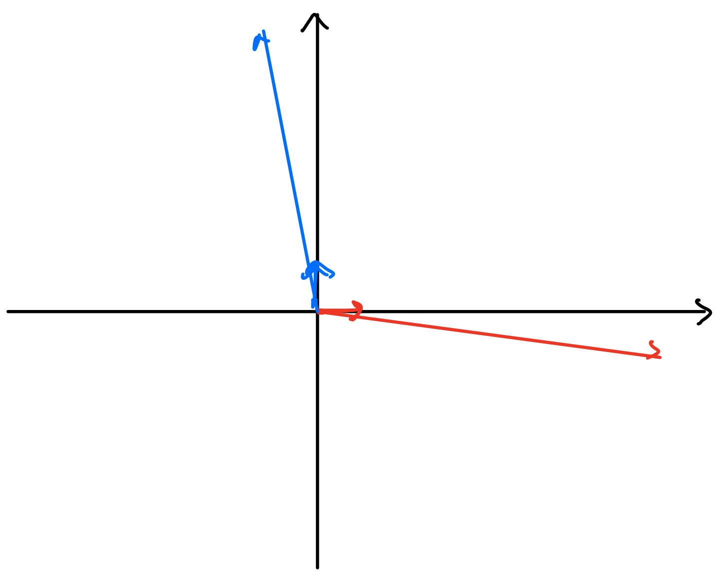
Rmk: Why do we want to choose different bases?

RECALL

e.g. $A = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}$, $T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 11 \\ -2 \end{bmatrix}$

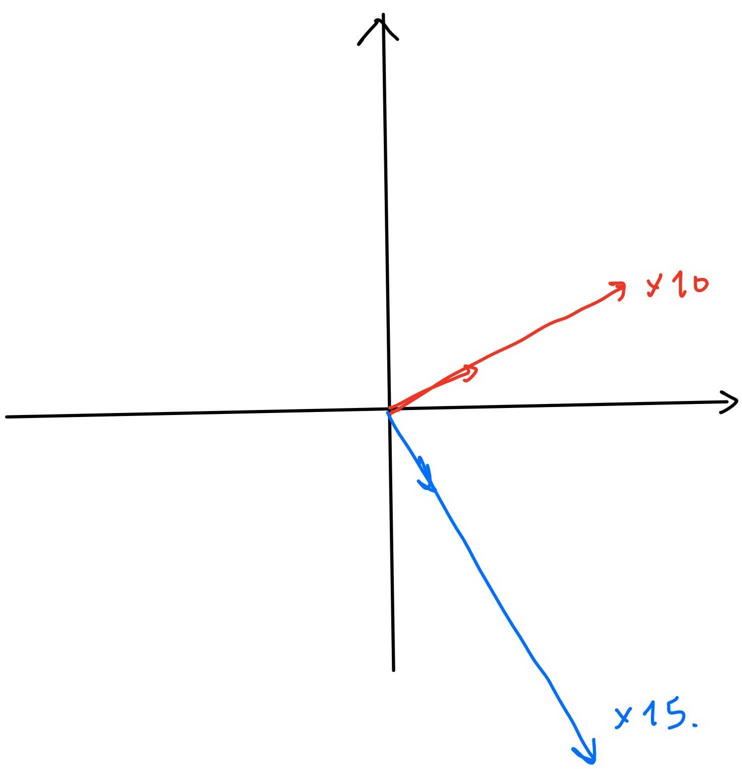
$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -2 \\ 14 \end{bmatrix}$



If we consider the basis $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} =: B$

$$T_A\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 15 \\ -30 \end{bmatrix} = 15 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



$\vec{x} \in \mathbb{R}^2$, $\vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

$$T_A(\vec{x}) = c_1 \underline{T_A \begin{bmatrix} 2 \\ 1 \end{bmatrix}} + c_2 \underline{T_A \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

$$= 10 c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 15 c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$[T_A(\vec{x})]_B = \begin{bmatrix} 10 c_1 \\ 15 c_2 \end{bmatrix}$$

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "T_A"] \arrow[d, shift right, "{[\ ]_{\mathcal{B}}}"'] & \mathbb{R}^2 \arrow[d, "{[\ ]_{\mathcal{B}}}"] \\
\mathbb{R}^2 \arrow[u, shift right, "{[\ ]_{\mathcal{B}}^{-1}}"'] & \mathbb{R}^2
\end{tikzcd}
\qquad
\boxed{A = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}}
$$

$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$.

$P_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "{T_{P_{\mathcal{B}}} \,=\, [\ ]_{\mathcal{B}}^{-1}}"] \arrow[rrr, bend right, "T_{\begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix}}"'] & \mathbb{R}^2 \arrow[r, "T_A"] & \mathbb{R}^2 \arrow[r, "{T_{P_{\mathcal{B}}^{-1}} \,=\, [\ ]_{\mathcal{B}}}"] & \mathbb{R}^2
\end{tikzcd}
$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longmapsto 10 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longmapsto 10 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 \\ -2 \end{bmatrix} \longmapsto 15 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \longmapsto 15 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T_{\begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix}} = T_{P_{\mathcal{B}}^{-1}} \circ T_A \circ T_{P_{\mathcal{B}}} = T_{P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}}.$$

$$\begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix} = P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}.$$

"diagonalization".

Def Say A and B (two square matrices) are <u>similar</u> if $\exists$ an invertible matrix P s.t.

$$A = PBP^{-1}. \qquad (\Leftrightarrow \quad P^{-1}AP = B)$$

Rmk: T_A and T_B are related by a change of basis given by the invertible matrix P.

Def Say a square matrix A is <u>diagonalizable</u> if it's similar to a diagonal matrix $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$,

i.e. $\exists\, P$ invertible s.t. $A = PDP^{-1}$.

Rmk: If A is diagonalizable, then A^k is easy to compute.

$$A^k = (PDP^{-1})^k = \underbrace{(PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})(\cancel{P}\cdots (PDP^{-1})}_{k}$$

$$= PD^kP^{-1}$$

$$= P \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} P^{-1}.$$

Suppose we have $A = PDP^{-1}$.

$P = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\Rightarrow \quad AP = PD$

$$AP = A\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} | & & | \\ A\vec{v}_1 & \cdots & A\vec{v}_n \\ | & & | \end{bmatrix}$$

$$PD = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$\Rightarrow \quad A\vec{v}_i = \lambda_i \vec{v}_i$ for each i.

Def: • Say $\vec{v} \neq \vec{0}$ is an eigenvector of A if $A\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$ $(\lambda \in \mathbb{C})$.

an eigenvalue of A

λ is an eigenvalue

$\iff \exists\, \vec{v} \neq \vec{0}$ s.t. $A\vec{v} = \lambda\vec{v}$.

$\iff \exists\, \vec{v} \neq \vec{0}$ s.t. $(A - \lambda I)\vec{v} = \vec{0}$.

$\iff \operatorname{Nul}(A - \lambda I) \supsetneqq \{0\}$

$\iff A - \lambda I$ is not invertible.

$\iff \det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22}-\lambda & & \\ & & \ddots & \\ & & & a_{nn}-\lambda \end{bmatrix}$$

e.g. $A = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}$

To find the eigenvalues of A, $\iff$ to find all λ s.t.

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 11-\lambda & -2 \\ -2 & 14-\lambda \end{bmatrix}$$

$$\parallel$$

$$(11-\lambda)(14-\lambda) - 4$$

$$\parallel$$

$$\lambda^2 - 25\lambda + 150 = (\lambda - 10)(\lambda - 15)$$

$\Rightarrow$ 10 and 15 are eigenvalues of A.

To find the eigenvectors associated to 10:

$$\mathrm{Nul}(A - 10I) = \mathrm{Nul}\left(\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\right) = \mathrm{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

Eigenvectors of 15:

$$\mathrm{Nul}(A - 15I) = \mathrm{Nul}\left(\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}\right) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$$

Def:

- λ is an eigenvalue $\Leftrightarrow$ $\mathrm{Nul}(A - \lambda I) \supsetneq \{\vec{0}\}$

  $\mathrm{Nul}(A - \lambda I)$ is called the eigenspace of λ

- characteristic polynomial of A is defined to be "$\det(A - \lambda I)$" (where λ is treated as the variable)

$$= \det\begin{bmatrix} a_{11}-\lambda & a_{12} & & \vdots \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & \vdots & \cdots & \\ a_{n1} & a_{n2} & & a_{nn}-\lambda \end{bmatrix}$$

$\uparrow$ poly. of deg n in λ.

- eigenvalues are the roots of the characteristic poly.

$$\det(A - \lambda I) = \prod_{i=1}^{k} (\lambda_i - \lambda)^{\mathrm{mult}(\lambda_i)}$$

$\mathrm{mult}(\lambda_i)$: multiplicity of the root λ_i.

where $\{\lambda_1, \dots, \lambda_k\}$ are distinct eigenvalues of A.

e.g.

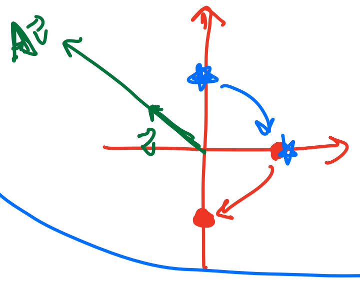
$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (2-\lambda)^2 (1-\lambda)$$

$\Rightarrow$ $\underset{\text{mult. 1}}{1}$ and $\underset{\text{mult. 2}}{2}$ are the eigenvalues of A

e.g. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



If $\exists\, \lambda \in \mathbb{R}$, $\vec{0} \neq \vec{v} \in \mathbb{R}^2$ s.t. $\boxed{A\vec{v}} = \lambda \vec{v}$

$$\det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$\Rightarrow$ eigenvalues are $\pm i \in \mathbb{C} \setminus \mathbb{R}$

Rmk: 0 is an eigenvalue of A $\iff$ $\det(A - 0I) = 0$

$\iff$ A is not invertible.

Thm An $n \times n$ matrix A is diagonalizable

$\iff$ $\exists$ an "eigenbasis" of A, i.e., $\exists$ a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of $\mathbb{R}^n$ $(\mathbb{C}^n)$, where each $\vec{v}_i$ is an eigenvector pf. ($\Rightarrow$) by our previous discussion.

($\Leftarrow$) Define $P = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$, $A\vec{v}_i = \lambda_i \vec{v}_i$ for some λ_i

$$AP = A\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} | & & | \\ A\vec{v}_1 & \cdots & A\vec{v}_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$= P\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\Rightarrow A = P\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}P^{-1}. \quad \square$$

Rmk: A, B similar. $\Rightarrow$ A and B have the same characteristic polynomial.

($\exists$ invertible P s.t. $A = PBP^{-1}$)

$\Rightarrow$ they have the same eigenvalues & multiplicities.

$$\det(A - \lambda I) = \det(PBP^{-1} - \lambda I)$$
$$= \det(PBP^{-1} - \lambda PP^{-1})$$
$$= \det(P(B - \lambda I)P^{-1})$$
$$= \cancel{\det(P)}\det(B - \lambda I)\cancel{\det(P^{-1})}$$
$$= \det(B - \lambda I).$$

Rmk: If A, B similar and A is diagonalizable $\Rightarrow$ B diagonalizable

$A = PBP^{-1}$ $\qquad$ $A = QDQ^{-1}$

$$B = P^{-1}AP = P^{-1}QDQ^{-1}P = (P^{-1}Q)D(P^{-1}Q)^{-1}$$

Rmk: A, B have the same char. poly $\not\Rightarrow$ A, B are similar.

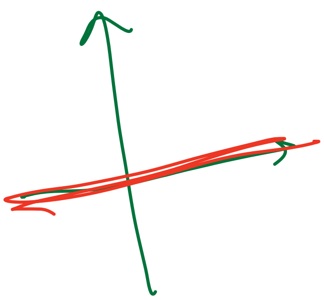
e.g. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$$\det(A - \lambda I) = (\lambda - 2)^2 = \det(B - \lambda I)$$

But A and B are not similar.

A is diagonalizable, but $\underbrace{B \text{ is not diagonalizable.}}_{\text{why?}}$

- 2 is the only eigenvalue of B.
- $\text{Nul}(B - 2I) = \text{Nul}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.



$\Rightarrow$ any eigenvector $\in \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$\Rightarrow$ there does not exist an eigenbasis of B.

$\Rightarrow$ B is not diagonalizable.

Rmk: Next time: we'll show that, for each eigenvalue λ,

- $1 \leq \dim \text{Nul}(A - \lambda I) \leq \text{multi}(\lambda)$,
- A is diagonalizable $\Leftrightarrow$ $\dim \text{Nul}(A - \lambda I) = \text{mult}(\lambda)$ for each eigenvalue λ.

Thm: Suppose $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A.
Suppose $\vec{v}_1, \ldots, \vec{v}_k$ are eigenvectors corresp. to $\lambda_1, \ldots, \lambda_k$.
$(A\vec{v}_i = \lambda_i \vec{v}_i)$

$\Longrightarrow \{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.

pf:

- Assume $\underline{a_1}\,\vec{v}_1 + \cdots + \underline{a_k}\,\vec{v}_k = \vec{0}$

- We can remove the terms with $a_i = 0$,

  So we can assume $a_i \neq 0$.

- $$A(a_1\vec{v}_1 + \cdots + a_k\vec{v}_k) = A\vec{0} = \vec{0}$$

$$\begin{aligned} &\parallel \\ &a_1 A\vec{v}_1 + \cdots + a_k A\vec{v}_k \\ &\parallel \\ &a_1\lambda_1\vec{v}_1 + \cdots + a_k\lambda_k\vec{v}_k. \end{aligned}$$

- $$\lambda_1(a_1\vec{v}_1 + \cdots + a_k\vec{v}_k) = \vec{0} \quad = a_1\lambda_1\vec{v}_1 + a_2\lambda_1\vec{v}_2 + \cdots + a_k\lambda_1\vec{v}_k.$$

$$a_1\lambda_1\vec{v}_1 + \cdots + a_k\lambda_k\vec{v}_k = \vec{0}$$

$$\Longrightarrow \underbrace{a_2(\lambda_1-\lambda_2)}_{\neq 0}\vec{v}_2 + \underbrace{a_3(\lambda_1-\lambda_3)}_{\neq 0}\vec{v}_3 + \cdots + \underbrace{a_k(\lambda_1-\lambda_k)}_{\neq 0}\vec{v}_k = \vec{0}$$

- Do this inductively, in the end, we'll find

$$\underbrace{b_k}_{\neq 0}\vec{v}_k = \vec{0}$$

Contradiction. $\square$