HW3 sol'n

#1: (a) converge (b) diverge ; (we discussed in class)

(c) $\lim_{n\to\infty} c_n = 0$. pf: $\forall \varepsilon>0$, let $N=\frac{1}{\varepsilon^2}$. then. $\forall n>N$, we have

$$|c_n - 0| = \left|\frac{\sin(2n)}{\sqrt{n}}\right| \le \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \varepsilon. \quad \square$$

(d) diverge. Observe that

$$d_1 = d_5 = d_9 = \cdots = d_{4n+1} = 1,$$
$$d_2 = d_4 = d_6 = \cdots = d_{2n} = 0,$$
$$d_3 = d_7 = d_{11} = \cdots = d_{4n+3} = -1.$$

The proof of divergent is similar to the proof of $(-1, 1, -1, 1, -1, 1, \cdots)$ diverges we did in class.

(e) $\lim_{n\to\infty}\left(\sqrt{n^2+4n} - n\right) = 2$. pf: $\forall \varepsilon>0$, let $N=\frac{2}{\varepsilon}$. Then $\forall n>N$,

$$\left|\left(\sqrt{n^2+4n}-n\right)-2\right| = \frac{4}{\sqrt{n^2+4n}+(n+2)} < \frac{4}{2n} < \varepsilon. \quad \square$$

(f) $\lim_{n\to\infty} \frac{2^n}{n!} = 0$. pf: Observe that $0 < \frac{2^n}{n!} = \frac{2\cdot 2 \cdots 2}{1\cdot 2 \cdots n} = 2\cdot 1 \cdot \frac{2}{3}\cdot\frac{2}{4}\cdots\frac{2}{n}$

$$< 2\left(\frac{2}{3}\right)^{n-2}.$$

Since $\lim_{n\to\infty} 2\left(\frac{2}{3}\right)^{n-2} = 0$, by squeeze lemma,

we have $\lim_{n\to\infty}\frac{2^n}{n!} = 0$. $\square$

#2: Since (b_n) and (a_n) only differ at finitely many terms,

$\exists M>0$ st. $b_n = a_n$ $\forall n>M$.

Since $\lim_{n\to\infty} a_n = a$, $\forall \varepsilon>0$, $\exists N>0$ st. $n>N \Rightarrow |a_n - a| < \varepsilon$.

Let $N_b := \max\{N, M\}$, Then $\forall n > N_b$, we have

$$|b_n - a| \underset{\text{since } n>M}{=} |a_n - a| \underset{\text{since } n>N}{<} \varepsilon. \quad \square$$

#3: ~~[illegible]~~, $\forall \varepsilon > 0$,

Since $\lim_{n\to\infty} a_n = a$, $\exists N_a > 0$ s.t. $n > N_a \Rightarrow |a_n - a| < \varepsilon$

Since $\lim_{n\to\infty} c_n = a$, $\exists N_c > 0$ s.t. $n > N_c \Rightarrow |c_n - a| < \varepsilon$

Define $N := \max\{N_a, N_c\}$, then $\forall n > N$, we have.

$$\underbrace{a - \varepsilon < a_n}_{\text{since } n > N_a} \leq b_n \leq \underbrace{c_n < a + \varepsilon}_{\text{since } n > N_c} \Rightarrow |b_n - a| < \varepsilon. \quad \square$$

#4: $\frac{n-1}{n} \leq a_n \leq \frac{n+1}{n}$, Since $\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$,

by squeeze lemma we have $\lim_{n\to\infty} a_n = 1$. $\square$

#5: Assume the contrary that $a > b$. Let $\varepsilon = \frac{a-b}{3} > 0$.

Since $\lim_{n\to\infty} a_n = a$, $\exists N_a > 0$ s.t. $n > N_a \Rightarrow |a_n - a| < \varepsilon$

Since $\lim_{n\to\infty} b_n = b$, $\exists N_b > 0$ s.t. $n > N_b \Rightarrow |b_n - b| < \varepsilon$.

Since $a_n \leq b_n$ for all but finitely many n, $\exists M > 0$ s.t. $a_n \leq b_n$ $\forall n > M$.

Take any $n > \max\{N_a, N_b, M\}$, then

$$a - \varepsilon \underset{\text{since } n > N_a}{<} a_n \underset{\text{since } n > M}{\leq} b_n \underset{\text{since } n > N_b}{<} b + \varepsilon$$

$\Rightarrow$ $a < b + 2\varepsilon = b + \frac{2}{3}(a-b) < a$. Contradiction. $\square$

#6: $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $n > N \Rightarrow |a_n - a| < \varepsilon$.

By triangle ineq., we have:

$$-|a_n - a| \leq |a_n| - |a| \leq |a_n - a|$$

$$\Rightarrow \big||a_n| - |a|\big| \leq |a_n - a| < \varepsilon. \Rightarrow \lim_{n\to\infty} |a_n| = |a|. \quad \square$$

Converse is NOT true: e.g. $a_n = (-1)^n$.

#7: By HW 2, #5, $\forall n \in \mathbb{N}$, $\exists a_n \in S$ s.t. $z - \frac{1}{n} < a_n \leq z$.

Then it's easy to check that (a_n) converges to z. $\square$

#8: Choose any c s.t. $b < c < 1$.

It's not hard to show that since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = b < c$, there exists $N > 0$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| < c$ $\forall n > N$.

$$\Rightarrow |a_{n+1}| < c|a_n| \quad \forall n > N.$$

$$\Rightarrow |a_{N+k}| < c^k |a_N| \quad \forall k \in \mathbb{N}.$$

Since $c < 1$, we have $\lim_{k\to\infty} c^k |a_N| = 0$.

By squeeze lemma, we have $\lim_{k\to\infty} |a_{N+k}| = 0$.

Hence $\lim_{n\to\infty} |a_n| = 0$. $\Rightarrow \lim_{n\to\infty} a_n = 0$. $\square$

(Why?)

#9: (a). $\exists M>0$ s.t. $|a_n| < M \quad \forall n$.

$\forall \varepsilon>0, \ \exists N>0$ s.t. $|b_n| = |b_n - 0| < \frac{\varepsilon}{M} \quad \forall n>N$.

$$\Rightarrow |a_nb_n - 0| = |a_nb_n| \le |a_n||b_n| < M\cdot\frac{\varepsilon}{M} = \varepsilon \quad \forall n>N.$$

$$\Rightarrow \lim_{n\to\infty} a_nb_n = 0. \quad \Box$$

(b) $a_n = n^2, \quad b_n = \frac{1}{n}. \qquad a_nb_n = n.$

(c) $a_n = (-1)^n, \quad b_n = 1$

#10. (a) Counterexample: $a_n = (-1)^n$.

(b) Claim: If $\lim_{n\to\infty} a_n^3 = a$, then $\lim_{n\to\infty} a_n = \sqrt[3]{a}$.

Case 1: $a=0$.

$\forall \varepsilon>0, \ \exists N>0$ s.t. $|a_n^3| < \varepsilon^3. \quad \forall n>N$.

$$\Rightarrow |a_n| < \varepsilon \quad \forall n>N.$$

Case 2: $a\neq 0$. Without lost of generality, assume that $a>0$.

Since $\lim_{n\to\infty} a_n^3 = a$, by the argument in class, $\exists N_0>0$

s.t. $a_n > 0 \quad \forall n > N_0$

$\forall \varepsilon>0$. Consider $\varepsilon\cdot a^{2/3} > 0$

$\exists N'>0$ s.t. $|a_n^3 - a| < \varepsilon\cdot a^{2/3} \quad \forall n>N'$.

Let $N = \max\{N_0, N'\}$, then $\forall n>N$, we have

$a_n>0$ since $n>N_0$

$$\varepsilon\cdot a^{2/3} > |a_n^3 - a| = |a_n - \sqrt[3]{a}|\,|a_n^2 + a_n\sqrt[3]{a} + a^{2/3}| = |a_n - \sqrt[3]{a}|(a_n^2 + a_n\sqrt[3]{a} + a^{2/3})$$

$$> |a_n - \sqrt[3]{a}|\cdot a^{2/3}. \quad \Rightarrow |a_n - \sqrt[3]{a}| < \varepsilon \quad \forall n>N. \quad \Box$$