HW9 sol'n

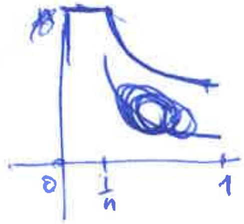
#1: (a). $\forall \varepsilon>0,\ \exists N>0$ s.t. $|f_n(x)-f(x)|<\varepsilon \quad \forall n>N,\ x\in X$.

Take $\varepsilon=1$, $\exists n$ s.t. $|f_n(x)-f(x)|<1 \quad \forall x\in X$.

Suppose f_n is bounded by M, i.e. $|f_n(x)|\le M \quad \forall x\in X$.

Then $|f(x)| \le |f_n(x)-f(x)|+|f_n(x)| < M+1 \quad \forall x\in X$. $\square$

(b) e.g. $f_n(x) := \begin{cases} n, & \text{if } x\in(0,\frac{1}{n}] \\ \frac{1}{x}, & \text{if } x\in[\frac{1}{n},1) \end{cases}$



$(f_n(x))$ are fcns on $(0,1)$,

which conv. pointwisely to $f(x)=\frac{1}{x}$ on $(0,1)$

each $f_n(x)$ is bounded, but $f(x)$ is unbounded.

#2: $\forall \varepsilon>0,\ \exists N>0$ s.t. $|f_n(x)-f(x)|<\frac{\varepsilon}{2} \quad \forall n>N,\ x\in X$

$\Rightarrow \forall n,m>N,\ \forall x\in X \quad |f_n(x)-f_m(x)| \le |f_n(x)-f(x)|+|f(x)-f_m(x)| < \varepsilon$. $\square$

#3. (a) $f(x)=\begin{cases} 0 & \text{if } x=0 \\ 1 & \text{if } x>0. \end{cases}$

(b) $\boxed{\text{Yes}}$. $\forall \varepsilon>0$, Take $N>0$ large s.t. $\frac{1}{1+Na}<\varepsilon$.

Then $\forall x\in[a,\infty)$, $n>N$, we have

$$|f_n(x)-f(x)| = \left|\frac{nx}{1+nx}-1\right| = \frac{1}{1+nx} \le \frac{1}{1+na} < \varepsilon. \quad \square$$

(c) $\boxed{\text{No}}$ If $f_n \to f$ unif. then f should be conti. $\divideontimes$. $\square$

#4. (a) $f(x) = \begin{cases} 1. & 0 < x < 1. \\ 1/2. & x = 1 \\ 0 & x > 1. \end{cases}$

(b) $\boxed{\text{Yes}}$ $\forall \varepsilon > 0$, Take $N > 0$ s.t. $a^N < \varepsilon$.

Then $\forall x \in [0, a]$, $n > N$, we have

$$|f_n(x) - f(x)| = \left| \frac{1}{1+x^n} - 1 \right| = \frac{x^n}{1+x^n} < x^n \le a^n < \varepsilon. \ \square$$

(c) $\boxed{\text{No}}$ If $f_n \to f$ unif., then f is conti. on $[0,1]$. $\times$. $\square$

#5. $\forall T > 0$, $\left(\frac{x^n}{n!}\right)^2 \le \frac{T^{2n}}{(n!)^2}$ on $x \in [-T, T]$.

The series $\sum \frac{T^{2n}}{(n!)^2}$ conv. since $\lim \left| \frac{\frac{T^{2(n+1)}}{((n+1)!)^2}}{\frac{T^{2n}}{(n!)^2}} \right| = \lim \left| \frac{T^2}{(n+1)^2} \right| = 0.$

By Weierstrass M-test, $\sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2$ conv. unif. on $[-T, T]$,

hence is conti. on $[-T, T]$.

Since f is conti. on $[-T, T]$ for any $T > 0$, it's conti. on $\mathbb{R}$. $\square$

#6. Define $g_n := f_n - f$ on X.

Since $(f_n(x))$ is decreasing and conv. to $f(x)$, $f(x) = \inf f_n(x)$.

In particular, we have $0 \le g_{n+1}(x) \le g_n(x)$ $\forall x$.

$\forall \varepsilon > 0$, Define $E_n := \{ x \in X \mid g_n(x) < \varepsilon \} \subset X$.

Since $0 \le g_{n+1}(x) \le g_n(x)$ $\forall x$, we have $E_1 \subset E_2 \subset E_3 \subset \cdots$.

- En is open: $g_n : X \to \mathbb{R}$ is conti., so $E_n = g_n^{-1}((-\infty, \varepsilon))$ is open.
- $X = \bigcup_{n=1}^{\infty} E_n$, since $f_n \to f$ pointwise.

$\Rightarrow$ X has a finite subcover. $\Rightarrow$ $X = E_N = E_{N+1} = E_{N+2} = \cdots$ for some N.

$\Rightarrow |f_n(x) - f(x)| < \varepsilon$ $\forall x \in X$, $\forall n \ge N$. $\square$

#7: (a) Let $f_1, \cdots, f_n$ be conti. fcns. on cpt metric space X.

$\Rightarrow$ they're unif. conti.

$\forall \varepsilon>0$, For each f_i, $\exists \delta_i > 0$ s.t. $|x-y|<\delta_i \Rightarrow |f_i(x)-f_i(y)|<\varepsilon$.

Let $\delta = \min\{\delta_1, \cdots, \delta_n\}$.

Then $|x-y|<\delta \Rightarrow |f_i(x)-f_i(y)|<\varepsilon \quad \forall\, 1\le i\le n$. □

(b). $\forall \varepsilon>0$, $\exists N>0$ s.t. $|f_n(x)-f(x)|<\frac{\varepsilon}{3} \quad \forall n>N,\ x\in X$.

f is conti. (since f_n conti and $(f_n)\to f$ unif.) $\Rightarrow$ f is unif conti. on X (X: cpt)

$\Rightarrow \exists \delta'>0$ s.t. $|x-y|<\delta' \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{3}$.

By part (a), $\exists \delta''>0$ s.t. $|x-y|<\delta'' \Rightarrow |f_i(x)-f_i(y)|<\varepsilon \quad \forall\, 1\le i\le N$.

Take $\delta=\min\{\delta',\delta''\}$. Then $\forall\ |x-y|<\delta$, $\forall f_n$.

- If $1\le n\le N$, then we have $|f_n(x)-f_n(y)|<\varepsilon$.
- If $n>N$, then

$$|f_n(x)-f_n(y)| \le |f_n(x)-f(x)|+|f(x)-f(y)|+|f_n(y)-f(y)|<\varepsilon.$$ □

#8: S is bounded by definition.

$\underline{S \text{ is closed}}$: Let $f\in \mathcal{C}([0,1],\mathbb{R})$ be a limit pt of S,

then $\exists (f_n)\in S$ s.t. $\lim d(f_n, f)=0$

$\Rightarrow d(f,0) \le d(f,f_n)+d(f_n,0) \le d(f,f_n)+1$.

$\Rightarrow d(f,0)\le 1$.

$\Rightarrow f\in S$. Hence S is closed. □

<u>S is not compact:</u>

Define
$$\Phi : \mathcal{C}([0,1],\mathbb{R}) \longrightarrow \mathbb{R}$$
$$f \longmapsto \left(\int_0^{\frac{1}{2}} f(x)\,dx\right) - \left(\int_{\frac{1}{2}}^1 f(x)\,dx\right).$$

It's easy to check that Φ is a conti. fcn. on $\mathcal{C}([0,1],\mathbb{R})$.

If S were compact, then Φ would have ~~attain~~ attain max. on S.

You can show that:

- $\sup_{f\in S} \Phi(f) = 1$.
- but $\Phi(f) < 1 \quad \forall f \in S$.

$\Rightarrow$ S is not compact. $\square$