HOMEWORK 8
MATH 104, SECTION 6

Office Hours (online): Tuesday and Wednesday 9:30-11am. Access link will be posted on the course website at the beginning of office hours.

Nima's Office Hours: Monday, Tuesday and Thursday 9:30am-1pm at 1010 Evans.

Reading

There will be reading assigned for each lecture. You should come to the class having read the assigned sections of the textbook.

Due March 12: Ross, Section 19, 21, 22

Due March 17: Ross, Section 24

Problem set (9 problems; due March 12)

Submit your homework at the beginning of the lecture on Thursday. *Late homework will not be accepted under any circumstances.*

You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any.

Write clearly! Mastering mathematical writing is one of the goals of this course.

You have to staple your work if it is more than one page.

(1) Find an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ that is discontinuous at every real number.

(2) Consider the function $f\colon (0,1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ where } p, q > 0,\ \gcd(p,q) = 1. \end{cases}$$

Find all the points in $(0,1)$ where f is continuous, and give a proof.

(3) (a) Let $f\colon (X, d_X) \to (Y, d_Y)$ be a uniformly continuous function (on the whole domain X). Suppose that (x_n) is a Cauchy sequence in X. Prove that $(f(x_n))$ is a Cauchy sequence in Y. (See Ross, Definition 13.2 for the definition of Cauchy sequences in metric spaces.)

(b) Find an example of a continuous function $f\colon (X, d_X) \to (Y, d_Y)$ and a Cauchy sequence (x_n) in X such that $(f(x_n))$ is not Cauchy in Y.

(4) Determine whether the following functions are uniformly continuous, and give proofs:

(a) $A(x) = \log x$ on $(0,1)$.

(b) $B(x) = x \sin x$ on $[0, \infty)$.

(c) $C(x) = \frac{1}{x^2+1}$ on $\mathbb{R}$.

(d) $D(x) = e^x$ on $[0, \infty)$.

(e) $E(x) = \sin(\frac{1}{x})$ on $(0, \infty)$.

(Hint: Problem 3(a) could be useful for proving non-uniform continuity.)

(5) Consider the function $f\colon [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$.

(a) Prove that f is not Lipschitz continuous on $[0, \infty)$, i.e. there does not exist $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|$$

holds for any $x, y \geq 0$.

(b) Prove that f is uniformly continuous on $[0, \infty)$.

(6) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function, i.e. there exists $L > 0$ such that $f(x + L) = f(x)$ holds for any $x \in \mathbb{R}$.

(a) Prove that f attains its supremum and infimum.

(b) Prove that f is uniformly continuous on $\mathbb{R}$.

(7) Let $f\colon \mathbb{Q} \to \mathbb{R}$ be a uniformly continuous function. Prove that there exists a uniformly continuous function $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that $\widetilde{f}(x) = f(x)$ for any $x \in \mathbb{Q}$. (Hint: You may want to mimic part of the proof of Ross, Theorem 19.5.)

(8) A Lipschitz continuous map $f\colon \mathbb{R} \to \mathbb{R}$ is called *contractive* if its Lipschitz constant is less than one, i.e. there exists $0 < K < 1$ such that

$$|f(x) - f(y)| \leq K|x - y|$$

holds for any $x, y \in \mathbb{R}$. In this problem, you'll show that any contractive map on $\mathbb{R}$ has a unique fixed point.

(a) Pick any $x_1 \in \mathbb{R}$. Construct a sequence inductively: $x_2 = f(x_1)$, $x_3 = f(x_2)$, $\ldots$ $x_{n+1} = f(x_n)$, $\ldots$. Prove that such sequence (x_n) is a Cauchy sequence, therefore is convergent.

(b) Moreover, prove that the limit x^* of (x_n) is a fixed point of f, i.e. $f(x^*) = x^*$.

(c) Prove that f has a unique fixed point.

(9) Prove that there does not exist a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ that takes on every value exactly twice. Is the same statement true if we replace twice by thrice?