Math H54 Final Exam December 17, 2019

Name: Solution

- You have 170 minutes to complete the exam (3:10pm – 6:00pm).
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using. Exception: if you are asked to prove a theorem, you are not allowed to use it!
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total		

1. (2 points each) True/False questions. You don't need to justify your answers. No partial credit.

(a) Let A, B be two real 3×3 matrices. If $AB = 0$ and $\det(A) = 3$, then $B = 0$.

$\boxed{\text{True}}$ $\det A = 3 \Rightarrow A$ is invertible

$\Rightarrow B = A^{-1} 0 = 0.$

(b) There exists a real 2×2 matrix $A \neq I$ such that $A^5 = I$.

Hint: What's the geometric interpretation of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$?

$\boxed{\text{True}}$ $\begin{bmatrix} \cos\frac{2\pi}{5} & -\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{bmatrix}^5 = I.$

(c) Let $T : V \to V$ be a linear transformation, and v_1, v_2, v_3 be three vectors in V. If $\{T(v_1), T(v_2), T(v_3)\}$ is a linearly dependent set, then $\{v_1, v_2, v_3\}$ is a linearly dependent set.

$\boxed{\text{False}}$ e.g. $V = \mathbb{R}^3$, $\{v_1, v_2, v_3\} = \{e_1, e_2, e_3\}$.

$T: \mathbb{R}^3 \to \mathbb{R}^3$ zero map.

$v \mapsto 0$

(d) If a real 2×2 matrix A satisfies $\langle \vec{v}, A\vec{v} \rangle = 0$ for any $v \in \mathbb{R}^2$. Then $A = 0$.

$\boxed{\text{False}}$ $A = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ satisfies $\langle v, Av \rangle = 0$ $\forall v \in \mathbb{R}^2$.

(e) Let A be a real $n \times n$ matrix. Let $\{\vec{x}_1(t), \ldots, \vec{x}_n(t)\}$ be a set of solutions of the homogeneous equation $\vec{x}\,'(t) = A\vec{x}(t)$. Suppose that $\{\vec{x}_1(1), \ldots, \vec{x}_n(1)\}$ is a linearly dependent set of vectors. Then $\{\vec{x}_1(t), \ldots, \vec{x}_n(t)\}$ is a linearly dependent set for any $t \in \mathbb{R}$.

$\boxed{\text{True}}$ $c_1 \vec{x}_1(1) + \cdots + c_n \vec{x}_n(1) = \vec{0}$ for some c_i not all zero.

Then both $c_1 \vec{x}_1(t) + \cdots + c_n \vec{x}_n(t)$ and 0 are sol'ns of:

$$\begin{cases} \vec{x}\,'(t) = A\vec{x}(t) \\ \vec{x}(1) = \vec{0}. \end{cases}$$

By uniqueness thm., $c_1 \vec{x}_1(t) + \cdots + c_n \vec{x}_n(t) = \vec{0}$ $\forall t \in \mathbb{R}$

2. (10 points; 3 parts) Let P_2 be the vector space consists of polynomials of degree less than or equal to 2. Define the linear transformation $T : P_2 \to \mathbb{R}^3$ by

$$T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}.$$

(a) Show that T is a linear transformation.

$$T(P_1+P_2) = \begin{bmatrix} (P_1+P_2)(-1) \\ (P_1+P_2)(0) \\ (P_1+P_2)(1) \end{bmatrix} = \begin{bmatrix} P_1(-1)+P_2(-1) \\ P_1(0)+P_2(0) \\ P_1(1)+P_2(1) \end{bmatrix} = T(P_1)+T(P_2)$$

Similarly, $T(cP) = cT(P)$.

(b) Is T injective? Surjective? Prove your answer.

$P_2 = \text{Span}\{1, x, x^2\}$ a basis.

$$T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T(x) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad T(x^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So T can be represented by $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ w.r.t. this basis

The determinant of this matrix is $2 (\neq 0)$, hence T is invertible.

So it's both injective and surjective.

(c) Find a polynomial p such that $T(p) = \begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix}$.

Hint: You should be solving a linear system with 3 equations and 3 unknowns.

Solve: $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix}$

$\Rightarrow \quad a = 3, \quad b = 4, \quad c = 5$

$\Rightarrow \quad p(x) = 3 + 4x + 5x^2. \qquad \square$

3. Let A be a real symmetric $n \times n$ matrix. Suppose that v and w are eigenvectors of A that correspond to distinct eigenvalues. Show that v and w are orthogonal (with respect to the standard inner product on $\mathbb{R}^n$).

Hint: Consider $\langle Av, w \rangle$.

Let $Av = \lambda_1 v$ and $Aw = \lambda_2 w$. , $\lambda_1 \neq \lambda_2$

Then

$$\lambda_1 \langle v, w \rangle = \langle Av, w \rangle = \langle v, A^T w \rangle$$

$$= \langle v, Aw \rangle \quad (A : \text{symmetric})$$

$$= \lambda_2 \langle v, w \rangle$$

$\Rightarrow \quad \langle v, w \rangle = 0. \quad \square$

4. (10 points) Let V be a finite-dimensional inner product space, and $W \subset V$ be a subspace. Let

$$W^\perp = \{v \in V : \langle v, w\rangle = 0 \text{ for any } w \in W\}$$

be the orthogonal complement of W. Prove that

$$\dim W + \dim W^\perp = \dim V.$$

Hint: You can prove it by showing that if $\{a_1, \ldots, a_n\}$ is a basis of W and $\{b_1, \ldots, b_m\}$ is a basis of $W^\perp$, then $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ is a basis of V.

Hint: You can also prove it via the rank-nullity theorem. (You don't have to prove the rank-nullity theorem.)

pf 1: Let $\{a_1, \cdots, a_n\}$ be basis of W and $\{b_1, \cdots, b_m\}$ basis of $W^\perp$.

Claim: $\{a_1, \cdots, a_n, b_1, \cdots, b_m\}$ is a basis of V.

- It spans V: any $v \in V$ has an orthogonal decomposition $v = w + z$, where $w \in W$ and $z \in W^\perp$.
- Linearly indep.: Suppose $\exists\, c_1, \cdots, c_n, d_1 \cdots, d_m \in \mathbb{R}$ s.t.

$$c_1 a_1 + \cdots + c_n a_n = d_1 b_1 + \cdots + d_m b_m = v.$$

Then $v \in W \cap W^\perp \Rightarrow \langle v, v\rangle = 0 \Rightarrow v = 0$.

Since $\{a_1, \cdots, a_n\}$ $\{b_1 \cdots b_m\}$ are l.i., $c_1 = \cdots = c_n = d_1 = \cdots = 0$. $\square$

pf 2: Consider $T : V \longrightarrow W$ given by the orthogonal projection.

It's clear that $\text{Image}(T) = W$ since $T(w) = w \ \forall w \in W$,

and $\text{kernel}(T) = W^\perp$.

Rank-Nullity thm $\Rightarrow$ $\dim W + \dim W^\perp = \dim V$. $\square$

5. Let $\mathcal{U}$ be the set of 5×5 matrices with the property that the sum of each row is zero and the sum of each column is zero.

(a) (3 points) Explain why $\mathcal{U}$ forms a vector space.

Check that the sum and scalar multiple of such matrices have the same property.

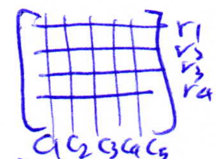
(b) (7 points) Find the dimension of $\mathcal{U}$ and prove it.

Hint: You may use the rank-nullity theorem, i.e. if $T : V \to W$ is a linear transformation, then $\dim V = \dim \mathrm{Kernel}(T) + \dim \mathrm{Image}(T)$.

Hint: 15 is not the correct answer.

$V := M_{5\times5}(\mathbb{R}) = \{\text{set of all real } 5\times5 \text{ matrices}\} \cong \mathbb{R}^{25}$

$W := \mathbb{R}^9$.



Define

$$\begin{aligned} T : V &\longrightarrow W \\ A &\longmapsto (c_1, c_2, \dots, c_5, r_1, \dots, r_4). \end{aligned}$$

where $c_1, \dots, c_5$ is the sum of each column of A,

and $r_1, \dots, r_4$ is the sum of the first 4-rows of A.

- T is a linear transformation (easy to check).
- T is surjective. (obviously we can get any r_1, r_2, r_3, r_4, and we can choose the last row to get any $c_1, c_2, \dots, c_5$)
- $\mathrm{Kernel}(T) = \{$the matrix w/ the desire property, i.e. sum of each row/column $= 0\} = \mathcal{U}$.

Hence $\dim \mathcal{U} = \dim V - \dim W = 25 - 9 = \boxed{16}$. $\square$

6. A square matrix A is called *unipotent* if $(A-I)^n = 0$ for some $n \in \mathbb{N}$.

(a) (5 points) Prove that if λ is an eigenvalue of an unipotent matrix, then $\lambda = 1$.
Hint: Suppose v is an eigenvector with eigenvalue λ. What's $(A-I)v$?

$$Av = \lambda v, \quad v \neq 0. \quad \Rightarrow \quad (A-I)v = (\lambda-1)v.$$

$$\Rightarrow \quad \underbrace{(A-I)^n v}_{=0} = (\lambda-1)^n v$$

$$\Rightarrow \quad \lambda = 1. \quad \square$$

(b) (5 points) Prove that if A is both unipotent and diagonalizable, then $A = I$.

Since 1 is the only eigenvalue, so if A is diagonalizable, then

$$A = P \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} P^{-1} = P I P^{-1} = I. \quad \square$$

7. (a) (5 points) Let A be a real $n \times n$ matrix, and let $\{\lambda_1, \ldots, \lambda_k\}$ be the set of distinct eigenvalues of A^2. Assume that λ_i is real and positive for any $1 \leq i \leq k$. Let S be the set of eigenvalues of A. Prove that:

(i) S is a subset of $\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, \sqrt{\lambda_2}, -\sqrt{\lambda_2}, \ldots, \sqrt{\lambda_k}, -\sqrt{\lambda_k}\}$;

(ii) For each $1 \leq i \leq k$, S contains at least one of $\pm\sqrt{\lambda_i}$.

Hint: Consider $(A^2 - \lambda I) = (A - \sqrt{\lambda}I)(A + \sqrt{\lambda}I)$ for Part (ii).

(i) Let $\mu \in S$, $Av = \mu v$, $v \neq 0$. $\Rightarrow$ $A^2 v = \mu^2 v$.

$\Rightarrow$ $\mu^2 = \lambda_i > 0$ for some $1 \leq i \leq k$.

$\Rightarrow$ $\mu = \sqrt{\lambda_i}$ or $-\sqrt{\lambda_i}$ for some i. $\square$

(ii) For each $1 \leq i \leq k$, we have

$$0 = \det(A^2 - \lambda_i I) = \det(A - \sqrt{\lambda_i} I) \det(A + \sqrt{\lambda_i} I)$$

$\Rightarrow$ at least one of $\pm\sqrt{\lambda_i}$ is an eigenvalue of A. $\square$

(b) (5 points) Find a 3×3 matrix A such that

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 9 \end{bmatrix}$$

How many such matrices are there?

Hint: Show that A is diagonalizable and has the same eigenspaces as A^2.

By Part (a), the set of eigenvalues of A contains exactly one of: $\{\pm 1\}, \{\pm 2\}, \{\pm 3\}$.

λ_1 - either 1 or -1
λ_2 - either 2 or -2
λ_3 - either 3 or -3.

eigenvecs: v_1, v_2, v_3.

$A v_1 = \lambda_1 v_1 \Rightarrow A^2 v_1 = \lambda_1^2 v_1 = v_1 \Rightarrow$ eigenspace of A for λ_1 = eigenspace of A^2 for 1

$A^2 v_2 = 4 v_2$

$A^2 v_3 = 9 v_3$. $\longrightarrow$ same for λ_2, λ_3.

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 20 \end{bmatrix}^{-1}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} \pm 1 & & \\ & \pm 2 & \\ & & \pm 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 20 \end{bmatrix}^{-1}$$

There are $2^3 = 8$ such matrices. $\square$

8. (10 points) Find a function $y(t)$ satisfying

$$y''(t) - 2y'(t) + 2y(t) = 2t - 2; \quad y(0) = 4, \quad y'(0) = 2.$$

$r^2 - 2r + 2 = 0 \Rightarrow r = 1 \pm i.$

$\Rightarrow$ homog. sol'n: $y(t) = c_1 e^t \cos t + c_2 e^t \sin t$

Observe that $y(t) = t$ solves $y'' - 2y' + 2y = 2t - 2.$

$4 = y(0) = 0 + c_1 \Rightarrow c_1 = 4.$

$y'(t) = 1 + c_1(e^t \cos t - e^t \sin t) + c_2(e^t \sin t + e^t \cos t)$

$2 = y'(0) = 1 + c_1 + c_2 \Rightarrow c_2 = -3$

$\Rightarrow y(t) = t + 4e^t \cos t - 3e^t \sin t \quad \square$

9. In this problem, you will find three functions $y_1(t), y_2(t), y_3(t)$ satisfying

$$y_1'(t) = 4y_2(t), \quad y_2'(t) = 4y_1(t) + 3y_3(t), \quad y_3'(t) = 3y_2(t)$$

and the initial conditions

$$y_1(0) = 0, \quad y_2(0) = -10, \quad y_3(0) = 0.$$

(a) (2 points) Write the differential equations as a matrix equation $\vec{y}\,'(t) = A\vec{y}(t)$ for some matrix A, and write the initial conditions as a vector equation $\vec{y}(0) = \vec{b}$ for some vector $\vec{b}$.

$$\vec{y}\,'(t) = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} \vec{y}(t) \quad , \quad \vec{y}(0) = \begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix}$$

(b) (3 points) Find a diagonalization of A.

$$\det \begin{bmatrix} -\lambda & 4 & 0 \\ 4 & -\lambda & 3 \\ 0 & 3 & -\lambda \end{bmatrix} = -\lambda^3 + 25\lambda \Rightarrow \lambda = -5, 0, 5.$$

$\underline{\lambda = -5}$: $\text{Nul} \begin{bmatrix} 5 & 4 & 0 \\ 4 & 5 & 3 \\ 0 & 3 & 5 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 4 \\ -5 \\ 3 \end{bmatrix} \right\}.$

$\underline{\lambda = 0}$: $\text{Nul} \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} \right\}$

$\underline{\lambda = 5}$: $\text{Nul} \begin{bmatrix} -5 & 4 & 0 \\ 4 & -5 & 3 \\ 0 & 3 & -5 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \right\}.$

$$A = \begin{bmatrix} 4 & -3 & 4 \\ -5 & 0 & 5 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} -5 & & \\ & 0 & \\ & & 5 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}^{-1}.$$

(c) (3 points) Find a fundamental matrix of $\vec{y}\,'(t) = A\vec{y}(t)$.

$$\begin{bmatrix} 4e^{-5t} & -3 & 4e^{5t} \\ -5e^{-5t} & 0 & 5e^{5t} \\ 3e^{-5t} & 4 & 3e^{5t} \end{bmatrix}.$$

(d) (2 points) Finish the problem, i.e. find the functions $y_1(t), y_2(t), y_3(t)$.

Find $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ s.t. $\begin{bmatrix} 4 & -3 & 4 \\ -5 & 0 & 5 \\ 3 & 4 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\Rightarrow \vec{y}(t) = \begin{bmatrix} 4e^{-5t} - 4e^{5t} \\ -5e^{-5t} - 5e^{5t} \\ 3e^{-5t} - 3e^{5t} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$ $\square$

10. (10 points; 2 parts)

(a) Find the Fourier sine series of the function $f(x) = x(x - \pi)$ on $0 < x < \pi$.
Hint: Integration by parts, with patience.

The Fourier sine series is $\sum_{n=1}^{\infty} b_n \sin(nx)$, where

$$b_n = \frac{2}{\pi}\int_0^{\pi} (x^2 - \pi x)\sin(nx)\,dx = \frac{2}{\pi}\left(\cancel{(x^2-\pi x)\frac{-\cos(nx)}{n}\Big|_0^{\pi}} - \int_0^{\pi} (2x-\pi)\frac{-\cos(nx)}{n}\,dx \right)$$

$$= \quad + \frac{2}{n\pi}\left(\cancel{(2x-\pi)\frac{\sin(nx)}{n}\Big|_0^{\pi}} - \int_0^{\pi} 2\,\frac{\sin(nx)}{n}\,dx \right)$$

$$= -\frac{4}{n^2\pi}\,\frac{-\cos nx}{n}\Big|_0^{\pi} = \frac{4\left((-1)^n - 1\right)}{n^3\pi}$$

(b) Find a function $u(x,t)$ satisfying

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$
$$u(0,t) = u(\pi,t) = 0, \quad t > 0,$$
$$u(x,0) = x(x-\pi), \quad 0 < x < \pi.$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4\left((-1)^n - 1\right)}{n^3 \pi} \sin(nx)\, e^{-3n^2 t}$$

$$= \sum_{k=0}^{\infty} \frac{-8}{(2k+1)^3 \pi} \sin\left((2k+1)x\right) e^{-3(2k+1)^2 t}$$