Invariants of categorical dynamical systems

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Goal of this talk: Explain reasons that “categorical dynamical systems” could be interesting.

A (discrete) dynamical system is a pair (X, ϕ) where

- $\phi \colon X \to X$ preserves certain mathematical structures on X.

We would like to study the long-term behavior of ϕ^n under large iterations.

Examples:

- A linear self-map $T \colon V \to V$ of a vector space V.
- A continuous self-map $f \colon X \to X$ of a compact metric space X.
- A holomorphic self-map $f \colon X \to X$ of a compact Kähler manifold X.
- An endofunctor $F \colon \mathcal{D} \to \mathcal{D}$ of a triangulated category $\mathcal{D}$.

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Examples of triangulated categories

Recall that a triangulated category is an additive category with a shift functor [1] and a collection of exact triangles

$$\cdots \to A \to B \to C \to A[1] \to \cdots$$

that satisfy a set of axioms.
(Analogy: Exact sequences $0 \to A \to B \to C \to 0$ in abelian categories.)
(C is the "mapping cone" of $A \to B$.)

Examples:

- $\mathcal{D}^b\mathrm{Coh}(X)$, where X is a smooth complex projective variety
  (objects: (complex of) holomorphic vector bundles on X)
- $\mathcal{D}^\pi\mathrm{Fuk}(Y)$, where Y is a symplectic manifold
  (objects: Lagrangian submanifolds in Y, morphisms: $L_1 \cap L_2$)

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Why categorical dynamical systems? (I)

Both holomorphic dynamics and symplectic dynamics can be discussed in the categorical settings.

- A holomorphic self-map $f\colon X \to X$ induces an endofunctor
$$\mathbb{L}f^*\colon \mathcal{D}^b\mathrm{Coh}(X) \to \mathcal{D}^b\mathrm{Coh}(X).$$
- A symplectomorphism $f\colon Y \to Y$ induces an autoequivalence
$$f_*\colon \mathcal{D}^\pi\mathrm{Fuk}(Y) \to \mathcal{D}^\pi\mathrm{Fuk}(Y).$$

Moreover, with homological mirror symmetry conjecture, one can consider "mixings" of holomorphic and symplectic dynamical systems on
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Why categorical dynamical systems? (II)

There is a parallel between Teichmüller theory and the theory of stability conditions on triangulated categories, developed by Bridgeland, Smith, Dimitrov, Haiden, Katzarkov, Kontsevich, etc.

Riemann surfaces	Triangulated categories
curve C	object E
$C_1 \cap C_2$	$\mathrm{Hom}(E_1, E_2)$
metric g	stability condition σ
geodesic	stable objects
length $\ell_g(C)$	mass $m_\sigma(E)$
slope of C	phase $\phi_\sigma(E)$
diffeomorphisms	autoequivalences
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Why categorical dynamical systems? (III)

Analogy with symbolic dynamics:

Subshifts	Triangulated categories
finite alpahbet $\mathcal{A}$	bounded t-structure
shift-invariant subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$	triangulated categories
shift map $\sigma\colon X \to X$	shift functor [1]
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One can also consider categorical analogues of results in symbolic dynamics.

Outline

(I). Entropy of holomorphic and symplectic dynamical systems, and mixings of them.

(II). Finite subgroups of $\mathrm{Aut}(\mathcal{D})$ acting on $\mathrm{Stab}(\mathcal{D})$.

(III). Shifting numbers and quasimorphisms on $\mathrm{Aut}(\mathcal{D})$.

Topological entropy...

... is hard to compute in general.

Let (X, d) compact metric space and $f\colon X \to X$ continuous. Consider

$$N(n,\epsilon) := \max\left\{\ell\colon \exists x_1,\ldots,x_\ell \text{ s.t. } \max_{0\le k\le n}\{d(f^k(x_i), f^k(x_j))\} \ge \epsilon \ \forall x_i, x_j\right\}$$

The topological entropy of f is defined to be

$$h_{\mathrm{top}}(f) := \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty}\frac{1}{n}\log N(n,\epsilon)\right) \in [0,\infty].$$

Basic properties:

- It's a topological invariant measuring the "complexity" of f.
- $f^n = \mathrm{id}_X \implies h_{\mathrm{top}}(f) = 0$.

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Example: Holomorphic maps on compact Kähler manifolds

One of the most fundamental results in (higher dimensional) complex dynamics is the following result of Gromov and Yomdin.

Theorem (Gromov, Yomdin)

If $f\colon X \to X$ is a surjective holomorphic map of a compact Kähler manifold, then

$$h_{\mathrm{top}}(f) = \log \rho(f^*)$$

where ρ is the spectral radius of $f^*\colon H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$.

Here is a geometric application of the topological entropy.

Theorem (Cantat)

If a compact complex surface X admits an automorphism of positive topological entropy, then X is either a torus, a K3 surface, an Enriques surface, or a rational surface.

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Categorical entropy

Let $F\colon \mathcal{D} \to \mathcal{D}$ be as before, and $G, G' \in \mathcal{D}$ be split generators.

Dimitrov, Haiden, Katzarkov, and Kontsevich defined:

$$h_{\mathrm{cat}}(F) := \lim_{n\to\infty} \frac{1}{n} \log \Big(\sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}(G, F^n G'[k]) \Big).$$

Basic properties:

- The limit exists, and is independent of the choice of G, G'.
- $F^n = [m] \implies h_{\mathrm{cat}}(F) = 0$.

Example: When $\mathcal{D} = \mathcal{D}^b\mathrm{Coh}(X)$:

- Kikuta–Takahashi: $h_{\mathrm{cat}}(\mathbb{L}f^*) = h_{\mathrm{top}}(f) = \log \rho(f^*) = \log \rho([\mathbb{L}f^*]_{H^*})$.
- $h_{\mathrm{cat}}(- \otimes L) = 0$.

However, there exist a Calabi–Yau manifold X and $F = T \circ (- \otimes L)$ with $h_{\mathrm{cat}}(F) > \log \rho([F])$, where T is mirror to certain Dehn twist.

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Nielsen–Thurston classification

- Σ: Riemann surface
- $\mathrm{MCG}(\Sigma) = \mathrm{Diff}(\Sigma)/\text{isotopy}$: mapping class group
- each mapping class is either:
  - finite order
  - reducible
  - pseudo-Anosov

For instance –

- elements of $\mathrm{MCG}(T^2) = \mathrm{SL}(2, \mathbb{Z})$ are either:
  - elliptic (finite order)
  - parabolic (Dehn twist)
  - hyperbolic (two transverse foliations: stretch one by $\lambda > 1$, compress the other by $1/\lambda$)

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Realization problem of finite subgroups

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Poincaré translation numbers vs Shifting numbers

$$0 \to \mathbb{Z} \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}^+(S^1) \to 1.$$

Poincaré translation number: $\rho(f) := \lim_{n\to\infty} \frac{f^{(n)}(x_0)-x_0}{n}$.

In the setting of triangulated categories, we have a central extension

$$0 \to \mathbb{Z} \to \mathrm{Aut}(\mathcal{D}) \to \mathrm{Aut}(\mathcal{D})/[1] \to 1.$$

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Theorem (F.–Filip, 2023)

- *The limit*

$$\tau^{\pm}(F) := \lim_{n\to\infty} \frac{\phi_\sigma^{\pm}(F^n G) - \phi_\sigma^{\pm}(G)}{n}$$

  always exists, and is independent of the choices of G and σ.

- *The function*

$$h_t(F) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}(G, F^n G[k]) e^{-kt} \right),$$

  is a convex function in t satisfying:
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Theorem (F., 2023)

Let X be an abelian variety. Then $\tau = \tau^{\pm} \colon \mathrm{Aut}(\mathcal{D}^b(X)) \to \mathbb{R}$ is a quasimorphism.

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Thank you for your attention!

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