- fixed pt.
- fixed orbit
-

Local study: only interesting @ fixed pt.

Global: Poincaré-Bendixson thm.

- Lorenz ^chaotic attractor (in a cpt set, but not periodic) ~~not fixed~~.

Let's consider a predator–prey model:

$$\begin{cases} x'(t) = (2-y)x. \\ y'(t) = (x-1)y. \end{cases}$$

e.g. $x(t) = $ # of rabbits at time t.

$y(t) = $ # of foxes at time t

In general, this is an autonomus eq^n.

$$\vec{x}'(t) = \vec{f}(\vec{x})$$

or equivalently,

$$x_1'(t) = f_1(x_1, \cdots, x_n)$$

$$\vdots$$

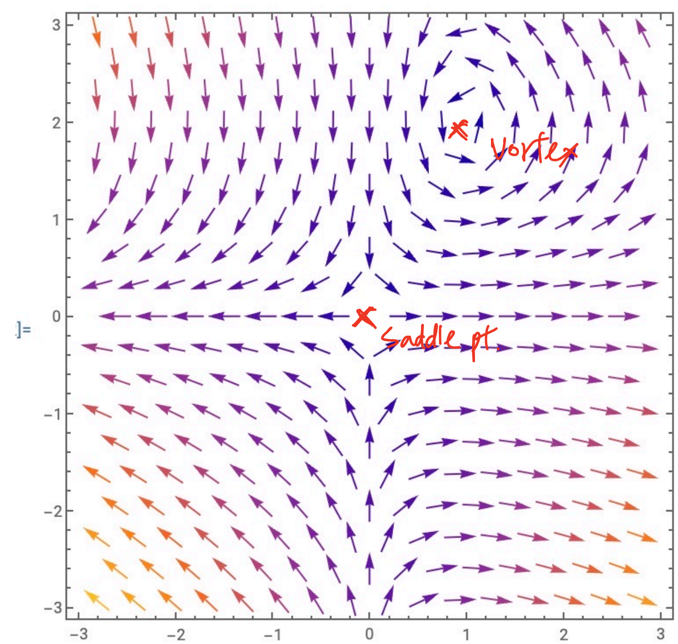
$$x_n'(t) = f_n(x_1, \cdots, x_n)$$

where the function $f = (f_1, \cdots, f_n)$ is independent of the time variable t.

We can draw a vector field where we put vector $\vec{f}(\vec{x})$ at the point $\vec{x}$.

Then solving the eq^n $\Longleftrightarrow$ finding integral curves of the vector field.

```
VectorPlot[{(2 - y) x, (x - 1) y}, {x, -3, 3}, {y, -3, 3}]
```



Rmk: We drew similar pictures before, for linear systems $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

In the example above, there are 2 special points: $(0,0)$, $(1,2)$.

One can check that they're the zeros of

$$\vec{F}(x,y) = \begin{bmatrix} (2-y)x \\ (x-1)y \end{bmatrix}.$$

Def: A point $x_0 \in \mathbb{R}^n$ is called a fixed point of $\vec{F}$ if $\vec{F}(x_0) = \vec{0}$.

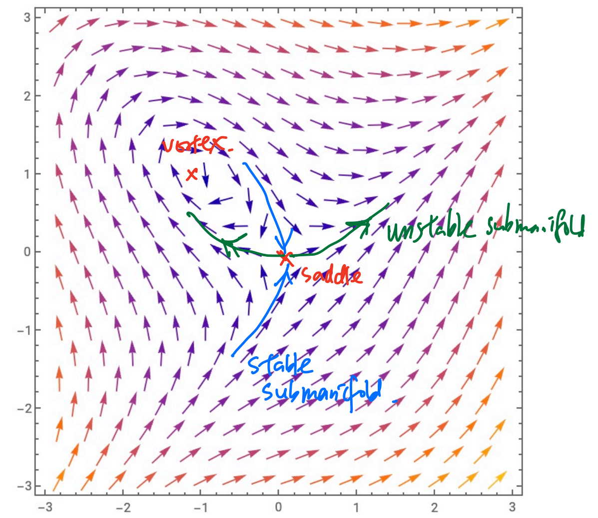
Fact: The unique sol^n of the IVP:

$$\vec{x}' = \vec{F}(\vec{x}), \quad \vec{x}(0) = x_0$$

is given by $\vec{x}(t) \equiv x_0$.

Next, we'd like to analyse the behavior of the flow lines near the fixed pts,

VectorPlot[{x + y^2, -y + x^2}, {x, -3, 3}, {y, -3, 3}]



The behavior near a fixed pt is essentially governed by the linear part of $\vec{F}$, i.e. the derivative $D\vec{F}(x_0) \in M_{n\times n}(\mathbb{R})$.

Recall that, if $f \in C^2$, then

$$\vec{F}(\vec{x}) = \vec{F}(\vec{x_0}) + D\vec{F}(\vec{x_0}) \cdot (\vec{x} - \vec{x_0}) + O(|\vec{x} - \vec{x_0}|^2)$$

near $\vec{x_0}$.

where

$$D\vec{F}(\vec{x_0}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{x_0}$$

Def: A fixed pt x_0 is called hyperbolic if the eigenvalues $\lambda_1, -, \lambda_n$ of $D\vec{f}(\vec{x}_0)$ have nonzero real parts. $Re(\lambda_i) \neq 0 \ \forall i$.

Say $Re(\lambda_1), \dots, Re(\lambda_s) < 0$

$Re(\lambda_{s+1}), \dots, Re(\lambda_n) > 0.$ $\quad 0 \leq s \leq n.$

Thm: Let x_0 be a hyperbolic fixed pt. of $\vec{f}$. Then

$\exists$ manifold $W^s(x_0)$ of dim. s, "stable submanifold"

manifold $W^u(x_0)$ of dim $n-s$. "unstable submanifold"

s.t. in a neighborhood of x_0,

$$x \in W^s(x_0) \iff \lim_{t \to \infty} \phi_t(x) = x_0$$

$$x \in W^u(x_0) \iff \lim_{t \to -\infty} \phi_t(x) = x_0$$

(Here $\phi_t(\tilde{x})$ denotes the sol$^{\underline{n}}$ of the IVP:

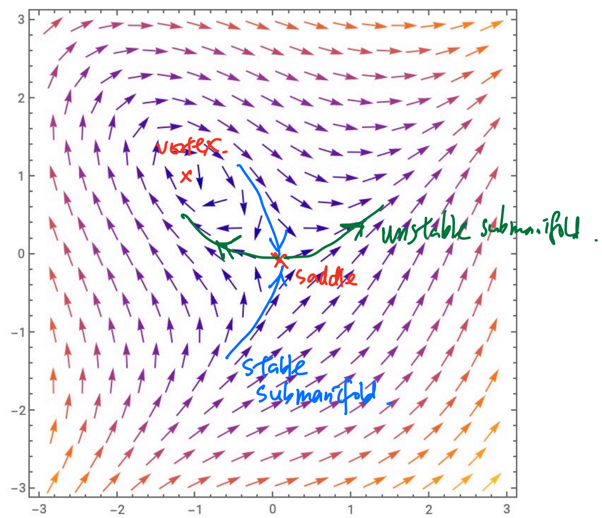
$\vec{x}'(t) = \vec{f}(\vec{x})$, $\vec{x}(0) = \tilde{x}$.)

Rmk: We'll not give a precise definition of manifolds here. We'll mostly consider the 2-dim'l cases. ($n=2$). In this case,

- if $s=0$ $\rightsquigarrow$ $W^u(x_0)$ is a neighbhd of x_0
  $\rightsquigarrow$ x_0 is a source
- if $s=2$ $\rightsquigarrow$ $W^s(x_0)$ is a nbd of x_0
  $\rightsquigarrow$ x_0 is a sink
- if $s=1$ $\rightsquigarrow$ $W^s(x_0)$, $W^u(x_0)$ are two curves passing through x_0.

where the tangent vectors at x_0 are the eigenvectors of $D\vec{f}(\vec{x}_0)$

```
VectorPlot[{x+y^2, -y+x^2}, {x, -3, 3}, {y, -3, 3}]
```



e.g. $\begin{cases} x' = x + y^2 \\ y' = -y + x^2 \end{cases}$

Let's analyze the stable & unstable submfd at $(x,y)=(0,0)$:

$$D\vec{f}(0) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow$ the unstable curve has tangent vector parallel to the x-axis. (@$(0,0)$)

the stable curve has tangent vector parallel to the y-axis (@$(0,0)$)

For instance,

$$W^u(x) = \{(x, g(x)) : |x| < \varepsilon\}$$

for some function g with $g(0)=0$, $g'(0)=0$.

We can find the Taylor expansion of g:

write

$$g(x) = a_2 x^2 + a_3 x^3 + \cdots$$

We know that

$$(x, g(x))' = (1, g'(x))$$

should be parallel to

$$(x + g(x)^2, \; -g(x) + x^2).$$

$$\Rightarrow \quad g'(x) = \frac{-g(x) + x^2}{x + g(x)^2}.$$

$-(a_2-1)x^2 - a_3 x^3 - a_4 x^4 - \cdots$

$x + a_2^2 x^4 + \cdots$

//

$2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \quad \Rightarrow \quad 2a_2 = 1 - a_2,$

$\cdots$

Note that we can also use the fact that

$$(1, g'(x)) \parallel (x+g(x)^2, -g(x)+x^2)$$

to deduce that $g'(0)$ should be 0.

(i.e. the unstable curve is tangent to x-axis @ 0)

Write $g(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

Then

$$g'(x) = \frac{-g(x)+x^2}{x+g(x)^2}$$

$-g(x)+x^2 = -a_1 x - (a_2-1)x^2 - a_3 x^3 - a_4 x^4 - \cdots$

$x+g(x)^2 = x + a_1^2 x^2 + \cdots$

$g'(x) = a_1 + 2a_2 x + \cdots$

$$\Rightarrow a_1 = -a_1$$

$$\Rightarrow a_1 = 0.$$

pf. of Thm, under assumptions: $s=n$, where $D\vec{f}(\vec{x}_0) = 0$ has real eigenvalues $\lambda_1, \ldots, \lambda_n < 0$ and eigenvectors $v_1, \ldots, v_n$

Write $D\vec{f}(\vec{x}_0) = PDP^{-1}$, where $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

We'd like to show:

$$\lim_{t\to\infty} \vec{x}(t) = \vec{0} \quad \text{if } \vec{x}(0) \text{ is close to } \vec{0}$$

Let $\vec{y}(t) := P^{-1}\vec{x}(t)$ and define

$$U(t) := |P^{-1}\vec{x}(t)|^2 = |\vec{y}(t)|^2$$

We'll show that $\lim_{t\to\infty} U(t) = 0$.

Claim: For $\vec{y}(0)$ close to $\vec{0}$, we have:

$$\frac{d\,U(t)}{dt} \leq -C\,U(t).$$

$$\left(\Rightarrow U(t) \leq U(0)e^{-Ct} \to 0 \text{ as } t\to\infty\right)$$

$$\frac{d\,|\vec{y}(t)|^2}{dt} = 2\langle P^{-1}\vec{x}, P^{-1}\vec{x}'\rangle$$

$$= 2\langle P^{-1}\vec{x}, P^{-1}\vec{f}(\vec{x})\rangle$$

$$= 2\langle P^{-1}\vec{x}, DP^{-1}\vec{x} + \overbrace{O(|\vec{x}|^2)}^{R(\vec{x})}\rangle$$

- $\langle P^{-1}\vec{x}, DP^{-1}\vec{x}\rangle = \langle \vec{y}, D\vec{y}\rangle$

$$\leq -\min_{1\leq i\leq n}|\lambda_i|\cdot|\vec{y}|^2$$

key step in the proof.

- Estimate: $\langle P^{-1}\vec{x}, R(\vec{x})\rangle$

$$R(\vec{x}) = O(|\vec{x}|^2)$$

$\forall \varepsilon>0$, $\exists \delta>0$ st.

$$|\vec{x}|\leq\delta \Rightarrow |R(\vec{x})| < \varepsilon\cdot|\vec{x}|.$$

$$|\langle P^{-1}\vec{x}, R(\vec{x})\rangle|^2 \leq \|\vec{y}\|\cdot\|R(\vec{x})\|$$

$$\leq \|\vec{y}\|\cdot\varepsilon\cdot\|\vec{x}\|$$

$P\vec{y}$

$$\leq \varepsilon\cdot\|P\|^2\,\|\vec{y}\|^2. \qquad \square$$

Rmk: modifications for general case:

① if $A = \begin{pmatrix}-a & b\\ -b & -a\end{pmatrix}$ has eigenvals $-a\pm ib$. $(a>0)$.

$$\Rightarrow \langle\vec{y}, A\vec{y}\rangle = -a\|\vec{y}\|^2.$$

so the same argument works.

② if $A = \begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$

$$\Rightarrow \langle\vec{y}, A\vec{y}\rangle = \left\langle \begin{bmatrix}y_1\\ y_2\end{bmatrix}, \begin{bmatrix}-y_1+y_2\\ -y_2\end{bmatrix}\right\rangle$$

$$= -y_1^2 + y_1y_2 - y_2^2$$

$$\leq -\frac{1}{2}\|\vec{y}\|^2$$

Same argument still works.

Rmk: The function $U(\vec{x}) = |P^{-1}\vec{x}|^2$ we considered in the proof, is called a Lyapunov function. It can be considered as a "distance" to the fixed point.

Def: Let $\vec{f} \in \mathcal{C}^0(D)$, $\vec{f}(0) = \vec{0}$. A function $V \in \mathcal{C}^1(D)$ is called a Lyapunov function if

- $V(\vec{0}) = 0$, $V(\vec{x}) > 0$ $\forall \vec{x} \in D \setminus \vec{0}$.
- $\dot{V}(\vec{x}) \leq 0$ in D,

where

$$\dot{V}(\vec{x}) := \langle \nabla V(\vec{x}), \vec{f}(\vec{x}) \rangle$$

$$= \lim_{t \to 0} \frac{V(\vec{x} + t\vec{f}(x)) - V(\vec{x})}{t}$$

Rmk: If $\vec{x}' = \vec{f}(\vec{x})$, then

$$\frac{d}{dt} V(\vec{x}(t)) = \dot{V}(\vec{x}(t))$$

$$\parallel$$

$$\frac{d}{dt}\left(V_1(x_1(t)), \cdots, V_n(x_n(t)) \right)$$

$$\parallel$$

$$\left(V_1'(x_1(t)) \cdot x_1'(t), \cdots, V_n'(x_n(t)) x_n'(t) \right)$$

$$\parallel$$

$$\langle \nabla V, x' \rangle = \langle \nabla V, \vec{f} \rangle =: \dot{V}$$

$\Rightarrow$ $\dot{V}$ = derivative of V along trajectories of $\vec{x}' = \vec{f}(\vec{x})$

Rmk: $\dot{V}(\vec{x}) \leq 0$ means that, the distance to the fixed pt is non-increasing as we move along in the trajectories

Thm (Lyapunov): $\vec{f} \in C(D)$, $\vec{f}(\vec{0}) = \vec{0}$,

Suppose $\exists$ Lyapunov fcn $V : D \to \mathbb{R}$

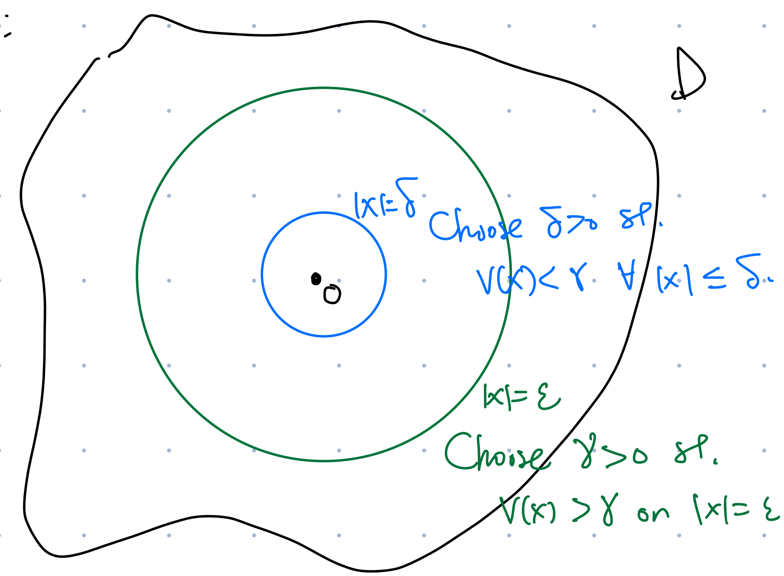
then

- $\dot{V} \le 0$ in $D \Rightarrow \vec{0}$ is a stable fixed point of $\vec{x}' = \vec{f}(\vec{x})$, i.e. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $x \in B_\delta(x_0)$, then $\phi_t(x) \in B_\varepsilon(x_0)$ $\forall t \ge 0$.

- $\dot{V} < 0$ in $D \setminus \{\vec{0}\} \Rightarrow \vec{0}$ is an asymptotically stable fixed pt of $\vec{x}' = \vec{f}(\vec{x})$, i.e. if is stable, and: $\exists \varepsilon > 0$ s.t. if $x \in B_\varepsilon(x_0)$, then $\phi_t(x) \to x_0$.

- $\dot{V} \le -\alpha V$ and $V(\vec{x}) \ge c|\vec{x}|^\beta$ in D $(\alpha, \beta, c > 0) \Rightarrow \vec{0}$ is an exponentially stable fixed pt., i.e. $\exists \varepsilon, \gamma, \tilde{c} > 0$ s.t. $x \in B_\varepsilon(x_0) \Rightarrow |\phi_t(x)| < \tilde{c} \cdot e^{-\gamma t}$ $\forall t > 0$.

pf:



This proves the first statement.

---

(2): $\dot{V} < 0 \Rightarrow t \mapsto V(\phi_x(t))$ is strictly decreasing

Claim: $\lim_{t \to \infty} V(\phi_x(t)) = 0$.

If not, say $\lim_{t \to \infty} V(\phi_x(t)) = \beta > 0$

$M := \{\vec{x} \in \overline{B_\varepsilon(0)} \mid \beta \le V(x) \le \gamma\}$ Choose any $\gamma > \beta$

is a compact subset of $\overline{B_\varepsilon(0)}$

$\Rightarrow \max\{\dot{V}(x) : x \in M\} = -\alpha < 0$. $\times$

(3) $c\,|\vec{x}|^{\beta} \leq V(\vec{x})$

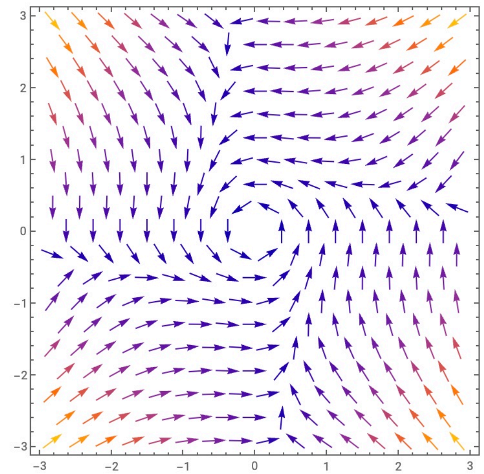
$$\phi' \leq -\alpha\,\phi.$$

$$\Rightarrow \phi(t) \leq \phi(0)e^{-\alpha t}$$

???

Walter's Book ?!!...

```
VectorPlot[{-y - x*y*y, x - y*x*x}, {x, -3, 3}, {y, -3, 3}]
```



$$\begin{cases} x' = -y - xy^2 \\ y' = x - x^2 y \end{cases}$$

Fixed pts: $-y - xy^2 = 0 = -y(xy+1)$ $\Rightarrow x = y = 0$

$x - x^2 y = 0 = x(1 - xy)$

$$D\vec{f}(0,0) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\Rightarrow$ NOT hyperbolic

(no stable/unstable submanifolds)

We can still use Lyapunov's method.!

Let $V(x,y) = x^2 + y^2$. Then

$$\dot{V}(x,y) = \langle \nabla V, f \rangle$$
$$= \langle (2x, 2y), (-y - xy^2, x - x^2 y) \rangle$$
$$= -4x^2y^2 \leq 0.$$

$\Rightarrow$ 0 is a <u>stable</u> fixed pt.

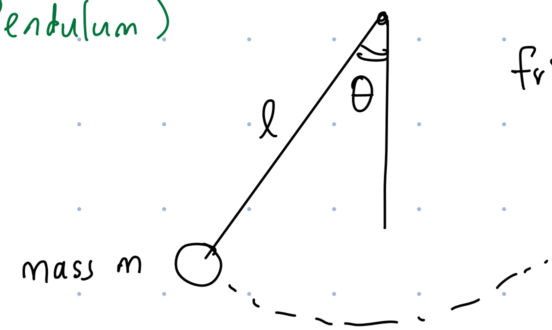
In fact, we can show that 0 is an asymptotically stable pt. !

Assume that $V(\vec{x}(t)) \searrow V_0 > 0$

$\Rightarrow$ the solution curve $\vec{x}(t)$ would converge to the circle $\{(x,y) : x^2 + y^2 = V_0\}$.

... ??? (Lecture Note)

e.g. (Pendulum)



friction k.

Force $\sim$ $k\ell \dfrac{d\theta}{dt} + mg\sin\theta$

Newton's law $\Rightarrow$ $\theta'' = -\dfrac{k}{m}\theta' - \dfrac{g}{\ell}\sin\theta$

Denote $\omega := \theta'$, then:

$$\begin{cases} \theta' = \omega \\ \omega' = -\dfrac{g}{\ell}\sin\theta - \dfrac{k}{m}\omega. \end{cases}$$

- fixed pts: $\omega = 0$, $\theta = n\pi$.
- Linearization at $(\theta, \omega) = (0,0)$:

$$\begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix}.$$ eigenvalues have negative real parts!

$\Rightarrow$ $(0,0)$ is a sink.

i.e. the pendulum will eventually stopped at $\theta = 0$, as expected.

(the eq^n has sinks at $\theta = n\pi$ $\forall n \in \mathbb{Z}$).

One can construct a Lyapunov fcn:

$$E = (\text{kinetic energy}) + (\text{potential energy})$$
$$= \frac{1}{2}mv^2 + mg\ell(1-\cos\theta)$$
$$= \frac{1}{2}m\ell^2\omega^2 + mg\ell(1-\cos\theta)$$

$E \geq 0$, and $E = 0 \Leftrightarrow \omega = 0$ and $\theta = n\pi$.

$$\dot{E} = mg\ell\sin\theta \cdot \omega + m\ell^2\omega \cdot \left(-\frac{g}{\ell}\sin\theta - \frac{k}{m}\omega\right)$$
$$= -k\ell^2\omega^2 \quad < 0 \quad \text{except } \omega = 0.$$

Rmk: If $k = 0$ (i.e. no friction), then $\dot{E} = 0$, so solutions stay on level sets of E.

DRAW GRAPHS!

Rmk: When $k=0$, we have:

$$E_\theta = mgl\sin\theta$$
$$E_\omega = ml^2\omega$$

so $\tilde{E} := E/ml^2$ satisfies:

$$\tilde{E}_\theta = \frac{g}{l}\sin\theta, \quad \tilde{E}_\omega = \omega.$$

This is an example of a Hamiltonian system.

i.e. a system for which $\exists\, H: \mathbb{R}^2 \to \mathbb{R}$

st. $\quad x' = \dfrac{\partial H}{\partial y}, \quad y' = -\dfrac{\partial H}{\partial x}.$

H can be regarded as the energy of the system, which is conserved over time:

$$\dot{H} = \frac{\partial H}{\partial x}\cdot x' + \frac{\partial H}{\partial y}\cdot y' = 0.$$

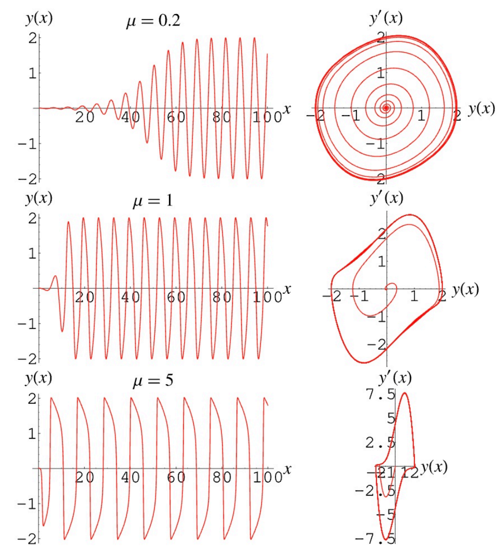
Van der Pol's eq^n:

$$\begin{cases} x' = y - x^3 + x \\ y' = -x. \end{cases}$$

$$\iff x'' = -x - 3x^2x' + x'$$

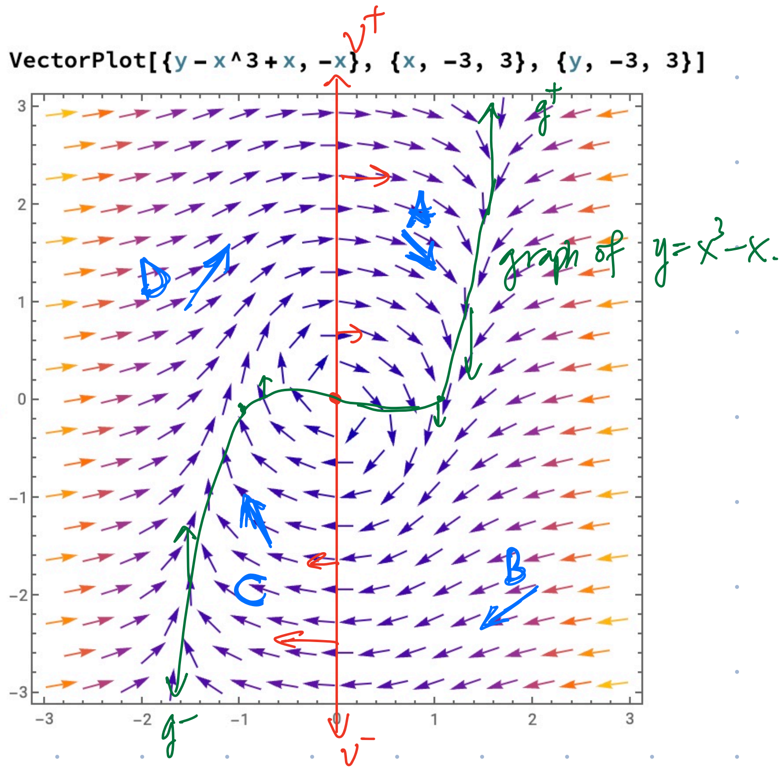
$$\iff x'' - (1-3x^2)x' + x = 0.$$

Rmk: This shows up often in electrical engineering $\quad y'' - \mu(1-y^2)y' + y = 0.$



Thm The van der Pol's eq^{n} has one periodic sol^{n}, and every non-equilibrium sol^{n}s tend to this periodic sol^{n}.

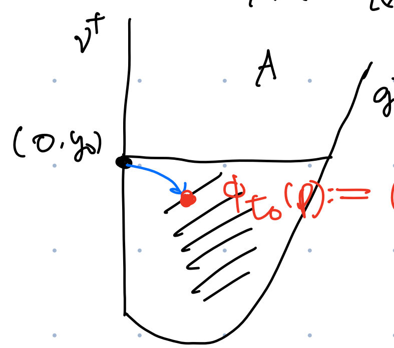
$$\begin{cases} x' = y - x^3 + x \\ y' = -x. \end{cases}$$



Lemma: $\forall\, p \in v^+$, $\exists\, t>0$ s.t. $\phi_t(p) \in g^+$.

pf:

- $x'(0) > 0 \Rightarrow \phi_t(p) \in A$ for $t>0$ small.
- Since $x'>0$, $y'<0$, the only way that $\phi_t(p)$ can leave the region $A \cap \{(x,y) \mid y < y_0\}$ is via crossing g^+



$\phi_{t_0}(p) := (a, y(t_0))$

- Define $T := \inf\{t>0 \mid \phi_t(p) \in g^+\}$.

Want: $T < +\infty$.

- $x(t) \geq a > 0 \;\; \forall\, t_0 \leq t \leq T$

$\Rightarrow y'(t) \leq -a \;\; \forall\, t \in [t_0, T]$.

$\Rightarrow T$ must be finite. □

Similarly, one can show that:

$\forall p \in g^+, \exists t>0$ s.t. $\phi_t(p) \in v^-$.

For each $y>0$, define $F(y) := \phi_t(0,y)$ where $t>0$ is the minimal s.t. $\phi_t(0,y) \in v^-$.

Similarly, for $y<0$, define $F(y) = \phi_t(0,y)$ where $t>0$ is minimal st. $\phi_t(0,y) \in v^+$.

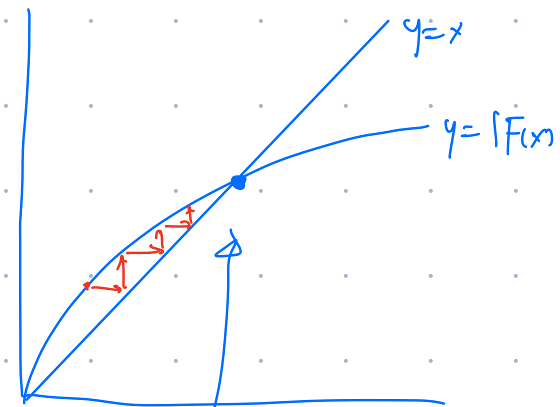
By symmetry, we have $F(-y) = -F(y)$.

Define the Poincaré first return map as:

$$P: v^+ \longrightarrow v^+$$
$$(0,y) \longmapsto (0, F^2(y))$$

Note that $P(p) = \phi_t(p)$ where $t>0$ is the minimal st. $\phi_t(p) \in v^+$.

We'll show that $x \mapsto |F(x)|$ look like:



and both $|F|$ and P would have a unique fixed pt.

$P(p) = |F(|F(p)|)|$

Define $p^* := (0, y^*) \in v^+$

st. $\exists t>0$ with $\phi_t(p^*) = (1,0)$,

and $\phi_s(p^*) \in A \quad \forall\, 0<s<t$.

Lemma: ① $P: \nu^+ \to \nu^+$ is continuous and increasing.

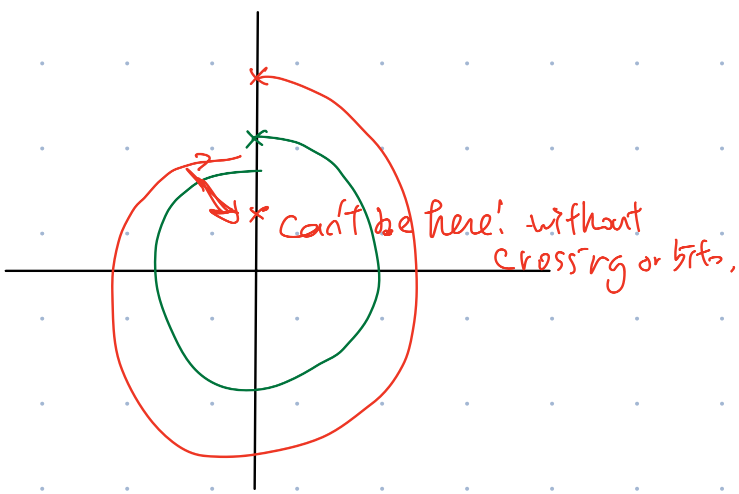
② $P(p) > p$ if $p \in [0, p^*]$.

③ $P(p) < p$ if p large

④ $P: \nu^+ \to \nu^+$ has a unique attracting fixed point $q \in [p^*, \infty)$.

pf: ① follows from $(t, x) \mapsto \phi_t(x)$ is continuous, and uniqueness of sol^n

$\Rightarrow$ orbits don't cross each other.

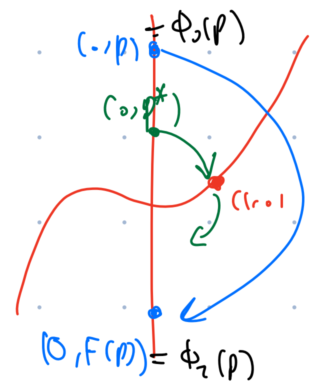


Claim: f is also increasing (by non-crossing)

⑤ $p \mapsto \delta(p) := |F(p)|^2 - p^2$ is strictly decreasing on $[p^*, \infty)$

⑥ $\delta(p) > 0$ for $p \in [0, p^*]$

⑦ $\delta(p) \to -\infty$ as $p \to +\infty$.



Define $u(x,y) = x^2 + y^2$

Step 1: Another description of $\delta(p)$.

$$\delta(p) = |F(p)|^2 - p^2 = u(\phi_\tau(p)) - u(\phi_0(p))$$

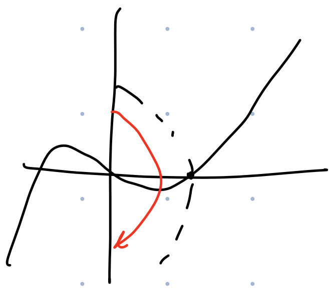
$$= \int_0^\tau \dot{u}(\phi_t(p))\, dt$$

$$2xx' + 2yy' = 2x(y - x^3 + x) + 2y(-x) = 2x^2(1 - x^2)$$

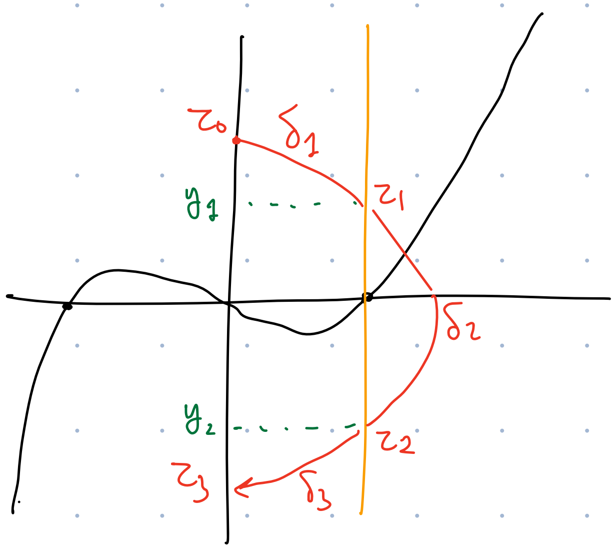
$$\Rightarrow \delta(p) = 2\int_0^{z} x(t)^2\,[1-x(t)^2]\,dt$$

Step 2: $\delta(p) > 0$ when $p \in (0, p^*]$

Since $0 < x(t) < 1$ in this region.



Step 3: $\delta(p)$ for $p > p^*$.



We'll show that $\delta_1, \delta_2, \delta_3 \downarrow$ for $p > p^*$

①
$$\frac{\delta_1}{2}(p) = \int_0^{z_1} x(t)^2\,[1-x(t)^2]\,dt$$
$$= \int_0^1 \frac{x^2(1-x^2)}{x'(t)}\,dx$$
$$= \int_0^1 \frac{x^2(1-x^2)}{y(x) - x^3 - x}\,dx.$$

As $p \uparrow$, $y(x) \uparrow \Rightarrow \delta_1(p) \downarrow$.

② δ_3: the same.

③ δ_2:
$$\int_{z_1}^{z_2} x(t)^2\,[1-x(t)^2]\,dt$$
$$= \int_{y_1}^{y_2} \frac{x(y)^2\,[1-x(y)^2]}{\underbrace{y'(t)}_{\boxed{-x}}}\,dy$$
$$= \int_{y_2}^{y_1} x(y)\,[1-x(y)^2]\,dy < 0 \text{ since } x>1.$$

As $p \nearrow$, we see that

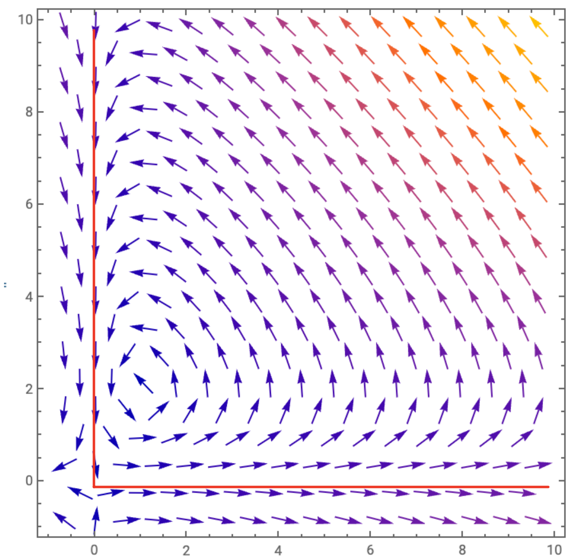
- $\delta_2 \searrow$, and
- $\delta_2 \to -\infty$ as $p \to +\infty$.

$\square$

Predator-prey model.

$$\begin{cases} x' = (A - By)x \\ y' = (Cx - D)y \end{cases} \qquad A, B, C, D > 0$$

```
VectorPlot[{(2 - y) x, (x - 1) y}, {x, -1, 10}, {y, -1, 10}]
Plot[x^3 - x, {x, -3, 3}]
```



Fixed pts: $(0,0)$, $\left(\frac{D}{C}, \frac{A}{B}\right)$

$(0,0)$: saddle pt.

$\left(\frac{D}{C}, \frac{A}{B}\right)$: linearization: $\begin{bmatrix} 0 & -\frac{BD}{C} \\ \frac{AC}{B} & 0 \end{bmatrix}$

eigenvalues: $\pm \sqrt{AD}\, i$.

Are the orbits periodic?

In fact, we can find the integral curves explicitly and show that they're closed.

Consider $H(x,y) = F(x) + G(y)$.

If $\{H(x,y) \equiv c\}$ are integral curves,

then $F' \cdot \dot{x} + G' \cdot \dot{y} = 0$

$$\underbrace{F' \cdot \dot{x} + G' \cdot \dot{y}}_{=\, x F'(x)(A - By) + y G'(y)(Cx - D)} = 0$$

$$\Rightarrow \quad \frac{x F'(x)}{Cx - D} = \frac{y G'(y)}{By - A} \equiv \text{const} = 1$$

$$\Rightarrow \quad F(x) = Cx - D \log x + \text{const}.$$

$$G(y) = By - A \log y + \text{const}.$$

$$\Rightarrow \quad H(x,y) = Cx - D \log x + By - A \log y + \text{const}.$$

Orbits can't escape from $\{x, y > 0\}$.

$\Rightarrow$ $H(x,y) \equiv c$ must be periodic.