Warmup question: If $a_n > 0\ \forall n$, and $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$

show that $a \geq 0$.

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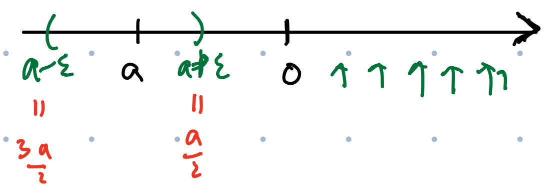
pf Assume the contrary that $a < 0$.



Since $\lim a_n = a$,

$\forall \varepsilon > 0,\ \exists N > 0$

s.t. $n > N \Rightarrow |a_n - a| < \varepsilon$

We can choose $\varepsilon > 0$ small enough, s.t. $a + \varepsilon < 0$. (e.g. $\varepsilon = \frac{-a}{2}$)

By the def^n of limit, we know that $\exists N > 0$

s.t. $n > N \Rightarrow |a_n - a| < \varepsilon$

$\Rightarrow a_n < a + \varepsilon = \frac{a}{2} < 0$

Contradiction. $\square$

Q: Is it true that if $a_n > 0\ \forall n$, and $\lim a_n = a$,

then $a > 0$?

No e.g. $(a_n = \frac{1}{n})$. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ $\lim a_n = 0$.

Def A seq. (a_n) is increasing if $a_n \leq a_{n+1}\ \forall n$. $(1, 2, 3, \dots)$

— — — decreasing if $a_n \geq a_{n+1}\ \forall n$ $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$

Both are called monotone $(-1, -2, \dots)$

Thm Any bounded monotone seq. converges

Rmk: • "bounded" can't be removed from the assumption

• Recall from last time: conv. $\Rightarrow$ bdd, but bdd $\not\Rightarrow$ conv.

Rmk: By the def^n of convergent, if we want to show a seq. is conv. normally, we have to first know/guess what the limit is.

pf Let's say (a_n) is a bounded increasing seq.

$(0.9, 0.99, 0.999, 0.9999, \cdots)$

$\lim = 1$

Consider the set $A = \{a_n \mid n \in \mathbb{N}\}$

$= \{a_1, a_2, a_3, \cdots\} \subseteq \mathbb{R}$

Since (a_n) is bdd, A is bdd, (in particular, A is bdd above)

By the least upper bound property of $\mathbb{R}$, the supremum of A exists in $\mathbb{R}$.

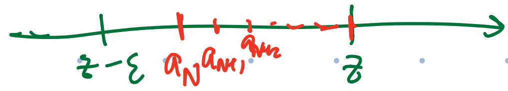
Denote $z := \sup A$

Claim: $z = \lim_{n\to\infty} a_n$

pf: In HW1 #4, we showed that if $z = \sup S$ then $\forall \varepsilon > 0$, $\exists a \in S$ s.t. $z - \varepsilon < a \le z$.

In our case, $S = \{a_1, a_2, a_3, \cdots\}$, $z = \sup S$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $z - \varepsilon < a_N \le z$



Then, since the seq. is increasing,

$\forall n > N$, we have $a_n \ge a_N > z - \varepsilon$

We also have $a_n \le z$ $\forall n$ since z is an upper bdd of S

$\Longrightarrow$ $\forall n > N$, $z - \varepsilon < a_n \le z$

$\Longrightarrow$ $\lim_{n\to\infty} a_n = z$. $\square$

Some limit theorems If $\lim a_n = a$, $\lim b_n = b$, then

1) $\forall r \in \mathbb{R}$, $\lim (r a_n) = ra$
2) $\lim (a_n + b_n) = a + b$.
3) $\lim (a_n b_n) = ab$
4) If $b_n \neq 0$ $\forall n$, $b \neq 0$, then $\lim \frac{a_n}{b_n} = \frac{a}{b}$

We'll prove these next time.

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e.g. $a_1 = 1$, $a_{n+1} = \sqrt{a_n + 1}$ $\forall n$.

We'll show the seq. is conv. and find the limit.

$a_1 = 1$, $a_2 = \sqrt{1+1} = \underbrace{\sqrt{2}}_{1.414}$, $a_3 = \underbrace{\sqrt{\sqrt{2}+1}}_{1.554}$, $a_4 \sim 1.598$, ...

$\boxed{\text{Guess}}$: (a_n) is increasing & bounded.

~~pf~~:

Observation $a_n \overset{??}{\leq} a_{n+1} = \sqrt{a_n + 1} \iff a_n^2 \overset{??}{\leq} a_n + 1$

$\Updownarrow$

$\underbrace{\frac{1-\sqrt{5}}{2}}_{\text{negative.}} \underbrace{\leq}_{\substack{\text{always} \\ \text{true}}} a_n \overset{??}{\leq} \frac{1+\sqrt{5}}{2}$

Since we know $a_n > 0$

Claim: $\forall n$, $a_n \leq a_{n+1} \leq \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$

pf By induction on n, $n=1$ is true.

Let's assume $\boxed{a_{n-1} \leq a_n \leq \frac{1+\sqrt{5}}{2}}$, inductive hypothesis.

we'd like to show: $\boxed{a_n \leq a_{n+1} \leq \frac{1+\sqrt{5}}{2}}$

true

since $a_n \le \frac{1+\sqrt5}{2}$

$$\sqrt{a_n+1} = a_{n+1} \le \frac{1+\sqrt5}{2}$$

$$\Updownarrow$$

$$a_n+1 \le \left(\frac{1+\sqrt5}{2}\right)^2 = \frac{6+2\sqrt5}{4} = \frac{3+\sqrt5}{2}$$

$$\Updownarrow$$

$$\underline{a_n \le \frac{1+\sqrt5}{2}}$$

Claim $\Rightarrow$ (a_n) is increasing & bdd.

$\xRightarrow{\text{Thm}}$ (a_n) converges.

How do we find the limit?

$$a_{n+1} = \sqrt{a_n+1}, \qquad \boxed{a_{n+1}^2 = a_n + 1} \quad \forall n$$

Say $\boxed{\lim_{n\to\infty} a_n = a}$ (we've already proved the limit exists)

$\lim(a_n+1) = a+1$ by the limit thm.

$\quad\|$

$\lim(a_{n+1}^2)$

$\quad\|$

$\lim(a_{n+1}a_{n+1}) = a^2$

$\Rightarrow$ the limit $a \in \mathbb{R}$ satisfies $a^2 = a + 1$.

$$a = \frac{1+\sqrt{5}}{2} \quad \text{or} \quad \cancel{\frac{1-\sqrt{5}}{2}}$$

$$\Rightarrow \lim a_n = \frac{1+\sqrt{5}}{2} \qquad \square$$

Ex: $a_1 = 1$ (continued fractions)

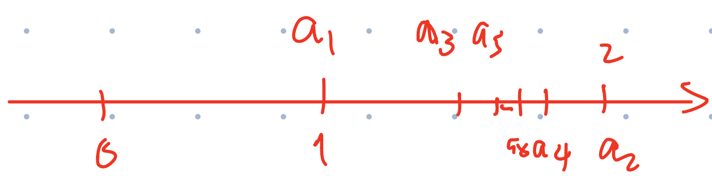
$$a_2 = 1 + \frac{1}{1} = 2$$

$$a_3 = 1 + \frac{1}{1+\frac{1}{1}} = 1 + \frac{1}{2} = 1.5$$

$$a_4 = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}} = 1 + \frac{1}{\frac{3}{2}} = \frac{5}{3} = 1.666\cdots$$

$\vdots$

$$a_{n+1} = 1 + \frac{1}{a_n}$$



You can prove that $\lim a_n = \frac{1+\sqrt{5}}{2}$

Ex: (a_n)

$(a_{2n-1}) = (a_1, a_3, a_5, \cdots)$

$(a_{2n}) = (a_2, a_4, a_6, \cdots)$

If $\lim a_{2n-1} = \lim a_{2n} = a$, then $\lim a_n = a$.

Important examples of limits that you should know:
Theorem 9.7, (Ross)
(try to prove them yourselves before looking at the proofs)

Some limit theorems If $\lim a_n = a$, $\lim b_n = b$, then
1) $\forall r \in \mathbb{R}$, $\lim (r a_n) = ra$
2) $\lim (a_n + b_n) = a + b$.
3) $\lim (a_n b_n) = ab$
4) If $b_n \neq 0 \ \forall n$, $b \neq 0$, then $\lim \frac{a_n}{b_n} = \frac{a}{b}$

pf of 1): $r \neq 0$
Want to show: if $\boxed{\lim a_n = a}$, then $\boxed{\lim r a_n = ra}$
$\forall \varepsilon > 0$, $\exists N > 0$
s.t. $n > N \Rightarrow \boxed{|r a_n - ra| < \varepsilon}$
$||$
$|r||a_n - a|$
$\Updownarrow$
$|a_n - a| < \frac{\varepsilon}{|r|}$

$\forall \tilde{\varepsilon} > 0 \ \exists \tilde{N} > 0$
s.t. $n > \tilde{N} \Rightarrow |a_n - a| < \tilde{\varepsilon}$

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pf: Since $\lim a_n = a$, $\forall \varepsilon > 0$, Consider the positive number $\frac{\varepsilon}{|r|} > 0$.
$\exists N > 0$
s.t. $n > N \Longrightarrow |a_n - a| < \frac{\varepsilon}{|r|}$
$\Downarrow$
$|r a_n - ra| < \varepsilon$

$\Rightarrow \lim ra_n = ra.$ $\square$

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pf of 2) $\lim a_n = a$, $\lim b_n = b$, want: $\lim (a_n+b_n) = a+b$

Obs: want $\forall \varepsilon>0$, $\exists N>0$

s.t. $n>N \Rightarrow \boxed{|(a_n+b_n)-(a+b)| < \varepsilon}$

$$|(a_n+b_n)-(a+b)| \leq |a_n-a| + |b_n-b| \quad < \varepsilon$$

If we have $|a_n-a| < \frac{\varepsilon}{2}$, $|b_n-b| < \frac{\varepsilon}{2}$, then

pf: $\boxed{\forall \varepsilon>0}$ Since $\lim a_n = a$, (consider the positive number $\frac{\varepsilon}{2} > 0$)

$\exists N_1 > 0$ s.t. $n > N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$

Since $\lim b_n = b$,

$\exists N_2 > 0$ s.t. $n > N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}$

$\Rightarrow$ Let $N = \max\{N_1, N_2\}$,

Then $\forall n > N$, $(\Rightarrow n > N_1,\ n > N_2)$

we have

$$|(a_n+b_n)-(a+b)| \leq |a_n-a| + |b_n-b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow \lim (a_n+b_n) = a+b.$ $\square$