HOMEWORK 13
MATH 104, SECTION 6

(1) Let (a_n) and (b_n) be two sequences of real numbers satisfying:
- The partial sums of (b_n) is bounded: there exists $L > 0$ such that $|b_1 + \cdots + b_k| < L$ for any k,
- $\lim a_n = 0$,
- $\sum |a_n - a_{n+1}|$ converges.

Prove that for any $k \in \mathbb{N}$, the series $\sum a_n^k b_n$ is convergent. (Hint: Try the same idea as in HW6, Problem 4.)

(2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} \frac{f(2x)-f(x)}{x} = 0$. Prove that $\lim_{x\to 0} \frac{f(x)}{x} = 0$. (Hint: Try to estimate $\frac{f(x)-f(x/2^n)}{x}$.)

(3) Recall that a collection of functions (f_n) on X is called *uniformly equicontinuous* on X if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any n, we have

$$|x - y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon.$$

Find all the functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfy the following conditions, and justify your answer:
- $f\colon \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$.
- The collection of functions $(f_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous on $\mathbb{R}$, where $f_n\colon \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x) \coloneqq f(nx)$.

(4) (a) Prove that the equation $x = \cos x$ has a unique root $x \in \mathbb{R}$.
(b) Define a sequence of real numbers (a_n) as follows: Let a_1 be any real number satisfying $0 < a_1 \leq 1$. Then define a_n recursively via

$$a_{n+1} \coloneqq \cos(a_n).$$

Prove that the sequence (a_n) is convergent.
(c) Define a sequence (a_n) as in (b). Prove that the series $\sum a_n$ is divergent.

(5) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that for any $r \in \mathbb{R}$, we have

$$\lim_{n\to\infty} f\left(\frac{r}{n}\right) = 0.$$

Prove or disprove: $\lim_{x\to 0} f(x) = 0$.

(6) Let (p_n) be a sequence of polynomials defined over real numbers, and let $f\colon \mathbb{R} \to \mathbb{R}$ be a real-valued function. Suppose that (p_n) converges uniformly to f on $\mathbb{R}$. Prove that f is also a polynomial.

(7) Let f and g be continuous functions on $[a,b]$ that are differentiable on (a,b). Suppose that $f(a) = f(b) = 0$. Prove that there exists $x \in (a,b)$ such that $g'(x)f(x) + f'(x) = 0$.

(8) Let f, g, h be continuous functions on $[a,b]$ that are differentiable on (a,b). Consider

$$F(x) = \det \begin{pmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{pmatrix}.$$

(a) Prove that F is also continuous on $[a,b]$ and differentiable on (a,b).

(b) Prove that there exists $x_0 \in (a,b)$ such that $F'(x_0) = 0$.

(c) Prove the following generalization of mean value theorem: If f and g are continuous functions on $[a,b]$ that are differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

(9) Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational, or } x = 0, \\ \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ where } p, q > 0,\ \gcd(p,q) = 1. \end{cases}$$

Prove that f is integrable on $[0,1]$, and compute $\int_0^1 f(x)dx$.

(10) Let F be an ordered field that contains the rational numbers $\mathbb{Q}$, where $0 \in \mathbb{Q} \subset F$ is the additive identity in F and $1 \in \mathbb{Q} \subset F$ is the multiplicative identity of F. There is a standard distance function on F:

$$d_{\text{std}}(x,y) := |x - y|_F,$$

where $|\cdot|_F$ is the absolute value on F. This gives a metric space structure on F.

(a) Prove that $\mathbb{Q}$ is a dense subset of F if and only if for any $x, y \in F$ such that $x < y$, there exists $q \in \mathbb{Q}$ such that $x < q < y$. (Recall that $E \subset X$ is *dense* if $\overline{E} = X$.)

(b) Suppose that $\mathbb{Q}$ is dense in F. Moreover, assume that any Cauchy sequence of rational numbers has a limit in F. Prove that F has the least upper bound property, i.e. any nonempty subset $S \subset F$ that is bounded above has the least upper bound. (Hint: You can try to construct two sequences of rational numbers (p_n) and (q_n) that converge to the same element in F, where each p_n is an upper bound of S and each q_n is not an upper bound of S. Then prove that the limit is the least upper bound of S.)