HOMEWORK 6
MATH 104, SECTION 2

Some ground rules:

- You have to submit your homework via **Gradescope** to the corresponding assignment. The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You're allowed to use any result that is proved in the lecture; but if you'd like to use other results, you have to prove them before using them.

Problem set (6 problems; due March 9 at 11am PT)

(1) Let E be a nonempty, closed, and bounded subset of $\mathbb{R}$. Prove that $\sup E$ and $\inf E$ both belong to E.

(2) Consider the following two functions on $\mathbb{R}$:

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

For each of the functions, prove or disprove that it is continuous at the point $x = 0$.

(3) Let $\epsilon > 0$ be a positive number. In each case, find a $\delta > 0$ (which should depend on ϵ) such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon \text{ holds.}$$

(a) $f(x) = \frac{1}{x}$; $x_0 = 1$.
(b) $f(x) = \sqrt{|x|}$; $x_0 = 0$.
(c) $f(x) = \sqrt{x}$; $x_0 = 1$.

(4) Suppose f, g are real-valued continuous functions on the closed interval $[a, b]$, and $f(a) < g(a)$ and $f(b) > g(b)$. Prove that $f(c) = g(c)$ for some $c \in (a, b)$.

(5) Prove the following generalization of Ross, Theorem 17.4: Let (X, d) be any metric space, and let $f, g : X \to \mathbb{R}$ be two real-valued functions that are continuous at $x_0 \in X$. Prove that the functions $f + g$ and fg are both continuous at x_0. Moreover, if $g(x_0) \neq 0$, then f/g is also continuous at x_0. (The proofs are very similar, so you can pick one of $f + g, fg, f/g$ and prove it.)

(6) Prove the following generalization of Ross, Theorem 17.5: Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be three metric spaces and let $f : X \to Y$ and $g : Y \to Z$ be two maps among them. Define the composite function $g \circ f : X \to Z$ via $(g \circ f)(x) := g(f(x))$. Prove that if f is continuous at $x_0 \in X$ and g is continuous at $f(x_0) \in Y$, then the composition $g \circ f$ is continuous at x_0.