Today: Mean value thm; chain rule.

Recall: $f: \overset{\text{open interval}}{I} \to \mathbb{R}$, $a \in I$.

We say f is differentiable at a if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists} \quad (=: f'(a))$$

Rmk: An equivalent definition. f is differentiable at a if $\forall$ seq. (x_n) that converges to a, and $x_n \neq a$ $\forall n$,

$$\lim_{n \to \infty} \frac{f(x_n) - f(a)}{x_n - a} = L \text{ for some } L \in \mathbb{R}$$

$= f'(a)$

Ex: Show that these two definitions are equivalent.

Thm f, g both differentiable at a,

Then cf, $f \pm g$, $f \cdot g$ are all differentiable at a, and if $g(a) \neq 0$, then f/g is also differentiable at a.

Pf: Let's first prove $f \cdot g$ is differentiable at a.

i.e. $$\boxed{\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} \text{ exists.}}$$

???

(what we know: $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ & $\lim_{x \to a} \frac{g(x) - g(a)}{x - a}$ exist.)

Idea: Somehow relate $f(x)g(x) - f(a)g(a)$

w/ $f(x) - f(a)$ & $g(x) - g(a)$

$$f(x)g(x) - f(a)g(a) = f(x)g(x) - \underline{f(x)g(a)} + \underline{f(x)g(a)} - f(a)g(a)$$
$$= f(x)\big(g(x)-g(a)\big) + g(a)\big(f(x)-f(a)\big)$$

$$\lim_{x\to a} \frac{f(x)g(x)-f(a)g(a)}{x-a} = \lim_{x\to a} \frac{f(x)\big(g(x)-g(a)\big)+g(a)\big(f(x)-f(a)\big)}{x-a}$$

($\lim_{x\to a} f(x) = f(a)$ since f is differentiable at a, therefore conti. at a.)

$$= f(a)g'(a) + g(a)f'(a).$$

by limit thm.

$\Rightarrow$ $\underline{f(x)g(x)}$ is differentiable at a,

and $(fg)'(a) = f(a)g'(a) + g(a)f'(a)$

(Leibniz rule).

$g(a) \neq 0$

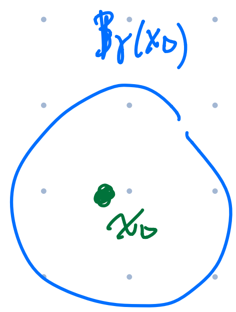
$$\lim_{x\to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} = \lim_{x\to a} \frac{f(x)g(a) - f(a)g(x)}{(x-a)\,g(x)\,g(a)}$$

$$\underline{f(x)}g(a) - f(a)g(x) = \underline{f(x)}g(a) - \underline{f(a)}g(a) + f(a)\underline{g(a)} - f(a)\underline{g(x)}$$
$$= \boxed{g(a)\big(f(x)-f(a)\big) - f(a)\big(g(x)-g(a)\big)}$$

$$\lim_{x\to a} \frac{g(a)\big(f(x)-f(a)\big) - f(a)\big(g(x)-g(a)\big)}{(x-a)\,g(x)\,g(a)}$$

$$= \frac{g(a)\,f'(a)}{g(a)^2} - \frac{f(a)\,g'(a)}{g(a)^2}$$

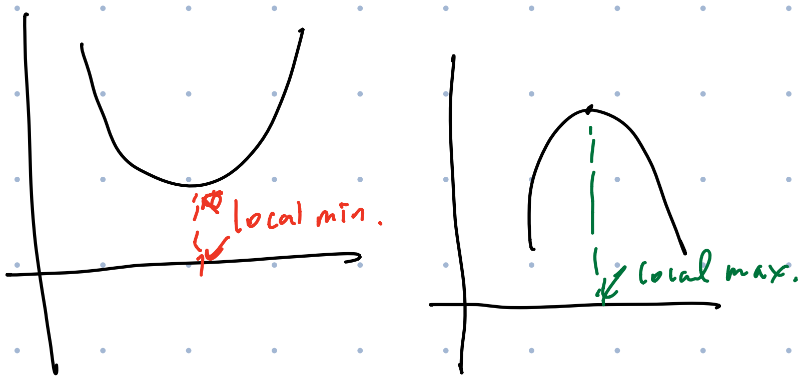
$$= \frac{f'(a)\,g(a) - g'(a)\,f(a)}{g(a)^2} \underset{\text{exists}}{\equiv} \left(\frac{f}{g}\right)'(a)$$



Def X: metric space. $f: X \longrightarrow \mathbb{R}$

Say $x_0 \in X$ is a <u>local min.</u> (max.) of f

if $\exists\, r > 0$ s.t. $f(x) \geq f(x_0)$ $(\leq)$ $\forall\, x \in B_r(x_0)$.



Thm $f: I \longrightarrow \mathbb{R}$ (I: open interval). Suppose f is differentiable on I.
(i.e. differentiable at any point in I).

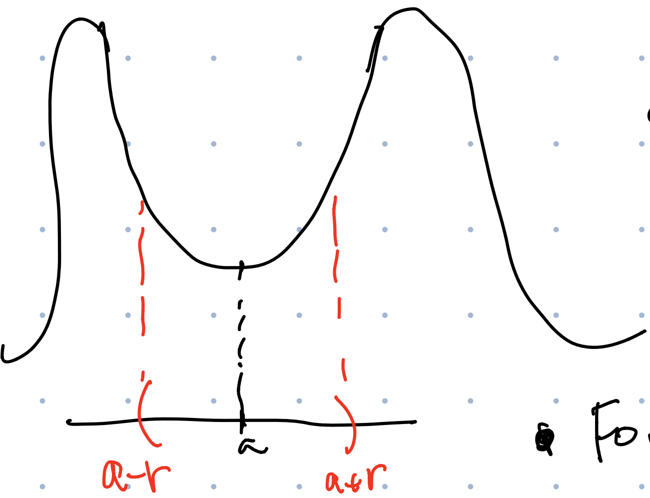
Suppose a is a local min. (max.) of f.

Then $f'(a) = 0$.

pf: Say a is a local min. of f.

$\exists\, r>0$ s.t. $f(x) \geq f(a)$ $\forall\, x \in B_r(a)$

i.e. $x \in (a-r, a+r)$



Since f is differentiable at a,

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \text{ exists.}$$

- For any $a-r<x<a$,

  we have $$\frac{\overbrace{f(x)-f(a)}^{\geq 0}}{\underbrace{x-a}_{<0}} \leq 0.$$

- For any $a<x<a+r$,

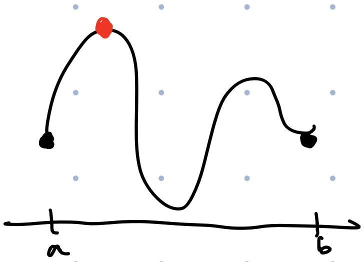
  we have $$\frac{\overbrace{f(x)-f(a)}^{\geq 0}}{\underbrace{x-a}_{>0}} \geq 0$$

$$\Rightarrow \quad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = 0 \qquad \square$$

Thm (Rolle) $f:[a,b] \longrightarrow \mathbb{R}$ conti. & differentiable on (a,b)

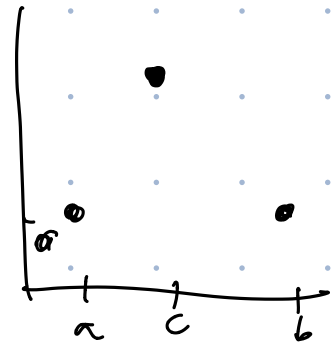
& $f(a)=f(b)$.

$\Rightarrow$ $\exists\, c\in(a,b)$ s.t. $f'(c)=0$



pf: ① $\exists\, c\in(a,b)$, s.t. $f(c)>f(a)=f(b)$.

By extreme value thm. (conti. map on cpt. set.)



max. of f on $[a,b]$ must be attended,

$\exists\, d\in(a,b)$ st. $f(d)=\max\limits_{x\in[a,b]} f(x)$.

$\Rightarrow$ d is a local max. of f.

$\Rightarrow f'(d)=0$

② $\exists\, c\in(a,b)$ st. $f(c)<f(a)=f(b)$

(Same proof, apply for the min. of f on $[a,b]$).

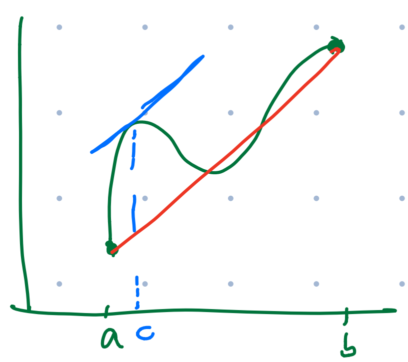
③ $f(a)=f(b)=f(x)\quad \forall x\in[a,b]$ (const fn).

$\Rightarrow \boxed{f'(x)}=0\quad \forall x\in(a,b)$. □

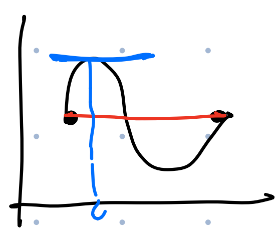
$$f'(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x}, \quad f(y)-f(x)=0$$

Thm (Mean value thm.) $f:[a,b]\to\mathbb{R}$ conti., differentiable on (a,b)

$\Rightarrow \exists\, c\in(a,b)$ s.t. $f'(c)=\dfrac{f(b)-f(a)}{b-a}$



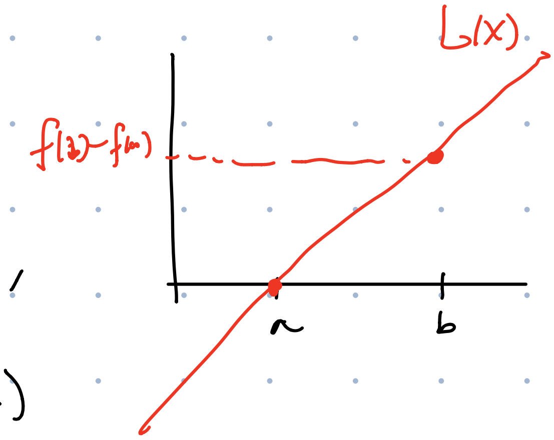
(Compare Rolle:



)

Idea: "tilt" the graph of f so that its values at the end points a, b coincide, and then apply Rolle's thm.

Pf. Define a linear fcn:



$$L(x) := (x-a)\frac{f(b)-f(a)}{b-a},$$

$$L(a) = 0, \quad L(b) = f(b) - f(a)$$

Consider: $\boxed{g(x) := f(x) - L(x)}$ $\rightarrow$ differentiable on (a,b)

Then $g(a) = f(a) - \underbrace{L(a)}_{=0} = f(a)$

$g(b) = f(b) - L(b) = f(a)$.

By Rolle, $\exists\, c \in (a,b)$ s.t. $g'(c) = 0$

$$g'(x) = f'(x) - L'(x) \quad \forall x \in (a,b).$$

$$0 = g'(c) = f'(c) - L'(c) = f'(c) - \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of the linear fcn } L}$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}. \quad \square$$

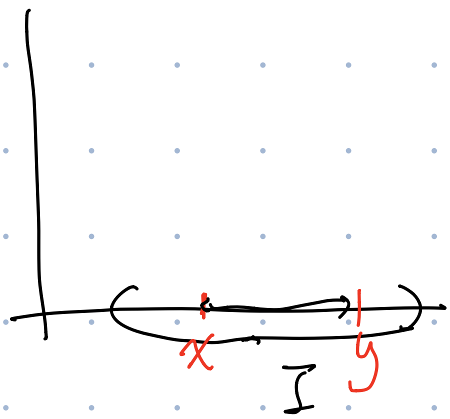
Some applications of MVT:

$$\boxed{\lim_{y\to x}\frac{f(y)-f(x)}{y-x}}$$
$$\|$$

Prop: $f: I \to \mathbb{R}$ differentiable & $f'(x)=0 \quad \forall x\in I$.

$\Rightarrow f$ is a const. fcn.

pf:



Want: $f(x)=f(y)$

By MVT, $\exists\, z\in(x,y)$

s.t. $f'(z) = \dfrac{f(y)-f(x)}{y-x}$

$\|$

0 by assumption.

$\Longrightarrow f(x)=f(y)$. $\square$

Recall: $f:(a,b)\to\mathbb{R}$ Lipschitz conti. if $\exists K>0$

s.t. $|f(x)-f(y)| < K|x-y| \quad \forall x,y\in(a,b)$

($\Longrightarrow$ unif. conti.)

Prop: $f:(a,b)\to\mathbb{R}$ differentiable. Then

f is Lip. conti. $\Longleftrightarrow$ $f':(a,b)\to\mathbb{R}$ is bounded.

pf ($\Longrightarrow$) $\left|\dfrac{f(x)-f(y)}{x-y}\right| < K \quad \forall x,y\in(a,b)$

$$f'(z) = \lim_{w \to z} \underbrace{\frac{f(w)-f(z)}{w-z}}$$

$$|f'(z)| \le K \quad \forall z \in (a,b).$$

($\Leftarrow$) Suppose f' is bounded, say $|f'(x)| < \underline{\underline{M}}$ $\forall x \in (a,b)$.

__Claim:__ $|f(x)-f(y)| < M \cdot |x-y| \quad \forall x, y \in (a,b)$.

PF: By MVT, $\exists\, z$ between x & y,

st. $f'(z) = \dfrac{f(x)-f(y)}{x-y}$

$$M > |f'(z)| = \left| \frac{f(x)-f(y)}{x-y} \right| . \quad \square$$

Thm (Chain Rule). $f: I \to \mathbb{R}$, $g: \underset{f(I)}{\overset{\cap}{J}} \to \mathbb{R}$

Suppose f is differentiable at $a \in I$,

and g - - - - - - - - - - : $f(a) \in J$.

$\boxed{(g \circ f)(a) = g(f(a))}$

$\Rightarrow$ $g \circ f : I \to \mathbb{R}$. is differentiable at a,

and $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

__Idea:__ need

$$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x-a} \text{ exists.}$$

$$\frac{g(f(x)) - g(f(a))}{x-a} = \underbrace{\frac{g(f(x)) - g(f(a))}{f(x)-f(a)}}_{\downarrow\, g'(f(a))} \cdot \underbrace{\frac{f(x)-f(a)}{x-a}}_{\downarrow\, f'(a)}$$

let $x \to a$, we know $f(x) \to f(a)$ (since f is contin. at a)

Why the argument above doesn't work ??

Issue: $f(x)$ could be equal to $f(a)$.

One situation that the above argument does go through is: if $\exists$ neighborhood $U \ni a$ s.t. $f(x) \neq f(a)$ $\forall x \in U \setminus \{a\}$.

The other case: $\forall \varepsilon > 0$, $\exists\, 0 < |x-a| < \varepsilon$ s.t. $f(x) = f(a)$.

We'll prove Chain Rule in this case next time!