HW6 sol'n.

#1. Let $\{U_\alpha\}_{\alpha \in I}$ be any open cover of E. Since $E^c \subset S$ is open,

so $\{U_\alpha\}_{\alpha\in I} \cup \{E^c\}$ is an open cover of S. Since S is compact,

the open cover has a finite subcover: $S \subset (\underbrace{U_1 \cup \cdots \cup U_n}_{\text{open subsets in the collection } \{U_\alpha : \alpha\in I\}} \cup E^c)$.

Claim: $E \subset (U_1 \cup \cdots \cup U_n)$.

pf: $\forall x \in E$, we have $x \in (U_1 \cup \cdots \cup U_n \cup E^c)$

since $x \notin E^c \Rightarrow x \in U_1 \cup \cdots \cup U_n$. $\square$

This proves that any open cover of E has a finite subcover. $\square$

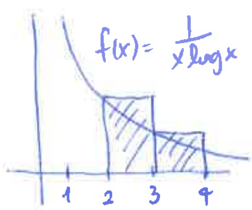
#2. (a) Diverge: $\lim\limits_{n\to\infty} \frac{(-1)^n(n-1)}{n} = 1 \neq 0$, so the series diverges.

(b) Converge: Root test: $\limsup \left(\frac{n^n}{(n+1)^{2n}}\right)^{1/n} = \limsup \frac{n}{(n+1)^2} = 0 < 1$.

(c) Converge: By Alternating series test.

(d) Converge: $\sum \frac{1}{(2n-1)^2} < \sum \frac{1}{n^2} = \frac{\pi^2}{6} < +\infty$.

(e) Diverge. Integral test:



$$\sum_{n=2}^{\infty} \frac{1}{n\log n} > \int_2^\infty \frac{dx}{x\log x} = \int_{\log 2}^{\infty} \frac{dt}{t} = +\infty.$$

(f) Converge. Root test: $\limsup \left(\frac{n}{e^{n^2}}\right)^{1/n} = \limsup \frac{n^{1/n}}{e^n} = 0 < 1$.

#3. Let $\sum_{n=1}^{\infty} a_n^{(i)} = A_i$ for $1 \le i \le k$.

$\forall \varepsilon > 0, \forall i, \exists N_i > 0$ s.t. $\left| \sum_{n=1}^{m} a_n^{(i)} - A_i \right| < \frac{\varepsilon}{k}$ $\forall m > N_i$.

Let $N = \max\{N_1, \dots, N_k\}$, Then $\left| \sum_{n=1}^{m} a_n^{(i)} - A_i \right| < \frac{\varepsilon}{k}$ $\forall m > N$ $\forall i$.

$$\Rightarrow \left| \sum_{n=1}^{m} b_n - \sum_{i=1}^{k} A_i \right| \underset{\substack{\uparrow \\ \text{triangle} \\ \text{inequality}}}{\le} \sum_{i=1}^{k} \left| \sum_{i=1}^{m} a_n^{(i)} - A_i \right| < \varepsilon \quad \forall m > N.$$

Hence $\sum_{n=1}^{\infty} b_n = \sum_{i=1}^{k} A_i$. $\square$

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e.g. $(a_n^{(1)}) = (-1, 1, -1, 1, \dots)$ $\quad \sum a_n^{(i)}$ diverges.

$(a_n^{(2)}) = (1, -1, 1, -1, \dots)$

$(a_n^{(1)} + a_n^{(2)}) = (b_n) = (0, 0, 0, 0, \dots)$ $\quad \sum b_n$ converges.

#4: By the Cauchy criterion, we need to show: $\forall \varepsilon > 0, \exists P > 0$ s.t. $\left| \sum_{n=M}^{N} a_n b_n \right| < \varepsilon$ $\forall N \ge M > P$.

- Since $\lim a_n = 0$, $\exists P_1 > 0$ s.t. $|a_n| < \frac{\varepsilon}{3L}$ $\forall n > P_1$.
- Since $\sum |a_{n+1} - a_n|$ is convergent, $\exists P_2 > 0$ s.t. $\sum_{n=M}^{N} |a_{n+1} - a_n| < \frac{\varepsilon}{3L}$ $\forall N \ge M > P_2$

Let $P = \max\{P_1, P_2\}$, Then $\forall N \ge M > P$, we have:

$$\left| \sum_{n=M}^{N} a_n b_n \right| = \left| \sum_{n=M}^{N-1} (a_n - a_{n+1}) s_n + a_N s_N - a_M s_{M-1} \right|$$

$$\le \sum_{n=M}^{N-1} |a_{n+1} - a_n| L + |a_N| L + |a_M| L$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \square$$

③

#5. $$\left(\sum_{n=1}^{N} \cos(n\theta)\right) + i\left(\sum_{n=1}^{N} \sin(n\theta)\right) = \sum_{n=1}^{N} e^{in\theta} = e^{i\frac{(N+1)\theta}{2}} \frac{\sin(N\theta/2)}{\sin(\theta/2)}$$

$$\Rightarrow \left|\left(\sum_{n=1}^{N} \cos(n\theta)\right) + i\left(\sum_{n=1}^{N} \sin(n\theta)\right)\right| \leq \frac{1}{\sin(\theta/2)}$$

$$\Rightarrow \left|\sum_{n=1}^{N} \cos(n\theta)\right| \leq \frac{1}{\sin(\theta/2)} \text{ and } \left|\sum_{n=1}^{N} \sin(n\theta)\right| \leq \frac{1}{\sin(\theta/2)}.$$

Think of $(\cos(n\theta))$ or $(\sin(n\theta))$ as (b_n) in Problem (4), and let $(a_n = \frac{1}{n})$. They satisfy the conditions in Problem (4), hence the series $\sum \frac{\cos n\theta}{n}$, $\sum \frac{\sin n\theta}{n}$ converge. □

#6. $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\left|\sum_{n=k}^{l} a_n\right| < \frac{\varepsilon}{2}$ $\forall l \geq k > N$.

Notice that $a_n \geq 0$ $\forall n$, (If $a_n < 0$ for some n, since (a_n) decreasing, $\sum a_n$ diverge)

$$\Rightarrow \frac{\varepsilon}{2} > a_{N+1} + a_{N+2} + \cdots + a_{N+M} \geq M a_{N+M} \geq 0. \quad \forall M \geq 0.$$

For any $M > N$, we have $\frac{\varepsilon}{2} > \frac{M}{N+M}\left((N+M)a_{N+M}\right) > \frac{1}{2}(N+M)a_{N+M} \geq 0.$

$\Rightarrow \forall n > 2N$, we have $0 \leq na_n < \varepsilon.$

$\Rightarrow \lim_{n\to\infty} na_n = 0.$ □

#7. (a) $\forall \varepsilon > 0$,

- $\exists N_1 > 0$ s.t. $\underbrace{|S_{km} - S_{lm}|}_{\left| \sum_{n=lm+1}^{km} a_n \right|} < \varepsilon/3 \quad \forall k \geq l \geq N_1$

- $\exists N_2 > 0$ s.t. $|a_n| < \varepsilon/3m \quad \forall n > N_2$.

Let $N = \max\{N_1 m, N_2\}$. Then $\forall x \geq y > N$,

$$\left| \sum_{n=y}^{x} a_n \right| = \Big| \underbrace{a_y + a_{y+1} + \cdots + a_{lm}}_{\text{at most } m \text{ terms}} + a_{lm+1} + \cdots + \underbrace{a_{km}}_{\substack{\text{the largest}\\ \text{multiple of } m \\ \text{that is} \leq x}} + \underbrace{a_{km+1} + \cdots + a_x}_{\text{at most } m \text{ terms}} \Big|$$

(the smallest multiple of m that is $\geq y$)

$$\leq |a_y + \cdots + a_{lm}| + |a_{lm+1} + \cdots + a_{km}| + |a_{km+1} + \cdots + a_x|$$

$$< m \cdot \frac{\varepsilon}{3m} + \frac{\varepsilon}{3} + m \cdot \frac{\varepsilon}{3m} = \varepsilon.$$

$\Rightarrow \sum a_n$ converges. $\square$

(b) $(a_n) = (-1, 1, -1, 1, -1, 1, \ldots)$

$(S_{2k}) = (0, 0, 0, 0, \ldots)$

(c) $(a_n) = (0, 1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \underbrace{-\frac{1}{8}, \ldots, -\frac{1}{8}}_{8 \text{ terms}}, \ldots)$

Then $S_4 = S_{16} = \cdots = 0$.

The subseq. $(S_{4^n})_{n=1}^{\infty}$ of (S_n) converges.

But $\sum a_n$ diverges. $\square$

#8. $s_k = \sum_{n=1}^{k} a_n$, $t_k = \sum_{n=1}^{k} |a_n|$. (s_k), (t_k) both converge.

$|s_k| \le t_k \quad \forall k$ by triangle inequality.

$\Rightarrow \quad -t_k \le s_k \le t_k \quad \forall k.$

$$\Rightarrow \quad -\lim_{k\to\infty} t_k \le \lim s_k \le \lim t_k$$

$$\begin{array}{ccccc} \| & & \| & & \| \\ -\sum_{n=1}^{\infty} |a_n| & & \sum_{n=1}^{\infty} a_n & & \sum_{n=1}^{\infty} |a_n|. \end{array} \quad \square$$

#9. Define (a_n), where:

- $(a_{2n-1}) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$.
- $(a_{2n}) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots)$.

Then $\lim a_n = 0$,

But $\sum (-1)^n a_n = -1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \frac{1}{6} \cdots$

$= -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots$ diverges. $\square$