1. Suppose that a meromorphic function f has two periods ω_1 and ω_2, with $\omega_2/\omega_1 \in \mathbb{R}$.

(a) Suppose ω_2/ω_1 is rational, say equal to p/q, where p and q are relatively prime integers. Prove that as a result the periodicity assumption is equivalent to the assumption that f is periodic with the simple period $\omega_0 = \frac{1}{q}\omega_1$. [Hint: Since p and q are relatively prime, there exist integers m and n such that $mq + np = 1$ (Corollary 1.3, Chapter 8, Book I).]

(b) If ω_2/ω_1 is irrational, then f is constant. To prove this, use the fact that $\{m - n\tau\}$ is dense in $\mathbb{R}$ whenever τ is irrational and m, n range over the integers.

(a) $\frac{\omega_2}{\omega_1} = \frac{p}{q}$, $\gcd(p,q) = 1$, $mp + nq = 1$, $m, n, p, q \in \mathbb{Z}$.

Suppose $f(z) = f(z+\omega_1) = f(z+\omega_2)$ $\forall z$,

Then. $f(z + \frac{1}{q}\omega_1) = f(z + \frac{mp+nq}{q} \cdot \omega_1)$

$$= f(z + m\omega_2 + n\omega_1) = f(z).$$

On the other hand, if $f(z + \frac{1}{q}\omega_1) = f(z)$ $\forall z$,

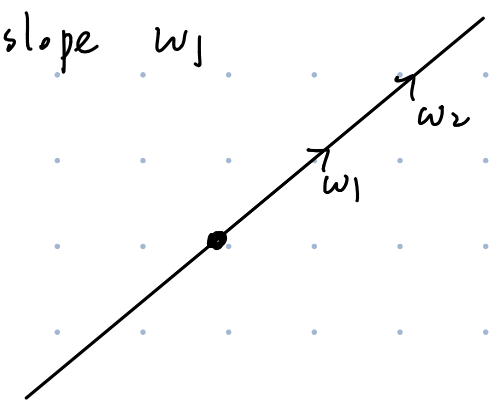
Then $f(z + \omega_1) = f(z)$.

$f(z + \omega_2) = f(z + \frac{p}{q}\omega_1) = f(z)$. $\square$

(b) $\{m - n \cdot \frac{\omega_2}{\omega_1} \mid m, n \in \mathbb{Z}\} \subseteq \mathbb{R}$ dense in $\mathbb{R}$.

$\Rightarrow$ f is constant on lines with slope ω_1

$\Rightarrow$ f is constant. $\square$



6. Prove that $\wp''$ is a quadratic polynomial in $\wp$.

- Laurent serie exp. of $\wp$ near $z=0$ is:

$$\wp(z) = \frac{1}{z^2} + * z^2 + \cdots$$

$$\Rightarrow \quad \wp'(z) = \frac{-2}{z^3} + * z + \cdots$$

$$\wp''(z) = \frac{6}{z^4} + * + \cdots$$

$$\wp(z)^2 = \frac{1}{z^4} + * + \cdots$$

$$\Rightarrow \quad \wp''(z) - 6\wp(z)^2 = \text{a holo. ell. fcn} \equiv \text{const.}$$

$\square$

(A) Let $\Lambda \subseteq \mathbb{C}$ be a lattice. Suppose z_1, z_2 are two complex numbers such that $\wp(z_1) \neq \wp(z_2)$ and $z_1, z_2, z_1 \pm z_2 \notin \Lambda$. In this problem, you'll prove the addition theorem for the $\wp$-function

$$\wp(z_1) + \wp(z_2) + \wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2.$$

(1) Let $f(z) = \wp'(z) - (a\wp(z) + b)$. There exists a unique pair of complex numbers a, b such that $f(z_1) = f(z_2) = 0$. Show that

$$a = \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}.$$

(2) By analyzing the poles of f in the fundamental domain, show that ~~$f(z_1+z_2) = 0$~~ $f(-z_1-z_2)=0$.

typo fixed on: 4/23.

(3) Consider the following polynomial of degree 3:

$$F(X) = 4X^3 - g_2 X - g_3 - (aX + b)^2.$$

Show that $\wp(z_1), \wp(z_2), \wp(z_1 + z_2)$ are the roots of F, then prove the addition theorem for the $\wp$-function.

(1) Find $a, b \in \mathbb{C}$, s.t. $\begin{cases} a\wp(z_1) + b = \wp'(z_1) \\ a\wp(z_2) + b = \wp'(z_2) \end{cases}$

Since $\wp(z_1) \neq \wp(z_2)$, there is a unique sol'n given by:

$$a = \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}, \quad b = \frac{\wp(z_1)\wp'(z_2) - \wp'(z_1)\wp(z_2)}{\wp(z_1) - \wp(z_2)}$$

(2) $f(z) = \wp'(z) - (a\wp(z) + b)$ has a pole at 0 of order 3. in the fundamental domain.

$\Longrightarrow$ there are 3 zeros w_1, w_2, w_3 in the fund. domain,

and $w_1 + w_2 + w_3 \in \Lambda$.

Since $f(z_1) = f(z_2) = 0$, $\Rightarrow f(-z_1 - z_2) = 0$.

(3) Recall that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$.

$$\underset{(a\wp+b)^2}{\|} \longleftarrow \text{at } z_1, z_2, -z_1-z_2 \text{ by part (2).}$$

$\Longrightarrow \wp(z_1), \wp(z_2), \wp(-z_1-z_2) = \wp(z_1+z_2)$ are the roots of $F(X) = 4X^3 - g_2 X - g_3 - (aX+b)^2$.

Consider the coeff. of X^2

$$\Longrightarrow \frac{a^2}{4} = \wp(z_1) + \wp(z_2) + \wp(z_1+z_2). \quad \Box$$

(B) Prove that

$$\sum_{1 \le n^2+m^2 \le R^2} \frac{1}{n^2+m^2} = 2\pi \log R + O(1) \quad \text{as } R \to \infty.$$

(This is part of Exercise 3 in the textbook.)

We'll show that $\exists\, M > 0$ const. s.t.

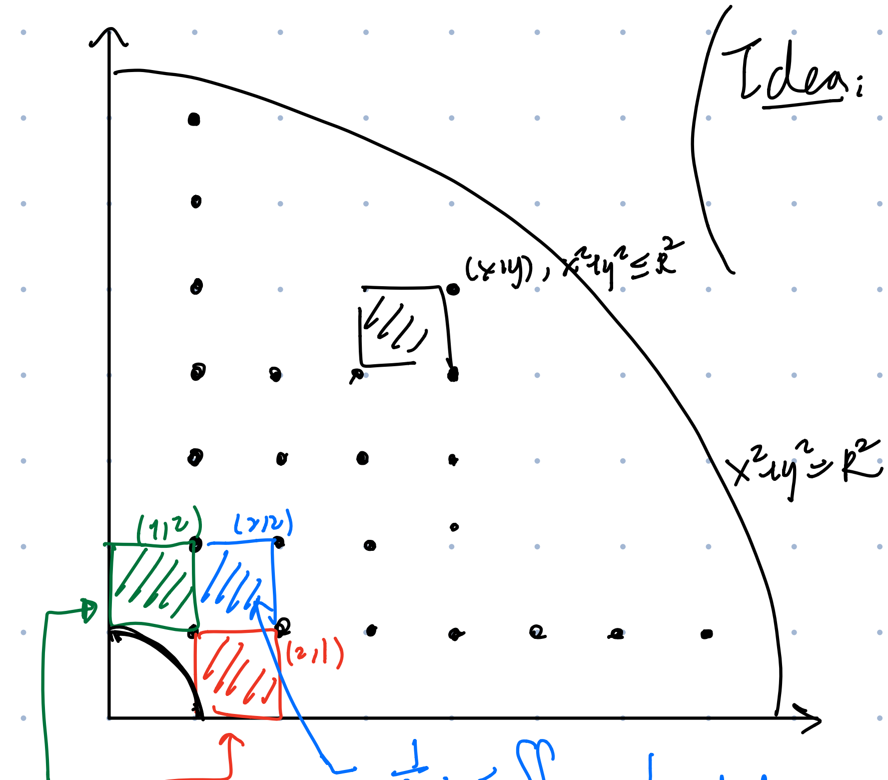
$$2\pi \log R - M < \sum_{1 \le n^2+m^2 \le R^2} \frac{1}{n^2+m^2} < 2\pi \log R + M$$

$\forall R \gg 0$.

It suffices to show that: $\exists M > 0$,

$$\frac{\pi}{2}\log R - M < \sum_{\substack{1 \le n^2+m^2 \le R^2 \\ n,m \ge 1}} \frac{1}{n^2+m^2} < \frac{\pi}{2}\log R + M \qquad \forall R \gg 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ is a finite number.



$$\frac{1}{2^2+2^2} < \iint_{\square} \frac{1}{x^2+y^2}\,dx\,dy$$

$$\frac{1}{1^2+2^2} < \iint_{\square} \frac{1}{x^2+y^2}\,dx\,dy$$

$$\frac{1}{2^2+1^2} < \iint_{\square} \frac{1}{x^2+y^2}\,dx\,dy$$

(Idea: Compare it with $\displaystyle\iint_{\substack{1 \le x^2+y^2 \le R^2 \\ x,y \ge 0}} \frac{1}{x^2+y^2}\,dx\,dy$)

We have:

$$\sum_{\substack{1 \le n^2+m^2 \le R^2 \\ n,m \ge 1}} \frac{1}{n^2+m^2} < 1 + \iint_{\substack{1 \le x^2+y^2 \le R^2 \\ x,y \ge 0}} \frac{1}{x^2+y^2}\,dx\,dy$$

$$= 1 + \iint_{\substack{1 \le r \le R \\ 0 \le \theta \le \frac{\pi}{2}}} \frac{1}{r^2}\, r\,dr\,d\theta$$

$$= 1 + \frac{\pi}{2}\log R$$

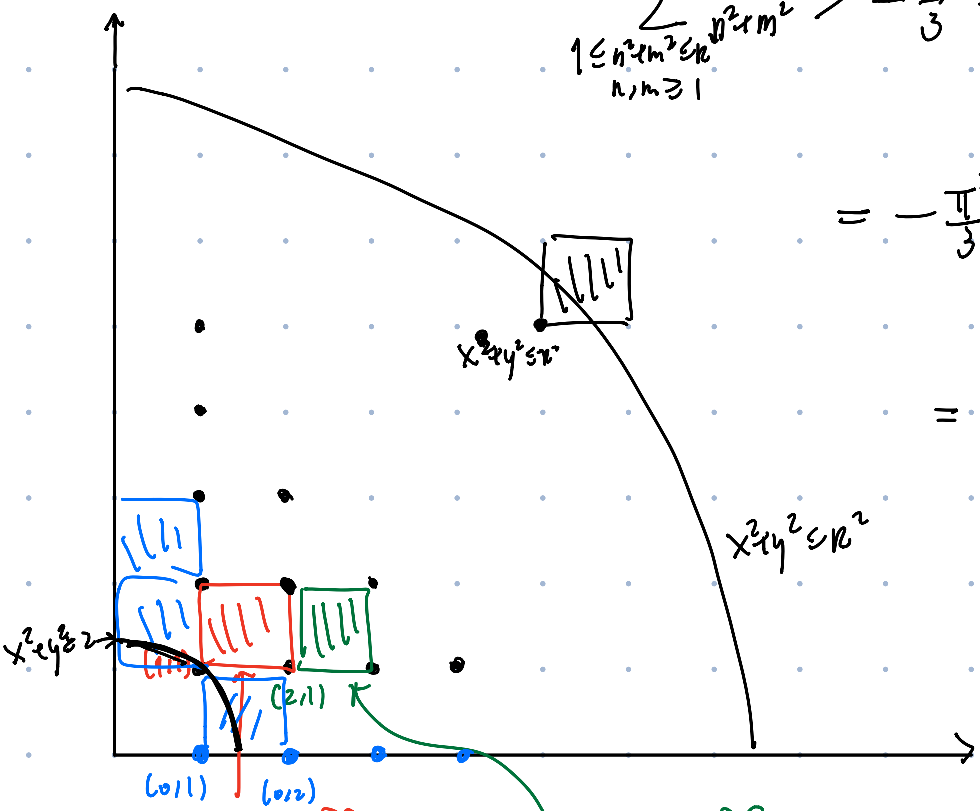
On the other hand:

$$\sum_{\substack{1\le n^2+m^2\le R^2\\ n,m\ge 1}} \frac{1}{n^2+m^2} > -\frac{\pi^2}{3} + \iint_{\substack{2\le x^2+y^2\le R^2\\ x,y>0}} \frac{1}{x^2+y^2}\,dx\,dy$$

$$= -\frac{\pi^2}{3} + \iint_{\substack{\sqrt{2}\le r\le R\\ 0\le\theta\le\frac{\pi}{2}}} \frac{1}{r^2}\, r\,dr\,d\theta$$

$$= -\frac{\pi^2}{3} + \frac{\pi}{2}\left(\log R - \log\sqrt{2}\right).$$

$\square$



$$\frac{1}{1^2+1^2} > \iint_{\boxed{(1,1)}} \frac{1}{x^2+y^2}$$

$$\frac{1}{2^2+1^2} > \iint_{\boxed{(2,1)}} \frac{1}{x^2+y^2}$$

(C) Let $\tau \in \mathbb{H}$ be an element in the upper half-plane. Denote

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right).$$

Prove that for any integers $a,b,c,d\in\mathbb{Z}$ with $ad-bc=1$ $\left(\text{i.e. } \begin{bmatrix} a & b\\ c & d\end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})\right)$,

$$\wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\wp(z,\tau).$$

$$\wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \frac{(c\tau+d)^2}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{\left(\frac{z}{c\tau+d}+m+n\cdot\frac{a\tau+b}{c\tau+d}\right)^2} - \frac{1}{\left(m+n\cdot\frac{a\tau+b}{c\tau+d}\right)^2}\right)$$

$$= (c\tau+d)^2\cdot\left[\frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}\left(\frac{1}{(z+m(c\tau+d)+n(a\tau+b))^2} - \frac{1}{(m(c\tau+d)+n(a\tau+b))^2}\right)\right]$$

$$= (c\tau+d)^2\,\wp(z,\tau)$$

since $\forall (x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}$, $\exists!\,(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}$ s.t. $\begin{cases} md+nb=x\\ mc+na=y\end{cases}$

$\left(\because \begin{bmatrix} a & b\\ c & d\end{bmatrix}\in SL_2(\mathbb{Z})\right)$