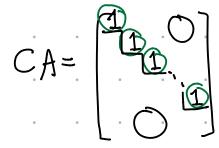
if Part. then
$$\vec{\delta} = \vec{B}(A\vec{x}) = (BA)\vec{x} = T_n \vec{x} = \vec{x}$$
.
This shows T_A is injective.

only if part. TA injective >> the reduced echelon form of A is:



We showed in class that elementary row operations can be realized as left multiplicate by invertible matrices. J. C: MXm invertible matrix sit.



Define an nxm matrix D as ?

$$D = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T_n \\ O_{n \times (mn)} \end{bmatrix}$$

One can easily check that

$$\mathbb{D} \cdot \left[\mathbb{D} \right] = \mathbb{I}_{n}.$$

Hence we can take B := DC. and get: BA=DCA=In. []

#2:

(
$$\Rightarrow$$
) For each $\vec{e}_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is happened in \mathbb{R}^m , $\vec{f}_i \in \mathbb{R}^n$ st. $\vec{f}_i = \vec{e}_i$.

Take $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we have $\vec{x} = T_m : (AB)\vec{x} = A(B\vec{x})$

(€)
$$\forall \vec{x} \in \mathbb{R}^m$$
, we have $\vec{x} = \mathbb{I}_m \vec{x} = (AB)\vec{x} = A(B\vec{x})$.
Hence T_A is surjective. \square

#3: S can't be surjective: Let A be the unique 3×2 matrix such that $T_A = S$. It's obvious that A can't have pivot in each vow.

Then SoT can't be surjective, therefore can't be inventible. [

(one can also show that T is NOT injective and get

the same conclusion.)

Suppose that $a_0 \vec{\nabla} + a_1 \vec{T}(\vec{v}) + \cdots + a_{k-1} \vec{T}^{k-1}(\vec{v}) = \vec{o}$.

Apply T^{k-1} on both sides, gets, $\vec{o} = T^{k-1}(\vec{o}) = T^{k-1}(a_0 \vec{\nabla} + a_1 \vec{T}(\vec{v}) + \cdots + a_{k-1} \vec{T}^{k-1}(\vec{v}))$ $= a_0 \vec{T}^{k-1}(\vec{v}) + a_1 \vec{T}^{k}(\vec{v}) + \cdots + a_{k-1} \vec{T}^{k-1}(\vec{v})$ $= a_0 \vec{T}^{k-1}(\vec{v}) + a_1 \vec{T}^{k}(\vec{v}) + \cdots + a_{k-1} \vec{T}^{k-1}(\vec{v})$ $= a_0 \vec{T}^{k-1}(\vec{v}) + a_1 \vec{T}^{k}(\vec{v}) + \cdots + a_{k-1} \vec{T}^{k-1}(\vec{v})$ $= a_0 \vec{T}^{k-1}(\vec{v}) + a_1 \vec{T}^{k}(\vec{v}) + \cdots + a_{k-1} \vec{T}^{k-1}(\vec{v})$ $= a_0 \vec{T}^{k-1}(\vec{v}) + a_1 \vec{T}^{k}(\vec{v}) + \cdots + a_{k-1} \vec{T}^{k-1}(\vec{v})$ $= a_0 \vec{T}^{k-1}(\vec{v}) + a_1 \vec{T}^{k}(\vec{v}) + \cdots + a_{k-1} \vec{T}^{k-1}(\vec{v})$ $= a_0 \vec{T}^{k-1}(\vec{v}) + a_1 \vec{T}^{k}(\vec{v}) + \cdots + a_{k-1} \vec{T}^{k-1}(\vec{v})$

$$\Rightarrow$$
 $q_0 = 0$.

Take any C+1. $\Rightarrow \alpha_{12} = \alpha_{13} = \dots = \alpha_{n_1} = 0.$ $\alpha_{21} = \alpha_{31} = \dots = \alpha_{n_1} = 0.$ By the same argument, one can show that any off-diagonal entry of A is zero. i.e.

A= (a1)

A= (an) • Take $B = \begin{bmatrix} 01 \\ 10 \\ 1 \end{bmatrix}$. Then AB = BA $\begin{bmatrix} a_{11} \\ a_{22} \\ \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ 0 a,, a₂₂ 0 ... By the same argument, one sees that ⇒ an= azz. $\alpha_{11} = \alpha_{22} = \cdots = \alpha_{nn}$

Hence
$$\text{tr}(AB) = \sum_{k=1}^{n} a_{ik} b_{kj}$$

 Hence $\text{tr}(AB) = \sum_{k=1}^{n} \sum_{k=1}^{n} a_{jk} b_{kk}$

$$(BA)_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj}$$
Hence $\text{tr}(BA) = \sum_{k=1}^{n} \sum_{k=1}^{m} b_{kk} a_{kk}$

If there were such AIB, then
$$tr(AB-BA) = tr(T_n) = n$$

$$tr(AB) - tr(BA) = 0 (by #6).$$

$$\frac{\#9:}{A=\begin{bmatrix} \cos\frac{2\pi}{h} & -\sin\frac{2\pi}{h} \\ \sin\frac{2\pi}{h} & \cos\frac{2\pi}{h} \end{bmatrix}}$$

#10: These are straight forward computations.

and
$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$
.