MATHEMATICS FROM EXAMPLES, SPRING 2023

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A note on the color code. Things in this color (blue) will not be tested in the exam; they are included either for the completeness of our discussions, or for providing further context for those of you who are interested in pursuing related topics.

A note on the references. If you have trouble with accessing the books or papers listed in the bibliography, please feel free to send me an email at: `ywfan@mail.tsinghua.edu.cn`. I can provide you with the electronic versions.

Suggestions. If you have any other suggestions, please feel free to send me an email at: `ywfan@mail.tsinghua.edu.cn`.

Lecture 1

1. Overview of the course

Examples in mathematics are like phenomena in physics. They play a vital role in the historical development of mathematics and are the driving force behind profound mathematical concepts and methods. Important theorems in modern mathematics often come from the understanding and research of some basic examples. The goal of this course is to provide the motivation and intuition behind abstract mathematical concepts by introducing some interesting examples. *Technical details are often omitted due to the nature of this course.*

Example 1.1. Let $x \in (0,1)\backslash\mathbb{Q}$ be an irrational number. It can be written uniquely as a continued fraction

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}}}$$

where $a_1, a_2, \ldots$ are positive integers. How often does a positive integer k appear in this expression?

It turns out that for any given k, the frequency of k appearing in the continued fraction expression of x is the same for *almost every* $x \in (0,1)\backslash\mathbb{Q}$. In fact, for almost every $x \in (0,1)\backslash\mathbb{Q}$, we have

$$\lim_{n\to\infty} \frac{\#\{i \mid a_i = k, 1 \le i \le n\}}{n} = \frac{1}{\log 2} \log\left(\frac{(k+1)^2}{k(k+2)}\right).$$

To prove this, we will introduce some basic ideas of *measure theory* and *ergodic theory*.

Example 1.2. Consider the following *necklace-splitting problem*. Two thieves have stolen a precious necklace (opened, with two ends), on which there are d kinds of stones (diamonds, sapphires, rubies, etc.), an even number of each kind. The thieves do not know the values of stones of various kinds, so they want to divide the stones of each kind evenly. They would like to achieve this by as few cuts as possible. The question is, what is the minimum amount of cuts to divide the stones of each kind evenly?

It is not hard to show that at least d cuts may be necessary: Place the stones of the first kind first, then the stones of the second kind, and so on. The *necklace theorem* shows that this is the worst, what can happen. In other words, d cuts is always sufficient. Surprisingly, all known proofs of this theorem are *topological*.

Example 1.3. Let $C \subseteq \mathbb{R}^2$ be a simple closed curve. One considers the following *Rectangular Peg Problems*.

- Does there always exist four points on C such that they form the vertices of a rectangle?
- Even harder question: Fix a rectangle R. Does there always exist four points on C such that they form the vertices of a rectangle which is similar to R?

The first question was answered positively by Vaughan in 1981, which uses some basic *topology*. The second question was also answered positively quite recently by Greene and Lobb [5]; their proof involves more advanced tools from *symplectic geometry*, which is beyond the scope of this course.

Example 1.4. Which positive integers n can be written as the sum of two squares $n = x^2 + y^2$?

To answer this question, it is natural to introduce the *ring* of *Gaussian integers* $\mathbb{Z}[i]$, since one has the factorization $x^2 + y^2 = (x + iy)(x - iy)$. The question then reduced to studying the properties of the ring $\mathbb{Z}[i]$.

Example 1.5. How many ways can a positive integer n be written as the sum of two (or more) squares?

The problem is closely related to the *Jacobi theta function*, which is a function defined for two complex variables $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$:

$$\theta(z;\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z) = \sum_{n=-\infty}^{\infty} q^{n^2} u^n$$

where $q = \exp(\pi i \tau)$ and $u = \exp(2\pi i z)$. By taking $z = 0$ we have

$$\theta(0;\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Let us define $r_2(n)$ to be the number of ways that n can be written as the sum of two squares; to be more precise,

$$r_2(n) = \#\{(x,y) \in \mathbb{Z}^2 \mid x^2 + y^2 = n\}.$$

It is not hard to see that

$$\theta(0;\tau)^2 = \sum_{n=0}^{\infty} r_2(n) q^n.$$

The problem then reduces to understand $\theta(0;\tau)^2$. It turns out that $\theta(0;\tau)^2$ is a *modular form of weight* 1 *for the congruence subgroup* $\Gamma_1(4) \subseteq \mathrm{SL}(2,\mathbb{Z})$, and we can use the theory of modular forms to obtain an explicit formula of $r_2(n)$.

Example 1.6. (to be continued...)

2. Measure theory and ergodic theory

Recall our motivating question: Let $x \in (0,1)\backslash\mathbb{Q}$ be an irrational number. It can be written uniquely as a continued fraction

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}}}$$

where $a_1, a_2, \ldots$ are positive integers. How often does a positive integer k appear in this expression? Below is the sketch of ideas toward answering this question.

- Define the *continued fraction map* $T\colon [0,1] \to [0,1]$ by $T(0) = 0$ and

  $$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \text{ for } x \neq 0,$$

  where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t. In other words, $T(x)$ is the fractional part $\{\frac{1}{x}\}$ of $\frac{1}{x}$.

- Observe that $a_n = k$ if and only if $T^{n-1}(x) \in (\frac{1}{k+1}, \frac{1}{k}]$. Hence
$$\frac{\#\{i \mid a_i = k, 1 \le i \le n\}}{n} = \frac{1}{n}\sum_{i=0}^{n-1} \chi_{(\frac{1}{k+1},\frac{1}{k}]}(T^i(x))$$
where χ is the characteristic function.
- Define the *Gauss measure* μ on $[0,1]$ to be
$$\mu(A) = \frac{1}{\log 2}\int_A \frac{1}{1+x}\,\mathrm{d}x \text{ for any measurable set } A \subseteq [0,1].$$
- Prove that the Gauss measure μ is *T-invariant* and *ergodic* .
- By *Birkhoff's pointwise ergodic theorem*, for almost every $x \in [0,1]\backslash\mathbb{Q}$ we have
$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_{(\frac{1}{k+1},\frac{1}{k}]}(T^i(x)) = \int \chi_{(\frac{1}{k+1},\frac{1}{k}]}\,\mathrm{d}\mu = \mu\left(\left(\frac{1}{k+1},\frac{1}{k}\right]\right).$$
- The conclusion then follows from a simple calculation
$$\frac{1}{\log 2}\int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{1}{1+x}\,\mathrm{d}x = \frac{1}{\log 2}\log\left(\frac{(k+1)^2}{k(k+2)}\right).$$

In order to understand this approach and appreciate the powerful tools provided by ergodic theory (in our case, the pointwise ergodic theorem), we will discuss the following topics in this section:

- basic measure theory;
- basic ergodic theory;
- ergodic theorems and applications.

Some references that might be helpful include [4] and [10].

2.1. **An outlook.** Consider a map $T\colon X \to X$. In ergodic theory, one studies how *typical* orbits $\{x, T(x), T^2(x), \ldots\}$ are distributed. We would be interested in properties like *frequencies of visits*, *equidistribution*, *mixing*, etc.

Here is a basic example. Let $A \subseteq X$ be a subset, and x be an element of X. The number of visits of orbit of x to the subset A up to time n is given by
$$\#\{0 \le k \le n-1 \mid T^k(x) \in A\}.$$

A convenient way to write this quantity is as follows. Let $\chi_A \colon X \to \mathbb{R}$ be the characteristic function of the subset A: $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \notin A$. Then we have

$$\sum_{k=0}^{n-1} \chi_A(T^k(x)) = \#\{0 \leq k \leq n-1 \mid T^k(x) \in A\}.$$

The *frequency* of visits up to time n is defined to be the average

$$\frac{1}{n}\sum_{k=0}^{n-1} \chi_A(T^k(x)) \in [0,1].$$

Question 2.1. *We are interested in the following questions.*

(a) Does the frequency of visits converge to a limit as n tends to infinity? (for all points of $x \in X$? or only for a typical point?)

(b) If the limit exists, what does the frequency converge to?

Another type of question concerns the equidistributioness. Let us consider specifically in the setting of the unit interval $[0,1]$. We say a sequence of points $\{x_n\}$ in $[0,1]$ is *equidistributed* if for all intervals $I \subseteq [0,1]$ we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_I(x_k) = \text{length}(I).$$

An equivalent definition is for all continuous functions $f \colon [0,1] \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(x_k) = \int_0^1 f(x)\,\mathrm{d}x.$$

So if we have a dynamical system $T \colon [0,1] \to [0,1]$ (or $T \colon S^1 \to S^1$, where $S^1 \cong \mathbb{R}/\mathbb{Z} \cong [0,1]/0 \sim 1$), we can ask whether orbits $\{x.T(x), T^2(x), \dots\}$ are equidistributed or not.

Example 2.2. Consider the *rotation* map

$$R_\alpha \colon S^1 \to S^1; \qquad x \mapsto x + \alpha \pmod 1.$$

If $\alpha \in \mathbb{Q}$ is a rational number, then every orbit of R_α is periodic, therefore cannot be equidistributed. If $\alpha \notin \mathbb{Q}$ is irrational, then one can show that every orbit of R_α is equidistributed (this is often thought of as the first ergodic theorem to have been proved).

Example 2.3. Consider the *doubling map*

$$T_2\colon S^1 \to S^1; \qquad x \mapsto 2x \pmod 1.$$

It is not hard to see that there is a dense subset of X for which the orbit of T_2 is periodic, therefore not equidistributed. However, it turns out that for *almost all* $x \in X$ the orbit of T_2 is equidistributed.

We may also have maps (e.g. the continued fraction map) where the orbits are not equidistributed for almost all $x \in X$.

To make these notions precise, we need to introduce some measure theory, which have the advantage of introducing a theory of integration that is suitable for our purposes.

2.2. **σ-algebras, measures, probability spaces.** Intuitively, a *measure* μ on a space X is a function on a collection of subsets of X, called *measurable sets*, which assigns to each measurable set A its *measure* $\mu(A) \geq 0$. You already know at least two natural examples of measures.

Example 2.4. Let $X = \mathbb{R}$. The Lebesgue measure λ on $\mathbb{R}$ assigns to each interval $[a, b]$ its length

$$\lambda([a, b]) = b - a = \int_a^b \mathrm{d}x.$$

Let $X = \mathbb{R}^2$. The Lebesgue measure λ on $\mathbb{R}^2$ assigns to each measurable set $A \subseteq \mathbb{R}^2$ its area

$$\lambda(A) = \int_A \mathrm{d}x\,\mathrm{d}y.$$

One might hope to assign a measure to all subsets of X. Unfortunately, if we want the measure to have reasonable and useful properties, this would lead to a contradiction in certain cases (we will see an example later). So we are forced to assign a measure only to a sub-collection of all subsets of X.

Let X be a set. Denote by $\mathbb{P}(X)$ the collection of all subsets of X.

Definition 2.5. A subset $\mathcal{B} \subseteq \mathbb{P}(X)$ is called a *σ-algebra* on X if

(a) the empty set $\emptyset \in \mathcal{B}$,

(b) $\mathcal{B}$ is closed under complementation: $A \in \mathcal{B}$ implies $X \backslash A \in \mathcal{B}$,

(c) $\mathcal{B}$ is closed under countable union: $A_1, A_2, \ldots \in \mathcal{B}$ implies $\cup_{n=1}^{\infty} A_i \in \mathcal{B}$.

Elements of the σ-algebra are called *measurable sets*.

Remark 2.6. Let $F \subseteq \mathbb{P}(X)$ be an arbitrary subset (may or may not be a σ-algebra). Then there exists a unique smallest σ-algebra which contains every set in F. It is called the *σ-algebra generated by F*.

An important example is the *Borel algebra* over any *topological space*: it is the σ-algebra generated by the *open sets*. For instance, the Borel algebra over $[0,1]$ is the σ-algebra generated by the collection of open sub-intervals of $[0,1]$.

Definition 2.7. Let X be a set and $\mathcal{B}$ be a σ-algebra on X. A function $\mu\colon \mathcal{B} \to \mathbb{R} \cup \{\infty\}$ is called a *measure* if

(a) $\mu(\emptyset) = 0$,
(b) (non-negativity) $\mu(E) \geq 0$ for all $E \in \mathcal{B}$,
(c) (countable additivity) for all countable collections $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{B}$, we have

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

The triple $(X, \mathcal{B}, \mu)$ is called a *measurable space*, and it is called a *probability space* if $\mu(X) = 1$.

Example 2.8. Let $X = [0,1]$ and let $\mathcal{B}$ be the Borel algebra on X, i.e. the σ-algebra generated by all open subintervals (a,b). There exists a measure (the *Lebesgue* measure) $\lambda\colon \mathcal{B} \to \mathbb{R}$ such that $\lambda((a,b)) = b - a$. The triple $(X, \mathcal{B}, \lambda)$ forms a probability space.

Remark 2.9. Given a probability space $(X, \mathcal{B}, \mu)$, one can regard X as the space of all possible *events*, and $\mu(A)$ gives the probability of an event occurs in a measurable subset $A \subseteq X$.

Example 2.10. Let us consider a discrete example. Let $X = \{1, \ldots, n\}$, and let $\mathcal{B} = \mathbb{P}(X)$ be the σ-algebra consists of all subsets of X. Choose any $0 \leq p_1, \ldots, p_n \leq 1$ such that $\sum p_i = 1$. Then one can define a measure $\mu\colon \mathcal{B} \to \mathbb{R}$ by

$$\mu(\{i_1, \ldots, i_k\}) = p_{i_1} + \cdots + p_{i_k}.$$

Remark 2.11. In this remark, we show that in general it is necessary to restrict the definition of measure on a subset $\mathcal{B} \subseteq \mathbb{P}(X)$, as opposed to defining it on the *whole* collection of subsets of X. Consider the Lebesgue measure $\lambda\colon \mathcal{B} \to \mathbb{R}_{\geq 0}$ on $X = \mathbb{R}$. It satisfies the following properties:

- λ has the countable additivity property in the definition of measure,
- if two subsets of A and B are related by a translation, then $\lambda(A) = \lambda(B)$,
- $\lambda([0, 1]) = 1$.

We show that unfortunately it is not possible to extend the definition of λ to *all* subsets of $\mathbb{R}$ that still satisfy these three properties.

Let us consider the example constructed by Vitali in 1905. A *Vitali set* is a subset $V \subseteq [0, 1]$ of real numbers such that, for each real number r, there is exactly one number $v \in V$ such that $v - r \in \mathbb{Q}$. Equivalently, V is constructed by choosing a representative in $[0, 1]$ of each element of the quotient group $\mathbb{R}/\mathbb{Q}$.

Let $q_1, q_2, \ldots$ be an enumeration of the rational numbers in $[-1, 1]$ (recall that $\mathbb{Q}$ is *countable*). Consider the translated sets $V_k = V + q_k$ for $k = 1, 2, \ldots$. It is not hard to show the following:

- V_k's are pairwise disjoint,
- $[0, 1] \subseteq \cup_{k=1}^{\infty} V_k \subseteq [-1, 2]$.

Assume the contrary that it is possible to extend the definition of Lebesgue measure to *all* subsets of $\mathbb{R}$ which satisfies the properties above. Then we have

$$1 \leq \sum_{k=1}^{\infty} \lambda(V_k) \leq 3.$$

Since Lebesgue measure is translation invariant, we have $\lambda(V_k) = \lambda(V)$, hence

$$1 \leq \sum_{k=1}^{\infty} \lambda(V) \leq 3.$$

But this is impossible: If $\lambda(V) = 0$ then $\sum_{k=1}^{\infty} \lambda(V) = 0$; if $\lambda(V) > 0$ then $\sum_{k=1}^{\infty} \lambda(V) = \infty$. Contradiction.

2.3. **Measure-preserving functions.**

Definition 2.12. Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be two probability spaces.

- A map $T\colon X \to Y$ is called *measurable* if $T^{-1}(A) \in \mathcal{B}$ for any $A \in \mathcal{C}$.
- Furthermore, a measurable function T is called *measure-preserving* if $\mu(T^{-1}(A)) = \nu(A)$ for any $A \in \mathcal{C}$.

- If $T\colon X \to X$ is measure-preserving, then we say $(X, \mathcal{B}, \mu, T)$ is a *measure-preserving system.*

Exercise. Let X be a topological space and $\mathcal{B}$ be the Borel σ-algebra on X (which is generated by open sets of X). Show that any *continuous* map $T\colon X \to X$ is measurable.

Exercise. To show a measurable map $T\colon X \to Y$ is measure-preserving, it is enough to check $\mu(T^{-1}(A)) = \nu(A)$ holds for a generating set of $\mathcal{C}$.

Example 2.13 (Rotation on S^1). Consider the circle $S^1 \cong \mathbb{R}/\mathbb{Z}$, which can be obtained by identifying the two endpoints of $[0, 1]$. One equips S^1 with the Lebesgue measure. It is easy to show that the rotation

$$R_\alpha\colon S^1 \to S^1; \qquad x \mapsto x + \alpha \pmod 1$$

is measure-preserving for any α.

Example 2.14 (Doubling map on S^1). Define the *doubling map*

$$T_2\colon S^1 \to S^1; \qquad x \mapsto 2x \pmod 1.$$

Let us show that it is measure-preserving. It is enough to check this on intervals: we have $\mu(T_2^{-1}(a, b)) = \mu(a, b)$ since

$$T_2^{-1}(a, b) = \left(\frac{a}{2}, \frac{b}{2}\right) \cup \left(\frac{a+1}{2}, \frac{b+1}{2}\right).$$

Note that the measure-preserving property cannot be seen by studying "forward iterates": $\mu(T_2(a, b)) \neq \mu(a, b)$ in general.

Example 2.15. Define the $(\frac{1}{2}, \frac{1}{2})$-measure $\mu_{(1/2,1/2)}$ on the finite set $\{1, 2\}$ by

$$\mu_{(1/2,1/2)}(\{1\}) = \mu_{(1/2,1/2)}(\{2\}) = \frac{1}{2}.$$

Consider the space of infinite product $X = \{1, 2\}^{\mathbb{N}}$, which models the set of possible outcomes of the infinitely repeated toss of a coin. Given a finite subset $I \subseteq \mathbb{N}$ and a map $a\colon I \to \{1, 2\}$, we define the *cylinder set* associated to I and a to be

$$I(a) = \{x \in X \mid x_j = a(j) \text{ for all } j \in I\},$$

i.e. one specifies the outcome of the j-th thows for all $j \in I$. We define $\mathcal{B}$ to be the σ-algebra generated by all cylinder sets, and define a measure $\mu\colon \mathcal{B} \to \mathbb{R}$ via

$$\mu(I(a)) = \left(\frac{1}{2}\right)^{\#|I|}.$$

Consider the *left shift map* $\sigma\colon X \to X$ defined by

$$\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

It is easy to see that $(X, \mathcal{B}, \mu, \sigma)$ is a measure-preserving system.

In fact, this system is *measurably isomorphic* to the doubling map T_2 on S^1, which roughly means that they are identical except on a measure zero set. Indeed, consider the map $\phi\colon X \to S^1 \cong [0,1]/0 \sim 1$ where

$$\phi(x_1, x_2, \ldots) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Then we have $\phi \circ \sigma = T_2 \circ \phi$. Below is the precise definition of the notion of measurably isomorphic.

Definition 2.16. We say two measure-preserving systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are *measurably isomorphic* if there exists $X' \in \mathcal{B}$ and $Y' \in \mathcal{C}$ such that:

- $\mu(X') = \nu(Y') = 1$,
- $T(X') \subseteq X'$, $S(Y') \subseteq Y'$,
- there exists a bijective map $\phi\colon X' \to Y'$ such that both ϕ and ϕ^{-1} are measurable and measure-preserving, and
- $\phi \circ T(x) = S \circ \phi(x)$ for any $x \in X'$.

Example 2.17 (Bernoulli shift). Consider the two-sided infinite set

$$\begin{aligned} X &= \{1, \ldots, n\}^{\mathbb{Z}} \\ &= \{x = (\ldots, x_{-1}, x_0, x_1, \ldots) \mid x_i \in \{1, \ldots, n\} \text{ for all } i\}. \end{aligned}$$

which gives the sample space of the outcome of throwing an n-sided die (each appears with probabilities $p_1, \ldots, p_n$) infinitely many times. Let us define a σ-algebra and a measure on X. Given a finite subset $I \subseteq \mathbb{Z}$ and a map $a\colon I \to \{1, \ldots, n\}$, we define the *cylinder set* associated to I and a to be

$$I(a) = \{x \in X \mid x_j = a(j) \text{ for all } j \in I\},$$

i.e. one specifies the outcome of the j-th thows for all $j \in I$. We define $\mathcal{B}$ to be the σ-algebra generated by all cylinder sets, and define a measure $\mu\colon \mathcal{B} \to \mathbb{R}$ via

$$\mu(I(a)) = \prod_{j \in I} p_{a(j)}.$$

Now, consider the left shift map $\sigma\colon X \to X$ defined by $\sigma(x)_i = x_{i+1}$. It clearly preserves the measure of all cylinder sets, hence $(X, \mathcal{B}, \mu, \sigma)$ is a measure-preserving system. The map σ is called the *Bernoulli shift*.

2.4. **Recurrence.** One of the central themes in ergodic theory is *recurrence*, which concerns how points in measurable dynamical systems return close to themselves under iterations.

Theorem 2.18 (Poincaré recurrence)**.** *Let $T\colon X \to X$ be a measure-preserving transformation on a probability space $(X, \mathcal{B}, \mu)$, and let $E \in \mathcal{B}$ be a measurable set with $\mu(E) > 0$. Then almost every point $x \in E$ returns to E infinitely many often under iterations of T. More precisely, there exists a measurable set $F \subseteq E$ such that $\mu(F) = \mu(E)$, and for every point $x \in F$ the sequence of points $\{T^n(x)\}_{n=1}^{\infty}$ returns to E infinitely many times.*

Proof. Let

$$B = \{x \in E \mid T^n(x) \notin E \text{ for all } n \geq 1\}.$$

It is an easy exercise to show that B is measurable. Using the definition of B, one can show that the sets $B, T^{-1}B, T^{-2}B, \dots$ are pairwise disjoint. Hence

$$\sum_{k=0}^{\infty} \mu(T^{-k}B) = \mu\left(\bigcup_{k=0}^{\infty} T^{-k}B\right) \leq \mu(X) = 1.$$

Therefore we have $\mu(B) = 0$, since T is measure-preserving.

Observe that the points of the union

$$\bigcup_{k=0}^{\infty} (T^{-k}B \cap E)$$

are precisely those points of E which do not return to E infinitely many often. Therefore, it suffices to show that the measure of the above union is zero.

$$\mu\left(\bigcup_{k=0}^{\infty} (T^{-k}B \cap E)\right) \leq \mu\left(\bigcup_{k=0}^{\infty} T^{-k}B\right) = \sum_{k=0}^{\infty} \mu(T^{-k}B) = 0$$

since $\mu(B) = 0$ and T is measure-preserving. □

Remark 2.19. The key step of the proof is to show that $\mu(B) = 0$, which is essentially the pigeon-hole principle: the sets $B, T^{-1}B, T^{-2}B, \dots$ are disjoint and with the same measure, so they can not fit into a space of finite measure ($\mu(X) = 1$) unless $\mu(B) = 0$. The recurrence property does not hold for spaces of infinite measure (can you give an example?).

Remark 2.20. If one further assumes that the map $T\colon X \to X$ is *ergodic*, then one can show that the *frequency* of return to the set E is precisely $\mu(E) > 0$.

2.5. Lebesgue integral.

Definition 2.21. Let $(X, \mathcal{B}, \mu)$ be a probability space. A function $f\colon X \to \mathbb{R}$ is called *measurable* if $f^{-1}(A) \in \mathcal{B}$ for any (Borel) measurable set $A \subseteq \mathbb{R}$.

We would like to define the *(Lebesgue) integral* $\int f \,\mathrm{d}\mu$ of measurable functions f. First, a function $g\colon X \to \mathbb{R}$ is called *simple* if

$$g(x) = \sum_{j=1}^{m} c_j \chi_{A_j}(x)$$

for some constants $c_j \in \mathbb{R}$ and *disjoint* measurable sets $A_j \in \mathcal{B}$. In this case, the integral of g is defined to be

$$\int g \,\mathrm{d}\mu = \sum_{j=1}^{m} c_j \mu(A_j).$$

Second, one can show that for any *non-negative* measurable function $f\colon X \to \mathbb{R}_{\geq 0}$, there exists a pointwise increasing sequence of simple functions $(g_n)_{n\geq 1}$ which converges to g_n pointwisely converges to f. This allows us to define

$$\int f \,\mathrm{d}\mu = \lim_{n\to\infty} \int g_n \,\mathrm{d}\mu.$$

A non-negative measurable function $f\colon X \to \mathbb{R}_{\geq 0}$ is called *integrable* if $\int f \,\mathrm{d}\mu < \infty$.

Finally, for a general measurable function $f\colon X \to \mathbb{R}$, one can decompose it into $f = f^+ - f^-$ where $f^+(x) = \max\{f(x), 0\}$. Both f^+, f^- are non-negative measurable functions. The function f is called *integrable* if both f^+, f^- are integrable, and its integral is defined to be

$$\int f \,\mathrm{d}\mu = \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\mu.$$

Notation. Let $(X, \mathcal{B}, \mu)$ be a measurable space. Define

$$L^1_\mu = \left\{ f\colon X \to \mathbb{R} : f \text{ is measurable and } \|f\|_1 := \int |f| \,\mathrm{d}\mu < \infty \right\}.$$

Similarly, define

$$L^2_\mu = \left\{ f\colon X \to \mathbb{R} : f \text{ is measurable and } \|f\|_2 := \left(\int |f|^2 \,\mathrm{d}\mu \right)^{1/2} < \infty \right\}.$$

The following theorem provides an important characterization of measure-preserving maps.

Theorem 2.22. *Let $(X, \mathcal{B}, \mu)$ be a probability space. A map $T\colon X \to X$ is measure-preserving if and only if*

$$\int f \,\mathrm{d}\mu = \int f \circ T \,\mathrm{d}\mu \qquad \textit{for all } f \in L^1_\mu.$$

Proof. First, we prove the "if" part. Take $f = \chi_B$ for any $B \in \mathcal{B}$, one gets

$$\mu(T^{-1}B) = \int \chi_{T^{-1}B} \,\mathrm{d}\mu = \int \chi_B \circ T \,\mathrm{d}\mu = \int \chi_B d\mu = \mu(B).$$

Conversely, if T is measure-preserving, then the integral equality holds for any simple functions. For any $f \in L^1_\mu$, one can take an increasing sequence (f_n) of simple functions such that $\lim f_n = f$ pointwise. Hence we also have $\lim f_n \circ T = f \circ T$. By dominated convergence theorem,

$$\int f \,\mathrm{d}\mu = \lim_{n\to\infty} \int f_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int f_n \circ T \,\mathrm{d}\mu \int f \circ T \,\mathrm{d}\mu$$

□

Remark 2.23. The Lebesgue integral is more general than the *Riemann integral*: The Lebesgue integral allows a countable infinity of discontinuities, while Riemann integral allows only a finite number of discontinuities. As an example, consider the set $A = \mathbb{Q} \cap [0, 1]$ of rational numbers in $[0, 1]$. It is an easy exercise of Riemann integral to show that the characteristic function

$\chi_A\colon [0,1] \to \mathbb{R}$ is not integrable. On the other hand, the set A is measurable and its Lebesgue measure is $\lambda(A) = 0$. Therefore, χ_A is Lebesgue measurable and

$$\int \chi_A \,\mathrm{d}\lambda = \lambda(A) = 0.$$

Lecture 2

2.6. Ergodicity.

Definition 2.24. Let $(X, \mathcal{B}, \mu)$ be a probability space. A measure-preserving transformation $T\colon X \to X$ is said to be *ergodic* if for any $B \in \mathcal{B}$,

$$T^{-1}B = B \implies \mu(B) = 0 \text{ or } \mu(B) = 1.$$

In words, it is impossible to split X into T-invariant subsets of positive measures.

Non-example. Consider the rotation map $R_\alpha(x) = x + \alpha \pmod 1$ on the circle S^1. It is not hard to show that if α is rational then R_α is not ergodic. For instance, when $\alpha = \frac{1}{2}$, the set $B = (0, \frac{1}{4}) \cup (\frac{1}{2}, \frac{3}{4})$ satisfies $R_\alpha^{-1}B = B$ but $\mu(B) = \frac{1}{2}$. We will see later that if α is irrational then R_α is ergodic.

Example 2.25. Let us show that the *Bernoulli shifts* σ are ergodic. First, we claim that the Bernoulli shifts are *mixing*, i.e.

$$\lim_{n\to\infty} \mu(B \cap \sigma^{-n}B') = \mu(B)\mu(B') \qquad \text{for all } B, B' \in \mathcal{B}.$$

It is easy to see that the statement is true if B and B' are both finite unions of cylinder sets. By Kolmogorov extension theorem (which we will not discuss here), for any measurable set B and any $\epsilon > 0$, there exists a finite union of cylinder sets A such that $\mu(A\Delta B) < \epsilon$. (Here $A\Delta B \coloneqq (A\backslash B) \cup (B\backslash A)$.) It is then an easy exercise to show the mixing property.

Second, we claim that mixing implies ergodic. Let $B = \sigma^{-1}B$ be a measurable σ-invariant set. By the mixing property, we have

$$\mu(B) = \lim_{n\to\infty} \mu(B \cap \sigma^{-n}B) = \mu(B)^2.$$

Hence $\mu(B) \in \{0, 1\}$.

Remark 2.26. As the proof above suggests, the concept of ergodicity is closely related to the idea of *mixing*, meaning, given a measurable set $A \subseteq X$, how the set $T^{-n}A$ is spread around the whole space X under large iterations n? It can be proved that a measure-preserving system $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if it is *weak-mixing* (a weaker condition than *mixing*), i.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}B) = \mu(A)\mu(B) \text{ for all } A, B \in \mathcal{B}.$$

A proof of this fact can be found in [4, Section 2.7].

The following theorem is very useful for proving a system is ergodic (or non-ergodic).

Theorem 2.27. *For a measure-preserving system $(X, \mathcal{B}, \mu, T)$, the following are equivalent.*

(a) T is ergodic.

(b) For any $f\colon X \to \mathbb{R}$ measurable, if $f \circ T = f$ almost everywhere, then f is constant almost everywhere.

Proof. It is easy to see that (b) implies (a): Suppose $T^{-1}B = B$. Take $f = \chi_B$. Then we have χ_B is constant almost everywhere, thus $\mu(B) \in \{0, 1\}$. A proof of (a) implies (b) can be found in [4, Proposition 2.14]. □

Remark 2.28. One can show that in the characterization theorem above, instead of considering all measurable functions, it is enough to consider only the integrable functions $f \in L^1_\mu$ or the square-integrable functions $f \in L^2_\mu$. More precisely, for a measure-preserving system $(X, \mathcal{B}, \mu, T)$, the following statements are all equivalent:

(a) T is ergodic.

(b) For any $f\colon X \to \mathbb{R}$ measurable, if $f \circ T = f$ almost everywhere, then f is constant almost everywhere.

(c) For any $f \in L^1_\mu$, if $f \circ T = f$ almost everywhere, then f is constant almost everywhere.

(d) For any $f \in L^2_\mu$, if $f \circ T = f$ almost everywhere, then f is constant almost everywhere.

Using this remark and some basic knowledge of *Fourier series*, one can easily show that the rotation maps and the doubling map of S^1 are ergodic. Let $f\colon S^1 \cong \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a square-integrable function, i.e. $f \in L^2(S^1)$. Results of

Fourier series imply that there exists a *unique* collection of complex numbers $\dots, c_{-2}, c_{-1}, c_0, c_1, c_2, \dots$, called the *Fourier coefficients* of f, such that

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x} \qquad \text{for a.e. } x \in \mathbb{R}/\mathbb{Z}.$$

Moreover, we have $\|f\|_2 = \sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$.

Example 2.29. Consider the rotation map $R_\alpha(x) = x + \alpha \pmod 1$ on the circle S^1 where α is irrational. By Remark 2.28, it suffices to show that for any $f \in L^2(S^1)$, if $f \circ R_\alpha = f$ almost everywhere, then f is constant almost everywhere. Let the Fourier series of f be $\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$. Then

$$\left(\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}\right) \circ R_\alpha = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n (x+\alpha)} = \sum_{n\in\mathbb{Z}} c_n e^{2\pi n \alpha} e^{2\pi i n x}.$$

By the uniqueness of the Fourier coefficients, we have

$$c_n \left(1 - e^{2\pi n \alpha}\right) = 0 \qquad \text{for all } n \in \mathbb{Z}.$$

Suppose α is irrational, then $1 - e^{2\pi n \alpha} \neq 0$ for all $n \in \mathbb{Z}\backslash\{0\}$, thus we have $c_n = 0$ for all $n \in \mathbb{Z}\backslash\{0\}$. Hence $f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x} = c_0$ is constant almost everywhere. (Can you identify at where this argument fails for α rational?)

Example 2.30. We show that the doubling map $T_2 \colon S^1 \to S^1$ is ergodic. Let $f \in L^2(S^1)$ with $f \circ T = f$ almost everywhere. Let $\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$ be the Fourier series of f, where $\|f\|_2^2 = \sum_{i\in\mathbb{Z}} |a_i|^2 < \infty$. Then

$$\left(\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}\right) \circ T_2 = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n (2x)} = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i (2n) x}$$

By the uniqueness of the Fourier coefficients, we have $c_n = c_{2n}$ for all $n \in \mathbb{Z}$. This implies that $c_n = 0$ for all $n \neq 0$. Hence f is a constant function almost everywhere.

2.7. **Ergodic theorems.** Let X be the *phase space* of a physical system (e.g. the points of X can represent configurations of positions and velocities of particles in a box). A measurable function $f \colon X \to \mathbb{R}$ represents an *observable* of the system, i.e. a quantity that can be measured (e.g. velocity, temperature, position, etc.). The value $f(x)$ is the measurement of the observable f that one gets when the system is in the state x. *Time evolution* of the system, if measured by discrete time units, can be given by a transformation $T\colon X \to X$, so that if $x \in X$ is the initial state of the system, then $T(x)$ is the state of the system after one time unit. The map T is measure-preserving if the system is in equilibrium.

In order to measure a physical quantity, one usually measures repeatedly in time and consider their average. The average of the first n measurements is given by

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x).$$

This quantity is called the *time average*. On the other hand, the *space average* of the observable f is simply

$$\int f \,\mathrm{d}\mu.$$

In physics, one would like to know the space average of the observable; but since experimentally it is easier to compute the time average, it is natural to ask whether the time average gives a good approximation of the space average as $n \to \infty$.

Boltzmann's Hypothesis was that for almost every initial state $x \in X$ the time averages of any observable f converge to the space average as time tends to infinity. Unfortunately, this is not true for general measure-preserving map T. On the other hand, *under the assumption that T is ergodic*, the conclusion of Boltzmann's Hypothesis is true, and this is exactly the content of Birkhoff's ergodic theorem. Finding the right condition under which Boltzmann's Hypothesis holds motivated the definition of ergodicity, and gave birth to the study of ergodic theory.

Theorem 2.31. *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system on a probability space, and let $f\colon X \to \mathbb{R}$ be an integrable function.*

(a) The limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) = f^*(x)$$

converges almost everywhere to a T-invariant integrable function f^*, where

$$\int f^* \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu.$$

(b) Moreover, if T is ergodic, then

$$f^*(x) = \int f \,\mathrm{d}\mu$$

almost everywhere.

A proof of the theorem can be found in [4, Section 2.6]. Note that the second part of the statement is an easy corollary of the first part using Theorem 2.27.

Remark 2.32. Note that for an ergodic system $(X, \mathcal{B}, \mu, T)$ and a measurable function $f\colon X \to \mathbb{R}$, the ergodic theorem only guarantees the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \int f \,\mathrm{d}\mu$$

almost everywhere; the equality may not be satisfied by *every* points of X. For instance, consider the doubling map $T_2\colon S^1 \to S^1$ which is ergodic. Choose any measurable function $f\colon S^1 \to \mathbb{R}$ such that $\int f \,\mathrm{d}\mu \neq f(0)$. Then the above equality is not satisfied at the point $x = 0 \in S^1$.

Example 2.33 (Frequency of visits). Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving ergodic system, and let $A \subseteq X$ be a measurable set with $\mu(A) > 0$. We would like to understand the frequency of visits:

$$\frac{\#\{0 \leq k \leq n-1 \mid T^k(x) \in A\}}{n} = \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(T^k(x)).$$

Applying Birkhoff's pointwise ergodic theorem to $f = \chi_A$, one gets

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(T^k(x)) = \int \chi_A \,\mathrm{d}\mu = \mu(A).$$

2.8. **Back to continued fractions.**

Definition 2.34. A *continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

denotes alternatively by $[a_0; a_1, a_2, a_3, \ldots]$, where $a_0 \in \mathbb{Z}_{\geq 0}$ and $a_n \in \mathbb{Z}_{>0}$ for all $n \geq 1$. This expression can be finite (when the represented number is rational) or infinite (when the represented number is irrational).

Exercise. Fix a sequence $(a_n)_{n \geq 0}$ where $a_0 \in \mathbb{Z}_{\geq 0}$ and $a_n \in \mathbb{Z}_{>0}$ for all $n \geq 1$. Denote the partial expressions as

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$$

where p_n, q_n are coprime positive integers. Then they satisfy the recursive relation

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, we have

$$p_{n+1} = a_{n+1} p_n + p_{n-1}, \qquad q_{n+1} = a_{n+1} q_n + q_{n-1}.$$

Also, by taking the determinants of the matrix equation, we get

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}.$$

Hence

$$\begin{aligned} \frac{p_n}{q_n} &= \frac{p_{n-1}}{q_{n-1}} + (-1)^{n+1} \frac{1}{q_{n-1} q_n} \\ &= a_0 + \frac{1}{q_0 q_1} - \frac{1}{q_1 q_2} + \cdots + (-1)^{n+1} \frac{1}{q_{n-1} q_n} \end{aligned}$$

by induction, and show that

$$x = \lim_{n \to \infty} [a_0; a_1, \ldots, a_n] = \lim_{n \to \infty} \frac{p_n}{q_n} = a_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{q_{n-1} q_n}.$$

Moreover, we have

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < \frac{p_{2n}}{q_{2n}} < \cdots < x < \cdots < \frac{p_{2n+1}}{q_{2n+1}} < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

The rational numbers $\frac{p_n}{q_n}$ are called the *convergents* of the continued fraction for x, and they provide very rapid rational approximation to x. We have

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

The numbers q_n and p_n grow exponentially as $n \to \infty$: using the recursive relation, one can show that both p_n and q_n are greater than $2^{(n-2)/2}$.

In fact, the continued fraction convergents provide the *optimal* rational approximants of an irrational number in the following sense.

Proposition 2.35. *Let $x > 0$ be an irrational number, $[a_0; a_1, \ldots]$ be its associated continued fraction, and $\frac{p_n}{q_n}$ be its convergents defined above. For any $1 \leq q < q_n$ and any $p_n > 0$, we have*

$$\left| x - \frac{p_n}{q_n} \right| < \left| x - \frac{p}{q} \right|.$$

Definition 2.36. Define the *continued fraction map* $T\colon [0,1] \to [0,1]$ by $T(0) = 0$ and

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \text{ for } x \neq 0,$$

where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t. In other words, $T(x)$ is the fractional part $\{\frac{1}{x}\}$ of $\frac{1}{x}$.

For our purpose, we would like to find a measure on $[0,1]$ such that the continued fraction map T is measure-preserving. Unfortunately, the usual Lebesgue measure on $[0,1]$ does not work. For instance,

$$T^{-1}\left(0, \frac{1}{2}\right) = \left(\frac{2}{3}, 1\right) \cup \left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{2}{7}, \frac{1}{3}\right) \cup \cdots,$$

which has measure strictly greater than $1/2$ with respect to the standard Lebesgue measure.

Definition 2.37. Define the *Gauss measure* μ on $[0,1]$ to be

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx \text{ for any measurable set } A \subseteq [0,1].$$

Exercise. The Gauss measure is “comparable” with the standard Lebesgue measure λ on $[0,1]$: Show that

$$\frac{\lambda(B)}{2\log 2} \leq \mu(B) \leq \frac{\lambda(B)}{\log 2} \qquad \text{for any measurable set } B \subseteq [0,1].$$

Proposition 2.38. *The continued fraction map T preserves the Gauss measure μ.*

Proof. It suffices to show it for $A = [0,b]$ for all $b > 0$. Observe that

$$T^{-1}[0,b] = \bigcup_{n=1}^{\infty} \left[\frac{1}{b+n}, \frac{1}{n}\right].$$

It is an easy exercise to show that

$$\begin{aligned}\mu(T^{-1}[0,b]) &= \frac{1}{\log 2}\sum_{n=1}^{\infty}\int_{\frac{1}{b+n}}^{\frac{1}{n}} \frac{1}{1+x}dx \\ &= \frac{1}{\log 2}\int_0^b \frac{1}{1+x}dx \\ &= \mu([0,b]).\end{aligned}$$

□

We now move on to prove the *ergodicity* of the continued fraction map T with respect to the Gauss measure. Notice that in terms of the continued fraction expansion, T behaves similar to the shift map in that

$$T([a_1, a_2, \dots]) = [a_2, a_3, \dots].$$

We therefore would like to pursue a method of proof similar to the proof of the ergodicity of Bernoulli shifts: we want to control the size of the *cylinder sets* and their *intersections*.

Exercise. Given an n-tuple $a = (a_1, \dots, a_n) \in \mathbb{Z}_{>0}^n$ of positive integers, define the cylinder set

$$I(a) = \{[x_1, x_2, \dots] \mid x_i = a_i \text{ for } 1 \leq i \leq n\} \subseteq [0,1].$$

- $I(a)$ is a subinterval of $[0,1]$ with length $\frac{1}{q_n(q_n+q_{n-1})}$, where $\frac{p_n}{q_n}$ is the convergent of $[a_1, \dots, a_n]$.

- Since $q_n \geq 2^{(n-2)/2}$, the length of $I(a) = I([a_1, \ldots, a_n])$ shrinks to zero as $n \to \infty$. Use this to show that the cylinder sets $I(a)$ for all possible strings of positive integers generate the Borel σ-algebra on $[0,1]$.

Proposition 2.39. *The continued fraction map T on $[0,1]$ is ergodic with respect to the Gauss measure μ.*

Proof. The key step of the proof is to show that

$$\mu(T^{-n}A \cap I(a)) \asymp \mu(A)\mu(I(a)) \qquad \text{for any measurable set } A, \tag{2.1}$$

i.e. there exist constants $C_1, C_2 > 0$ which are independent of the choice of A (but may depend on $I(a)$), such that

$$C_1\mu(T^{-n}A \cap I(a)) \leq \mu(A)\mu(I(a)) \leq C_2\mu(T^{-n}A \cap I(a)).$$

We first prove that T is ergodic assuming (2.1). Let $B \subseteq [0,1]$ be a measurable set with $T^{-1}B = B$. By (2.1) we have

$$\mu(B \cap I(a)) \asymp \mu(B)\mu(I(a)).$$

Since the cylinder sets generate the Borel σ-algebra of A, we have

$$\mu(B \cap A) \asymp \mu(B)\mu(A) \qquad \text{for any measurable set } A.$$

By applying this to $A = X \backslash B$, we obtain $\mu(B)\mu(X\backslash B) = 0$, which concludes the proof.

We now proceed to prove (2.1). Recall that the Gauss measure μ is comparable with the Lebesgue measure λ, thus it suffices to show

$$\lambda(T^{-n}A \cap I(a)) \asymp \lambda(A)\lambda(I(a)) \qquad \text{for any measurable set } A$$

As usual, it suffices to show it for any interval $A = [d,e]$. It is an exercise to show that $T^{-n}A \cap I(a)$ is an interval with endpoints given by

$$\frac{p_n + p_{n-1}d}{q_n + q_{n-1}d} \qquad \text{and} \qquad \frac{p_n + p_{n-1}e}{q_n + q_{n-1}e}.$$

Therefore

$$\begin{aligned}\lambda(T^{-n}A \cap I(a)) &= \frac{e-d}{(q_n + q_{n-1}d)(q_n + q_{n-1}e)} \\ &= \lambda(A)\lambda(I(a))\frac{q_n(q_n + q_{n-1})}{(q_n + q_{n-1}d)(q_n + q_{n-1}e)} \\ &\asymp \lambda(A)\lambda(I(a)).\end{aligned}$$

□

Example 2.40. This answers our motivating question: By applying Birkhoff's pointwise ergodic theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\#\{i \mid a_i = k, 1 \le i \le n\}}{n} &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{(\frac{1}{k+1},\frac{1}{k}]}(T^i(x)) \\
&= \int \chi_{(\frac{1}{k+1},\frac{1}{k}]} \,\mathrm{d}\mu \\
&= \mu\left(\left(\frac{1}{k+1},\frac{1}{k}\right]\right) \\
&= \frac{1}{\log 2}\int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{1}{1+x}\,\mathrm{d}x = \frac{1}{\log 2}\log\left(\frac{(k+1)^2}{k(k+2)}\right)
\end{aligned}$$

for almost every $x \in (0,1)$.

Example 2.41. The following result also is an application of the pointwise ergodic theorem: for almost every $x \in (0,1)$, the rate of approximation of the continued fractions is given by

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| = \frac{-\pi^2}{6\log 2}.$$

3. Topology

3.1. **The Borsuk–Ulam theorem.** Let us consider the following *continuous* version of the necklace splitting problem. We say a probability measure μ on $[0,1]$ is *continuous* if $\int_0^x \mathrm{d}\mu$ is continuous in x.

Question 3.1. *Let $\mu_1, \ldots, \mu_n$ be continuous probability measures on $[0,1]$. Does there exist a partition of $[0,1]$ into $n+1$ intervals $I_0, \ldots, I_n$ and signs $\epsilon_0, \ldots, \epsilon_n \in \{\pm 1\}$ such that*

$$\sum_{j=0}^{n} \epsilon_j \cdot \mu_i(I_j) = 0 \quad \textit{for all } 1 \le i \le n \ ?$$

Remark 3.2. In the original necklace splitting problem, the n measures μ_i corresponds to the n kinds of precious stones, the interval $[0,1]$ is separated into $n+1$ subintervals by n cuts, and the signs ±1 determine the corresponding portion of the necklace belongs to which one of the two thieves. An affirmative answer to the above continuous version would imply an affirmative answer to the original necklace splitting problem. For more details, cf. [7].

There is a clever way to encode the divisions of the necklace by points of the *n-dimensional sphere* S^n. With every point of the sphere

$$S^n = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\right\}$$

we associate a division of the interval $[0,1]$ into $n+1$ parts, of lengths $x_0^2, \ldots, x_n^2$; i.e. we cut the interval at the points $0 = z_0 \leq z_1 \leq \cdots z_n \leq z_{n+1} \leq 1$. The sign ϵ_j for the j-th interval $[z_{j-1}, z_j]$ is chosen as $\operatorname{sign}(x_j)$. This defines a continuous map $g\colon S^n \to \mathbb{R}^n$, where its i-th component is given by

$$g_i(x) = \sum_{j=0}^{n} \operatorname{sign}(x_j) \cdot \mu_i\left([z_{j-1} - z_j]\right).$$

The function g clearly satisfies $g(-x) = -g(x)$ for all $x \in S^n$. We would like to show that $g(x) = 0$ for some $x \in S^n$. It follows directly from the *Borsuk–Ulam theorem.*

Theorem 3.3 (Borsuk–Ulam)**.** *Let $f\colon S^n \to \mathbb{R}^n$ be a continuous map. Then there exists an $x \in S^n$ such that $f(-x) = f(x)$.*

For instance, the case $n = 2$ can be illustrated by saying that at any moment, there is always a pair of antipodal points on the Earth's surface with equal temperatures and equal pressures.

Lecture 3

Exercise. For any $n \geq 1$, the following statements are equivalent:

- For every continuous map $f\colon S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ such that $f(-x) = f(x)$.
- For every *antipodal* continuous map $f\colon S^n \to \mathbb{R}^n$ (antipodal means $f(-x) = -f(x)$ for all $x \in S^n$), there exists $x \in S^n$ such that $f(x) = 0$.
- There is no antipodal map $f\colon S^n \to S^{n-1}$.
- There is no continuous map $f\colon B^n \to S^{n-1}$ that is antipodal on the boundary, i.e. satisfies $f(-x) = -f(x)$ for all $x \in S^{n-1} = \partial B^n$.

Remark 3.4. As a direct corollary, there is no continuous map $f\colon B^n \to S^{n-1}$ that is the *identity* on the boundary $\partial B^n = S^{n-1}$, which implies the *Brouwer fixed point theorem.*

As an another corollary of the Borsuk–Ulam theorem, one can show the following *ham sandwich theorem.* The informal statement that gave the ham sandwich theorem its name is this: "For every sandwich made of ham, cheese, and bread, there is a planar cut that simultaneously halves the ham, the cheese, and the bread."

Theorem 3.5 (Ham sandwich theorem)**.** *For any compact sets $A_1, \dots, A_n \subseteq \mathbb{R}^n$, there exists a hyperplane dividing each of them into two subsets of equal measure.*

One can prove a more general version of ham sandwich theorem in terms of measures. We say a measure on $\mathbb{R}^n$ is a *finite Borel measure* if all open subsets of $\mathbb{R}^n$ are measurable and $0 < \mu(\mathbb{R}^n) < \infty$. For instance, for any compact set $A \subseteq \mathbb{R}^n$, one can define a finite Borel measure μ_A by $\mu_A(X) \coloneqq \lambda(X \cap A)$.

Theorem 3.6 (Ham sandwich theorem for measures)**.** *For any finite Borel measures $\mu_1, \dots, \mu_n$ on $\mathbb{R}^n$, there exists a hyperplane h such that*

$$\mu_i(h^+) = \frac{1}{2}\mu_i(\mathbb{R}^n) \qquad \text{for } 1 \leq i \leq n$$

where h^+ denotes one of the half-spaces defined by h.

Proof. Let $u = (u_0, \dots, u_n)$ be a point of the sphere $u \in S^n$. If at least one of the components $u_1, \dots, u_n$ is nonzero, we assign u the half-space

$$h^+(u) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid u_1 x_1 + \cdots + u_n x_n \leq u_0\}.$$

It is clear that antipodal points of S^n correspond to opposite half-spaces. For $u = (\pm 1, 0, \dots, 0) \in S^n$, we have by the same formula

$$h^+((+1, 0, \dots, 0)) = \mathbb{R}^n,$$
$$h^+((-1, 0, \dots, 0)) = \emptyset.$$

Define a continuous function $f\colon S^n \to \mathbb{R}^n$ where the i-th component is

$$f_i(u) \coloneqq \mu_i(h^+(u)).$$

By the Borsuk–Ulam theorem, there exists $x \in S^n$ such that $f(-x) = f(x)$. Then the boundary of the half space $h^+(x)$ is the desired hyperplane. □

Let us discuss the proof of the Borsuk–Ulam theorem. For $n = 1$, the theorem follows easily from the intermediate value theorem. One can prove the $n = 2$ case using some basic knowledge of *fundamental groups* of topological spaces. We will be discussing this in more details in later subsections.

For $n \geq 3$, the proofs usually are more involved (we will only discuss the case of $n = 2$ later); let us sketch a proof here.

- Assume the contrary that there exists an antipodal map $f\colon S^n \to S^{n-1}$. This descends to a continuous map $g\colon \mathbb{RP}^n \to \mathbb{RP}^{n-1}$. Here $\mathbb{RP}^n \cong S^n/\mathbb{Z}_2$ is the n-dimensional *real projective space.*
- One can show that such g induces an isomorphism $g_*\colon \pi_1(\mathbb{RP}^n) \to \pi_1(\mathbb{RP}^{n-1})$ between the *fundamental groups.*
- By the *Poincaré–Hurewicz theorem*, we have an isomorphism $g_*\colon H_1(\mathbb{RP}^n, \mathbb{Z}) \to H_1(\mathbb{RP}^{n-1}, \mathbb{Z})$ between the *homology groups.*
- By the *universal coefficient theorem*, we have an induced *ring homomorphism* between the *cohomology rings*

$$\mathbb{F}_2[b]/b^n \cong H^*(\mathbb{RP}^{n-1}, \mathbb{F}_2) \xrightarrow{g^*} H^*(\mathbb{RP}^n, \mathbb{F}_2) \cong \mathbb{F}_2[a]/a^{n+1}$$

  which sends $b \mapsto a$. But then we get that $b^n = 0$ is sent to $a^n \neq 0$, a contradiction.

Remark 3.7. The real projective space $\mathbb{RP}^n$ is the topological space that parametrizes the 1-dimensional subspaces of $\mathbb{R}^{n+1}$. It can be defined by quotienting the scaling action:

$$\mathbb{RP}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\mathbb{R}^*.$$

Thus $\mathbb{RP}^n$ can also be formed by identifying antipodal points of S^n. It is a smooth compact manifold, and is a special case of *Grassmannians* $\mathrm{Gr}(k, n+1)$ which parametrizes the k-dimensional subspaces of $\mathbb{R}^{n+1}$.

In the following, we will introduce the notion of *fundamental groups* of topological spaces, and prove the Borsuk–Ulam theorem for $n = 2$. A nice reference in which you can find all these notions mentioned above is a book by Hatcher [6].

3.2. **Fundamental groups.**

Let us start with recalling the definition of *topological spaces* and *continuous maps* between them.

Definition 3.8. A *topology* on a set X is a collection τ of subsets of X satisfying the following axioms:

- The empty set and X itself belong to τ.

- Any arbitrary (finite or infinite) union of members of τ belongs to τ.
- The intersection of any finite number of members of τ belongs to τ.

Members of τ are called *open subsets* of X (with respect to this topology).

Definition 3.9. A map $f\colon X \to Y$ between topological spaces is called *continuous* if

$$U \subseteq Y \text{ is an open subset } \implies f^{-1}(U) \subseteq X \text{ is an open subset.}$$

The map f is called a *homeomorphism* if it is bijective, and both f and f^{-1} are continuous. In this case, X and Y are said to be *homeomorphic.*

The *fundamental groups* of topological spaces will be defined in terms of *loops* and their deformations.

Definition 3.10. Let X be a topological space.

- A *path* in X is a continuous map $\gamma\colon I \to X$ where $I = [0, 1]$.
- Its *inverse path* $\gamma^{-1}\colon I \to X$ is defined by $\gamma^{-1}(t) = \gamma(1 - t)$.
- A path is called a *loop* if $\gamma(0) = \gamma(1)$. It can be considered as a map $\gamma\colon S^1 \to X$, with *basepoint* $x_0 = \gamma(0) = \gamma(1)$.
- If $\gamma(t) = x_0 \in X$ for all $t \in [0, 1]$, then such γ is called a *constant path*, and denoted by i_{x_0}.
- If γ_1 and γ_2 are two loops satisfying $\gamma_1(1) = \gamma_2(0)$, we define their *composition* or *product path* to be

$$(\gamma_1 \cdot \gamma_2)(s) = \begin{cases} \gamma_1(2s), & 0 \le s \le 1/2 \\ \gamma_2(2s - 1), & 1/2 \le s \le 1 \end{cases}$$

Definition 3.11. Two paths γ_0, γ_1 with the same endpoints x_0, x_1 are called *homotopic* if there exists a continuous map $F\colon I \times I \to X$ such that

- $F(s, 0) = \gamma_0(s)$ and $F(s, 1) = \gamma_1(s)$ for all $s \in [0, 1]$.
- $F(0, t) = x_0$ and $F(1, t) = x_1$ for all $t \in [0, 1]$.

In this case, we will denote $\gamma_0 \simeq \gamma_1$.

Example 3.12. Any two paths γ_0, γ_1 in $\mathbb{R}^n$ having the same endpoints x_0, x_1 are homotopic via the linear homotopy $F(s, t) = (1 - t)\gamma_0(s) + t\gamma_1(s)$.

Exercise. The relation of homotopy on paths with fixed endpoints is an *equivalence relation*, i.e.

- $\gamma \simeq \gamma$.

- If $\gamma_1 \simeq \gamma_2$, then $\gamma_2 \simeq \gamma_1$.
- If $\gamma_1 \simeq \gamma_2$ and $\gamma_2 \simeq \gamma_3$, then $\gamma_1 \simeq \gamma_3$.

We denote the homotopy class of γ as $[\gamma]$.

Exercise. Let $\gamma_1, \gamma_2, \beta_1, \beta_2$ be paths in X. Suppose $\gamma_1 \simeq \gamma_2$, $\beta_1 \simeq \beta_2$, and $\gamma_1(1) = \gamma_2(1) = \beta_1(0) = \beta_2(0)$. Prove that $\gamma_1 \cdot \beta_1 \simeq \gamma_2 \cdot \beta_2$.

This shows that the *composition* (or *product*) can be defined on homotopy classes:

$$[\gamma] \cdot [\beta] \coloneqq [\gamma \cdot \beta].$$

Exercise. This exercise shows that the product on homotopy classes has *associativity.* Let $\gamma_1, \gamma_2, \gamma_3$ be paths in X satisfying $\gamma_1(1) = \gamma_2(0)$ and $\gamma_2(1) = \gamma_3(0)$. Prove that

$$([\gamma_1] \cdot [\gamma_2]) \cdot [\gamma_3] = [\gamma_1] \cdot ([\gamma_2] \cdot [\gamma_3]).$$

Note that the equality is not true without considering their homotopy classes: $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \neq \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$ in general.

Exercise. Let γ be a path from x_0 to x_1 in X. Prove that

$$[\gamma] \cdot [\gamma^{-1}] = [i_{x_0}], \quad [\gamma^{-1}] \cdot [\gamma] = [i_{x_1}], \quad [\gamma] \cdot [i_{x_1}] = [\gamma] = [i_{x_0}] \cdot [\gamma].$$

We are now ready to define the fundamental group.

Definition 3.13. The *fundamental group* of X at the basepoint x_0, denoted by $\pi_1(X, x_0)$, is defined to be the set of all homotopy classes $[\gamma]$ of loops $\gamma \colon I \to X$ with basepoint x_0, where

- the group structure given by the product $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$,
- the identity element is $[i_{x_0}]$,
- the inverse of an element $[\gamma]$ is given by $[\gamma^{-1}]$.

Example 3.14. Hold a mug in your hand. Now, without letting go of the mug and without spilling the coffee, see if you can rotate the mug *two full turns* and return your hand, arm, and cup to their original positions. If you can do that, can you do the same trick with only *one* full turn? (*No!*)
Continuously rotating a mug is equivalent to following a path in $\mathrm{SO}(3)$, the space of rotations in $\mathbb{R}^3$, and if you start and end the mug in the same orientation, you have traced a loop in $\mathrm{SO}(3)$. The reason this trick works for 2 twists but not 1 twist could be explained by the fact that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$.

Proposition 3.15. *Suppose X is path-connected, i.e. for any two points $x_0, x_1 \in X$, there exists a path $\gamma\colon I \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then the isomorphic class of the fundamental group $\pi_1(X, x_0)$ is independent of the choice of the basepoint x_0, i.e. for any two points $x_0, x_1 \in X$ we have $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. Let γ be a path connecting x_0 and x_1. It is easy to check that

$$\pi_1(X, x_0) \to \pi_1(X, x_1); \qquad [\beta] \mapsto [\gamma^{-1}] \cdot [\beta] \cdot [\gamma]$$

and

$$\pi_1(X, x_1) \to \pi_1(X, x_0); \qquad [\beta] \mapsto [\gamma] \cdot [\beta] \cdot [\gamma^{-1}]$$

are group homomorphisms inverse with each other. Thus $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. □

Proposition 3.16. *A continuous map $f\colon X \to Y$ induces a group homomorphism*

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0)); \qquad [\gamma] \mapsto [f \circ \gamma].$$

Proof. One can verify that the map preserves homotopy equivalences and compositions. The proposition then follows easily. □

3.3. Fundamental group of a circle and applications.

Consider the circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} = \{(\cos(2\pi s), \sin(2\pi s)) \in \mathbb{R}^2 \mid s \in \mathbb{R}\}$$

and choose a basepoint $x_0 = (1, 0) \in S^1$.

Theorem 3.17. *The fundamental group $\pi_1(S^1, x_0) \cong \mathbb{Z}$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos(2\pi s), \sin(2\pi s))$.*

Note that $[\omega]^n = [\omega_n]$ where $\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$ for all $n \in \mathbb{Z}$. The theorem is therefore equivalent to the statement that every loop in S^1 based at $(1, 0)$ is homotopic to ω_n for a unique $n \in \mathbb{Z}$.

The main idea is to compare paths in S^1 with paths in $\mathbb{R}$ via the map

$$p\colon \mathbb{R} \to S^1; \qquad s \mapsto (\cos(2\pi s), \sin(2\pi s)).$$

Consider the path $\widetilde{\omega_n}(s) = ns$ in $\mathbb{R}$, which starts at 0 and ends at ns. The relation $\omega_n = p\widetilde{\omega_n}$ is expressed by saying that $\widetilde{\omega_n}$ is a *lift* of ω_n.

Definition 3.18. Let X be a topological space. A *covering space* of X consists of a space $\widetilde{X}$ and a map $p\colon \widetilde{X} \to X$ such that: for each point $x \in X$ there is an open neighborhood U of x such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p.

Example 3.19. Here are some basic examples of covering spaces of S^1.

- The map $p\colon \mathbb{R} \to S^1$ where $s \mapsto (\cos(2\pi s), \sin(2\pi s))$ is a covering map.
- The map $S^1 \to S^1$ where $(\cos(2\pi s), \sin(2\pi s)) \mapsto (\cos(2\pi ns), \sin(2\pi ns))$ is a covering map for any nonzero integer n. In terms of complex numbers, the map can be expressed as $z \mapsto z^n$.

Exercise. Below are two basic (yet important) facts about covering spaces $p\colon \widetilde{X} \to X$.

(a) For each path $f\colon I \to X$ starting at a point $x_0 \in X$ and each $\widetilde{x_0} \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f}\colon I \to \widetilde{X}$ of f starting at $\widetilde{x_0}$.

(b) For each homotopy $F\colon I \times I \to X$ starting at a point $x_0 \in X$ and each $\widetilde{x_0} \in p^{-1}(x_0)$, there is a unique lifted homotopy $\widetilde{F}\colon I \times I \to \widetilde{X}$ of f starting at $\widetilde{x_0}$.

Proof of Theorem 3.17. Let $f\colon I \to S^1$ be a loop at the basepoint $x_0 = (1, 0)$. We would like to show that it is homotopic to ω_n for a unique $n \in \mathbb{Z}$. By (a) there is a unique lift $\widetilde{f}$ of the loop f starting at 0. Note that the path $\widetilde{f}$ ends at some integer n since $p\widetilde{f}(1) = f(1) = x_0$. Recall that $\widetilde{f}$ and $\widetilde{\omega_n}$ are homotopic since they can be linearly homotopic with each other in $\mathbb{R}$. Thus $[f] = [\omega_n]$.

To show that n is uniquely determined by $[f]$, suppose there is $\omega_m \simeq \omega_n$ for some $m, n \in \mathbb{Z}$. Let F be a homotopy from ω_m to ω_n. By (b) it lifts to a homotopy $\widetilde{F}$ starting at 0, therefore the endpoints of $\widetilde{\omega_m}$ and $\widetilde{\omega_n}$ coincide. Hence $m = n$. □

Remark 3.20. For a covering space $p\colon \widetilde{X} \to X$, a homeomorphism $d\colon \widetilde{X} \to \widetilde{X}$ is called a *deck transformation* if $p \circ d = p$. Together with the composition of maps, the set of deck transformation forms a group $\mathrm{Deck}(p)$. For instance, for the n-sheeted covering space $S^1 \to S^1$ given by $z \mapsto z^n$, the deck transformations are the rotations of S^1 through angles that are multiples of $2\pi/n$, so the deck transformation group is $\mathbb{Z}/n\mathbb{Z}$. Similarly, the deck transformation group of the covering space $\mathbb{R} \to S^1$ is isomorphic to $\mathbb{Z} \cong \pi_1(S^1)$.

The covering space $p\colon \mathbb{R} \to S^1$ where $s \mapsto (\cos(2\pi s), \sin(2\pi s))$ is the *universal cover* of S^1: any covering space of S^1 can be covered by the universal cover. For instance, the covering space $S^1 \xrightarrow{z^n} S^1$ can be covered by $p_n\colon \mathbb{R} \to S^1$ where $s \mapsto (\cos(2\pi s/n), \sin(2\pi s/n))$; we have $z^n \circ p_n = p$. The deck transformation group of p_n is given by $n\mathbb{Z}$. In general, there is a one-to-one correspondence:

$$\{\text{covering space of } X\} \leftrightarrow \{\text{subgroups of } \pi_1(X)\}$$

where a covering space $p\colon \widetilde{X} \to X$ corresponds to the subgroup $p_*(\pi_1(\widetilde{X}))$ of $\pi_1(X)$. Moreover, the deck transformation group of p is isomorphic to $N(p_*(\pi_1(\widetilde{X})))/p_*(\pi_1(\widetilde{X}))$, where $N(p_*(\pi_1(\widetilde{X})))$ is the normalizer subgroup of $p_*(\pi_1(\widetilde{X}))$ in $\pi_1(X)$.

Theorem 3.21 (Borsuk–Ulam in dimension 2)**.** *There is no antipodal map $f\colon S^2 \to S^1$.*

Proof. Assume the contrary that such map f exists. Define a loop η circling the equator

$$\eta\colon I \to S^2; \qquad s \mapsto (\cos(2\pi s), \sin(2\pi s), 0),$$

and consider the loop $g = f \circ \eta\colon I \to S^1$.

On the one hand, the loop η in S^2 is homotopic to a constant map, thus so is the loop g in S^1. In other words, $[g] = 0$ in $\pi_1(S^1) \cong \mathbb{Z}$.

On the other hand, since $f(-x) = -f(x)$, we have

$$g\left(s + \frac{1}{2}\right) = -g(s) \qquad \text{for all } s \in \left[0, \frac{1}{2}\right].$$

Let $\widetilde{g}\colon I \to \mathbb{R}$ be a lift of g. Then for each $s \in \left[0, \frac{1}{2}\right]$ we have

$$\widetilde{g}\left(s + \frac{1}{2}\right) = \widetilde{g}(s) + \frac{q}{2} \qquad \text{for some odd integer } q.$$

Note that q depends continuously on $s \in \left[0, \frac{1}{2}\right]$, so it must be a constant for all $s \in \left[0, \frac{1}{2}\right]$ since it is of integer value. In particular, we have

$$\widetilde{g}(1) = \widetilde{g}(0) + q.$$

Thus $[g] \neq 0$ in $\pi_1(S^1) \cong \mathbb{Z}$ since q is odd. Contradiction. □

Theorem 3.22 (Fundamental theorem of algebra)**.** *Every non-constant polynomial with complex coefficients has a root in* $\mathbb{C}$*.*

Proof. Consider a complex polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Assume the contrary that $p(z)$ has no roots in $\mathbb{C}$, then for each $r \geq 0$

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}$$

defines a loop in S^1 based at 1. As r varies, f_r is a homotopy of loops in S^1 based at 1. Since f_0 is the trivial loop, we have $[f_r] = 0$ in $\pi_1(S^1)$ for all $r \geq 0$.

On the other hand, for r sufficiently large, on the circle $|z| = r$ we have

$$|z^n| > (|a_0| + \cdots + |a_{n-1}|)|z^{n-1}| \geq |a_{n-1}z^{n-1} + \cdots + a_0|.$$

Thus the polynomial $p_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$ has no zero on the circle $|z| = r$ when $0 \leq t \leq 1$. Replacing p by p_t in the formula above and letting t go from 1 to 0, one obtains a homotopy from the loop f_r to the loop $\omega_n(s) = e^{2\pi ins}$, thus $[f_r] = [\omega_n]$ in $\pi_1(S^1)$. We then conclude that $n = 0$. □

3.4. **The rectangular peg problem.** Let $C \subseteq \mathbb{R}^2$ be a continuous simple closed curve. Does there always exist four points on C such that they form the vertices of a rectangle? Below is the sketch of ideas toward answering this question (affirmatively).

Lecture 4

- Denote M the *moduli space* of unordered pairs of points in C: each (unordered) pair of points c_1, c_2 in C corresponds to a unique point in M.
- Observe that M is naturally topologically equivalent to a Möbius strip, where its boundary can be identified with the curve C.
- Define a continuous function $f_C \colon M \to \mathbb{R}^3$ which sends a pair of points $c_1 = (x_1, y_1), c_2 = (x_2, y_2)$ on the curve C to the point

  $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right) \in \mathbb{R}^3$$

  where the first two coordinates give the midpoint of c_1, c_2, and the third coordinate is the distance between c_1 and c_2.
- Observe that the rectangular peg problem has an affirmative answer for a curve C if and only if f_C is not injective.

- Observe that one gets the *real projective plane* $\mathbb{RP}^2$ by gluing the Möbius strip with a disk along their boundaries.
- Assume the contrary that there exists a curve C such that f_C is injective. Then one gets an embedding of the real projective plane $\mathbb{RP}^2$ into $\mathbb{R}^3$.
- Use topological tools to show that there is no embedding of $\mathbb{RP}^2$ into $\mathbb{R}^3$. This concludes the proof.

One way to show the last statement, namely there is no embedding of $\mathbb{RP}^2$ into $\mathbb{R}^3$, is by consider the *orientability* of the real projective plane $\mathbb{RP}^2$. It is known that $\mathbb{RP}^2$ is *non-orientable*: this can be rigorously proved by computing the homology groups of $\mathbb{RP}^2$. On the other hand, assume the contrary that there exists an embedding of $\mathbb{RP}^2$ into $\mathbb{R}^3$, then the image would bound a compact region in $\mathbb{R}^3$ (by the *generalized Jordan curve theorem*). The outward-pointing normal vector field would then give an orientation of $\mathbb{RP}^2$. Contradiction.

4. Algebra

Which positive integers n can be written as the sum of two squares? To answer this question, it is convenient to consider the factorization in the *ring* of *Gaussian integers* $\mathbb{Z}[i]$:

$$n = x^2 + y^2 = (x + iy)(x - iy).$$

One would also like to study other number rings; for instance, to understand the Diophantine equation $n = x^2 - 5y^2$, one would like to do factorizations in the ring $\mathbb{Z}[\sqrt{5}]$.

It is important to be aware that not all number rings have the same properties. For instance, the ring of Gaussian integers $\mathbb{Z}[i]$ is a *Unique Factorization Domain (UFD)*, but the ring $\mathbb{Z}[\sqrt{5}]$ is not: there are factorizations

$$(3 + \sqrt{5})(3 - \sqrt{5}) = 4 = 2 \cdot 2$$

where $3 \pm \sqrt{5}$ and 2 are all *irreducible* elements of $\mathbb{Z}[\sqrt{5}]$, so there are two truly different factorizations of 4 in $\mathbb{Z}[\sqrt{5}]$.

We will begin our discussions with the general notion of *rings*, then gradually specialized to commutative rings, integral domains, unique factorization domains, principal ideal domains, Euclidean domains. It turns out that the ring of Gaussian integers $\mathbb{Z}[i]$ is an *Euclidean domain* (a condition stronger than UFD), which will allow us to completely classify the integers that can be written as the sum of two squares. A nice reference for this part (and abstract algebra in general) is a book of Artin [1].

4.1. **Rings.**

Definition 4.1. A *ring* is a set R equipped with two binary operations $+$ (addition) and $\cdot$ (multiplication) satisfying:

(1) R is an abelian group under addition, namely:
- $(a+b)+c = a+(b+c)$ for all $a, b, c \in R$.
- $a+b = b+a$ for all $a, b \in R$.
- There is an element $0 \in R$ such that $a+0 = a$ for all $a \in R$.
- For each $a \in R$ there exists $-a \in R$ such that $a+(-a) = 0$.

(2) R is a monoid under multiplication, namely:
- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- There is an element $1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in R$.

(3) Multiplication is distributive with respect to addition, namely:
- $a \cdot (b+c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$.
- $(b+c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in R$.

Note that the multiplication symbol $\cdot$ is often omitted: for instance, ab means $a \cdot b$.

Definition 4.2. A ring R is said to be *commutative* if $ab = ba$ for all $a, b \in R$.

Non-example. The set of 2×2 real matrices forms a ring under the standard matrix additions and multiplications. It is not commutative.

Remark 4.3. Whether a ring is commutative has profound implications on its behavior. *Commutative algebra*, the theory of commutative rings, is a major branch of ring theory. Its development has been greatly influenced by problems and ideas of *algebraic number theory* and *algebraic geometry.* If you are interested, a standard textbook on commutative algebra is [2].

Commutative rings resemble familiar number systems, and various definitions for commutative rings are designed to formalize properties of the integers.

Definition 4.4. A nonzero commutative ring R is called an *integral domain* if the product of any two nonzero elements is nonzero.

Non-example. The quotient ring $\mathbb{Z}/6\mathbb{Z}$ is a commutative ring, but is not an integral domain.

Non-example. The quotient ring $\mathbb{Z}[x]/(x^2-1)$ is a commutative ring, but is not an integral domain.

In order to introduce the definition of unique factorization domain, we need to define the notion of *units*.

Definition 4.5. An element $u \in R$ is called a *unit* if there exists $v \in R$ such that $uv = vu = 1$. In other words, a unit is an invertible element for the multiplication of the ring.

Example 4.6. Here are some basic examples:

- The units of $\mathbb{Z}$ are 1 and -1.
- The units of $\mathbb{Z}[i]$ are 1, -1, i, and $-i$.
- The units of $M_2(\mathbb{R})$ are all invertible matrices.
- The ring $\mathbb{Z}[\sqrt{3}]$ has infinitely many units: for instance, $(2+\sqrt{3})$ and its powers are units of the ring. In general, the ring of integers in a number field can be determined by the *Dirichlet's unit theorem.*

Definition 4.7. An element of an integral domain R is called *irreducible* if it is not a unit, and is not the product of two non-unit elements.

Remark 4.8. An element of an integral domain R is called *prime* if, whenever $a \mid bc$ (i.e. $bc = ax$ for some $x \in R$), then $a \mid b$ or $a \mid c$. In an integral domain, every prime element is irreducible, but the converse is not true in general. For instance, in the ring $\mathbb{Z}[\sqrt{-5}]$, it can be shown that 3 is irreducible. However, it is not a prime element since

$$3 \mid (2+\sqrt{-5})(2-\sqrt{-5}) = 9$$

but 3 does not divide either of the two factors.

Definition 4.9. An integral domain R is said to be a *unique factorization domain* (or UFD for short) if every nonzero element $x \in R$ can be written as a product

$$x = up_1 \cdots p_n$$

where u is a unit and p_i's are irreducible, and this representation is unique in the following sense: If we also have

$$x = vq_1 \cdots q_m$$

where v is a unit and q_i's are irreducible, then $m = n$, and there exists a bijective map $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$ such that $p_i = w_i q_{\sigma(i)}$ for some units w_i.

Non-example. The quadratic ring $\mathbb{Z}[\sqrt{-5}]$ is an integral domain, but is not a UFD:

$$2 \cdot 3 = 6 = (1+\sqrt{-5})(1-\sqrt{5}).$$

One can show that $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$ are all irreducible, and the only units of $\mathbb{Z}[\sqrt{-5}]$ is ± 1, therefore these truly are two different factorizations.

One important class of examples of UFDs are given by *principal ideal domains* (PID).

Definition 4.10. An ideal I of a commutative ring R is an additive subgroup of R which is closed under multiplications: more precisely,

- $(I, +)$ is a subgroup of $(R, +)$.
- For every $r \in R$ and $x \in I$, the product rx is in I.

An ideal is called *principal* if it can be generated by a single element, i.e. it is of the form $xR = \{xr \mid r \in R\}$.

Definition 4.11. An integral domain R is called a *principal ideal domain* (PID) if every ideal of R is principal.

Non-example. $\mathbb{Z}[x]$ is a UFD, but is not a PID: for instance, the ideal $\langle 2, x\rangle$ can not be generated by a single polynomial.

Theorem 4.12. *Every PID is a UFD.*

Proof. Let R be a PID. First, we show that R satisfies the *ascending chain condition* (ACC) on ideals; namely, whenever there are ideals

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

then there is some $N > 0$ such that $I_n = I_N$ for all $n \geq N$. Consider the union

$$I = \bigcup_{n \geq 1} I_n$$

which is also an ideal of R. Thus $I = (a)$ for some $a \in I$, and there exists $N > 0$ such that $a \in I_N$. This shows that R satisfies ACC.

Second, we show that every irreducible elements of R is prime. Let $a \in R$ be an irreducible element. Suppose $a \mid bc$ for some $b, c \in R$. We would like to show that $a \mid b$ or $a \mid c$ holds. Let us consider the ideal (a, b). Since R is PID, there exists $x \in R$ such that $(x) = (a, b)$. In particular, $a = xy$ for some $y \in R$. Since a is irreducible, x or y has to be a unit.

- If y is a unit, then $(a) = (x) = (a, b)$, thus $a \mid b$ as desired.
- If x is a unit, then $(1) = (x) = (a, b)$, so there exists $c, d \in R$ such that $ac + bd = 1$. Multiplying both sides with c, one gets $ac^2 + bcd = c$. Note that the left hand side is a multiple of a since $a \mid bc$, thus we obtain $a \mid c$.

Now we are ready to show that R is a UFD. First, we show that any nonzero nonunit element of R can be written as a product of irreducible elements. Assume the contrary that there exists nonzero nonunit element of R that cannot be written as a product of irreducibles. Denote the collection of such elements by S. Since R satisfies ACC, there exists $r \in S$ such that $(r) \nsubseteq (s)$ for any $s \in S \backslash \{r\}$. In particular, r is not irreducible, so it can be written as $r = xy$ for some nonunit elements $x, y \in R$. Since $(r) \subseteq (x)$ and $(r) \subseteq (y)$, we have $x, y \notin S$, therefore x and y both can be written as a product of irreducibles. But then we get $r = xy$ can also be written as a product of irreducibles. Contradiction.

Finally, we show that the factorization is unique. Suppose

$$a = up_1 \cdots p_n = vq_1 \cdots q_m$$

where u, v are units and p_i, q_i's are irreducibles (therefore are primes by what we proved earlier). Then $p_1 \mid vq_1 \cdots q_m$, thus it must divide some q_j. Since p_1 and q_j are both primes, they are the same up to a unit. We may continue this process and match each prime factor on both sides. □

Definition 4.13. An integral domain R is said to be a *Euclidean domain* if there exists a function $N\colon R\backslash\{0\} \to \mathbb{Z}_{\geq 0}$ (called a *norm function*) such that:

- For all nonzero elements $a, b \in R$, there exists $q, r \in R$ such that $a = qb + r$ and either $r = 0$ or $N(r) < N(b)$.
- For all nonzero elements $a, b \in R$ we have $N(a) \leq N(ab)$.

Non-example. The ring $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is a PID, but is not a Euclidean domain.

Example 4.14. Here are some basic examples of Euclidean domains.

- The ring of integers $\mathbb{Z}$, with $N(a) = |a|$.

- The ring of Gaussian integers $\mathbb{Z}[i]$, with $N(a+ib) = a^2 + b^2$ (we will discuss more details later).
- The ring of polynomials $\mathbb{R}[x]$ over $\mathbb{R}$ (can be replaced by any *field*), with $N(P) = \deg(P)$.

Theorem 4.15. *Every Euclidean domain is a PID.*

Proof. Let R be a Euclidean domain. Let $I \subseteq R$ be a nonzero ideal. Then there exists a nonzero element $a \in I$ such that $N(a)$ is minimal among all elements of the ideal. We claim that $I = (a)$. For any $b \in I$, there exists $q, r \in R$ such that $b = qa + r$ where $r = 0$ or $N(r) < N(a)$. Since $a, b \in I$, we have $r \in I$, thus $N(r) \geq N(a)$ by the minimality. Therefore we have $r = 0$ and $b \in (a)$. □

4.2. **Ring of Gaussian integers.**

Definition 4.16. The norm function on the ring of Gaussian integers $\mathbb{Z}[i]$ is defined to be

$$N(a+ib) = (a+ib)(a-ib) = a^2 + b^2.$$

Exercise. Here are some basic properties of the norm function.

- $N(\alpha) = 0$ if and only if $\alpha = 0$.
- $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in \mathbb{Z}[i]$.
- $N(\alpha) = 1$ if and only if α is a unit of $\mathbb{Z}[i]$.
- $\{1, -1, i, -i\}$ are the only units of $\mathbb{Z}[i]$.

Theorem 4.17. *$\mathbb{Z}[i]$ is a Euclidean domain.*

Proof. Let a, b be nonzero elements of $\mathbb{Z}[i]$. Observe that the set $b\mathbb{Z}[i]$ forms a lattice of squares with side length $|b| = \sqrt{N(b)}$. Then the distance between a and the lattice point closest to it (say bq) is no bigger than $|b|/\sqrt{2}$. Let $r = a - bq \in \mathbb{Z}[i]$. Then

$$N(r) = |r|^2 \leq \frac{|b|^2}{2} = \frac{N(b)}{2} < N(b).$$

□

Lemma 4.18. *If $\pi \in \mathbb{Z}[i]$ is such that $N(\pi)$ is a prime number, then π is a prime in $\mathbb{Z}[i]$.*

Proof. If $\pi = \alpha\beta$ in $\mathbb{Z}[i]$, then $N(\pi) = N(\alpha)N(\beta)$. So either $N(\alpha)$ or $N(\beta)$ is 1, which means that either α or β is a unit. □

Lemma 4.19. *Let q be a prime number with $q = 3$ (mod 4). Then q is a prime in $\mathbb{Z}[i]$.*

Proof. If $q = \alpha\beta$ in $\mathbb{Z}[i]$, then $q^2 = N(\alpha)N(\beta)$. Note that $q = N(\alpha) = a^2 + b^2$ is impossible since $q = 3 \pmod 4$. Thus either $N(\alpha)$ or $N(\beta)$ is 1. □

Lemma 4.20. *Let p be a prime number with $p = 1$ (mod 4). Then there exists a Gaussian prime π such that $p = \pi\overline{\pi}$.*

Proof. First, we claim that there exists an integer $c \in \mathbb{Z}$ such that $c^2 = -1 \pmod p$. This can be easily proved by assuming the fact that the multiplicative group $\mathbb{Z}_p^*$ of the finite field $\mathbb{Z}_p$ is cyclic. Let a be a generator of the multiplicative group $\mathbb{Z}_p^*$ (which has $p-1$ elements), i.e.

$$\mathbb{Z}_p^* = \{1, a, a^2, \ldots, a^{p-2}\}.$$

Observe that -1 is the unique order two element of $\mathbb{Z}_p^*$, thus $a^{\frac{p-1}{2}} = -1 \pmod p$. The claim then follows from the assumption that $p = 1 \pmod 4$.

By the claim, we have $p \mid (c+i)(c-i)$ in $\mathbb{Z}[i]$. It is easy to show that p does not divide $c+i$ or $c-i$. Therefore p is *not* a Gaussian prime. Hence there exists nonunit elements $\alpha, \beta \in \mathbb{Z}[i]$ such that $p = \alpha\beta$. By comparing the norms on both sides, we obtain $N(\alpha) = N(\beta) = p$. Therefore both α and β are Gaussian primes. It is then easy to check that they are complex conjugate with each other. □

Lecture 5

Proposition 4.21. *Up to multiplying by units, all the Gaussian primes are the following:*

- $1+i$ *(which is of norm 2),*
- π *and* $\overline{\pi}$, *where* $p = \pi\overline{\pi}$ *is a prime number with* $p = 1$ *(mod 4) (the norms of* π *and* $\overline{\pi}$ *are both* p*),*
- q, *where* q *is a prime number with* $q = 3$ *(mod 4) (which is of norm* q^2*).*

Proof. Let α be a Gaussian prime. Then we can find a Gaussian prime π in the above list so that $\pi \mid N(\alpha) = \alpha\overline{\alpha}$. So either π or $\overline{\pi}$ divides α. Thus α is also in the above list. □

4.3. **Applications.** Let us apply the arithmetic of $\mathbb{Z}[i]$ to solve a classic problem: finding all *Pythagorean triples.* A Pythagorean triples is $(x, y, z) \in \mathbb{Z}^3_{>0}$ where $x^2 + y^2 = z^2$. It suffices to only look for *primitive* Pythagorean triples, i.e. $\gcd(x, y, z) = 1$. Also, observe that x and y cannot both be odd, so may assume that x is odd and y is even.

Theorem 4.22. *Let $(x, y, z) \in \mathbb{Z}^3_{>0}$ be a primitive Pythagorean triples with x odd and y even. Then there exists coprime integers a, b with $a > b > 0$ and $a \neq b$ (mod 2) such that*

$$x = a^2 - b^2, \quad y = 2ab, \quad z = a^2 + b^2.$$

Proof. Let $\alpha = x + iy \in \mathbb{Z}[i]$, so $N(\alpha) = x^2 + y^2 = z^2$. The idea is to show that α is a *square* in $\mathbb{Z}[i]$; writing $\alpha = (a + ib)^2$ gives the desired result. We have

$$z^2 = N(\alpha) = (x + iy)(x - iy).$$

We claim that $x + iy$ and $x - iy$ are coprime in $\mathbb{Z}[i]$. Assume the contrary that there exists a Gaussian integer π that divides both $x + iy$ and $x - iy$. Then it also divides $2x$ and $2y$. Since x, y are coprime, π has to divide 2. Therefore $\pi = 1 + i$ (up to a unit). But $1 + i$ does not divide $x + iy$ since $x \neq y$ (mod 2). Contradiction.

Hence $x + iy$ and $x - iy$ are coprime in $\mathbb{Z}[i]$. As their product is a square, unique factorization in $\mathbb{Z}[i]$ implies that each of them is a square (up to a unit). Using $-1 = i^2$, each of them must be a square or i times a square.

If $x + iy = i(a + ib)^2$, then $x = -2ab$ which contradicts with the assumption that x is odd. Therefore $x + iy$ is a square. □

Next, we solve the sum of two squares problem.

Theorem 4.23. *Let $n = a \cdot b^2$ be an integer with a square-free. Then n can be written as a sum of two squares if and only if no prime $q = 3$ (mod 4) divides a.*

Proof. The "if" part: For each prime p dividing a, there is a Gaussian prime π_p such that $p = \pi_p \overline{\pi}_p$. Let $x + iy = b \cdot \prod_{p|a} \pi_p$. Then $x^2 + y^2 = n$.

The "only if" part: Suppose $n = x^2 + y^2 = (x + iy)(x - iy)$. If a prime $q = 3$ (mod 4) divides n, as it is a Gaussian prime, it divides $x + iy$ or $x - iy$, which implies that q divides both $x + iy$ and $x - iy$. Thus q^2 divides n. The statement can then be proved by induction on b. □

In the upcoming section, we will use the theory of *modular forms* to count the number

$$r_2(n) = \#\{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1^2 + x_2^2 = n\}.$$

Here is a sketch of the main idea. One can show that

$$E_1^\chi(q) = \frac{1}{4} + \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(d) \right) q^n \in M_1(\Gamma_1(4)), \qquad \text{where } \chi(d) = \begin{cases} 1 & \text{if } d = 1 \pmod 4 \\ -1 & \text{if } d = 3 \pmod 4 \\ 0 & \text{if } d \text{ is even} \end{cases}$$

and the space $M_1(\Gamma_1(4))$ of modular form of weight 1 for the group $\Gamma_1(4) \subseteq \mathrm{SL}(2, \mathbb{Z})$ is one-dimensional, therefore is generated by the function $E_1^\chi(q)$. On the other hand, one can also show that

$$\theta(q)^2 = \sum_{n=0}^{\infty} r_2(n) q^n \in M_1(\Gamma_1(4)).$$

Thus $\theta(q)^2$ is a scalar multiple of $E_1^\chi(q)$. The coefficient of the constant term of $\theta(q)^2$ is $r_2(0) = 1$, while the coefficient of the constant term of $E_1^\chi(q)$ is $1/4$. Hence one obtains

$$\theta(q)^2 = 4E_1^\chi(q).$$

By comparing the coefficients on both sides, we get an explicit formula for $r_2(n)$:

$$r_2(n) = 4 \sum_{d|n} \chi(d).$$

Let us give another proof of the formula using the properties of the ring of Gaussian integers. The number $r_2(n)$ can also be interpreted as the number of Gaussian integers with norm n. Thus

$$\sum_{n \geq 1} \frac{r_2(n)}{n^s} = \sum_{0 \neq \alpha \in \mathbb{Z}[i]} \frac{1}{N(\alpha)^s}.$$

Denote the set of all Gaussian primes (up to units) by $\mathcal{P}$. Then we have

$$\begin{aligned}\sum_{0\neq\alpha\in\mathbb{Z}[i]}\frac{1}{N(\alpha)^s} &= 4\prod_{\pi\in\mathcal{P}}\frac{1}{1-N(\pi)^{-s}}\\ &= 4\cdot\frac{1}{1-2^{-s}}\cdot\prod_{p=1(\text{mod }4)}\frac{1}{(1-p^{-s})^2}\prod_{q=3(\text{mod }4)}\frac{1}{1-q^{-2s}}\\ &= \zeta(s)\cdot L(s,\chi).\end{aligned}$$

Here $\zeta(s)$ is the Riemann zeta function

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_{p\in\mathbb{Z}\text{ prime}}\frac{1}{1-p^{-s}}$$

and $L(s,\chi)$ is the Dirichlet L-series

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\prod_{p\in\mathbb{Z}\text{ prime}}\frac{1}{1-\chi(p)p^{-s}}.$$

So we have

$$\frac{1}{4}\sum_{n\geq 1}\frac{r_2(n)}{n^s}=\left(\sum_{m\geq 1}\frac{1}{m^s}\right)\left(\sum_{d=1}^{\infty}\frac{\chi(d)}{d^s}\right).$$

Thus

$$\frac{1}{4}r_2(n)=\sum_{md=n}\chi(d)=\sum_{d|n}\chi(d).$$

5. Complex analysis and modular forms

The German mathematician Martin Eichler once stated that there were five fundamental operations of mathematics: addition, subtraction, multiplication, division, and *modular forms*. In this unit, we will start with discussing the basic concepts of complex analysis, then move on to the discussions of *elliptic functions* and *modular forms*. We will mention many applications along the way, and solve the sums of four squares problem at the end. Some references that might be helpful include [3], [8], and [9].

5.1. **Some applications of modular forms.** Let us discuss the j-*invariant* first. Classically, the j-invariant was studied as a parameterization of *elliptic curves* over $\mathbb{C}$. Every elliptic curve E over $\mathbb{C}$ is a complex torus, and thus can be identified with a rank 2 lattice. This lattice can be rotated and scaled (which preserve the isomorphism class), so that it is generated by 1 and $\tau \in \mathbb{H}$. This lattice corresponds to the elliptic curve

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau),$$

where

$$g_2(\tau) = \frac{4\pi^4}{3}E_4(\tau), \qquad g_3(\tau) = \frac{8\pi^6}{27}E_6(\tau),$$

and

$$E_4(\tau) = 1 + 240\sum_{r\geq 1}\sigma_3(r)q^r, \qquad E_6(\tau) = 1 - 504\sum_{r\geq 1}\sigma_5(r)q^r$$

are *Eisenstein series* (which are *modular forms* of weight 4 and 6, respectively), where $q = e^{2\pi i\tau}$ and $\sigma_k(r) = \sum_{d|r} d^k$. The isomorphic class of elliptic curves is uniquely determined by the j-invariant

$$j(\tau) = 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}.$$

It is the *unique* (up to scalar multiplication) holomorphic function on $\mathbb{H}$ that is invariant under the $\mathrm{SL}(2,\mathbb{Z})$-action and has a simple pole at infinity. In fact, any meromorphic modular function (i.e. invariant under $\mathrm{SL}(2,\mathbb{Z})$-action) on $\mathbb{H}$ is a rational function of $j(\tau)$.

The j-invariant has many interesting and surprising applications. For instance, let us consider

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925...$$

which is very close to an integer. This remarkable phenomenon can be easily deduced using the fact that

$$j\left(\frac{1+\sqrt{-163}}{2}\right) \in \mathbb{Z}.$$

together with the q-expansion of the j-function

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + O(q^3), \quad \text{where } q = e^{2\pi i\tau}.$$

Consider primitive positive-definite quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$, where $a, b, c \in \mathbb{Z}$, $\gcd(a, b, c) = 1$, $a > 0$, and $D = b^2 - 4ac < 0$. There is a natural notion of *equivalence* between two such quadratic forms, essentially given by change of variables. One can show that two such quadratic forms are equivalent if and only if $D = D'$ and

$$j\left(\frac{b + \sqrt{-D}}{2a}\right) = j\left(\frac{b' + \sqrt{-D}}{2a'}\right).$$

For each possible discriminant D there are only finitely many equivalence classes, thus we get a finite set of j-values for each discriminant. The big theorem is that these values are the solutions of a monic algebraic equation with integer coefficients. In particular, when there is only one equivalence class for D, the j-invariant of the corresponding quadratic form must be an integer. The above phenomenon then follows from the fact that all positive-definite integer quadratic forms of discriminant $D = -163$ are equivalent (to $x^2 - xy + 41y^2$). In fact, 163 is the largest number satisfying this property; other numbers are: $1, 2, 3, 7, 11, 19, 43, 67$; for instance, we also have

$$e^{\pi\sqrt{67}} \approx \mathbb{Z} + 0.0000013; \qquad e^{\pi\sqrt{43}} \approx \mathbb{Z} + 0.00022.$$

These results on the j-function are one of the starting points of the theory of *complex multiplications* of elliptic curves.

Another surprising result is a connection between the j-function and the *monster group*.

Theorem 5.1. *Every finite simple group is isomorphic to one of the following groups:*

- *a member of one of three infinite classes of:*
  - *the cyclic groups of prime order,*
  - *the alternating groups A_n for $n \geq 5$,*
  - *the groups of Lie type*
- *one of the 27 sporadic groups.*

Among the 27 sporadic groups, the *monster group* M has the largest order of roughly 8×10^{53}. The minimal dimension of a faithful complex representation of the monster group is 196883, which happens to be very close to one of the coefficients in the q-expansion

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots.$$

In fact, the dimensions of the irreducible representations of M are: $r_1 = 1$, $r_2 = 196883$, $r_3 = 21296876$, $r_4 = 842609326$, $r_5 = 18538750076$, etc., and the coefficients of the q-expansion of j-function satisfies

$$\begin{aligned}
196884 &= r_1 + r_2 \\
21493760 &= r_1 + r_2 + r_3 \\
864299970 &= 2r_1 + 2r_2 + r_3 + r_4 \\
20245856256 &= 3r_1 + 3r_2 + r_3 + 2r_4 + r_5 \\
&\dots
\end{aligned}$$

Very roughly, this can be explained by the fact that there exists a *vertex operator algebra* which admits an infinite-dimensional graded representation of the monster group, whose graded dimensions are the coefficients of the j-function. The precise content of this statement and their detailed properties (Conway–Norton conjecture) are proved by Borcherds, who won the Fields Medal in 1998 in part for his solution of the conjecture.

Let us consider a more elementary application of modular forms. Consider the functions

$$\sigma_3(r) = \sum_{d|r} d^3 \quad \text{and} \quad \sigma_7(r) = \sum_{d|r} d^7.$$

They satisfy a relation

$$\sigma_7(r) = \sigma_3(r) + 120 \sum_{p+q=r} \sigma_3(p)\sigma_3(q).$$

This is not an easy statement to prove. Using the fact that

$$E_4(\tau) = 1 + 240 \sum_{r\geq 1} \sigma_3(r) q^r \quad \text{and} \quad E_8(\tau) = 1 + 480 \sum_{r\geq 1} \sigma_7(r) q^r$$

are modular forms of weight 4 and 8, respectively; together with the fact the space of modular forms of weight 8 is one-dimensional, one deduces $E_4(\tau)^2 = E_8(\tau)$. The above relation then follows from comparing the coefficients of both sides of the equation.

5.2. **A crash course on complex analysis.** We recall in this subsection some theorems of complex analysis that will be useful and necessary for our discussions of modular forms later. The proofs of these theorems can be found in any textbook on complex analysis, for instance [9].

Let $U \subseteq \mathbb{C}$ be an open subset of the complex plane. A function $f\colon U \to \mathbb{C}$ is called *holomorphic* if for every $z_0 \in U$, the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists.}$$

In other words, it is holomorphic if the derivative in the "complex sense" exists. If the limit exists, it will be denoted by $f'(z_0) \in \mathbb{C}$. This is exactly the complex analogue of the *differentiable* functions over $\mathbb{R}$. However, holomorphic functions possess many nicer properties than differentiable functions.

Example 5.2. Holomorphic functions satisfy the "local determine global" principle. Namely, suppose there are two holomorphic functions f, g on a (connected) open set $U \subseteq \mathbb{C}$ such that their values agree on an open subset $V \subseteq U$, i.e. $f(z) = g(z)$ for all $z \in V$. Then, no matter how small the open subset V is, we would have $f(z) = g(z)$ for all $z \in U$.

This is not true for smooth functions over $\mathbb{R}$. For instance, the smooth function

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is identical with the zero function on $\mathbb{R}_{<0}$, but they are obvious not identical on the whole real line.

Example 5.3. Another important result is that if $f\colon U \to \mathbb{C}$ is holomorphic, then its derivative $f'\colon U \to \mathbb{C}$ is automatically holomorphic as well. This implies that any holomorphic is infinitely differentiable, i.e. $f, f', f'', f''', \ldots$ exist. Moreover, for any $z_0 \in U$ the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

converges in a neighborhood of z_0, and the limit coincides with $f(z)$. These are again not true for differentiable functions over $\mathbb{R}$.

These results, together with other basic theorems in complex analysis, including Liouville's theorem, Morera's theorem, residue formula, argument principle, etc., essentially all are corollaries of a single theorem, the *Cauchy integral theorem.* To state the theorem, we need to define the notion of *path integrals.*

Definition 5.4. A *parametrized smooth curve* in $U \subseteq \mathbb{C}$ is a map

$$\gamma\colon [a,b] \to U; \qquad \gamma(t) = x(t) + iy(t)$$

such that

- $x(t), y(t)$ are differentiable, and $x'(t), y'(t)$ are continuous,
- $\gamma'(t) = (x'(t), y'(t)) \neq (0,0)$ for all $t \in (a,b)$.

Example 5.5. $\gamma\colon [0,\pi] \to \mathbb{C}$ where $\gamma(t) = e^{it} = \cos(t) + i\sin(t)$ parametrizes the upper half of the unit circle (going counterclockwise). Note that there are infinitely many ways to represent a curve. For instance, $\gamma'\colon [0,2\pi] \to \mathbb{C}$ where $\gamma'(s) = e^{is/2}$ also parametrizes the upper half of the unit circle with the same orientation.

Definition 5.6. Two parametrized smooth curves $\gamma\colon [a,b] \to \mathbb{C}$ and $\gamma'\colon [c,d] \to \mathbb{C}$ are said to be *equivalent* if there exists a smooth bijective map $\varphi\colon [c,d] \to [a,b]$ so that $\gamma(\varphi(s)) = \gamma'(s)$ and $\varphi'(s) > 0$ for all $s \in [c,d]$.

Note that the condition $\varphi'(s) > 0$ guarantees that the two curves have the same orientations (going in the same direction).

Definition 5.7. A *piecewise parametrized smooth curve* in $U \subseteq \mathbb{C}$ is a continuous map

$$\gamma\colon [a,b] \to U$$

such that there exists $a < p_1 < \cdots < p_n < b$ so that

$$\gamma|_{[a,p_1]}, \ldots, \gamma_{[p_n,b]}$$

are parametrized smooth curves.

Definition 5.8. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise parametrized smooth curve on an open set $U \subseteq \mathbb{C}$, and let $f\colon U \to \mathbb{C}$ be a continuous function. The *integral of f along* γ is defined to be

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t.$$

Exercise. Show that if γ and γ' are equivalent, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma'} f(z)\,\mathrm{d}z \quad \text{for any } f.$$

In other words, the integral depends only on the underlying curve (and its orientation). (Hint: This essentially follows from the change of variables of integrals.)

Exercise. Show that if γ and γ' parametrizes the same curve but with opposite orientations, then

$$\int_\gamma f(z)\,\mathrm{d}z = -\int_{\gamma'} f(z)\,\mathrm{d}z \qquad \text{for any } f.$$

The following is perhaps the most important (yet simple) example of path integrals.

Example 5.9. Consider the unit circle parametrizes counterclockwisely $\gamma\colon [0,2\pi] \to \mathbb{C}$ where $\gamma(t) = e^{it}$. The function $f(z) = \frac{1}{z}$ is continuous (in fact, holomorphic) on $\mathbb{C}\backslash\{0\}$, so it makes sense to compute the path integral of f along the unit circle.

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it}\,\mathrm{d}t = 2\pi i.$$

Remark 5.10. In general, let γ be a curve, not necessarily simple (i.e. may have self-intersections), that does not pass through the origin. Then the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,\mathrm{d}z \in \mathbb{Z}$$

is always an integer, which gives the *winding number* of γ around the origin.

Exercise. Let $F\colon U \to \mathbb{C}$ be a holomorphic function on an open set U, and let γ be a piecewise smooth curve in U, starting at w_1 and ending at w_2. Then

$$\int_\gamma F'(z)\,\mathrm{d}z = F(w_2) - F(w_1).$$

In particular, $\int_{S^1} z^n\,\mathrm{d}z = 0$ unless $n = -1$.

Remark 5.11. The previous example and exercise suggest that the log function $\log z$ is not well-defined on $\mathbb{C}\backslash\{0\}$. Indeed, it is only possible to define $\log z$ on the *universal cover* of $\mathbb{C}\backslash\{0\}$.

We now state the Cauchy integral theorem.

Theorem 5.12 (Cauchy integral theorem)**.** *Let γ be a simple closed curve in $\mathbb{C}$. Suppose f is holomorphic on an open set containing γ* and its interior*, then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Corollary 5.13. *Let γ be a simple closed curve in $\mathbb{C}$ (oriented counterclockwisely). Suppose f is holomorphic on an open set containing γ and its interior, except at the points $z_1, \ldots, z_k$ in the interior of γ where f is not defined. Choose any small loops $\gamma_1, \ldots, \gamma_k$ (oriented counterclockwisely) that lie in the interior of γ, so that γ_i contains only one of the z_i. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^{k} \int_{\gamma_i} f(z)\,\mathrm{d}z.$$

In other words, to compute $\int_\gamma f(z)\,\mathrm{d}z$, it suffices to compute the integrals $\int_{\gamma_i} f(z)\,\mathrm{d}z$ around the *singularities* (where f is not defined) $z_1, \ldots, z_k$. These integrals are completely determined by the local behavior of f near the singular points.

Theorem 5.14 (Laurent series expansion)**.** *Let $z_0 \in \mathbb{C}$ and $R > 0$. Suppose f is a holomorphic function on the open set $0 < |z - z_0| < R$. For each $n \in \mathbb{Z}$, define*

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z$$

where γ is counterclockwise around a simple closed curve enclosing z_0 inside the open set $0 < |z - z_0| < R$. Then the series

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

converges and coincides with $f(z)$ for any $0 < |z - z_0| < R$.

Remark 5.15. In particular, if f is holomorphic on the whole neighborhood $|z - z_0| < R$, then $a_{-n} = 0$ for any $n > 0$ by the Cauchy integral theorem, so the series above gives the power series expansion near z_0. In particular, for each $n \geq 0$ the n-th derivative of f at z_0 is

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z. \tag{5.1}$$

This is the *Cauchy integral formula*.

Lecture 6

Remark 5.16. Logically speaking, the theorem on Laurent series expansion is a consequence of the Cauchy integral formula, which, is ultimately a consequence of the Cauchy integral theorem that we started with. Let us sketch the proof of

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z \tag{5.2}$$

assuming the Cauchy integral theorem (here f is holomorphic on z_0 and its neighborhood); and in the next remark, we sketch the proof of the Cauchy integral theorem. The idea is to write

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z = \frac{1}{2\pi i} \int_\gamma \frac{f(z) - f(z_0)}{z - z_0} \, \mathrm{d}z + \frac{1}{2\pi i} \int_\gamma \frac{f(z_0)}{z - z_0} \, \mathrm{d}z.$$

By Cauchy integral theorem, the second term is $f(z_0)$, so it suffices to show that the first term is zero. This follows from the following two observations. First, the function $(f(z) - f(z_0))/(z - z_0)$ is bounded (say, by $M > 0$) near z_0 since f is holomorphic at z_0. Second, again by Cauchy integral theorem, we have

$$I_1 = \frac{1}{2\pi i} \int_\gamma \frac{f(z) - f(z_0)}{z - z_0} \, \mathrm{d}z = \frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{f(z) - f(z_0)}{z - z_0} \, \mathrm{d}z$$

for any circle γ_ϵ of radius $\epsilon > 0$ centered at z_0. Thus

$$|I_1| \leq \frac{1}{2\pi} \cdot M \cdot \mathrm{length}(\gamma_\epsilon) = M \cdot \epsilon \quad \text{for any} \quad \epsilon > 0.$$

Hence $I_1 = 0$. The fact that holomorphic functions are indefinitely differentiable, and the general Cauchy integral formula (5.1) are both easy consequences of (5.2).

Remark 5.17. In this remark, we sketch the proof of the Cauchy integral theorem. Let us discuss only the case where the curve γ is a *triangle*. The general case would follow from this case together with certain limiting process, which we omit here. Let us denote the interior of γ, which is a triangle, by $T^{(0)}$. One can divide $T^{(0)}$ into four sub-triangles, so that the path integral of f along γ equals to the sum of the path integrals along the boundary of these four sub-triangles. Therefore, at least one of the four sub-triangles, say $T^{(1)}$, satisfies

$$\left|\int_{\gamma=\partial T^{(0)}} f(z)\,\mathrm{d}z\right| \leq 4\left|\int_{\partial T^{(1)}} f(z)\,\mathrm{d}z\right|.$$

Continue this process indefinitely, we obtain a sequence of triangles

$$\cdots \subseteq T^{(2)} \subseteq T^{(1)} \subseteq T^{(0)}$$

where the diameter $d^{(n)}$ and perimeter $p^{(n)}$ is decreased by half in each step, and

$$\left|\int_{\gamma=\partial T^{(0)}} f(z)\,\mathrm{d}z\right| \leq 4^n\left|\int_{\partial T^{(n)}} f(z)\,\mathrm{d}z\right|.$$

Since each triangle is a compact subset, the sequence would converge to a unique point, say z_0. Using the condition that f is holomorphic, for any $\epsilon > 0$ there exists a $\delta > 0$ so that

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| < \epsilon \cdot (z - z_0) \quad \text{for all} \quad z \in B_\delta(z_0).$$

Choose n large enough so that $T^{(n)} \subseteq B_\delta(z_0)$, then we have

$$\begin{aligned}\left|\int_{\partial T^{(n)}} f(z)\,\mathrm{d}z\right| &= \left|\int_{\partial T^{(n)}} (f(z) - f(z_0) - (z - z_0)f'(z_0))\,\mathrm{d}z\right| \text{ (why?)}\\ &\leq p^{(n)} \cdot \sup_{z\in\partial T^{(n)}} |f(z) - f(z_0) - (z - z_0)f'(z_0)|\\ &< p^{(n)} \cdot \epsilon \cdot d^{(n)} = \epsilon \cdot \frac{p^{(0)}}{2^n} \cdot \frac{d^{(0)}}{2^n}.\end{aligned}$$

Thus

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq 4^n \cdot \epsilon \cdot \frac{p^{(0)}}{2^n} \cdot \frac{d^{(0)}}{2^n} = \epsilon \cdot p^{(0)} \cdot d^{(0)} \quad \text{for any} \quad \epsilon > 0.$$

Hence $\int_\gamma f(z)\,\mathrm{d}z = 0$.

Definition 5.18. The residue of f at a singular point z_0 is defined to be

$$\operatorname{Res}(f, z_0) := a_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z.$$

Notation. Let n be a positive integer. Let f be a holomorphic function on $0 < |z - z_0| < R$, and a_n be the coefficients of its Laurent series expansion defined earlier. We say

- f has a *zero of order* n at z_0 if $a_n \neq 0$ and $a_m = 0$ for all $m < n$.
- f has a *pole of order* n at z_0 if $a_{-n} \neq 0$ and $a_m = 0$ for all $m < -n$.

Example 5.19. Suppose f has a *simple pole* at z_0, i.e. $(z - z_0)f(z)$ can be extended to a holomorphic function on the whole neighborhood $|z - z_0| < R$. Then a_{-n} for any $n \geq 2$ by the Cauchy integral theorem, so the Laurent series expansion of f near the point z_0 is given by

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

Therefore a_{-1} can be computed by the limit

$$a_{-1} = \lim_{z \to z_0} f(z)(z - z_0).$$

Similarly, suppose f has a pole of order n at z_0 (i.e. $(z - z_0)^n f(z)$ can be extended to a holomorphic function on the whole neighborhood $|z - z_0| < R$, but $(z - z_0)^{n-1} f(z)$ cannot), then its residue can be computed by

$$\operatorname{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z) \right).$$

Example 5.20. Let $a > 0$ be a positive real number. The function

$$f(z) = \frac{e^{iz}}{z^2 + a^2}$$

is holomorphic except at $\pm ia$. It is clear that both $\pm ia$ are simple poles of f.

$$\operatorname{Res}(f, ia) = \lim_{z \to ia} \frac{e^{iz}}{z^2 + a^2}(z - ia) = \lim_{z \to ia} \frac{e^{iz}}{z + ia} = \frac{e^{-a}}{2ia}.$$

Example 5.21. How to compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} \, \mathrm{d}x \ = ?$$

Let $R \gg 0$ and let γ_R parametrizes the upper half of the circle $|z| = R$ going counterclockwisely. Then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,\mathrm{d}x &= \lim_{R\to\infty}\int_{-R}^{R}\frac{\cos x}{x^2+a^2}\,\mathrm{d}x \\ &= \operatorname{Re}\left(\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{iz}}{z^2+a^2}\,\mathrm{d}z\right) \\ &= \operatorname{Re}\left(2\pi i\cdot\operatorname{Res}\left(\frac{e^{iz}}{z^2+a^2}, ia\right) - \lim_{R\to\infty}\int_{\gamma_R}\frac{e^{iz}}{z^2+a^2}\,\mathrm{d}z\right) \\ &= \frac{\pi e^{-a}}{a} - \operatorname{Re}\left(\lim_{R\to\infty}\int_{\gamma_R}\frac{e^{iz}}{z^2+a^2}\,\mathrm{d}z\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left|\int_{\gamma_R}\frac{e^{iz}}{z^2+a^2}\,\mathrm{d}z\right| &= \left|\int_0^{\pi}\frac{e^{iRe^{i\theta}}}{R^2e^{2i\theta}+a^2}\cdot iRe^{i\theta}\,\mathrm{d}\theta\right| \\ &\le \int_0^{\pi}\frac{1}{|R^2-a^2|}\cdot R\,\mathrm{d}\theta \longrightarrow 0 \quad \text{as } R\to\infty.\end{aligned}$$

Thus we get

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi e^{-a}}{a}.$$

Theorem 5.22 (Liouville)**.** *Let f be a bounded ($|f(z)| < M$ for all z) and entire (holomorphic on the whole complex plane $\mathbb{C}$) function. Then f is a constant function.*

Proof. It suffices to show that the derivative $f'(z_0)$ is zero for all $z_0 \in \mathbb{C}$. Let $\gamma_R(z_0)$ be the circle of radius R centered at the point z_0. By the Cauchy integral formula, we have

$$\begin{aligned}|f'(z_0)| &= \frac{1}{2\pi}\left|\int_{\gamma_R(z_0)}\frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z\right| \\ &< \frac{1}{2\pi}\cdot\frac{M}{R^2}\cdot 2\pi R = \frac{M}{R}.\end{aligned}$$

The inequality $|f'(z_0)| < \frac{M}{R}$ holds for all $R > 0$. Thus $f'(z_0) = 0$. □

The fundamental theorem of algebra is a simple corollary of the Liouville theorem.

Corollary 5.23 (Fundamental theorem of algebra)**.** *Any non-constant complex polynomial $p(z)$ has a root in $\mathbb{C}$.*

Proof. Assume the contrary that $p(z)$ has no roots in $\mathbb{C}$. Then $\frac{1}{p(z)}$ is an entire function. It is not hard to show that $\frac{1}{p(z)}$ is a bounded function on $\mathbb{C}$. By Liouville theorem, it can only be the constant function. □

Finally, we state the *argument principle*, which claims that the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

counts the number of zeros minus the number of poles in the interior of γ. To illustrate this, let us start with two basic examples.

Example 5.24. Consider $f(z) = z^n$, which has a zero of order n at the point 0. Let γ be any simple closed curve enclosing 0 (oriented counterclockwisely). Then

$$\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \int_\gamma \frac{nz^{n-1}}{z^n}\,\mathrm{d}z = n\int_\gamma \frac{1}{z}\,\mathrm{d}z = 2\pi i \cdot n.$$

Example 5.25. Consider $f(z) = z^{-n}$, which has a pole of order n at the point 0. Let γ be any simple closed curve enclosing 0 (oriented counterclockwisely). Then

$$\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \int_\gamma \frac{-nz^{-n-1}}{z^{-n}}\,\mathrm{d}z = (-n)\int_\gamma \frac{1}{z}\,\mathrm{d}z = 2\pi i \cdot (-n).$$

In general, we have the following theorem.

Theorem 5.26 (Argument principle)**.** *Let γ be a simple closed curve. Suppose f is a holomorphic function on an open set containing γ and its interior, except at finitely many poles. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = (\#\ \textit{zeros of } f(z)\ \textit{inside } \gamma) - (\#\ \textit{poles of } f(z)\ \textit{inside } \gamma)\,.$$

Here the numbers are counted with multiplicities, i.e. an order n zero is counted as n zeros, and an order n pole is counted as n poles.

5.3. **Elliptic functions.** We discuss the *elliptic functions* in this subsection; some of the aspects of elliptic functions are closely related to modular forms and will be useful later.

Definition 5.27. Let $\omega_1, \omega_2 \in \mathbb{C}$ be two complex numbers that are linearly independent over $\mathbb{R}$ (i.e. they span the vector space $\mathbb{C} \cong \mathbb{R}^2$). We say a function f on $\mathbb{C}$ is *elliptic* (with respect to ω_1, ω_2) if

$$f(z) = f(z+\omega_1) = f(z+\omega_2) \quad \text{for all} \quad z \in \mathbb{C}.$$

The parallelogram with vertices $0, \omega_1, \omega_2, \omega_1+\omega_2$ is called the *fundamental domain.* It is easy to see that the values of an elliptic function on $\mathbb{C}$ is determined by its value on the fundamental domain.

Exercise. Show that the only *holomorphic* elliptic functions are the constant functions. (Hint: Liouville's theorem.)

Therefore, it is more interesting to consider the *meromorphic* elliptic functions. (A function on $U \subseteq \mathbb{C}$ is called *meromorphic* if for any $z_0 \in U$, the function f either is holomorphic at z_0 or has a pole at z_0.) Here is a rough idea of a way to construct such functions. Let g be a meromorphic function on $\mathbb{C}$. Then

$$f(z) = \sum_{m,n\in\mathbb{Z}} g(z+m\omega_1+n\omega_2)$$

must be elliptic, provided that the series on the right hand side converges. Suppose we have $|g(z)| < \frac{C}{|z|^\alpha}$ for $|z| \gg 0$. Observe that for a fix $z \in \mathbb{C}$, the number of points of the form $z+m\omega_1+n\omega_2$ in the annulus $R \leq |z+m\omega_1+n\omega_2| < R+1$ is roughly (constant)$\cdot R$. Thus

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} |g(z+m\omega_1+n\omega_2)| &= \sum_{R=0}^{\infty} \sum_{\substack{m,n\in\mathbb{Z} \\ R\leq|z+m\omega_1+n\omega_2|<R+1}} |g(z+m\omega_1+n\omega_2)| \\ &\approx \sum_{R=0}^{\infty} \frac{C}{R^\alpha}\cdot R\end{aligned}$$

Hence, if $\alpha > 2$, then the series $\sum_{m,n\in\mathbb{Z}} g(z+m\omega_1+n\omega_2)$ converges absolutely. Let us summarize this as the following example.

Example 5.28. Let $C > 0$ be a constant and $\alpha > 2$. If $|g(z)| < \frac{C}{|z|^\alpha}$ for all $|z| \gg 0$, then

$$f(z) = \sum_{m,n\in\mathbb{Z}} g(z + m\omega_1 + n\omega_2)$$

is a meromorphic elliptic function.

For instance, one can take $g(z) = \frac{1}{(z-\alpha)(z-\beta)(z-\gamma)}$ for some $\alpha, \beta, \gamma \in \mathbb{C}$. Then

$$f(z) = \sum_{m,n\in\mathbb{Z}} g(z + m\omega_1 + n\omega_2)$$

is a meromorphic elliptic function, which has 3 poles in the fundamental domain.

Question 5.29. *Do there exist meromorphic elliptic functions with only* 1 *or* 2 *poles in the fundamental domain?*

Notation. We denote the lattice

$$\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

To answer this question, let us first establish the following basic (yet important) fact about elliptic functions.

Theorem 5.30. *Let f be a meromorphic elliptic function with respect to Λ. Assume that f has no zeros or poles on the boundary of the fundamental domain. Then*

(a) The number of zeros of f in the fundamental domain coincides with the number of poles of f in the fundamental domain.

(b) The sum of the zeros of f (which is a complex number) minus the sum of the poles of f in the fundamental domain is an element in Λ.

Here the zeros and poles are counted with multiplicities.

Proof. The first statement follows directly from the argument principle. To show the second the statement, we first claim a general statement that the sum of the zeros of f minus the sum of the poles of f in an area enclosed by a loop γ (oriented counterclockwisely) is given by

$$\frac{1}{2\pi i}\int_\gamma z \cdot \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

The computation of the integral boils down to computing the integral around the zeros and poles of f. As an example, say z_0 is a zero of order n of f. Then $f(z) = (z - z_0)^n h(z)$ where h is holomorphic in a neighborhood of z_0 with $h(z_0) \neq 0$. Let γ_0 be a small loop centered at the zero z_0. Then we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_0} z\cdot\frac{f'(z)}{f(z)}\,\mathrm{d}z &= \frac{1}{2\pi i}\int_{\gamma_0} z\cdot\frac{n(z-z_0)^{n-1}h(z)+(z-z_0)^n h'(z)}{(z-z_0)^n h(z)}\,\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{\gamma_0} z\cdot\frac{n}{z-z_0}\,\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{\gamma_0} (z-z_0)\cdot\frac{n}{z-z_0}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{\gamma_0} z_0\cdot\frac{n}{z-z_0}\,\mathrm{d}z\\ &= 0 + nz_0 = nz_0.\end{aligned}$$

Similar computations work for the poles. Therefore, to show the second statement, one needs to show that

$$\begin{aligned}&\frac{1}{2\pi i}\int_0^{\omega_1} z\cdot\frac{f'(z)}{f(z)}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{\omega_1}^{\omega_1+\omega_2} z\cdot\frac{f'(z)}{f(z)}\,\mathrm{d}z\\ &\qquad + \frac{1}{2\pi i}\int_{\omega_1+\omega_2}^{\omega_2} z\cdot\frac{f'(z)}{f(z)}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{\omega_2}^{0} z\cdot\frac{f'(z)}{f(z)}\,\mathrm{d}z \in \Lambda.\end{aligned}$$

Observe that

$$\frac{1}{2\pi i}\int_{\omega_1}^{\omega_1+\omega_2} z\cdot\frac{f'(z)}{f(z)}\,\mathrm{d}z - \frac{1}{2\pi i}\int_0^{\omega_2} z\cdot\frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{\omega_1}{2\pi i}\int_0^{\omega_2}\frac{f'(z)}{f(z)}\,\mathrm{d}z$$

by the periodicity of f. Define $\eta(t) = f(\omega_2 t)$ for $t \in [0,1]$, which parametrizes a closed curve (not necessarily simple) in $\mathbb{C}\backslash\{0\}$. Then

$$\frac{1}{2\pi i}\int_0^{\omega_2}\frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_0^1 \frac{f'(\omega_2 t)}{f(\omega_2 t)}\cdot\omega_2\,\mathrm{d}t = \frac{1}{2\pi i}\int_0^1\frac{\eta'(t)}{\eta(t)}\,\mathrm{d}t = \frac{1}{2\pi i}\int_\eta \frac{1}{z}\,\mathrm{d}z \in \mathbb{Z}.$$

Hence

$$\frac{1}{2\pi i}\int_{\omega_1}^{\omega_1+\omega_2} z\cdot\frac{f'(z)}{f(z)}\,\mathrm{d}z - \frac{1}{2\pi i}\int_0^{\omega_2} z\cdot\frac{f'(z)}{f(z)}\,\mathrm{d}z \in \omega_1\mathbb{Z}.$$

Similarly, one can show that

$$\frac{1}{2\pi i}\int_{\omega_2}^{\omega_1+\omega_2} z\cdot\frac{f'(z)}{f(z)}\,\mathrm{d}z - \frac{1}{2\pi i}\int_0^{\omega_1} z\cdot\frac{f'(z)}{f(z)}\,\mathrm{d}z \in \omega_2\mathbb{Z}.$$

This concludes the proof. □

Remark 5.31. In fact, one can show that given $z_1, \dots, z_n, p_1, \dots, p_n$ in the fundamental domain satisfying $\sum z_i = \sum p_i$, there exists a meromorphic elliptic function f with zeros at $z_1, \dots, z_n$ and poles at $p_1, \dots, p_n$.

Corollary 5.32. *There is no meromorphic elliptic function with exactly* 1 *pole in the fundamental domain (counted with multiplicity).*

Proof. Assume the contrary that f is a meromorphic elliptic function with exactly 1 pole in the fundamental domain, say at p_0. By the first part of the theorem, there is exactly 1 zero in the fundamental domain as well, say at z_0. The second part of the theorem then implies that $z_0 = p_0$, which is impossible since a point cannot be a zero and a pole of f simultaneously. □

It turns out that there exist meromorphic elliptic functions with exactly 2 poles in the fundamental domain. One of such functions is the *Weierstrass ℘-function.* Recall that the naive construction using

$$\sum_{\lambda \in \Lambda} \frac{1}{(z+\lambda)^2}$$

fails, because the series does not converge. One way to get around this is to consider

$$\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} = \frac{-z^2 - 2z\lambda}{(z+\lambda)^2\lambda^2}$$

which is now of degree -3 in λ. Indeed, one can show that the series

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)$$

converges, and is defined to be the *Weierstrass ℘-function.* Now, because the right hand side is not symmetric with respect to all $\lambda \in \Lambda$, we have to show that it is indeed an elliptic function.

Proposition 5.33. *The function $\wp(z)$ is elliptic with respect to Λ.*

Proof. It is clear that the derivative $\wp'(z)$ is elliptic (the asymmetry of $\wp$ is caused by the terms $\frac{1}{\lambda^2}$, which will be annihilated by the derivative in z), so

$$\wp'(z) = \wp'(z+\omega_1) = \wp'(z+\omega_2).$$

Therefore, the function $\wp(z) - \wp(z+\omega_1)$ is a constant function in z, say

$$\wp(z) - \wp(z+\omega_1) = C.$$

Using the fact that $\wp(z)$ is an even function ($\wp(-z) = \wp(z)$), we have

$$C = \wp(-\omega_1/2) - \wp(\omega_1/2) = 0.$$

Thus $\wp(z) = \wp(z+\omega_1)$. Similarly, one can show that $\wp(z) = \wp(z+\omega_2)$. □

The proposition shows that $\wp(z)$ is an elliptic meromorphic function. It has exactly two poles in the fundamental domain (which is given by the double pole at the origin). By the theorem we proved earlier, $\wp(z)$ should also have two zeros in the fundamental domain.

Question 5.34. *What are the zeros of $\wp(z)$ (in the fundamental domain)?*

It turns out that the answer to this simple question is much harder than it appears to be.

Theorem 5.35 (Eichler–Zagier). *Let Λ_τ be the lattice generated by 1 and $\tau \in \mathbb{H} = \{x+iy \mid y > 0\}$. Then the zeros of $\wp(z,\tau)$ in the fundamental domain are given by*

$$\frac{1}{2} \pm \left(\frac{\log(5+2\sqrt{6})}{2\pi i} + 144\pi i\sqrt{6} \int_\tau^{i\infty} (\sigma - \tau) \frac{E_4(\sigma)^3}{E_6(\sigma)^{3/2} j(\sigma)} \, \mathrm{d}\sigma \right)$$

where E_4, E_6, j are the modular forms and functions we will discuss further.

Remark 5.36. Recall that the trick to make the series $\sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^2}$ converges was to consider

$$\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} = \frac{-z^2 - 2z\lambda}{(z+\lambda)^2\lambda^2},$$

which becomes of degree -3 in λ. One can apply the same method to the series $\sum_{\lambda\in\Lambda} \frac{1}{z-\lambda}$. Since

$$\frac{1}{z-\lambda} = \frac{-1}{\lambda}\left(1 + \frac{z}{\lambda} + \frac{z^2}{\lambda^2} + \cdots\right),$$

the expression

$$\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \quad \text{is of degree 3 in } \lambda.$$

This gives the *Weierstrass ζ-function*

$$\zeta(z) = \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right).$$

Since the Weierstrass ζ-function has only one simple pole in the fundamental domain of Λ, so it cannot be an elliptic function. On the other hand, its derivative

$$\zeta'(z) = -\wp(z)$$

is elliptic. Hence $\zeta'(z) = \zeta'(z+\omega_1)$, thus $\zeta(z+\omega_1) - \zeta(z)$ is a constant function. Similarly, $\zeta(z+\omega_2) - \zeta(z)$ also is a constant function.

Remark 5.37. By the last property, for any $a, b \in \mathbb{C}$, the function $\zeta(z-a) - \zeta(z-b)$ is an elliptic function, which has exactly two poles in the fundamental domain (a, b modulo Λ).

Let us compute the Laurent series expansion of the Weierstrass $\wp$-function near $z = 0$. First, since for any $|w| < 1$ we have

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n, \qquad \text{thus} \qquad \frac{1}{(1-w)^2} = \sum_{n=0}^{\infty} (n+1) w^n.$$

Thus

$$\frac{1}{(z-\lambda)^2} = \frac{1}{\lambda^2 \left(1 - \frac{z}{\lambda}\right)^2} = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \sum_{n=1}^{\infty} (n+1) \left(\frac{z}{\lambda}\right)^n.$$

Therefore

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + \sum_{\lambda \neq 0} \left(\frac{1}{\lambda^2} \sum_{n=1}^{\infty} (n+1) \left(\frac{z}{\lambda}\right)^n \right) \\ &= \frac{1}{z^2} + \sum_{n=1}^{\infty} \left(\left(\sum_{\lambda \neq 0} \frac{1}{\lambda^{n+2}} \right) (n+1) z^n \right) \end{aligned}$$

For each $n \geq 3$, define the *Eisenstein series* of Λ as

$$\widetilde{E_n}(\Lambda) = \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^n}.$$

Note that $\widetilde{E_n}(\Lambda) = 0$ if n is odd. Thus we have

$$\wp(z) = \frac{1}{z^2} + 3\widetilde{E_4}z^2 + 5\widetilde{E_6}z^4 + \cdots .$$

Remark 5.38. Using the fact that the only holomorphic elliptic functions are constant functions, one can deduce many identities about $\wp(z)$. For instance, one can prove the following proposition.

Proposition 5.39. *$\wp'(z)^2$ can be expressed as a cubic polynomial of $\wp(z)$.*

Proof. Compute the first few terms of the Laurent series of:

$$\wp'(z) = \frac{-2}{z^3} + 6\widetilde{E_4}z + 20\widetilde{E_6}z^3 + \cdots$$

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{24\widetilde{E_4}}{z^2} - 80\widetilde{E_6} + \cdots$$

$$\wp(z)^3 = \frac{1}{z^6} + \frac{9\widetilde{E_4}}{z^2} + 15\widetilde{E_6} + \cdots$$

Thus

$$\wp'(z)^2 - 4\wp(z)^3 + 60\widetilde{E_4}\wp(z) = -140\widetilde{E_6} + \cdots$$

is a holomorphic elliptic function, therefore is a constant. Hence

$$\wp'(z)^2 = 4\wp(z)^3 - 60\widetilde{E_4}\wp(z) - 140\widetilde{E_6}.$$

□

Remark 5.40. The proposition is closely related to the cubic equation of *elliptic curves*. There is a map

$$\mathbb{C}/\Lambda \longrightarrow \{y^2 = 4x^3 - 60\widetilde{E_4}x - 140\widetilde{E_6}\} \subseteq \mathbb{C}^2; \quad z \mapsto (\wp(z), \wp'(z)).$$

Remark 5.41. In fact, one can show that

$$\wp'(z)^2 = 4\left(\wp(z) - \wp\left(\frac{\omega_1}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_2}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_1+\omega_2}{2}\right)\right).$$

Theorem 5.42. *Any meromorphic elliptic function can be expressed as a rational polynomial in $\wp(z)$ and $\wp'(z)$.*

Proof. Let f be a meromorphic elliptic function. By considering

$$f(z) = \left(\frac{f(z) + f(-z)}{2}\right) + \left(\frac{f(z) - f(-z)}{2}\right),$$

it suffices to prove the theorem for *even* meromorphic elliptic functions and *odd* meromorphic elliptic functions. Up to multiplying $\wp'(z)$ (which is an odd function), it suffices to prove the theorem only for *even* meromorphic elliptic functions.

We claim that any even meromorphic elliptic function f is a rational polynomial in the Weierstrass $\wp$-function $\wp(z)$.

- Up to multiplying $\wp(z) - \wp(z_0)$, one reduces to the case where the poles of f are at Λ.
- Let $f(z) = \frac{a_{-2n}}{z^{2n}} + \cdots$ be the Laurent series expansion near $z = 0$. Then

  $$f(z) - a_{-2n}\wp(z)^n = \frac{\star}{z^{2n-2}} + \cdots$$

  is also an even meromorphic elliptic function.
- Continue this process inductively, one finds $a_{-2}, a_{-4}, \ldots, a_{-2n}$ so that

  $$f(z) - a_{-2n}\wp(z)^n - a_{-2(n-1)}\wp(z)^{n-1} - \cdots - a_{-2}\wp(z)$$

  is a holomorphic elliptic function, therefore is a constant function. Thus $f(z)$ can be expressed as a polynomial in $\wp(z)$.

□

Exercise. Let $\tau \in \mathbb{H}$ be an element in the upper half-plane $\mathbb{H}$. Denote the lattice $\langle 1, \tau\rangle$ as Λ_τ. The Weierstrass $\wp$-function depends on the choice of the lattice. We denote

$$\wp(z, \tau) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda_\tau \setminus \{0\}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}\right).$$

Prove that for any integers $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$, we have

$$\wp\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2\wp(z, \tau).$$

Lecture 7

5.4. **Modular functions and modular forms.** Informally, modular functions (resp. modular forms) are functions (resp. differential forms) define on the *moduli space of complex torus*, or equivalently, on the *moduli space of lattices in* $\mathbb{C}$ up to the equivalence $\Lambda_1 \sim \Lambda_2$ if $\Lambda_1 = c\Lambda_2$ for some $c \in \mathbb{C}\backslash\{0\}$. Since any lattice is equivalent to a lattice of the form $\Lambda_\tau = \langle 1, \tau\rangle$ for some $\tau \in \mathbb{H}$, the modular functions or forms can be regarded as functions on $\mathbb{H}$ that behaves nicely under the $\mathrm{SL}(2,\mathbb{Z})$-action (since the $\mathrm{SL}(2,\mathbb{Z})$-actions preserve lattices). Recall that $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ acts on $\tau \in \mathbb{H}$ by

$$g \cdot \tau = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$

Remark 5.43. One can check easily that

$$\mathrm{Im}(g \cdot \tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}.$$

Therefore the $\mathrm{SL}(2,\mathbb{Z})$ action preserves the set $\mathbb{H}$. Note that the element $-\mathrm{id} \in \mathrm{SL}(2,\mathbb{Z})$ acts trivially on $\mathbb{H}$, so one can also consider the $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathrm{SL}(2,\mathbb{Z})/\{\pm\mathrm{id}\}$ action on $\mathbb{H}$.

Let

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

One has:

$$T(\tau) = \tau + 1; \qquad S(\tau) = -1/\tau; \qquad S^2 = (ST)^3 = I.$$

Consider the set

$$D = \left\{ z \in \mathbb{H} \colon |z| \geq 1 \text{ and } -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \right\}.$$

We will show that D is a *fundamental domain* for the action of $\mathrm{PSL}(2,\mathbb{Z})$ on the upper half plane $\mathbb{H}$.

Theorem 5.44. *More precisely, we have:*

(a) For every $\tau \in \mathbb{H}$, there exists $g \in \mathrm{PSL}(2,\mathbb{Z})$ such that $g \cdot \tau \in D$.

(b) Suppose $\tau' = g\tau$ for some $\tau, \tau' \in D$ and $g \in \mathrm{PSL}(2,\mathbb{Z})\backslash\{I\}$, then:

- *either $\mathrm{Re}(\tau) = \pm 1/2$ and $\tau = \tau' \pm 1$,*

- *or* $|\tau| = 1$ *and* $\tau' = -1/\tau$.

(c) Let $\tau \in D$ *and let* $H_\tau = \{g \in \mathrm{PSL}(2,\mathbb{Z}) \mid g\tau = \tau\}$ *be the stabilizer of* τ*. Then:*

- $H_\tau = \langle S\rangle \cong \mathbb{Z}_2$ *if* $\tau = i$.
- $H_\tau = \langle ST\rangle \cong \mathbb{Z}_3$ *if* $\tau = e^{2\pi i/3} (= \omega)$.
- $H_\tau = \langle TS\rangle \cong \mathbb{Z}_3$ *if* $\tau = e^{\pi i/3} = (-1/\omega)$.
- $H_\tau = \{I\}$ *otherwise.*

Theorem 5.45. *The group* $\mathrm{PSL}(2,\mathbb{Z})$ *is generated by* S *and* T*.*

Let us prove both theorems together.

Proof. Let $G \subseteq \mathrm{PSL}(2,\mathbb{Z})$ be the subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ generated by S and T. Let $\tau \in \mathbb{H}$. We will show that there exists $g \in G = \langle S, T\rangle$ so that $g\tau \in D$, which proves the first statement. Recall that

$$\operatorname{Im}\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix} \cdot \tau\right) = \frac{\operatorname{Im}(\tau)}{|c\tau + d|^2}.$$

Since c, d are integers, the number of pairs (c, d) such that $|c\tau + d|$ is less than a given number is *finite*. Therefore, there exists $g \in G$ such that $\operatorname{Im}(g\tau)$ is maximum. Now, choose an integer n so that $T^n g\tau$ has real part between $-\frac{1}{2}$ and $\frac{1}{2}$. Then the element $\tau' = T^n g\tau \in D$: indeed, it suffices to show that $|\tau'| \geq 1$; but if $|\tau'| < 1$, then the element $-1/\tau'$ would have imaginary part strictly greater than $\operatorname{Im}(\tau')$, contradiction. Thus $T^n g \in G$ has the desired property.

We now prove the second and third statements of the first theorem. Let $\tau \in D$ and $g \in \mathrm{PSL}(2,\mathbb{Z})$ so that $g\tau \in D$. By replacing (τ, g) by $(g\tau, g^{-1})$ if necessary, one may assume that $\operatorname{Im}(g\tau) \geq \operatorname{Im}(\tau)$, i.e. $|c\tau + d| \leq 1$. This is clearly impossible if $|c| \geq 2$, leaving the cases $c = 0, \pm 1$. If $c = 0$, then $d = \pm 1$ and g is the translation by $\pm b$. This is only possible for $\operatorname{Re}(\tau) = \pm 1/2$ and $g = T^{\pm 1}$. (Also, note that $T^{\pm 1}$ do not fix any point on D.)

If $c = 1$, then we have $|\tau + d| \leq 1$.

- If $d = 0$, then $|\tau| \leq 1$ hence $|\tau| = 1$ since $\tau \in D$. On the other hand, $ad - bc = 1$ implies $b = -1$, hence $g\tau = a - \frac{1}{\tau} \in D$. This is only if:
  - $a = 0$; in which case $g = S$, which sends $\{|\tau| = 1\} \cap D$ to itself. (Note that S has a unique fixed point $i \in D$.)

  - $a = 1$ and $\tau = -1/\omega$, which gives rise to the element $TS \in H_{-1/\omega}$.
  - $a = -1$ and $\tau = \omega$, which gives rise to the element $ST \in H_{\omega}$.
- If $d \neq 0$, then the only d and τ that satisfies $|\tau + d| \leq 1$ are:
  - $d = 1$ and $\tau = \omega$, which gives rise to the element $(ST)^2 \in H_{\omega}$.
  - $d = -1$ and $\tau = -1/\omega$, which gives rise to the element $(TS)^2 \in H_{-1/\omega}$.

This concludes the proof of the first theorem.

It remains to prove that $G = \mathrm{PSL}(2, \mathbb{Z})$. Let $g \in \mathrm{PSL}(2, \mathbb{Z})$. Choose any interior point of D, say $z_0 = 2i$. Consider the element $gz_0 \in \mathbb{H}$. By (a), there exists an element $g' \in G$ such that $g'gz_0 \in D$. By (c), we must have $g'g = I$. Thus $g \in G$. □

Remark 5.46. By the second theorem, to check the invariance of $\mathrm{SL}(2, \mathbb{Z})$, it suffices to check the invariance under the actions by T and S. For instance, a function $f \colon \mathbb{H} \to \mathbb{C}$ satisfies $f(\frac{a\tau+b}{c\tau+d}) = f(\tau)$ for all $\tau \in \mathbb{H}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ is equivalent to it satisfies $f(\tau) = f(\tau + 1) = f(\frac{-1}{\tau})$ for all $\tau \in \mathbb{H}$.

Remark 5.47. One can show that

$$\mathrm{PSL}(2, \mathbb{Z}) = \left\langle S, T \mid S^2 = (ST)^3 = 1 \right\rangle,$$

or equivalently, G is the *free product* of the cyclic group of order 2 generated by S and the cyclic group of order 3 generated by ST.

Remark 5.48. It turns out that the j-function we saw earlier, which can be written as

$$j(\tau) = \frac{\left(1 + 240 \sum_{n=1}^{\infty} \left(\sum_{d|n} d^3\right) q^n\right)^3}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}} \quad \text{where} \quad q = e^{2\pi i \tau}$$

is the *simplest* non-constant holomorphic function on $\mathbb{H}$ invariant under $\mathrm{SL}(2, \mathbb{Z})$-action! It is really the simplest in the sense that *any* holomorphic function $f \colon \mathbb{H} \to \mathbb{C}$ satisfying $f(\tau) = f(\tau + 1) = f(\frac{-1}{\tau})$ for all $\tau \in \mathbb{H}$ can be written as a polynomial in $j(\tau)$.

Because of the fact stated in the previous remark, there are not many interesting functions that are $SL(2, \mathbb{Z})$-invariant on the nose. On the other hand, there are many interesting functions on $\mathbb{H}$ (such as the Eisenstein series) that satisfies a slightly modified condition. These are the *modular forms.*

Definition 5.49 (non-precise version)**.** Let k be a positive integer. A holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ is called a *modular form of weight k* if

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \text{for all } \tau \in \mathbb{H} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

Or equivalently, $f(\tau+1) = f(\tau)$ and $f(-1/\tau) = \tau^k f(\tau)$ hold for all $\tau \in \mathbb{H}$.

Exercise. Let k be an odd integer. Show that the only modular form of weight k is the zero function. (Hint: Consider the action by $-\mathrm{id} \in \mathrm{SL}(2,\mathbb{Z})$.)

Remark 5.50. The notion of modular "*forms*" comes from the following observation. Consider the differential form $f(\tau)d\tau$ on $\mathbb{H}$. One can ask whether it is invariant under the $\mathrm{SL}(2,\mathbb{Z})$-action. Since

$$f\left(\frac{a\tau+b}{c\tau+d}\right) d\left(\frac{a\tau+b}{c\tau+d}\right) = f\left(\frac{a\tau+b}{c\tau+d}\right)\frac{d\tau}{(c\tau+d)^2},$$

we find that $f(\tau)d\tau$ is $\mathrm{SL}(2,\mathbb{Z})$-invariant if and only if $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 f(\tau)$, which is equivalent to $f(\tau)$ is a modular form of weight 2. In general, the differential form $f(\tau)(d\tau)^{k/2}$ is $\mathrm{SL}(2,\mathbb{Z})$-invariant if and only if $f(\tau)$ is a modular form of weight k.

Remark 5.51. One way to produce non-trivial modular function is that, if one can find two modular forms f_1, f_2 of the same weight which are linearly independent, then their ratio f_1/f_2 would give a (meromorphic) modular function. The j-function actually arises from the quotient of two modular forms of weight 12, which, as we will see later, is the smallest weight whose space of modular forms has dimension greater than one.

Recall the following facts about the Weierstrass $\wp$-function, with respect to a lattice $\Lambda_\tau = \langle 1, \tau\rangle$:

- We have the following expansion

$$\wp(z,\tau) = \frac{1}{z^2} + 3\widetilde{E_4}(\tau)z^2 + 5\widetilde{E_6}(\tau)z^4 + 7\widetilde{E_8}(\tau)z^6 + \cdots$$

  where $\widetilde{E_{2k}}(\tau) = \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^{2k}}$ is the Eisenstein series.

- For any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ we have

$$\wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 \wp(z,\tau).$$

It is then not hard to deduce that

$$\widetilde{E_{2k}}(\tau) \text{ is a modular form of weight } 2k \text{ for any } k \geq 2.$$

Remark 5.52. Like modular functions, modular forms are also rare. In fact, we will show later that any modular form can be written as a polynomial in $\widetilde{E_4}$ and $\widetilde{E_6}$! In particular, the space of modular forms of weight k is always finite dimensional for any k.

Definition 5.53 (Precise version)**.** Let k be a positive integer. A holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ is called a *modular form of weight* k if

- $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ for all $\tau \in \mathbb{H}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$.
- $f(\tau)$ is bounded as $\mathrm{Im}(\tau) \to \infty$.

Remark 5.54. Since f is invariant under $\tau \mapsto \tau+1$, it is convenient to introduce the variable $q = \exp(2\pi i\tau)$, where f can be considered as a function in q on the punctured unit disk $\mathbb{D}_1^\times(0) = \{q\colon 0 < |q| < 1\}$. Then the second condition that$f(\tau)$ is bounded as $\mathrm{Im}(\tau) \to \infty$ is equivalent to $f(q)$ is bounded near $q = 0$. This actually is equivalent to f can be extended holomorphic to the whole unit disk $\mathbb{D}_1(0) = \{q\colon |q| < 1\}$.

Exercise. Show that the Eisenstein series $\widetilde{E_{2k}}(\tau)$ is a modular form of weight $2k$ for all $k \geq 2$.

Proposition 5.55. *Let $k \geq 4$ even and Im$(\tau) > 0$. We have*

$$\widetilde{E_k}(\tau) = 2\zeta(k) + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{r=1}^{\infty} \sigma_{k-1}(r) e^{2\pi i\tau r}.$$

Here $\sigma_{k-1}(r) = \sum_{d|r} d^{k-1}$ is the divisor function.

Proof. Recall that

$$\widetilde{E_k}(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k} = 2\zeta(k) + 2\sum_{n\geq 1}\sum_{m\in\mathbb{Z}} \frac{1}{(m+n\tau)^k}.$$

The right hand side can be computed by the *Poisson summation formula*. Let f be a function with certain appropriate regularity and decay conditions, one can define its Fourier transform as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x,$$

and the *Poisson summation formula* states that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

By applying the Poisson summation formula to the function $f(z) = (\tau+z)^{-k}$, one obtains

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1}e^{2\pi i m\tau}.$$

Thus

$$\begin{aligned}\widetilde{E_k}(\tau) &= 2\zeta(k) + 2\sum_{n\geq 1}\sum_{m\in\mathbb{Z}}\frac{1}{(m+n\tau)^k}\\ &= 2\zeta(k) + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!}\sum_{n\geq 1}\sum_{\ell=1}^{\infty}\ell^{k-1}e^{2\pi i\ell(n\tau)}\\ &= 2\zeta(k) + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!}\sum_{r\geq 1}\sigma_{k-1}(r)e^{2\pi i\tau r}.\end{aligned}$$

□

Notation. The *Bernoulli numbers* B_n are a sequence of rational numbers, which can be defined via

$$\frac{x}{e^x-1} = \sum_{k\geq 0}\frac{B_k x^k}{k!}.$$

The first few Bernoulli numbers are:

k	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	$\cdots$

We have

$$B_{2n} = \frac{2(-1)^{n+1}(2n)!}{(2\pi)^{2n}}\zeta(2n).$$

Hence, for $k \geq 4$ even we have

$$\widetilde{E_k}(\tau) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{r\geq 1}\sigma_{k-1}(r)q^r\right), \qquad \text{where } q = e^{2\pi i\tau}.$$

One can normalized the series $\widetilde{E_k}(\tau)$ as

$$E_k(\tau) \coloneqq 1 - \frac{2k}{B_k}\sum_{r\geq 1}\sigma_{k-1}(r)q^r$$

which still is a modular form of weight k for any $k \geq 4$ even. By plugging in the first few Bernoulli numbers, one obtains

$$E_4(\tau) = 1 + 240\sum_{r\geq 1}\sigma_3(r)q^r; \qquad E_6(\tau) = 1 - 504\sum_{r\geq 1}\sigma_5(r)q^r;$$

$$E_8(\tau) = 1 + 480\sum_{r\geq 1}\sigma_7(r)q^r; \qquad E_{10}(\tau) = 1 - 264\sum_{r\geq 1}\sigma_9(r)q^r.$$

Theorem 5.56 (Valence formula)**.** *Let $f\colon \mathbb{H} \to \mathbb{C}$ be a nonzero modular form of weight k. Then*

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\omega(f) + \sum_{\tau\in\mathbb{H}'/\mathrm{SL}(2,\mathbb{Z})} v_\tau(f) = \frac{k}{12}.$$

Here the notion $v_z(f)$ denotes the order of zero at z; the summation runs through the orbits in $\mathbb{H}/SL(2,\mathbb{Z})$ other than those of i and ω.

Before proving this theorem, let us see how useful it is in studying the space of modular forms. Denote M_k the (complex vector) space of modular forms of weight k.

Corollary 5.57. *We have*

(a) $M_k = \{0\}$ if $k < 0$ or $k = 2$.

(b) $M_0 \cong \mathbb{C}$ consists only of constant functions.

(c) $M_4 = \langle E_4\rangle$ is a one-dimensional vector space generated by the Eisenstein series E_4, which has simple zeros at the orbit of ω and has no other zeros.

(d) $M_6 = \langle E_6\rangle$, which has simple zeros at the orbit of i and has no other zeros.

(e) $M_8 = \langle E_8 \rangle$, *which has double zeros at the orbit of* ω *and has no other zeros. In particular, we have* $E_8 = E_4^2$.

(f) $M_{10} = \langle E_{10} \rangle$, *which has simple zeros at the orbits of* ω *and* i *and has no other zeros. In particular, we have* $E_{10} = E_4 E_6$.

Proof. Part (a) follows directly from the valence formula. To prove (b), let f be a modular form of weight 0. Since the constant function $g = f(2i)$ is a modular form of weight 0 (the point $2i$ can be chosen arbitrarily), so is $f - g \in M_0$. But $f - g$ now has a zero at $2i$, by the valence formula, one must have $f = g$.

To prove (c), observe that any modular form of weight 4 has simple zeros at the orbit of ω and has no other zeros. Therefore, given any two modular forms f_1, f_2 of weight 4, the ratio f_1/f_2 is a modular form of weight zero, therefore a constant function. The remaining statements can be proved similarly. □

Proposition 5.58. *The smallest weight* k *that admits linearly independent modular forms is* $k = 12$. *In fact,* M_{12} *is of two dimension, generated by* $M_{12} = \langle E_4^3, E_6^2 \rangle$.

Proof. Let us denote

$$\Delta = \frac{E_4^3 - E_6^2}{1728} \in M_{12}$$

which is a modular form of weight 12, and its q-expansion has vanishing constant term (in fact, $\Delta(q) = q +$ (higher order terms)). Therefore $v_\infty(\Delta) = 1$. By the valence formula, we find that Δ has no other zeros except at $\tau = \infty$ (equivalently, at $q = 0$).

Let $f \in M_{12}$, with its q-expansion given by

$$f(q) = a_0 + a_1 q + \cdots .$$

Observe that $f - a_0 E_4^3 \in M_{12}$, which has a zero at $q = 0$. By the same argument as in the previous corollary, one deduces that $f - a_0 E_4^3$ is a constant multiple of Δ. Thus $f \in \langle E_4^3, E_6^2 \rangle$.

Finally, it is clear that E_4^3 and E_6^2 are linearly independent since the locations of their zeros are different. □

Theorem 5.59. *Any modular form is a polynomial in* E_4 *and* E_6. *In other words, the space* M_k *has a basis given by*

$$M_k = \left\langle M_4^a M_6^b \mid a, b \geq 0,\ 4a + 6b = k \right\rangle .$$

Proof. We prove the statement by induction on k. The statement is true for $k \leq 12$ by our previous discussions. Now, let $f \in M_k$ where $k > 12$ an even integer. Choose $a, b \geq 0$ so that $4a + 6b = k$. Then

$$f - f(\infty) \cdot E_4^a E_6^b \in M_k$$

has a zero at $\tau = \infty$. Therefore we have

$$\frac{f - f(\infty) \cdot E_4^a E_6^b}{\Delta} \in M_{k-12},$$

which can be written as a polynomial in E_4 and E_6 by induction hypothesis, thus so can f. □

Definition 5.60. The j-function is defined to be

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} = 1728 \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2}.$$

It satisfies the following properties:

- $j(\tau)$ is a modular function, i.e. invariant under the $\mathrm{SL}(2, \mathbb{Z})$-action.
- $j(\tau)$ is holomorphic on $\mathbb{H}$, and has a simple pole at $\tau = \infty$.
- $j(\tau)$ has zeros of order 3 at ω and its $\mathrm{SL}(2, \mathbb{Z})$-orbit.
- $j(\tau) - 1728 = \frac{E_6(\tau)^2}{\Delta(\tau)}$ has zeros of order 2 at i and its orbit.

A perhaps unexpected application of the j-function is a proof of the *little Picard theorem.*

Theorem 5.61 (Little Picard theorem)**.** *Let $f \colon \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose f omits (at least) two values, i.e. there exists z_1, z_2 so that $f^{-1}(z_1) = f^{-1}(z_2) = \emptyset$. Then f is a constant function.*

Proof. First, we claim that the j-function defines a bijection between $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$ onto $\mathbb{C}$. To see this, one has to show that for all $\lambda \in \mathbb{C}$, the modular form $E_4(\tau)^3 - \lambda\Delta(\tau)$ has a unique zero $\tau \in \mathbb{H}$ up to the $\mathrm{PSL}(2, \mathbb{Z})$-action. This follows directly from the valence formula.

Suppose f is an entire function which omits two values. Up to composing f with an invertible linear map, one can assume that it omits $\{0, 1728\}$. The map $\mathbb{H}'/\mathrm{PSL}(2, \mathbb{Z}) \to \mathbb{C} \backslash \{0, 1728\}$ is biholomorphic (which admits an inverse). The composition

$$\mathbb{C} \xrightarrow{f} \mathbb{C} \backslash \{0, 1728\} \to \mathbb{H}'/\mathrm{PSL}(2, \mathbb{Z}) \xrightarrow{1/z} \mathbb{D}_1(0)$$

is therefore a bounded entire function, thus is a constant map by Liouville's theorem. □

Theorem 5.62. *Any meromorphic modular function is a rational polynomial of $j(\tau)$.*

Proof. Let f be a meromorphic modular function. By suitably multiplying $j(\tau)-j(z_0)$, one can assume that f is holomorphic on $\mathbb{H}$. Write the q-expansion of f near $q=0$ as

$$f(q) = a_{-n}q^{-n} + \cdots .$$

Then $f(q) - a_{-n}j(q)^n$ is also holomorphic on $\mathbb{H}$, and with pole at $q=0$ of order strictly less than n:

$$f(q) - a_{-n}j(q)^n = b_{-(n-1)}q^{-(n-1)} + \cdots .$$

By continuing this process, one deduces that there exist constants $a_{-n}, \ldots, a_{-1}$ so that

$$f(q) - a_{-n}j(q)^n - \cdots - a_{-1}j(q)$$

is modular and holomorphic on $\mathbb{H} \cup \{\infty\}$, therefore is a constant function. □

Remark 5.63. A more natural way to understand the above theorem is that, the j-function defines an isomorphism of the *compactification* $\overline{\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})}$ onto the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. A meromorphic modular function is nothing but a meromorphic function on $\overline{\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})}$. The above theorem amounts to the well-known fact that the only meromorphic functions on $\mathbb{CP}^1$ are the rational functions.

Now we return to the proof of the valence formula. Before that, let us state the following "arc version" of the Cauchy integral formula.

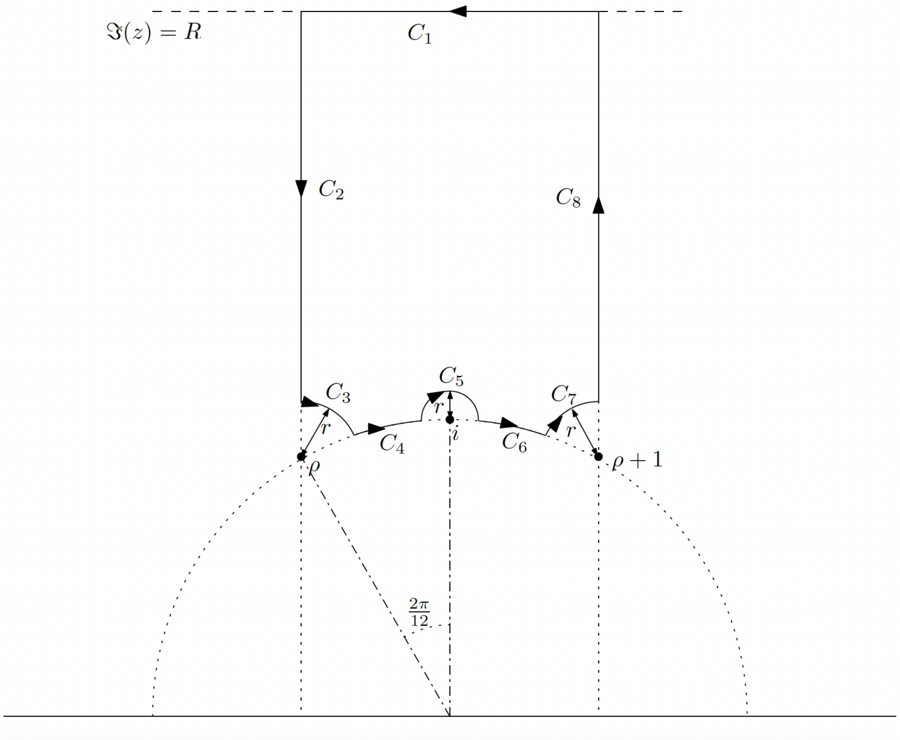
Exercise. Let f be a holomorphic function on a neighborhood of z_0. Let $0 < \theta_r \leq 2\pi$ be a number depending on $r > 0$, which satisfies $\lim_{r\to 0}\theta_r = \theta$ where $0 < \theta \leq 2\pi$. Let $C(z_0, r, \theta_r)$ be an arc of a circle, of radius r and angle θ_r around the point z_0. Then we have

$$\lim_{r\to 0}\int_{C(z_0,r,\theta_r)} \frac{f(z)}{z-z_0}\,\mathrm{d}z = \theta i f(z_0).$$

Similarly, we also have the "arc version" of the argument principle

$$\lim_{r\to 0}\int_{C(z_0,r,\theta_r)} \frac{f'(z)}{f(z)}\,\mathrm{d}z = \theta i v_{z_0}(f).$$

Proof of the valence formula. Consider the following closed loop in the fundamental domain. Let C be the union of these paths, which forms a closed loop.



Let $f\colon \mathbb{H} \to \mathbb{C}$ be a modular form of weight k. Choose $R > 0$ large enough so that f has no zeros in $\{z \in \mathbb{H}\colon |\mathrm{Re}(z)| < \frac{1}{2},\ \mathrm{Im}(z) > R\}$. By the argument principle, we have

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{\tau \in \mathbb{H}'/\mathrm{SL}(2,\mathbb{Z})} v_\tau(f).$$

Thus, to prove the valence formula, it suffices to show that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{k}{12} - v_\infty(f) - \frac{1}{2}v_i(f) - \frac{1}{3}v_\omega(f).$$

(The ρ in the figure is our ω.)

(a) Integration along C_1: Consider the change of variable $q = \exp(2\pi i z)$. Then, in terms of the q-coordinate, the path C_1 becomes a loop around

$q = 0$ of radius $\exp(-2\pi R)$ traveling *clockwisely.* By the argument principle, we have

$$\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)}\,\mathrm{d}z = -v_\infty(f).$$

(b) Integrations along C_2 and C_8: Since f is a modular form, it satisfies $f(z) = f(z+1)$. Therefore the integrations along C_2 and C_8 canceled with each other.

(c) Integration along C_5: By the arc version of the argument principle, we have

$$\frac{1}{2\pi i}\int_{C_5}\frac{f'(z)}{f(z)}\,\mathrm{d}z \xrightarrow{r\to 0} -\frac{1}{2}v_i(f).$$

(d) Integrations along C_3 and C_7: By the arc version of the argument principle, we have

$$\frac{1}{2\pi i}\int_{C_3}\frac{f'(z)}{f(z)}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{C_7}\frac{f'(z)}{f(z)}\,\mathrm{d}z \xrightarrow{r\to 0} 2\cdot\left(-\frac{1}{6}v_\omega(f)\right) = -\frac{1}{3}v_\omega(f).$$

(e) Integrations along C_4 and C_6: This is the most interesting part of the computation, where the weight k of the modular form gets involved. Observe that the map $S\colon z \mapsto -\frac{1}{z}$ sends C_4 to $-C_6$. The modularity of f implies that $f(z) = z^{-k}f(S(z))$. Thus

$$f'(z) = -kz^{-k-1}f(S(z)) + z^{-k}f'(S(z))S'(z),$$

hence

$$\frac{f'(z)}{f(z)} = -\frac{k}{z} + \frac{f'(S(z))S'(z)}{f(S(z))}.$$

Therefore

$$\frac{1}{2\pi i}\int_{C_4}\frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{C_4} -\frac{k}{z}\,\mathrm{d}z - \frac{1}{2\pi i}\int_{C_6}\frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

Again by the arc version of the argument principle, we have

$$\frac{1}{2\pi i}\int_{C_4}\frac{f'(z)}{f(z)}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{C_6}\frac{f'(z)}{f(z)}\,\mathrm{d}z = -\frac{1}{2\pi i}\int_{C_4}\frac{k}{z}\,\mathrm{d}z = \frac{k}{12}.$$

This completes the proof. □

5.5. **Sum of four squares.** Let us return to our motivating problem, the sum of squares problem. We would like to understand the counting

$$r_k(n) = \#\{(x_1, \dots, x_k) \in \mathbb{Z}^2 \mid x_1^2 + \cdots + x_k^2 = n\}.$$

Consider the *theta function*

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i \tau n^2} = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \quad (q = \exp(2\pi i \tau)).$$

It is not hard to see that

$$\theta(\tau)^k = \sum_{n=0}^{\infty} r_k(n) q^n.$$

The problem then reduces to understanding the coefficients of powers of the theta function. It turns out that the theta function satisfies certain modular properties, which will allow us to write down an explicit formula for $r_k(n)$. The key fact is that θ satisfies the following transformation formula.

Lemma 5.64.

$$\theta\left(\frac{-1}{4\tau}\right) = \sqrt{-2i\tau}\,\theta(\tau).$$

Proof. The proof uses again the *Poisson summation formula.* Consider the function $f(x) = e^{2\pi i \tau x^2}$. Then we have $\theta(\tau) = \sum_{n\in\mathbb{Z}} f(n)$. Its Fourier transform is

$$\begin{aligned}\hat{f}(n) &= \int_{\mathbb{R}} f(x) e^{-2\pi i x n}\, \mathrm{d}x \\ &= \int_{\mathbb{R}} \exp\left(2\pi i \tau \left(x - \frac{n}{2\tau}\right)^2 - \frac{\pi i}{2\tau} n^2\right) \mathrm{d}x \\ &= e^{-\frac{\pi i}{2\tau} n^2} \frac{1}{\sqrt{-2i\tau}}.\end{aligned}$$

By the Poisson summation formula, we have

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) = \theta\left(\frac{-1}{4\tau}\right) \frac{1}{\sqrt{-2i\tau}}.$$

□

Corollary 5.65.

$$\theta\left(\frac{\tau}{4\tau+1}\right) = \sqrt{4\tau+1}\theta(\tau).$$

Proof.

$$\begin{aligned}\theta\left(\frac{\tau}{4\tau+1}\right) &= \theta\left(-\frac{1}{4\left(\frac{-1}{4\tau}-1\right)}\right) = \sqrt{2i\left(\frac{1}{4\tau}+1\right)}\theta\left(\frac{-1}{4\tau}-1\right)\\ &= \sqrt{2i\left(\frac{1}{4\tau}+1\right)}\theta\left(\frac{-1}{4\tau}\right) = \sqrt{2i\left(\frac{1}{4\tau}+1\right)}\sqrt{-2i\tau}\theta(\tau)\\ &= \sqrt{4\tau+1}\theta(\tau).\end{aligned}$$

□

Thus, the function $\theta(\tau)^{2k}$ satisfies the following:

- $\theta(\tau+1)^{2k} = \theta(\tau)^{2k}$.
- $\theta\left(\frac{\tau}{4\tau+1}\right)^{2k} = (4\tau+1)^k\theta(\tau)^{2k}$.

Definition 5.66. Let $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{Z})$ be a subgroup. We say a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ is a *modular form of weight k with respect to* Γ if

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \text{for all } \tau \in \mathbb{H} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma.$$

The lemma above shows that the function $\theta(\tau)^{2k}$ is a modular form of weight k with respect to the group

$$\Gamma_1(4) := \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \right\rangle.$$

Let us focus on the case of $2k = 4$, i.e. the sum of *four* squares problem. The following theorem involves more detailed study of the modular curve $\mathbb{H}/\Gamma_1(4)$, for which the proof we omit.

Theorem 5.67. *The space $M_2(\Gamma_1(4))$ of modular forms of weight 2 with respect to $\Gamma_1(4)$ is 2-dimensional, with basis given by*

$$M_2(\Gamma_1(4)) = \mathrm{Span}\{E_2(\tau) - 2E_2(2\tau), E_2(\tau) - 4E_2(4\tau)\},$$

where

$$E_2(\tau) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n, \quad \sigma_1(n) = \sum_{d|n} d.$$

Now, by comparing the first two coefficients of the q-expansions of $\theta(\tau)^4$ and the basis functions $E_2(\tau) - 2E_2(2\tau)$ and $E_2(\tau) - 4E_2(4\tau)$, one obtains

$$\theta(\tau)^4 = 8(E_2(\tau) - 4E_2(4\tau)).$$

Therefore,

$$r_4(n) = 8\left(\sum_{d|n} d - 4\sum_{d|\frac{n}{4}} d\right) = 8\sum_{\substack{d|n \\ 4\nmid d}} d.$$

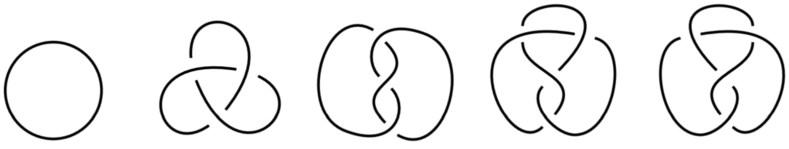
This is the *Jacobi four-square theorem.*

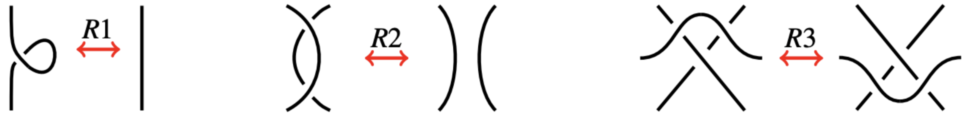
6. Knot theory and invariants

Definition 6.1. An *oriented knot* $K \subseteq \mathbb{R}^3$ is a subset of the form $K = f(S^1)$ where $f\colon S^1 \to \mathbb{R}^3$ is a smooth embedding. We say two knots K_0, K_1 are *equivalent* if there is a smooth map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ so that $K_0 = F|_{S^1\times\{0\}}$, $K_1 = F|_{S^1\times\{1\}}$, and $K_t = F|_{S^1\times\{t\}}$ is a knot for each t.

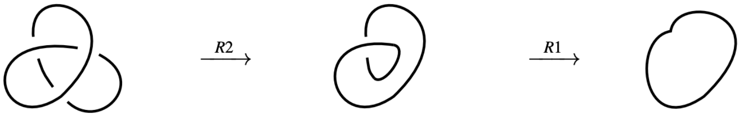
One can generalize the notion of oriented knots to oriented links. An *oriented n-component link* in $L \subseteq \mathbb{R}^3$ is a subset of the form $L = f(\coprod^n S^1)$ where $f\colon \coprod^n S^1 = S^1 \coprod \cdots \coprod S^1 \to \mathbb{R}^3$ is a smooth embedding. The notion of equivalence between links can be defined in the same way.

To draw pictures of a link, we consider its image under a linear projection $\mathbb{R}^3 \to \mathbb{R}^2$. Note that any given link can be represented by different planar diagrams. We say two diagrams are related by *Reidemeister moves* if we can obtain the second diagram by applying the three moves (R1, R2, or R3) to some small regions of the first diagram. It is easy to see that if two diagrams are related by Reidemeister moves, then they represent equivalent links.





Example 6.2. This example shows how a diagram of a knot with 3 crossings can be transformed into the *unknot* by a sequence of Reidemeister moves.



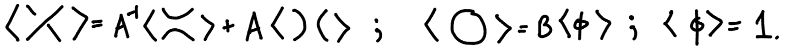
Theorem 6.3 (Reidemeister, 1932)**.** *Two diagrams represent the equivalent link if and only if they can be related by a sequence of Reidemeister moves.*

It would be nice to have some ways of telling when two diagrams represent different links. Here, the idea is to define certain *link invariants*: one would like to associate certain invariants (numbers, polynomials, or other objects) to each planar diagram, so that two diagrams have the same invariant if they are related by Reidemeister moves.

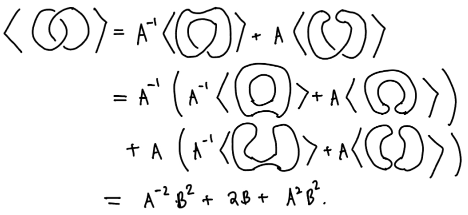
Let us try to define a link invariant with the help of the *Kauffman bracket*, which is a function

$$\langle\cdot\rangle : \mathcal{D} \to \mathbb{Z}[A^{\pm 1}, B]$$

(let $\mathcal{D}$ denote the set of all link diagrams up to isotopy), satisfying the local relations:



Example 6.4. One can compute the Kauffman bracket of the *Hopf link* as follows.



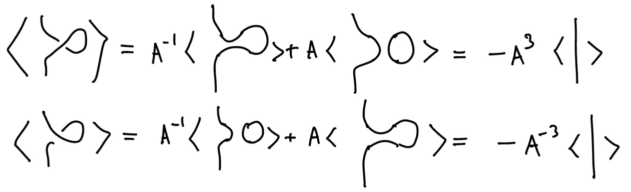
In order to obtain a link invariant, one needs to consider how the bracket changes under Reidemeister moves.

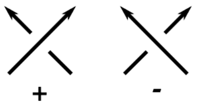
Exercise. In order for the bracket to be invariant under R2 and R3 (the second and third Reidemeister moves), we must set

$$B = -A^{-2} - A^2.$$

It remains to consider the R1 move. Disappointingly, we find that the bracket is *not* invariant under the R1 move, and hence *not* a link invariant. But it is very close, and indeed there is a fix for our problem. The fix involves paying attention to the *orientations*. There are two possible crossings, which we refer to as *positive* and *negative* crossings. We denote $n_\pm(D)$ the number of positive/negative crossings of a planar diagram D, and define the *writhe* of D to be

$$w(D) = n_+(D) - n_-(D).$$





It is clear that the writhe is invariant under R2 and R3 moves. On the other hand, an R1 move will either increase or decrease the writhe by 1. We can use it to counteract the change in the Kauffman bracket under an R1 move, and therefore obtain a link invariant.

Definition 6.5. Let D be an oriented link diagram. Its *unreduced Jones polynomial* is defined to be

$$\overline{V}(D) = (-A^3)^{-w(D)} \langle D \rangle \in \mathbb{Z}[A^{\pm 1}].$$

It is an invariant of oriented links.

Example 6.6. With this definition, we have $\overline{V}(\emptyset) = 1$, and $\overline{V}(\bigcirc) = -A^{-2} - A^2$. More generally, $\overline{V}(\bigcirc^n) = (-A^{-2} - A^2)^n$, where $\bigcirc^n$ denotes the n-component unlink.

Note that for any non-empty link diagram, every complete resolution will have at least one circle, therefore $\langle \cdot \rangle$ is always divisible by $B = -A^{-2} - A^2$. One considers the following normalization.

Definition 6.7. A non-empty link L has a *reduced Jones polynomial* $V(L)$ defined by

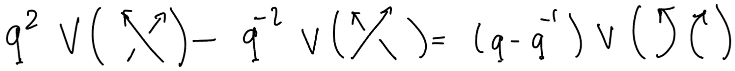
$$V(L) = \frac{\overline{V}(L)}{-A^{-2} - A^{-1}} \in \mathbb{Z}[A^{\pm 1}].$$

Here are some basic properties of Jones polynomial (which we will not prove here).

Remark 6.8. Let L be a non-empty link diagram. Then we have $V(L) \in \mathbb{Z}[A^{\pm 2}]$. (In other words, the coefficients of any odd power of A are zero.) Therefore, we generally use the variable

$$q = -A^{-2}.$$

So we have $V(L) \in \mathbb{Z}[q^{\pm 1}]$. Moreover, the Jones polynomial satisfies the *oriented skein relation*



Remark 6.9. Our strategy of checking invariance under the Reidemeister moves is an effective way of proving that some quantity is a link invariant, but it is a terrible way of finding such invariants to start with. The definition of the Jones polynomial we have given is completely elementary, but remained undiscovered for 100 years after mathematicians first started thinking about knots. Jones arrived at his original definition by thinking about something entirely different – representations of von Neumann algebras.

After Jones's discovery, Witten realized that the Jones polynomial should fit into a much broader theory of invariants of 3-manifolds defined using Chern–Simons theory. His work launched an entire industry devoted to the study of these *quantum invariants*, both in physics and mathematics. It is worth knowing that Witten's approach assigns a polynomial invariant of knots to each complex Lie algebra $\mathfrak{g}$ equipped with a representation; the Jones polynomial corresponds to the vector representation of $\mathfrak{sl}_2$.

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