#1: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Sol^n: Row operations:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & & \\ 0 & 1 & 4 & & 1 & \\ 0 & 0 & 1 & & & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & \\ & 1 & 4 & & 1 & \\ & & 1 & & & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & & & 4 & -2 & 5 \\ & 1 & & & 1 & -4 \\ & & 1 & & & 1 \end{array}\right].$$

$\underbrace{\qquad}_{A^{-1}}$ $\square$

#2: A, B: $n \times n$ matrices,

Prove: AB invertible $\Rightarrow$ A, B invertible.

Sol^n: AB invertible $\Leftrightarrow$ $\exists\, C$ s.t. $ABC = CAB = I_n$.

By the invertibility criterions proved in class:

$A(BC) = I_n \Rightarrow A$ is invertible.

$(CA)B = I_n \Rightarrow B$ is invertible. $\square$

#3. A, B: $n \times n$. $C = AB$.

Denote the columns of A as $\vec{a}_1, \dots, \vec{a}_n$.

— B as $\vec{b}_1, \dots, \vec{b}_n$

— C as $\vec{c}_1, \dots, \vec{c}_n$.

One of the following statements is true for any A, B,

Find out which one and prove it:

① $\text{Span}\{\vec{c}_1, \cdots, \vec{c}_n\} \subseteq \text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$

② $\text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\} \subseteq \text{Span}\{\vec{c}_1, \cdots, \vec{c}_n\}$

③ $\text{Span}\{\vec{c}_1, \cdots, \vec{c}_n\} \subseteq \text{Span}\{\vec{b}_1, \cdots, \vec{b}_n\}$

④ $\text{Span}\{\vec{b}_1, \cdots, \vec{b}_n\} \subseteq \text{Span}\{\vec{c}_1, \cdots, \vec{c}_n\}$.

Sol^n: ① is true.

Recall that each column of AB is a linear comb of the columns of A, i.e.

$$\vec{c}_i \in \text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\} \quad \forall i.$$

In fact, $\vec{c}_i = b_{1i}\vec{a}_1 + \cdots + b_{ni}\vec{a}_n$,

To prove ①, Suppose $\vec{v} \in \text{Span}\{\vec{c}_1, \cdots, \vec{c}_n\}$,

i.e. $\exists\, d_1, \cdots, d_n \in \mathbb{R}$ s.t. $\vec{v} = d_1\vec{c}_1 + \cdots + d_n\vec{c}_n$.

Then

$$\begin{aligned}\vec{v} &= d_1\vec{c}_1 + \cdots + d_n\vec{c}_n \\ &= d_1(b_{11}\vec{a}_1 + \cdots + b_{n1}\vec{a}_n) + \cdots + d_n(b_{1n}\vec{a}_1 + \cdots + b_{nn}\vec{a}_n) \\ &= (d_1b_{11} + \cdots + d_nb_{1n})\vec{a}_1 + \cdots + (d_1b_{n1} + \cdots + d_nb_{nn})\vec{a}_n \\ &\in \text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}.\end{aligned}$$

This proves $\text{Span}\{\vec{c}_1, \cdots, \vec{c}_n\} \subseteq \text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$. $\square$