This week: System of 1st-order ordinary diff'l eq'ns

e.g.
$$\begin{cases} x_1'(t) = 3x_1(t) + 2x_2(t) + \sin t \\ x_2'(t) = x_1(t) + x_2(t) + e^t \end{cases} \quad \text{or} \quad \vec{x}'(t) = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} \sin t \\ e^t \end{bmatrix}$$

(w/ initial condition $\vec{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.)

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} : \underset{\subseteq \mathbb{R}}{I} \to \mathbb{R}^2$$

More complicated example: Three body problem (no general closed-form sol'n exist).

eq'n of motion for three gravitationally interacting bodies:

- $\vec{r}_i(t) = (x_i(t), y_i(t), z_i(t))$, m_i: mass; $i = 1, 2, 3$.

$$\begin{cases} \vec{r}_1'' = -g m_2 \dfrac{\vec{r}_1 - \vec{r}_2}{\|\vec{r}_1 - \vec{r}_2\|^3} - g m_3 \cdot \dfrac{\vec{r}_1 - \vec{r}_3}{\|\vec{r}_1 - \vec{r}_3\|^3}. \\ \vec{r}_2'' = -g m_1 \dfrac{\vec{r}_2 - \vec{r}_1}{\|\vec{r}_2 - \vec{r}_1\|^3} - g m_3 \dfrac{\vec{r}_2 - \vec{r}_3}{\|\vec{r}_2 - \vec{r}_3\|^3}. \\ \vec{r}_3'' = -g m_1 \dfrac{\vec{r}_3 - \vec{r}_1}{\|\vec{r}_3 - \vec{r}_1\|^3} - g m_2 \dfrac{\vec{r}_3 - \vec{r}_2}{\|\vec{r}_3 - \vec{r}_2\|^3} \end{cases}$$

e.g.
$$\boxed{y^{(n)}(t) + p_{n-1}\, y^{(n-1)}(t) + \cdots + p_0\, y(t) = 0}$$

(auxiliary eq'n: $\underbrace{\boxed{r^n + p_{n-1} r^{n-1} + \cdots + p_0 = 0}}_{e^{rt}}$)

hard to solve for n large!!

Trick.
$$\begin{cases} x_1(t) = y(t) \\ x_2(t) = y'(t) \\ \vdots \\ x_n(t) = y^{(n-1)}(t) \end{cases} \qquad \begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \end{aligned}$$

Then $\begin{cases} x_n' = y^{(n)}(t) \\ \quad = -p_{n-1} y^{(n-1)}(t) - \cdots - p_0 y \\ \quad = -p_{n-1} x_n - \cdots - p_0 x_1 \end{cases}$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = -p_0 x_1 - p_1 x_2 - \cdots - p_{n-1} x_n . \end{cases}$$

Thm (existence & uniqueness).

open interval (e.g. $(0,1)$, $(0,+\infty)$, $\mathbb{R}$!)

Let $A(t): I \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{R})$, $\vec{f}(t): I \longrightarrow \mathbb{R}^n$
$I \subset \mathbb{R}$ conti. fcn.

Then for any $\vec{x}_0 \in \mathbb{R}^n$ and $t_0 \in I$,

$\exists!$ $\vec{x}(t): I \longrightarrow \mathbb{R}^n$

s.t. $\begin{cases} \vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$

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e.g. $\vec{x}'(t) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \vec{x}(t)$

$$\begin{bmatrix} \vec{x}_1'(t) \\ \vdots \\ \vec{x}_n'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1(t) \\ \vdots \\ \lambda_n x_n(t) \end{bmatrix}$$

What are the sol^{n}s to ?? ??

e.g.
$$\begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e^{\lambda_2 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_n t} \end{bmatrix}$$

& any linear combin^{n} is also a sol^n

General sol^n:
$$c_1 \begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_n t} \end{bmatrix}$$

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e.g. $\vec{x}'(t) = A\vec{x}(t)$, where $A = PDP^{-1}$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

e.g. $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$

$$\begin{cases} x_1'(t) = 2x_1(t) - 3x_2(t) \\ x_2'(t) = x_1(t) - 2x_2(t) \end{cases}$$

$$\boxed{\vec{x}'(t) = PDP^{-1}\vec{x}(t)}$$

$\Downarrow$

$$P^{-1}\vec{x}'(t) = DP^{-1}\vec{x}(t) \Longrightarrow \vec{y}'(t) = D\vec{y}(t)$$

$(P^{-1}\vec{x}(t))' = \vec{y}$

Fact: $B\vec{x}'(t) = (B\vec{x}(t))' = \begin{bmatrix} b_{11}x_1(t) + \cdots + b_{1n}x_n(t) \\ \vdots \\ b_{n1}x_1(t) + \cdots + b_{nn}x_n(t) \end{bmatrix}'$

$$B\begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = x_1'(t)\vec{b}_1 + \cdots + x_n'(t)\vec{b}_n$$

$[\vec{b}_1 \cdots \vec{b}_n]$

$$= x_1'(t)\begin{bmatrix} b_{11} \\ \vdots \\ \vdots \end{bmatrix} + \cdots + x_n'(t)\begin{bmatrix} b_{1n} \\ \vdots \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}x_1'(t) + \cdots + b_{1n}x_n'(t) \\ \vdots \\ b_{n1}x_1'(t) + \cdots + b_{nn}x_n'(t) \end{bmatrix}$$

B is a const. matrix

$$= \begin{bmatrix} (b_{11}x_1(t) + \cdots + b_{1n}x_n(t))' \\ \vdots \\ (b_{n1}x_1(t) + \cdots + b_{nn}x_n(t))' \end{bmatrix}$$

$\Rightarrow$ general sol^{n}

$$\vec{y}(t) = c_1 \begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_n t} \end{bmatrix}$$

$\Rightarrow$

$$\vec{x}(t) = P\,\vec{y}(t)$$

$$P = [\vec{v}_1 \cdots \vec{v}_n]$$

eigenvectors

$$= [\vec{v}_1 \cdots \vec{v}_n] \left(c_1 \begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_n t} \end{bmatrix} \right)$$

$$= c_1 e^{\lambda_1 t} \vec{v}_1 + \cdots + c_n e^{\lambda_n t} \vec{v}_n.$$

For $\vec{x}' = A\vec{x}$, where A is diagonalizable $A = PDP^{-1}$.

$e^{\lambda_i t} \vec{v}_i$ is a sol^{n}

eigenvalue eigenvector

Def $\{\vec{x}_1(t), \dots, \vec{x}_k(t)\}$ l.d. if $\exists\, c_1, \dots, c_k$ not all 0

s.t. $c_1\vec{x}_1(t) + \dots + c_k\vec{x}_k(t) = \vec{0}$

Otherwise, called l.i.

Def The Wronskian of $\{\vec{x}_1(t), \dots, \vec{x}_n(t)\}$, is defined to be the fcn

$$W[\vec{x}_1, \dots, \vec{x}_n](t) := \det\begin{pmatrix} \vec{x}_1(t) & \cdots & \vec{x}_n(t) \end{pmatrix}$$

Rmk. $\{\vec{x}_1(t), \vec{x}_n(t)\}$ l.i. $\Leftrightarrow$ $\exists\, t \in I$ s.t. $W[\vec{x}_1 \cdots \vec{x}_n](t) \neq 0$

($\{\vec{x}_1(t), \dots, \vec{x}_n(t)\}$ l.d. then $W[\vec{x}_1 \cdots \vec{x}_n](t) = 0 \quad \forall t$)

Prop. If $\{\vec{x}_1(t), \dots, \vec{x}_n(t)\}$ l.i. sol^{n} of $\underline{\vec{x}' = A\vec{x}}$.

then $W[\vec{x}_1, \dots, \vec{x}_n](t) \neq 0 \quad \forall t$.

Pf: Suppose $\exists\, t_0$ s.t. $\boxed{W[\vec{x}_1, \dots, \vec{x}_n](t_0) = 0}$

$\Updownarrow$

$\{\vec{x}_1(t_0), \dots, \vec{x}_n(t_0)\}$ l.d.

$\exists\, c_1, \dots, c_n$ not all 0

s.t. $c_1\vec{x}_1(t_0) + \dots + c_n\vec{x}_n(t_0) = \vec{0}$.

Consider: $\boxed{c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t)}$

satisfy $\quad \| \quad$ uniqueness thm.

$$\begin{cases} \vec{x}' = A\vec{x} \\ \vec{x}(t_0) = \vec{0} \end{cases}$$

satisfy $\quad \vec{0}$

Contradict w/ $\{\vec{x}_1, \dots, \vec{x}_n\}$ l.i. $\square$

Thm Let $\{\vec{x}_1, \dots, \vec{x}_n\}$ l.i. sol^{n} to $\vec{x}' = A\vec{x}$.

Then any sol^{n} can be written as $c_1\vec{x}_1 + \dots + c_n\vec{x}_n$ for some $c_1, \dots, c_n \in \mathbb{R}$

pf Let $\vec{x}$ be a sol^{n} to $\vec{x}' = A\vec{x}$

Pick any t_0,

$$\{\underbrace{\vec{x}_1(t_0), \dots, \vec{x}_n(t_0)}_{\text{l.i.}}\}, \quad \boxed{\vec{x}(t_0)}$$

(By Prop) $\Downarrow$

forms a basis of $\mathbb{R}^n$

$\Downarrow$

$\exists\, c_1, \dots, c_n \in \mathbb{R}$

s.t. $\vec{x}(t_0) = c_1\vec{x}_1(t_0) + \dots + c_n\vec{x}_n(t_0) = \vec{w}$

Claim: $\vec{x}(t) = \boxed{c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t)}$ $\forall t$.

Consider

$$\begin{cases} \vec{x}' = A\vec{x}, \\ \vec{x}(t_0) = \vec{w} \end{cases}$$

satisfy

By uniqueness thm. Δ

Notion: Suppose $\{\vec{x}_1(t), \dots, \vec{x}_n(t)\}$ l.i. sol^{n},

then we can define a fundamental matrix:

$$X(t) = \begin{bmatrix} \vec{x}_1(t) & \cdots & \vec{x}_n(t) \end{bmatrix}$$

$c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t)$

$\parallel$

$\Longrightarrow$ any sol^{n} can be written as $X(t)\vec{c}$

where $\vec{c}$ is a const. vector in $\mathbb{R}^n$

<u>Rmk</u> If $X(t)$ and $Y(t)$ are both fundamental matrix of $\vec{x}' = A\vec{x}$,

then $\exists M$ const invertible $n\times n$, s.t.

$$X(t)\cdot M = Y(t)$$

<u>eg:</u> $\underline{\vec{x}' = A\vec{x} \quad A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}}$ $\quad 1, -1$

$$\det\begin{pmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$$

$$\mathrm{Nul}(A - 1I) = \mathrm{Nul}\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$$

$$\mathrm{Nul}(A + 1I) = \mathrm{Nul}\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\left\{ e^{t}\begin{bmatrix} 3 \\ 1 \end{bmatrix},\ e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \text{ l.i. sol}^{n}$$

$$\vec{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \longleftarrow \begin{bmatrix} c_1 \cdot 3e^{t} + c_2 \cdot e^{-t} \\ c_1 \cdot e^{t} + c_2 \cdot e^{-t} \end{bmatrix}$$

$\|$

$$\begin{bmatrix} 3c_1 + c_2 \\ c_1 + c_2 \end{bmatrix}$$

Matrix exponential:

(naively: $\vec{x}' = A\vec{x}$. $\quad x' = cx$, $x = e^{ct}$)

$A \in M_{n\times n}(\mathbb{R})$, $t \in \mathbb{R}$.

Define

$$e^{tA} := \mathbb{I} + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \cdots\cdots$$

$$\left(e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \right.$$

Taylor expansion of $\exp(x)$ at $x=0$)

Fact e^{tA} always converge., so $e^{tA} \in M_{n\times n}(\mathbb{R})$

e.g. $A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$

$$e^{tA} = \mathbb{I} + \begin{bmatrix} ta_{11} & 0 \\ 0 & ta_{22} \end{bmatrix} + \begin{bmatrix} \frac{t^2}{2}a_{11}^2 & 0 \\ 0 & \frac{t^2}{2}a_{22}^2 \end{bmatrix} + \cdots\cdots$$

$$= \begin{bmatrix} e^{ta_{11}} & 0 \\ 0 & e^{ta_{22}} \end{bmatrix}$$

$$\forall A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix},$$

$$(I)_{ij} + \underbrace{(A)_{ij}}_{=\, a_{ij}} + \frac{1}{2}(A^2)_{ij} + \frac{1}{3!}(A^3)_{ij} + \cdots \quad \text{conv.} \quad \forall\, ij$$

Fact:

- $e^{(t_1+t_2)A} = e^{t_1 A} \cdot e^{t_2 A}$
- e^{tA} is invertible, and $(e^{tA})^{-1} = e^{-tA}$
- $\boxed{\frac{d}{dt} e^{tA} = A e^{tA}}$

$\Rightarrow$ e^{tA} is a fundamental matrix to $\vec{x}' = A\vec{x}$.

e.g. $\boxed{\vec{x}' = A\vec{x}, \qquad A = PDP^{-1}}$

$\{e^{\lambda_1 t}\vec{v}_1, \ldots, e^{\lambda_n t}\vec{v}_n\}$ l.i. sol^n

$$e^{tA} \cdot P = \begin{bmatrix} e^{\lambda_1 t}\vec{v}_1 & \cdots & e^{\lambda_n t}\vec{v}_n \end{bmatrix} \text{ fund. mat.}$$

$$e^{tA} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

$$= I + tPDP^{-1} + \frac{t^2}{2}PD^2P^{-1} + \frac{t^3}{3!}PD^3P^{-1} + \cdots$$

$$= P\left(\underline{I + tD + \frac{t^2}{2}D^2 + \frac{t^3}{3!}D^3 + \cdots}\right)P^{-1}$$

$$= P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

$$= \begin{bmatrix} e^{\lambda_1 t}\vec{v}_1 & \cdots & e^{\lambda_n t}\vec{v}_n \end{bmatrix} P^{-1}$$

<u>Next time</u>: use generalized eigenvectors to deal w/ the non-diagonalizable case.