This week: series & convergence tests.

Coming next: finish "compact $\Leftrightarrow$ sequentially compact".

Study continuous maps between metric spaces.

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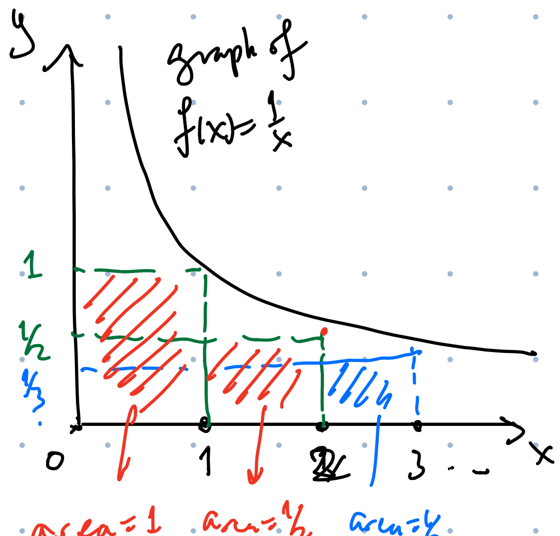
e.g. $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. (a_n) diverges

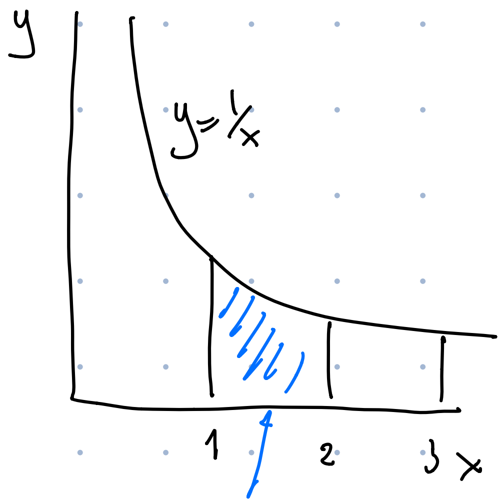
$$\geq 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{1/2} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{1/2} + \cdots$$

e.g. $a_n = \log(n)$ $\quad |a_{n+1} - a_n| = \log(n+1) - \log n$

$$= \log\left(\frac{n+1}{n}\right) \to 0 \quad \text{as } n \to \infty$$

($\frac{n+1}{n} \to 1$)





$$\int_1^2 \frac{1}{x}\,dx = \log 2 - \log 1$$
$$= \log 2$$

<u>Notation:</u> (a_k) seq. of real numbers, $n \geq m$

$$\sum_{k=m}^{n} a_k := a_m + a_{m+1} + \cdots + a_n$$

e.g. $$\sum_{k=1}^{n} \frac{1}{3^k} = \frac{1}{3^1} + \frac{1}{3^2} + \cdots + \frac{1}{3^n} = S$$

How to compute this?

$$S \cdot \frac{1}{3} = \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}}$$

$$S - S \cdot \frac{1}{3} = \frac{1}{3} - \frac{1}{3^{n+1}}$$

$$S = \frac{\frac{1}{3} - \frac{1}{3^{n+1}}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}\left(1 - \frac{1}{3^n}\right)}{\frac{2}{3}} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right).$$

Def: $\sum_{k=l}^{\infty} a_k = a_l + a_{l+1} + a_{l+2} + \cdots$ is called series

We say the series converges if the sequence

$$\left(\sum_{k=l}^{l} a_k, \sum_{k=l}^{l+1} a_k, \sum_{k=l}^{l+2} a_k, \cdots \right) \text{ converges.}$$

$$\begin{matrix} \| & \| & \| \\ a_l & a_l + a_{l+1} & a_l + a_{l+1} + a_{l+2} \end{matrix}$$

"partial sums of the series"

(otherwise, the series is divergent)

If the seq. $\left(\sum_{k=l}^{l} a_k, \sum_{k=l}^{l+1} a_k, \cdots \right)$ converges to $s \in \mathbb{R}$, then we denote

$$\sum_{k=l}^{\infty} a_k = s.$$

e.g. $$\sum_{k=1}^{\infty} \frac{1}{3^k} = \lim_{n \to \infty} \underbrace{\boxed{\frac{1}{2}\left(1 - \frac{1}{3^n}\right)}}_{= \sum_{k=1}^{n} \frac{1}{3^k}} = \frac{1}{2}$$

Q: $(a_1, a_2, a_3, \ldots)$ seq. in $\mathbb{R}$

$$\sum_{k=1}^{\infty} a_k, \qquad \sum_{k=2}^{\infty} a_k$$

(1) If $\boxed{\sum_{k=1}^{\infty} a_k \text{ is convergent,}}$ then do we know $\sum_{k=2}^{\infty} a_k$ also convergent?

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$\vdots$

$\lim_{n\to\infty} s_n = s$ exists.

$t_2 = a_2 = s_2 - a_1$

$t_3 = a_2 + a_3 = s_3 - a_1$

$t_4 = a_2 + a_3 + a_4 = s_4 - a_1$

$\vdots$

$\lim_{n\to\infty} t_n = s - a_1$ by limit thm.

Rmk: The starting term of the series doesn't effect its convergence, (but the sum will be effected). So, it makes sense to talk about "converge of $\sum a_k$" without specifying the starting term.

Ex: For which $r \in \mathbb{R}$ does the series $\sum r^k$ converge?

($\Leftrightarrow |r| < 1$)

Cauchy criterion:

$$\sum_{k=l}^{\infty} a_k \text{ conv.} \overset{\text{def}^b}{\Longleftrightarrow} \text{the partial sums}$$

$(s_l, s_{l+1}, s_{l+2}, \cdots)$ is conv.

where:

$s_l = a_l$

$s_{l+1} = a_l + a_{l+1}$

$s_{l+2} = a_l + a_{l+1} + a_{l+2}$

$\vdots$

$\overset{\text{Cauchy criterion}}{\Longleftrightarrow}$ $\forall \varepsilon > 0$, $\exists N > 0$. s.t.

$$\underbrace{|s_n - s_m|}_{} < \varepsilon \quad \forall n > m > N.$$

$$s_n = \sum_{k=l}^{n} a_k \qquad s_m = \sum_{k=l}^{m} a_k$$

so $|s_n - s_m| = \left| \sum_{k=m+1}^{n} a_k \right|$

Ex

$\Longleftrightarrow$ $\forall \varepsilon > 0$, $\exists N > 0$ s.t.

$$\Bigg| \underbrace{\sum_{k=m}^{n}}_{\uparrow} a_k \Bigg| < \varepsilon \quad \forall n \geq m > N.$$

sum of any finite number of consecutive terms of (a_k) after the N-th term.

Prop: If $\sum a_k$ is conv., then $\lim_{k\to\infty} a_k = 0$

pf: By Cauchy criterion, $\forall \varepsilon>0$, $\exists N>0$ s.t.

$$\left|\sum_{k=m}^{n} a_k\right| < \varepsilon \quad \forall\, n\geq m>N.$$

(apply to the case where $n=m>N$.)

$$\Longrightarrow \quad |a_n|<\varepsilon \quad \forall\, n>N$$

$$\Longrightarrow \quad \lim a_n = 0. \quad \square$$

Rmk: The converse is not true:

$$\text{"} \lim_{k\to\infty} a_k = 0 \text{"} \quad \not\Longrightarrow \quad \text{"} \sum a_k \text{ is conv."}$$

(counterexample: $a_k = \frac{1}{k}$, $\sum a_k$ div.)

Rmk: e.g. we can use this to show that if $|r|\geq 1$ then $\sum r^k$ div.

Comparison test: $\begin{cases} \bullet\ \sum a_k \text{ series, suppose } a_k\geq 0\ \forall k. \\ \bullet\ \sum a_k \text{ converges} \\ \bullet\ \sum b_k \text{ series, where } |b_k|\leq a_k\ \forall k. \end{cases}$

$$\Longrightarrow \quad \sum b_k \text{ is conv.}$$

~~idea~~: Need: $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $$\left| \sum_{k=m}^{n} b_k \right| < \varepsilon \quad \forall n \geq m > N$$

$$\wedge\!\!|$$

$$\sum_{k=m}^{n} |b_k| \leq \boxed{\sum_{k=m}^{n} a_k}$$

Pf: Since $\sum a_k$ conv., so $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon \quad \forall n \geq m > N$$

$$\|$$

$$\sum_{k=m}^{n} a_k \geq \sum_{k=m}^{n} |b_k| \geq \left| \sum_{k=m}^{n} b_k \right|$$

$\Longrightarrow$ $\sum b_k$ conv. $\square$

Def: $\sum a_k$ is called <u>absolutely convergent</u> if $\sum |a_k|$ conv.

Ex: If $\sum a_k$ is abs. conv., then $\sum a_k$ conv.

Rmk: conv. $\not\Rightarrow$ abs. conv.

A series which is conv., but not abs. conv., is called "conditionally conv.".

(cf. Wikipedia) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \cdots$

Ratio test: $a_k \neq 0 \quad \forall k.$

1) $\sum a_k$ abs. conv. If $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1.$

2) $\sum a_k$ div. if $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1.$

Root test: $\alpha := \limsup_{k\to\infty} |a_k|^{1/k}$

1) $\sum a_k$ abs. conv. If $\alpha < 1$

2) $\sum a_k$ div. if $\alpha > 1.$

e.g. $\boxed{\sum \frac{1}{k}}$ $\boxed{\text{div.}}$

$$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{k+1}}{\frac{1}{k}} = \frac{k}{k+1} \to 1 \text{ as } k\to\infty$$

Root/Ratio test don't give any info

$$\left(\frac{1}{k}\right)^{1/k} = \frac{1}{k^{1/k}} \to 1 \text{ as } k\to\infty$$

e.g. $\boxed{\sum \frac{1}{k^2}} = ?? \frac{\pi^2}{6}$ $\boxed{\text{conv.}}$

Euler formula

$$\frac{a_{k+1}}{a_k} = \left(\frac{k}{k+1}\right)^2 \to 1 \text{ as } k\to\infty$$

$$\left(\frac{1}{k^2}\right)^{1/k} \to 1 \text{ as } k\to\infty$$

$\alpha_1, \alpha_2, \ldots, \alpha_n$

$p(x) = c(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

Euler's idea: (not a formal proof)

$$\frac{\sin x}{x} = \boxed{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots}$$

has zeros at $\pm\pi, \pm 2\pi, \ldots$

$-\frac{1}{6}$

Compare the coeff. of x^2

$$= \left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots$$

(need complex analysis to make sense of this)

$$-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\ldots\right) = \boxed{\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots}$$