This week: Basis, dimension of a vector space.

Next Tue: Q & A for 1^{st} midterm.

Next Thu: 1^{st} midterm.

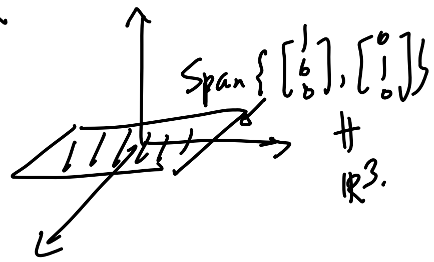
- More details & practice problems will be announced later this week.

Def: V: vector space, say $\mathcal{B} = \{\vec{v}_1, \cdots, \vec{v}_n, \cdots\} \subseteq V$ is a **basis** of V if
- $\mathcal{B}$ is a linearly independent set,
- $\text{Span}\{\underbrace{\vec{v}_1, \cdots, \vec{v}_n, \cdots}_{\mathcal{B}}\} = V$.

$\boxed{\begin{array}{l} \text{If } \vec{v}_1, \cdots, \vec{v}_n \in \mathcal{B}, \\ \quad c_1, \cdots, c_n \in \mathbb{R} \\ \quad \text{satisfy: } c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0} \\ \text{then } \quad c_1 = c_2 = \cdots = c_n = 0. \end{array}}$

e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$

is NOT a basis since



e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$ is NOT a basis since

they're linearly dependent: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{0}$

e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$ is a basis.

Consider $V = \mathbb{R}^n$, $\{\vec{v}_1, \cdots, \vec{v}_k\} \subseteq V$.

- **If $k < n$:** $\text{Span}\{\vec{v}_1, \cdots, \vec{v}_k\} \neq \mathbb{R}^n$

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{bmatrix}_{n \times k}$$

$n > k$ $\Downarrow$ can't have pivot in each row.

- <u>If $k > n$</u>: $\{\vec{v}_1, \dots, \vec{v}_k\}$ is l.d.

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{bmatrix}_{n\times k}$$

$k > n \Downarrow$ can't have pivot in each column

- <u>If $k = n$</u>: $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq \mathbb{R}^n$ basis. $\iff A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}_{n\times n}$ is invertible,

  - $\{\vec{v}_1, \dots, \vec{v}_n\}$ l.i. $\iff T_A$ is inj. $\iff$ A invertible
  - $\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = \mathbb{R}^n \iff T_A$ is surj. $\iff$

b/c A is a square matrix

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e.g. $\text{Poly}_{\le n} = \{$ all polynomials of degree $\le n \}$.

$$= \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_0, \dots, a_n \in \mathbb{R} \}.$$

$\{1, x, x^2, \dots, x^n\} \subseteq \text{Poly}_{\le n}$ forms a basis:

- <u>l.i.</u>: suppose $c_0 \cdot 1 + c_1 x + \cdots + c_n \cdot x^n = 0$ for some $c_0, \dots, c_n \in \mathbb{R}$

  $\Rightarrow c_0 = \cdots = c_n = 0$.

- <u>they span $\text{Poly}_{\le n}$</u>: $\forall f \in \text{Poly}_{\le n}$, $\exists\, a_0, \dots, a_n \in \mathbb{R}$ s.t. $f = a_0 + a_1 x + \cdots + a_n x^n$.

  $\in \text{Span}\{1, x, \dots, x^n\}$.

<u>Thm</u>: Suppose $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of a v.s. V.
Then $\forall \vec{x} \in V$, $\exists!\, c_1, \dots, c_n \in \mathbb{R}$ s.t. $\vec{x} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n$.

pf: • such $c_1, \dots, c_n$ exists b/c $\mathcal{B}$ spans V.

• If $\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$
$= c_1' \vec{v}_1 + \dots + c_n' \vec{v}_n$.

then $(c_1 - c_1')\vec{v}_1 + (c_2 - c_2')\vec{v}_2 + \dots + (c_n - c_n')\vec{v}_n = \vec{0}$

Since $\{\vec{v}_1, \dots, \vec{v}_n\}$ l.i., we have
$$c_1 - c_1' = 0$$
$$c_2 - c_2' = 0$$
$$\vdots$$
$$c_n - c_n' = 0. \quad \square$$

Def: These $c_1, \dots, c_n$ are called the coordinates of $\vec{x}$ relative to the basis $\mathcal{B}$.

$$[\vec{x}]_{\mathcal{B}} := \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$
the coordinate vector of $\vec{x}$ relative to $\mathcal{B}$

$$V \xrightarrow{[\ ]_{\mathcal{B}}} \mathbb{R}^n$$
$$\vec{x} \longmapsto [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$
the coordinate mapping with respect to $\mathcal{B}$.

where $\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$,
and $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$.

Prop: $[\ ]_{\mathcal{B}}$ is linear and bijective.

pf: • linear: $[\vec{x} + \vec{y}]_{\mathcal{B}} \overset{??}{=} [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \qquad \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$\vec{x} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n \qquad \vec{y} = d_1 \vec{v}_1 + \cdots + d_n \vec{v}_n$$

$$\vec{x} + \vec{y} = (c_1 + d_1)\vec{v}_1 + \cdots + (c_n + d_n)\vec{v}_n$$

$$\Rightarrow \quad [\vec{x} + \vec{y}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix}$$

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e.g. $\underline{\mathrm{Poly}_{\leq n}}$ $\qquad \mathcal{B} = \{1, x, \ldots, x^n\}$ $\mathcal{B}$ a basis

$$\mathrm{Poly}_{\leq n} \xrightarrow{[\ ]_B} \mathbb{R}^{n+1}$$

$$\begin{array}{ccc} \mathrm{Poly}_{\leq n} & & \\ \cup\!\!\!| & & \\ a_0 + a_1 x + \cdots + a_n x^n & \longmapsto & \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \\ \| & & \\ a_0 \cdot \underset{\uparrow}{1} + a_1 \cdot \underset{\uparrow}{x} + \cdots + a_n \cdot \underset{\uparrow}{x^n} & & \end{array}$$

vectors in $\mathcal{B}$.

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e.g. $V = \mathbb{R}^2 \qquad \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$.

$$\mathbb{R}^2 = V \xrightarrow{[\ ]_B} \mathbb{R}^2$$

$$\vec{x} \longmapsto [\vec{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$$\|$$

$$2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 7 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} 7 \\ 1 \end{bmatrix}_B = ?$$

find c_1, c_2 s.t. $\begin{bmatrix} 7 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

In general, $V = \mathbb{R}^n$, $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ basis of V.

$$\mathbb{R}^n = V \xrightarrow{[\ ]_B} \mathbb{R}^n$$

$$\vec{x} \longmapsto [\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

$$\vec{x} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n$$

$$= \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow [\vec{x}]_B = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}^{-1} \vec{x}$$

Denote $P_B := \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$

then $[\vec{x}]_B = P_B^{-1} \vec{x}$.

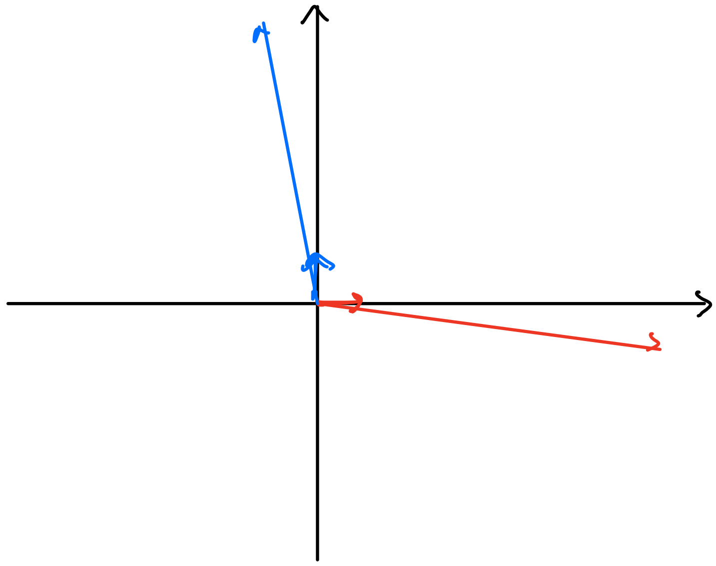
In other words, $[\ ]_{\mathcal{B}} = T_{P_B^{-1}}$.

Rmk: Why do we want to choose different bases?

e.g. $A = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}$, $T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 11 \\ -2 \end{bmatrix}$$

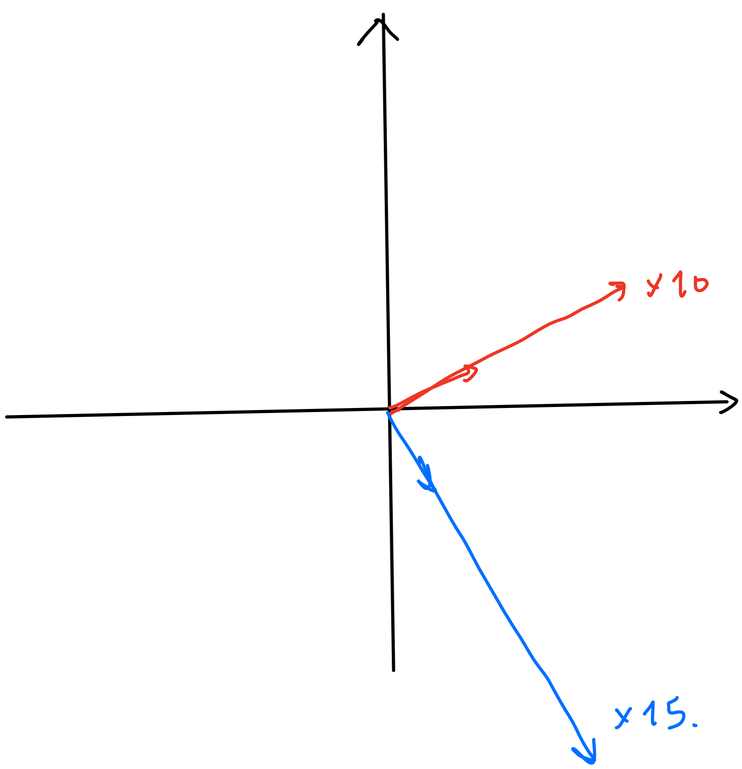
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -2 \\ 14 \end{bmatrix}$$



If we consider the basis $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} =: B$

$$T_A\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 15 \\ -30 \end{bmatrix} = 15 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



$\vec{x} \in \mathbb{R}^2$, $\vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

$$T_A(\vec{x}) = c_1 \underline{T_A \begin{bmatrix} 2 \\ 1 \end{bmatrix}} + c_2 \underline{T_A \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

$$= 10 c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 15 c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$[T_A(\vec{x})]_B = \begin{bmatrix} 10 c_1 \\ 15 c_2 \end{bmatrix}$$

$$
\boxed{A = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}}
$$

$$
\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}.
$$

$$
P_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}
$$

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "T_A"] \arrow[d, shift right, "{[\ ]_{\mathcal{B}}}"'] & \mathbb{R}^2 \arrow[d, "{[\ ]_{\mathcal{B}}}"] \\
\mathbb{R}^2 \arrow[u, shift right, "{[\ ]_{\mathcal{B}}^{-1}}"'] & \mathbb{R}^2
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "{[\ ]_{\mathcal{B}}^{-1} = T_{P_{\mathcal{B}}}}"] \arrow[rrr, bend right, "T_{\begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix}}"'] & \mathbb{R}^2 \arrow[r, "T_A"] & \mathbb{R}^2 \arrow[r, "{[\ ]_{\mathcal{B}} = T_{P_{\mathcal{B}}^{-1}}}"] & \mathbb{R}^2
\end{tikzcd}
$$

$$
\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longmapsto 10 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longmapsto 10 \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 \\ -2 \end{bmatrix} \longmapsto 15 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \longmapsto 15 \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

$$
T_{\begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix}} = T_{P_{\mathcal{B}}^{-1}} \circ T_A \circ T_{P_{\mathcal{B}}} = T_{P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}}.
$$

$$
\begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix} = P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}.
$$

"diagonalization".

A: $m \times n$ $\quad \underset{\displaystyle \{\vec{x} \mid A\vec{x} = \vec{0}\}}{\underset{\Vert}{\mathrm{Nul}(A)}} \subseteq \mathbb{R}^n$, $\quad \underset{\displaystyle \mathrm{Span}\{\text{columns of } A\}.}{\underset{\Vert}{\mathrm{Col}(A)}} \subseteq \mathbb{R}^m$

Find a basis of $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$.

$\boxed{\mathrm{Nul}(A)}$:

$$A \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

$$[A \mid \vec{0}] \rightsquigarrow [B \mid \vec{0}]$$

$$\mathrm{Nul}(A) = \mathrm{Nul}(B)$$

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix} \right\}$$

forms a basis of $\mathrm{Nul}(B)$