Is $1 = \boxed{0.9999\cdots}$?

What's this?

To make sense of this, we have to introduce the notion of limit.

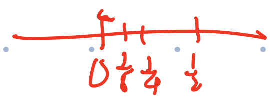
(we'll think of "0.999..." as the limit of a sequence of real numbers $0.9,\ 0.99,\ 0.999,\ \ldots$)

Def: A sequence of real numbers is a list of real numbers indexed by $\mathbb{N}$: $(a_1, a_2, a_3, \ldots)$ where each $a_i \in \mathbb{R}$

(Alternatively, a sequence is a function $\mathbb{N} \longrightarrow \mathbb{R}$)

$$\begin{aligned} 1 &\longmapsto a_1 \\ 2 &\longmapsto a_2 \\ &\ \vdots \end{aligned}$$

e.g. $\boxed{(\underset{c_1}{0.9},\ \underset{c_2}{0.99},\ \underset{c_3}{0.999},\ \ldots\ \underset{(c_n)}{\ldots})}$ $\lim_{n\to\infty} c_n = 1$



$\boxed{(a_n)} = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots \right)$, where $a_n = \frac{1}{2^n}$ $\lim_{n\to\infty} a_n = 0$

$(b_n) = (-1, 1, -1, 1, \ldots)$, where $b_n = (-1)^n$

Intuition

~~Def~~ We say $a \in \mathbb{R}$ is the limit of a seq. (a_n)

if a_n gets very close to a, as n gets very large.

Def: We say $a \in \mathbb{R}$ is the limit of a seq. (a_n) if

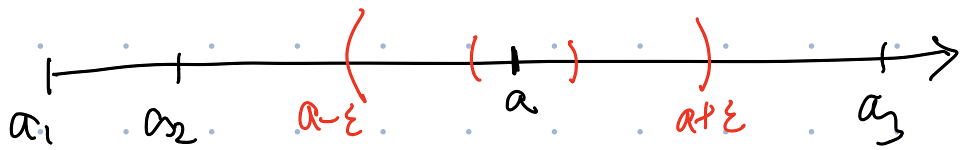
$\forall \boxed{\varepsilon} > 0,\ \exists N > 0$ s.t. $n > N \Rightarrow \underbrace{|a_n - a| < \varepsilon}$

Could be very small

$\Updownarrow$

$a - \varepsilon < a_n < a + \varepsilon$

all (a_n) falls into this nbhd except the first N terms



e.g. $(a_n) = \left(\frac{1}{2^n}\right)$ We guess: the limit of $(a_n) = 0$

e.g. $\varepsilon = \frac{1}{10}$, What N can make

$\boxed{\text{“} n > \underset{=3}{N} \Rightarrow \underbrace{|\overset{\frac{1}{2^n}}{a_n} - a| < \varepsilon}_{= \frac{1}{2^n}} \text{”}}$ true??

$\varepsilon = \frac{1}{100}$, What N can make

“$n > \underset{= 7}{N} \Rightarrow |a_n - a| < \varepsilon$” hold??

$\frac{1}{2^7} = \frac{1}{128}$

Rmk: usually, N depends on ε

Let's proof: $\boxed{\lim_{n\to\infty} \frac{1}{2^n} = 0}$

We need to show that for ANY $\varepsilon > 0$,

we can find an $N > 0$

s.t. $n > N \Rightarrow \left| \frac{1}{2^n} - 0 \right| < \varepsilon$

$= \frac{1}{2^n}$

$$\frac{1}{2^n} < \varepsilon \iff \boxed{n > \log_2\left(\frac{1}{\varepsilon}\right)}$$

$$\Updownarrow$$

$$2^n > \frac{1}{\varepsilon}$$

We can choose $N = \log_2\left(\frac{1}{\varepsilon}\right) > 0.$

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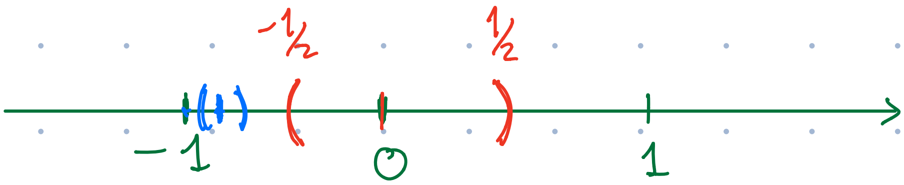
e.g. $(a_n) = ((-1)^n) = (-1, 1, -1, 1, \dots)$

We want to show that (a_n) has no limit.



Can 0 be the limit of (a_n)?

$\forall \varepsilon > 0, \ \exists N > 0$ s.t. $n > N \Rightarrow |a_n - 0| < \varepsilon$

pf: ($\lim a_n$ doesn't exist):

Prove by contradiction.

Assume that $\lim_{n \to \infty} a_n = a \in \mathbb{R}$ exists.

Then $\forall \varepsilon > 0, \ \exists N > 0$

s.t. $n > N \Rightarrow |a_n - a| < \varepsilon$

$$\left.\begin{array}{l} \bullet \text{ If } n \text{ odd and } n > N \Rightarrow |-1-a| < \varepsilon \\ \bullet \text{ If } n \text{ even and } n > N \Rightarrow |1-a| < \varepsilon \end{array}\right\} \text{true for any } \varepsilon > 0.$$

Take $\varepsilon = \frac{1}{2}$, $|-1-a| < \frac{1}{2}$, $|1-a| < \frac{1}{2}$

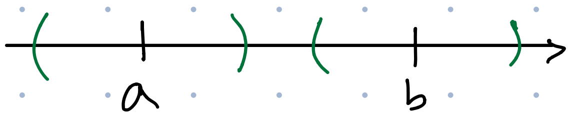
$$1 = \frac{1}{2} + \frac{1}{2} > |-1-a| + |1-a| \underset{\substack{\uparrow \\ \text{triangle} \\ \text{ineq.}}}{\geq} |(-1-a)-(1-a)| = 2.$$

Contradiction $\square$

Prop: The limit of a sequence is unique (if exists).

Pf: Say $a \neq b$ s.t. they're both a limit of (a_n)

$a < b$



$\forall \varepsilon > 0$, $\exists N_a > 0$ s.t. $n > N_a \Rightarrow |a_n - a| < \varepsilon$.

$\exists N_b > 0$ s.t. $n > N_b \Rightarrow |a_n - b| < \varepsilon$.

Now, if we choose $\underline{\varepsilon = \frac{b-a}{3} > 0}$

for any $n > \max\{N_a, N_b\}$,

we have
$$\left.\begin{array}{c} |a_n - a| < \varepsilon \\ \text{and} \\ |a_n - b| < \varepsilon \end{array}\right\}$$

$$\underset{\frac{2}{3}(b-a)}{\underset{\parallel}{\varepsilon + \varepsilon}} > |a_n - a| + |a_n - b| \geq b - a > 0$$

Contradiction. $\square$

Def If $a \in \mathbb{R}$ is the limit of a seq. (a_n), then we denote

$$\lim_{n\to\infty} a_n = a,$$

and we say the seq. (a_n) <u>converges</u>.

If (a_n) has no limit, then we say (a_n) <u>diverges</u>.

e.g. $(a_n) = \left(\dfrac{4n^2 - 7n}{n^2+1}\right)$

$\left(\text{If we } \dfrac{4n^2-7n}{n^2+1} = \dfrac{4 - 7/n}{1 + 1/n^2} \right)$ Guess: $\boxed{\lim a_n = 4}$

<u>We Need to show:</u>

pf: $\forall \varepsilon > 0$, we can find $N > 0$

s.t. $n > N \Rightarrow |a_n - 4| < \varepsilon$

$$\left| \frac{4n^2 - 7n}{n^2+1} - 4 \right| = \left| \frac{-7n - 4}{n^2+1} \right|$$

$$= \frac{7n+4}{n^2+1}$$

try to find some simpler expression that bound $\frac{7n+4}{n^2+1}$ from <u>above</u>

$$\frac{7n+4}{n^2+1} < \frac{7n+4}{n^2} \leq \frac{11n}{n^2} = \boxed{\frac{11}{n}} < \varepsilon$$

$n > \frac{11}{\varepsilon}$

$\forall \varepsilon > 0$, Choose $N = \frac{11}{\varepsilon} > 0$.

Then $n > N$,

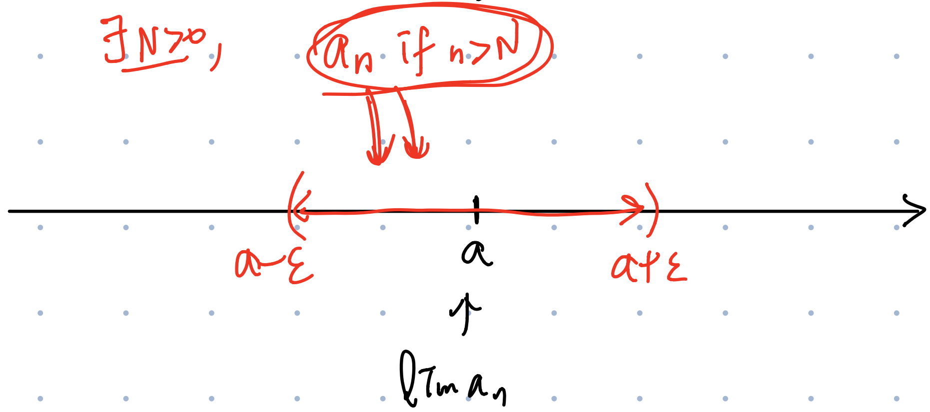
$|a_n - 4| = \frac{7n+4}{n^2+1} < \frac{11}{n} < \frac{11}{N} = \varepsilon$.

$\square$

Def: We say a seq. $(a_n) \subseteq \mathbb{R}$ is <u>bounded</u> if $\exists M > 0$ s.t. $|a_n| < M \quad \forall n \in \mathbb{N}$.

Prop: If $(a_n) \subseteq \mathbb{R}$ converges, then it's bounded.

~~Pf:~~



pf: Say $a = \lim_{n\to\infty} a_n$,

Take $\varepsilon = 1$,

$\exists N > 0$ s.t. $n > N \Rightarrow |a_n - a| < 1$

$$\Rightarrow |a_n| \leq |a_n - a| + |a| < 1 + |a| \quad \forall n > N$$

Define

$$M := \max\{|a_1|, |a_2|, \dots, |a_N|, 1 + |a|\} + 1/27$$

Claim: $|a_n| < M \quad \forall n \in \mathbb{N}$. $\square$

Rmk: (a_n) bounded $\not\Rightarrow$ (a_n) converge

e.g. $(-1)^n$