Today: Classification results of mero. ell. fcns.

① Any meromorphic elliptic fcn is a rational poly. of $\wp, \wp'$.

② Given $z_1, \cdots, z_n, p_1, \cdots, p_n \in$ F.D. s.t. $\sum z_i = \sum p_i$

$\exists$ f: mero. ell. s.t. f has zeros at z_i, poles at p_i.

Rmk: $\Lambda \subseteq \mathbb{C}$, $\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$, $\omega_1, \omega_2 \in \mathbb{C}$

$f(z) = f(z+\lambda)$, $\forall \lambda \in \Lambda$, $z \in \mathbb{C}$.

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

- $$\frac{1}{z-\lambda} = \frac{1/\lambda}{z/\lambda - 1} = \frac{-1/\lambda}{1 - z/\lambda} = -\frac{1}{\lambda}\left(1 + \frac{z}{\lambda} + \frac{z^2}{\lambda^2} + \cdots\right)$$

- $$\frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right) \quad \text{convergent}$$

__Weierstrass ζ-fcn:__ $$\zeta(z) := \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right),$$

has a simple pole at Λ.

But $\zeta(z)$ is __NOT elliptic__:

- $$\zeta'(z) = \frac{-1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{-1}{(z-\lambda)^2} + \frac{1}{\lambda^2} \right) = -\wp(z) \quad \text{ell.}$$

- $\zeta'(z) = \zeta'(z+\omega_1)$

$\Longrightarrow$ $\zeta(z) - \zeta(z+\omega_1) \equiv$ const. c_1

Similarly. $\zeta(z) - \zeta(z+\omega_2) \equiv$ another const. c_2

Rmk: $\forall a, b \in \mathbb{C}$, $\underline{\zeta(z-a) - \zeta(z-b)}$ elliptic $\nearrow$ has exactly 2 poles in the F.D.

$$\begin{pmatrix} \zeta(z+w_1-a) - \zeta(z+w_1-b) \\ = (\zeta(z-a)-c_1) - (\zeta(z-b)-c_1) = \zeta(z-a) - \zeta(z-b) \end{pmatrix}$$

<u>Laurent series exp. of $\wp(z)$ near $z=0$</u>:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{\underbrace{(z-\lambda)^2}_{*}} - \frac{1}{\lambda^2} \right)$$

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n, \quad \text{for } |w| < 1.$$

$$\frac{1}{(1-w)^2} = \sum_{n=0}^{\infty} (n+1) w^n$$

$$\nearrow \quad \frac{1}{(z-\lambda)^2} = \frac{1}{\lambda^2 \left(\frac{z}{\lambda} - 1\right)^2} = \frac{1}{\lambda^2 \left(1 - \frac{z}{\lambda}\right)^2} = \frac{1}{\lambda^2} \sum_{n=0}^{\infty} (n+1) \left(\frac{z}{\lambda}\right)^n.$$

$$= \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \sum_{n=1}^{\infty} (n+1) \frac{z^n}{\lambda^n}$$

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \neq 0} \sum_{n=1}^{\infty} (n+1) \frac{z^n}{\lambda^{n+2}}$$

$$= \frac{1}{z^2} + \sum_{\lambda \neq 0} \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+2}} \cdot (n+1) \cdot z^n$$

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty} \underbrace{\left(\sum_{\lambda \neq 0} \frac{1}{\lambda^{n+2}} \right)}_{\parallel} (n+1) z^n.$$

$$E_{n+2}(\Lambda) = \sum_{\substack{k, \ell \in \mathbb{Z} \\ (k,\ell) \neq (0,0)}} \frac{1}{(k w_1 + \ell w_2)^{n+2}} \quad \neq 0 \text{ only if } n \text{ is even.}$$

Eisenstein series of Λ

$\boxed{\text{Write } n = 2k, \ k \geq 1}$

$$= \frac{1}{z^2} + \sum_{k=1}^{\infty} E_{2k+2}(\Lambda) \cdot (2k+1) z^{2k}$$

$$= \frac{1}{z^2} + 3E_4 z^2 + 5E_6 z^4 + 7E_8 z^6 + \cdots$$

Recall: $\boxed{\text{holo. ell. fcn} \Rightarrow \text{const.}}$

$\rightsquigarrow$ lots of identities about $\wp(z)$.

e.g.: $\wp'(z)^2$ can be expressed as a cubic poly. in $\wp(z)$

$$\wp'(z) = \frac{-2}{z^3} + 6E_4 z + 20 E_6 z^3 + \cdots$$

$$\wp'(z)^2 = \frac{4}{z^6} + \frac{-24E_4}{z^2} - 80E_6 + \cdots$$

$$\wp(z)^3 = \frac{1}{z^6} + \frac{9E_4}{z^2} + 15E_6 + \cdots$$

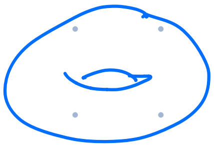
$$\wp'(z)^2 - 4\wp(z)^3 = \frac{-60E_4}{z^2} - 140E_6 + \cdots$$

$$\underbrace{\wp'(z)^2 - 4\wp(z)^3 + 60E_4\wp(z)}_{\text{elliptic}} = \underbrace{-140E_6 + \cdots}_{\text{holo.}}$$

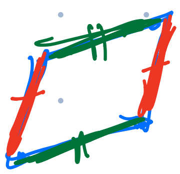
$$= -140E_6$$

$$\Rightarrow \quad \wp'(z)^2 = 4\wp(z)^3 - 60E_4\wp(z) - 140E_6. \qquad \square$$

Rmk: closely related to the cubic eq^{ns} of ell. curves:



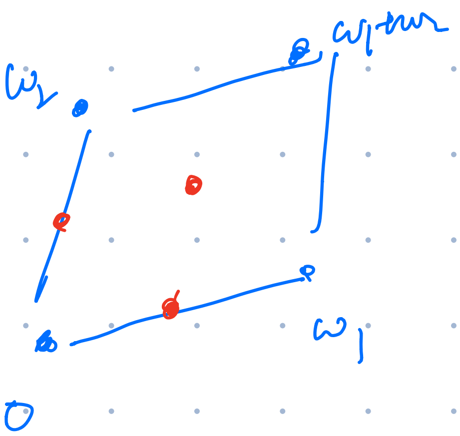
$$\mathbb{C}/\Lambda \longrightarrow \{y^2 = 4x^3 - 60E_4 x - 140E_6\} \subset \mathbb{C}^2_{x,y}$$



$$z \longmapsto (\wp(z), \wp'(z)).$$

Rmk: It's not hard to show:

$$\wp'(z)^2 = 4\left(\wp(z) - \wp\left(\tfrac{\omega_1}{2}\right)\right)\left(\wp(z) - \wp\left(\tfrac{\omega_2}{2}\right)\right)\cdot\left(\wp(z) - \wp\left(\tfrac{\omega_1+\omega_2}{2}\right)\right)$$



(Thm 9.1.7)

Thm Any mero. ell. fun. is a rational poly in $\wp(z)$, and $\wp'(z)$

pf. f = any mero ell. fun.

$$f(z) = \underbrace{\frac{f(z)+f(-z)}{2}}_{\text{even } f_e} - \underbrace{\frac{f(z)-f(-z)}{2}}_{\text{odd } f_o}$$

$\to$ reduce to the case where f is an $\begin{cases}\text{even mero. ell fun} \\ \text{odd mero ell. fun.}\end{cases}$

$\boxed{\wp' - \text{odd fun.}}$ $\rightsquigarrow$

$\rightsquigarrow$ reduce to the case where f is an $\underline{\text{even}}$ mero ell. fun.

Claim: Any even mero. ell. fun^f is a rational poly. in $\wp(z)$.

- get rid of poles $w \notin \Lambda$ that are not lattice pts Λ.
  by multiplying $\boxed{\wp(z) - \boxed{\wp(w)}}$

  complex numbs.

  has zero at w, has pole at Λ

$\rightsquigarrow$ reduce to the case where $\underline{f}$ only has poles at Λ.

- Suppose near $z=0$,

$$f(z) = \frac{a_{-2n}}{z^{2n}} + \cdots\cdots$$

$$\wp(z) = \frac{1}{z^2} + \cdots\cdots$$

$$\wp(z)^n = \frac{1}{z^{2n}} + \cdots\cdots$$

$$\underbrace{f(z) - a_{-2n}\,\wp(z)^n}_{\text{even, ell. mero. fun}} = \frac{*}{z^{2n-2}} + \cdots$$

$\rightsquigarrow$ $\exists\; a_{-2}, a_{-4}, \ldots, a_{-2n}$ so that

$$\underbrace{f(z) - a_{-2n}\,\wp(z)^n - a_{-2n+2}\,\wp(z)^{n-1} - \cdots - a_{-2}\,\wp(z)}_{} = \text{holo} \;\; \| \;\; \text{Const.}$$

$\Rightarrow$ $f(z)$ is a poly in $\wp(z)$. $\square$

Thm: Given $z_1, \dots, z_n, p_1, \dots, p_n \in$ F.D. s.t. $\sum z_i = \sum p_i$

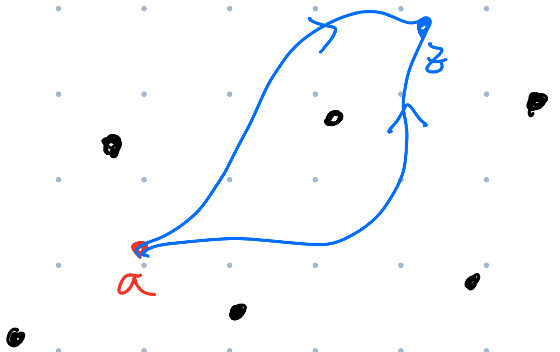
$\exists f$: mero. ell. s.t. f has zeros at z_i, poles at p_i.

pf: We'll construct certain fcn that has simple zero at Λ, no other zeros/poles.

Recall $\zeta(z)$ has simple poles at Λ.

Define Weierstrass σ-fcn: $\sigma(z) := \exp\left(\int_a^z \zeta(w)\,dw\right)$

a ← any base pt.



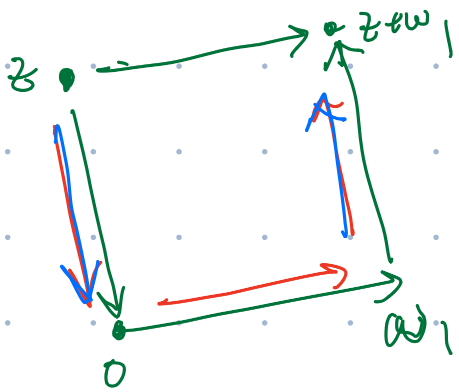
- $\sigma(z)$ has zero of order 1 at Λ, no other zeros/poles
- $\sigma(z+\omega_1) = \exp\left(\int_a^{z+\omega_1} \zeta(w)\,dw\right)$

$$= \exp\left(\underline{\int_a^z \zeta(w)\,dw} + \int_z^{z+\omega_1} \zeta(w)\,dw\right)$$

$$= \sigma(z) \cdot \underbrace{\exp\left(\int_z^{z+\omega_1} \zeta(w)\,dw\right)}_{\displaystyle \| \atop \displaystyle \exp\left(\int_z^0 \zeta(w)\,dw + \int_0^{\omega_1} \zeta(w)\,dw + \int_{\omega_1}^{z+\omega_1} \zeta(w)\,dw\right)}$$

$$\exp\left(\int_z^0 \zeta(w)\,dw + \int_0^{\omega_1} \zeta(w)\,dw + \int_{\omega_1}^{z+\omega_1} \zeta(w)\,dw\right) = \exp(C_1 z + D_1)$$

C_1, D_1 const.



$$\int_{\omega_1}^{z+\omega_1} \zeta(w)\,dw - \int_0^z \zeta(w)\,dw.$$

$$= \int_0^z \Big(\underbrace{\boxed{\zeta(w+\omega_1) - \zeta(w)}}_{C_1} \Big)\, dw.$$

$$= C_1 z$$

Recall: $\zeta(w+\omega_1) - \zeta(w) = C_1$ const. fn.

- $$\prod_{i=1}^n \sigma(z-z_i) \cdot \prod_{i=1}^n \sigma(z-p_i)^{-1}$$ WANT: is elliptic

$$\Big\downarrow\; z \longmapsto z+\omega_1$$

$$\prod_{i=1}^n \sigma(z-z_i) \cdot \prod_{i=1}^n \sigma(z-p_i)^{-1}$$

- $$\boxed{\prod_{i=1}^n \exp\big(C_1(z-z_i)+D_1\big) \cdot \prod_{i=1}^n \exp\big(C_1(z-p_i)+D_1\big)^{-1}}$$

$$= \exp\Big(\sum_{i=1}^n \big(C_1(\cancel{z}-z_i)+\cancel{D_1}\big) - \sum_{i=1}^n \big(C_1(\cancel{z}-p_i)+\cancel{D_1}\big) \Big)$$

$$= \exp(0)$$ using $\sum z_i = \sum p_i$

$$= 1$$

$\square$

Rmk: Analyze w/ 1-periodic fcn (trigonometric fcn)

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \neq 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \qquad \frac{1}{(\sin z)^2} = \sum \frac{1}{(z-n\pi)^2}$$

$$\Big\downarrow \int \qquad\qquad\qquad\qquad \Big\downarrow \int$$

$$\zeta(z) = \frac{1}{z} + \sum_{\lambda \neq 0} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right) \qquad \frac{1}{\tan z} = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n\pi} + \frac{1}{n\pi} \right)$$

$$\Big\downarrow \int \qquad\qquad\qquad\qquad \Big\downarrow \int$$

$$\Big\downarrow \exp \qquad\qquad\qquad\qquad \Big\downarrow \exp$$

$$\sigma(z) \qquad\qquad \sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2} \right)$$

simple zeros at Λ \qquad simple zeros at $2\pi\mathbb{Z}$

Def: Say $\Lambda_1 \cong \Lambda_2$ are equivalent if

$\exists\, c \in \mathbb{C} \setminus \{0\}$ s.t. $c\Lambda_1 = \Lambda_2$.

Rmk: Any Λ $= \{m\omega_1 + n\omega_2 \mid \cdots\}$ is equivalent to $\Lambda_\tau = \{m + n\tau \mid m, n \in \mathbb{Z}\} \subseteq \mathbb{C}$, where $\tau \in \mathbb{H}$.