HW 13 Sol'n

#1. WTS: $\sum a_n^k b_n$ conv.

Following the hint of HW6, #4, define

$$S_N := b_1 + \cdots + b_N,$$

and write

$$\left|\sum_{n=M}^{N} a_n^k b_n\right| = \left|\sum_{n=M}^{N-1} (a_n^k - a_{n-1}^k) S_n + a_N^k S_N - a_M^k S_{M-1}\right|$$

$$\leq \sum_{n=M}^{N-1} |a_n^k - a_{n-1}^k||S_n| + |a_N^k||S_N| + |a_M^k||S_{M-1}|$$

$$< L\left(\sum_{n=M}^{N-1} |a_n - a_{n-1}||a_n^{k-1} + \cdots + a_{n-1}^{k-1}| + |a_N^k| + |a_M^k|\right)$$

$\left(\forall \varepsilon > 0, \exists N_0 > 0 \text{ s.t. } |a_n| < \varepsilon \quad \forall n > N_0\right)$ $\left(\text{Also, } \exists N_1 > 0 \text{ s.t. } \sum_{n=M}^{N} |a_n - a_{n-1}| < \varepsilon \quad \forall N > M > N_1\right)$

For any $N > M > \max\{N_0, N_1\}$, we have

$$\left|\sum_{n=M}^{N} a_n^k b_n\right| < L\left(k\varepsilon^{k-1} \sum_{n=M}^{N-1} |a_n - a_{n-1}| + 2\varepsilon^k\right)$$

$$< L(k+2)\varepsilon^{k-1}.$$

Then $\sum a_n^k b_n$ conv. follows from Cauchy criterion. □

#2. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x| < \delta \Rightarrow \dfrac{|f(2x) - f(x)|}{|x|} < \varepsilon$

For any $|x| < \delta$, we have

$$\left|\frac{f(x) - f(\frac{x}{2^n})}{x}\right| \leq \left|\frac{f(x) - f(\frac{x}{2})}{x}\right| + \cdots + \left|\frac{f(\frac{x}{2^{n-1}}) - f(\frac{x}{2^n})}{x}\right|$$

$$= \frac{1}{2}\left|\frac{f(x) - f(\frac{x}{2})}{\frac{x}{2}}\right| + \frac{1}{4}\left|\frac{f(\frac{x}{2}) - f(\frac{x}{4})}{\frac{x}{4}}\right| + \cdots + \frac{1}{2^n}\left|\frac{f(\frac{x}{2^{n-1}}) - f(\frac{x}{2^n})}{\frac{x}{2^n}}\right| < \varepsilon.$$

Let $n \to \infty$, since $\lim_{x \to 0} f(x) = 0$, we obtain $\left|\dfrac{f(x)}{x}\right| < \varepsilon$.

Hence $\lim_{x \to 0} \dfrac{f(x)}{x} = 0$. □

#3: Such f is a const. fcn.

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(nx) - f(ny)| < \varepsilon$. $\forall n$

In particular, $|x| < \delta \Rightarrow |f(nx) - f(0)| < \varepsilon$. $\forall n$.

~~Since f is cont. at 0, $\exists \delta > 0$ s.t. $|x| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$~~

For any $x_0 \in \mathbb{R}$, choose n large enough s.t. $\left|\frac{x_0}{n}\right| < \delta$.

Then we have $|f(x_0) - f(0)| < \varepsilon$.

This ineq. holds for any $\varepsilon > 0$,

hence $f(x_0) = f(0)$ $\forall x_0 \in \mathbb{R}$. $\square$

#4:

(a). • Since $\cos x \in [-1, 1]$, the root of $x = \cos x$ can only be in the range $[-1, 1]$.

• Since $x \leq 0 < \cos x$ $\forall x \in [-1, 0]$, the root can only be in the range $(0, 1]$.

•

• Let $f(x) = x - \cos x$, $f(0) = -1 < 0$, $f(1) > 0$.

So f has at least one root in $(0, 1]$ by IVT.

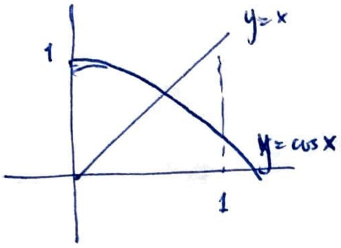
• $f'(x) = 1 + \sin x > 0$ for $x \in (0, 1]$,

so f has exactly one root. $\square$

(b) Call the unique root of $x=\cos x$: $x_0 \in (0,1)$

Suppose $a_1 > x_0$:

Claim. $a_1 > a_3 > a_5 > \cdots > x_0 > \cdots > a_6 > a_4 > a_2 > 0$.



- It's clear that $a_1 > x_0 \Rightarrow x_0 > a_2$, (and $a_1 < x_0 \Rightarrow x_0 < a_2$)
- By taking derivative of $g(x) := \cos(\cos x) - x$, one can ~~check~~ show that g is monotonically decreasing.

Thus $\forall\, a > x_0$, we have $g(a) < g(x_0) = 0$.

$\Rightarrow a_1 > a_3 > x_0$

Hence $\{a_{2n+1}\}, \{a_{2n}\}$ are both bdd. monotone seq., therefore convergent.

- $a_{2n+1} = \cos(\cos a_{2n-1}) \Rightarrow \underbrace{\lim_{n\to\infty} a_{2n+1}}_{=A} = \cos(\cos \underbrace{\lim_{n\to\infty} a_{2n-1}}_{=A})$

$\Rightarrow A = x_0$.

- Similarly, one can show that $\lim_{n\to\infty} a_{2n} = x_0 = \lim_{n\to\infty} a_{2n+1}$.

Therefore (a_n) is convergent, and the limit is x_0. $\square$

(c) $\sum a_n$ div. since $\lim a_n = x_0 \neq 0$. $\square$

#5. False.

$$f(x) = \begin{cases} 1, & x = \frac{1}{\sqrt{p}} \text{ for some prime number } p > 0. \\ 0, & \text{o/w}. \end{cases}$$

#6

Claim: $\exists N_0 > 0$ s.t. the polynomials $P_n(x)$ $\forall n > N_0$ have the same degree.

pf Otherwise, $\forall N > 0$, $\exists M > N$

s.t. $P_N(x)$ and $P_M(x)$ have diffent degrees.

then $\sup_{x \in \mathbb{R}} |P_N(x) - P_M(x)| = \infty$.

This contradicts w/ the Cauchy criterion for unif. conv. $\square$

Using the same argument, one can show that

$\exists N_1 > 0$ s.t. the coefficients of $P_n(x)$ $\forall n > N_1$ are all identical.

i.e. $P_{N_1+1}(x)$, $P_{N_1+2}(x)$, ... are all identical poly.

Hence the unif. limit. $f(x)$ is obviously a poly. $\square$

#7 Consider $h(x) := e^{g(x)} f(x)$.

$h(a) = h(b) = 0$,

By MVT, $\exists x \in (a,b)$ s.t. $h'(x) = 0$

$$h'(x) = g'(x) e^{g(x)} h(x) + e^{g(x)} f'(x)$$

$\Rightarrow g'(x) f(x) + f'(x) = 0$. $\square$

#8.

(a) is obvious.

(b) The statements follows from MVT since $F(a) = F(b) = 0$.

(c)

$$F'(x) = f'(x)\big(g(a)h(b) - g(b)h(a)\big)$$
$$+\, g'(x)\big(h(a)f(b) - h(b)f(a)\big)$$
$$+\, h'(x)\big(f(a)g(b) - f(b)g(a)\big).$$

Let $h \equiv 1$. By (b), $\exists\, c \in (a,b)$ s.t.

$$0 = F'(c) = f'(c)\big(g(a) - g(b)\big) + g'(c)\big(f(b) - f(a)\big). \quad \square$$

#9. It's clear that the lower sum of any partition P

$$L(f,P) = 0$$

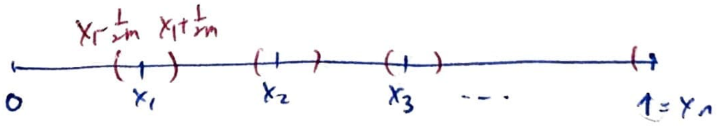
Hence $L(f) = \sup_P L(f,P) = 0$.

WTS: $U(f) = \inf_P U(f,P) = 0$, (therefore f is int. & $\int f = 0$)

pf: $\forall \varepsilon > 0$, the set

$$B_\varepsilon := \left\{ \frac{p}{q} \in [0,1] \cap \mathbb{Q} \;\middle|\; (p,q) = 1,\ \frac{1}{q} \geq \frac{\varepsilon}{2} \right\}$$

is a finite set. say $B_\varepsilon = \{0 < x_1 < x_2 \leq \cdots \leq x_n = 1\}$.



Choose m large enough s.t.
- $\frac{1}{m} < \frac{\varepsilon}{2n}$
- $x_i + \frac{1}{2m} < x_{i+1} - \frac{1}{2m}$.

Consider the partition $P_\varepsilon = \left\{0,\ x_1 - \frac{1}{2m},\ x_1 + \frac{1}{2m},\ x_2 - \frac{1}{2m},\ x_2 + \frac{1}{2m},\ \ldots,\ 1\right\}$

Then $0 \le U(f, P_\varepsilon) \le \underbrace{\underbrace{\frac{1}{m}}_{\substack{\text{length of} \\ (x_i - \frac{1}{2m},\, x_i + \frac{1}{2m})}} \cdot \underbrace{1}_{\sup f} \cdot \underbrace{n}_{\substack{\#\text{ of intervals of} \\ \text{the form } (x_i - \frac{1}{2m},\, x_i + \frac{1}{2m})}}}_{\substack{\text{subintervals containing} \\ B_\varepsilon}} + \underbrace{\frac{\varepsilon}{2}}_{\substack{\sup f \text{ for} \\ \text{intervals don't} \\ \text{contain } B_\varepsilon}} \cdot 1 < \varepsilon.$ □

#10.

(a) "$\mathbb{Q} \subset F$ dense $\Longrightarrow$ $\forall x<y$ in F, $\exists q \in \mathbb{Q}$ st. $x<q<y$".

pf Suppose $\exists x<y$ in F st. $\nexists q \in \mathbb{Q}$ st. $x<q<y$.

Then $\frac{x+y}{2} \notin \mathbb{Q}$ is not a limit pt of $\mathbb{Q}$.

$\Rightarrow$ $\mathbb{Q} \subset F$ is NOT dense. $\times$. □

"$\forall x<y$ in F, $\exists q \in \mathbb{Q}$ st. $x<q<y$ $\Longrightarrow$ $\mathbb{Q} \subset F$ dense"

pf. $\forall x \in F$, $\exists q_n \in \mathbb{Q}$ st. $x < q_n < x + \frac{1}{n}$.
$\forall n$,

$\Rightarrow$ $\lim q_n = x$.

$\Rightarrow$ $x \in \overline{\mathbb{Q}}$. □

(b) Start with an upper bound $p_1 \in \mathbb{Q}$ of S and a $q_1 \in \mathbb{Q}$ which is not an upper bound of S.

Consider $\frac{p_1+q_1}{2} \in \mathbb{Q}$

If $\frac{p_1+q_1}{2}$ is an upper bound of S, define $p_2 := \frac{p_1+q_1}{2}$ and $q_2 := q_1$

If $\frac{p_1+q_1}{2}$ is not an upper bound of S, define $p_2 := p_1$ and $q_2 := \frac{p_1+q_1}{2}$

Construct (p_n), (q_n) by proceeding the constructⁿ.

Then $(p_n) \subseteq \mathbb{Q}$ is a decreasing seq.

$(q_n) \subset \mathbb{Q}$ — increasing seq.

and $\lim_{n\to\infty} |p_n - q_n| = 0$.

Moreover, $(p_1, q_1, p_2, q_2, p_3, q_3, \cdots)$ is Cauchy by our construction.

Hence it converges to $x \in F$ by assumption.

<u>Claim</u>: $x \in F$ is the least upper bound of S.

pf • If x is not an upper bound of S,

then $\exists s \in S$ st. $x < s$.

Since each p_n is upper bound of S and $\lim p_n = x$,

$\forall \varepsilon > 0$, $\exists N > 0$ st. $n > N \Rightarrow |p_n - x| < \varepsilon$.

Pick $\varepsilon = s - x > 0$,

then $x > p_n - \varepsilon \geq s - \varepsilon = x$. $\forall n > N$. $*$. $\square$

• If x is not the least upper bound of S, i.e.

$\exists\, y$ upper bound of S st. $y < x$.

Since each q_n is not an upper bound of S and $\lim q_n = x$,

Let $\varepsilon = x - y > 0$, $\exists N > 0$ st. $n > N \Rightarrow |q_n - x| < \varepsilon$.

$\Rightarrow x < q_n + \varepsilon \leq y + \varepsilon = x$. $*$. $\square$