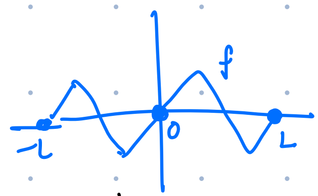
Recap: Heat eq'n:



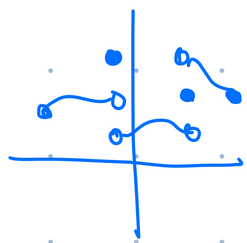
$$\begin{cases} u_t(x,t) = \beta u_{xx}(x,t). & (\beta > 0.) \\ u(0,t) = u(L,t) = 0. & \forall t \geq 0. \\ u(x,0) = f(x). & \forall x \in [0,L]. \end{cases} \quad \left(\begin{array}{l} f(0) = f(L) = 0 \\ f: \text{continuous.} \end{array} \right)$$

- If $f(x) = \sum_n c_n \sin\left(\frac{n\pi}{L} x\right)$,

  then $u(x,t) = \sum_n c_n \sin\left(\frac{n\pi}{L} x\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$ is the sol^{n}.

- In general, $f(x)$ can't be written as a <u>finite</u> sum of these sine functions. But one can find an infinite sequence $\{c_n\} \subseteq \mathbb{R}$ s.t. $\lim_{N\to\infty} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right) = f(x) \quad \forall x \in [0,L]$

  using <u>Fourier series</u>.



Def: Let f be a (piecewise) continuous fcn on $[-L,L]$, the <u>Fourier series of f</u> is:

$$\tilde{f}(x) := \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L} \right).$$

where $a_k = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{k\pi x}{L}\, dx$, $b_k = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{k\pi x}{L}\, dx$.

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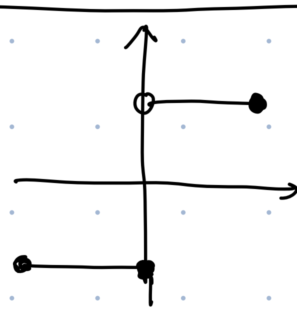
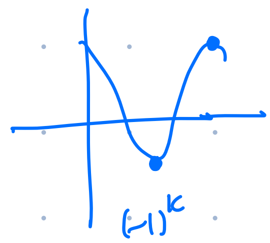
e.g.: $L = \pi$, $f(x) = \begin{cases} -1 & \text{if } -\pi \leq x \leq 0 \\ 1 & \text{if } 0 < x \leq \pi \end{cases}$



$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}\,dx$$

$$= \frac{1}{L}\left(\int_{-L}^{0} -\cos\frac{k\pi x}{L}\,dx + \int_{0}^{L}\cos\frac{k\pi x}{L}\,dx\right)$$

$$= 0$$



$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx.$$

$$= \frac{2}{\pi}\int_{0}^{\pi}\sin\frac{k\cancel{\pi}x}{\cancel{\pi}}\,dx.$$

$$\left.\frac{-\cos kx}{k}\right|_{0}^{\pi} = \frac{-\overset{(-1)^k}{\cos(k\pi)}+1}{k}$$

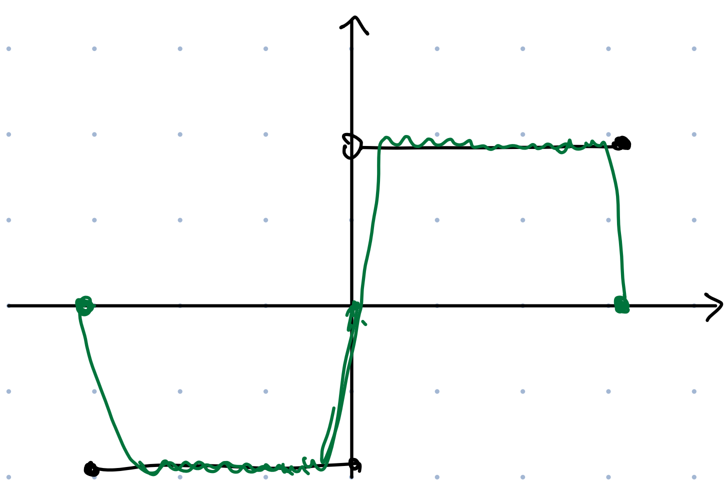
$$= \frac{-(-1)^k+1}{k}$$

$$= \begin{cases} 0 & \text{if } k \text{ even} \\ 2/k & \text{if } k \text{ odd} \end{cases}$$

$$= \begin{cases} 0 & \text{if } k \text{ even} \\ \frac{4}{k\pi} & \text{if } k \text{ odd} \end{cases}$$

$$\tilde{f}(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$$

$$= \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right)$$



Thm: If f and f' are piecewise continuous on $[-L,L]$,

- if $x \in (-L,L)$, then

$$\lim_{N\to\infty}\left(\frac{a_0}{2}+\sum_{k=1}^{N}\left(a_k\cos\frac{k\pi x}{L}+b_k\sin\frac{k\pi x}{L}\right)\right)=\frac{1}{2}\left(\underbrace{f(x^-)}_{\text{left-hand limit at } x}+\underbrace{f(x^+)}_{\text{right-hand limit at } x}\right)$$

- If $x = \pm L$, then

$$\lim_{N\to\infty}\left(\cdots \quad \_\_ \; \cdot \; \_\_ \; \cdots \right)=\frac{1}{2}\left(f(L^-)+f(-L^+)\right)$$

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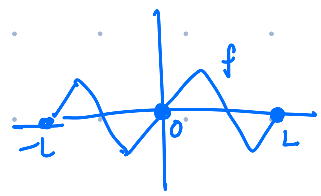
e.g. $f(x)=x^2$. Compute its Fourier series on $[-\pi,\pi]$.

$$\tilde{f}(x)=\frac{\pi^2}{3}+\sum_{n\geq 1}\frac{(-1)^n}{n^2}\cos(nx). = f(x) \text{ by Thm.}$$

$\Longrightarrow$ Euler's formula: $\displaystyle\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{6}.$

Back to the heat eq'n.:



We were given f, conti. on $[0,L]$,

$f(0)=f(L)=0$.

Let's define the odd-extension $f_{odd}(x)$ on $[-L,L]$,

where
$$f_{odd}(x)=\begin{cases} f(x) & \text{if } x\in[0,L]. \\ -f(x) & \text{if } x\in[-L,0]. \end{cases}$$

Then, the Fourier coeff. of f_{odd} are:

$$a_k=\frac{1}{L}\int_{-L}^{L}\underbrace{f_{odd}(x)}_{\text{odd}}\underbrace{\cos\frac{k\pi x}{L}}_{\text{even}}\,dx=0$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\sin\frac{k\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_{0}^{L} f_{odd}(x)\sin\frac{k\pi x}{L}\,dx. = \boxed{\frac{2}{L}\int_{0}^{L} f(x)\sin\frac{k\pi x}{L}\,dx.}$$

The Fourier series of f_{odd} is:

$$\tilde{f}_{odd}(x) = \sum_{k\geq 1} b_k \sin\frac{k\pi x}{L} \;=\; f(x) \text{ for } x\in[0,L]$$

Thm.

(Then

$$u(x,t) = \sum_{k\geq 1} b_k \sin\left(\frac{k\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$ is the sol^n

of the heat eq^n)

Rmk: the only difficulty for solving the heat eq^n now reduces to computing the Fourier sine coefficients b_k.

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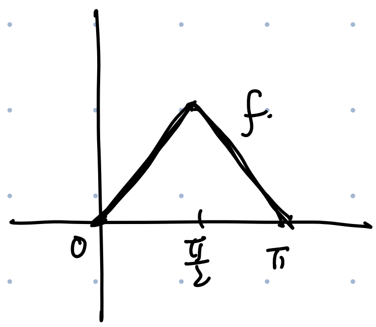
e.g.

$$\begin{cases} u_t = 2u_{xx} & 0\leq x\leq \pi,\ t\geq 0. \\ u(0,t) = u(\pi,t) = 0 & \\ u(x,0) = f(x) = \begin{cases} x, & 0\leq x\leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2}\leq x\leq \pi. \end{cases} \end{cases}$$



$$b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx. = \frac{2}{\pi}\Bigg(\underbrace{\int_0^{\frac{\pi}{2}} x\sin(kx)\,dx}_{(I)} + \underbrace{\int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin(kx)\,dx}_{(II)}\Bigg)$$

$$(I) = x\,\frac{-\cos(kx)}{k}\Bigg|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{-\cos(kx)}{k}\,dx.$$

$$= \frac{\pi}{2} \cdot \frac{-\cos\left(\frac{k\pi}{2}\right)}{k} + \frac{1}{k} \int_0^{\frac{\pi}{2}} \cos(kx)\, dx.$$

$$= \frac{\pi}{2} \frac{-\cos\left(\frac{k\pi}{2}\right)}{k} + \frac{1}{k} \left. \frac{\sin(kx)}{k} \right|_0^{\frac{\pi}{2}}$$

$$= \boxed{\frac{\pi}{2} \frac{-\cos\left(\frac{k\pi}{2}\right)}{k}} + \frac{1}{k} \cdot \frac{\sin\left(\frac{k\pi}{2}\right)}{k}$$

$$(\mathrm{II}) = \pi \int_{\frac{\pi}{2}}^{\pi} \sin(kx)\, dx - \int_{\frac{\pi}{2}}^{\pi} x \sin(kx)\, dx.$$

$$= \pi \left. \frac{-\cos(kx)}{k} \right|_{\frac{\pi}{2}}^{\pi} - \left(x \left. \frac{-\cos(kx)}{k} \right|_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} \frac{-\cos(kx)}{k}\, dx \right)$$

$$= \frac{\pi}{k} \left(\cancel{-\cos(k\pi)} + \cos\left(\frac{k\pi}{2}\right) \right) + \frac{1}{k} \left(\cancel{\pi \cos(k\pi)} - \frac{\pi}{2} \cos\left(\frac{k\pi}{2}\right) \right)$$

$$- \frac{1}{k^2} \left(\cancel{\sin(k\pi)}^{\,0} - \sin\left(\frac{k\pi}{2}\right) \right)$$

$$= \boxed{\frac{\pi}{2k} \cos\left(\frac{k\pi}{2}\right)} - \frac{1}{k^2} \left(\cancel{\sin(k\pi)}^{\,0} - \sin\left(\frac{k\pi}{2}\right) \right)$$

$$(\mathrm{I}) + (\mathrm{II}) = \frac{2}{k^2} \sin\left(\frac{k\pi}{2}\right) = \begin{cases} \frac{2}{k^2} & \text{if } k \equiv 1 \pmod 4 \\ 0 & \text{if } 2 \mid k \\ \frac{-2}{k^2} & \text{if } k \equiv 3 \pmod 4 \end{cases}$$

$$b_k = \begin{cases} 4/k^2\pi & \text{if } k \equiv 1 \pmod 4 \\ -4/k^2\pi & \text{if } k \equiv 3 \pmod 4 \\ 0 & \text{if } 2 \mid k. \end{cases}$$

$\square$

Maximum principle for heat equation:

If $u(x,t)$ is a sol^{n} of the heat eq^{n}; $\left(u_t = \beta u_{xx},\ u(0,t) = u(L,t) = 0\right)$

then
$$\max_{\substack{t \geq 0 \\ x \in [0,L]}} u(x,t) = \max_{x \in [0,L]} u(x,0).$$

We can use the max. principle to give another proof of the uniqueness of sol^{n}:

Suppose u_1, u_2 are sol^{n} $*$ $\begin{cases} u_t = \beta u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$

$W := u_1 - u_2$. $\begin{cases} w_t = \beta w_{xx} \\ w(0,t) = w(L,t) = 0 \\ w(x,0) = 0 \end{cases}$

Max. principle $\Rightarrow$ $w(x,t) \leq \underbrace{0}_{\max\limits_{x \in [0,L]} w(x,0)}$ $\forall x \in [0,L],\ t \geq 0$

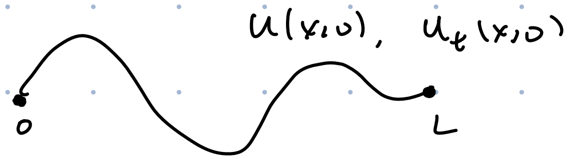
On the other hand, we can consider

$\tilde{W} := u_2 - u_1$.

max. principle $\Rightarrow$ $\tilde{w}(x,t) \leq 0$ $\forall x, t$.

$\Longrightarrow$ $W = \tilde{W} = 0$ $\square$

Wave equation:



$$\begin{cases} u_{tt} = \alpha^2 u_{xx} & 0 \le x \le L, \quad t \ge 0 \\ u(0,t) = u(L,t) \equiv 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Uniqueness via energy method:

u_1, u_2

$w = u_1 - u_2$

$$\begin{cases} w_{tt} = \alpha^2 w_{xx} \\ w(0,t) = w(L,t) \equiv 0 \\ w(x,0) = 0 \\ w_t(x,0) = 0 \end{cases}$$

$$E(t) := \frac{1}{2} \int_0^L \Big(\underbrace{\alpha^2 w_x^2}_{\text{potential energy}} + \underbrace{w_t^2}_{\text{kinetic energy}} \Big) \, dx$$

- $\frac{dE}{dt} = 0$ (use $w_{tt} = \alpha^2 w_{xx}$).
- $E(0) = 0$

$\Rightarrow \quad E \equiv 0.$

$\Rightarrow \quad w_x(x,t) = w_t(x,t) = 0$

$\Rightarrow \quad w \equiv 0.$