Last time:

- linear system $\leftrightsquigarrow$ associated augmented matrix.

they have the same solution set.

elementary row operations

reduced echelon form.

- How to read the sol^{n}s of a reduced echelon form?

Case 1: The last column is pivot.

$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

has no solution!!

Case 2: The last column is not pivot.

$$\begin{array}{c} \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 2 & 0 & 4 & 7 \\ 0 & 0 & 1 & 3 & 0 & 5 & 8 \\ 0 & 0 & 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \end{array}$$

$$\begin{cases} x_2 + 2x_4 + 4x_6 = 7 \\ x_3 + 3x_4 + 5x_6 = 8 \\ x_5 + 6x_6 = 9 \end{cases} \Longleftrightarrow \begin{cases} x_2 = 7 - 2x_4 - 4x_6. \\ x_3 = 8 - 3x_4 - 5x_6 \\ x_5 = 9 - 6x_6 \end{cases}$$

Observation:
- for the "non-pivot variables" (x_1, x_4, x_6), we can assign them to be any real numbers.
- for each choice of $x_1, x_4, x_6 \in \mathbb{R}$, there is a unique choice of x_2, x_3, x_5 that makes $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{bmatrix}$ a sol^n.

In other words, the sol^n set is given by:

$$\left\{ \vec{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix} \in \mathbb{R}^6 : \begin{array}{l} x_1, x_4, x_6 \text{ any real numbers, and,} \\ x_2 = 7 - 2x_4 - 4x_6 \\ x_3 = 8 - 3x_4 - 5x_6 \\ x_5 = 9 \quad\;\; - 6x_6 \end{array} \right\}.$$

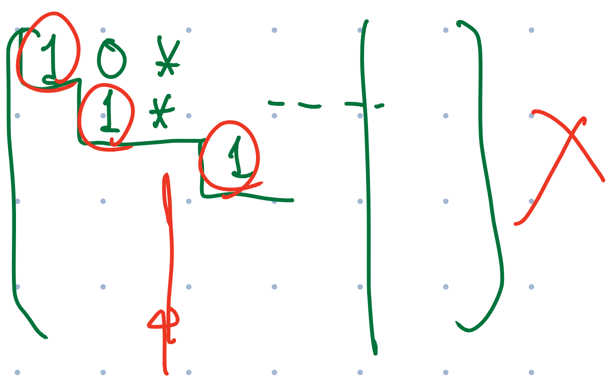
Summary:

- a linear system has a sol^n $\Leftrightarrow$ the last column of it's augmented matrix is NOT pivot.
- when it has sol^n, the sol^n is unique
  $\Leftrightarrow$ every column in the coefficient matrix is pivot (contains a pivot position)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ & & & 0 \\ & 0 & & 0 \\ & & & 0 \end{array}\right]$$

$\Updownarrow$

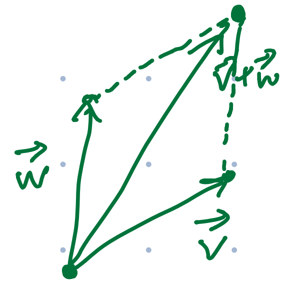
no free variables.

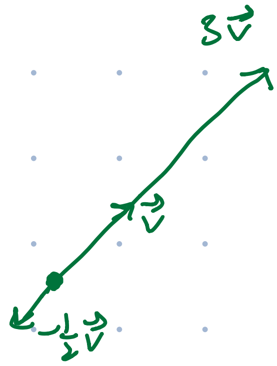


Q: How do we characterize the vectors $\vec{b}$ s.t. $[A \mid \vec{b}]$ has sol^n?

§ linear combinations of vectors.

Def. $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ in $\mathbb{R}^n$.





Sum. $\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}$ in $\mathbb{R}^n$

Scalar multiplication: $c \in \mathbb{R}$, $c\vec{v} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}$ in $\mathbb{R}^n$

Def Let $\vec{v}_1, \dots, \vec{v}_k$ in $\mathbb{R}^n$; $c_1, \dots, c_k \in \mathbb{R}$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \quad (\text{vector in } \mathbb{R}^n)$$

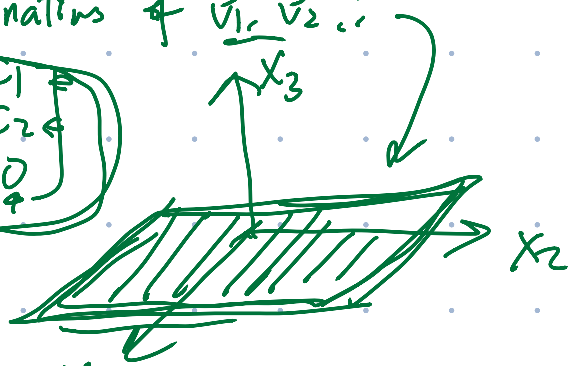
is a **linear combination** of $\vec{v}_1, \dots, \vec{v}_k$ (of weight $c_1, \dots, c_k$)

e.g. $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Q: What are all the linear combinations of $\vec{v}_1, \vec{v}_2$??

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}$$

$\mathbb{R}$



Q: What are all the linear comb of $\vec{v}_1$??

$$c_1\vec{v}_1 = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}.$$

Def. $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} := \{\text{linear comb. of } \vec{v}_1, \dots, \vec{v}_k\}$

$$= \{c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}$$

e.g. $\vec{v_1} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

$\left[\begin{array}{cc|c} 1 & 2 & \\ -2 & 5 & \vec{b} \\ -5 & 6 & \end{array}\right]$ has a sol^n

$\Updownarrow$

Q: Is $\boxed{\vec{b} \in \text{Span}\{\vec{v_1}, \vec{v_2}\}}$??

$\Updownarrow$

$\boxed{\exists\ c_1, c_2 \in \mathbb{R},\ \text{s.t.}\ \vec{b} = c_1\vec{v_1} + c_2\vec{v_2}}$

(there exists)

$\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ -2c_1 + 5c_2 \\ -5c_1 + 6c_2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right]$. has a sol^n.

$\begin{cases} c_1 + 2c_2 = 7 \\ -2c_1 + 5c_2 = 4 \\ -5c_1 + 6c_2 = -3 \end{cases}$

augmented matrix $\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right]$.

in $\mathbb{R}^n$

Observation: $\vec{b} \in \text{Span}\{\vec{v_1}, \dots, \vec{v_k}\}$

$\Longleftrightarrow \left[\begin{array}{cccc|c} | & | & & | & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_k} & \vec{b} \\ | & | & & | & | \end{array}\right]$ has a sol^n.

Def (matrix-vector product) A: $\underset{\text{\#rows}}{m} \times \underset{\text{\#columns}}{n}$ matrix $= \begin{bmatrix} | & & | \\ \vec{a_1} & \cdots & \vec{a_n} \\ | & & | \end{bmatrix}$ $\vec{a_i} \in \mathbb{R}^m$

$\vec{x} \in \mathbb{R}^n$. $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

Define $A\vec{x} := x_1\vec{a_1} + \cdots + x_n\vec{a_n}$. $\in \mathbb{R}^m$.

linear comb. of columns of A w/ wts given by $\vec{x}$.

e.g. $\underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}}\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 8\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 9\begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 7\times 1 + 8\times 3 + 9\times 5 \\ 7\times 2 + 8\times 4 + 9\times 6 \end{bmatrix} \in \mathbb{R}^2$

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ doesn't make sense.

Rmk Given $A = [\vec{a}_1 \cdots \vec{a}_n]$: $m\times n$ matrix, $\vec{b} \in \mathbb{R}^m$,

Then "$\exists\, \vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$"

$\Leftrightarrow$ $\vec{b} \in \mathrm{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$

$\Leftrightarrow$ $[\vec{a}_1 \cdots \vec{a}_n \mid \vec{b}]$ has a sol^{n}.

> Given a matrix $\underset{m\times n}{A}$, define a function.
>
> $$T_A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
> $$\vec{x} \longmapsto A\vec{x}$$

Rmk The map T_A encodes the info of the solvability of linear eq^{n} $A\vec{x} = \vec{b}$:

$[A \mid \vec{b}]$ $\exists$ sol^{n} $\Leftrightarrow$ $\vec{b}$ is in the image of T_A

Prop. $A: m\times n$, $T_A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$\vec{x} \longmapsto A\vec{x}$.

- $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$, we have $T_A(\vec{u}+\vec{v}) = T_A(\vec{u}) + T_A(\vec{v})$
- $\forall c \in \mathbb{R}$, we have $T_A(c\vec{v}) = c\,T_A(\vec{v})$

("linear transformation")

$$A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$$

pf • $T_A(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\cdot\begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$, where $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

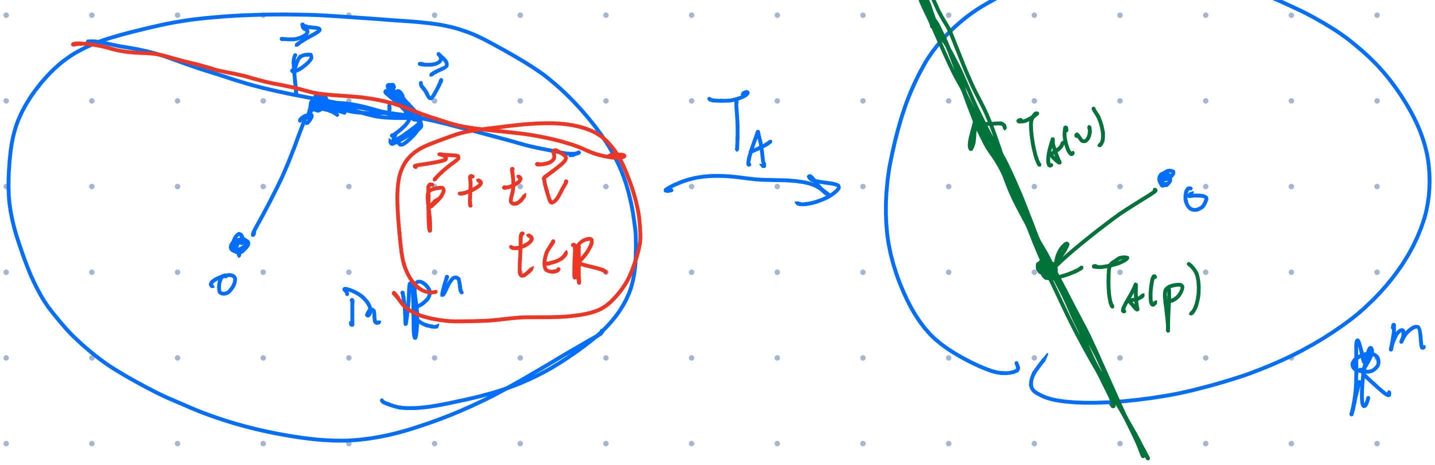
$$= (u_1+v_1)\vec{a}_1 + \cdots + (u_n+v_n)\vec{a}_n$$

$$\|$$

$T_A(\vec{u}) + T_A(\vec{v})$

$$= A\vec{u} + A\vec{v} = (u_1\vec{a}_1 + \cdots + u_n\vec{a}_n) + (v_1\vec{a}_1 + \cdots + v_n\vec{a}_n)$$

T_A sends a line in $\mathbb{R}^n$ to a line in $\mathbb{R}^m$.



$$T_A(\vec{p}+t\vec{v}) = T_A(\vec{p}) + T_A(t\vec{v})$$
$$= \boxed{T_A(\vec{p}) + t\,T_A(\vec{v})}$$

In general, $f: X \longrightarrow Y$

X ← domain

$T_A: \mathbb{R}^n \to \mathbb{R}^m$

- $f(X) = \{y \in Y \mid y = f(x) \text{ for some } x \in X\} \subseteq Y$ image of f.

- **Surjective** (onto): $f(X) = Y$. i.e.

  $\forall y \in Y, \exists x \in X$ s.t. $f(x) = y$.

  $\forall$: for all; $\exists$: There exists; s.t.: such that.

- **Injective**: $\forall x_1, x_2 \in X$, $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

Q: When is $T_A: \mathbb{R}^n \to \mathbb{R}^m$ surjective?

$\Leftrightarrow \forall \vec{b} \in \mathbb{R}^m, \exists \vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$.

$\Leftrightarrow \forall \vec{b} \in \mathbb{R}^m$, the system $[A \mid \vec{b}]$ has a sol^{n}

e.g.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & 8 \\ 3 & 5 & 8 & 13 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Does $\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & b_1 \\ 2 & 3 & 5 & 8 & b_2 \\ 3 & 5 & 8 & 13 & b_3 \end{array}\right]$ always have sol^{n} $\forall \vec{b}$?

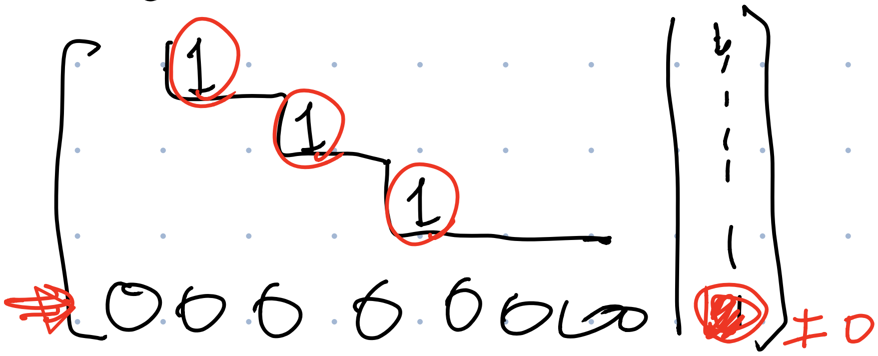
$\downarrow \vdots$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - 2(b_2 - 2b_1) \end{array}\right]$$

If we choose $\vec{b}$ s.t. $\neq 0$,

then $[A|\vec{b}]$ has no sol^{n}.

More generally,



<u>Observation</u>: If A doesn't have pivot in each row,
then $\exists\, \vec{b} \in \mathbb{R}^m$ s.t. $A\vec{x} = \vec{b}$ has no sol^{n}.
($\Leftrightarrow$ T_A is not surjective)

Conversely, if A has pivot in each row,

then $[A|\vec{b}]$ doesn't have pivot in the last column $\forall\, \vec{b} \in \mathbb{R}^m$.

$\Rightarrow$ T_A is surjective (i.e. $[A|\vec{b}]$ has sol^{n} $\forall\, \vec{b}$)

Thm. Let A: $m \times n$ matrix. Then the following are equivalent:

- the map $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is surjective.
  $\vec{x} \longmapsto A\vec{x}$
- $A\vec{x} = \vec{b}$ has a sol^{n} $\forall \vec{b} \in \mathbb{R}^m$. (doesn't have to be unique)
- A has pivot in each row. ($\Rightarrow m \le n$)
- $\forall \vec{b} \in \mathbb{R}^m$, $\vec{b}$ is a linear comb. of the columns $\vec{a}_1, \dots, \vec{a}_n$ of A.
- $\operatorname{Span}\{\vec{a}_1, \dots, \vec{a}_n\} = \mathbb{R}^m$.

$$\begin{bmatrix} \\ \quad A \quad \\ \\ \end{bmatrix} \cdot \begin{bmatrix} \\ \qquad \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \qquad \\ \\ \end{bmatrix}$$

$m \times n$ $\qquad$ $n \times p$ $\qquad\qquad$ $m \times p$

$\qquad\qquad\qquad\quad p = 1$ $\qquad\qquad\quad p = 1$

$$T_A : M_{n \times p}(\mathbb{R}) \longrightarrow M_{m \times p}(\mathbb{R})$$

$$A := \begin{bmatrix} 1 & 1 \end{bmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^1$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x + y$$