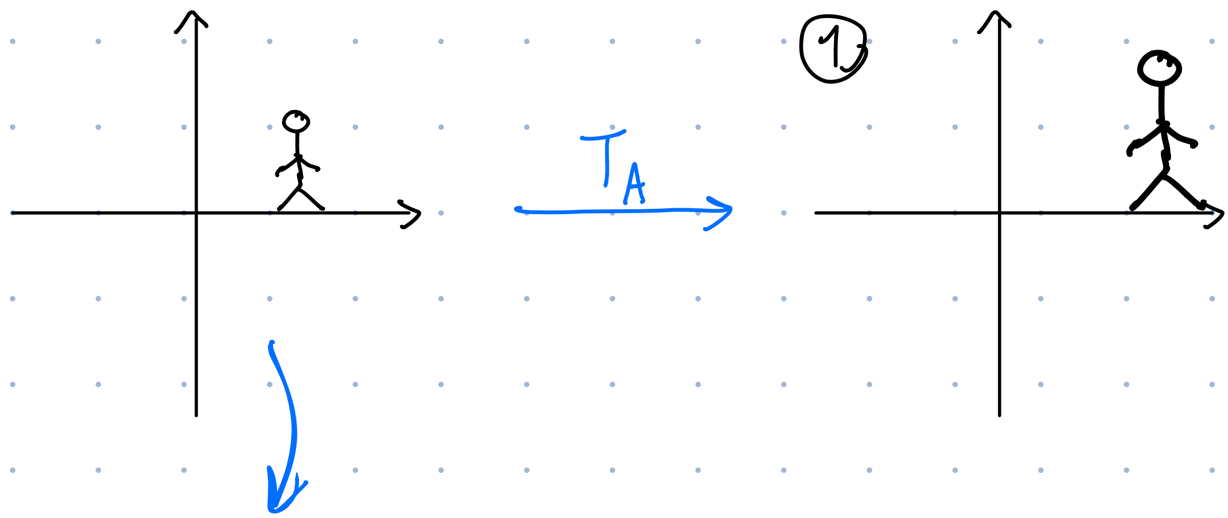
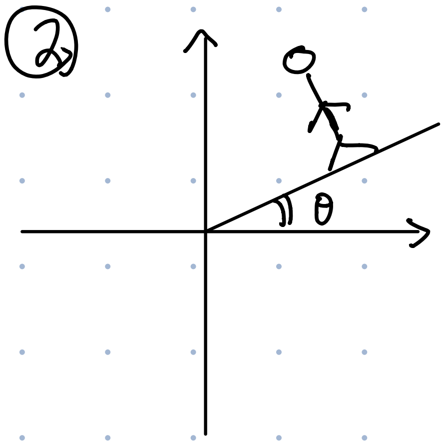
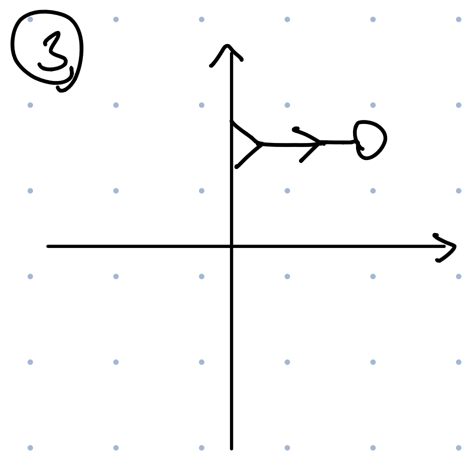
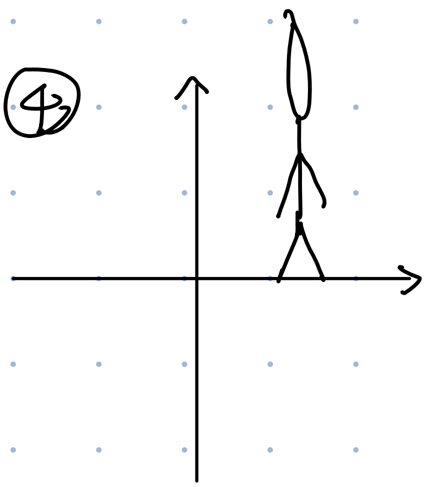
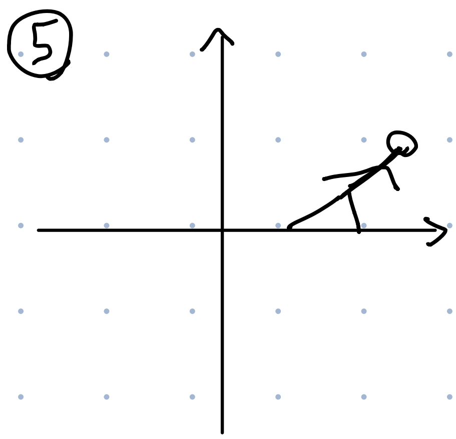
Find a 2x2 matrix A such that T_A looks like:

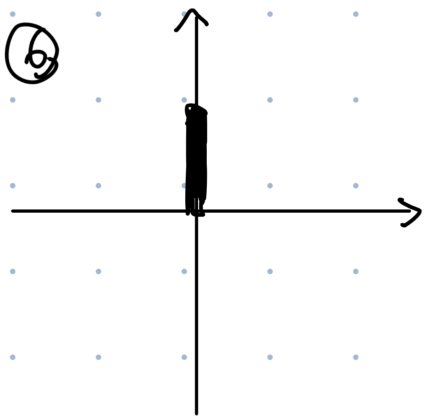












Announcement: (see course website).

- change of OH.
- Quizzes. (usually on Monday)
- 1st HW due next Tuesday.
- 1st Quiz next Wednesday

Last time:

- $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$ $m \times n$. $\rightsquigarrow$ $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\vec{x} = \begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix} \longmapsto A\vec{x} := x_1\vec{a}_1 + \cdots + x_n\vec{a}_n.$$

- T_A is surjective $\Leftrightarrow$ $[A \mid \vec{b}]$ has sol^{n} $\forall \vec{b} \in \mathbb{R}^m$.

$\Leftrightarrow$ $\operatorname{Span}\{\vec{a}_1, \dots, \vec{a}_n\} = \mathbb{R}^m$

$\Leftrightarrow$ A has pivot in each row.

Q: When is T_A injective?

A: $m \times n$ matrix, $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\vec{x} \longmapsto A\vec{x}.$$

Injective: $\forall \vec{x}_1, \vec{x}_2$ in $\mathbb{R}^n$, $\vec{x}_1 \neq \vec{x}_2$ then $A\vec{x}_1 \neq A\vec{x}_2$

Idea. Suppose $\exists$ $\vec{b}$ in $\mathbb{R}^m$ s.t.

$A\vec{x}_1 = \vec{b}$, $A\vec{x}_2 = \vec{b}$ for some $\vec{x}_1 \neq \vec{x}_2$ in $\mathbb{R}^n$.

$$A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = \vec{0}$$

$$\| $$

$$A(\vec{x}_1 - \vec{x}_2)$$

$$\#$$

$$\vec{0}$$

Matrix-vector product is compatible w/ sum & scalar mult.

$\Rightarrow$ $\exists$ $\vec{y} := \vec{x}_1 - \vec{x}_2 \neq \vec{0}$ s.t. $A\vec{y} = \vec{0}$

$\Leftrightarrow$ The linear system $A\vec{x} = \vec{0}$ has a nontrivial sol^{n}, i.e. $\vec{x} \neq \vec{0}$.

$A = \begin{bmatrix} \vec{a}_1 \cdots \vec{a}_n \end{bmatrix}$ $\qquad A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$

$\Longleftrightarrow \ \exists \ \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \neq \vec{0}$ s.t. $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{0}$

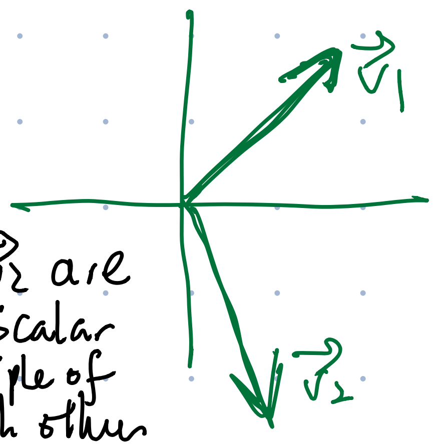
Def: $\{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is <u>linearly dependent</u> if $\exists \ c_1, \ldots, c_k \in \mathbb{R}$ not all zero s.t.

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

Otherwise, it's called <u>linearly independent</u>.

e.g. $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ $\qquad 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \vec{0}$

e.g.



$\vec{v}_1, \vec{v}_2$ are not scalar multiple of each other

l.i.: $\quad c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$

Suppose c_1, c_2 are not all 0,

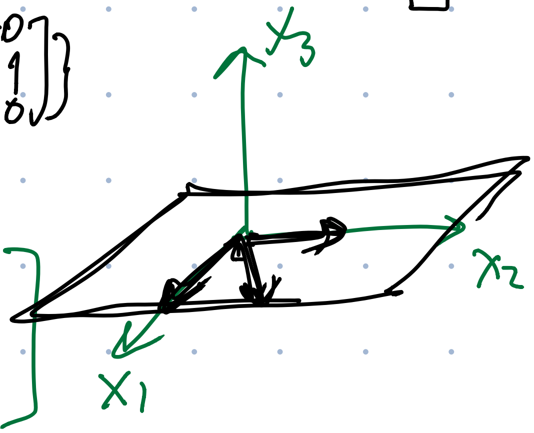
Say $c_1 \neq 0$.

$$\vec{v}_1 = \frac{-c_2}{c_1}\vec{v}_2 \qquad \times$$

$\square$

e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ $\quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$-\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



Thm. $\{\vec{v}_1, \dots, \vec{v}_k\}$ l.d.

$\iff \exists\, 1 \le i \le k$, s.t. $\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$

pf ($\Leftarrow$) $\vec{v}_i = a_1\vec{v}_1 + \dots + a_{i-1}\vec{v}_{i-1} + a_{i+1}\vec{v}_{i+1} + \dots + a_k\vec{v}_k$

for some $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k \in \mathbb{R}$

$\Rightarrow a_1\vec{v}_1 + \dots + a_{i-1}\vec{v}_{i-1} - \vec{v}_i + a_{i+1}\vec{v}_{i+1} + \dots + a_k\vec{v}_k = \vec{0}$

$\Rightarrow \{\vec{v}_1, \dots, \vec{v}_k\}$ is l.d.

($\Rightarrow$) $\exists\, a_1, \dots, a_k$ not all 0

$a_1\vec{v}_1 + \dots + a_k\vec{v}_k = \vec{0}$

Suppose $a_i \neq 0$.

$$\vec{v}_i = -\frac{1}{a_i}\left(a_1\vec{v}_1 + \dots + a_{i-1}\vec{v}_{i-1} + a_{i+1}\vec{v}_{i+1} + \dots + a_k\vec{v}_k\right)$$

$\in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$. $\square$

Exercise. $\vec{v}_1 \in \text{Span}\{\vec{v}_2, \dots, \vec{v}_k\}$,

Then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_2, \dots, \vec{v}_k\}$

Thm. A: $m \times n$ matrix. The following are equivalent:

1): $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is injective.
$\quad \vec{x} \mapsto A\vec{x}$

2) $A\vec{x} = \vec{b}$ has at most one sol^n $\forall \vec{b} \in \mathbb{R}^m$.

3) $A\vec{x} = \vec{0}$ has no nontrivial sol^n

4) the columns of A are l.i.

5) A has pivots in each of the columns

3) $\Leftrightarrow$ 4) by definition of l.i.

pf: 1) $\Leftrightarrow$ 2) by definition of injectivity.

2) $\Rightarrow$ 3) obvious

$\boxed{3) \Rightarrow 2)}$: Suppose $\exists \vec{b}$ s.t. $A\vec{x}_1 = \vec{b} = A\vec{x}_2$ for some $\vec{x}_1 \neq \vec{x}_2$
$\Rightarrow A(\vec{x}_1 - \vec{x}_2) = \vec{0}$ $\times$.

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3) $\Rightarrow$ $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & & & \\ & 1 & & \vec{0} \\ & & 1 & \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & & 0 & 0 \\ & 1 & & 0 \\ & & 1 & 0 \\ 0 & & & 0 \\ & & & 0 \end{array}\right]$

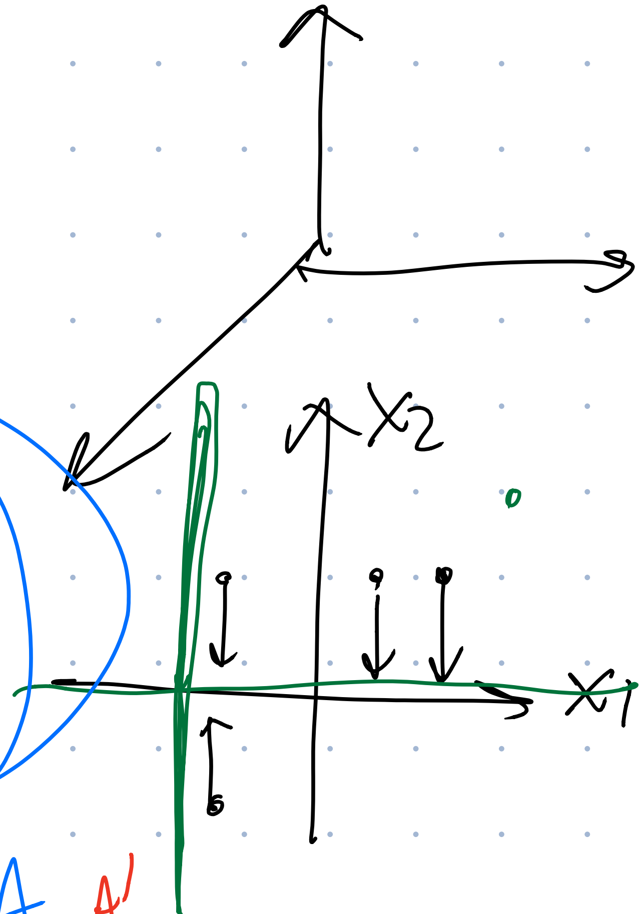
no nontrivial sol^n
$\Rightarrow$ there is no free variable

$\boxed{3) \Leftrightarrow 5)}$

$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & & 0 & b_1' \\ 0 & 1 & & b_2' \\ 0 & 0 & 1 & b_3' \\ & 0 & & \boxed{b_4'} \end{array}\right] \Rightarrow$ at most 1 sol^n $\forall \vec{b} \in \mathbb{R}^m$.

e.g. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$$



$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

Can you find a 2×2 A, A'

s.t. $T_A = T \overset{??}{=} T_{A'}$?? $= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

~~for any x_1, x_2~~

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§ Linear transformations between Euclidean spaces

Def. A fcn $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called a <u>linear transformation</u>

If

- $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$
- $T(c\vec{v}) = cT(\vec{v})$

$\vec{v}_1, \vec{v}_2, \vec{v} \in \mathbb{R}^n$

$c \in \mathbb{R}$

e.g. A: $m \times n$ matrix $\quad T_A: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transf^{n}

$$\vec{x} \mapsto A\vec{x}$$

Rmk: T is linear $\Rightarrow$ • $T(\vec{0}) = \vec{0}$.

• $T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = c_1 T(\vec{v}_1) + \cdots + c_k T(\vec{v}_k)$

Thm Let $T: \mathbb{R}^n \to \mathbb{R}^m$ linear transform^{n}

Then $\underbrace{\exists!}_{\text{there exists a unique...}}$ A: $m \times n$ matrix s.t. $T_A(\vec{v}) = T(\vec{v}) \ \forall \vec{v} \in \mathbb{R}^n$

("standard matrix")

In fact, the i-th column of A is given by

$T(\vec{e}_i)$ where $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← i-th component.

pf: Let's first let

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

Let's check that $\underline{A\vec{v}} = T(\vec{v})$ for any $\vec{v} \in \mathbb{R}^n$

Write $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$A\vec{v} = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1 T(\vec{e}_1) + \cdots + v_n T(\vec{e}_n)$$

$$\underset{T \text{ is linear}}{=} T(v_1\vec{e}_1 + \cdots + v_n\vec{e}_n) = T(\vec{v})$$

$$\begin{bmatrix} v_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \vec{v}$$

Let's prove the uniqueness of A:

Suppose $\exists\, A, B$: $m \times n$ matrix s.t.

$$A\vec{v} = B\vec{v} \quad \text{for any } \vec{v} \in \mathbb{R}^n.$$

e.g. $\vec{v} = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ }

$$A\vec{e}_1 = B\vec{e}_1$$

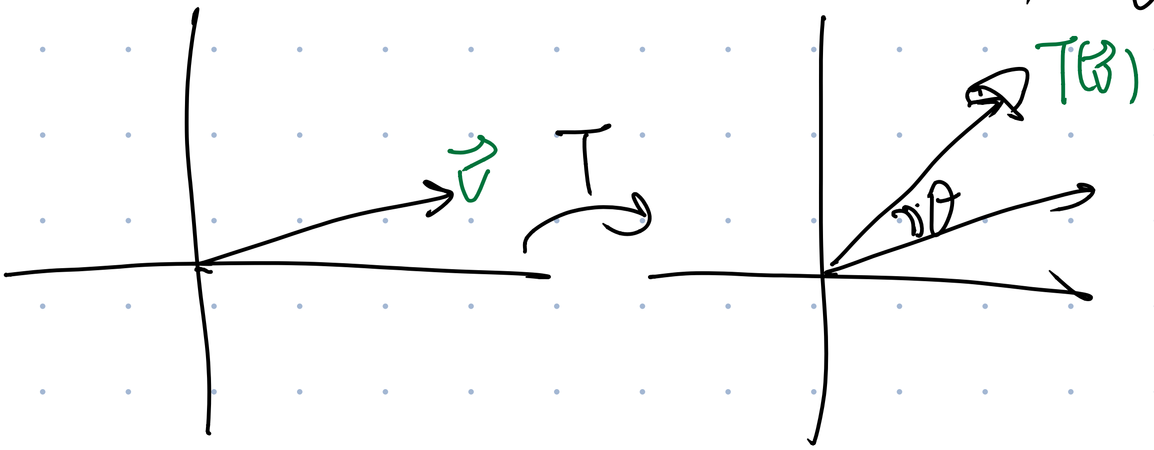
$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{a}_1 \qquad \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{b}_1$$

$\Rightarrow \vec{a}_1 = \vec{b}_1$, i.e. the first column of A = the first column of B.

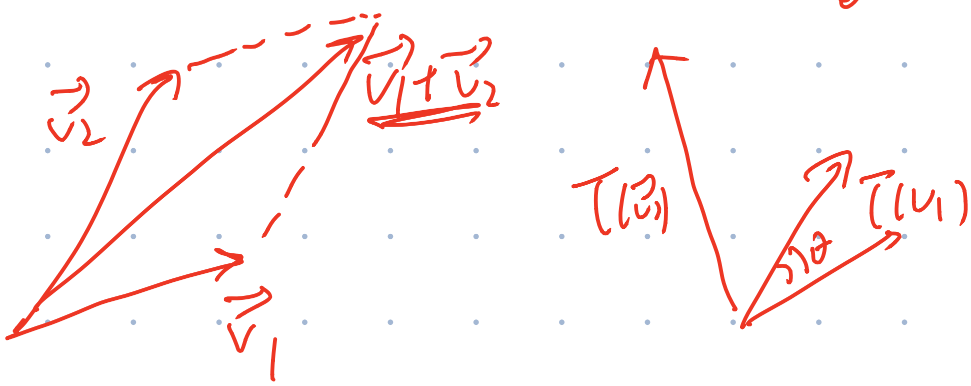
By the same argument, we can show that i-th column of A = i-th column of B $\forall i$

by plugging in $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. $\Rightarrow A = B$. $\square$

e.g. $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ rotation by angle θ



Check: T is linear.



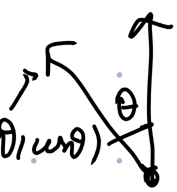
rotation preserves parallelograms.

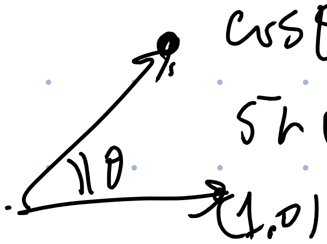
How to find A s.t. $T_A = T$??

$\Downarrow$

$$\begin{bmatrix} | & | \\ T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ | & | \end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

$(-\sin\theta, \cos\theta)$





$\cos\theta$

$\sin\theta$