HOMEWORK 2
MATH 104, SECTION 6

Office Hours: I'll be out of town for a conference next week. So I'll be holding office hours on Monday (1/27) 8-11am, and my office hours on Tuesday (1/28) and Wednesday (1/29) are cancelled.

Reading

There will be reading assigned for each lecture. You should come to the class having read the assigned sections of the textbook.

Due January 30: Ross, Section 8

Due February 4: Ross, Section 9

Problem set (10 problems)

(1) Prove that $\sqrt{2}$ is not a rational number.

(2) Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$. (Read Ross, Section 1 if you're not familiar with mathematical induction.)

(3) For any ordered field F and any $a \in F$, one can define the notion of absolute value $|a| \in F$ of a (c.f. Ross, Definition 3.3). Prove that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

holds for any $a_1, a_2, \ldots, a_n \in F$.

(4) Formulate the definition of the greatest lower bound $\inf A$ of a set of real numbers. State a "greatest lower bound property" for $\mathbb{R}$ and show that it is equivalent to the least upper bound property of $\mathbb{R}$. (c.f. Ross, Section 4)

(5) Let $S \subset \mathbb{R}$ be a nonempty subset which is bounded above, and let $z = \sup S$. Prove that for any $\epsilon > 0$, there exists $a \in S$ such that $z - \epsilon < a \leq z$. Can $a \in S$ always be found so that $z - \epsilon < a < z$?

(6) Let $x, y \in \mathbb{R}$. Suppose that $x < y + \epsilon$ for any $\epsilon > 0$. Prove that $x \leq y$.

(7) Prove that $\sup\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 1$.

(8) Let

$$A = \{m + n\sqrt{2} | m, n \in \mathbb{Z} \text{ and } m + n\sqrt{2} > 0\}.$$

Prove that $\inf A = 0$.

(9) Define $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Prove that the sequence (a_n) diverges.

(10) (a) Suppose that (a_n) is a convergent sequence. Show that the subsequences $(a_{2n}) = (a_2, a_4, a_6, \cdots)$ and $(a_{2n-1}) = (a_1, a_3, a_5, \cdots)$ both converge.

(b) Let (a_n) be a sequence such that both (a_{2n}) and (a_{2n-1}) converge. Is it guaranteed that (a_n) converges?

(c) Let (a_n) be a sequence such that (a_{2n}), (a_{2n-1}), and (a_{3n}) converge. Prove that (a_n) is a convergent sequence.