(1) Determine whether each of the following series converges or not. Prove your answers.

$$(a) \sum \frac{(-1)^n(n-1)}{n}; \ (b) \sum \frac{n^n}{(n+1)^{2n}}; \ (c) \sum \frac{(-1)^n}{n^{1/12}};$$

$$(d) \sum \frac{1}{(2n-1)^2}; \ (e) \sum \frac{1}{n \log n}; \ (f) \sum n e^{-n^2}.$$

(a) diverge: $\left(\frac{(-1)^n(n-1)}{n}\right)$ doesn't conv. to 0.

(b) converge: root test: $\limsup_{n\to\infty} \left(\frac{n^n}{(n+1)^{2n}}\right)^{1/n} = \limsup_{n\to\infty} \frac{n}{(n+1)^2} = 0 < 1.$

(c) converge: by alternating series test.

(d) converge: $\left|\frac{1}{(2n-1)^2}\right| \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2} < +\infty.$ (Comparison test)

(e) diverge: integral test: $\int_2^\infty \frac{dx}{x \log x} = \int_{\log 2}^\infty \frac{dt}{t} = +\infty$

(f): converge: root test: $\limsup_{n\to\infty} \left(\frac{n}{e^{n^2}}\right)^{1/n} = \limsup \frac{n^{1/n}}{e^n} = 0 < 1.$

(2) Prove the triangle inequality for series: if $\sum a_n$ converges absolutely, then

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|.$$

By triangle inequality, we have: $-\sum_{n=1}^{k} |a_n| \leq \sum_{n=1}^{k} a_n \leq \sum_{n=1}^{k} |a_n|.$

Take limit $k \to +\infty$, we obtain:

$$-\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n|.$$

limit exists since $\sum a_n$ conv. abs.

$\Rightarrow \left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|.$ $\square$

(3) Show that the monotonicity assumption in alternating series test is necessary: find a sequence of positive real numbers (a_n) with $\lim a_n = 0$, but $\sum(-1)^n a_n$ diverges.

Define (a_n) where:

- $(a_{2n-1}) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots)$
- $(a_{2n}) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots)$

Then $\lim a_n = 0$.

But $\sum (-1)^n a_n = -1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \frac{1}{6} - \dots$

$= -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \dots$ diverges. $\square$

(4) Let $(a_n^{(1)})_{n=1}^\infty, (a_n^{(2)})_{n=1}^\infty, \dots, (a_n^{(k)})_{n=1}^\infty$ denote k sequences of real numbers. (For instance, the first sequence is $(a_1^{(1)}, a_2^{(1)}, \dots, a_n^{(1)}, \dots)$.) Define another sequence $(b_n)_{n=1}^\infty$ where the n-th term is defined to be

$$b_n = a_n^{(1)} + a_n^{(2)} + \cdots a_n^{(k)}.$$

Suppose that the series $\sum_{n=1}^\infty a_n^{(i)}$ converges for each $i = 1, 2, \dots, k$. Prove that

(a) the series $\sum_{n=1}^\infty b_n$ also converges; moreover,

(b)

$$\sum_{n=1}^\infty b_n = \sum_{i=1}^k \Big(\sum_{n=1}^\infty a_n^{(i)}\Big).$$

This is a discrete version of *Fubini's theorem.*

- Let $\sum_{n=1}^{\infty} a_n^{(i)} = A_i$ for each $1 \le i \le k$.

- $\forall \varepsilon > 0, \forall 1 \le i \le k, \exists N_i > 0$

  s.t. $\left| \sum_{n=1}^{m} a_n^{(i)} - A_i \right| < \frac{\varepsilon}{k}$. $\forall m > N_i$.

- Let $N := \max\{N_1, \dots, N_k\} > 0$. Then $\forall m > N$, we have:

$$\left| \sum_{n=1}^{m} b_n - \sum_{i=1}^{k} A_i \right| = \left| \sum_{i=1}^{k} \left(\sum_{n=1}^{m} a_n^{(i)} - A_i \right) \right|$$

$$\le \sum_{i=1}^{k} \left| \sum_{n=1}^{m} a_n^{(i)} - A_i \right| < \varepsilon.$$ Hence, $\sum_{n=1}^{\infty} b_n = \sum_{i=1}^{k} A_i$. $\square$

(5) Let (a_n) and (b_n) be two sequences of real numbers. Assume that they satisfy the following three properties:

(a) The partial sums of (b_n) is bounded: there exists $L > 0$ such that $|s_k| = |b_1 + \cdots + b_k| < L$ for any k;

(b) $\lim a_n = 0$;

(c) $\sum |a_{n+1} - a_n|$ is convergent.

Prove that the series $\sum a_n b_n$ is convergent. This is known as *Abel's theorem*.
(Hint: Show that $\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} a_n(s_n - s_{n-1}) = \sum_{n=M}^{N-1}(a_n - a_{n+1})s_n + a_N s_N - a_M s_{M-1}$, then try to apply the assumptions.)

For any $\varepsilon > 0$,

- Since $\lim a_n = 0$, $\exists P_1 > 0$ s.t. $|a_n| < \frac{\varepsilon}{3L}$ $\forall n > P_1$.
- Since $\sum |a_{n+1} - a_n|$ conv., $\exists P_2 > 0$ s.t. $\sum_{n=M}^{N} |a_{n+1} - a_n| < \frac{\varepsilon}{3L}$ $\forall N \geq M > P_2$.
- Let $P := \max\{P_1, P_2\} > 0$. Then $\forall N \geq M > P$, we have:

$$\left|\sum_{n=M}^{N} a_n b_n\right| = \left|\sum_{n=M}^{N-1} (a_n - a_{n+1}) s_n + a_N s_N - a_M s_{M-1}\right|$$

$$\leq \sum_{n=M}^{N-1} |a_{n+1} - a_n| \cdot L + |a_N| \cdot L + |a_M| \cdot L$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

By Cauchy criterion, $\sum a_n b_n$ is therefore convergent. $\square$

(6) (optional; basic knowledge of complex numbers required) Show that the series

$$\sum \frac{\cos(n\theta)}{n} \text{ and } \sum \frac{\sin(n\theta)}{n}$$

are convergent for any $0 < \theta < 2\pi$.
(Hint: Show that

$$\left(\sum_{n=1}^{N} \cos(n\theta)\right) + i\left(\sum_{n=1}^{N} \sin(n\theta)\right) = \sum_{n=1}^{N} e^{in\theta} = e^{i\theta}\frac{1 - e^{iN\theta}}{1 - e^{i\theta}} = e^{i(N+1)\theta/2}\frac{\sin(N\theta/2)}{\sin(\theta/2)}$$

and use the previous problem.)

Euler's formula

$$1 - e^{i\theta} = 1 - \cos\theta - i\sin\theta = 2\sin^2\left(\frac{\theta}{2}\right) - i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$= -2i\sin\frac{\theta}{2} \cdot e^{i\frac{\theta}{2}}.$$

Hint $\Rightarrow$ $$\left|\sum_{n=1}^{N}\cos(n\theta) + i\sum_{n=1}^{N}\sin(n\theta)\right| \le \frac{1}{\sin(\frac{\theta}{2})}$$

$\forall\, 0<\theta<2\pi.$

$$\Rightarrow \left|\sum_{n=1}^{N}\cos(n\theta)\right| \le \frac{1}{\sin(\frac{\theta}{2})} \text{ and } \left|\sum_{n=1}^{N}\sin(n\theta)\right| \le \frac{1}{\sin(\frac{\theta}{2})}$$

Now, think of $(\cos(n\theta))$ or $(\sin(n\theta))$ as (b_n) in #4, and let $(a_n = \frac{1}{n})$.

They satisfy the conditions in #4, hence $\sum \frac{\cos n\theta}{n}$, $\sum \frac{\sin n\theta}{n}$ conv.

$\square$