HOMEWORK 5
MATH 104, SECTION 2

Some ground rules:

- You have to submit your homework via **Gradescope** to the corresponding assignment. The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You're allowed to use any result that is proved in the lecture; but if you'd like to use other results, you have to prove them before using them.

Problem set (6 problems; due March 2 at 11am PT)

(1) Determine whether each of the following series converges or not. Prove your answers.

$$(a)\ \sum \frac{(-1)^n(n-1)}{n};\ (b)\ \sum \frac{n^n}{(n+1)^{2n}};\ (c)\ \sum \frac{(-1)^n}{n^{1/12}};$$

$$(d)\ \sum \frac{1}{(2n-1)^2};\ (e)\ \sum \frac{1}{n\log n};\ (f)\ \sum ne^{-n^2}.$$

(2) Prove the triangle inequality for series: if $\sum a_n$ converges absolutely, then

$$\Big|\sum_{n=1}^{\infty} a_n\Big| \le \sum_{n=1}^{\infty} |a_n|.$$

(3) Show that the monotonicity assumption in alternating series test is necessary: find a sequence of positive real numbers (a_n) with $\lim a_n = 0$, but $\sum (-1)^n a_n$ diverges.

(4) Let $(a_n^{(1)})_{n=1}^{\infty}, (a_n^{(2)})_{n=1}^{\infty}, \ldots, (a_n^{(k)})_{n=1}^{\infty}$ denote k sequences of real numbers. (For instance, the first sequence is $(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}, \ldots)$.) Define another sequence $(b_n)_{n=1}^{\infty}$ where the n-th term is defined to be

$$b_n = a_n^{(1)} + a_n^{(2)} + \cdots a_n^{(k)}.$$

Suppose that the series $\sum_{n=1}^{\infty} a_n^{(i)}$ converges for each $i = 1, 2, \ldots, k$. Prove that

(a) the series $\sum_{n=1}^{\infty} b_n$ also converges; moreover,

(b)

$$\sum_{n=1}^{\infty} b_n = \sum_{i=1}^{k} \Big(\sum_{n=1}^{\infty} a_n^{(i)}\Big).$$

This is a discrete version of *Fubini's theorem*.

(5) Let (a_n) and (b_n) be two sequences of real numbers. Assume that they satisfy the following three properties:

(a) The partial sums of (b_n) is bounded: there exists $L > 0$ such that $|s_k| = |b_1 + \cdots + b_k| < L$ for any k;

(b) $\lim a_n = 0$;

(c) $\sum |a_{n+1} - a_n|$ is convergent.

Prove that the series $\sum a_n b_n$ is convergent. This is known as *Abel's theorem.* (Hint: Show that $\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} a_n (s_n - s_{n-1}) = \sum_{n=M}^{N-1} (a_n - a_{n+1}) s_n + a_N s_N - a_M s_{M-1}$, then try to apply the assumptions.)

(6) (optional; basic knowledge of complex numbers required) Show that the series

$$\sum \frac{\cos(n\theta)}{n} \text{ and } \sum \frac{\sin(n\theta)}{n}$$

are convergent for any $0 < \theta < 2\pi$.

(Hint: Show that

$$\left(\sum_{n=1}^{N} \cos(n\theta)\right) + i \left(\sum_{n=1}^{N} \sin(n\theta)\right) = \sum_{n=1}^{N} e^{in\theta} = e^{i\theta} \frac{1 - e^{iN\theta}}{1 - e^{i\theta}} = e^{i(N+1)\theta/2} \frac{\sin(N\theta/2)}{\sin(\theta/2)}$$

and use the previous problem.)