HOMEWORK 1
MATH H54, FALL 2021

PART I (NO NEED TO TURN IN)

This part of the homework provides some routine computational exercises. You don't have to turn in your solutions for this part, but being able to do the computations is vitally important for the learning process, so you definitely should do these practices before you start doing Part II of the homework.

The following exercises are selected from the corresponding sections of the UC Berkeley custom edition of Lay, Nagle, Saff, Snider, *Linear Algebra and Differential Equations*.

- **Exercise 1.1**: 11, 19, 31
- **Exercise 1.2**: 9, 19,
- **Exercise 1.3**: 13, 17, 19
- **Exercise 1.4**: 7, 11, 15
- **Exercise 1.5**: 9, 13, 21
- **Exercise 1.7**: 7, 31, 33–38
- **Exercise 1.8**: 5, 17, 23, 33
- **Exercise 1.9**: 5, 15, 21, 27, 35

PART II (DUE SEPTEMBER 7, 11AM)

Some ground rules:

- Please submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW1**.
- The submission should be a **single PDF file**.
- Late homework will not be accepted/graded under any circumstances.
- Make sure the writing in your submission is clear enough. Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own, and acknowledge the students with whom you worked.
- **For True/False questions**: You have to prove the statement if your answer is "True"; otherwise, you have to provide an explicit counterexample and justification.
- You are allowed to use any result that is proved in the lecture. But if you would like to use other results, you have to prove it first before using it.

Problems (next page):

(1) Prove that any system of linear equations

$$\begin{cases} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n = \ b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n = \ b_2 \\ \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

has either no solution, a unique solution, or infinitely many solutions. (Hint: Prove that if $\vec{u}, \vec{v}$ are solutions of the linear system, then so is $t\vec{u} + (1-t)\vec{v}$ for any $t \in \mathbb{R}$. By the way, what's the geometric interpretation of $t\vec{u} + (1-t)\vec{v}$?)

(2) We say two matrices are *row equivalent* if there is a sequence of elementary row operations that transforms one matrix to the other. Prove that if the augmented matrices of two linear systems are row equivalent, then the linear systems have the same solution set.

(3) For any $k \geq 2$, prove that $\{\vec{v_1}, \ldots, \vec{v_k}\}$ in $\mathbb{R}^n$ is linearly independent if and only if $\{\vec{v_2}, \ldots, \vec{v_k}\}$ is linearly independent and $\vec{v_1}$ is not in $\operatorname{span}\{\vec{v_2}, \ldots, \vec{v_k}\}$. (Hint for the "if" part: Suppose $a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_k\vec{v_k} = \vec{0}$. Consider two cases: $a_1 = 0$ or $a_1 \neq 0$.)

(4) True/False: If a system of linear equations has more equations than variables (i.e. $m > n$ in the notation of Problem (1)), then the system has no solutions.

(5) True/False: If $m < n$, then m vectors in $\mathbb{R}^n$ can not span all of $\mathbb{R}^n$.

(6) Let A be an $n \times n$ matrix. Suppose that there exists $\vec{b} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} \in \mathbb{R}^n$. Prove that the columns of A must span all of $\mathbb{R}^n$.

(7) Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that $\{\vec{v_1}, \ldots, \vec{v_k}\} \subseteq \mathbb{R}^n$ spans $\mathbb{R}^n$. Prove that T is a *zero transformation* (i.e. $T(\vec{x}) = \vec{0}$ for any $\vec{x} \in \mathbb{R}^n$) if and only if $T(\vec{v_i}) = 0$ for all $1 \leq i \leq k$.

(8) Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that $\{\vec{v_1}, \ldots, \vec{v_k}\} \subseteq \mathbb{R}^n$ is linearly dependent. Prove that $\{T(\vec{v_1}), \ldots, T(\vec{v_k})\} \subseteq \mathbb{R}^m$ also is linearly dependent.

(9) Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that there exists a linearly independent set $\{\vec{v_1}, \ldots, \vec{v_k}\} \subseteq \mathbb{R}^n$ such that $\{T(\vec{v_1}), \ldots, T(\vec{v_k})\} \subseteq \mathbb{R}^m$ is linearly dependent. Prove that $T(\vec{x}) = \vec{0}$ has a non-zero solution $\vec{x} \neq \vec{0}$.

(10) Let $T_1\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T_2\colon \mathbb{R}^m \to \mathbb{R}^p$ be two linear transformations. Prove that the composition

$$T_2 \circ T_1\colon \mathbb{R}^n \to \mathbb{R}^p, \text{ which sends } \vec{x} \mapsto T_2(T_1(\vec{x}))$$

also is a linear transformation.

(11) Consider the plane $P = \{x_3 = 1\} \subseteq \mathbb{R}^3$. (Here x_3 is the third coordinate of $\mathbb{R}^3$.)
  (a) Find a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the image of P is a plane.
  (b) Find a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the image of P is a line.
  (c) Find a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the image of P is a point.
  (d) Prove that there doesn't exist a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the image of P is the whole $\mathbb{R}^3$.
  (e) Prove that there doesn't exist a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the image of P is the hyperboloid $\{x_1^2 + x_2^2 - x_3^2 = 1\}$.

Part III (Extra credit problems, Due September 14, 11am)

The following problems worth up to 2 extra points in total (out of 10), so you can potentially get 12/10 for this homework.

Please submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW1_extra_credit**, which is separated from Part II.

Let A be a $m \times n$ matrix. Denote the columns of A as $\vec{v}_1, \ldots, \vec{v}_n$.

(1) Suppose that A is of reduced echelon form, with pivots at the $i_1, \ldots, i_k$-th columns where $1 \leq i_1 < \cdots < i_k \leq n$. Prove that the numbers $i_1, \ldots, i_k$ can be characterized by the linear dependence relations among the columns of A as follows:

  (a) By definition, we have
$$i_1 = \min\{\ell \mid 1 \leq \ell \leq n,\ \vec{v}_\ell \neq \vec{0}\}.$$

  (b) Prove that
$$i_2 = \min\{\ell \mid 1 \leq \ell \leq n,\ \text{there exists two vectors in } \vec{v}_1, \ldots, \vec{v}_\ell \text{ that are linearly independent}\}.$$

  (c) Formulate similar characterizations of $i_3, \ldots, i_k$ using the linear dependence relations among the columns of A, and prove them.

(2) Prove that elementary row operations do not effect the linear dependence relations among the columns of a matrix. More precisely, suppose that A and A' can be related via a sequence of elementary row operations. Denote the columns of A' by $\vec{v'_1}, \ldots, \vec{v'_n}$. Then for any subset $\{j_1, \cdots, j_p\} \subseteq \{1, \ldots, n\}$ we have
$$\text{“}\{\overrightarrow{v_{j_1}}, \ldots, \overrightarrow{v_{j_p}}\} \text{ is linearly independent ”} \Leftrightarrow \text{“}\{\overrightarrow{v'_{j_1}}, \ldots, \overrightarrow{v'_{j_p}}\} \text{ is linearly independent ”}.$$

(3) Using (1) and (2), show that the pivot positions of a matrix are unique, i.e. independent of the elementary row operations performed in the row reduction process.

(4) Using (3), prove a stronger statement: the reduced echelon form of a matrix is unique, i.e. independent of the elementary row operations performed in the row reduction process.