3/19/2020

Recall $f(x)=\sum a_n x^n$ w/ r.o.c. $R>0$

We proved $\sum a_n x^n$ conv. unif. on $[-R', R']$
$\forall\ 0<R'<R$

$\Rightarrow$ $f(x)$ is conti. on $(-R,R)$

Abel's thm Suppose $f(x)=\sum a_n x^n$ is conv. at $x=R$, then f is conti. at $x=R$.

pf Suppose $f(x)=\sum a_n x^n$ has r.o.c. $=1$, and $\sum a_n x^n$ conv. at $x=1$.

WTS: $f(x)$ is conti. on $[0,1]$

$\Uparrow$

$\sum a_n x^n$ conv. unif. to f on $[0,1]$

$\Updownarrow$

unif Cauchy, i.e.

$$\boxed{\forall \varepsilon>0,\ \exists N>0 \text{ s.t. } \left|\sum_{k=m}^{n} a_k x^k\right|<\varepsilon \quad \forall n\geq m>N,\ \forall x\in[0,1]}$$

WLOG, we can subtract f by a const.,

s.t. $f(1) = 0 = \sum a_n$

$$\sum_{k=m}^{n} a_k x^k = \sum_{k=m}^{n} (s_k - s_{k-1}) x^k \quad \Big| \quad s_k = \sum_{l=0}^{k} a_l$$

$$= \sum_{k=m}^{n} s_k x^k - \sum_{k=m}^{n} s_{k-1} x^k$$

$$= \boxed{\sum_{k=m}^{n} s_k x^k} - x \sum_{k=m}^{n} s_{k-1} x^{k-1}$$

$$= \boxed{\sum_{k=m}^{n-1} (1-x) s_k x^k} + s_n x^n - s_{m-1} x^m$$

Since $\sum a_n = 0$, $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $|s_n| < \frac{\varepsilon}{3}$ $\forall n > N$. $\quad x \in [0,1]$

$\Rightarrow$ • $|s_n x^n| = |s_n||x|^n < \frac{\varepsilon}{3} \cdot 1 = \frac{\varepsilon}{3}$ $\quad \forall n > N$

• $|s_{m-1} x^m| < \frac{\varepsilon}{3}$ $\quad \forall m > N$.

• $\left|(1-x) \sum_{k=m}^{n-1} s_k x^k\right| \le (1-x) \cdot \frac{\varepsilon}{3} \sum_{k=m}^{n-1} x^k$

$$= \cancel{(1-x)} \cdot \frac{\varepsilon}{3} \cdot \frac{\boxed{x^m (1-x^{n-m})}}{\cancel{1-x}} < \frac{\varepsilon}{3}$$

$\Rightarrow$ $\forall n \ge m > N$, $\forall x \in [0,1]$,

we have $\left|\sum_{k=m}^{n} a_k x^k\right| < \varepsilon$. $\square$

Thm $f(x) = \sum a_n x^n$ has r.o.c. $R > 0$

Then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has r.o.c. $R > 0$, and

f is differentiable on $(-R, R)$,

and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

pf

$$\limsup |n a_n|^{1/n} = \big(\underbrace{\lim n^{1/n}}_{\substack{\shortparallel \\ 1}}\big)\big(\limsup |a_n|^{1/n}\big) = \limsup |a_n|^{1/n}$$

$\Rightarrow \sum n a_n x^{n-1}$ also has r.o.c. $R > 0$.

$$g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

On $|x| < R$, we have (we proved last time)

$$\int_0^x g(t)\,dt = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$$

Since g is conti. on $(-R, R)$, by

Fundamental Theorem of Calculus,

$\int_0^x g(t)\,dt$ is differentiable, and

$$\left(\int_0^x g(t)\,dt\right)' = g(x)$$

$\Rightarrow$ f is differentiable and $f'(x) = g(x)$. $\square$

Q: How to approx. a conti. fcn. on $[a,b]$ by simpler functions, e.g. polynomials?

Weierstrass Approximation Thm every conti. fcn. $f(x)$ on $[a,b]$ can unif. approx. by polynomials, i.e.

$\exists\ P_n(x)$ polynomials s.t. $P_n(x) \to f(x)$ unif. on $[a,b]$

Def X: metric space, $E \subset X$ subset.

We say E is dense in X, if $\overline{E} = X$

$$\overline{E} = E \cup \{\text{limit pts of } E\}$$

e.g. $X = \mathbb{R}$, with standard d.

$E = \mathbb{Q}$ is dense in $\mathbb{R}$

We ~~can~~ define

- $\mathcal{C}([a,b];\mathbb{R})$ - the set of real-valued conti. fcn on $[a,b]$

$$d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

metric space

- $(f_n) \to f$ unif. $\Leftrightarrow$ $f_n \to f$ in the metric space $\mathcal{C}([a,b];\mathbb{R})$

Weierstrass Approx. thm

$\Leftrightarrow$ $\{$poly. ~~functions~~$\} \subset \underline{\mathcal{C}([a,b];\mathbb{R})}$

is a dense subset.

finite dim'l v.s./$\mathbb{R}$

vector space/$\mathbb{R}$ (∞-dim'l)

One can replace $[a,b]$ by cpt metric space X.

X: cpt metric space

$\mathcal{C}(X;\mathbb{R})$ - real-valued conti. fn on X.

metric $d(f,g) := \sup_{x \in X} |f(x) - g(x)|$

Def Say $A \subset \mathcal{C}(X;\mathbb{R})$ is a <u>subalgebra</u> of $\mathcal{C}(X;\mathbb{R})$ if

- A is a $\mathbb{R}$-vector subspace of $\mathcal{C}(X;\mathbb{R})$
- A is closed under multiplication

Def Say $A \subset \mathcal{C}(X;\mathbb{R})$ <u>separate points</u> if $\forall\, x, y$ in X, $x \neq y$.

$\exists\, f \in A$ s.t. $f(x) \neq f(y)$

<u>Stone-Weierstrass thm</u> X - cpt metric space

$A \subset \mathcal{C}(X;\mathbb{R})$ is a subalgebra, separate pts, contains $\mathbb{1}$ (constant fun 1),

$\Rightarrow$ A is dense in $\mathcal{C}(X;\mathbb{R})$

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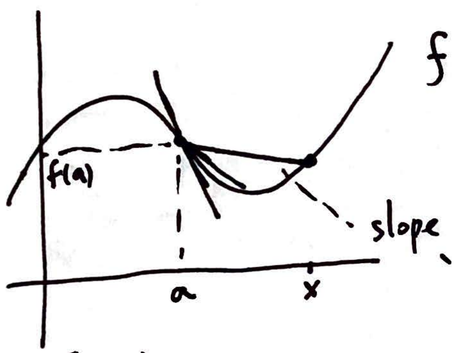
§ Differentiation

<u>Derivative</u> "$f'(a)$".

"rate of change of $f(x)$ at $x=a$"



$\dfrac{f(x)-f(a)}{x-a}$

(Ross, §20)

<u>Limit of fcns</u>: Say "$\lim\limits_{x\to a} F(x) = L$" if

$$\boxed{\forall \varepsilon>0,\ \exists \delta>0 \text{ s.t. } 0<|x-a|<\delta \Rightarrow |F(x)-L|<\varepsilon}$$

<u>Note</u>. In this definition, ~~f(a) may~~ F doesn't have to define at a.

<u>Note</u> (HW) $\Leftrightarrow$ $\boxed{\forall \text{ seq. } (x_n) \text{ st. } \lim x_n = a,\ x_n \neq a\ \forall n, \text{ we have } \lim\limits_{n\to\infty} F(x_n) = L}$

<u>Def</u>. $f: I \to \mathbb{R}$, I: open interval containing $a \in \mathbb{R}$

We say f is <u>differentiable at a</u> if

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} \text{ exists and is finite}$$

In this case, we denote $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

e.g. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$

<u>$f'(a) = ?$</u>

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{x^n - a^n}{\cancel{x-a}}$$

$x^n - a^n = (\cancel{x-a})(x^{n-1} + x^{n-2}a + \cdots + a^{n-1})$

$$= \lim_{x \to a} (x^{n-1} + x^{n-2}a + \cdots + a^{n-1})$$

$$= n a^{n-1}$$

<u>Thm</u> If f is differentiable at a, then it's conti. at a.

pf. f is diff^{ble} at $a \Leftrightarrow \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists and finite

- f conti. at $a \Leftrightarrow \lim_{x \to a} f(x) = f(a) \Leftrightarrow \lim_{x \to a} (f(x)-f(a)) = 0$

$$f(x)-f(a) = \frac{f(x)-f(a)}{x-a} \cdot (x-a) \quad \forall x \neq a$$

Take $\lim_{x \to a}$. Then $\lim_{x \to a} (f(x)-f(a)) = \lim_{x \to a} \left(\frac{f(x)-f(a)}{x-a}\right) \lim_{x \to a} (x-a) = 0$ $\square$

Thm f, g both differentiable at a.

Then cf, $f+g$, fg are diff^{ble} at a.

& if $g(a) \neq 0$, then f/g is also diff^{ble} at a.

pf We'll prove "$\underline{fg}$"

$$\boxed{\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}} = \lim_{x\to a}\frac{\overbrace{f(x)g(x)-f(x)g(a)}^{f(x)(g(x)-g(a))}+\overbrace{f(x)g(a)-f(a)g(a)}^{g(a)(f(x)-f(a))}}{x-a}$$

we have: $\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a)$,

$\lim_{x\to a}\frac{g(x)-g(a)}{x-a} = g'(a)$

$$\lim_{x\to a} f(x)\cdot\frac{g(x)-g(a)}{x-a} + g(a)\cdot\frac{f(x)-f(a)}{x-a}$$

$$(fg)'(a) = \underline{f(a)\cdot g'(a) + g(a)\cdot f'(a)}.$$

(product rule of derivatives)