Math H54 Second Midterm (Practice Exam) November 7, 2019

Name: Solution

- You have 80 minutes to complete the exam.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	20	
2	20	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	100	

1. (4 points each) Determine if each statement is TRUE or FALSE. Give a short justification for 'TRUE'; give a counterexample and justify it for 'FALSE'.

(a) If an $n \times n$ matrix A does not have n distinct eigenvalues, then A is not diagonalizable.

False. e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) For any matrix A, the matrix AA^T is diagonalizable.

True: AA^T is symmetric, hence diagonalizable.

(c) If $\langle \vec{v}_1, \vec{w} \rangle = \langle \vec{v}_2, \vec{w} \rangle$ for any $\vec{w} \in V$, then $\vec{v}_1 = \vec{v}_2$.

True. $\Downarrow$

$\langle \vec{v}_1 - \vec{v}_2, \vec{w} \rangle = 0 \quad \forall \vec{w} \in V.$

$\Rightarrow \langle \vec{v}_1 - \vec{v}_2, \vec{v}_1 - \vec{v}_2 \rangle = \|\vec{v}_1 - \vec{v}_2\|^2 = 0$

$\Rightarrow \vec{v}_1 = \vec{v}_2$

(d) The eigenvalues of an orthogonal matrix are all real.

False. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(e) If 3 is an eigenvalue of A, then 9 must be an eigenvalue of A^2.

True. If $A\vec{v} = 3\vec{v}$ for some $\vec{v} \neq \vec{0}$,

then $A^2\vec{v} = A(3\vec{v}) = 3A\vec{v} = 9\vec{v}$.

2. Consider the inner product space $\mathcal{C}([-1,1])$ of continuous functions on $[-1,1]$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

(a) (10 points) Construct an orthonormal basis (with respect to this inner product) for the subspace $\mathbb{P}_2$ of polynomials with real coefficients of degree at most two.

Start with $\{\underset{f_1}{1}, \underset{f_2}{x}, \underset{f_3}{x^2}\}$, then apply Gram-Schmidt:

$\|f_1\|^2 = \int_{-1}^{1} 1\, dx = 2$. $\rightsquigarrow$ define $\underline{\underline{\hat{f}_1 := \frac{1}{\sqrt{2}}}}$ (unit vector in $\mathcal{C}([-1,1])$)

$\tilde{f}_2 := f_2 - \langle f_2, \hat{f}_1\rangle \hat{f}_1$, Note that $\underline{\langle f_2, \hat{f}_1\rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} x\, dx = 0}$.

$= x$,

$\|\tilde{f}_2\|^2 = \int_{-1}^{1} x^2 dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3}$., $\rightsquigarrow$ $\underline{\underline{\hat{f}_2 := \sqrt{\frac{3}{2}}\, x}}$.

$\tilde{f}_3 := f_3 - \langle f_3, \hat{f}_1\rangle \hat{f}_1 - \langle f_3, \hat{f}_2\rangle \hat{f}_2$

$$\left(\begin{array}{l} \bullet\ \langle f_3, \hat{f}_1\rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} x^2 dx = \frac{1}{\sqrt{2}}\cdot\frac{2}{3} \\ \bullet\ \langle f_3, \hat{f}_2\rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}}\, x^3 dx = 0. \end{array} \right)$$

$= x^2 - \frac{1}{3}$.

$\|\tilde{f}_3\|^2 = \int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx = \frac{8}{45}$, $\rightsquigarrow$ $\hat{f}_3 := \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$.

$$\boxed{\left\{ \frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\, x,\ \sqrt{\frac{45}{8}}(x^2 - \tfrac{1}{3}) \right\}}$$

(b) (10 points) Find the best approximation (with respect to this inner product) to $f(x) = x^5$ by polynomials in $\mathbb{P}_2$.

$$\operatorname{Proj}_{\mathbb{P}_2} x^5 = \underbrace{\langle x^5, \tfrac{1}{\sqrt{2}} \rangle}_{= 0} \tfrac{1}{\sqrt{2}} + \langle x^5, \sqrt{\tfrac{3}{2}}\, x \rangle \sqrt{\tfrac{3}{2}}\, x + \underbrace{\langle x^5, \sqrt{\tfrac{45}{8}}(x^2 - \tfrac{1}{3}) \rangle}_{= 0} \sqrt{\tfrac{45}{8}}(x^2 - \tfrac{1}{3})$$

$$= \sqrt{\tfrac{3}{2}}\, x \int_{-1}^{1} \sqrt{\tfrac{3}{2}}\, x^6 \, dx$$

$$= \frac{3}{2} x \cdot \frac{2}{7}$$

$$= \frac{3}{7} x. \quad \square$$

3. (10 points) Let

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & 3 \\ 1 & 0 & -1 \end{pmatrix}.$$

(a) Find all eigenvalues of A.

$$\det \begin{pmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & 3 \\ 1 & 0 & -1-\lambda \end{pmatrix} = -\lambda(\lambda^2+2\lambda) = -\lambda^2(\lambda+2)$$

eigenvalues are: 0, -2

multiplicity 2.

(b) Find a basis for the eigenspace associated to each eigenvalue.

$\underline{\lambda=0}$: $\mathrm{Nul}(A-0I) = \mathrm{Nul}\begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & 3 \\ 1 & 0 & -1 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}.$

$\underline{\lambda=-2}$: $\mathrm{Nul}(A+2I) = \mathrm{Nul}\begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}.$

(c) Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain the reason.

<u>No</u>. $\lambda=0$ has multiplicity 2, but

$\dim \underbrace{\mathrm{Nul}(A-\lambda I)}_{\text{eigenspace}} = 1 < 2$. $\square$

4. (10 points) Consider the symmetric matrix

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{pmatrix} = (2-\lambda)(\lambda^2-4) = -(\lambda-2)^2(\lambda+2)$$

$\lambda = \underset{(\text{mult. } 2)}{2}$ or -2

$\underline{\lambda=2}$: $\mathrm{Nul}(A-2I) = \mathrm{Nul}\begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix} = \mathrm{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}.$

$\underline{\lambda=-2}$: $\mathrm{Nul}(A+2I) = \mathrm{Nul}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} = \mathrm{Span}\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}.$

Take $P = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & -2 \end{bmatrix}$ $\square$

5. (10 points) Suppose a matrix A is similar to $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. For each, give a proof or a counterexample.

Let $\underline{A = PBP^{-1}}$.

(a) $A^2 = A$.

<u>True</u> $A^2 = PBP^{-1}PBP^{-1} = PB^2P^{-1} = PBP^{-1} = A$

$\left(B^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = B\right)$

(b) $\det A = 0$.

<u>True</u> $\det A = \det PBP^{-1} = \det P \cdot \det B \cdot \det P^{-1}$

$= \det B = 0.$

(c) $\mathrm{tr}A = 1$.

<u>True</u> $\mathrm{tr}\, A = \mathrm{tr}\, PBP^{-1} = \mathrm{tr}\, PP^{-1}B = \mathrm{tr}\, B = 1$

(d) $\lambda = 1$ is an eigenvalue of A.

<u>True</u> $\lambda = 1$ is an eigenvalue of B: $B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\Rightarrow A\left(P\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = (PBP^{-1})P\begin{bmatrix} 1 \\ 1 \end{bmatrix} = PB\begin{bmatrix} 1 \\ 1 \end{bmatrix} = P\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \lambda = 1$ is an eigenvalue of A.

(e) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A.

<u>False</u> e.g. Take $P = \begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix}$, $P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix}$.

then $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$

$\begin{pmatrix} -3 \\ -3 \end{pmatrix}$

$\begin{pmatrix} -6 \\ -18 \end{pmatrix}$

6. Let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in an inner product space. Define the *Gram determinant* by $G(\vec{v}_1, \ldots, \vec{v}_n) = \det(\langle \vec{v}_i, \vec{v}_j \rangle)$, i.e. the determinant of the matrix with entry $\langle \vec{v}_i, \vec{v}_j \rangle$ in the (i, j)-th position.

(a) (3 points) If $\vec{v}_1, \ldots, \vec{v}_n$ are orthogonal, compute their Gram determinant.

$$G(\vec{v}_1, \ldots, \vec{v}_n) = \det \begin{pmatrix} \langle \vec{v}_1, \vec{v}_1 \rangle & \cdots & \langle \vec{v}_1, \vec{v}_n \rangle \\ \vdots & & \vdots \\ \langle \vec{v}_n, \vec{v}_1 \rangle & \cdots & \langle \vec{v}_n, \vec{v}_n \rangle \end{pmatrix} = \det \begin{pmatrix} \|\vec{v}_1\|^2 & 0 & 0 & \cdots \\ 0 & \|\vec{v}_2\|^2 & & \\ \vdots & & \ddots & \\ & & & \|\vec{v}_n\|^2 \end{pmatrix} = \|\vec{v}_1\|^2 \cdots \|\vec{v}_n\|^2.$$

(b) (7 points) Show that $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent if and only if their Gram determinant is not zero.

Observe that $\begin{pmatrix} \langle \vec{v}_1, \vec{v}_1 \rangle & \cdots & \langle \vec{v}_1, \vec{v}_n \rangle \\ \vdots & & \vdots \\ \langle \vec{v}_n, \vec{v}_1 \rangle & \cdots & \end{pmatrix} = A^T A$, where $A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$

$$G(\vec{v}_1, \ldots, \vec{v}_n) = \det(A^T A) = \det(A^T) \cdot \det(A) = \det(A)^2$$

So $G(\vec{v}_1, \ldots, \vec{v}_n) \neq 0 \iff \det A \neq 0$

$\iff \{\vec{v}_1, \ldots, \vec{v}_n\}$ are l.i. $\square$

7. (10 points) Suppose that A is a real $n \times n$ symmetric matrix with two equal eigenvalues. Show that for any $\vec{v} \in \mathbb{R}^n$, the vectors $\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}$ are linearly dependent.

Let $Q = \begin{bmatrix} \vec{v} & A\vec{v} & \cdots & A^{n-1}\vec{v} \end{bmatrix}_{n\times n}$

A is symmetric $\Rightarrow$ diagonalizable. $A = PDP^{-1}$, P- invertible, D- diagonal

Then $$P^{-1}Q = P^{-1}\begin{bmatrix} \vec{v} & A\vec{v} & \cdots & A^{n-1}\vec{v} \end{bmatrix}$$

$$= P^{-1}\begin{bmatrix} \vec{v} & PDP^{-1}\vec{v} & \cdots & PD^{n-1}P^{-1}\vec{v} \end{bmatrix}$$

$$= \begin{bmatrix} P^{-1}\vec{v} & DP^{-1}\vec{v} & \cdots & D^{n-1}P^{-1}\vec{v} \end{bmatrix}$$

Let $P^{-1}\vec{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$= \begin{bmatrix} x_1 & \lambda_1 x_1 & \cdots & \lambda_1^{n-1}x_1 \\ \vdots & \vdots & & \vdots \\ x_n & \lambda_n x_n & \cdots & \lambda_n^{n-1}x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix}}$$

Since A has 2 equal eigenvalues $\Rightarrow$ is not invertible

$\Rightarrow \det(P^{-1}Q) = 0$

$\Rightarrow \det(Q) = 0$

$\Rightarrow \{\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}\}$ are l. dependent. $\square$

8. (10 points) Show that the only real matrix that is orthogonal, symmetric, and has all positive eigenvalues is the identity matrix.

A: orthogonal $\Rightarrow$ $A^{-1} = A^T$

symmetric $\Rightarrow$ $A = A^T$

$\Rightarrow$ $A^2 = I$.

Symmetric $\Rightarrow$ orthogonal diagonalizable. $A = PDP^T$.

P - orthogonal

D - diagonal

$I = A^2 = PD^2P^T$ $\Rightarrow$ $D^2 = I$ $\Rightarrow$ entries are ± 1.

$\Rightarrow$ entries are 1 because A has only positive eigenvalues.

$\Rightarrow$ $D = I$.

$\Rightarrow$ $A = PDP^T = PP^T = I$. $\square$