FINAL EXAM SOLUTION
MATH H54, FALL 2021

Problem 1: (10 points) Find the unique triple of real-valued functions $x_1(t), x_2(t), x_3(t)\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$x_1'(t) = x_2'(t) = x_3'(t) = x_1(t) + 2x_2(t) + 3x_3(t) \quad \text{for all} \quad t \in \mathbb{R}$$

and the initial conditions

$$x_1(0) = 6 \quad \text{and} \quad x_2(0) = x_3(0) = 0.$$

Solution: There is a diagonalization

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 6 & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}^{-1}.$$

Therefore general solutions of the differential equation can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 e^{6t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

Plug in $t = 0$, one sees that $c_1 = c_2 = c_3 = 1$ is the unique set of coefficients that satisfies the initial condition. Hence the solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{6t} + 5 \\ e^{6t} - 1 \\ e^{6t} - 1 \end{bmatrix}.$$

Problem 2: (10 points) Find the unique real-valued function $y(t)\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$y''(t) + 2y'(t) + y(t) = \frac{e^{-t}}{t^2+1}$$

and the initial conditions

$$y(0) = 5 \quad \text{and} \quad y'(0) = 4.$$

Solution: First, $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ form a basis of the space of solutions of the homogeneous equation $y'' + 2y' + y = 0$. Consider

$$\int \frac{-y_2(t)\frac{e^{-t}}{t^2+1}}{y_1(t)y_2'(t) - y_2(t)y_1'(t)}\,dt = \int \frac{\frac{-te^{-2t}}{t^2+1}}{e^{-2t}}\,dt = -\int \frac{t}{t^2+1}\,dt = -\frac{1}{2}\log(t^2+1) + \text{constant}$$

and

$$\int \frac{y_1(t)\frac{e^{-t}}{t^2+1}}{y_1(t)y_2'(t) - y_2(t)y_1'(t)}\,dt = \int \frac{\frac{e^{-2t}}{t^2+1}}{e^{-2t}}\,dt = \int \frac{1}{t^2+1}\,dt = \tan^{-1}(t) + \text{constant}.$$

By the variation of parameters method, general solution of the non-homogeneous equation $y''(t) + 2y'(t) + y(t) = \frac{e^{-t}}{t^2+1}$ is therefore of the form

$$y(t) = c_1 e^{-t} + c_2 te^{-t} - \frac{1}{2}\log(t^2+1)e^{-t} + \tan^{-1}(t)te^{-t}.$$

It's not hard to see that $y(0) = c_1$, and $y'(0) = -c_1 + c_2$. Therefore $c_1 = 5$ and $c_2 = 9$, so the solution is given by

$$y(t) = 5e^{-t} + 9te^{-t} - \frac{1}{2}\log(t^2+1)e^{-t} + \tan^{-1}(t)te^{-t}.$$

Problem 3: (10 points) Prove that the matrices

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{nn} & \cdots & a_{n2} & a_{n1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{2n} & \cdots & a_{22} & a_{21} \\ a_{1n} & \cdots & a_{12} & a_{11} \end{bmatrix}$$

are similar.

Solution:

$$\begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} = \begin{bmatrix} a_{nn} & \cdots & a_{n2} & a_{n1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{2n} & \cdots & a_{22} & a_{21} \\ a_{1n} & \cdots & a_{12} & a_{11} \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}.$$

Problem 4: (10 points) Let $T\colon V \to V$ be a linear transformation of a (possibly infinite dimensional) vector space V. Suppose there exists a positive integer k such that $\mathrm{Ker}(T^k) = \mathrm{Ker}(T^{k+1})$. Prove that $\mathrm{Ker}(T^\ell) = \mathrm{Ker}(T^{\ell+1})$ for any integer $\ell \geq k$.
(Reminder: $T^n = T \circ T \circ \cdots \circ T$ denotes the composition of T with itself n times; for instance, $T^2 = T \circ T$.)

Solution: We prove the statement by induction on ℓ. The statement is obviously true for $\ell = k$. Now suppose $\mathrm{Ker}(T^\ell) = \mathrm{Ker}(T^{\ell+1})$. We would like to show that $\mathrm{Ker}(T^{\ell+1}) = \mathrm{Ker}(T^{\ell+2})$. It is clear that $\mathrm{Ker}(T^{\ell+1}) \subseteq \mathrm{Ker}(T^{\ell+2})$. On the other hand, if $\vec{v} \in \mathrm{Ker}(T^{\ell+2})$, i.e. $T^{\ell+2}(\vec{v}) = \vec{0}$, then we have $T(\vec{v}) \in \mathrm{Ker}(T^{\ell+1}) = \mathrm{Ker}(T^\ell)$, hence $T^{\ell+1}(\vec{v}) = \vec{0}$.

Problem 5: (10 points) Recall that the cofactor matrix $C(A)$ of an $n \times n$ matrix A is the matrix with entries given by

$$C(A)_{ij} = (-1)^{i+j} \det(A_{ij}),$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and the j-th column of A.

Suppose A is a real orthogonal matrix. Prove that one of the following statements must be true:

(i) $C(A) = A$,

(ii) $C(A) = -A$.

(Hint: the inverse formula.)

Solution: Since A is orthogonal, it is invertible and $A^T = A^{-1}$. Hence $1 = \det(\mathbb{I}) = \det(A^T A) = \det(A^T)\det(A) = \det(A)^2$, therefore $\det(A) = \pm 1$. The inverse formula we proved in class states that $A^{-1} = \frac{1}{\det(A)} C(A)^T$. Thus $C(A) = \pm A$.

Problem 6: (10 points) Let A and B be two $n \times n$ matrices. Suppose that $AB = BA$. Prove that

$$\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B) - \operatorname{rank}(AB).$$

Solution: First, we have $\operatorname{Col}(A+B) \subseteq \operatorname{Col}(A) + \operatorname{Col}(B)$ since each column of $A+B$ is a linear combination of a column in A and a column in B. Second, we observe that $\operatorname{Col}(AB) \subseteq \operatorname{Col}(A) \cap \operatorname{Col}(B)$ since $\operatorname{Col}(AB) \subseteq \operatorname{Col}(A)$ and $\operatorname{Col}(AB) = \operatorname{Col}(BA) \subseteq \operatorname{Col}(B)$ by the assumption. Therefore

$$\begin{aligned}
\operatorname{rank}(A+B) &= \dim \operatorname{Col}(A+B) \\
&\leq \dim(\operatorname{Col}(A) + \operatorname{Col}(B)) \\
&= \dim \operatorname{Col}(A) + \dim \operatorname{Col}(B) - \dim(\operatorname{Col}(A) \cap \operatorname{Col}(B)) \\
&\leq \dim \operatorname{Col}(A) + \dim \operatorname{Col}(B) - \dim(\operatorname{Col}(AB)) \\
&= \operatorname{rank}(A) + \operatorname{rank}(B) - \operatorname{rank}(AB).
\end{aligned}$$

Problem 7: (10 points) Let A be an $n \times n$ matrix with $\operatorname{rank}(A) = 1$, where $n \geq 2$. Prove that the following statements are equivalent:

(i) A is diagonalizable,

(ii) the trace of A is non-zero.

(Reminder: the trace of a square matrix is the sum of its diagonal entries.)

Solution: By the rank-nullity theorem, we have $\dim \operatorname{Nul}(A) = n - 1 \geq 1$. Hence 0 is an eigenvalue of A with multiplicity $\operatorname{mult}(0) \geq \dim \operatorname{Nul}(A - 0\mathbb{I}) = n - 1$. We claim that both statements (i) and (ii) are equivalent to the following statement:

$$\text{(iii)} \ \operatorname{mult}(0) = n - 1\,.$$

"(i) $\Leftrightarrow$ (iii)": First, if A is diagonalizable, then the multiplicity of each eigenvalue coincides with the dimension of its eigenspace; in particular, $\operatorname{mult}(0) = \dim \operatorname{Nul}(A - 0\mathbb{I}) = n - 1$. Conversely, suppose $\operatorname{mult}(0) = n - 1$. Then A has another eigenvalue $\lambda \neq 0$ with $\operatorname{mult}(\lambda) = 1$, and $\{0, \lambda\}$ is the set of all eigenvalues of A. Observe that the multiplicities of both eigenvalues 0 and λ coincide with the dimensions of their eigenspaces respectively, therefore A is diagonalizable.

"(ii) $\Leftrightarrow$ (iii)": If $\operatorname{mult}(0) = n - 1$, then A has another eigenvalue $\lambda \neq 0$ with $\operatorname{mult}(\lambda) = 1$. Recall that the trace of A coincides with the sum of its eigenvalues (counted with multiplicities). Hence $\operatorname{tr}(A) = (n-1) \cdot 0 + 1 \cdot \lambda = \lambda \neq 0$. On the other hand, if "$\operatorname{mult}(0) = n - 1$" is not true, then we must have $\operatorname{mult}(0) = n$ since we know that $\operatorname{mult}(0) \geq n - 1$. Therefore 0 is the only eigenvalue of A and $\operatorname{tr}(A) = n \cdot 0 = 0$.

Problem 8: (10 points) Let A be a real symmetric matrix. Prove that there exist matrices B_1 and B_2 that are both real symmetric positive definite, such that $A = B_1 - B_2$.

Solution: Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of A. Choose $\mu > 0$ large enough so that $\mu + \lambda_i > 0$ for any $1 \leq i \leq k$. Then $B_1 = A + \mu\mathbb{I}$ and $B_2 = \mu\mathbb{I}$ satisfy the desired properties, since they are both symmetric and their eigenvalues are all positive.

Problem 9: (10 points) Let A be a real $n \times m$ matrix, and B be a real $p \times n$ matrix. Note that $\mathrm{Col}(A)$ and $\mathrm{Nul}(B)$ are both subspaces of $\mathbb{R}^n$.

Suppose that $\mathrm{Col}(A) = \mathrm{Nul}(B)$. Prove that the $n \times n$ matrix $AA^T + B^T B$ is invertible. (Hint: consider inner products.)

Solution: Suppose $(AA^T + B^T B)\vec{v} = \vec{0}$. Then we have $0 = \vec{v}^T(AA^T + B^T B)\vec{v} = ||A^T\vec{v}||^2 + ||B\vec{v}||^2$. Hence $\vec{v} \in \mathrm{Nul}(A^T) \cap \mathrm{Nul}(B)$. Recall that $\mathrm{Nul}(A^T) = \mathrm{Col}(A)^\perp$. Therefore $\vec{v} \in \mathrm{Col}(A)^\perp \cap \mathrm{Col}(A) = \{\vec{0}\}$, so $\vec{v} = \vec{0}$.

Problem 10: (10 points)

(i) Let A be a real symmetric positive semi-definite $n \times n$ matrix. Prove that the subset
$$S_A := \{\vec{v} \in \mathbb{R}^n \colon \vec{v}^T A \vec{v} = 0\} \subseteq \mathbb{R}^n$$
is a subspace of $\mathbb{R}^n$.

(ii) Find a real symmetric matrix B such that S_B (defined as in Part (i)) is not a subspace of $\mathbb{R}^n$ (i.e. find a counterexample of the previous statement if the positive semi-definite assumption is removed.)

Solution: (i) Write $A = PDP^T$ where P is orthogonal and D is diagonal with non-negative entries. One can arrange the eigenvalues so that the first k diagonal entries of D are positive (say they are $\lambda_1, \dots, \lambda_k > 0$) and the remaining diagonal entries of D are zero ($0 \le k \le n$).

Let $\vec{v} \in S_A$ and write $P^T\vec{v} = \vec{w} = [w_1\ w_2\ \cdots\ w_n]^T$. Then we have
$$0 = \vec{w}^T D \vec{w} = \lambda_1 w_1^2 + \cdots + \lambda_k w_k^2,$$
hence $w_1 = \cdots = w_k = 0$. Conversely, if the first k components of $P^T\vec{v}$ are all zero for some vector $\vec{v}$, then the same computation shows that $\vec{v} \in S_A$. This proves that "$\vec{v} \in S_A$" is equivalent to "$P^T\vec{v} \in \operatorname{Span}\{e_{k+1}, \dots, e_n\}$", which is equivalent to "$\vec{v} \in \operatorname{Span}\{Pe_{k+1}, \dots, Pe_n\}$". Hence we have $S_A = \operatorname{Span}\{Pe_{k+1}, \dots, Pe_n\}$, which is a subspace of $\mathbb{R}^n$.

(ii) Consider $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $S_B = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 : v_1^2 - v_2^2 = 0 \right\}$ is not a subspace of $\mathbb{R}^2$: for instance, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are in S_B, but their sum $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is not.

Remark: For Part (i), one can also prove the statement by showing directly that S_A satisfies the definition of subspace. If one tries to prove it in this way, the most crucial (and the only non-trivial) part is to show that if $\vec{v}_1$ and $\vec{v}_2$ are in S_A then so is $\vec{v}_1 + \vec{v}_2$. Note that it's not possible to prove it without the assumption that A is positive semi-definite. (The statement also holds if A is negative semi-definite, but it is not true if A is indefinite, see Part (ii).) Here is one possible argument:
$$\begin{aligned} 0 &\le (\vec{v}_1 + \vec{v}_2)^T A (\vec{v}_1 + \vec{v}_2) + (\vec{v}_1 - \vec{v}_2)^T A (\vec{v}_1 - \vec{v}_2) \\ &= (\vec{v}_1^T A \vec{v}_1 + \vec{v}_2^T A \vec{v}_1 + \vec{v}_1^T A \vec{v}_2 + \vec{v}_2^T A \vec{v}_2) + (\vec{v}_1^T A \vec{v}_1 - \vec{v}_2^T A \vec{v}_1 - \vec{v}_1^T A \vec{v}_2 + \vec{v}_2^T A \vec{v}_2) \\ &= 2(\vec{v}_1^T A \vec{v}_1 + \vec{v}_2^T A \vec{v}_2) = 0. \end{aligned}$$
The first inequality follows from the assumption that A is positive semi-definite, and the last equality follows from $\vec{v}_1, \vec{v}_2 \in S_A$. Therefore, again by the positive semi-definiteness of A, we have $\vec{v}_1 + \vec{v}_2$ and $\vec{v}_1 - \vec{v}_2$ are both in S_A.