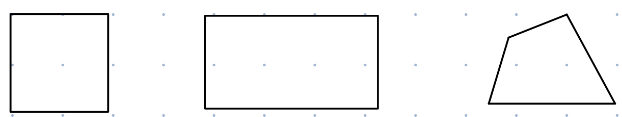
ALGEBRAIC COMBINATORICS II, SUMMER 2024

1. Overview of the course

Lecture 1

We will explore the *symmetries* of various *geometric spaces* in this course. The spaces that we will consider include: the Euclidean spaces $\mathbb{R}^2$, $\mathbb{R}^3$, the spheres S^1, S^2, the hyperbolic space $\mathbb{H}^2$, and some of their interesting subsets.

Question 1.1. *Which of the following shapes is more "symmetric"?*



Question 1.2. *How to define "symmetries"?*

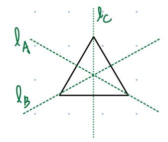
Each of the geometric spaces that we will consider ($\mathbb{R}^2$, $\mathbb{R}^3$, S^1, S^2, $\mathbb{H}^2$, etc.) has a natural metric (i.e. distance $d(x, y)$ between any two points x, y). The symmetries that we are interested in are the *isometries* (i.e. distance-preserving functions) of these spaces. For instance, an isometry of $\mathbb{R}^2$ is a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $d(f(x), f(y)) = d(x, y)$ for any $x, y \in \mathbb{R}^2$.

Definition 1.3. Let $S \subseteq \mathbb{R}^2$ be a subset of $\mathbb{R}^2$. An isometry $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called a *symmetry of* S if we have $f(S) = S$, i.e.

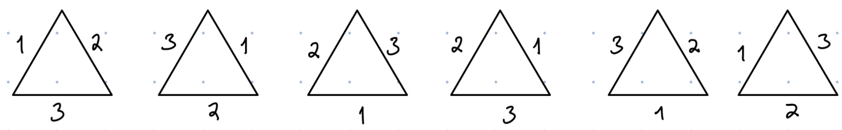
- for any $p \in S$, we have $f(p) \in S$; and
- for any $q \in S$, there exists $p \in S$ such that $f(p) = q$.

Example. Let us look at an easy example: an equilateral triangle. It has two kinds of symmetries:

- Rotational symmetries: one can rotate the triangle by $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, or 2π without changing its appearance.
- Reflection symmetries: there are three "mirror lines" through which we can reflect the shape without changing its appearance.

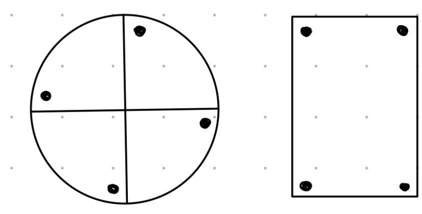


The easiest way to study the symmetries of a shape is by *counting*. In this example, it's easy to check that there are 6 symmetries. If we put labels on the edges of the triangle, then the effect of these symmetries look like:



However, counting alone is usually not good enough.

Example. Both of the following shapes have 4 symmetries. The shape on the



left has 4 rotational symmetries (by $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π), but no reflection symmetries. In contrast, the shape on the right has 2 rotational symmetries and 2 reflection symmetries. How can we distinguish them?

As we'll see later in this course, *group theory* provides rigorous tools to describe the symmetries of shapes. For any shape (or any geometric object), the set of its symmetries has a natural *group structure*. In the example above, although the sets of symmetries of both shapes have 4 elements, but their underlying group structures are different, and that's how we can tell them apart (e.g. consider the *orders* of elements in these two groups).

Another important tool that we will encounter is basic *linear algebra*, in particular *matrices* or *matrix groups*. The reason is that certain matrix groups $(\mathrm{O}(2,\mathbb{R}),\ \mathrm{O}(3,\mathbb{R}),\ \mathrm{SL}(2,\mathbb{R}),\ \mathrm{SL}(2,\mathbb{C}),\ \text{etc.})$ act naturally as isometries on the spaces that we are interested in like $\mathbb{R}^2$, $\mathbb{R}^3$, S^1, S^2, $\mathbb{H}^2$. For instance, you'll show in the homework that any isometry of the Euclidean space $\mathbb{R}^n$ is a composition of a translation and a linear transformation.

2. A crash course on basic group theory

2.1. **Binary operators.** Before discussing the actual definition of a *group*, let us first consider a more general notion of *binary operators*.

Definition 2.1. Let S be a set. A *binary operator* on S is a function

$$\circ\colon S \times S \to S.$$

Example. Addition on the set of positive integers (denoted by $\mathbb{N}$), or the set of integers (denoted by $\mathbb{Z}$), or the set of rational numbers (denoted by $\mathbb{Q}$) or the set of real numbers (denoted by $\mathbb{R}$), is a binary operator. Same for multiplication.

Non-example. Subtraction on the set of positive integers is *not* a binary operator. Division on the set of integers is *not* a binary operator.

Definition 2.2. Let $(S, \circ)$ be a set with a binary operator. We say an element $e \in S$ is an *identity element* if $e \circ a = a \circ e = a$ for any $a \in S$.

Example. The element $0 \in \mathbb{Z}$ is an identity element of $(\mathbb{Z}, +)$. The element $1 \in \mathbb{Z}$ is an identity element of $(\mathbb{Z}, \times)$.

Non-example. $(\mathbb{N}, +)$ has no identity element.

Exercise. Prove that any set with a binary operator $(S, \circ)$ has at most one identity element.

Definition 2.3. Let $(S, \circ, e)$ be a set with a binary operator and an identity element. We say an element $a' \in S$ is an *inverse* of $a \in S$ if $a \circ a' = a' \circ a = e$.

Example. For $(\mathbb{Z}, +)$, the inverse of $a \in \mathbb{Z}$ is given by $-a$. For $(\mathbb{R}, \times)$, the inverse of $a \in \mathbb{R}$ is given by $1/a$, provided that $a \neq 0$.

Non-example. For $(\mathbb{Z}, \times)$, any element $a \in \mathbb{Z}$ has no inverse unless $a = \pm 1$.

Definition 2.4. Let $(S, \circ)$ be a set with a binary operator. We say $(S, \circ)$ is *associative* if $(a \circ b) \circ c = a \circ (b \circ c)$ holds for any $a, b, c \in S$.

Exercise. Let $(S, \circ, e)$ be a set with an associative binary operator and an identity element. Prove that any element in S has at most one inverse.

Most of the examples that we'll be discussing are associative. Here is a non-example (which we will not encounter in this course):

Non-example. The cross product $\times$ on $\mathbb{R}^3$ is *not* associative. Rather, it satisfies the *Jacobi identity*

$$\vec{v}_1 \times (\vec{v}_2 \times \vec{v}_3) + \vec{v}_2 \times (\vec{v}_3 \times \vec{v}_1) + \vec{v}_3 \times (\vec{v}_1 \times \vec{v}_2) = 0$$

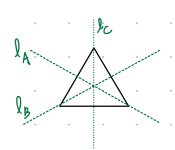
Definition 2.5. Let $(S, \circ)$ be a set with a binary operator. We say $(S, \circ)$ is *commutative* if $a \circ b = b \circ a$ for any $a, b \in S$.

Warning. Many of the examples that we'll consider are *not* commutative.

Non-example. Consider the set of all six geometric transformations that give the symmetries of an equilateral triangle:

$$S = \left\{\text{rotate } 0, \text{ rotate } \frac{2\pi}{3}, \text{ rotate } \frac{4\pi}{3}, \text{ reflect along } \ell_A, \text{ reflect along } \ell_B, \text{ reflect along } \ell_C\right\}.$$

(note: rotations are typically assumed to be counterclockwise)

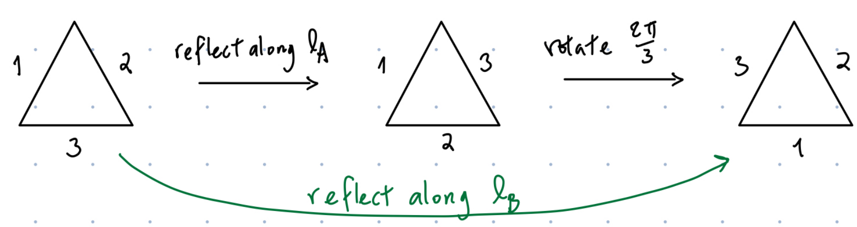


There is a binary operation on S given by *composing* these geometric transformations:

$$\circ \colon S \times S \to S,$$

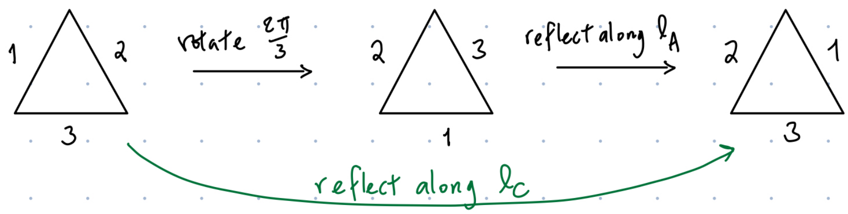
where $a \circ b \in S$ is the transformation given by "do b, and then do a". For instance, we have

$$\left(\text{rotate } \frac{2\pi}{3}\right) \circ \left(\text{reflect along } \ell_A\right) = \text{reflect along } \ell_B.$$



On the other hand, by reversing the order one gets

$$\Big(\text{reflect along } \ell_A\Big) \circ \Big(\text{rotate } \frac{2\pi}{3}\Big) = \text{reflect along } \ell_C.$$



This shows that $(S, \circ)$ is *not* commutative.

Non-example. Another important class of groups that we will discuss is the *matrix groups*. They are *not* commutative in most cases.

2.2. **Groups.**

Definition 2.6. Let $(G, \circ)$ be a set with a binary operator. It is called a *group* if it satisfies the following conditions:

(1) It is associative.
(2) It has the identity element (which will usually be denoted by e, e_G, 1, or 1_G).
(3) Any element $a \in G$ has an inverse (which will be denoted by $a^{-1} \in G$).

Remark 2.7. Here are some notions that we will be using frequently:

- If a group $(G, \circ)$ is commutative, then it is called an *abelian group*.
- We'll use $|G|$ to denote the number of elements in the set G, and will call it the *order* of G. Note that the order of a group could be infinite in general.
- We quite often would omit "$\circ$", and simply denote $a \circ b$ by ab, denote $a \circ a$ by a^2, denote $a \circ a \circ a$ by a^3, and so on.

Example. Consider the set of integers modulo n

$$\mathbb{Z}/n\mathbb{Z} := \Big\{\bar{0}, \bar{1}, \ldots, \overline{n-1}\Big\}.$$

Addition and multiplication are well-defined on $\mathbb{Z}/n\mathbb{Z}$. It's not hard to show that $(\mathbb{Z}/n\mathbb{Z}, +)$ is an abelian group of order n, with the identity given by $\bar{0}$.

Example. Consider the subset of $\mathbb{Z}/n\mathbb{Z}$ consisting of elements that are coprime with n:

$$(\mathbb{Z}/n\mathbb{Z})^* := \Big\{\overline{m} \in \mathbb{Z}/n\mathbb{Z}\colon \gcd(m, n) = 1\Big\}.$$

It's not hard to show that $((\mathbb{Z}/n\mathbb{Z})^*, \times)$ is an abelian group, with the identity given by $\bar{1}$.

Example. The set of all integers $\mathbb{Z}$ under addition is an example of an abelian group with infinite order.

Example. The set $\{0\}$ under addition is an example of a group with only one element (a trivial group).

Example. Let G_1 and G_2 be two groups. Consider the set

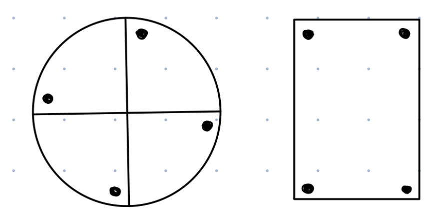
$$G_1 \times G_2 := \Big\{(g_1, g_2)\colon g_1 \in G_1 \text{ and } g_2 \in G_2\Big\}.$$

Define a binary operator on $G_1 \times G_2$ as follows:

$$(g_1, g_2) \circ (g_1', g_2') := (g_1 \circ g_1', g_2 \circ g_2').$$

It's not hard to show that $(G_1 \times G_2, \circ)$ is also a group. It's called the *direct product* of G_1 and G_2.

Example. Let's come back to the following examples again. As discussed ear-



lier, the symmetries of a shape form a group, where the binary operation is given by composition. The symmetry group of the first shape is

$$G_1 := \Big\{\text{rotate } 0,\ \text{rotate } \frac{\pi}{2},\ \text{rotate } \pi,\ \text{rotate } \frac{3\pi}{2}\Big\}.$$

One thing we might notice about this group is that all elements of the group can be obtained by taking one element of the set, and combining it different number of times. Let's denote rotate $\frac{\pi}{2}$ by a. Then G_1 can be rewritten as

$$G_1 = \{e, a, a^2, a^3\}.$$

Notice that $a^4 = e$ since rotate 2π is the same as rotate 0, i.e. the identity map. The same is true for $\mathbb{Z}/4\mathbb{Z}$ (under addition) if one lets $a = \bar{1}$ and note that $a^4 = \bar{4} = \bar{0} = e$ in $\mathbb{Z}/4\mathbb{Z}$. In fact, we'll see that the symmetry group of the first shape and $\mathbb{Z}/4\mathbb{Z}$ are *isomorphic*, which means that they are essentially the same group.

On the other hand, the symmetry group of the second shape is

$$G_2 := \Big\{\text{rotate } 0,\ \text{rotate } \pi,\ \text{reflect along } \ell_1,\ \text{reflect along } \ell_2\Big\}.$$

It's not hard to see that there is no element $a \in G_2$ such that $G_2 = \{e, a, a^2, a^3\}$. Therefore, G_2 and G_1 are not isomorphic. In fact, one can show that G_2 is isomorphic to the direct product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

2.3. **Homomorphisms.** For any mathematical structure (like groups), it is crucially important to understand how two structures of the same type (like two groups) are related in a meaningful way. Functions that bridge such two structures are called *homomorphisms*. (In the Ancient Greek language, "homos" means "same", and "morphe" means "form" or "shape".) In general, a homomorphism is a function between two mathematical structures of the same type, that preserves the operations of the structures.

Definition 2.8. Let G and H be two groups. A function $f\colon G \to H$ is called a *homomorphism* if for any $g_1, g_2 \in G$ we have

$$f(g_1 g_2) = f(g_1) f(g_2)$$

Furthermore, a homomorphism that is both injective and surjective is called an *isomorphism*. In this case, we'll use the notation "$G \cong H$".

In other words, a homomorphism is a function that is compatible with the binary operations on the two groups.

Exercise. Let $f\colon G \to H$ be a group homomorphism. Prove that

- It preserves the identity: $f(e_G) = e_H$.
- It preserves the inverses: $f(g^{-1}) = f(g)^{-1}$ for any $g \in G$.

Example. We considered the symmetry group

$$G_1 := \Big\{\text{rotate } 0,\ \text{rotate } \frac{\pi}{2},\ \text{rotate } \pi,\ \text{rotate } \frac{3\pi}{2}\Big\} = \{e, a, a^2, a^3\}$$

where $a^4 = e$. One can define a function

$$G_1 \to \mathbb{Z}/4\mathbb{Z}$$

by sending $e \mapsto \bar{0}$, $a \mapsto \bar{1}$, $a^2 \mapsto \bar{2}$, and $a^3 \mapsto \bar{3}$. It's an easy exercise to show that this function is an isomorphism.

Lecture 2

Remark 2.9. A convenient way to present a group is by choosing elements that *generate* the group (which means that any element of the group can be written as a product of some of these generators and their inverses), and a set of *relations* among these generators. For instance, $\mathbb{Z}/4\mathbb{Z}$ can be presented by

$$\mathbb{Z}/4\mathbb{Z} = \left\langle a\colon a^4 = e \right\rangle,$$

which means that one can find an element $a \in \mathbb{Z}/4\mathbb{Z}$ such that any element in $\mathbb{Z}/4\mathbb{Z}$ can be written as a power of a, and it satisfies $a^4 = e$ (it's not hard to see that a can be chosen to be $\bar{1}$ or $\bar{3}$ in this case).

Definition 2.10. A group G that can be generated by a single element g is called a *cyclic* group (i.e. any element of G is of the form g^k for some $k \in \mathbb{Z}$).

Definition 2.11. Let g be an element in a group G. If there exists a positive integer n such that $g^n = e$, then the smallest possible n satisfying $g^n = e$ is called the *order* of g. If such n does not exist, then we say g is of infinite order.

Exercise. Let G be a cyclic group, and say it can be generated by an element $g \in G$.

- If g is of finite order, say $\mathrm{order}(g) = n$. Prove that $G \cong \mathbb{Z}/n\mathbb{Z}$.
- If g is of infinite order, then prove that $G \cong \mathbb{Z}$.

Therefore, any cyclic group is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for some positive integer n.

Exercise. Prove that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is not a cyclic group.

Example. Let D_n be the symmetry group of a regular n-gon. It is not hard to show that D_n is generated by rotation by $2\pi/n$ (which we'll denote by r), and a reflection (which we'll denote by s). The group D_n is of order $2n$, with elements given by

$$D_n = \left\{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\right\}.$$

The generators r and s satisfy the relations $r^n = s^2 = 1$ and $s^{-1}rs = r^{-1}$.

$$\begin{aligned} D_n &= \left\langle r, s \mid r^n = s^2 = 1,\ s^{-1}rs = r^{-1} \right\rangle \\ &= \left\langle r, s \mid r^n = s^2 = (rs)^2 = 1 \right\rangle. \end{aligned}$$

Remark 2.12. Since D_n is not commutative, it is not isomorphic to the direct product $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. On the other hand, it is isomorphic to the *semi-direct product* of its order 2 *subgroup* $\langle s \rangle$ and its order n *normal subgroup* $\langle r \rangle$: $D_n \cong \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z}$. We'll introduce these notations later on.

2.4. **Subgroups.**

Definition 2.13. Let G be a group. We say a subset $H \subseteq G$ is a *subgroup* if:

(1) it is closed under the binary operation of G: for any $a, b \in H$, we have $ab \in H$;
(2) it contains the identity element of G: $e_G \in H$;
(3) it is closed under taking inverse: for any $a \in H$, we have $a^{-1} \in H$.

Exercise. A subgroup $H \subseteq G$ is itself a group, with the binary operator and the identity element inherit from G.

Example. For any group G, the subset $\{e_G\} \subseteq G$ is always a subgroup, called the *trivial* subgroup of G. Also, the group G itself is a subgroup of G.

Example. For any positive integer n, the subset $n\mathbb{Z} \subseteq (\mathbb{Z}, +)$ is a subgroup.

Theorem 2.14. *Let G be a finite subgroup of* $\mathrm{O}(2, \mathbb{R})$*. Then G is isomorphic to either a cyclic group or a dihedral group.*

Proof. Any element of $\mathrm{O}(2, \mathbb{R})$ acts naturally on the unit circle $S^1 \subseteq \mathbb{R}^2$. Let $g \in G$ be a non-identity element. It is not hard to show that g is either a rotation (when g does not fix any point of S^1), or a reflection (when g fixes at least a point of S^1).

First, suppose that all elements of G are rotations. Write $r_\theta \in \mathrm{O}(2, \mathbb{R})$ for the counterclockwise rotation by θ, where $0 \le \theta < 2\pi$. Choose $r_\phi \in G$ with the smallest positive ϕ (it is possible since G is finite). We claim that G is the cyclic group generated by r_ϕ. Let $r_\theta \in G$, and write $\theta = m\phi + \psi$ where $m \in \mathbb{N}$ and $0 \le \psi < \phi$. Then $r_\psi = (r_\phi)^{-m} r_\theta \in G$. Therefore $\psi = 0$ by the minimality of ϕ. Hence $r_\theta = (r_\phi)^m$.

Second, suppose G contains a reflection s. Let $H \subseteq G$ be the subgroup consisting of rotations (including the identity). By the first case, we have $H = \{1, r, \cdots, r^{n-1}\}$ for some positive integer n. Consider any other reflection $s' \in G$. One can show that the composition of any two reflections is a rotation, hence $ss' \in H$. So $s' = sr^k$ for some $0 \leq k \leq n-1$. This shows that

$$G = \{1, r, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\}.$$

It is easy to show that a rotation r and a reflection s satisfy the relation $sr = r^{-1}s$. Hence we get $G \cong D_n$. □

If H is a subgroup of G, then one can break G up into pieces, each of which looks like H. These pieces are called *cosets* of H, and they arise by "multiplying" H by elements of G.

Definition 2.15. Let G be a group and $H \subseteq G$ be a subgroup. A *left coset* of H in G is a subset of the form

$$gH = \{gh \mid h \in H\} \text{ for some } g \in G.$$

The element g is called a *representative* of the coset gH. The collection of all left cosets is denoted by G/H. Its order $|G/H|$ is called the *index* of H in G, and will sometimes be denoted by $[G : H]$.

Similarly, a *right coset* is a subset of the form

$$Hg = \{hg \mid h \in H\} \text{ for some } g \in G.$$

The collection of all right cosets is denoted by $H\backslash G$.

Example. Consider the subgroup $n\mathbb{Z} \subseteq (\mathbb{Z}, +)$. Since the group $(\mathbb{Z}, +)$ is abelian, its left cosets and right cosets are identical. It is clear that the subgroup has exactly n cosets $\bar{0}, \bar{1}, \ldots, \overline{n-1}$, where $\bar{i} = i + n\mathbb{Z}$ consists of integers $\equiv i$ modulo n. Hence $n\mathbb{Z} \subseteq \mathbb{Z}$ is a subgroup of index n.

Exercise. The representative of a coset is *not* unique. In fact, show that a coset gH can be represented by any element of the form gh where $h \in H$.

Proposition 2.16. *Let $H \subseteq G$ be a subgroup. Prove that for any two cosets aH and bH, we have:*

- *either aH and bH are disjoint: $aH \cap bH = \emptyset$,*
- *or aH and bH are exactly the same: $aH = bH$.*

Proof. Suppose aH and bH are not disjoint. Then there exists $h_1, h_2 \in H$ such that $ah_1 = bh_2$. For any $h \in H$, we have

$$ah = a(h_1 h_1^{-1})h = b(h_2 h_1^{-1} h) \in bH.$$

Hence $aH \subseteq bH$. Similarly, one can show that $bH \subseteq aH$. Therefore $aH = bH$. □

Theorem 2.17 (Lagrange)**.** *Let G be a finite group, and $H \subseteq G$ be a subgroup. Then $|G|$ is divisible by $|H|$. Moreover, we have $|G| = |H|[G : H]$.*

Proof. Since $g \in gH$, any element of G belongs to a left coset of H. Then the previous proposition shows that G is the disjoint union of the left cosets of H. Since each coset has exactly $|H|$ elements, we can conclude that $|G| = |H|[G : H]$. □

Lecture 3

Exercise. Consider the subgroup $\mathbb{Z} \subseteq (\mathbb{R}, +)$. The set of cosets $\mathbb{R}/\mathbb{Z}$ can be identified with S^1, the unit circle in $\mathbb{R}^2$: Points of the circle are of the form $e^{2\pi i\theta}$ where $\theta \in \mathbb{R}$. Show that the map $t \mapsto e^{2\pi it}$ gives a bijection between $\mathbb{R}/\mathbb{Z}$ and S^1.

Exercise. Let G be a finite group and g be an element of G. Prove that the order of g divides the order of G.

Remark 2.18*.* In the example $n\mathbb{Z} \subseteq (\mathbb{Z}, +)$, one can notice that the set of all cosets $\{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$ also has a natural group structure inherits from the group structure on $(\mathbb{Z}, +)$: one defines $\bar{i} + \bar{j}$ to be $\overline{i+j}$.

However, the set of all left cosets does *not* always admit a group structure! Let $H \subseteq G$ be a subgroup and $a, b \in G$ be two elements in G. It is tempting to define a group structure on G/H simply by declaring "$aH \circ bH = (ab)H$". In order for this definition to make sense, we need to show that, if a' is a representative of aH and b' is a representative of bH, then $a'b'H = abH$. This is equivalent to, for any $a, b \in G$ and $h_1, h_2 \in H$, one needs $ah_1bh_2H = abH$, or equivalently, $b^{-1}h_1b \in H$. This is equivalent to the condition that for any $g \in G$ one needs $gH = Hg$, i.e. the left and right cosets of H in G coincide, which is *not* true in general.

Definition 2.19. A subgroup $H \subseteq G$ is called *normal* if $gH = Hg$ for any $g \in G$.

By the previous remark, if $H \subseteq G$ is a normal subgroup, then the set of (left) cosets G/H admits a group structure inherit from G: let aH and bH be two cosets, then $aH \circ bH \coloneqq (ab)H$ gives a well-defined group structure on G/H. The resulting group G/H is called the *quotient group*.

Theorem 2.20 (First isomorphism theorem)**.** *Let $f\colon G \to H$ be a group homomorphism. Define*

$$\mathrm{Ker}(f) \coloneqq \{g \in G \mid f(g) = 1_H\} \subseteq G$$

and

$$\mathrm{Im}(f) \coloneqq \{h \in H \mid h = f(g) \text{ for some } g \in G\} \subseteq H.$$

Then

(1) $\mathrm{Ker}(f)$ *is a normal subgroup of* G*.*
(2) $\mathrm{Im}(f)$ *is a subgroup of* H*.*
(3) There is an isomorphism between $G/\mathrm{Ker}(f)$ *and* $\mathrm{Im}(f)$*.*

Proof. It is not hard to show that $\mathrm{Ker}(f) \subseteq G$ and $\mathrm{Im}(f) \subseteq H$ are subgroups (exercise). To show that $\mathrm{Ker}(f) \subseteq G$ is normal, one needs to show that for any $g \in \mathrm{Ker}(f)$ and $g' \in G$, we have $g'gg'^{-1} \in \mathrm{Ker}(f)$. This is true because

$$f(g'gg'^{-1}) = f(g')f(g)f(g'^{-1}) = f(g')f(g')^{-1} = 1_H.$$

Now we define a map $\overline{f}$ from $G/\mathrm{Ker}(f)$ to $\mathrm{Im}(f)$: For any coset $g\mathrm{Ker}(f)$, we define $\overline{f}(g\mathrm{Ker}(f)) \coloneqq f(g)$. This is a well-defined function on the set of cosets $G/\mathrm{Ker}(f)$, because any representative of $g\mathrm{Ker}(f)$ is of the form gg' for some $g' \in \mathrm{Ker}(f)$, and we have $f(gg') = f(g)f(g') = f(g)$. It is not hard to check that $\overline{f}\colon G/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ is a surjective group homomorphism. It is also injective: if $\overline{f}(g_1\mathrm{Ker}(f)) = \overline{f}(g_2\mathrm{Ker}(f))$, then we have $f(g_1) = f(g_2)$, or equivalently $g_2^{-1}g_1 \in \mathrm{Ker}(f)$. Hence the cosets $g_1\mathrm{Ker}(f) = g_2\mathrm{Ker}(f)$ coincide. □

Example. From Homework 1, we know that for any $f \in \mathrm{Isom}(\mathbb{R}^n)$, there exists a unique pair of an orthogonal matrix A and a vector $\vec{v}$ such that

$$f(\vec{x}) = A\vec{x} + \vec{v} \text{ for any } \vec{x} \in \mathbb{R}^n.$$

This gives a function

$$\pi\colon \mathrm{Isom}(\mathbb{R}^n) \to \mathrm{O}(n, \mathbb{R}), \quad f \mapsto A.$$

The function π is in fact a group homomorphism: suppose $f_1(\vec{x}) = A_1\vec{x} + \vec{v}_1$ and $f_2(\vec{x}) = A_2\vec{x} + \vec{v}_2$, then

$$f_1(f_2(\vec{x})) = A_1(A_2\vec{x} + \vec{v}_2) + \vec{v}_1 = (A_1A_2)\vec{x} + (A_1\vec{v}_2 + \vec{v}_1).$$

Hence $\pi(f_1f_2) = A_1A_2$. The kernel of π is an isometry of the form $f(\vec{x}) = \vec{x}+\vec{v}$, which is simply the translation by $\vec{v}$. Hence $\mathrm{Ker}(\pi) = T(n,\mathbb{R}) \cong \mathbb{R}^n$. This shows that the group of translations $T(n,\mathbb{R})$ is normal in $\mathrm{Isom}(\mathbb{R}^n)$. The homomorphism π is clearly surjective, so we have an isomorphism

$$\mathrm{Isom}(\mathbb{R}^n)/T(n,\mathbb{R}) \cong \mathrm{O}(n,\mathbb{R}).$$

Lecture 4

2.5. **Symmetry groups.** For any set X, a *permutation* of X is a bijective function $f\colon X \to X$. The *symmetric group S_X defined over* X is the set of all permutations of X, equipped with the group structure given by compositions. In particular, when X is a finite set of n elements $\{1, 2, \ldots, n\}$, its symmetric group would be denoted by S_n. It is not hard to see that $|S_n| = n!$.

Remark 2.21. Symmetric groups arise naturally when we discuss the symmetry groups of Platonic solids. Let G be the symmetry group of a tetrahedron T. It is not hard to see that any symmetry of T sends a vertex of T to a vertex (not necessarily the same one); in other words, it gives rise to a permutation of the four vertices of T. This gives a group homomorphism $\rho\colon \mathrm{Aut}(T) \to S_4$. Note that ρ is injective (why?), hence the symmetry group $\mathrm{Aut}(T)$ is isomorphic to a subgroup of the symmetric group S_4.

Any element of S_n can be represented by Cauchy's "two-line notation". Let $\sigma \in S_n$ be a permutation of the set $\{1, 2, \ldots, n\}$. Then we'll write

$$\sigma = \begin{bmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n). \end{bmatrix}$$

As usual, the composition $\sigma_1\sigma_2 \in S_n$ is given by $k \mapsto \sigma_1(\sigma_2(k))$, i.e. first apply σ_2 then apply σ_1. For instance, verify that

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Permutations are also often written in *cycle notation* ("decomposition into disjoint cycles"). To write down $\sigma \in S_n$ in cycle notation, one proceeds as follows:

- Write an open bracket then select an arbitrary element $x \in \{1, \dots, n\}$, and write down: $(x$
- Then trace the orbit of x: write down its value under successive applications of σ: $(x\ \sigma(x)\ \sigma^2(x) \cdots$
- Repeat until the value return to x, and write down a closing parenthesis rather than x: $(x\ \sigma(x)\ \sigma^2(x) \cdots)$
- Continue with any element y that is not yet written down, and proceed in the same way: $(x\ \sigma(x)\ \sigma^2(x) \cdots)(y\ \sigma(y) \cdots)$
- Repeat until all elements of $\{1, \dots, n\}$ are written in one of the cycles.

For instance,

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 6 & 2 & 3 & 5 \end{bmatrix} = (1)(24)(365) = (24)(365).$$

Here $\sigma(1) = 1$ forms an 1-cycle, which is often omitted.

A 2-cycle is called a *transposition*. An important fact is that any element $\sigma \in S_n$ can be written as a product of transpositions. To see this, it suffices to show that any cycle can be written as a product of transpositions, as any σ is a product of cycles. This can be easily verified:

$$(i_1 i_2 \cdots i_k) = (i_1 i_k)(i_1 i_{k-1}) \cdots (i_1 i_2).$$

It is not hard to see that there is no unique way to represent a permutation by a product of transpositions. For instance, $(123) = (13)(12) = (12)(23) = (12)(23)(13)(13)$. However, the *parity* (i.e. even or odd) of the numbers of transpositions of such representations is unique. (For instance, (123) can not be written as the product of odd number of transpositions.) This permits the *parity of a permutation* to be a well-defined notion.

The key idea of the proof is to define a group homomorphism

$$\operatorname{sgn}\colon S_n \to \{+1, -1\} \text{ (under multiplication)}$$

so that all transpositions map to -1. Indeed, if we can find such a homomorphism, then for any representation $\sigma = \tau_1 \cdots \tau_k$ where τ_i's are transpositions, we have

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau_1) \cdots \operatorname{sgn}(\tau_k) = (-1)^k.$$

This shows that the parity of k is independent of the choice of the decomposition. The proof is left as a homework problem.

Definition 2.22. The subset of S_n consisting of all *even* permutations will be denoted by A_n. It is a *normal subgroup* of S_n since it is the kernel of the group homomorphism sgn. The group A_n is called the *alternating group* (of n elements).

Exercise. Show that $A_n \subseteq S_n$ is a normal subgroup of index 2; it has two cosets, one of them consists of all even permutations, the other consists of all odd permutations.

2.6. **Group actions.** We will be interested in groups G that act as symmetries of a set X (for instance, the symmetry group of a tetrahedron acting on the set of its vertices). Let us introduce the formal definition of group actions.

Definition 2.23. We say that a group G *acts on a set* X if there is a map

$$G \times X \to X; \quad (g, x) \mapsto g \cdot x$$

satisfying:

- $e_G \cdot x = x$ for any $x \in X$,
- $g \cdot (h \cdot x) = (gh) \cdot x$ for any $g, h \in G$ and $x \in X$.

The dot " $\cdot$ " is sometimes omitted when the context is clear.

Exercise. Show that to give a group action of G on X is equivalent to give a group homomorphism $\rho\colon G \to S_X$. (Hint: Relate them by $g \cdot x = \rho(g)(x)$.)

Example. The symmetric group S_n acts on the set $\{1, \dots, n\}$.

Example. $\mathrm{Isom}(\mathbb{R}^n)$ acts on $\mathbb{R}^n$.

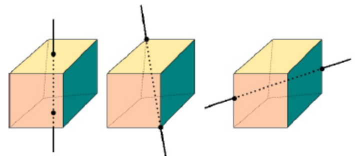
Example. $\mathrm{O}(n, \mathbb{R})$ acts on the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$, where

$$S^{n-1} = \{\vec{x} \in \mathbb{R}^n \mid ||\vec{x}|| = 1\}.$$

Example. The dihedral group D_n acts on the set of vertices of a regular n-gon, which gives a group homomorphism $D_n \to S_n$. Similarly, the symmetry group of a Platonic solid P acts on the set of its vertices.

Example. Let C be a cube in $\mathbb{R}^3$ centered at the origin. Denote $\mathrm{Aut}^+(C)$ the *rotational symmetric group* of C. Each element of $\mathrm{Aut}^+(C)$ is a rotation that fixes a line through the origin, and sends the cube C to itself. For instance:

- identity map;
- rotate $\pi/2, \pi, 3\pi/2$ along the first (left-most) line: there are 3 such lines, so this gives in total 9 elements of $\mathrm{Aut}^+(C)$;



- rotate $2\pi/3, 4\pi/3$ along the second line: there are 4 such lines, so this gives in total 8 elements of $\operatorname{Aut}^+(C)$;
- rotate π along the third line: there are 6 such lines, so this gives in total 6 elements of $\operatorname{Aut}^+(C)$.

Hence $|\operatorname{Aut}^+(C)|$ is at least 24.

On the other hand, observe that $\operatorname{Aut}^+(C)$ gives an action on the set of the four main diagonals of C, therefore induces a group homomorphism

$$\rho\colon \operatorname{Aut}^+(C) \to S_4.$$

One can show that ρ is injective (this is not a trivial observation: one needs to show that the antipodal map $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3)$ is *not* a rotation). Now, combining with the fact that $|\operatorname{Aut}^+(C)| \geq 24$, we can conclude that ρ is an isomorphism $\operatorname{Aut}^+(C) \cong S_4$.