Math H54 Second Midterm November 7, 2019

Name: Solution

- You have 80 minutes to complete the exam.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total	100	

1. (4 points each) Determine if each statement is TRUE or FALSE, and justify your answer.

(a) If A and B are similar matrices, then they have the same eigenvectors.

FALSE. e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$.

Then $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of A,

but not an eigenvector of B.

(b) Any diagonalizable matrix is orthogonally diagonalizable.

FALSE e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable ($\because$ eigenvalues distinct),

but not orthogonally diagonalizable since $A \neq A^T$.

(c) If A is an $n \times n$ symmetric matrix, then $\langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A\vec{w} \rangle$ for any $\vec{v}, \vec{w} \in \mathbb{R}^n$.

TRUE

$$\underbrace{\langle Av, w \rangle = \langle v, A^T w \rangle}_{\text{true in general}} \underbrace{= \langle v, Aw \rangle}_{A:\text{ symmetric}}$$

(d) If A is an $n \times n$ orthogonal matrix, then $\langle A\vec{v}, A\vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle$ for any $\vec{v}, \vec{w} \in \mathbb{R}^n$.

TRUE

$$\langle Av, Aw \rangle = \langle v, A^T A w \rangle \underbrace{=}_{A^T A = I} \langle v, w \rangle$$

(e) $||\vec{v} + \vec{w}||^2 + ||\vec{v} - \vec{w}||^2 = 2||\vec{v}||^2 + 2||\vec{w}||^2$ holds for any $\vec{v}, \vec{w}$ in any inner product space.

TRUE

$$\begin{aligned} \|v+w\|^2 + \|v-w\|^2 &= \langle v+w, v+w \rangle + \langle v-w, v-w \rangle \\ &= (\|v\|^2 + \|w\|^2 + 2\langle v, w \rangle) + (\|v\|^2 + \|w\|^2 - 2\langle v, w \rangle) \\ &= 2(\|v\|^2 + \|w\|^2) \end{aligned}$$

2. (10 points) Let

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 2 \end{pmatrix}.$$

(a) Find all eigenvalues of A.

(Part (b) and (c) are on the next page.)

Char. poly: $\det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & 1 & 0 \\ -1 & 1-\lambda & 0 \\ 2 & -2 & 2-\lambda \end{pmatrix}$

$= (2-\lambda)\left((-1-\lambda)(1-\lambda)+1\right)$

$= (2-\lambda)\,\lambda^2.$

$\Rightarrow$ eigenvalues are: 0 and 2

$\uparrow$

(w/ mult. 2).

(b) Find a basis for the eigenspace associated to each eigenvalue.

Eigenspace of 0: $\mathrm{Nul}(A-0I) = \mathrm{Nul}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 2 \end{pmatrix}$

$= \mathrm{Nul}\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$

Eigenspace of 2: $\mathrm{Nul}(A-2I) = \mathrm{Nul}\begin{pmatrix} -3 & 1 & 0 \\ -1 & -1 & 0 \\ 2 & -2 & 0 \end{pmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ (you don't need to compute P^{-1}). If not, explain the reason.

No.

Because

$\dim \mathrm{Nul}(A - 0I) = 1 < 2 =$ Multiplicity of 0.

3. (10 points) Consider the quadratic form on $\mathbb{R}^3$

$$Q(\vec{x}) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.$$

(a) Find a symmetric 3×3 matrix A such that $Q(\vec{x}) = \vec{x}^T A \vec{x}$ for any $\vec{x} \in \mathbb{R}^3$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. (Part (c) is on the next page.)

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^3 + 2 - 3(1-\lambda)$$

$$= -\lambda^2(\lambda - 3)$$

Eigenvalues are 0 and 3.

<u>Eigenspace of 0:</u> $\mathrm{Nul}(A - 0I) = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$.

Find orthonormal basis: via Gram-Schmidt:

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}; \quad w_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}$$

✓

$\rightsquigarrow$ normalize $V_2 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$

<u>Eigenspace 3:</u> $\mathrm{Nul}(A - 3I) = \mathrm{Nul}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

$\rightsquigarrow$ normalize $V_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

Take $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(c) Is Q positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite? Give a brief explanation.

Positive semi-definitte.

- has eigenvalues > 0 and $= 0$, but no < 0. $\square$

4. (10 points) Find the QR decomposition of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In other words, find a matrix Q with orthonormal columns and an upper triangular square matrix R with positive entries on its diagonal, so that $A = QR$.

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix},$$

$$w_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \rightsquigarrow \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$w_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\rangle \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ -1/2 \end{bmatrix} \rightsquigarrow \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Take $Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, and

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

5. (10 points) Consider the inner product space $C([0,1])$ of continuous functions on $[0,1]$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

(a) For which values of real constants α, β, γ are the polynomials 1, $x+\alpha$, and $x^2+\beta x+\gamma$ pairwise orthogonal? (Recall that f and g are orthogonal if $\langle f, g\rangle = 0$.) (Recall that the antiderivative of x^n is $\frac{x^{n+1}}{n+1}$.)

(Part (b) is on the next page.)

$$\langle 1, x+\alpha\rangle = \int_0^1 x+\alpha \, dx = \frac{1}{2} + \alpha \quad \Rightarrow \quad \boxed{\alpha = -\tfrac{1}{2}.}$$

$$\langle 1, x^2+\beta x+\gamma\rangle = \int_0^1 x^2+\beta x+\gamma \, dx = \frac{1}{3} + \frac{1}{2}\beta + \gamma \quad = 0$$

$$\langle x-\tfrac{1}{2}, x^2+\beta x+\gamma\rangle = \int_0^1 x^3 + (\beta-\tfrac{1}{2})x^2 + (\gamma - \tfrac{\beta}{2})x - \tfrac{\gamma}{2} \, dx$$

$$= \frac{1}{4} + \frac{1}{3}(\beta - \tfrac{1}{2}) + \frac{1}{2}(\gamma - \tfrac{\beta}{2}) - \cancel{\tfrac{\gamma}{2}} \quad = 0$$

$$= \frac{1}{12}\beta + \frac{1}{12} \quad \Rightarrow \quad \boxed{\begin{aligned} &\beta = -1 \\ \Rightarrow\ &\gamma = \tfrac{1}{6} \end{aligned}}$$

(b) Describe how you would construct an orthonormal basis for the subspace $\mathbb{P}_2$ of polynomials of degree at most two, using the result of Part (a). You don't need to do calculations for this problem.

$$\left\{ \frac{1}{\langle 1,1\rangle^{1/2}},\ \frac{x-\frac{1}{2}}{\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle^{1/2}},\ \frac{x^2-x+\frac{1}{6}}{\langle x^2-x+\frac{1}{6}, x^2-x+\frac{1}{6}\rangle^{1/2}} \right\}$$

(normalize to get unit vectors)

6. (10 points) Let $(V, \langle -, - \rangle)$ be an inner product space, and let $T : V \to V$ be a linear transformation. Suppose that $||T(\vec{x})|| = ||\vec{x}||$ for every $\vec{x} \in V$. Prove that

$$\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$$

holds for any $\vec{x}, \vec{y} \in V$.
(Hint: Consider $||T(\vec{x}+\vec{y})||^2 = ||\vec{x}+\vec{y}||^2$.)

$$\begin{array}{cc} \| & \| \\ \langle Tx+Ty, Tx+Ty \rangle & \langle x+y, x+y \rangle \\ \| & \| \\ \|T(x)\|^2 + \|T(y)\|^2 & \|x\|^2 + \|y\|^2 \\ + 2\langle T(x), T(y) \rangle & + 2\langle x, y \rangle \end{array}$$

Since $\|T(x)\| = \|x\| \ \forall x$

$\Rightarrow \langle T(x), T(y) \rangle = \langle x, y \rangle \ \forall x, y. \ \square$

7. (10 points) Show that the quadratic form $Q(\vec{x}) = \vec{x}^T M \vec{x}$ associated to a symmetric matrix M is positive definitie if and only if $M = A^T A$ for some A with linearly independent columns.
(Hint 1: For the '$\Longleftarrow$' direction, consider $\langle A\vec{x}, A\vec{x}\rangle$.)
(Hint 2: For the '$\Longrightarrow$' direction, consider the 'square root' of a diagonal matrix.)

"$\Longleftarrow$" $Q(x) = x^T M x = x^T A^T A x = \langle Ax, Ax \rangle$.

A has l.i. columns $\Rightarrow$ If $x \neq 0$ then $Ax \neq 0$.

So if $x \neq 0$, then $Ax \neq 0$, and

$$Q(x) = \langle Ax, Ax \rangle > 0.$$

Hence Q is positive definite.

"$\Longrightarrow$" $M = PDP^T$, where the diagonal entries of D are > 0.

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

Consider $\tilde{D} := \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$.

Define $A = \tilde{D} P^T$. Then

- A has l.i. columns since $\tilde{D}$ and P^T are invertible,
- $A^T A = P \tilde{D}^T \tilde{D} P^T = PDP^T = M$. $\square$

8. We say an $n \times n$ matrix P is a *projection matrix* if it satisfies $P^2 = P$. In this problem, you will prove that projection matrices are diagonalizable.

(a) (5 points) Show that the column space of P coincides with the eigenspace of P associated with eigenvalue 1, i.e. $\mathrm{Col}(P) = \mathrm{Nul}(P - I)$.

(Part (b) is on the next page.)

$x \in \mathrm{Col}(P) \iff \exists\, y$ s.t. $Py = x$

$\Rightarrow x = Py = P^2 y = Px$

$\Rightarrow x \in \mathrm{Nul}(P - I)$.

$x \in \mathrm{Nul}(P-I) \iff Px = x$

$\Rightarrow$ $x \in \mathrm{Col}(P)$. $\square$

(b) (5 points) Show that P is diagonalizable via the rank theorem.
(Rank theorem: $\dim \mathrm{Col}(P) + \dim \mathrm{Nul}(P) = n$.)
(Hint: Suppose $0 \neq \vec{v} \in \mathrm{Nul}(P)$. Is v an eigenvector of P?)

$\mathrm{Nul}(P) =$ eigenspace of P for eigenvalue 0.

$\mathrm{Col}(P) =$ eigenspace of P for eigenvalue 1 by part (a).

Since $\dim \mathrm{Col}(P) + \dim \mathrm{Nul}(P) = n$.

$\Rightarrow$ The union of a basis of $\mathrm{Nul}(P)$ and a basis of $\mathrm{Col}(P)$ forms an eigenbasis of P

$\Rightarrow$ P is diagonalizable. $\square$

9. Let $M_{2\times 2}(\mathbb{R})$ be the set of all 2×2 real matrices. It is a vector space with the standard matrix addition and scalar multiplication.

(a) (5 points) Consider the function $\langle -,-\rangle : M_{2\times 2}(\mathbb{R}) \times M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by:

$$\langle A, B\rangle := \operatorname{tr}(AB^T).$$

Here $A, B \in M_{2\times 2}(\mathbb{R})$ and tr denotes the trace of a matrix (sum of diagonal entries).

Prove that the function $\langle -,-\rangle$ defines an inner product on the vector space $M_{2\times 2}(\mathbb{R})$.

(Part (b) is on the next page.)

- $\underline{\langle A, B\rangle = \langle B, A\rangle:}$

$$\langle A, B\rangle = \operatorname{tr}(AB^T) = \operatorname{tr}((AB^T)^T)$$
$$\langle B, A\rangle = \operatorname{tr}(BA^T)$$

- $\underline{\langle cA_1 + A_2, B\rangle = c\langle A_1, B\rangle + \langle A_2, B\rangle:}$

$$\langle cA_1 + A_2, B\rangle = \operatorname{tr}((cA_1 + A_2)B^T)$$
$$= c\operatorname{tr}(A_1B^T) + \operatorname{tr}(A_2B^T) = c\langle A_1, B\rangle + \langle A_2, B\rangle$$

- $\underline{\langle A, A\rangle > 0 \text{ unless } A = 0:}$

$$\langle A, A\rangle = \operatorname{tr}(AA^T) = \operatorname{tr}\begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix}$$
$$= a^2+b^2+c^2+d^2.$$

$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$\square$

(b) (5 points) Construct an orthonormal basis (with respect to this inner product) for the subspace of $M_{2\times 2}(\mathbb{R})$ spanned by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$$\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle = \mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) = 1. \quad \rightarrow \text{unit vector.}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \mathrm{tr}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\rightsquigarrow$ normalize $\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}$.

An orthonormal basis: $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \right\}$. $\square$