#1: ($\Leftarrow$): The only nontrivial part is to show that
$\langle \vec{v}, \vec{v} \rangle_A = \vec{v}^T A \vec{v} > 0$ for any $\vec{v} \neq \vec{0}$.

Write $A = PDP^T$, P: orthogonal, D: diagonal, with positive diagonal entries $\lambda_1, \dots, \lambda_n > 0$.

Then
$$\langle \vec{v}, \vec{v} \rangle_A = \vec{v}^T P D P^T \vec{v} = (P^T \vec{v})^T D (P^T \vec{v}).$$

Write $P^T \vec{v} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$. Then $\langle \vec{v}, \vec{v} \rangle_A = \lambda_1 w_1^2 + \dots + \lambda_n w_n^2$.

which is positive unless $w_1 = \dots = w_n = 0$, in which case $\vec{v} = \vec{0}$. $\square$

($\Rightarrow$): $\langle \vec{x}, \vec{y} \rangle_A = \langle \vec{y}, \vec{x} \rangle_A$ for any $\vec{x}, \vec{y} \in \mathbb{R}^n$.

$$\underset{\vec{x}^T A \vec{y}}{\overset{\langle \vec{x}, \vec{y} \rangle_A}{\parallel}} \qquad \underset{\vec{y}^T A \vec{x} = \vec{x}^T A^T \vec{y}.}{\overset{\langle \vec{y}, \vec{x} \rangle_A}{\parallel}}$$

It's not hard to show that this implies $A = A^T$, i.e. A is symmetric.

Then $A = PDP^T$, P: orthogonal, D: diagonal $= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

We have $\forall \vec{v} \neq \vec{0}$,
$$0 < \langle \vec{v}, \vec{v} \rangle_A = (P^T \vec{v})^T D (\underbrace{P^T \vec{v}}_{= \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}}) = \lambda_1 w_1^2 + \dots + \lambda_n w_n^2.$$

It's not hard to show that this implies $\lambda_1, \dots, \lambda_n > 0$. $\square$

#2: The proof of ($\Rightarrow$) is essentially the same as the proof of ($\Rightarrow$) in #1.

The proof of ($\Leftarrow$) is essentially the same as the proof of ($\Leftarrow$) in #1.

#3: (a) $\vec{x}^T A \vec{x} = \vec{x}^T(-A^T)\vec{x} = -\vec{x}^T A^T \vec{x}$

$= -(\vec{x}^T A^T \vec{x})^T = -\vec{x}^T A \vec{x}$. $\square$

(b) $\vec{x}^T A^2 \vec{x} = \vec{x}^T(-A^T) A \vec{x} = -\|A\vec{x}\|^2 \le 0$. $\square$

(c) By (b), $I - A^2$ is positive definite.

Hence $0 \neq \det(I - A^2) = \det(I-A)\det(I+A)$. $\square$

#4: Suppose $(A+B)\vec{x} = \vec{0}$. Then

$$0 = \vec{x}^T(A+B)\vec{x} = \vec{x}^T A \vec{x} + \vec{x}^T B \vec{x} \underset{\#3(a)}{=} \vec{x}^T B \vec{x}$$

$\Rightarrow \vec{x} = \vec{0}$ since B is positive definite. $\square$

#5: (a) $A = PDP^T$, P: orthogonal, $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $\lambda_1, \dots, \lambda_n > 0$.

Since P, D, P^T are all invertible, so is A, and

$A^{-1} = (P^T)^{-1} D^{-1} P^{-1}$, $(P^T)^{-1}$ is an orthogonal matrix

and the eigenvalues of A^{-1} are the diagonal entries of D^{-1}, which are $\lambda_1^{-1}, \dots, \lambda_n^{-1} > 0$.

(b). $\forall \vec{x} \neq \vec{0}$, we have

$$\vec{x}^T(A+B)\vec{x} = \underbrace{\vec{x}^T A \vec{x}}_{>0} + \underbrace{\vec{x}^T B \vec{x}}_{>0} > 0.$$ $\square$

#6: (a) $\vec{x}^T A^T A \vec{x} = \|A\vec{x}\|^2 \geq 0$, $\vec{x}^T A A^T \vec{x} = \|A^T\vec{x}\|^2 \geq 0$.

(b)

($\Rightarrow$): Suppose A is an $m \times n$ matrix.

$\underset{n\times n}{A^T A}$, $\underset{m\times m}{AA^T}$ are invertible. hence

$\operatorname{rank}(A^T A) = n$, $\operatorname{rank}(AA^T) = m$.

On the other hand,

$$\operatorname{rank}(A^T A) \leq \min\{\operatorname{rank}(A^T), \operatorname{rank}(A)\} \leq \min\{m, n\},$$

and similarly $\operatorname{rank}(AA^T) \leq \min\{m,n\}$.

Hence we must have $m = n$ and $\operatorname{rank}(A) = n$,

hence A is a square matrix and is invertible. □

($\Leftarrow$) NOT hard to prove. □

#7: (a) Write $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, then

$$f(A) = \operatorname{tr}(A^2) = \sum_{i=1}^{n} a_{ii}^2 + 2 \sum_{1 \leq j < k \leq n} a_{ij} a_{ji}$$

is a quadratic form.

(b). Notice that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has signature $(1, 0, 1)$.

Hence the signature of f is

$$\left(n + \frac{n(n-1)}{2}, 0, \frac{n(n-1)}{2}\right). \square$$