Last Time:

- Vector spaces, subspaces, $\mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$
- linear transformat'ns b/w vector spaces, kernel, image

e.g. A: matrix. $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.
$\vec{x} \longmapsto A\vec{x}$.

$$\mathrm{Ker}(T_A) = \mathrm{Nul}(A), \quad \mathrm{Im}(T_A) = \mathrm{Col}(A)$$

Def: $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$ (v.s.) is <u>linearly dependent</u> if $\exists\, c_1, \dots, c_n$ not all 0

s.t. $c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{0}$

Otherwise, is called <u>linearly independent</u>.

Today: basis, dimension of a vector space.

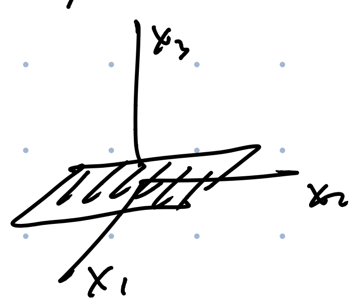
(why consider basis? why consider different bases?
how to find a basis?)

Def V: v.s. $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$ is a <u>basis</u> if

they're l.i. and $\mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$.

e.g.: • $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$ not a basis

(b/c it doesn't span $\mathbb{R}^3$)



- $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} \subseteq \mathbb{R}^3$ not a basis (b/c they're l.d.)

- $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\} \subseteq \mathbb{R}^3$ is a basis

Consider $V = \mathbb{R}^n$, $\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq V$

- If $k < n$, $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subsetneq V$

$$\begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}$$

can't have pivots in each row.

- If $k > n$, $\{\vec{v}_1, \dots, \vec{v}_k\}$ l.d.

$$\begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}$$

can't have pivots in each column

- If $k = n$, $\{\vec{v}_1, \dots, \vec{v}_n\}$ a basis?

$$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}_{n \times n}$$

$\{\vec{v}_1, \dots, \vec{v}_n\}$ l.i. $\iff$ A has pivots in each column.

$\Updownarrow$

$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = \mathbb{R}^n \iff A$ has pivots in each row

$\Updownarrow$

A invertible

e.g. $\text{Poly}_{\le n} = \{\text{all poly. w/ deg} \le n\}$

$\{1, x, x^2, \cdots, x^n\}$ is a basis of $\text{Poly}_{\le n}$?

① l.i.?

$$\boxed{c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 + \cdots + c_n \cdot x^n = 0.}$$

$\Rightarrow c_0 = c_1 = \cdots = c_n = 0.$

$\Rightarrow$ the set is l.i.

② Span $\text{Poly}_{\le n}$?

$\forall\, p(x) \in \text{Poly}_{\le n}$.

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ &= a_0 \cdot 1 + a_1 \cdot x + \cdots + a_n \cdot x^n \\ &\in \text{Span}\{1, x, \cdots, x^n\}. \end{aligned}$$

$\square$

Thm $\mathcal{B} = \{\vec{v}_1, \cdots, \vec{v}_n\}$ a basis of V.

Then $\forall\, \vec{x} \in V$, $\exists!\, c_1, \cdots, c_n \in \mathbb{R}$ s.t. $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$

pf

- existence of $c_1, \cdots, c_n$: $\text{Span}\{\vec{v}_1, \cdots, \vec{v}_n\} = V$
- uniqueness: $\left.\begin{aligned} \vec{x} &= c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \\ &= c_1'\vec{v}_1 + \cdots + c_n'\vec{v}_n \end{aligned}\right\}$

$$\vec{0} = (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) - (c_1'\vec{v}_1 + \cdots + c_n'\vec{v}_n)$$
$$= (c_1 - c_1')\vec{v}_1 + (c_2 - c_2')\vec{v}_2 + \cdots (c_n - c_n')\vec{v}_n$$

$\{\vec{v}_1, \ldots, \vec{v}_n\}$ l.i. $\Rightarrow$ $c_1 - c_1' = 0,\ c_2 - c_2' = 0, \ldots$

$\Rightarrow$ $c_1 = c_1',\ c_2 = c_2', \ldots$

$\square$

Def These $c_1, \ldots, c_n$ are called the coordinats of $\vec{x}$ relative to the basis B.

$\vec{x} \in V$

$$[\vec{x}]_B := \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

the coordinate vector of $\vec{x}$ relative to B.

or B-coordinate vector of $\vec{x}$.

$$\begin{aligned} V &\xrightarrow{[\ ]_B} \mathbb{R}^n \\ \vec{x} &\longmapsto \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \end{aligned}$$

coordinate mapping w.r.t. B.

where $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$

$B = \{\vec{v}_1, \ldots, \vec{v}_n\}$.

Prop $[\ ]_B$ is linear and bijective. $\rightarrow$ Clear.

pf • linear: $[\vec{x}+\vec{y}]_B \overset{??}{=} [\vec{x}]_B + [\vec{y}]_B$

$$\begin{bmatrix} c_1+d_1 \\ \vdots \\ c_n+d_n \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$
$$\vec{y} = d_1\vec{v}_1 + \cdots + d_n\vec{v}_n$$
$$\Rightarrow \vec{x}+\vec{y} = (c_1+d_1)\vec{v}_1 + \cdots + (c_n+d_n)\vec{v}_n$$
$$\Rightarrow [\vec{x}+\vec{y}]_B = \begin{bmatrix} c_1+d_1 \\ c_2+d_2 \\ \vdots \\ c_n+d_n \end{bmatrix}$$

e.g. $\text{Poly}_{\leq n}$ $\{1, x, \ldots, x^n\} = B$ is a basis

$$\text{Poly}_{\leq n} \xrightarrow{[\ ]_B} \mathbb{R}^{n+1}$$
$$a_0 + a_1x + \cdots + a_nx^n \longmapsto \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$
$$\parallel$$
$$a_0 \cdot 1 + a_1 \cdot x + \cdots + a_n \cdot x^n$$

e.g. $V = \mathbb{R}^2$, $B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

$$V = \mathbb{R}^2 \xrightarrow{[\ ]_B} \mathbb{R}^2$$
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} 2 \\ 1 \end{bmatrix}_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\vec{x} \longmapsto [\vec{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\vec{x} = 2\begin{bmatrix}2\\1\end{bmatrix} + 3\begin{bmatrix}1\\-2\end{bmatrix}$$

$$\begin{bmatrix}7\\1\end{bmatrix} \longmapsto \begin{bmatrix}7\\1\end{bmatrix}_B = ?? \begin{bmatrix}c_1\\c_2\end{bmatrix}$$

$$\begin{bmatrix}7\\1\end{bmatrix} = \underline{c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-2\end{bmatrix}}$$

$$= \begin{bmatrix}2 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & -2\end{bmatrix}^{-1}\begin{bmatrix}7\\1\end{bmatrix}$$

In general, $V = \mathbb{R}^n$,

$B = \boxed{\{\vec{v}_1, \dots, \vec{v}_n\}}$ $\qquad P_B = \begin{bmatrix}| & & |\\ \vec{v}_1 & \cdots & \vec{v}_n\\ | & & |\end{bmatrix}$

$\underline{\text{invertible}}$

$$V = \mathbb{R}^n \xrightarrow{[\ ]_B} \mathbb{R}^n$$

$$\vec{x} \longmapsto [\vec{x}]_B = \begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix} = P_B^{-1}\vec{x}.$$

$T_{P_B^{-1}}$

$$\vec{x} = \underline{c_1\vec{v}_1 + \cdots + c_n\vec{v}_n}$$

$$= \underbrace{\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}}\underbrace{\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}} \qquad \Rightarrow \quad \underline{[\vec{x}]_B = P_B^{-1}\vec{x}.}$$

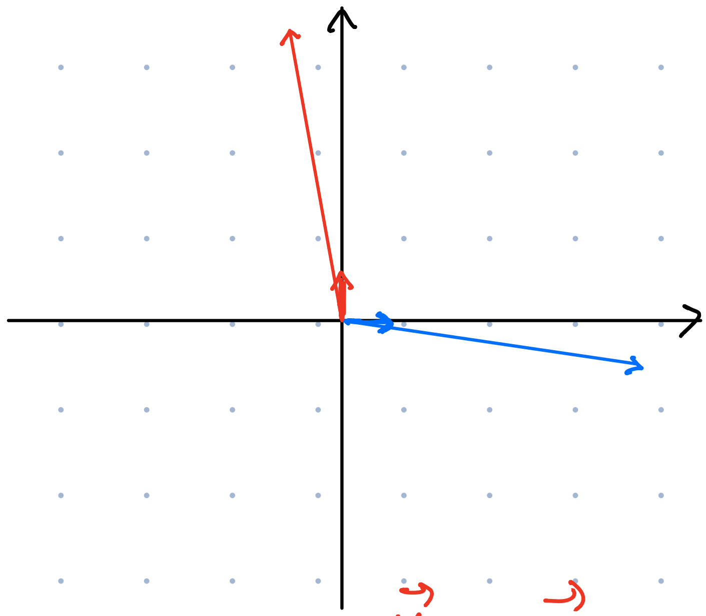
$$= P_B\,[\vec{x}]_B.$$

Rmk: Why do we want to consider different basis?

e.g. Consider $A = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}$, and $T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Say we want to better understand T_A geometrically.

$$T_A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 11 \\ -2 \end{bmatrix}, \quad T_A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 14 \end{bmatrix}.$$



If we consider the basis $\left\{ \overset{\vec{v}_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ -2 \end{bmatrix}} \right\}$.

$$T_A\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix} = 10\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

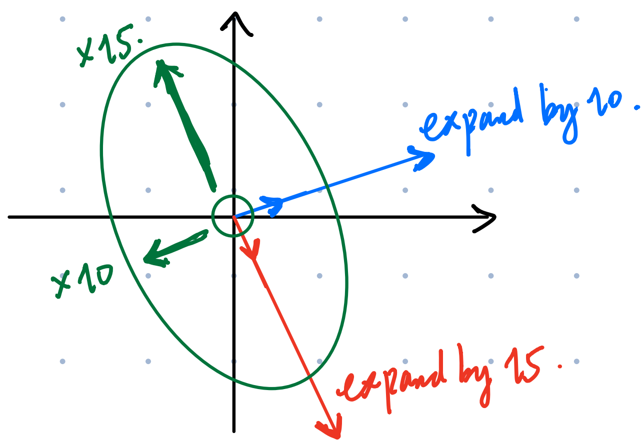
$$T_A\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 15 \\ -30 \end{bmatrix} = 15\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$\vec{x} \in \mathbb{R}^2$

$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$

$\Downarrow$

$T_A(\vec{x}) = 10c_1\vec{v}_1 + 15c_2\vec{v}_2$



It's easier to understand T using this basis!!

e.g.

$B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "T_A"] \arrow[d, "{[\ ]_B}"'] & \mathbb{R}^2 \arrow[d, "{[\ ]_B}"] \\ \mathbb{R}^2 & \mathbb{R}^2 \end{tikzcd}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 11 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} -2 \\ 14 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\mathbb{R}^2 \xrightarrow{[\ ]_B^{-1}} \mathbb{R}^2 \xrightarrow{T_A} \mathbb{R}^2 \xrightarrow{[\ ]_B} \mathbb{R}^2$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longmapsto 10\begin{bmatrix} 2 \\ 1 \end{bmatrix} \longmapsto 10\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 \\ -2 \end{bmatrix} \longmapsto 15\begin{bmatrix} 1 \\ -2 \end{bmatrix} \longmapsto 15\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T_{\begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix}} = [\ ]_B \circ T_A \circ [\ ]_B^{-1}$$

$$= T_{P_B^{-1}} \circ T_A \circ T_{P_B}$$

$$= T_{P_B^{-1} A P_B}.$$

What we've done is:

$A = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}$,

We found $P_B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, s.t.

$$P_B^{-1} A P_B = \begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix}$$

diagonalization

Spanning set thm

If $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$, $\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$, then there exists a subset of $\{\vec{v}_1, \dots, \vec{v}_n\}$ that gives a basis of V.

pf ① If $\underline{\{\vec{v}_1, \dots, \vec{v}_n\}}$ l.i. $\Rightarrow \{\vec{v}_1, \dots, \vec{v}_n\}$ a basis

② $\underline{\text{If } \{\vec{v}_1, \dots, \vec{v}_n\} \text{ l.d.}}$

$\Rightarrow \exists i$ s.t. $\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}$

$\Rightarrow \underline{\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\}} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}$

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$$

Continue this process on the remaining vectors.

b/c there is only finitely many vectors in $\{\vec{v}_1, \dots, \vec{v}_n\}$.

so we'll stop in finitely many steps. $\square$

Find a basis of Nul(A) and Col(A):

$\boxed{A: m \times n}$ $\quad \text{Nul}(A) = \{\vec{x} \mid A\vec{x} = 0\} \subseteq \mathbb{R}^n \quad \text{Col}(A) = \text{Span}\{\text{columns of } A\} \subseteq \mathbb{R}^m.$

(Using reduced echelon form)

Nul(A):

$$A \xrightarrow{\text{row reduction}} \begin{array}{ccccc} & x_2 & & x_4 & x_5 \\ & \downarrow & & \downarrow & \downarrow \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

$$\underline{\mathrm{Nul}(A)} = \underline{\mathrm{Nul}(B)}$$

$$\{\vec{x} \mid A\vec{x}=0\} = \{\vec{y} \mid B\vec{y}=0\}$$

b/c $A\vec{x}=\vec{0} \iff B\vec{x}=\vec{0}$

b/c A, B are related by row operations.

$$\mathrm{Nul}(B) = \left\{ \vec{x} \in \mathbb{R}^5 : x_2, x_4, x_5 \in \mathbb{R},\ x_1 = -2x_2 - 3x_4 - 5x_5,\ x_3 = -4x_4 - 6x_5 \right\}$$

$$= \left\{ \begin{bmatrix} -2x_2 - 3x_4 - 5x_5 \\ x_2 \\ -4x_4 - 6x_5 \\ x_4 \\ x_5 \end{bmatrix} \,\middle|\, x_2, x_4, x_5 \in \mathbb{R} \right\}$$

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathrm{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$x_2 = 1$, $x_4 = 0$, $x_5 = 0$

$x_2 = 0$, $x_4 = 1$, $x_5 = 0$

<u>they're l.i.!</u>