Math 104 (Section 4) Second Midterm November 5, 2019

Name: Solution

- You have 70 minutes to complete the exam.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	~~30~~ 25	
2	25	
3	~~25~~ 20	
4	~~20~~ 30	
Total	100	

1. (a) (~~20~~ 15 points) Consider the power series $\sum_{n=1}^{\infty} a_n x^n$ where

$$a_n = \begin{cases} 2^n & \text{if } n = 2k \text{ for some } k \in \mathbb{N}, \\ 4^n & \text{otherwise.} \end{cases}$$

Find the exact interval of convergence of this power series. You need to justify your answer.

- Compute $\limsup_{n\to\infty} |a_n|^{1/n}$:

  $\{|a_n|^{1/n} : n \in \mathbb{N}\} = \{4, 2, 4, 2, \dots\}$.

  Hence $\limsup |a_n|^{1/n} = 4$.

- So the radius of convergence of $\sum a_n x^n$ is $1/4$.

- Plug in $x = 1/4$:

  $\sum a_n (1/4)^n = 1 + (\frac{1}{2})^2 + 1 + (\frac{1}{2})^4 + \cdots$ diverges.

- Plug in $x = -1/4$:

  $\sum a_n (-1/4)^n = -1 + (\frac{1}{2})^2 - 1 + (\frac{1}{2})^4 - 1 + \cdots$ diverges.

So the exact interval of convergence is $(-1/4, 1/4)$. $\square$

(b) (5 points; incomplete definition gets 0 points) Let (f_k) be a sequence of real-valued functions defined on an interval I, and f be a real-valued function defined on I. Write down the precise definition of (f_k) converges uniformly to f on I.

$\forall \varepsilon > 0$, $\exists N > 0$

s.t. $|f_k(x) - f(x)| < \varepsilon$ for any $x \in I$ and any $k > N$.

(c) (5 points) Let $f_k(x) = \sum_{n=1}^{k} a_n x^n$ be a degree k polynomial, where a_n is defined as in Part (a). Let I be the exact interval of convergence you find in Part (a). Then the series of functions $f(x) = \sum_{n=1}^{\infty} a_n x^n$ is a well-defined function on I, and f_n converges pointwise to f on I.

Prove or disprove the following statement: For (f_k) and f defined above, the sequence of functions (f_k) converges uniformly to f on the interval of convergence I.

(Hint: You may use the theorem that if the series of functions $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on a set S, then $\lim_{n\to\infty} \sup\{|g_n(x)| : x \in S\} = 0$.)

If $f_k \to f$ converged uniformly on I, then by the theorem in the hint,

$$\lim_{n\to\infty} \sup\{|a_n x^n| : x \in I\} = 0.$$

However, in our case we have: ~~$\sup\{|a_n x^n| : x \in (-\frac{1}{4}, \frac{1}{4})\}$~~

$$\sup\{|a_n x^n| : x \in (-\tfrac{1}{4}, \tfrac{1}{4})\} = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

$\Rightarrow \lim_{n\to\infty} \sup\{|a_n x^n| : x \in (-\frac{1}{4}, \frac{1}{4})\}$ doesn't exist.

Therefore, (f_k) does <u>not</u> converge uniformly to f on I. $\square$

2. (a) (5 points; incomplete definition gets 0 points) Let f be a real-valued function on the closed interval $[0,1]$. Write down the precise definition of f is continuous on $[0,1]$.

$\forall x_0 \in [0,1]$ and $\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $|x - x_0| < \delta$ and $x \in [0,1] \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Or: $\forall x_0 \in [0,1]$ and $\forall$ sequence (x_n) in $[0,1]$ that converges to x_0,

we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

(b) (20 points) Let f be a real-valued continuous and bounded function on $[0,1]$. Prove that f assumes its maximum values on $[0,1]$, i.e. there exists $a \in [0,1]$ such that $f(x) \le f(a)$ for any $x \in [0,1]$.
(Hint: Let $M = \sup\{f(x) : x \in [0,1]\}$. Since f is bounded, $M \in \mathbb{R}$. Prove that there exists $a \in [0,1]$ such that $f(a) = M$.)

- $\forall n \in \mathbb{N}$, since $M - \frac{1}{n}$ is not an upper bound of $\{f(x) : x \in [0,1]\}$, $\exists x_n \in [0,1]$ s.t. $f(x_n) > M - \frac{1}{n}$. So $M - \frac{1}{n} < f(x_n) \le M$.
- By Bolzano-Weierstrass thm, (x_n) has a convergent subseq. (x_{k_n}) that converges to a point $a \in [0,1]$.
- By the continuity of f, we have $\lim_{n\to\infty} f(x_{k_n}) = f(a)$.
- On the other hand, since $M - \frac{1}{k_n} < f(x_{k_n}) \le M$, we have $\lim_{n\to\infty} f(x_{k_n}) = M$ by squeeze theorem.

$\Rightarrow$ We find that the point $a \in [0,1]$ satisfies $f(a) = M$. $\square$

3. (a) (5 points; incomplete definition gets 0 points) Let f be a real-valued function on $S \subset \mathbb{R}$. Write down the precise definition of f is uniformly continuous on S.

$\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. If $|x-y| < \delta$ and $x, y \in S$ then $|f(x) - f(y)| < \varepsilon$.

(b) (~~20~~ 15 points) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two uniformly continuous functions on $\mathbb{R}$. Prove that the composite $g \circ f : \mathbb{R} \to \mathbb{R}$ (which maps x to $g(f(x))$) is also uniformly continuous on $\mathbb{R}$.
(Hint: You only need to use the definition of uniformly continuous to prove this statement.)

- By uniform continuity of g,
  $\forall \varepsilon > 0$, $\exists \eta > 0$
  s.t. $|u - v| < \eta \Rightarrow |g(u) - g(v)| < \varepsilon$. —— $(*)$
- By uniform continuity of f, for the $\eta > 0$,
  $\exists \delta > 0$
  s.t. $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \eta$.
  $\overset{(*)}{\Longrightarrow} |g(f(x)) - g(f(y))| < \varepsilon$. $\square$

4. There are four statements below:

(I) For $n \in \mathbb{N}$, define the continuous function $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$. Then the sequence (f_n) converges uniformly to a function on $[0,1]$.

(II) The function $f(x) = x^2$ is uniformly continuous on the open interval $(0,1)$.

(III) Let (a_n) and (b_n) be sequences of nonnegative numbers such that the series $\sum a_n$ and $\sum b_n$ converge. Then the series $\sum \sqrt{a_n b_n}$ also converges.

(IV) The function $f(x) = \frac{1}{x}$ is uniformly continuous on the open interval $(0,1)$.

(a) ($\cancel{10}$ 15 points) Choose a statement that is true and prove it. *You are not allowed to choose more than one statement.*

My statement is (II) or (III)

(II): $\forall \varepsilon > 0$, take $\delta = \varepsilon/2$. Then for any $x, y \in (0,1)$ and $|x-y| < \delta$, we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x-y||x+y| \leq 2|x-y| < 2\delta = \varepsilon. \quad \square$$

(III): Observe that $\sqrt{a_n b_n} \leq a_n + b_n$ for any $a_n, b_n \geq 0$.

$\sum a_n$ and $\sum b_n$ converge $\Rightarrow$ $\sum (a_n + b_n)$ converges.

By comparison test, we have $\sum \sqrt{a_n b_n}$ converges. $\square$

(b) (~~10~~ 15 points) Choose a statement that is false. Give an explicit counterexample and justify it. *You are not allowed to choose more than one statement.*

My statement is (I) or (IV).

(I). $\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1. \end{cases}$

Hence f_n converges pointwise to a discontinuous function on $[0,1]$.

Therefore the convergence is not uniform. $\square$

(IV). Take $\varepsilon = 1$, For any $\delta > 0$,

Consider $x = \min\{\frac{1}{3}, \delta\}$ and $y = \frac{3}{2}\min\{\frac{1}{3}, \delta\} = \frac{3}{2}x$

Then we have:

- $x, y \in (0,1)$
- $|x-y| \leq \delta/2 < \delta$.
- $|f(x) - f(y)| = \left|\frac{1}{x} - \frac{2}{3}\frac{1}{x}\right| = \left|\frac{1}{3x}\right| \geq 1 = \varepsilon$.

Hence $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$. $\square$