HOMEWORK 8
MATH H54

Yu-Wei's Office Hours: Sunday 1-2:30pm and Friday 12-1:30pm (PDT)
Michael's Office Hours: Monday 12-3pm (PDT)

Part I (no need to turn in)

This part of the homework provides some routine computational exercises. You don't have to turn in your solutions for this part, but being able to do the computations is vitally important for the learning process, so you definitely should do these practices before you start doing Part II of the homework.

The following exercises are from the corresponding sections of the UC Berkeley custom edition of Lay, Nagle, Saff, Snider, *Linear Algebra and Differential Equations*.

- **Exercise 7.1**: 17, 23, 29, 35, 36
- **Exercise 7.2**: 5, 7, 13

Part II (Due November 3, 8am PDT)

Some ground rules:

- You have to submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW8**.
- The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own.
- You're allowed to use any result that is proved in the lecture. But if you'd like to use other results, you have to prove it first before using it.

Problems:

(1) Let $p(\lambda)$ be the characteristic polynomial of an $n \times n$ orthogonal matrix. Prove that $\lambda^n p(\lambda^{-1}) = \pm p(\lambda)$.

(2) Let $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ be a real 2×2 symmetric invertible matrix. Prove the following statements.

  (a) A is positive definite if $\det A > 0$ and $a > 0$.
  (b) A is negative definite if $\det A > 0$ and $a < 0$.
  (c) A is indefinite if $\det A < 0$.

(Hint: HW5 Problem 4.)

(3) (a) Prove that if B is an $m \times n$ matrix, then $B^T B$ is positive semidefinite.
(b) Prove that if B is an $n \times n$ invertible matrix, then $B^T B$ is positive definite.
(Hint: Consider the associate quadratic forms.)

(4) Prove that if A a positive definite matrix, then there exists a positive definite matrix B such that $A = B^T B$. (Hint: Write $A = PDP^T$, and write $D = CC^T$, where C is another diagonal matrix.)

(5) Let A and B be $n \times n$ positive definite matrices. Prove that $A + B$ also is positive definite. (Hint: Consider the associate quadratic forms.)

(6) Let A be an positive definite matrix. Prove that A^{-1} also is positive definite. (Hint: Consider eigenvalues.)

(7) Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and corresponding orthonormal eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. Prove that

$$\lambda_1 = \min_{\vec{x} \neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} \quad \text{and} \quad \lambda_n = \max_{\vec{x} \neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}.$$

(Hint: Write $A = PDP^T$.)

(8) Let A be a real symmetric matrix. Prove that

$$\operatorname{rank}(A) \cdot \operatorname{tr}(A^2) \geq (\operatorname{tr} A)^2.$$

(Hint: Consider eigenvalues and use the Cauchy–Schwartz inequality.)

(9) Let A be a real skew-symmetric matrix, i.e. $A = -A^T$. Prove that A^2 is a symmetric, negative semidefinite matrix. (Hint: Consider the associate quadratic form.)

(10) Let A be a real skew-symmetric matrix. Prove that $\mathbb{I} + A$ is invertible. (Hint: Use Problem 9.)