k}| |s_{n}| + |a_{n}^{k}| |s_{n}| + |a_{m}^{k}| |s_{m-1}|$$

$$< \left[\sum_{n=m}^{N-1} |a_n - a_{n-1}| |a_n + - a_{n-1}| + |a_n| + |a_n| \right]$$

$$\left| \sum_{n=m}^{N} a_{n}^{k} b_{n} \right| < \left| \sum_{n=m}^{N-1} |a_{n} - a_{n-1}| |a_{n} + a_{n-1}| + |a_{n}| + |a_{n}| \right|
< \left| \sum_{n=m}^{N-1} |a_{n} - a_{n-1}| + |a_{n}| + |a_{n}| \right|
< \left| \sum_{n=m}^{N-1} |a_{n} - a_{n-1}| + |a_{n}| \right|
< \left| \sum_{n=m}^{N} |a_{n} - a_{n-1}| + |a_{n}| \right|$$