Report on "COUNTING SPECIAL LAGRANGIAN CLASSES AND SEMISTABLE MUKAI VECTORS FOR K3 SURFACES" for Geometriae Dedicata

This paper delivers on the promise of its title. It performs various lattice-point counts for lattices of signature $(2, 19)$, and more generally $(2, n)$. These lattices naturally arise on the two sides of mirror symmetry, as either the algebraic Mukai lattice of an algebraic K3 surface, or as the Lagrangian lattice of a symplectic K3 surface. The agreement of their asymptotics is, in a sense, a verification of a prediction of mirror symmetry. Ultimately, the paper boils down to relatively standard lattice-point counting techniques.

The background of the paper is mathematically quite fancy—concerning Fukaya categories, derived categories, stability conditions, etc. Beyond these motivations, there is relatively little geometry in the paper. But overall, the paper is well written, the results seem correct, and I think the paper is worthy of publication. It is quite interesting that certain bounds necessary to perform the lattice-point count arise from the axioms of Bridgeland stability conditions (in particular, the support property).

A direction worth further mathematical exploration would be the dynamics of the $\widetilde{\mathrm{GL}}_2(\mathbb{R})^+$ action on the moduli space of stability conditions, and the ways in which it might parallel the flat-surfaces story (the magic wand theorem, orbit closures, Siegel-Veech constants, volumes, etc.).

Here are some concrete editorial comments:

(p. 1) turn $\to$ turns (possibly also rephrase this sentence)

(p. 2) for all flat surface $\to$ for all flat surfaces

(p. 2) a Cesàro $\to$ Cesàro (or asymptotics $\to$ asymptotic)

(p. 2) Ricci-flat metric $\to$ the (or a) Ricci-flat metric

(p. 2) "Let g be a Ricci-flat metric on X" is redundant given the previous sentence.

(p. 2) There is some issue with exposition about the twistor family. If the Ricci-flat metric g is fixed, as at the beginning of the paragraph, then the positive-definite 3-plane is uniquely determined to be the cohomology classes of the 3-dimensional space of parallel 2-forms.

Similarly, fixing a complex structure on X without fixing g, the 3-space should again not be arbitrary: It must contain $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$. Since you *are* fixing X it seems in (1.2) and a given twistor family, you should remove the phrase "any positive definite 3-plane" and replace with the positive definite 3-plane discussed above.

This issue is also present in the statement of Theorem 1.3 on p. 5.

(p. 3) Perhaps remove the parentheses around "(For example..."

(p. 3) Maybe $\mathcal{P}_\sigma(\phi)$ instead of $\mathcal{P}(\phi)$? (Though it is clear from context)

(p. 6) bases $\to$ basis

(p. 6) isomorphism $\to$ isometry

(p. 6) $\mathbb{C}\Omega_J \to \mathbb{C}[\Omega_J]$, unless this is too pedantic and the authors want to conflate Ω_J with its cohomology class.

(p. 7) Perhaps include a citation for the Torelli theorem

(p. 7) The phrasing of the period mapping $\varphi_{\mathbb{C}}\big|_{H^{2,0}(X)}$ is slightly confusing. More accurate would be $X \mapsto \varphi_{\mathbb{C}}(H^{2,0}(X)) \in \Omega(\Lambda_{\mathbb{C}})$ or some such.

(p. 7) "surjective but not injective..." perhaps add "...on the moduli space of marked K3 surfaces" and possibly "and generically 2-to-1" if desired.

(p. 7) "isomorphism classes": really "numerical classes" would be more accurate but they are the same in this case of course.

(p. 7) "no line bundle is numerically trivial". Of course, instead it should be "only the trivial bundle is numerically trivial."

(p. 7) "whose associated metric is Ricci-flat" Perhaps worth adding a short definition such as $\omega_J \wedge \omega_J = \Omega_J \wedge \overline{\Omega}_J$ or some such.

(p. 8) The exposition of the twistor family is (only slightly) inaccurate. Fixing $\omega_t \in \mathbb{S}^2$, there is still a S^1-worth of compatible choices for Ω_t (and not a unique Ω_t as suggested by the text). The space of orthonormal frames is $SO(3)$, but the point is that there is still a family of complex manifolds on $SO(3)/S^1 = \mathbb{S}^2$ because $e^{i\theta}\Omega_t$ are all holomorphic 2-forms for the same complex structure.

(p. 9) descends $\rightarrow$ descend, or switch the ordering in the sentence below (2.21).

(p. 10) Perhaps it is worth mentioning that the fundamental group of $\mathrm{GL}^+(2;\mathbb{R})$ is $\mathbb{Z}$, with the generator of the fundamental group acting by $e^{2\pi i}$.

(p. 10) After Definition 2.1, perhaps include a reference that $U(\mathcal{D})$ is connected?

(p. 10) In the discussion after (2.25) "the vectors whose real and imaginary part spans positive 2-planes in $N(\mathcal{D}) \otimes \mathbb{R}$" is closely related to the period domain associated to the numerical Grothendieck group. Perhaps this is worth mentioning.

(p. 10) Presumably (2.29) should say $\langle \varphi, \delta \rangle \notin \mathbb{R}_{\leq 0}$

(p. 11) I am a little suspicious about the commentary after Lemma 2.2. Presumably, special Lagrangians representing $\gamma \in \mathrm{Lag}(X, \omega)$ are constructed via hyperKähler rotation of an algebraic curve representing a class in Neron-Severi. Thus, for instance, when $\gamma = \gamma_1 + \gamma_2$ with both $\gamma_1, \gamma_2 \in \mathrm{Lag}(X, \omega)$, $\gamma_i^2 = 2$ and $\gamma_1 \cdot \gamma_2 = 1$, one might expect that a special Lagrangian representing γ would be the union of two (-2)-curves meeting at a node (in particular, not irreducible). Furthermore, why does Lemma 2.2 *characterize* special Lagrangians? The statement is only a containment, not an equality. Though this also surprised me, wouldn't the hyperKähler rotation trick give an equality?

- (p. 12) (grammar) Not all objects in $K(\mathcal{F})$ are geometric $\to$ An object in $K(\mathcal{F})$ need not be geometric
- (p. 12) "It is not known whether $N(\mathcal{F}(X))$ can be identified with $\mathrm{Lag}(X, \omega)$." A short elaboration on this issue might be interesting. Presumably there is a group homomorphism from the former to the latter? Depending on the status of containment vs equality in Lemma 2.2, one might even have surjectivity?
- (p. 12) There should be a period after $Z_\sigma \colon N(\mathcal{D}) \to \mathbb{C}$
- (p. 13) a word of additional explanation about the attainment of the systole could be useful. Perhaps even as a short lemma (if the authors view it as sufficiently nontrivial).
- (p. 13) of in $\to$ in
- (p. 13) The argument in the paragraph at the bottom of p. 13 might be best suited in a proposition environment. Or perhaps the whole discussion of the (-2)-class contribution could be put into a proposition?
- (p. 17) It really seems from Equation (4.2) that Lemma 2.2 but have been an *equality* rather than a containment.
- (p. 18) typo: "allow all"
- (p. 19) typo: "dialate"
- (p. 14-19) After the initial version of the lattice-point counting argument is given, it seems that the same argument is repeated over and over (perhaps with minor variation) in these pages. Perhaps there is a way to reduce this repetitiveness, but consistently citing a sufficiently general version introduced as its own theorem.

(p. 19) (This comment isn't about suggesting a change) I have never heard the term "Sylvester's law of inertia" for the well-definedness of signature!