

HOMEWORK 10
MATH 104, SECTION 6

Office Hours (via Zoom): Tuesday and Wednesday 9:30-11am.

PROBLEM SET (10 PROBLEMS; DUE APRIL 9)

Submit your homework before the lecture on Thursday. *Late homework will not be accepted under any circumstances.* You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any.

Write clearly! Mastering mathematical writing is one of the goals of this course.

- (1) Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that the derivative $f'(x)$ exists for any $x \in \mathbb{R}$, but $f': \mathbb{R} \rightarrow \mathbb{R}$ is not a continuous function.

- (2) Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that the Taylor series for f about $x = 0$ converges on \mathbb{R} , but it does not coincide with f on any open interval containing 0.

- (3) Consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

One can regard \mathbb{R}^2 and \mathbb{R} as metric spaces via the standard distance functions:

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Prove that:

- (a) For any fixed $x \in \mathbb{R}$, the function $f_x: \mathbb{R} \rightarrow \mathbb{R}$ that sends y to $f(x, y)$ is continuous. Similarly, for any fixed $y \in \mathbb{R}$, the function $f_y: \mathbb{R} \rightarrow \mathbb{R}$ that sends x to $f(x, y)$ is also continuous.
- (b) $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is not a continuous function.

- (4) We say a function $f: (a, b) \rightarrow \mathbb{R}$ is *strictly increasing* if $f(x) < f(y)$ for any $a < x < y < b$. Suppose f is differentiable on (a, b) .
- (a) Prove or disprove: If f is strictly increasing, then $f'(x) > 0$ for any $x \in (a, b)$.
 - (b) Prove or disprove: If $f'(x) > 0$ for any $x \in (a, b)$, then f is strictly increasing. (Hint: Mean value theorem.)
- (5) Consider the function $f(x) = \log(1 + x)$ on $(-1, \infty)$.
- (a) Compute the Taylor series for f about $x = 0$.
 - (b) Let $R_n(x)$ be the remainder of the Taylor series in part (a), i.e.

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} (x - 0)^k.$$

Use Taylor's theorem to show that $\lim_{n \rightarrow \infty} R_n(1) = 0$, then obtain the formula

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

- (6) Prove that the equation $e^x = 1 - x$ has a unique solution in \mathbb{R} .
- (7) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $|f(x) - f(y)| \leq |x - y|^2$ for any $x, y \in \mathbb{R}$. Prove that f is a constant function.
- (8) Let $f: (a, b) \rightarrow \mathbb{R}$ be an unbounded differentiable function. Prove that the derivative $f': (a, b) \rightarrow \mathbb{R}$ is also unbounded.
- (9) Let $f: [0, 1] \rightarrow \mathbb{R}$ be a continuous function and is differentiable on $(0, 1)$. Suppose that f satisfies:
- $f(0) = 0$.
 - There exists $M > 0$ such that $|f'(x)| \leq M|f(x)|$ for any $x \in (0, 1)$.
- Prove that $f(x) = 0$ for any $x \in [0, 1]$.
- (10) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and fix a point $x_0 \in \mathbb{R}$. Consider the sequence $\{x_n\} \subset \mathbb{R}$ defined iteratively by $x_{n+1} = f(x_n)$. Suppose that $\lim_{n \rightarrow \infty} x_n = \ell \in \mathbb{R}$ converges, and suppose that $f'(\ell)$ exists. Prove that $|f'(\ell)| \leq 1$.