HOMEWORK 11
MATH 104, SECTION 6

Office Hours (via Zoom): Tuesday and Wedneday 9:30-11am.

PROBLEM SET (9 PROBLEMS; DUE APRIL 16)

Submit your homework before the lecture on Thursday. *Late homework will not be accepted under any circumstances.* You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any.

Write clearly! Mastering mathematical writing is one of the goals of this course.

*****Warning: You're not allowed to use Riemann–Lebesgue theorem in this problem set.*****

(1) Define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } \; 1-2^{-2k} \le x \le 1-2^{-(2k+1)} \; \text{ for } \; k=0,1,2,\dots \\ 0 & \text{if } \; 1-2^{-(2k+1)} < x < 1-2^{-(2k+2)} \; \text{ for } \; k=0,1,2,\dots \\ 0 & \text{if } \; x=1 \end{cases}$$

Prove that f is integrable on $[0,1]$, and compute $\int_0^1 f(x)dx$.

(2) For a bounded function $f\colon [0,1] \to \mathbb{R}$, define

$$R_n \coloneqq \frac{1}{n}\sum_{k=1}^{n} f\Big(\frac{k}{n}\Big).$$

(a) Prove that if f is integrable, then $\lim_{n\to\infty} R_n = \int_0^1 f(x)dx$.

(b) Find an example of f that is not integrable, but $\lim_{n\to\infty} R_n$ exists.

(3) Suppose that $f\colon [a,b] \to \mathbb{R}$ is integrable. Prove that the function $|f|\colon [a,b] \to \mathbb{R}$ which sends x to $|f(x)|$ is also integrable, and

$$\Big|\int_a^b f(x)dx\Big| \le \int_a^b |f(x)|dx.$$

(4) Let f be a positive and continuous function on $[0,1]$. Compute

$$\int_0^1 \frac{f(x)}{f(x)+f(1-x)}dx.$$

(5) Let $f\colon [a,b] \to \mathbb{R}$ be an integrable function. Prove that
$$\lim_{n\to\infty} \int_a^b f(x)\sin(nx)dx = 0.$$
Hint: First show that the statement is true for *step functions* (see Wikipedia for the definition of step functions). Then show that there exists a step function $S(x)$ such that $0 \leq \int_a^b (f(x) - S(x))dx < \epsilon$.

(6) Let $(\mathcal{C}[0,1], d_\infty)$ be the metric space of continuous functions on $[0,1]$, where the distance function is defined by
$$d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$
Consider the function $T\colon (\mathcal{C}[0,1], d_\infty) \to (\mathcal{C}[0,1], d_\infty)$ defined by
$$(Tf)(x) \coloneqq \int_0^x f(t)dt.$$
Prove that:

(a) T is not a contraction, i.e. there does not exist $0 < K < 1$ such that
$$d_\infty(Tf, Tg) \leq K \cdot d_\infty(f,g)$$
holds for any $f, g \in \mathcal{C}[0,1]$.

(b) T has a unique fixed point, i.e. there is a unique $f \in \mathcal{C}[0,1]$ satisfies $Tf = f$.

(c) T^2 is a contraction.

(7) Let f be a continuous function on $[a,b]$. Suppose that
$$\int_a^b x^n f(x)dx = 0$$
for $n = 0, 1, 2, \ldots$. Prove that $f(x) = 0$ for any $x \in [a,b]$.

(8) Let f, g be integrable functions on $[a,b]$. Prove that
$$\Big(\int_a^b f(x)g(x)\Big)^2 \leq \Big(\int_a^b f(x)^2 dx\Big)\Big(\int_a^b g(x)^2 dx\Big).$$
When does the equality hold?
Hint: Consider $\int_a^b (\int_a^b (f(x)g(y) - f(y)g(x))^2 dx)dy$.

(9) Let A be the set of integrable functions on $[0,1]$ such that
$$\int_0^1 f(x)dx = 3 \quad\text{and}\quad \int_0^1 xf(x)dx = 2.$$
Compute
$$\min_{f\in A} \int_0^1 f^2(x)dx$$
and find a function which attains the minimum.
Hint: Use the previous problem in a clever way.