(1) Let (a_n) be a sequence with the property that its subsequences (a_{2n}), (a_{2n-1}), and (a_{3n}) are all convergent. Prove that (a_n) is convergent.

Let $\lim a_{2n} = A$, $\lim a_{2n-1} = B$, $\lim a_{3n} = C$.

- Since (a_{6n}) is a subseq. of both (a_{2n}) and (a_{3n}), we have:

$$A = \lim a_{2n} = \lim a_{6n} = \lim a_{3n} = C$$

- Similarly, (a_{6n-3}) is a subseq. of both (a_{2n-1}) and (a_{3n}), hence $B = C$.
- Therefore, we have $A = B$.
- $\forall \varepsilon > 0$,

since $\lim\limits_{n\to\infty} a_{2n} = A$, $\exists N_1 > 0$ s.t. $|a_{2n} - A| < \varepsilon$ $\forall n > N_1$.

since $\lim\limits_{n\to\infty} a_{2n-1} = B = A$, $\exists N_2 > 0$ s.t. $|a_{2n-1} - A| < \varepsilon$ $\forall n > N_2$.

Define $N = \max\{2N_1, 2N_2 - 1\} > 0$. then we have

$$|a_n - A| < \varepsilon \quad \forall n > N.$$

Hence $\lim a_n = A$. $\square$

(2) Let (a_n) be a bounded sequence. Prove that

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n).$$

$$\limsup_{n\to\infty}(-a_n) = \lim_{N\to\infty}\left(\sup\{-a_n : n > N\}\right)$$

$$= \lim_{N\to\infty}\left(-\inf\{a_n : n > N\}\right)$$

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$$= -\lim_{N\to\infty}\left(\inf\{a_n : n > N\}\right)$$

$$= -\liminf_{n\to\infty} a_n. \ \square$$

(3) Prove that $\limsup |a_n| = 0$ if and only if $\lim a_n = 0$.

$$\limsup_{n\to\infty} |a_n| = 0. \iff \lim_{N\to\infty}\left(\sup\{|a_n| : n > N\}\right) = 0.$$

$$\iff \forall \varepsilon > 0,\ \exists N > 0 \text{ s.t. } |a_n| < \varepsilon \ \forall n > N.$$

$$\iff \lim a_n = 0. \quad \square$$

(4) Let (a_n) be a sequence of nonzero real numbers. Assume that $\limsup \left|\frac{a_{n+1}}{a_n}\right| = L$ is finite. You'll prove $\limsup(|a_n|^{1/n}) \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$ in this problem. Using similar argument, you can show that

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf_{n\to\infty}(|a_n|^{1/n}) \leq \limsup_{n\to\infty}(|a_n|^{1/n}) \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|,$$

which will be important for us later on in the course.

(a) Let L' be any number bigger than L. Prove that there exists $N > 0$ such that $\left|\frac{a_{n+1}}{a_n}\right| < L'$ for any $n > N$.

(b) Prove that for any $n > N$, we have $|a_n| < (L')^{n-N}|a_N|$.

(c) Prove that $\limsup(|a_n|^{1/n}) \leq L'$.

(d) Finally, prove that $\limsup(|a_n|^{1/n}) \leq L$.

(a) $$L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{N\to\infty}\left(\sup\left\{\left|\frac{a_{n+1}}{a_n}\right| : n > N\right\}\right)$$

Hence $\forall L' > L$, $\exists N > 0$ s.t. $\sup\left\{\left|\frac{a_{n+1}}{a_n}\right| ; n > N\right\} < L'$

$$\Longrightarrow \left|\frac{a_{n+1}}{a_n}\right| < L' \quad \forall n > N.$$

(b) $\forall n > N$, we have. $|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \cdot \left|\frac{a_{n-1}}{a_{n-2}}\right| \cdots \left|\frac{a_{N+1}}{a_N}\right| \cdot |a_N| < (L')^{n-N} \cdot |a_N|$

(c) $\forall n > N$, $$|a_n|^{1/n} < L' \cdot \big(\underbrace{(L')^{-N}}_{\text{Independent of } n.} \cdot |a_N|\big)^{1/n}$$

$$\Longrightarrow \limsup |a_n|^{1/n} \leq L' \cdot \limsup\left((L')^{-N} \cdot |a_N|\right)^{1/n} = L'.$$

(d) Since $\limsup |a_n|^{1/n} \leq L'$ for any $L' > L$.

we have $\limsup |a_n|^{1/n} \leq L$. (HW1 # 5). $\square$

(5) Let (a_n) and (b_n) be bounded sequences.

(a) Prove that $(a_n + b_n)$ is bounded.

(b) Prove that

$$(\liminf_{n\to\infty} a_n)+(\liminf_{n\to\infty} b_n) \leq \liminf_{n\to\infty}(a_n+b_n) \text{ and } (\limsup_{n\to\infty} a_n)+(\limsup_{n\to\infty} b_n) \geq \limsup_{n\to\infty}(a_n+b_n).$$

(c) Find an example of (a_n) and (b_n) such that

$$(\liminf_{n\to\infty} a_n) + (\liminf_{n\to\infty} b_n) < \liminf_{n\to\infty}(a_n + b_n).$$

(a) Follows from triangle ineq.

(b) We prove "$\liminf a_n + \liminf b_n \leq \liminf (a_n+b_n)$". the proof of the other statement is similar.

- For any $N>0$, we have

$$\inf\{a_n : n>N\} + \inf\{b_n : n>N\} \leq a_n + b_n \quad \forall n>N.$$

- Hence $\inf\{a_n : n>N\} + \inf\{b_n : n>N\} \leq \inf\{a_n+b_n : n>N\}$.

- Therefore:

$$\liminf a_n + \liminf b_n = \lim_{N\to\infty}\left(\inf\{a_n : n>N\}\right) + \lim_{N\to\infty}\left(\inf\{b_n : n>N\}\right)$$

$$= \lim_{N\to\infty}\left(\inf\{a_n : n>N\} + \inf\{b_n : n>N\}\right)$$

$$\text{(HW1\#8)} \leq \lim_{N\to\infty}\left(\inf\{a_n+b_n : n>N\}\right)$$

$$= \liminf (a_n+b_n). \quad \square$$

(c) $(a_n) = (0,1,0,1,0,1,\dots)$

$(b_n) = (1,0,1,0,1,0,\dots)$

(6) (a) Let (a_n) be a sequence such that $|a_{n+1}-a_n| < C^n$ for all n for some constant $0 < C < 1$. Prove that (a_n) is a Cauchy sequence, therefore is convergent.

(b) Let (a_n) be a sequence such that $|a_{n+1}-a_n| < \frac{1}{n}$ for all n. Is it true that such (a_n) is always convergent?

(a) $\forall n < m$, we have:

$$|a_n - a_m| \leq |a_n - a_{n+1}| + \cdots + |a_{m-1} - a_m|$$

$$< C^n + C^{n+1} + \cdots + C^{m-1} < \frac{C^n}{1-C}.$$

Since $0 < C < 1$, $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\frac{C^N}{1-C} < \varepsilon$.

Then $\forall n, m \geq N$, $|a_n - a_m| < \frac{C^{\min\{n,m\}}}{1-C} < \frac{C^N}{1-C} < \varepsilon$.

Hence (a_n) is Cauchy $\square$

(b) No. e.g. $(a_n) = \left(\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right)\right)$.

Idea (of why it diverges):

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$> 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{1/2} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{1/2} + \cdots$$