Prop: If y_1, y_2 are sol$^{\underline{s}}$ to $y'' + by' + cy = 0$,
and suppose $\det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = 0$ for some $t_0 \in \mathbb{R}$

Then $\{y_1, y_2\}$ is linearly dependent.

pf: (relies on the existence & uniqueness thm.)

Case 1: $y_1(t_0) \neq 0$.

Denote $k := \dfrac{y_2(t_0)}{y_1(t_0)}$, consider $y_3(t) := k\, y_1(t)$.

We'll show that $y_3 = y_2$ via the existence & uniqueness thm.

Consider the following initial value problem:

$$\begin{cases} y'' + by' + cy = 0 \\ y(t_0) = y_2(t_0), \quad y'(t_0) = y_2'(t_0) \end{cases}$$

- $y_3(t_0) = k\, y_1(t_0) = \dfrac{y_2(t_0)}{y_1(t_0)} \cdot y_1(t_0) = y_2(t_0)$
- $y_3'(t_0) = k\, y_1'(t_0) = \dfrac{y_2(t_0)}{y_1(t_0)}\, y_1'(t_0) = y_2'(t_0)$

By the assumption, we have $y_1(t_0)\, y_2'(t_0) = y_2(t_0)\, y_1'(t_0)$

Case 2: $y_1(t_0) = 0$, $y_1'(t_0) \neq 0$

Consider $k := \dfrac{y_2'(t_0)}{y_1'(t_0)}$, $y_3(t) := k\, y_1(t)$

- $y_3(t_0) = k\, y_1(t_0) = \dfrac{y_2'(t_0)}{y_1'(t_0)}\, y_1(t_0) = y_2(t_0)$

- $y_3'(t_0) = k\, y_1'(t_0) = \dfrac{y_2'(t_0)}{y_1'(t_0)} y_1'(t_0) = y_2'(t_0)$.

Again by uniqueness, we have $y_2 = y_3 = k y_1$,

Case 3: $y_1(t_0) = 0$, $y_1'(t_0) = 0$

Consider $\begin{cases} y'' + by' + cy = 0 \\ y(t_0) = 0, \quad y'(t_0) = 0. \end{cases}$

y_1 is a sol^n to this initial value problem.

Also, the zero function is a sol^n.

By uniqueness $y_1 \equiv 0$. $\square$

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pf. $\det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = 0$,

$\exists\, c_1, c_2 \in \mathbb{R}$ s.t. $c_1 \begin{bmatrix} y_1(t_0) \\ y_1'(t_0) \end{bmatrix} + c_2 \begin{bmatrix} y_2(t_0) \\ y_2'(t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Consider $y_3(t) := c_1 y_1(t) + c_2 y_2(t)$

$y_3(t_0) = 0$, $y_3'(t_0) = 0$

$\Longrightarrow$ $y_3 \equiv 0$.

$y'' + by' + cy = 0 \quad \rightsquigarrow \quad$ auxiliary eq^n: $r^2 + br + c = 0$.

① two distinct real roots r_1, r_2.

$\rightsquigarrow \{e^{r_1 t}, e^{r_2 t}\}$ are l. i. sol$^{\text{ns}}$.

② root r_0 with multiplicity 2.

$\rightsquigarrow \{e^{r_0 t}, t e^{r_0 t}\}$ are l. i. sol$^{\text{ns}}$.

Claim: $y(t) = t e^{r_0 t}$ is a sol$^{\text{n}}$ to $y'' + by' + cy = 0$ in this case.

pf: $y'(t) = e^{r_0 t} + r_0 t e^{r_0 t}$

$= (1 + r_0 t) e^{r_0 t}$

$y''(t) = r_0 e^{r_0 t} + (1 + r_0 t) r_0 e^{r_0 t}$

$= (2r_0 + r_0^2 t) e^{r_0 t}$.

$y'' + by' + cy = (2r_0 + r_0^2 t + b + br_0 t + ct) e^{r_0 t}$

$$= \Big(\underbrace{(2r_0 + b)}_{=0} + \underbrace{(r_0^2 + br_0 + c)}_{=0} t\Big) e^{r_0 t}$$

r_0 is the double root of $r^2 + br + c = 0$.

i.e. $r^2 + br + c = (r - r_0)^2$

$= r^2 - 2rr_0 + r_0^2$

i.e. $b = -2r_0$, $c = r_0^2$.

③ $r^2 + br + c = 0$ has complex roots. $\alpha \pm i\beta$.

$$e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))$$

$$e^{(\alpha - i\beta)t} = e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right)$$

$\Rightarrow \quad \{e^{\alpha t}\cos(\beta t),\ e^{\alpha t}\sin(\beta t)\}$ are l.i. sol$^{\underline{n}}$s.

$$y(t) = e^{\alpha t}\cos(\beta t)$$

$$y'(t) = \alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t)$$

$$y''(t) = \alpha^2 e^{\alpha t}\cos(\beta t) - \alpha\beta e^{\alpha t}\sin(\beta t) - \alpha\beta e^{\alpha t}\sin(\beta t) - \beta^2 e^{\alpha t}\cos(\beta t)$$

$$y'' + by' + cy = e^{\alpha t}\cos(\beta t)\left[\alpha^2 - \beta^2 + b\alpha + c\right] + e^{\alpha t}\sin(\beta t)\left[-2\alpha\beta - b\beta\right]$$

$$r^2 + br + c = (r - (\alpha + i\beta))(r - (\alpha - i\beta))$$

$$= r^2 - 2r\alpha + (\alpha^2 + \beta^2)$$

$$\Rightarrow \boxed{b = -2\alpha, \quad c = \alpha^2 + \beta^2.}$$

Non-homog. 2^{nd} order diff'l eq$^{\underline{n}}$s

e.g. $\quad y'' + by' + cy = t. \qquad c \neq 0.$

Consider $\quad y(t) = A_0 + A_1 t.$

$$y'(t) = A_1, \quad y''(t) = 0$$

$$b A_1 + c(A_0 + A_1 t) = t \qquad \frac{b}{c} A_1 + c A_0 = 0$$

$$A_1 = \frac{1}{c}, \quad A_0 = \frac{-b}{c^2}$$

$$y(t) = \frac{-b}{c^2} + \frac{1}{c} t. \longleftarrow$$ a sol^n to the non-homg eq^n.

eg. $y'' + b y' + \overset{\overset{0}{\#}}{c} y = \overset{\overset{0}{\#}}{a_k} t^k + a_{k-1} t^{k-1} + \cdots + a_0.$

$$y(t) = A_k t^k + A_{k-1} t^{k-1} + \cdots + A_0.$$

Try to find $A_k, \dots, A_0$ s.t. $y(t)$ satisfies the eq^n.

$$
\begin{array}{rlllll}
c & y(t) = & c A_k t^k + c A_{k-1} t^{k-1} & + c A_{k-2} t^{k-2} & + \cdots & + c A_0 \\
b & y'(t) = & b k A_k t^{k-1} & + b(k-1) A_{k-1} t^{k-2} & + \cdots & + b A_1 \\
+ & y''(t) = & & k(k-1) A_k t^{k-2} & + \cdots & + 2A_2 \\
\hline
& & a_k t^k + a_{k-1} t^{k-1} & + & \cdots & + a_0
\end{array}
$$

$$
\begin{bmatrix}
\overset{\overset{0}{\#}}{c} & 0 & 0 & \cdots & 0 \\
bk & c & 0 & \cdots & 0 \\
k(k-1) & b(k-1) & c & 0 \cdots & 0 \\
0 & (k-1)(k-2) & b(k-2) & c & 0 \cdots 0 \\
 & \ddots & & \ddots & \ddots \\
\end{bmatrix}
\begin{bmatrix} A_k \\ A_{k-1} \\ \vdots \\ \vdots \\ A_0 \end{bmatrix}
=
\begin{bmatrix} a_k \\ a_{k-1} \\ \vdots \\ \vdots \\ a_0 \end{bmatrix}
$$

Invertible. $\uparrow$

e.g.: $y'' + by' + cy = e^{r_0 t}$ for some $r_0 \in \mathbb{R}$.

Guess: $y(t) = Ae^{r_0 t}$, try to find A

$y'(t) = A r_0 e^{r_0 t}$

$y''(t) = A r_0^2 e^{r_0 t}$.

$$y'' + by' + cy = A(r_0^2 + br_0 + c)e^{r_0 t}.$$

Take $A = \dfrac{1}{r_0^2 + br_0 + c}$.

Need: $r_0^2 + br_0 + c \neq 0$

Then $y(t) = Ae^{r_0 t}$ is a solⁿ to the eqⁿ.

Rmk: When r_0 is a root of $r^2 + br + c = 0$

this argument doesn't work. ($Ae^{r_0 t}$ is never a solⁿ in this case)

Guess: $c \cdot y(t) = cAte^{r_0 t}$.

$b\, y'(t) = bA[e^{r_0 t} + r_0 t e^{r_0 t}]$

$= bA(1 + r_0 t)e^{r_0 t}$.

$y''(t) = A[r_0 e^{r_0 t} + (1 + r_0 t) r_0 e^{r_0 t}]$

$= A[2r_0 + r_0^2 t]e^{r_0 t}$.

$$y'' + by' + cy = Ae^{r_0 t}\left[b + 2r_0 + t\underbrace{(c + br_0 + r_0^2)}_{=0} \right]$$

$\Leftrightarrow$ r_0 is not a double root of $r^2 + br + c = 0$.

If $b + 2r_0 \neq 0$, then we can take

$$A = \frac{1}{b + 2r_0},$$

and $y(t) = Ate^{r_0 t}$ is a sol^n to

$$y'' + by' + cy = e^{r_0 t}.$$

Last case: r_0 is the double root of $r^2 + br + c = 0$

Ex: Show that $y(t) = \frac{1}{2}t^2 e^{r_0 t}$

is a sol^n to $y'' + by' + cy = e^{r_0 t}$.

Summary $\quad y'' + by' + cy = e^{r_0 t}$.

- $r_0^2 + br_0 + c \neq 0 \Rightarrow \frac{1}{r_0^2 + br_0 + c} e^{r_0 t}$ is a sol^n
- $r_0^2 + br_0 + c = 0$, $2r_0 + b \neq 0 \Rightarrow \frac{1}{2r_0 + b} te^{r_0 t}$ is a sol^n
- $r_0^2 + br_0 + c = 2r_0 + b = 0 \Rightarrow \frac{1}{2} t^2 e^{r_0 t}$ is a sol^n.