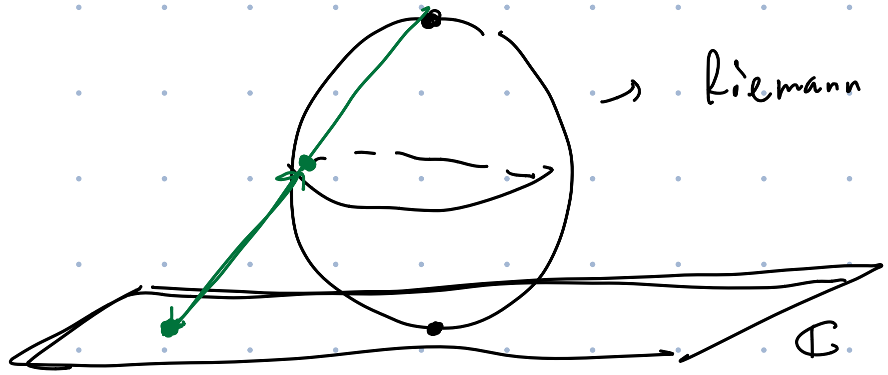
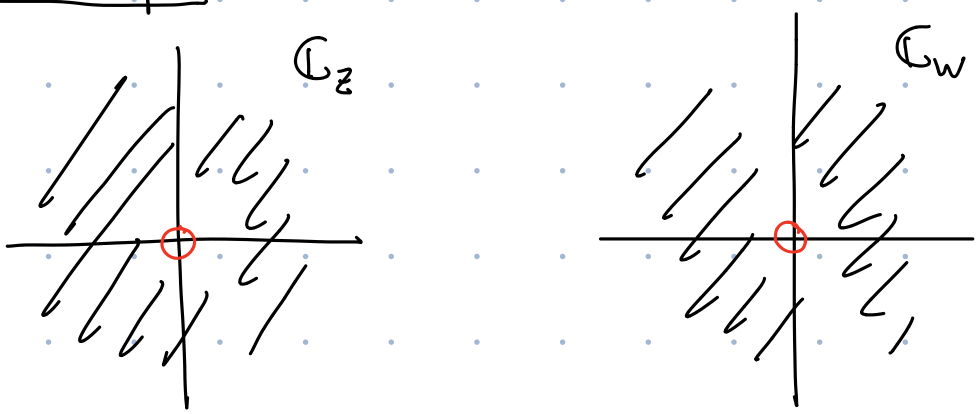
extended complex plane $\mathbb{C} \cup \{\infty\}$

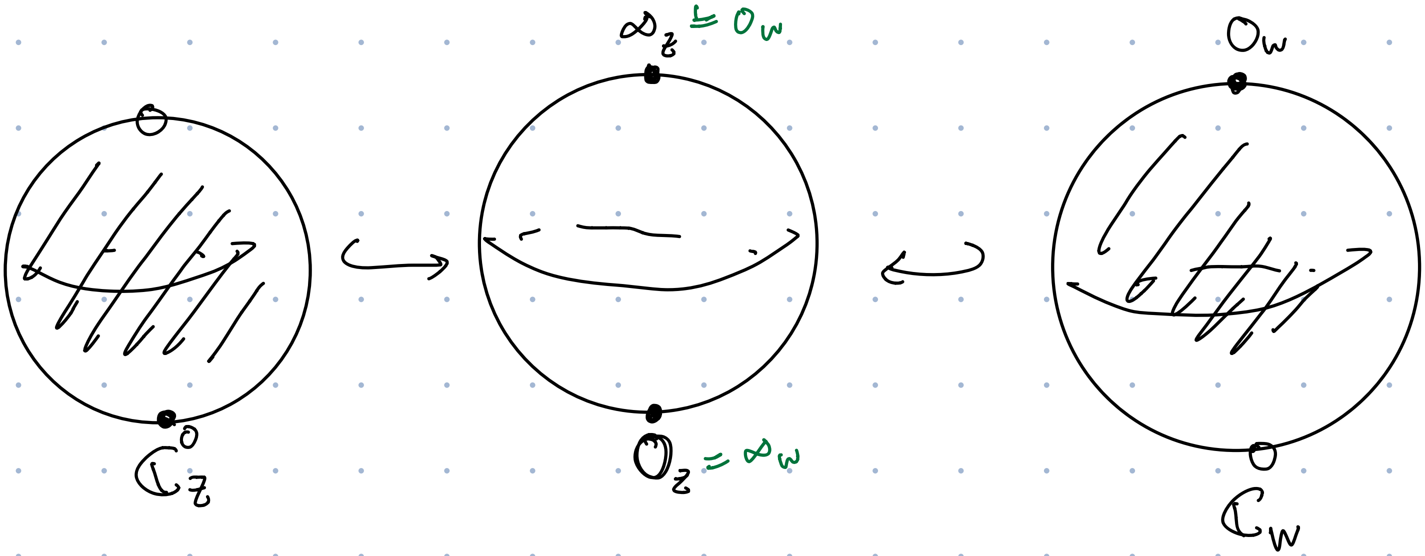


<u>Another description:</u>

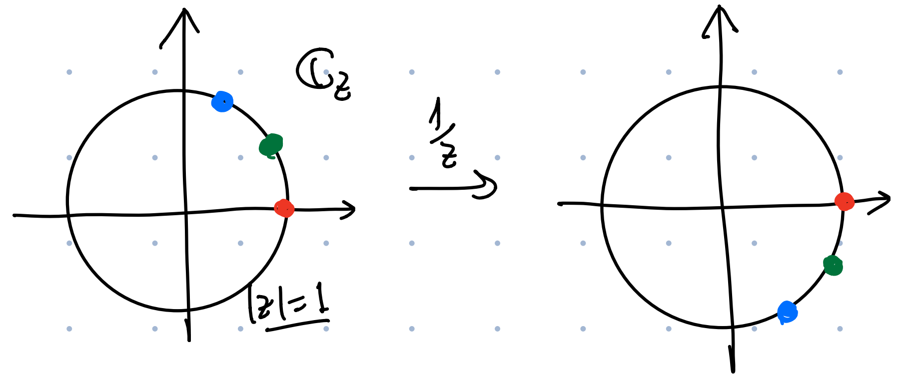


glue $\mathbb{C}_z \setminus \{0\}$ with $\mathbb{C}_w \setminus \{0\}$ via the identification.

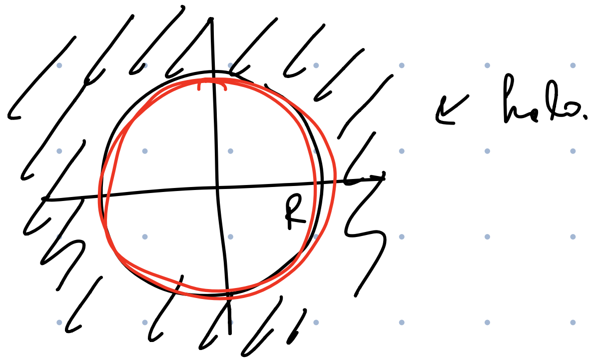
$$z \longleftrightarrow 1/w.$$

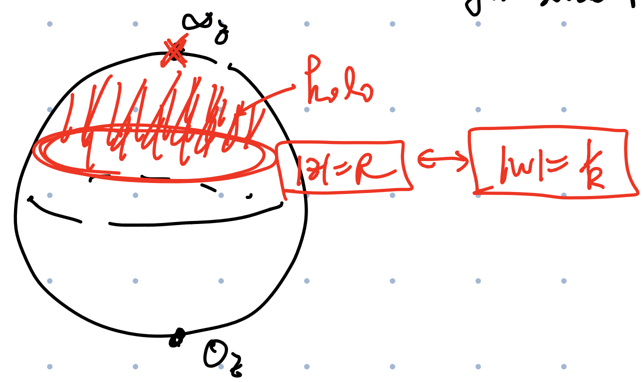


In particular, $\{|z|=1\} \subseteq \mathbb{C}_z$ is identified w/ $\{|w|=1\} \subseteq \mathbb{C}_w$



If f is a fcn on $\mathbb{C}$, suppose it's holo. on $\{|z|>R\} \subseteq \mathbb{C}$ for some $R>0$





Then $F(z) := f(\frac{1}{z})$ has an isolated singularity at 0.

(b/c F is holo. in $\mathbb{D}^{\times}_{1/R}(0)$).

($\Leftrightarrow$ "f has an isolated sing. at ∞")

Rmk: Analyzing the iso. sing. at ∞ is sometimes very useful.

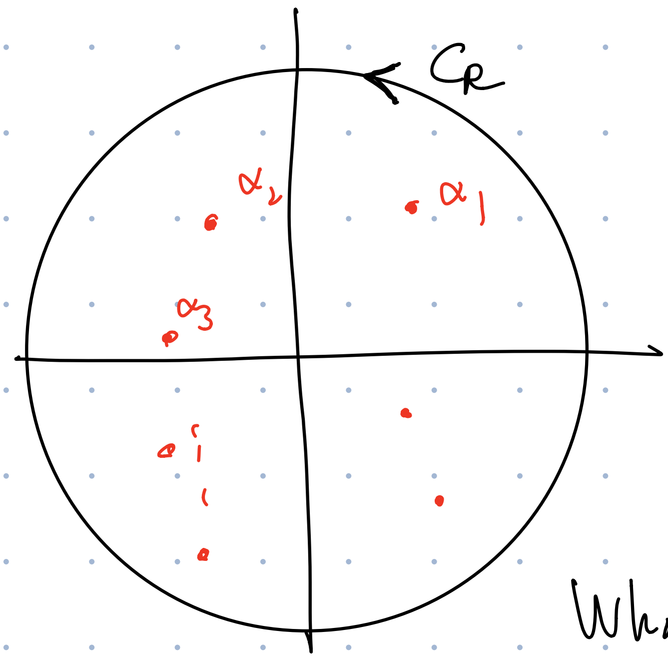
e.g. (HW) f: entire inj. $\Rightarrow$ linear

(we prove this by analyzing the type of sing at ∞)

$\boxed{\text{e.g.}}$ $p(z) = (z-\alpha_1)\cdots(z-\alpha_n)$, $|\alpha_i| < R \;\; \forall i$ $\quad n \geq 2$

$\boxed{\alpha_i \neq \alpha_j}$

Then $\displaystyle\int_{C_R} \frac{1}{p(z)}\, dz = 0$



Res. thm.

$$\int_{C_R} \frac{1}{p(z)}\,dz = 2\pi i \sum_{i=1}^{n} \boxed{\operatorname*{Res}_{z=\alpha_i}\left(\frac{1}{p(z)}\right)}$$

What's $\boxed{\operatorname*{Res}_{z=\alpha_1} \frac{1}{(z-\alpha_1)\boxed{(z-\alpha_2)\cdots(z-\alpha_n)}}} = ??$

$$\| \qquad \frac{1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_n)}$$

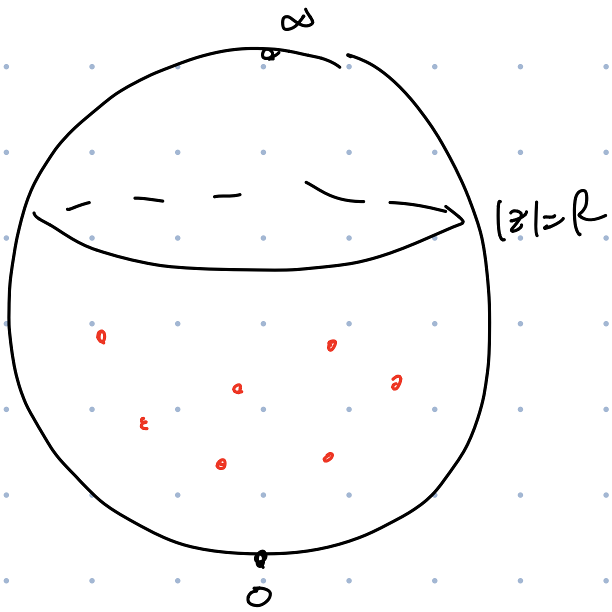
$n \geq 2$

$$\boxed{\frac{1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_n)} + \frac{1}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\cdots(\alpha_2-\alpha_n)} + \cdots + \frac{1}{(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\cdots(\alpha_n-\alpha_{n-1})} = 0\ ?!?}$$

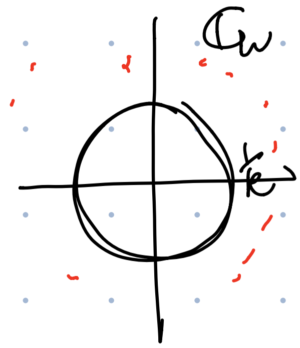
e.g. $n=2$: $\quad \frac{1}{\alpha_1-\alpha_2} + \frac{1}{\alpha_2-\alpha_1} = 0$

$n=3$: $\quad \boxed{\frac{1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}} + \frac{1}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{1}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}$

$$= \frac{\alpha_2-\alpha_3-(\alpha_1-\alpha_3)+\alpha_1-\alpha_2}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)} = 0.$$



$$\int_{\substack{C_R \\ |z|=R}} \frac{1}{p(z)}\,dz = -\int_{|w|=1/R} \frac{1}{p(\frac{1}{w})}\,d\left(\frac{1}{w}\right)$$



$w = \frac{1}{z}$

$$= -\int_{|w|=1/R} \frac{1}{p(\frac{1}{w})} \frac{-1}{w^2}\,dw$$

doesn't have zero in $\mathbb{D}^{\times}_{1/R}(0)$.

$p(z)$ has roots α_i, $|\alpha_i| < R$

$p(\frac{1}{w})$ has roots $\frac{1}{\alpha_i}$, $\left|\frac{1}{\alpha_i}\right| > \frac{1}{R}$

so $\frac{1}{p(\frac{1}{w})} \cdot \frac{-1}{w^2}$ is holo. in $\mathbb{D}^{\times}_{1/R}(0)$ (in the w-plane)

$\boxed{\text{Analyze the sing. of } \frac{-1}{p(\frac{1}{w})\cdot w^2} \text{ at } w=0\text{:}}$

What type of sing. does $\frac{-1}{p(\frac{1}{w})\cdot w^2}$ have at $w=0$??

$\boxed{p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0}$ $\quad ((z-\alpha_1)\cdots(z-\alpha_n))$

$$p\left(\frac{1}{w}\right) = \frac{1}{w^n} + a_{n-1}\frac{1}{w^{n-1}} + \dots + a_0$$

$$= \frac{1}{w^n}\left(1 + a_{n-1}w + a_{n-2}w^2 + \dots + a_0 w^n\right)$$

$$\frac{-1}{p(\frac{1}{w})\cdot w^2} = \frac{-1}{\frac{1}{w^n}\boxed{(1+a_{n-1}w+\cdots+a_0w^n)}\cdot w^2}$$

nonvanishing holo. near $w=0$

$$= \frac{-w^{n-2}}{1+a_{n-1}w+\cdots+a_0w^n} \qquad n-2 \geq 0$$

$\Longrightarrow \dfrac{-1}{p(\frac{1}{w})w^2}$ has removable sing at $w=0$.

$$\Longrightarrow \int_{|z|=\frac{1}{R}} \frac{-1}{p(\frac{1}{w})w^2} = 0 \qquad \square$$

Def: Say $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic (holo.) if

- $f_z: \mathbb{C}_z \to \hat{\mathbb{C}}$ meromorphic (holo.) (has isolated poles, no essential sing.)
- $f_w: \mathbb{C}_w \to \hat{\mathbb{C}}$ meromorphic (holo.)

e.g.

	zeros	poles
$f(z)=\frac{1}{z}$	∞	0
$F(w)=w$	simple	simple

$f(z) = z(z+1)$ $\quad 0, -1$ simple zeros $\quad \infty$ double pole

$\frac{1}{w}\left(\frac{1}{w}+1\right) = \frac{w+1}{w^2}$

$f(z) = \frac{z+1}{z+2}$ $\quad -1$ simple zero $\quad -2$ simple pole

$\frac{\frac{1}{w}+1}{\frac{1}{w}+2} = \frac{1+w}{1+2w} \to 1$ as $w \to 0$

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e.g. p, q poly in z

$f(z) = \frac{p(z)}{q(z)}$ $\quad$ zeros of p $\to$ zeros of f $\quad \to \#\, n$

zeros of q $\to$ poles of f $\quad \to m$

At ∞ $\quad p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_n \neq 0$,

$q(z) = b_m z^m + \dots + b_0$, $b_m \neq 0$

$$\frac{p(\frac{1}{w})}{q(\frac{1}{w})} = \frac{a_n \frac{1}{w^n} + \dots + a_0}{b_m \frac{1}{w^m} + \dots + b_0}$$

If $n > m$, $\quad \frac{\boxed{a_n + a_{n-1} w + \dots + a_0 w^n}}{\boxed{b_m w^{n-m} + \dots + b_0 w^n}}$ $\quad a_n \neq 0$, $b_m \neq 0$

nonvanishing holo near $w=0$.

$\}$ has a pole of order $n-m$ at $w=0$

Ex: $\frac{p(z)}{q(z)}$ meromorphic,

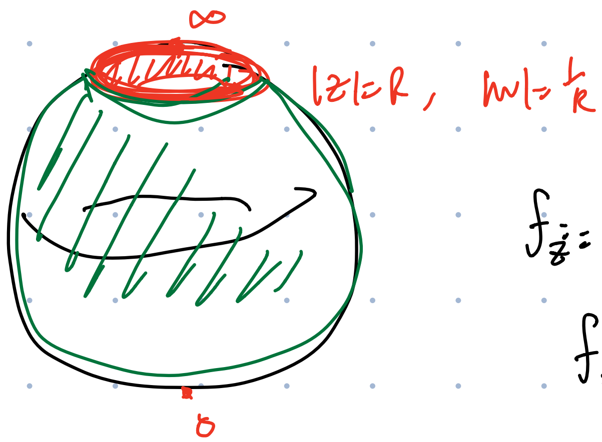
and # of zeros in $\hat{\mathbb{C}}$ = # of poles in $\hat{\mathbb{C}}$

Thm 1) The only holo fcns on $\hat{\mathbb{C}}$ are the constant fcns.

2) The only mero. fcns on $\hat{\mathbb{C}}$ are the rational fcns $\left(\frac{p(z)}{q(z)}\right)$.

Sketch:

1)



$f_z : \mathbb{C}_z \to \mathbb{C}$ holo.

$f_z\left(\boxed{\{|z| \le R\}}\right)$ is bdd.

↑ cpt

$f_w\left(\{|w| \le \frac{1}{R}\}\right)$ is bdd.

$\Rightarrow f$ is entire & bdd

$\underset{\text{Liouville}}{\Rightarrow}$ $f \equiv$ const.