# MATHEMATICS FROM EXAMPLES, SPRING 2023

INSTRUCTOR: YU-WEI FAN

*Course Description.* Examples in mathematics are like phenomena in physics. They play a vital role in the historical development of mathematics and are the driving force behind profound mathematical concepts and methods. Important theorems in modern mathematics often come from the understanding and research of some basic examples. The goal of this course is to provide the motivation and intuition behind abstract mathematical concepts by introducing some interesting examples.

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