Today: Root test, Ratio test, Integral test, Alternating series test.

Ratio test: $a_k \neq 0 \ \forall k$.

1) $\sum a_k$ abs. conv. If $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$.

2) $\sum a_k$ div. if $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$.

Root test: $\alpha := \limsup_{k\to\infty} |a_k|^{1/k}$

1) $\sum a_k$ abs. conv. If $\alpha < 1$

2) $\sum a_k$ div. If $\alpha > 1$.

Assuming the root test is true, we prove the ratio test:

recall from HW:

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{1/n} \leq \limsup |a_n|^{1/n} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

pf of root test:

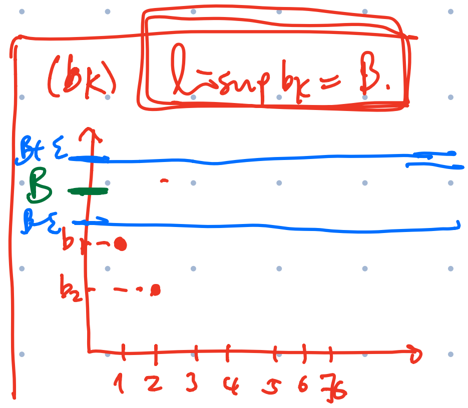
1) Suppose $\overset{\alpha}{\overset{\|}{\limsup}} |a_k|^{1/k} < 1$, $\boxed{\text{want: } \sum |a_k| \text{ conv.}}$

- Choose $\varepsilon > 0$ small enough s.t. $\alpha + \varepsilon < 1$
- $\alpha = \limsup |a_k|^{1/k} = \lim_{N\to\infty} \left(\sup\{|a_k|^{1/k} : k > N\}\right)$
- $\exists N > 0$ s.t. $\sup\{|a_k|^{1/k} : k > N\} < \alpha + \varepsilon$.

$\Longrightarrow$ $|a_k| < (\alpha+\varepsilon)^k \quad \forall k > N$.

- Since $0 < \alpha + \varepsilon < 1$, $\sum (\alpha+\varepsilon)^k$ is conv.
  $\uparrow$
  geometric series

$\Longrightarrow$ $\sum |a_k|$ conv. $\square$

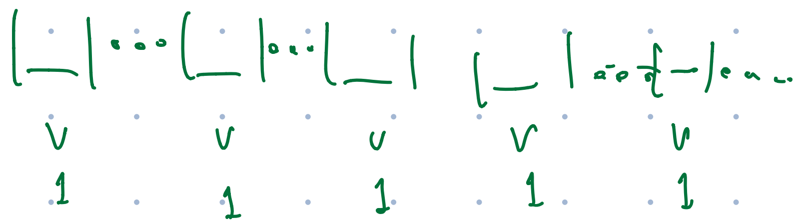


2) $\alpha = \limsup |a_k|^{1/k} > 1$, $\boxed{\text{Want: } \sum a_k \text{ divergent.}}$

- $\exists$ subseq. (a_{k_n}) of (a_k) s.t. $\lim_{n\to\infty} |a_{k_n}|^{1/k_n} = \alpha > 1$.
- Choose $\varepsilon > 0$ small enough s.t. $\alpha - \varepsilon > 1$.
- $\exists\, N > 0$ s.t. $|a_{k_n}|^{1/k_n} > \alpha - \varepsilon > 1$. $\forall n > N$.

$$\Rightarrow \quad |a_{k_n}| > (\alpha-\varepsilon)^{k_n} > 1 \quad \forall n > N.$$

$\overset{??}{\Longrightarrow}$ $\sum a_k$ is divergent. $\square$



We proved last time: "$\sum a_n$ conv. $\Longrightarrow \lim a_n = 0$"

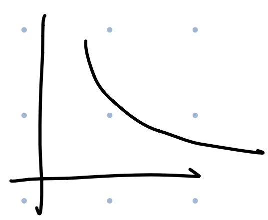
"If (a_n) doesn't conv. to 0, then $\sum a_n$ div."

<u>Integral test</u>: $f : [1,\infty) \longrightarrow \mathbb{R}_{>0}$ be a decreasing positive function.
(integrable)

(If $x < y$, then $f(x) \geq f(y)$)

Let $a_k = f(k)$ $\forall k \in \mathbb{N}$.

and consider $\sum a_k$



1) $\sum a_k$ conv. if the improper integral $\int_1^\infty f(x)\,dx < +\infty$

2) $\sum a_k$ div. if the improper integral $\boxed{\int_1^\infty f(x)\,dx} = +\infty$

$\lim_{N\to+\infty} \int_1^N f(x)\,dx$

e.g. $\sum \frac{1}{n}$.

Consider $f(x)=\frac{1}{x}$.

$$\int_1^N \frac{1}{x}\,dx = \log N - \cancel{\log 1} \longrightarrow +\infty \quad \text{as } N\to+\infty$$

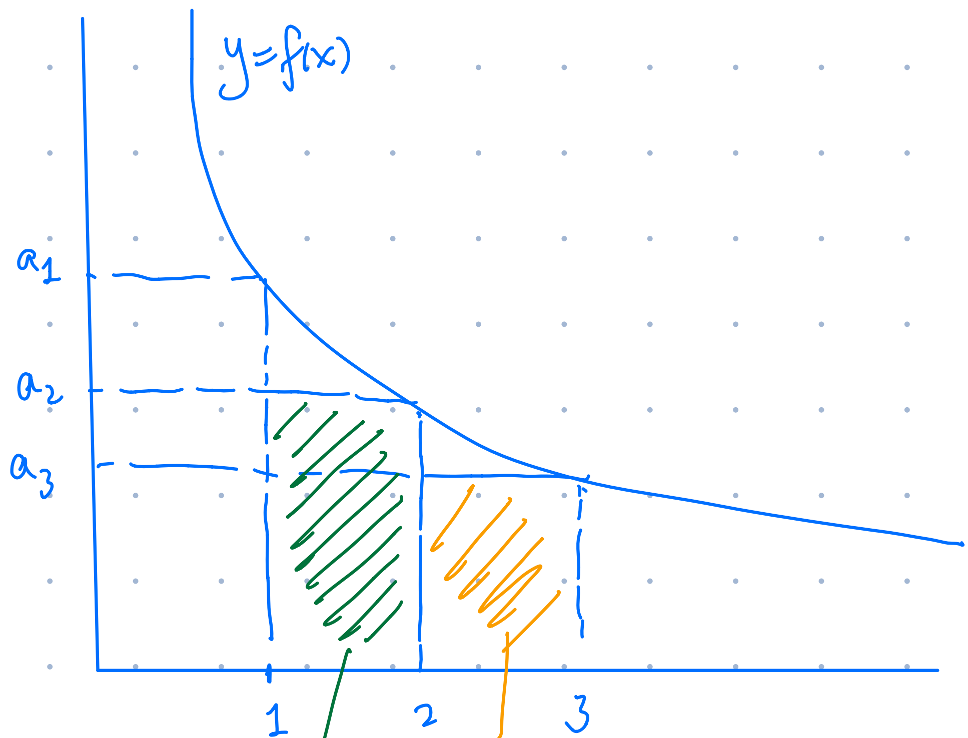
$\Longrightarrow \sum \frac{1}{n}$ is div.

e.g. $\sum \frac{1}{n^2} \overset{\text{conv.}}{=\!=}$ $f(x)=\frac{1}{x^2}$

$$\int_1^N \frac{1}{x^2}\,dx = \frac{-1}{N} - \frac{-1}{1} = 1-\frac{1}{N} \to 1$$

as $N\to+\infty$

pf:



Area $= a_2$ $\quad$ Area $= a_3$

f is decreasing $\longrightarrow$ $\le$ $\quad$ $\le$

$\int_1^2 f(x)\,dx$ $\quad$ $\int_2^3 f(x)\,dx$

$\|$ $\quad$ $\|$

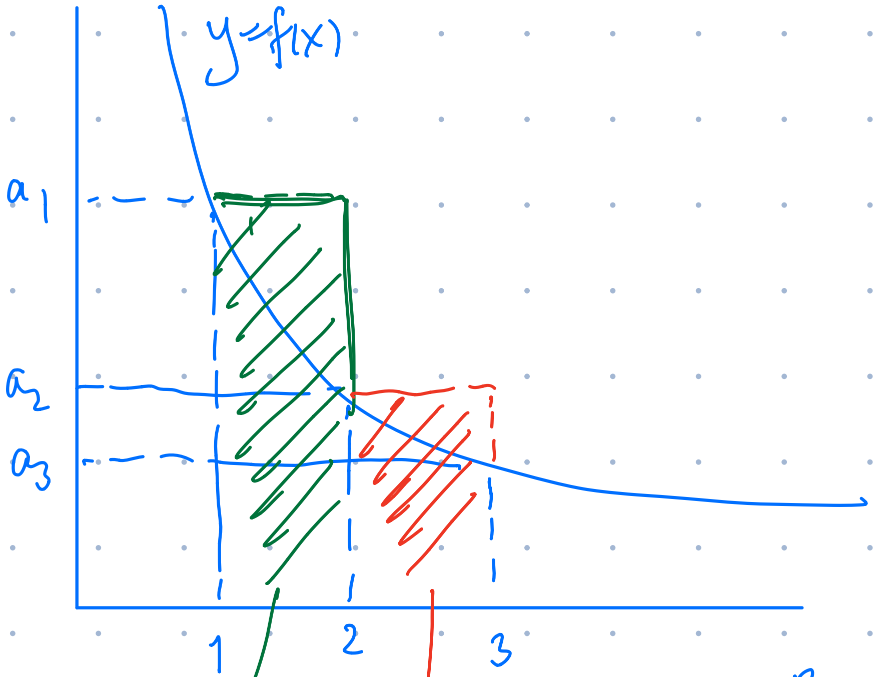
b_2 $\quad$ b_3

- $a_n \le b_n \quad \forall n \ge 2$.
- $\sum b_n$ conv.

  $\left(\int_1^\infty f(x)\,dx = \lim_{N\to\infty} \boxed{\int_1^N f(x)\,dx}\right)$ by assumption conv.

  $\boxed{\int_1^N f(x)\,dx} = (b_2+b_3+\cdots+b_N)$
- $a_n, b_n \ge 0$ b/c f is positive

$\Longrightarrow$ By comparison test, $\sum a_n$ conv. $\square$



area $= a_1$ $\geq$ $\int_1^2 f(x)\,dx$ area $= a_2$ $\geq$ $\int_2^3 f(x)\,dx$

By comparison test, if $\int_1^\infty f(x)\,dx = +\infty$, then $\sum a_n$ div. $\square$

<u>Alternating series test</u>: If $a_1 \geq a_2 \geq a_3 \geq \cdots \cdots \geq 0$ and $\lim a_n = 0$

then $\sum (-1)^n a_n$ is convergent.

e.g. $\underbrace{-1 + \frac{1}{2}}_{S_1 \quad S_2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$ conv. (Later: $= -\log 2$)

<u>Idea</u>: $-a_1 + a_2 - a_3 + a_4 - \cdots -$

$$\boxed{S_k = \sum_{n=1}^{k} (-1)^n a_n}$$

pf: Ex: $S_1 \le S_3 \le S_5 \le \cdots\cdots \le S_6 \le S_4 \le S_2$ — $(*)$

(follows from $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$)

$\boxed{\text{Want: } \sum (-1)^n a_n \text{ conv.}} \iff \boxed{(S_n) \text{ conv.}}$

By $(*)$, $\lim_{n\to\infty} (S_{2n+1}) = A$, $\lim_{n\to\infty} (S_{2n}) = B$ exist.

It suffices to show $A = B$:

By limit thm.,

$$A - B = \lim_{n\to\infty} \Big(\underbrace{\boxed{S_{2n+1} - S_{2n}}}_{} \Big) \quad \left(= (-1)^{2n+1} a_{2n+1}\right)$$

$$= 0 \quad \text{since } \lim a_n = 0$$

$\square$

Lemma 2: If K is seq. cpt, then

$\forall \varepsilon > 0$, $\exists$ finite set $F \subseteq K$

s.t. $$K \subseteq \bigcup_{x \in F} B_\varepsilon(x)$$

pf Suppose the statement is not true, i.e.

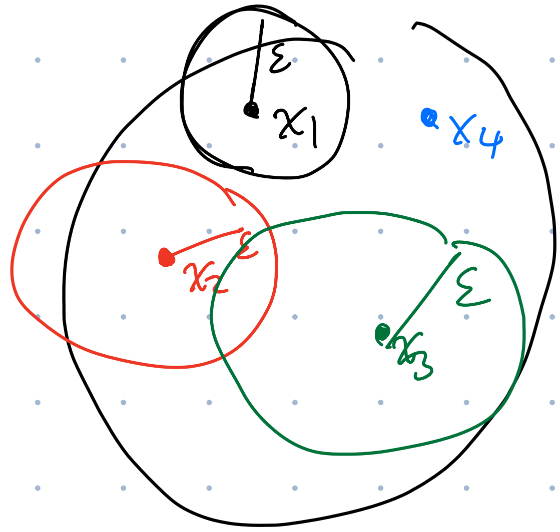
$\boxed{\exists \varepsilon > 0 \text{ s.t. } \forall \text{ finite } F \subseteq K, \quad K \nsubseteq \bigcup_{x\in F} B_\varepsilon(x)}$

(Want: a contradiction, i.e. find a seq. in K which has no conv. subseq.)

- Pick any $x_1 \in K$,

  $K \nsubseteq B_\varepsilon(x_1)$,

$\exists\ x_2 \in K, \quad x_2 \notin B_\varepsilon(x_1)$



- Consider $\{x_1, x_2\} \subseteq K$,

$$K \nsubseteq \bigcup_{i=1}^{2} B_\varepsilon(x_i)$$

$$\exists\ x_3 \in K, \quad x_3 \notin \bigcup_{i=1}^{2} B_\varepsilon(x_i)$$

- Consider $\{x_1, x_2, x_3\} \subseteq K$

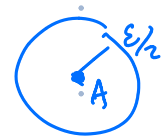
$$K \nsubseteq \bigcup_{i=1}^{3} B_\varepsilon(x_i)$$

$$\exists\ x_4 \in K \text{ s.t. } \quad x_4 \notin \bigcup_{i=1}^{3} B_\varepsilon(x_i)$$

Proceed this construction inductively, we can get:

- (x_n) in K
- $d(x_i, x_j) \geq \varepsilon \quad \forall i \neq j$

$(x_{k_n}) \to A$



$\Downarrow$

(x_n) has no conv. subseq.

$\Rightarrow K$ is not seq. cpt. $\square$