#1: Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 1 & 2 & 3 \end{bmatrix}$. Diagonalizable?

Sol^{n}: NOT diagonalizable.

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 2 & 1 \\ 0 & 4-\lambda & 0 \\ 1 & 2 & 3-\lambda \end{pmatrix} = (4-\lambda)\left((3-\lambda)^2 - 1\right)$$

$$= (4-\lambda)^2(2-\lambda)$$

$$\mathrm{Nul}(A - 4I) = \mathrm{Nul}\begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & -1 \end{pmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

$$1 = \dim \mathrm{Nul}(A - 4I) < \mathrm{mult}(4) = 2. \quad \square$$

#2: Compute $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{2020} = ?$

Sol^{n}:

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

$$\mathrm{Nul}(A - 3I) = \mathrm{Nul}\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$\mathrm{Nul}(A + 1I) = \mathrm{Nul}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}.$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{2020} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3^{2020} & \\ & (-1)^{2020} \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3^{2020} & 1 \\ 3^{2020} & -1 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3^{2020}+1}{2} & \frac{3^{2020}-1}{2} \\ \frac{3^{2020}-1}{2} & \frac{3^{2020}+1}{2} \end{pmatrix}. \quad \square$$

#3. Let A be an $n \times n$ matrix. Suppose A has n distinct eigenvalues. Is it guaranteed that A is diagonalizable?

Sol^n: Yes.

$\Rightarrow$ each eigenvalue has multiplicity 1.

$\Rightarrow$ dim. of each eigenspace $= 1 =$ multiplicity.

$\Rightarrow$ diagonalizable. $\square$