HINTS OF HW6
MATH 185

14. Consider $g(z) := f(1/z)$ which is holomorphic on $\mathbb{C}\backslash\{0\}$. Analyze its singularity at the point 0.

(1) Prove that the the singularity is not an essential singularity, using the Casorati–Weierstrass theorem and the open mapping theorem.

(2) Prove that the singularity is not removable, using Liouville theorem.

This would imply that $g(z)$ has a pole at $z = 0$. Therefore, near $z = 0$, g can be written as

$$g(z) = \frac{a_{-n}}{z^n} + \cdots + \frac{a_{-1}}{z} + h(z),$$

where h is holomorphic in a neighborhood of $z = 0$. Consider

$$\widetilde{f}(z) := f(z) - (a_{-n}z^n + \cdots + a_{-1}z),$$

and show that $\widetilde{f}$ is entire and bounded.

16. Rouché's theorem.

17.(a) First, show that $f(z) = 0$ for some $z \in \mathbb{D}$. (Assume not, then $g(z) = 1/f(z)$ is holomorphic in a neighborhood of $\overline{\mathbb{D}}$. Apply maximum modulus principle to f and g to get a contradiction.) Second, use Rouché's theorem to show that for any $w_0 \in \mathbb{D}$, the equation $f(z) = w_0$ has a solution in $\mathbb{D}$.