Math H54 Second Midterm November 7, 2019

Name: ____

- You have 80 minutes to complete the exam.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total	100	

1. (4 points each) Determine if each statement is TRUE or FALSE, and justify your answer.

   (a) If A and B are similar matrices, then they have the same eigenvectors.

   (b) Any diagonalizable matrix is orthogonally diagonalizable.

   (c) If A is an $n \times n$ symmetric matrix, then $\langle A\vec{v}, \vec{w}\rangle = \langle \vec{v}, A\vec{w}\rangle$ for any $\vec{v}, \vec{w} \in \mathbb{R}^n$.

   (d) If A is an $n \times n$ orthogonal matrix, then $\langle A\vec{v}, A\vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle$ for any $\vec{v}, \vec{w} \in \mathbb{R}^n$.

   (e) $||\vec{v} + \vec{w}||^2 + ||\vec{v} - \vec{w}||^2 = 2||\vec{v}||^2 + 2||\vec{w}||^2$ holds for any $\vec{v}, \vec{w}$ in any inner product space.

2. (10 points) Let

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 2 \end{pmatrix}.$$

(a) Find all eigenvalues of A.

(Part (b) and (c) are on the next page.)

(b) Find a basis for the eigenspace associated to each eigenvalue.

(c) Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ (you don't need to compute P^{-1}). If not, explain the reason.

3. (10 points) Consider the quadratic form on $\mathbb{R}^3$

$$Q(\vec{x}) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.$$

(a) Find a symmetric 3×3 matrix A such that $Q(\vec{x}) = \vec{x}^T A\vec{x}$ for any $\vec{x} \in \mathbb{R}^3$.

(b) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.
(Part (c) is on the next page.)

(c) Is Q positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite? Give a brief explanation.

4. (10 points) Find the QR decomposition of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In other words, find a matrix Q with orthonormal columns and an upper triangular square matrix R with positive entries on its diagonal, so that $A = QR$.

5. (10 points) Consider the inner product space $\mathcal{C}([0,1])$ of continuous functions on $[0,1]$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

(a) For which values of real constants α, β, γ are the polynomials 1, $x+\alpha$, and $x^2+\beta x+\gamma$ pairwise orthogonal? (Recall that f and g are orthogonal if $\langle f, g\rangle = 0$.) (Recall that the antiderivative of x^n is $\frac{x^{n+1}}{n+1}$.)
(Part (b) is on the next page.)

(b) Describe how you would construct an orthonormal basis for the subspace $\mathbb{P}_2$ of polynomials of degree at most two, using the result of Part (a). You don't need to do calculations for this problem.

6. (10 points) Let $(V, \langle -, - \rangle)$ be an inner product space, and let $T : V \to V$ be a linear transformation. Suppose that $||T(\vec{x})|| = ||\vec{x}||$ for every $\vec{x} \in V$. Prove that

$$\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$$

holds for any $\vec{x}, \vec{y} \in V$.
(Hint: Consider $||T(\vec{x} + \vec{y})|| = ||\vec{x} + \vec{y}||$.)

7. (10 points) Show that the quadratic form $Q(\vec{x}) = \vec{x}^T M \vec{x}$ associated to a symmetric matrix M is positive definitie if and only if $M = A^T A$ for some A with linearly independent columns.
(Hint 1: For the '$\Longleftarrow$' direction, consider $\langle A\vec{x}, A\vec{x} \rangle$.)
(Hint 2: For the '$\Longrightarrow$' direction, consider the 'square root' of a diagonal matrix.)

8. We say an $n \times n$ matrix P is a *projection matrix* if it satisfies $P^2 = P$. In this problem, you will prove that projection matrices are diagonalizable.
   (a) (5 points) Show that the column space of P coincides with the eigenspace of P associated with eigenvalue 1, i.e. $\mathrm{Col}(P) = \mathrm{Nul}(P - I)$.
   (Part (b) is on the next page.)

(b) (5 points) Show that P is diagonalizable via the rank theorem.
(Rank theorem: $\dim \mathrm{Col}(P) + \dim \mathrm{Nul}(P) = n$.)
(Hint: Suppose $0 \neq \vec{v} \in \mathrm{Nul}(P)$. Is v an eigenvector of P?)

9. Let $M_{2\times 2}(\mathbb{R})$ be the set of all 2×2 real matrices. It is a vector space with the standard matrix addition and scalar multiplication.
   (a) (5 points) Consider the function $\langle -, - \rangle : M_{2\times 2}(\mathbb{R}) \times M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by:
   $$\langle A, B \rangle := \operatorname{tr}(AB^T).$$
   Here $A, B \in M_{2\times 2}(\mathbb{R})$ and tr denotes the trace of a matrix (sum of diagonal entries).
   Prove that the function $\langle -, - \rangle$ defines an inner product on the vector space $M_{2\times 2}(\mathbb{R})$.
   (Part (b) is on the next page.)

(b) (5 points) Construct an orthonormal basis (with respect to this inner product) for the subspace of $M_{2\times 2}(\mathbb{R})$ spanned by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.