Today: Simply connectedness, complex logarithm, complex roots.

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pf of $\boxed{f}$ mero. fns on $\hat{\mathbb{C}}$ are rat'l poly. $\boxed{p(z)/q(z)}$ (sketch):

- $\hat{\mathbb{C}}$ is cpt set.
- The sing. pts of a mero. fn are isolated,

set of isolated pts in a cpt set is always finite.

- $\Rightarrow$ set of poles is a finite set

- Suppose f has a pole at ∞

i.e. $\boxed{f\left(\frac{1}{w}\right)}$ has a pole at $w=0$

$$= \underbrace{\frac{a_{-n}}{w^n} + \frac{a_{-n+1}}{w^{n-1}} + \cdots + \frac{a_{-1}}{w}} + \text{holo. fn} \quad (\text{near } w=0)$$

- $\underbrace{f\left(\frac{1}{w}\right) - \left(\frac{a_{-n}}{w^n} + \cdots + \frac{a_{-1}}{w}\right)}$ holo. near $w=0$.

- $f(z) - \left(a_{-n} z^n + \cdots + a_{-1} z\right)$ holo. near $z=\infty$.

Similarly, for other poles $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$

locally near each α_i,
$$f(z) = \underbrace{\boxed{\frac{\text{const}}{(z-\alpha_i)^{*}} + \cdots + \frac{*}{z-\alpha_i}}}_{f_{prin,\alpha_i}(z)} + \text{holo} \ldots$$

$$\Rightarrow f(z) - \left(\frac{*}{(z-\alpha_i)^{*}} + \cdots + \frac{*}{z-\alpha_i}\right) \text{ holo. near } \alpha_i$$

$$\Rightarrow \underbrace{f(z) - \left(\underbrace{f_{prin,\alpha_1}(z) + \cdots + f_{prin,\alpha_n}(z)}_{\text{rat'l fn}}\right)} \text{ holo. in } \hat{\mathbb{C}} \equiv \text{const.}$$

$\square$

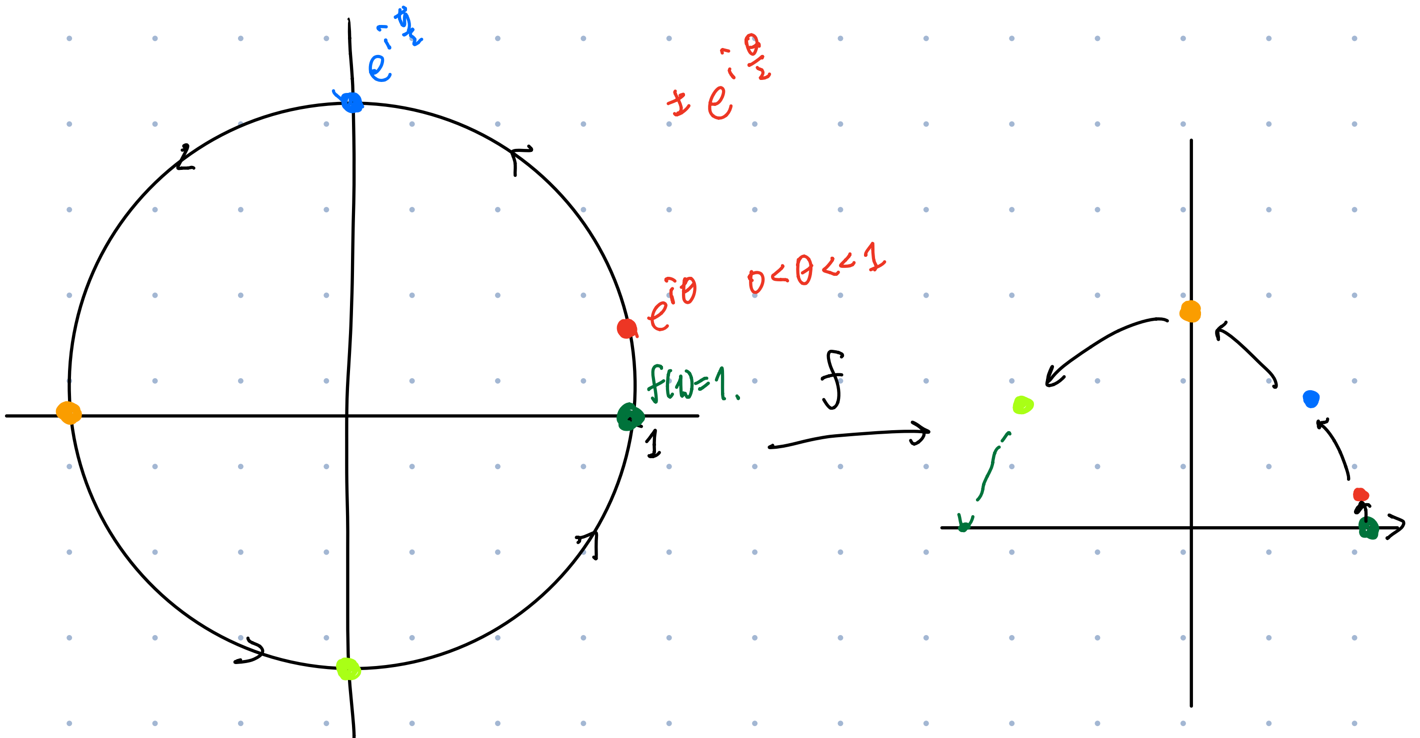
"Complex square root"

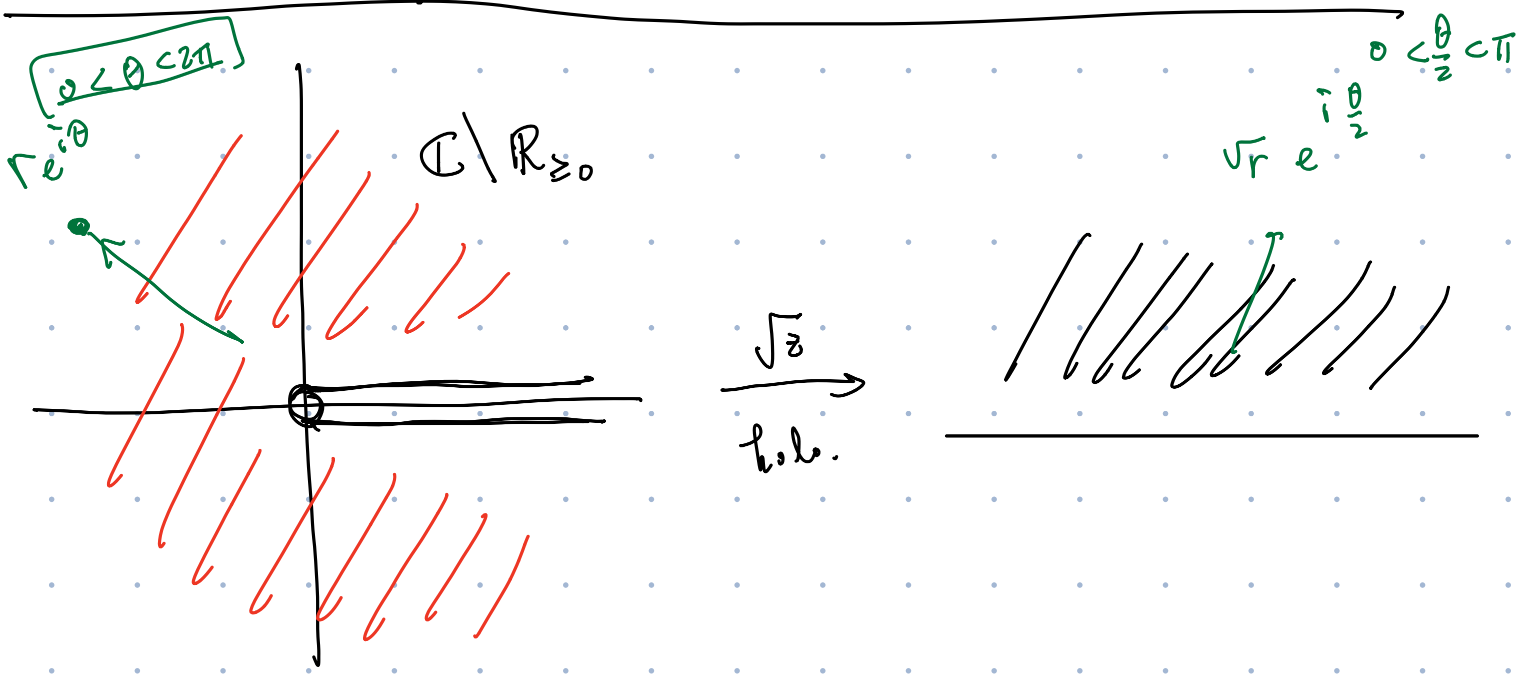
Q: Does there exist an entire fn: $f: \mathbb{C} \to \mathbb{C}$

s.t. $\quad f(z)^2 = z \quad \forall z \in \mathbb{C}$

e.g. $\quad z = re^{i\theta}$, $\quad f(z) = \sqrt{r}\, e^{i\frac{\theta}{2}}, \; -\sqrt{r}\, e^{i\frac{\theta}{2}}$

$z \neq 0$, $r > 0$ $\qquad$ the roots of $y^2 = z$



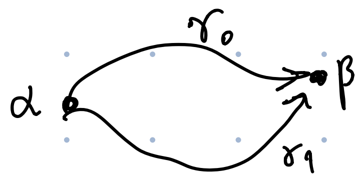


(More generally, the square root fcn can be defined on any simply connected domain not containing 0.)

(**Recall:** Ω is simply connected

$\iff$ $\boxed{\text{any holo. fcn } f \text{ on } \Omega \text{ has a primitive } F}$ $(F' = f)$)

Def: $\Omega \subseteq \mathbb{C}$ open.



γ_0, γ_1 curves in Ω w/ the same end points.

$\gamma_0, \gamma_1 : [a,b] \longrightarrow \Omega$, $\quad \gamma_0(a) = \gamma_1(a) = \alpha \in \Omega$

$\gamma_0(b) = \gamma_1(b) = \beta \in \Omega$

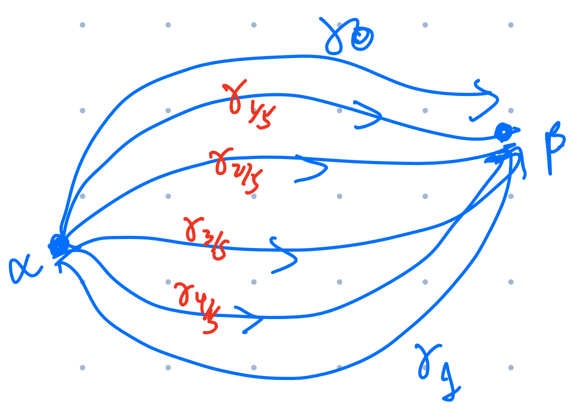
Say γ_0, γ_1 are homotopic in Ω if

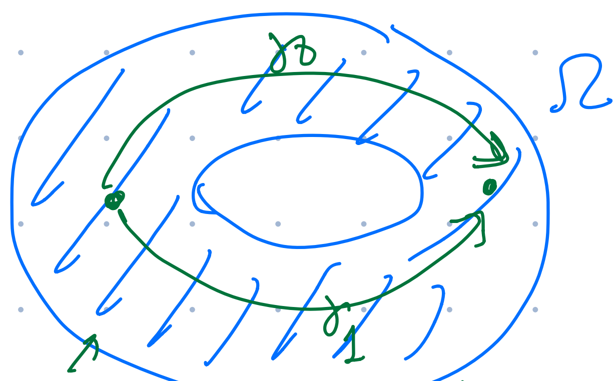
$\exists\ \gamma : [a,b] \times [0,1] \longrightarrow \Omega$ conti. fcn.

$(t, s) \longmapsto \gamma(t,s)$.

st.
- $\gamma(0,s) = \alpha$, $\gamma(1,s) = \beta$ $\forall s \in [0,1]$,
- $\gamma(t,0) = \gamma_0(t)$, $\gamma(t,1) = \gamma_1(t)$ $\forall t \in [a,b]$.

(think of γ as a family of parametrized curves γ_s.)



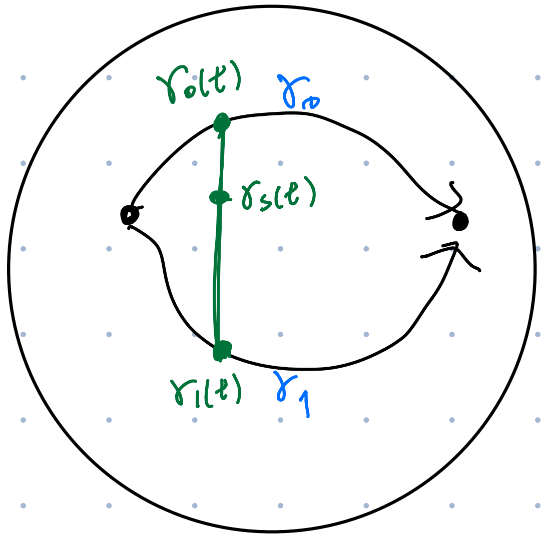


(γ_0, γ_1 are not homotopic)

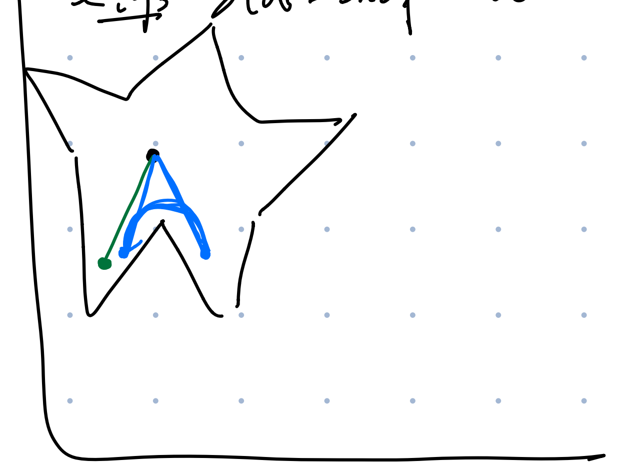
Def. A domain is simply connected if any two curves w/ the same endpoints are homotopic.

e.g. $\mathbb{D}$

or any convex set in $\mathbb{C}$



e.g. "star-shaped domain"



$$\gamma(t,s) := (1-s)\,\gamma_0(t) + s\,\gamma_1(t) \qquad \forall\, s\in[0,1],\ t\in[a,b]$$

Thm (contour integral is deformation invariant)

$f: \Omega \to \mathbb{C}$ holo., $\gamma_0, \gamma_1 \subseteq \Omega$ homotopic.

Then $\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$.

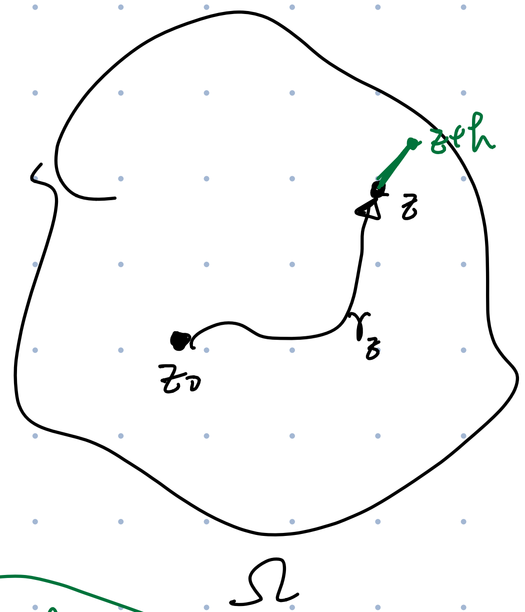
Thm Ω = ~~simply~~ connected (simply connected), $f: \Omega \to \mathbb{C}$ holo.

Then f has a primitive in Ω,

(i.e. $\exists\, F: \Omega \to \mathbb{C}$ holo. s.t. $F' = f$)

pf: Choose any $z_0 \in \Omega$,

$\forall z \in \Omega$, Choose any path γ_z in Ω connecting z_0 & z



Define
$$F(z) := \int_{\gamma_z} f(z)\,dz,$$

(it's a well-defined fcn on z b/c Ω is simply connected & the deformation invar. of contour integrals)

Goal: $\displaystyle\lim_{h\to 0} \frac{F(z+h)-F(z)}{h}$ exists & $= f(z)$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\frac{1}{h}\int_{[z,\,z+h]} f(z)\,dz$$

we proved before (only uses f is conti.)

$\square$

Complex Logarithm:

Q: Does there exist a holo. fcn. $f : \mathbb{C}^* \to \mathbb{C}$

s.t.
$$e^{f(z)} = z \qquad \forall z \in \mathbb{C}^* \ ?$$

NO

Thm. Ω simply connected, $0 \notin \Omega$.

- $z_0 \in \Omega$, $z_0 = re^{i\theta}$, $r>0$, $\theta \in \mathbb{R}$

Then: $\exists!$ $F : \Omega \to \mathbb{C}$ holo $= \log_\Omega(z)$

s.t.

- $e^{F(z)} = z \quad \forall z \in \Omega$.
- $F(z_0) = \log r + i\theta$

$\boxed{e^{*} = z_0} = re^{i\theta} = e^{\log r + i\theta}$

$\boxed{\log r + i(\theta + 2\pi\mathbb{Z})}$

pf: $\frac{1}{z}$ holo. in Ω $\Longrightarrow$ $\exists$ primitive F on Ω

simply connected

$$\begin{cases} \bullet \; F'(z) = \frac{1}{z} \quad \forall z \in \Omega \\ \bullet \; F(z_0) = \log r + i\theta \end{cases}$$

Uniqueness. $F, \tilde{F}$

$\boxed{F(z) - \tilde{F}(z)} = G(z) : \Omega \to \mathbb{C}$

- $G(z_0) = 0$
- $e^{G(z)} = e^{F(z) - \tilde{F}(z)} = e^{F(z)}/e^{\tilde{F}(z)} = 1$

$\Rightarrow G(z) = 2\pi i \mathbb{Z} \quad \forall z$

$$\left(z e^{-F(z)}\right)' = e^{-F(z)} - z F'(z) e^{-F(z)}$$
$$= e^{-F(z)} \cdot \left(1 - zF'(z)\right) = 0$$

$\Rightarrow$ $z \cdot e^{-F(z)} \equiv \text{const.} = 1$ $\Rightarrow$ $\boxed{e^{F(z)} = z}$ $\square$

- $z_0 \cdot e^{-F(z_0)} = re^{i\theta} e^{-(\log r + i\theta)} = 1.$

Thm: Ω = simply connected, $f : \Omega \to \mathbb{C}$ nonvanishing holo.

Then $\exists$ $g : \Omega \to \mathbb{C}$ holo s.t. $f(z) = e^{g(z)} \quad \forall z \in \Omega$

Idea: $(f \cdot e^{-g})' = f' e^{-g} - f g' e^{-g}$

$f' = f g'$ $\quad g' = \boxed{\frac{f'}{f}}$

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pf: $\frac{f'}{f}$ holo. in Ω $\Longrightarrow$ $\exists$ primitive g on Ω

$\uparrow$ simply conn. $\qquad g' = \frac{f'}{f}$

- $g' = \frac{f'}{f}$
- Choose $g(z_0)$ s.t. $\boxed{e^{g(z_0)} = f(z_0)}$.

$\Rightarrow$ $(f e^{-g})' = 0$

$\Rightarrow$ $f(z) e^{-g(z)} = \text{const.} \equiv 1$

$\Rightarrow$ $e^{g(z)} = f(z)$ $\square$

(complex n-th root)

Thm Ω simply conn. $0 \notin \Omega$.

$z_0 \in \Omega$. $z_0 = re^{i\theta}$, $r > 0$, $\theta \in \mathbb{R}$.

Then $\exists!$ $F: \Omega \to \mathbb{C}$ holo. s.t.

- $F(z)^n = z$
- $F(z_0) = r^{1/n} e^{i\frac{\theta}{n}}$

Pf: Define $F(z) = e^{\frac{\mathrm{Log}_\Omega(z)}{n}}$

$$F(z)^n = e^{\mathrm{Log}_\Omega(z)} = z \quad \forall z \in \Omega.$$

$$F(z_0) = e^{\frac{\mathrm{Log}_\Omega(z_0)}{n}} = e^{\frac{\log r + i\theta}{n}} = r^{1/n} e^{i\theta/n}.$$
