HW 2 sol'ns.

#1: Assume the contrary that $\sqrt{2} \in \mathbb{Q}$. Then $\exists\, a, b \in \mathbb{Z} \setminus \{0\}$ s.t. $\sqrt{2} = \frac{a}{b}$.

$\Rightarrow a^2 = 2b^2 \Rightarrow$ a is even, $a = 2a'$ for some $a' \in \mathbb{Z}$.

$\Rightarrow 2a'^2 = b^2 \Rightarrow$ b is even, $b = 2b'$ for some $b' \in \mathbb{Z}$.

$\Rightarrow a'^2 = 2b'^2 \Rightarrow \cdots$

By this argument, one can show that 2^N divides a and b for any $N \in \mathbb{N}$, which is impossible. Hence $\sqrt{2} \notin \mathbb{Q}$ $\square$

#2: Prove by induction. The statement is true for $n = 1$.

Assume the statement is true for n, then

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = (1 + \cdots + n)^2 + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = (n+1)^2\left(\frac{n^2}{4} + n + 1\right)$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2 = (1 + \cdots + n + (n+1))^2. \quad \square$$

#3: Prove by induction.

Claim: $|a_1 + a_2| \le |a_1| + |a_2|$:

By the definition of absolute value, we have: $-|a_1| \le a_1 \le |a_1|$, $-|a_2| \le a_2 \le |a_2|$

$\Rightarrow -(|a_1| + |a_2|) \le a_1 + a_2 \le |a_1| + |a_2|$

$\Rightarrow |a_1 + a_2| \le |a_1| + |a_2|$.

Hence the statement is true for $n = 2$.

Assume the statement is true for n. then

$$|a_1 + \cdots + a_n + a_{n+1}| \le |a_1 + \cdots + a_n| + |a_{n+1}| \quad (\text{using } n=2 \text{ case})$$

$$\le |a_1| + \cdots + |a_n| + |a_{n+1}|. \quad \square$$

#4: See Ross, § 4.1 ~ 4.5.

#5: • Suppose that $\exists \varepsilon > 0$ s.t. $z - \varepsilon < a \le z$ doesn't hold for any $a \in S$, then $a \le z - \varepsilon \ \forall a \in S$, which contradicts with z is the least upper bound.

• No. e.g.: $S = \{z\}$ set of only one real number. then $\sup S = z$, But $\forall \varepsilon > 0$, there is no $a \in S$ so that $z - \varepsilon < a < z$.

#6: Assume the contrary that $x > y$.
Let $\varepsilon = \frac{x-y}{2} > 0$, then $y + \varepsilon = y + \frac{x-y}{2} = \frac{x+y}{2} < x$.
contradicts with the assumption.

Hence $x \le y$. $\square$

#7: 1 is obviously an upper bound of $A = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$.
Suppose that $\exists z < 1$ s.t. z is an upper bound of A.
Let $\varepsilon = 1 - z > 0$.
Then $z \ge 1 - \frac{1}{n} \ \forall n \in \mathbb{N} \Rightarrow n \le \frac{1}{\varepsilon} \ \forall n \in \mathbb{N}$,
which is impossible since $\mathbb{N}$ is not bounded above. $\square$

#8: It suffices to show that $\forall \varepsilon > 0$, $\exists m, n \in \mathbb{Z}$ s.t. $0 < m + n\sqrt{2} < \varepsilon$. Choose an $N \in \mathbb{N}$ st. $N > \frac{1}{\varepsilon}$.
Consider the set $\{\{\sqrt{2}\}, \{2\sqrt{2}\}, \cdots, \{(N+1)\sqrt{2}\}\} \subset (0,1)$,
where $\{\alpha\} := \alpha - \lfloor \alpha \rfloor$ is the fractional part of α.

By pigeon hole principle. $\exists\, a, b \in \{1, \dots, N+1\}$ s.t.

$$0 < \{a\sqrt{2}\} - \{b\sqrt{2}\} < \frac{1}{N} < \varepsilon.$$

Note that $\begin{array}{l}\{a\sqrt{2}\} = a\sqrt{2} - z_1 \\ \{b\sqrt{2}\} = b\sqrt{2} - z_2\end{array}$ for some integers $z_1, z_2 \in \mathbb{Z}$.

Hence $0 < \underbrace{(a-b)}_{\in\, \mathbb{Z}}\sqrt{2} - \underbrace{(z_1 - z_2)}_{\in\, \mathbb{Z}} < \varepsilon$ $\square$

#9: For $n = 2^N$, we have

$$a_n = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\downarrow} + \underbrace{\frac{1}{5} + \dots + \frac{1}{8}}_{\downarrow} + \dots + \underbrace{\frac{1}{2^{N-1}+1} + \dots + \frac{1}{2^N}}_{\downarrow}$$

$$> 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{} + \underbrace{\left(\frac{1}{8} + \dots + \frac{1}{8}\right)}_{4 \times \frac{1}{8}} + \dots + \underbrace{\left(\frac{1}{2^N} + \dots + \frac{1}{2^N}\right)}_{2^{N-1} \times \frac{1}{2^{N-1}}} = 1 + \frac{N}{2}$$

$\Rightarrow$ (a_n) is not bounded, hence doesn't converge. $\square$

#10:

(a) $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $|a_n - a| < \varepsilon$ $\forall n > N$.

$\Rightarrow$ $|a_{2n} - a| < \varepsilon$ $\forall n > N$ and $|a_{2n-1} - a| < \varepsilon$ $\forall n > N$.

Hence (a_{2n}) and (a_{2n-1}) both converge, and have the same limit as (a_n).

(b) No. e.g. $(0, 1, 0, 1, 0, 1, \dots)$.

(c) The argument in (a) shows that any subsequence of a convergent sequence also converges, and converges to the same limit.

Let $\lim_{n\to\infty} a_{2n} = A$, $\lim_{n\to\infty} a_{2n-1} = B$, $\lim_{n\to\infty} a_{3n} = C$.

Since (a_{6n}) is a subsequence of both (a_{2n}) and (a_{3n}),

we have $A = \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{6n} = \lim_{n\to\infty} a_{3n} = C$.

Also, (a_{6n-3}) is a subsequence of both (a_{2n-1}) and (a_{3n}),

hence $B = \lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} a_{6n-3} = \lim_{n\to\infty} a_{3n} = C$.

$\Longrightarrow$ $A = C = B$. $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n-1} = A$.

$\forall \varepsilon > 0$, $\exists N_1 > 0$ s.t. $|a_{2n} - A| < \varepsilon$ $\forall n > N_1$

$\exists N_2 > 0$ s.t. $|a_{2n-1} - A| < \varepsilon$ $\forall n > N_2$.

Define $N = \max\{2N_1, 2N_2 - 1\}$. then we have

$|a_n - A| < \varepsilon$ $\forall n > N$.

Hence (a_n) converges. $\square$