1. Using Euler's formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i},$$

show that the complex zeros of $\sin \pi z$ are exactly at the integers, and that they are each of order 1.

Calculate the residue of $1/\sin \pi z$ at $z = n \in \mathbb{Z}$.

- $\sin(\pi z) = 0 \iff e^{i\pi z} = e^{-i\pi z} \iff e^{2i\pi z} = 1.$
- write $z = x + iy$, then

$$1 = e^{2\pi i z} = e^{2\pi i (x+iy)} = e^{-2\pi y} \left(\cos(2\pi x) + i \sin(2\pi x)\right)$$

$$\iff y = 0 \text{ and } x \in \mathbb{Z}. \quad \square$$

- to show that $\sin(\pi z)$ has zero of order 1 at $z = n \in \mathbb{Z}$, it's equivalent to show that $(\sin(\pi z))' \big|_{z=n} \neq 0$.

$$(\sin(\pi z))' \big|_{z=n} = \pi \cos(\pi z) \big|_{z=n} = \pi (-1)^n \neq 0. \quad \square$$

- $\forall n \in \mathbb{Z}$, $\sin(\pi z) = (z-n) f(z)$ for some non-vanishing holo. fcn near n.

Then $\dfrac{1}{\sin(\pi z)} = \dfrac{1}{z-n} \dfrac{1}{f(z)}$, and the residue at n is $\dfrac{1}{f(n)}$.

- $(\sin(\pi z))' = f(z) + (z-n) f'(z)$

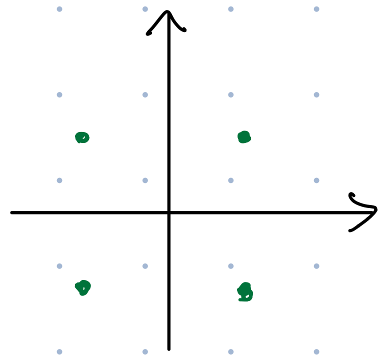
$$\Rightarrow f(n) = \pi \cos(\pi n) = \pi (-1)^n.$$

$$\Rightarrow \operatorname*{Res}_{z=n} \frac{1}{\sin(\pi z)} = \frac{(-1)^n}{\pi}. \quad \square$$

2. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

Where are the poles of $1/(1+z^4)$?



Poles of $\dfrac{1}{1+z^4}$: $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$.

Each of the poles is simple (of order one).

$$\operatorname*{Res}_{z=e^{i\pi/4}} \frac{1}{1+z^4} = \lim_{z\to e^{i\pi/4}} \frac{z-e^{i\pi/4}}{1+z^4} = \left.\frac{1}{4z^3}\right|_{z=e^{i\pi/4}}$$

$$= \frac{-e^{i\pi/4}}{4}$$

Similarly, $\displaystyle \operatorname*{Res}_{z=e^{i3\pi/4}} \frac{1}{1+z^4} = \left.\frac{1}{4z^3}\right|_{z=e^{i3\pi/4}} = \frac{-e^{i3\pi/4}}{4}$

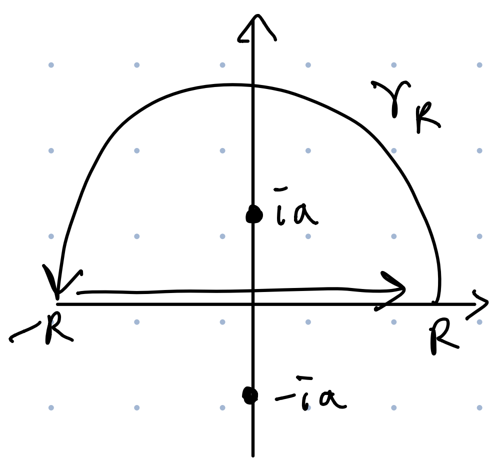
By the same argument we did in class,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi i\left(\frac{-e^{i\pi/4}-e^{i3\pi/4}}{4}\right) = 2\pi i \frac{-\sqrt{2}i}{4} = \frac{\pi}{\sqrt{2}}.$$

$\square$

3. Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \pi\frac{e^{-a}}{a}, \qquad \text{for } a>0.$$



$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{\cos x}{x^2+a^2}\,dx$$

$$= \operatorname{Re}\left(\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{iz}}{z^2+a^2}\,dz\right)$$

$$= \operatorname{Re}\left(2\pi i \operatorname*{Res}_{z=ia}\frac{e^{iz}}{z^2+a^2} - \underbrace{\lim_{R\to\infty}\int_{\gamma_R}\frac{e^{iz}}{z^2+a^2}\,dz}_{=0 \text{ by Jordan's Lemma}}\right)$$

- ia is a simple pole of $\dfrac{e^{iz}}{z^2+a^2}$,

hence $\displaystyle \operatorname*{Res}_{z=ia}\frac{e^{iz}}{z^2+a^2} = \left.\frac{e^{iz}}{z+ia}\right|_{z=ia} = \frac{e^{-a}}{2ia}$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,dx = \operatorname{Re}\left(2\pi i\cdot\frac{e^{-a}}{2ia}\right) = \pi\frac{e^{-a}}{a}. \quad \square$$

6. Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot \pi.$$

By the argument we did in class, it suffices to compute the residue of $\frac{1}{(1+z^2)^{n+1}}$ at $z=i$. (this is a pole of order $\underline{n+1}$)

$$\operatorname*{Res}_{z=i} \frac{1}{(1+z^2)^{n+1}} = \frac{1}{n!}\left(\frac{d}{dz}\right)^n \left((z-i)^{n+1}\frac{1}{(1+z^2)^{n+1}}\right)\Bigg|_{z=i}$$

$$= \frac{1}{n!}\left(\frac{d}{dz}\right)^n \left(\frac{1}{(z+i)^{n+1}}\right)\Bigg|_{z=i}$$

$$\left[\left(\frac{d}{dz}\right)^n \left((z+i)^{-n-1}\right) = \left(\frac{d}{dz}\right)^{n-1}\left[-(n+1)(z+i)^{-n-2}\right] = \cdots\cdots \right.$$

$$\left. = (-1)^n (n+1)(n+2)\cdots(2n)(z+i)^{-2n-1}\right]$$

$$= \frac{1}{n!}(-1)^n\cdot(n+1)(n+2)\cdots(2n)\,(2i)^{-2n-1}$$

$$= \frac{-i}{2^{2n+1}\cdot n!}(n+1)(n+2)\cdots(2n)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{n+1}} = \frac{2\pi}{2^{2n+1}\cdot n!}\cdot\frac{(2n)!}{n!}$$

$$= \frac{(2n)!}{(2\cdot4\cdot6\cdot8\cdots 2n)(2\cdot4\cdot6\cdot8\cdots 2n)}\pi$$

$$= \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{2\cdot4\cdot6\cdot8\cdots 2n}\cdot\pi. \qquad \square$$

13. Suppose $f(z)$ is holomorphic in a punctured disc $D_r(z_0) - \{z_0\}$. Suppose also that

$$|f(z)| \le A|z - z_0|^{-1+\epsilon}$$

for some $\epsilon > 0$, and all z near z_0. Show that the singularity of f at z_0 is removable.

- Consider $g(z) := f(z) \cdot (z - z_0)$, which is holo. in $D_r^\times(z_0)$.
- We have:

$$|g(z)| \le A|z - z_0|^{\varepsilon} \text{ for all } z \text{ near } z_0.$$

$\Rightarrow$ g is bounded near z_0.

$\Rightarrow$ g has removable singularity at z_0.

- Let G be the holo. fun. on $\mathbb{D}_r(z_0)$ s.t. $G(z) = g(z)$ $\forall z \in \mathbb{D}_r^\times(z_0)$.

Then $G(z_0) = \lim_{z \to z_0} g(z) = 0$ since $|g(z)| \le A|z - z_0|^{\varepsilon}$ $\forall z$ near z_0

Hence $G(z) = (z - z_0) F(z)$ for some holo. fun. F on $\mathbb{D}_r(z_0)$.

- Therefore $F(z) = f(z)$ $\forall z \in \mathbb{D}_r^\times(z_0)$.

$\Rightarrow$ f has removable sing. at z_0. $\square$