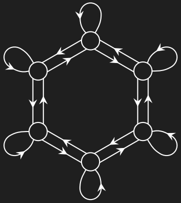

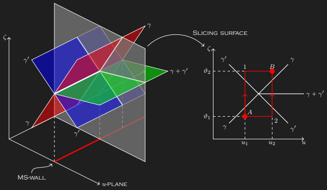

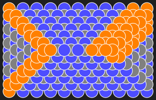

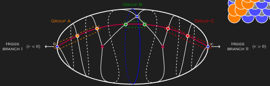

Quiver BPS Algebras

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February 24th, 2022

Motivation

Based on:

- D.G. and Masahito Yamazaki 2008.07006
- D.G., Wei Li and and Masahito Yamazaki 2106.01230
- D.G., Wei Li and and Masahito Yamazaki 2108.10286

Enum.Geometry — Non-perturbative phenomena

Categorification — Wall-crossing

New Algebras & Reps — **BPS states** — Integrability

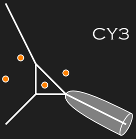
Modularity — Black hole physics

[Nakajima; Kontsevich, Soibelman; Alday, Gaiotto, Tachikawa; Douglass, Moore; Schifman, Vasserot, ...]

UV

Type II

D6/D4/D2/D0 $\longrightarrow$

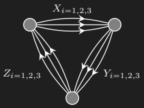


4d $\mathcal{N}=1$ SYM

$\downarrow$

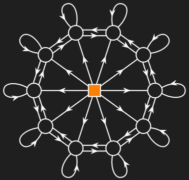
1d $\mathcal{N}=4$ QM

IR



$$+ \quad W = \epsilon^{ijk}\,\mathrm{Tr}\, Z_i Y_j X_k$$

Quiver BPS Algebras



Q_0 – QUIVER VERTICES

Q_1 – QUIVER ARROWS

Q_2 – SUPERPOTENTIAL

$a, b \in Q_0$

$|a| = (|a \to a| + 1) \bmod 2$

$I, J \in Q_1$

$$e^{(a)}(z) = \sum_{n \in \mathbb{Z}_{\geqslant 0}} \frac{e_n^{(a)}}{z^n},$$

$$f^{(a)}(z) = \sum_{n \in \mathbb{Z}_{\geqslant 0}} \frac{e_n^{(a)}}{z^n},$$

$$\psi^{(a)}(z) = \sum_{n \in \mathbb{Z}} \frac{\psi_n^{(a)}}{z^n},$$

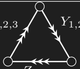
$h_I \in \mathbb{C}$ – EQUIV. WEIGHTS, FLAVOR CHARGE

BOND FACTOR: $\varphi^{a \Leftarrow b}(u) \equiv \dfrac{\prod_{I \in \{a \to b\}} (u + h_I)}{\prod_{J \in \{b \to a\}} (u - h_J)}$

$$\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\,, \\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{a \Leftarrow b}(z-w)\,e^{(b)}(w)\,\psi^{(a)}(z)\,, \\
e^{(a)}(z)\,e^{(b)}(w) &\simeq (-1)^{|a||b|}\varphi^{a \Leftarrow b}(z-w)\,e^{(b)}(w)\,e^{(a)}(z)\,, \\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{a \Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,, \\
f^{(a)}(z)\,f^{(b)}(w) &\simeq (-1)^{|a||b|}\varphi^{a \Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,, \\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\simeq -\delta^{a,b}\,\frac{\psi^{(a)}(z) - \psi^{(a)}(w)}{z-w}\,,
\end{aligned}$$

$\simeq$ – EQUIVALENT UP TO $z^n w^{m \geqslant 0}$, $z^{n \geqslant 0} w^m$

Quiver BPS Algebras II

$\mathbb{C}^3$		 $X_{1,2,3}$ $W = \mathrm{Tr}\ X_1\,[X_2, X_3]$	$\mathsf{Y}\left(\widehat{\mathfrak{gl}}_1\right)$
Conifold		$X_{1,2}$  $Y_{1,2}$ $W = \mathrm{Tr}\ (Y_2X_2Y_1X_1 - Y_2X_1Y_1X_2)$	$\mathsf{Y}\left(\widehat{\mathfrak{gl}}_{1\|1}\right)$
$xy = z^n w^m$	...	...	$\mathsf{Y}\left(\widehat{\mathfrak{gl}}_{n\|m}\right)$
$K_{\mathbb{P}^2}$		 $X_{1,2,3}$, $Y_{1,2,3}$, $Z_{1,2,3}$ $W = \mathrm{Tr}\ \epsilon^{ijk} Z_i Y_j Z_k$	$\mathsf{Y}\,(K_{\mathbb{P}^2})$???

$\Big[$ $\mathsf{E}_\tau(Q,W)$  $\xrightarrow{\text{dim. red.}}$ $\mathsf{T}_\beta(Q,W)$  $\xrightarrow{\text{dim. red.}}$ $\mathsf{Y}(Q,W)$ $\bullet$ $\Big] \times \left(\text{toric Calabi-Yau 3-fold}\right)$

D8-D6-D4-D2-D0	D7-D5-D3-D1	D6-D4-D2-D0
$\vartheta_{11}(z\|\tau)$	$2\sinh(\beta z/2)$	z
3d $\mathcal{N}=2$	2d $\mathcal{N}=(2,2)$	1d $\mathcal{N}=4$
Ell. Cohomology	K-theory	Cohomology

 $\leftrightsquigarrow$ (generalized cohomology theories)???

Localization

[Denef '02]

[Witten'82, Gaiotto-Moore-Witten'15,...]

$$\psi_i \rightsquigarrow dx^i, \quad \psi_i^\dagger \rightsquigarrow \iota_{\partial/\partial x^i}, \quad \mathcal{Q}_\alpha,\ \bar{\mathcal{Q}}_{\dot\alpha} \rightsquigarrow \text{differentials}, \quad \mathcal{H} \rightsquigarrow \text{Laplacian}$$

de Rham, Dolbeault, equivariant, sup. twist

$$\mathcal{Q} = e^{-h}\left(\overset{\text{de Rham}}{d} + \overset{\text{Dolbeault}}{\bar\partial} + \overset{\text{equivariant}}{\iota_V} + \overset{\text{sup. twist}}{dW\wedge}\right) e^{h}$$

Morse height function

BPS states:

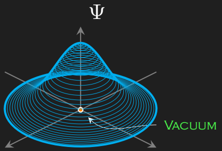
$$\mathscr{H}_{\text{BPS}} = H_G^*\left(\text{Target Space}, \mathcal{Q}\right) \approx \bigoplus_{p\in\mathcal{I}} \mathbb{C}\Psi_p$$

$$\mathcal{I} = \{\text{crit. fixed points}\} = \{\text{classical vacua}\}$$

$$\mathcal{Q}^\dagger \sim \sum_i \left(d\bar{x}^i \partial_{\bar{x}^i} + \omega_i x^i \iota_{\partial/\partial x^i}\right)$$

$$\text{Eul} = \bigwedge_i \left(\omega_i - |\omega_i|\, dx^i \wedge d\bar{x}^i\right) e^{-|\omega_i||x_i|^2} =$$

$$= \prod_i \omega_i \times \exp\left(-\left\{\mathcal{Q}^\dagger, \sum_i \frac{|\omega_i|}{\omega_i}\bar{x}^i\, dx^i\right\}\right)$$



Fixed Points

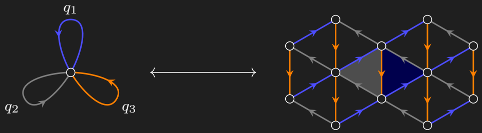
D-term + F-term:

$$\sum_{x\in Q_0}\sum_{I\in\{a\to x\}} q_I {q_I}^\dagger - \sum_{y\in Q_0}\sum_{J\in\{y\to a\}} q_J^\dagger q_J = \zeta_a \mathrm{Id}_{d_a\times d_a}, \quad \forall\ a\in Q_0;$$

$$\Phi_b q_I - q_I \Phi_a - \mu_I q_I = 0, \quad \forall\ a,b\in Q_0,\ I\in\{a\to b\};$$

$$\partial_{q_I} W = 0, \quad \forall\ I\in Q_1.$$

Periodic quiver:



$$W = \sum_{\text{faces}} (-1)^{\text{ori}}\,\mathrm{Tr}\prod_{\text{loop}} q = \triangle - \triangle = \mathrm{Tr}\ q_1\,[q_2,q_3]$$

Constraints on flavor charges (masses):

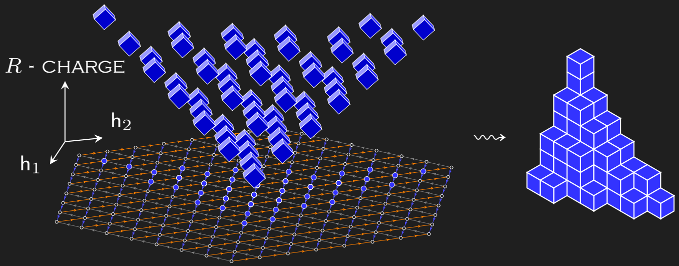
$$\left.\begin{array}{lll} \text{Loop:} & \sum_{\text{loop}} h_I = 0, & \forall\text{faces;} \\ \text{Vertex:} & h_I \sim h_I - \epsilon_a + \epsilon_b, & \forall I\in\{a\to b\}. \end{array}\right\} h_I = x_I\hbar_1 + y_I\hbar_2.$$

Equivariant toric action on CY3:

$$(z_1,z_2,z_3) \longmapsto \left(e^{\hbar_1}z_1, e^{\hbar_2}z_2, e^{-\hbar_1-\hbar_2}z_3\right)$$

Crystals

Quiver path algebra: $\mathbb{C}Q/\langle dW\rangle \rightsquigarrow \prod q$ – "Baryons"
Crystal = Possible baryons:



$\square$ – atom of a crystal

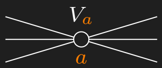
Color of $\square$ denoted $\hat{\square} \in Q_0$ is a color of atom projection to $(\mathsf{h}_1, \mathsf{h}_2)$

Melting rule: K – molten crystal

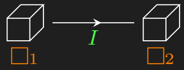
> For any atom $\square$ such that $I \cdot \square \in \mathsf{K}$ for some arrow I, then $\square$ is also contained in K

[Szendroi; Mozgovoy, Reyneke; Nagao, Nakajima; Ooguri-Yamazaki; Jafferis, Chuang, Moore; Sulkowski; Aganagic, Schaeffer; Aganagic, Vafa; ...]

Euler classes



$$\mathcal{V} = \bigoplus_{a\in Q_0} V_a$$



Quiver representation in crystal basis:

$$V_a = \bigoplus_{\square\in \mathrm{K},\hat{\square}=a} \mathbb{C}|\square\rangle,\ a\in Q_0, \qquad \begin{aligned} q_I &= \langle q_I\rangle + \delta q_I \\ \langle q_I\rangle &= \begin{cases} 1, & \text{link present} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

G-action:

$$\delta q_{I\in\{a\to b\}} \mapsto \delta q_{I\in\{a\to b\}} + g_a\langle q_I\rangle - \langle q_I\rangle g_b, \quad g_a\in\mathfrak{gl}(d_a,\mathbb{C}),\ g_b\in\mathfrak{gl}(d_b,\mathbb{C})$$

Fixed point structure:

$$\underbrace{\text{Fixed Point K}}_{\text{Baryons, }\langle q_I\rangle} + \underbrace{(\text{Tangent Space}/G)}_{\text{Mesons, }\delta q_I}$$

IR meson flavor charges:

$$h_{\text{eff}}\left(\langle\square_2|\delta q_I|\square_1\rangle\right) = h_{\square_2} - h_{\square_1} - h_I$$

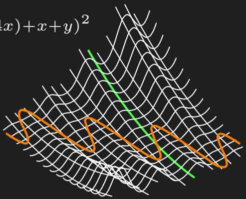
Meson space:

$$\mathcal{M}_{\text{meson}} = \text{Span}\left\{q_\alpha, h_\alpha\right\}_{\alpha=1}^{N_{\text{meson}}}, \quad \text{Eul}(\mathcal{M}) \sim \prod_\alpha h_\alpha$$

However it is singular!

(REGULARIZED) EULER CLASSES

$U(x,y)=(\sin(4x)+x+y)^2$



$$\Psi_{\mathrm{UV}} \longrightarrow \Psi_{\mathrm{IR}} \sim \left[\begin{array}{c}\text{FREE}\\ \text{PARTICLE ON CLASS.}\\ \text{VACUUM LOCUS}\end{array}\right] \times \left[\ \text{FLUCTUATIONS}\ \right]$$

$$m\,(\text{FLUCTUATION}) \sim |h|$$

NEED TO ADD HIGHER LOOPS!

$$\begin{array}{lccc} \mathcal{Q} = & e^{-sh}\left(d+\bar{\partial}+\iota_{sV}+s\,dW\wedge\right)e^{sh} & \Rightarrow & e^{-s_1 h}\left(d+\bar{\partial}+\iota_{s_1 V}+s_2\,dW\wedge\right)e^{s_1 h} \\ \text{IR:} & s \to \infty & \Rightarrow & s_1 \to \infty \text{ THEN } s_2 \to \infty \end{array}$$

SUPERPOTENTIAL FOR MASSLESS MODES:

$$W \sim A\phi_1\phi_2 \quad \Rightarrow \quad \Psi_{\mathrm{IR}}\,(\phi_1\phi_2) \sim (-A)(A) \sim (-1)$$

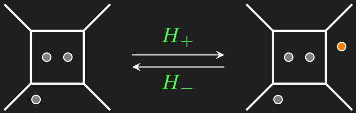
CONJECTURE: [G.-YAMAZAKI '20]

$$\mathcal{N} = \mathrm{Span}\{q_\alpha, h_\alpha\}_{\alpha=1}^{N}$$

$$\boxed{\widetilde{\mathrm{Eul}}(\mathcal{N}) = (-1)^{\left\lfloor \sum_{\alpha:\ h_\alpha=0} \frac{1}{2} \right\rfloor} \prod_{\alpha:\ h_\alpha\neq 0} h_\alpha}$$

Hecke modification

Adding/deleting branes → Hecke modifications:



[Nakajima'99; Kontsevich-Soibelman'11; ...]

Fourier-Mukai transform:

$$\begin{tikzcd} \mathrm{Rep}(Q,\vec{d}) \arrow[rr, shift left, "e"] & & \mathrm{Rep}(Q,\vec{d}') \arrow[ll, shift left, "f"] \\ & \mathrm{Rep}(Q,\vec{d}) \times \mathrm{Rep}(Q,\vec{d}') \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$$\vec{d}' = \vec{d} + \vec{1}_{a\in Q_0}$$

$$\vec{1}_a := \left(0,\ldots,0,\overset{a^{\mathrm{th}}\ \mathrm{place}}{1},0,\ldots,0\right)$$

with a kernel given by $\mathcal{O}_{\mathcal{I}}$ where $\mathcal{I}$ is an incidence locus:

$$\mathcal{I} = \left\{ \mathrm{Rep}(Q,\vec{d}) \xhookrightarrow[\text{homo}]{} \mathrm{Rep}(Q,\vec{d}') \right\}$$

Homomorphism of quiver reps: $\{\tau_a\}_{a\in Q_0}$, $\tau_a : V_a \to V_a{}'$

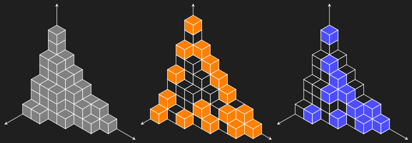
so that

$$\begin{tikzcd} V_a \arrow[r, "I"] \arrow[d, "\tau_a"'] & V_b \arrow[d, "\tau_b"] \\ V_a' \arrow[r, "I'"'] & V_b' \end{tikzcd}$$

commutes $\forall I \in Q_1$

Matrix elements

Fixed point: $\mathcal{I} = \{\mathrm{K} \subset \mathrm{K}'\}$,
denote corresponding Euler class as $\widetilde{\mathrm{Eul}}(\mathrm{K},\mathrm{K}')$
Vacant positions:



A) K B) Add (K) C) Rem (K)

$$e|\mathrm{K}\rangle = \sum_{\square\in\mathrm{Add}(\mathrm{K})} [\mathrm{K}\to\mathrm{K}+\square]|\mathrm{K}+\square\rangle \qquad f|\mathrm{K}\rangle = \sum_{\square\in\mathrm{Rem}(\mathrm{K})} [\mathrm{K}\to\mathrm{K}-\square]|\mathrm{K}-\square\rangle$$

$$[\mathrm{K}\to\mathrm{K}+\square] = \frac{\widetilde{\mathrm{Eul}}(\mathrm{K})}{\widetilde{\mathrm{Eul}}(\mathrm{K},\mathrm{K}+\square)} \qquad [\mathrm{K}\to\mathrm{K}-\square] = \frac{\widetilde{\mathrm{Eul}}(\mathrm{K})}{\widetilde{\mathrm{Eul}}(\mathrm{K}-\square,\mathrm{K})}$$

Numerical results ($[a\to b\to c] := [a\to b]\cdot[b\to c]$):

$$[\mathrm{K}+\boxed{a}_1 \to \mathrm{K}+\boxed{a}_1+\boxed{b}_2 \to \mathrm{K}+\boxed{b}_2] = [\mathrm{K}+\boxed{a}_1 \to \mathrm{K} \to \mathrm{K}+\boxed{b}_2]\;,$$

$$\frac{[\mathrm{K}\to\mathrm{K}+\boxed{b}_2\to\mathrm{K}+\boxed{a}_1+\boxed{b}_2]}{[\mathrm{K}\to\mathrm{K}+\boxed{a}_1\to\mathrm{K}+\boxed{a}_1+\boxed{b}_2]} = \varphi^{a\Leftarrow b}\left(h_{\boxed{a}_1} - h_{\boxed{b}_2}\right)\;,$$

$$\frac{[\mathrm{K}+\boxed{a}_1+\boxed{b}_2\to\mathrm{K}+\boxed{b}_2\to\mathrm{K}]}{[\mathrm{K}+\boxed{a}_1+\boxed{b}_2\to\mathrm{K}+\boxed{a}_1\to\mathrm{K}]} = \varphi^{a\Leftarrow b}\left(h_{\boxed{a}_1} - h_{\boxed{b}_2}\right)\;,$$

$$[\mathrm{K}\to\mathrm{K}+\boxed{a}\to\mathrm{K}] = \operatorname*{res}_{t=h_{\boxed{a}}} \Psi_{\mathrm{K}}^{(a)}(t)$$

$$,\quad \Psi_{\mathrm{K}}^{(a)}(z) = \left(\prod_{I\in\{a\to a\}} \frac{1}{-h_I}\right)\times \times \prod_{\boxed{b}\in\mathrm{K}} \varphi^{a\Leftarrow b}\left(z - h_{\boxed{b}}\right)$$

Spectral parameters

$$\mathcal{N}=4\ \text{SQM} \xleftarrow{\text{dim.red.}} \mathcal{N}=1\ \text{4d SYM}$$

Vector multiplet: A_0, $X_{i=1,2,3}$, ψ_α, D
Notice for $\Phi_a = A_{1,a} - \mathrm{i}A_{2,a}$, $a \in Q_0$:

$$[\mathcal{Q}, \Phi_a] = 0$$

therefore $\mathrm{Tr}\ \Phi_a{}^k$, $k \in \mathbb{Z}$ is a BPS operator.
For example, for a resolvent:

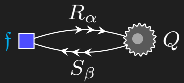
$$\mathrm{Tr}\ (z-\Phi_a)^{-1}\,|\mathrm{K}\rangle = \left(\sum_{\boxed{a}\in\mathrm{K}} \frac{1}{z-h_{\boxed{a}}}\right)|\mathrm{K}\rangle$$

Eventually, we define:

$$\begin{aligned} e^{(a)}(z) &= \left[\mathrm{Tr}\ (z-\Phi_a)^{-1}, e\right] \\ f^{(a)}(z) &= -\left[\mathrm{Tr}\ (z-\Phi_a)^{-1}, f\right] \\ \psi^{(a)}(z) &= \exp\left(\sum_{b\in Q_0} \mathrm{Tr}\,\log\varphi^{a\Leftarrow b}\,(z-\Phi_a)\right) \end{aligned}$$

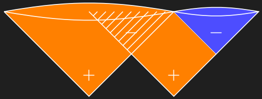
$e^{(a)}(z)$, $f^{(a)}(z)$, $\psi^{(a)}(z)$ are a basis of a quiver Yangian

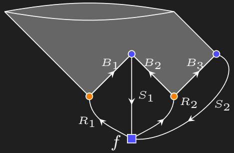
Framing and New Reps

Framing Node ≈ “Frozen” Gauge Node: 

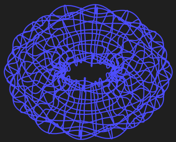
R – positive crystal 

, S – negative crystal 





$$\Delta W = \mathrm{Tr}\left[S_1(B_1R_1 - B_2R_2) + S_2B_3R_2\right], \quad \psi^{(a)}(z)|\varnothing\rangle = \frac{\prod\limits_{I\in\{a\to\mathfrak{f}\}}(-z-h_I)}{\prod\limits_{J\in\{\mathfrak{f}\to a\}}(z-h_J)}|\varnothing\rangle$$

Even “bizarre” crystals: 

However the resulting Rep may be **reducible**...

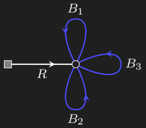
Example

From CY3 to CY2 [Rapcak-Soibelman-Yang-Zhao'18]

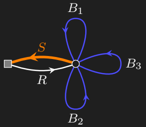
D4 wrapping $\mathbb{C}^2 \subset \mathbb{C}^3$ wrapped by D6

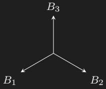
$\mathrm{Hilb}^n(\mathbb{C}^2)$ — $\mathrm{Hilb}^n(\mathbb{C}^3)$

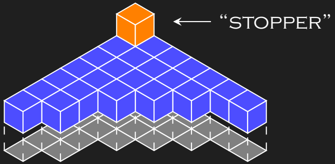
ordinary partitions — plane partitions

D6 on $\mathbb{C}^3$:  ,

$$W = \mathrm{Tr}\ B_3\,[B_1, B_2]$$

Extra D4:  ,





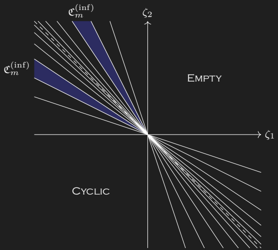
$$\Delta W = \mathrm{Tr}\ S B_3 R$$

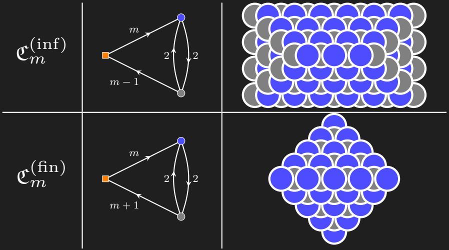
$$W = \mathrm{Tr}\ B_3\,([B_1, B_2] + RS)$$

$$\partial_{B_3} W = [B_1, B_2] + RS = 0 \quad \rightsquigarrow \quad \text{ADHM for } \mathrm{Hilb}^n(\mathbb{C}^2)$$

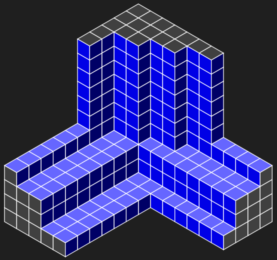
Other Examples

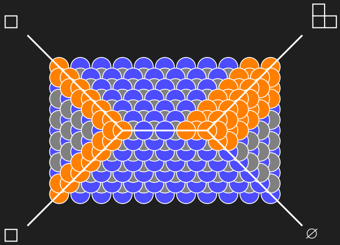
Wall-crossing: [Nagao-Nakajima'08; Aganagic-Schaeffer'10; ...]





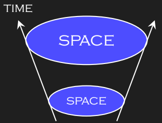
Open BPS counting:





And more exotic things...

Generalized Cohomology



Field Theory $\approx$ Mechanics (Space $\to$ Fields)

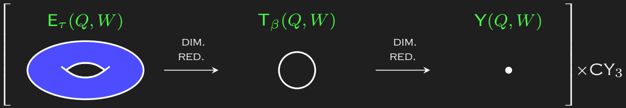
Ψ [Space $\to$ Fields]

Suppose the space is given by a torus with coordinate z:

$$V = \int d^2z \, \left(D_z\phi(z,\bar{z}) + h\,\phi(z,\bar{z})\right) \frac{\delta}{\delta\phi(z,\bar{z})}$$

Expand $\phi(z,\bar{z})$ over Fourier modes: $h \to h_{m,n} = h + n + m\tau$

$$\mathrm{Eul} \sim \prod_{\alpha} \prod_{m,n\in\mathbb{Z}} (h+n+m\tau) \sim \prod_{\alpha} \vartheta_{11}\left(h_\alpha|\tau\right)$$



D8-D6-D4-D2-D0	D7-D5-D3-D1	D6-D4-D2-D0
$\zeta(z) \leftarrow$??? $\vartheta_{11}(z\|\tau)$	$2\sinh\beta z$	z
3d $\mathcal{N}=2$	2d $\mathcal{N}=(2,2)$	1d $\mathcal{N}=4$

Yes if $\zeta(-z) = -\zeta(z)$

Generalized Cohomology II

Generalized Euler class and bond factor: [Yang-Zhao'14]

$$\widetilde{\mathrm{Eul}}_{\zeta}\left(\mathcal{N}\right)=(-1)^{\left\lfloor\sum\limits_{a:\ h_a=0}\frac{1}{2}\right\rfloor}\prod_{a:\ h_a\neq 0}\zeta(h_a),\quad \varphi^{a\Leftarrow b}(u)\equiv\frac{\prod\limits_{I\in\{a\to b\}}\zeta(u+h_I)}{\prod\limits_{J\in\{b\to a\}}\zeta(u-h_J)}$$

x – 1st Chern class $c_1^E(\mathcal{O}(1))$
Formal group law for gen. coho $E^*(\mathbb{CP}^{\infty})=E^*(\mathrm{pt})\left[[x]\right]$

$$c_1^E(\mathcal{L}\otimes\mathcal{L}')=F\left(c_1^E(\mathcal{L}),c_1^E(\mathcal{L}')\right)$$

$$\begin{array}{l}F(x,0)=x,\ F(0,y)=y,\\ F(x,y)=F(y,x),\\ F(x,F(y,z))=F(F(x,y),z);\end{array}$$

Logarithm:
$\ell_F(F(x,y))=\ell_F(x)+\ell_F(y)$

Flavor charges h behave linearly under bundle tensor multiplication, therefore:

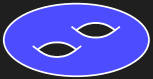
$$\boxed{\zeta(u)=\ell_F^{-1}(u)}$$

Generalized genus:

$$\zeta^{-1}(u)=u+\frac{\phi(\mathbb{CP}^2)}{3}u^3+\frac{\phi(\mathbb{CP}^4)}{5}u^5+\ldots,\quad \phi:\Omega_*^{SO}\to\mathbb{Q}$$

Other questions

How to mimic GCT in physical terms?
What about higher genus surfaces?

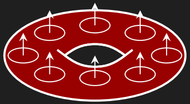
 $\rightsquigarrow$ (GCT???)

Flavor charges are promoted to Wilson lines wrapping cycles of Σ:

$$\text{Spectral parameter: } \vec{z} \in \mathcal{J}(\Sigma) \cong H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$$

$$\zeta(z) \rightsquigarrow \Theta_{\text{char???}}\left(\vec{z}\,\middle|\,\text{period matrix}\right)$$

Fluxes:



$$\Phi = \frac{1}{2\pi}\int_{\Sigma} F \in \mathbb{Z}$$

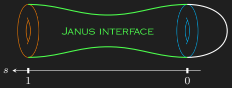
We have to extend our coordinate space:

$$(\operatorname{Re} h,\ \operatorname{Im} h,\ R\text{-charge}, \Phi) \rightsquigarrow \text{4d crystals???}$$

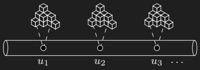
$$\Sigma \times \mathrm{CY}_3 \leftrightsquigarrow \mathrm{CY}_4???$$

INTEGRABILITY

[WORK IN PROGRESS...]



$$Z_i(s=1) = \sum_j M_{i,j}(1,0) Z_j(s=0)$$



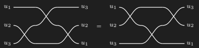
[BULLIMORE-KIM-LUKOWSKI'17]

$\mathrm{Re}\, u_2 > \mathrm{Re}\, u_1$

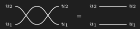
$\mathrm{Re}\, u_1 > \mathrm{Re}\, u_2$

$$R_{12}(u_{12}) = \frac{u_{12}}{u_{12}+\hbar}\mathrm{Id} + \frac{\hbar}{u_{12}+\hbar}P_{12}$$

YBE+UNITARITY:



,



TRANSFER MATRIX: $T(z) := \mathrm{Tr}' R_{0n}(z-u_n)\dots R_{01}(z-u_1)$, $T(z) : \mathcal{F}^{\otimes n} \to \mathcal{F}^{\otimes n}$

$$[T(u), T(v)] = 0$$

BAES FOR OFF-SHELL BETHE VECTORS FOLLOWS
FROM THE BETHE/GAUGE CORRESPONDENCE

Open Problems

- Wall-crossing, what about non-crystal phases?
- Calabi-Yau 4- and 5-folds (???), 6-folds(???), ...
- Non-toric Calabi-Yau n-folds
- Generalized cohomology

Thank you for your attention