Today: liminf, limsup (cont'd), Cauchy seq.

HW1 #5 "$x < y+\varepsilon\ \forall \varepsilon>0$" $\Rightarrow$ "$x \le y$"

"$x < y+\varepsilon\ \forall \varepsilon>0$" not true $\Leftrightarrow$ "$\exists\, \varepsilon>0$ s.t. $x \ge y+\varepsilon$"

"(a_n) is convergent" $\Leftrightarrow$ "$\exists\, a\in\mathbb{R}$, s.t. $\forall \varepsilon>0$, $\exists N>0$ s.t. $n>N \Rightarrow |a_n-a|<\varepsilon$"

"(a_n) is divergent" $\Leftrightarrow$ "$\forall a\in\mathbb{R}$, $\exists\, \varepsilon>0$ st. $\forall N>0$, $\exists\, n>N$ st. $|a_n-a|\ge\varepsilon$"

Ex: they're counter statement.

Recall: (a_n) bounded seq. in $\mathbb{R}$,

$$I_N := \inf\{a_n : n>N\},\quad S_N := \sup\{a_n : n>N\}.$$

$$\liminf_{n\to\infty} a_n := \lim_{N\to\infty} I_N,\quad \limsup_{n\to\infty} a_n := \lim_{N\to\infty} S_N$$

$$I_1 \le I_2 \le \cdots \le I_N \le \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n \le S_N \le \cdots \le S_1 \quad \forall N.$$

Rmk: (a_n) may not converge, (i.e. $\lim a_n$ may not exist) but $\liminf a_n$, $\limsup a_n$ always exist.

$(a_n) = (-1, 1, -1, 1, -1, 1, -1, \cdots)$
$\liminf a_n = -1$, $\limsup a_n = 1$, (a_n) div.

Thm (a_n) is bounded.

$$\lim a_n \text{ exists} \Leftrightarrow \liminf a_n = \limsup a_n.$$

In this case, $\liminf a_n = \limsup a_n = \lim a_n$.

pf: ($\Rightarrow$) Suppose $\lim a_n = a \in \mathbb{R}$ exists.

$\forall \varepsilon > 0$, $\exists N > 0$ s.t.

$$|a_n - a| < \varepsilon \quad \forall n > N.$$

$$\Updownarrow$$

$$a - \varepsilon < a_n < a + \varepsilon \quad \forall n > N.$$

$I_N = \inf\{a_n : n > N\} \geq a - \varepsilon$ since $a_n > a - \varepsilon \ \forall n > N$

$S_N = \sup\{a_n : n > N\} \leq a + \varepsilon$ since $a_n < a + \varepsilon \ \forall n > N$

$$a - \varepsilon \leq I_N \leq \liminf a_n \leq \limsup a_n \leq S_N \leq a + \varepsilon$$

$$\Rightarrow \quad \boxed{a - \varepsilon \leq \liminf a_n \leq \limsup a_n \leq a + \varepsilon} \ \forall \varepsilon > 0$$

$$\Rightarrow \quad a \leq \liminf a_n \leq \limsup a_n \leq a$$

$$\Rightarrow \quad \liminf a_n = \limsup a_n = \underline{a} = \lim a_n \qquad \square$$

($\Leftarrow$) $\underline{\boxed{\liminf a_n} = \boxed{\limsup a_n}} = a$

$$\boxed{\liminf a_n} = \boxed{\lim_{N \to \infty} I_N}, \qquad \limsup a_n = \underline{\lim_{N \to \infty} S_N}$$

$\forall \varepsilon > 0$,

$\exists N_1 > 0$ s.t. $|I_N - a| < \varepsilon \quad \forall N > N_1$

$\exists N_2 > 0$ s.t. $\boxed{|S_N - a| < \varepsilon \quad \forall N > N_2}$

If $N > N_1$, then

$$\left|\inf\{a_n : n > N\} - a\right| < \varepsilon$$

$$\Rightarrow \quad \underbrace{\inf\{a_n : n > N\}}_{a_n \,\geq\, \cdot \ \ \forall n > N} > a - \varepsilon$$

If $N > N_2$, then

$$\left|\sup\{a_n : n > N\} - a\right| < \varepsilon$$

$$\Rightarrow \quad \underbrace{\sup\{a_n : n > N\}}_{a_n \,\leq\, \cdot \ \ \forall n > N} < a + \varepsilon$$

Choose any $N > \max\{N_1, N_2\}$, then

- $a_n > a - \varepsilon \quad \forall n > N$
- $a_n < a + \varepsilon \quad \forall n > N$

$$\Rightarrow \quad |a_n - a| < \varepsilon \quad \forall n > N$$

$$\Rightarrow \quad \lim a_n = a. \quad \square$$

Rmk: "$\liminf a_n = \limsup a_n$" intuitively means:

when N larger

elements in $\boxed{\{a_n : n > N\}}$ are close to each other

$I_N \qquad S_N$

Another proof: $\boxed{I_N} \leq \boxed{a_{N+1}} \leq \boxed{S_N}$

$I_N = \inf\{a_n : n > N\}$, $\quad S_N = \sup\{a_n : n > N\}$

Since $\lim I_N = \liminf a_n = a$

$$\lim S_N = \limsup a_n = a$$

By squeeze lemma, $\Longrightarrow$ $\lim a_{N+1}$ exists $= a$. $\square$

Def. (a_n) is called a <u>Cauchy seq.</u> If $\exists N > 0$

s.t. $$|a_n - a_m| < \varepsilon \quad \forall\, n, m > N.$$

Rmk: Cauchy condition guarantees the existence of the limit (we'll show that Cauchy $\Leftrightarrow$ Convergent), without knowing what the limit is!!

Thm: (a_n) Cauchy $\Leftrightarrow$ (a_n) convergent.

pf: $(\Leftarrow)$ $\lim a_n = a$, so $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $|a_n - a| < \varepsilon/2 \quad \forall n > N$.

So, $n, m > N$, we have: $|a_n - a_m| \leq |a_n - a| + |a_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$

Augustin-Louis Cauchy

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"Cauchy" redirects here. For the lunar crater, see Cauchy (crater). For the statistical distribution, see Cauchy distribution. For the condition on sequences, see Cauchy sequence.

Baron **Augustin-Louis Cauchy** FRS FRSE (/koʊˈʃiː/;[1] French: [ogystɛ̃ lwi koʃi]; 21 August 1789 – 23 May 1857) was a French mathematician, engineer, and physicist who made pioneering contributions to several branches of mathematics, including mathematical analysis and continuum mechanics. He was one of the first to state and rigorously prove theorems of calculus, rejecting the heuristic principle of the generality of algebra of earlier authors. He almost singlehandedly founded complex analysis and the study of permutation groups in abstract algebra.

A profound mathematician, Cauchy had a great influence over his contemporaries and successors;[2] Hans Freudenthal stated: "More concepts and theorems have been named for Cauchy than for any other mathematician (in elasticity alone there are sixteen concepts and theorems named for Cauchy)."[3] Cauchy was a prolific writer; he wrote approximately eight hundred research articles and five complete textbooks on a variety of topics in the fields of mathematics and mathematical physics.

Augustin-Louis Cauchy

Cauchy around 1840. Lithography by Zéphirin Belliard after a painting by Jean Roller.

Born	21 August 1789 Paris, France
Died	23 May 1857 (aged 67) Sceaux, France
Nationality	French

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($\Rightarrow$) (a_n) Cauchy (want to prove: convergent)

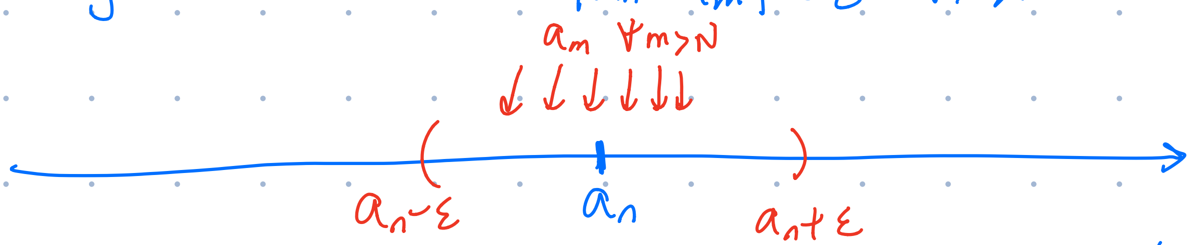
→ ① (a_n) bdd

→ ② $\liminf a_n = \limsup a_n$ ($\longrightarrow$ (a_n) convergent)

Lemma: (a_n) Cauchy $\Rightarrow$ (a_n) bounded.

Idea: $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $|a_n - a_m| < \varepsilon$ $\forall n, m > N$.

Choose any $n > N$, then $|a_n - a_m| < \varepsilon$ $\forall m > N$



Then follow the same as the proof of "convergent $\Rightarrow$ bounded".

Lemma: (a_n) Cauchy $\Rightarrow$ $\liminf a_n = \limsup a_n$.

pf: $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $|a_n - a_m| < \varepsilon \quad \forall n, m > N$.

$$\Longrightarrow \quad a_n < a_m + \varepsilon \quad \forall n, m > N.$$

Fix $n > N$ for now,

$$a_n < a_m + \varepsilon \quad \forall m > N.$$

$$\Rightarrow \quad a_n \leq \underbrace{\inf\{a_m : m > N\}}_{= I_N} + \varepsilon$$

So:

$$a_n \leq I_N + \varepsilon \quad \forall n > N.$$

$$\Rightarrow \quad \underbrace{\sup\{a_n : n > N\}}_{= S_N} \leq I_N + \varepsilon$$

So $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $\underset{\limsup a_n \leq}{S_N} \leq \underset{\leq \liminf a_n + \varepsilon}{I_N + \varepsilon}$.

$$\Longrightarrow \quad \limsup a_n \leq \liminf a_n + \varepsilon \quad \forall \varepsilon > 0$$

$$\Longrightarrow \quad \limsup a_n \le \liminf a_n$$

$$\rotatebox{90}{$\le$}$$

$$\liminf a_n$$

$$\Longrightarrow \quad \liminf a_n = \limsup a_n. \qquad \square$$

Def A <u>subsequence</u> of (a_n) is a sequence $(a_{\underline{k_1}}, a_{\underline{k_2}}, a_{\underline{k_3}}, \dots)$ s.t. $1 \le k_1 < k_2 < k_3 < \cdots$

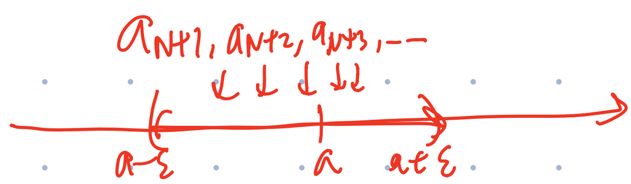
infinitely many

<u>e.g.</u> $(a_1, a_3, a_5, a_7, \dots)$

$(a_2, a_4, a_6, \dots)$

<u>Q</u>: Suppose (a_n) is convergent, $\lim a_n = \underline{a}$

Is it true that any subseq. of (a_n) is convergent ??



<u>Yes</u>. and they all conv. to a

$\forall \varepsilon > 0, \exists N > 0$ s.t. $|a_n - a| < \varepsilon \ \forall n > N$.

For any subseq. (a_{k_m}), $\exists M > 0$ s.t. $k_M > N$

Then for any $n > M$, we have $k_n > k_M > N$

$\Rightarrow \quad |a_{k_n} - a| < \varepsilon \quad \forall n > M.$

Rmk:

$(a_{2n-1}), (a_{2n})$ conv. $\not\Rightarrow$ (a_n) conv.

$(-1, 1, -1, 1, -1, \ldots)$

$\lim a_{2n-1} = \boxed{-1}$, $\lim a_{2n} = \boxed{1}$, (a_n) div.

$\lim a_{2n-1} = \liminf a_n$, $\lim a_{2n} = \limsup a_n$

<u>Bolzano-Weierstrass thm:</u> Any bounded sequence has a convergent subsequence.