§ Geometric meaning of determinants.

Thm: Suppose A: invertible.

- A: 2×2, the area of the parallegram spanned by the rows of A is $|\det(A)|$.

- A: 3×3, the volume of the parallpiped spaned by the rows of A is $|\det(A)|$

Rmk: The same statements hold for columns, since $\det(A) = \det(A^T)$

pf: • Clear if A is 1×1 matrix

• prove by induction:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & \end{bmatrix}, \quad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots \\ a_{12} & \ddots & & \\ a_{13} & & & \\ \vdots & & & \end{bmatrix}$$

$\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots$ $\qquad$ $\det A^T = a_{11} C'_{11} + a_{12} C'_{12} + \cdots$

$$(-1)^{1+1} \det \begin{bmatrix} a_{22} & a_{23} & a_{24} & \cdots \\ a_{32} & \ddots & & \\ a_{42} & & & \\ \vdots & & & \end{bmatrix} \qquad (-1)^{1+1} \det \begin{bmatrix} a_{22} & a_{32} & a_{42} & \cdots \\ a_{23} & & & \\ a_{24} & & & \\ \vdots & & & \end{bmatrix}$$

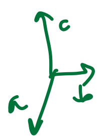
transpose to each other, of size $(n-1)\times(n-1)$

pf: $\underset{3\times 3}{A} \xrightarrow[\text{row operation}]{\text{elementary}} I_3$

$$\begin{bmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \begin{bmatrix} a & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

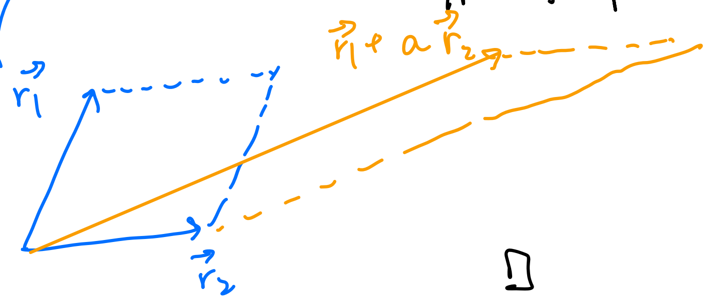
If we only use these 2 types of row operations, then we can reduce A to a diagonal matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$



Claim: • these two types of operations do not change the volume of the parallepiped spanned by the rows.

• these two types of operations do not change $|\det(A)|$.



$\square$

Cramer's rule:

A: $n\times n$ invertible, $\forall \vec{b}\in\mathbb{R}^n$, $A\vec{x}=\vec{b}$ has a unique sol^{n}.

given by:

$$x_i = \frac{\det A_i(\vec{b})}{\det A} \qquad \text{(}i\text{-th entry of the sol}^{n}\ \vec{x}\text{)}$$

where

$$A_i(\vec{b}) = \begin{bmatrix} | & & | & \boxed{|} & | & & | \\ \vec{a}_1 & \cdots & \vec{a}_{i-1} & \boxed{\vec{b}} & \vec{a}_{i+1} & \cdots & \vec{a}_n \\ | & & | & \boxed{|} & | & & | \end{bmatrix}$$

columns of A

pf: Consider

$$I_i(\vec{x}) = \begin{bmatrix} 1 & & & x_1 & & & \\ & \ddots & & \vdots & & & \\ & & 1 & & & & \\ & & & x_i & & & \\ & & & & 1 & & \\ & & & \vdots & & \ddots & \\ & & & x_n & & & 1 \end{bmatrix}, \qquad \det I_i(\vec{x}) = x_i$$

$$A\, I_i(x) = A \begin{bmatrix} 1 & & & x_1 & & & \\ & \ddots & & \vdots & & & \\ & & 1 & & & & \\ & & & x_i & & & \\ & & & & 1 & & \\ & & & \vdots & & \ddots & \\ & & & x_n & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} | & & | & | & | & & | \\ \vec{a}_1 & \cdots & \vec{a}_{i-1} & \vec{b} & \vec{a}_{i+1} & \cdots & \vec{a}_n \\ | & & | & | & | & & | \end{bmatrix} = A_i(\vec{b})$$

$$\det A_i(\vec{b}) = \det\left(A\, I_i(\vec{x})\right)$$

$$= \det(A)\,\det\left(I_i(\vec{x})\right)$$

$$= \det(A)\cdot x_i$$

$$\Rightarrow \quad x_i = \frac{\det A_i(\vec{b})}{\det(A)}. \qquad \square$$

Inverse formula: A invertible.

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots \\ C_{12} & & \ddots & \\ C_{13} & & & \ddots \\ \vdots & & & \end{pmatrix}$$

$\uparrow$

cofactors of A:

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

pf:

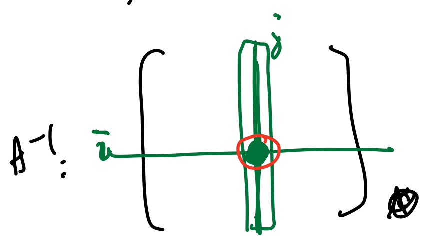
$$A A^{-1} = I = \begin{bmatrix} | & & | \\ \vec{e}_1 & \cdots & \vec{e}_m \\ | & & | \end{bmatrix}$$

$\Rightarrow$ the 1^{st} column of A^{-1} is the sol^{n} to $A\vec{x} = \vec{e}_1$.

the i^{th} column of A^{-1} ———— $A\vec{x} = \vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

So we can write down each column of A^{-1} by Cramer's rule.

Let's write down (i,j)-th entry of A^{-1}:



We should look at the $\boxed{j\text{-th column of } A^{-1}}$, and look at it's i-th entry

$\|$ the sol^{n} to $A\vec{x} = \vec{e}_j$.

by Cramer's rule,

$$\frac{\det A_i(\vec{e}_j)}{\det A} = \frac{C_{ji}}{\det(A)} \quad \square$$

$$A_i(\vec{e}_j) = \begin{bmatrix} a_{11} & & a_{1\,i-1} & 0 & a_{1\,i+1} & \\ \vdots & \cdots & | & \vdots & \vdots & \cdots \\ & & & 1 & & \\ & & & \vdots & & \\ a_{n1} & & a_{n\,i-1} & 0 & a_{n\,i+1} & \end{bmatrix}$$

located at the j-th row & i-th column

$$\det A_i(\vec{e}_j) = (-1)^{i+j} \det \left[A_{ji} \right] = C_{ji}$$

2x2 example: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc \neq 0$.

we know the inverse

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

§ Vector spaces

Def: A <u>vector space</u> is a set V (whose elements are called <u>vectors</u>) with two operations: <u>addition</u> and <u>scalar multiplication</u>, s.t.:

1) closed under operations:

$$\forall \vec{v}_1, \vec{v}_2 \in V, \ c \in \mathbb{R}, \text{ we have } \vec{v}_1 + \vec{v}_2 \in V, \quad c\vec{v}_1 \in V.$$

2) commutativity & associativity of addition:

$$\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1, \quad (\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$$

3) $\exists$ additive identity

$$\exists \vec{0} \in V \text{ s.t. } \vec{v} + \vec{0} = \vec{v} \quad \forall \vec{v} \in V.$$

4) $\exists$ additive inverse

$$\forall \vec{v} \in V, \ \exists \vec{w} \in V \text{ s.t. } \vec{v} + \vec{w} = \vec{0}.$$

5) compatibility w/ scalar multiplicat°.

$$c(\vec{v}_1 + \vec{v}_2) = c\vec{v}_1 + c\vec{v}_2, \quad (c_1 + c_2)\vec{v} = c_1\vec{v} + c_2\vec{v},$$

$$c_1(c_2\vec{v}) = (c_1 c_2)\vec{v}, \quad 1 \cdot \vec{v} = \vec{v}.$$

"$A \subseteq B$" "$A \subset B$" A is a subset of B

"$A \in B$" A is an element in the set B

Examples of vector spaces:

- $\mathbb{R}^n$, w/ standard addition & scalar mult°.
- $Poly = \{$polynomials with real coefficients$\}$

  $\qquad = \{a_0 + a_1 x + \cdots + a_n x^n$ for some $a_0, \ldots, a_n \in \mathbb{R}\}$
- $Poly_{\leq n} = \{$polynomials with degree $\leq n\}$
- $M_{m \times n}(\mathbb{R}) = \{$matrices of size $m \times n\}$
- $\{0\}$

- S: set, ~~Consider~~ $\{f: S \to \mathbb{R}\}$ is a vector space, where for $f_1, f_2: S \to \mathbb{R}$,

$$f_1 + f_2 : S \longrightarrow \mathbb{R}$$
$$x \longmapsto f_1(x) + f_2(x)$$

- $\mathcal{C}(\mathbb{R}) = \{\text{continuous functions } \mathbb{R} \to \mathbb{R}\}$

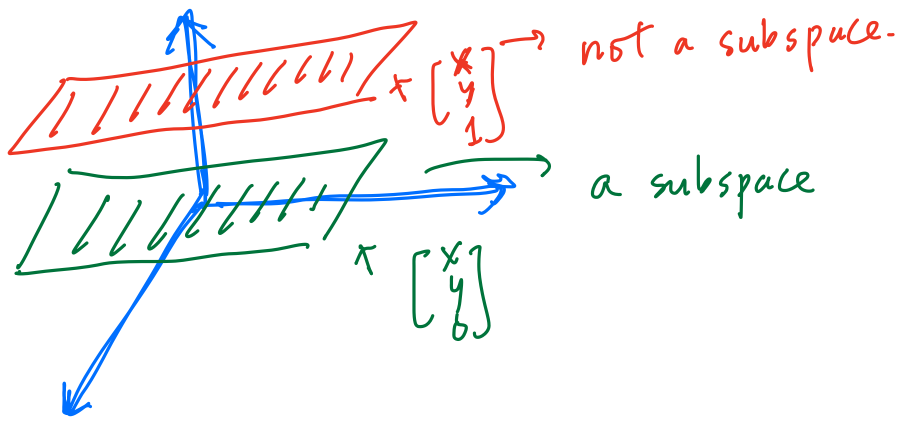
  $\mathcal{C}([0,1]) = \{\text{continuous functions } [0,1] \to \mathbb{R}\}$

Def: A subspace of a vector space (v.s.) V is a subset $H \subseteq V$ s.t.

- $\vec{0} \in H$
- $\forall\, \vec{v}_1, \vec{v}_2 \in H,\ c \in \mathbb{R}$, we have $\vec{v}_1 + \vec{v}_2,\ c\vec{v}_1 \in H$

Ex: A subspace of a v.s. is also a vector space.

e.g. $\mathbb{R}^3$



e.g. $\mathrm{Poly}_{\le n} \subseteq \mathrm{Poly}$

e.g. $\{\vec{0}\} \subseteq V$, $\quad V \subseteq V$

Def $\vec{v}_1, \dots, \vec{v}_k \in V$ ← v.s.

Ex: this is a subspace of V.

$$\mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_k\} := \{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}$$

e.g. $A: m\times n = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$

<u>null space</u> $\mathrm{Nul}(A) := \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} \subseteq \mathbb{R}^n$

<u>column space</u> $\mathrm{Col}(A) := \mathrm{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} \subseteq \mathbb{R}^m$

<u>Ex</u>: $\mathrm{Nul}(A) \subseteq \mathbb{R}^n$, and $\mathrm{Col}(A) \subseteq \mathbb{R}^m$ are subspaces

<u>Rmk</u>: T_A inj. $\Leftrightarrow$ $\mathrm{Nul}(A) = \{\vec{0}\}$.

T_A surj. $\Leftrightarrow$ $\mathrm{Col}(A) = \mathbb{R}^m$.

<u>Def</u>: V, W v.s.

A function $T: V \to W$ is a <u>linear transformation</u>

if

- $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) \quad \forall\, \vec{v}_1, \vec{v}_2 \in V$
- $T(c\vec{v}) = cT(\vec{v}) \quad \forall\, c \in \mathbb{R},\ \vec{v} \in V.$

<u>e.g.</u> $T: \mathrm{Poly} \longrightarrow \mathrm{Poly}$ is linear.

$$a_0 + a_1 x + \cdots + a_n x^n \longmapsto a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}$$

<u>Def</u>: <u>kernel</u> of T: $\mathrm{Ker}(T) := \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\} \subseteq V$

<u>Image/range</u> of T: $\mathrm{Im}(T) = T(V)$

$$:= \{\vec{w} \in W \mid \exists\, \vec{v} \in V \text{ s.t. } T(\vec{v}) = \vec{w}\} \subseteq W$$

<u>Ex</u>: Kernel & image are <u>subspaces</u> in V & W (HW3 Q7 is a generalization of this)