

**SECOND MIDTERM PRACTICE PROBLEMS**  
**MATH 104, SECTION 2**

- (1) Let  $(a_n)_{n=1}^\infty$  be a sequence of real numbers and  $s_k = a_1 + \cdots + a_k$  be the  $k$ -th partial sum.
- (a) Suppose that  $\lim a_n = 0$ , and there exists a  $m \in \mathbb{N}$  such that the sequence  $(s_{mk})_{k=1}^\infty = (s_m, s_{2m}, s_{3m}, \dots)$  converges. Prove that  $\sum a_n$  converges.
  - (b) Find an example where  $(s_{2k})_{k=1}^\infty$  converges and  $(a_n)$  doesn't converge to 0.
  - (c) Find an example where  $\lim a_n = 0$ , and there is a subsequence  $(s_{k_n})$  of  $(s_n)$  that converges, but  $\sum a_n$  diverges.
- (2) (a) Show that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $f(x) = 0$  for all  $x \in \mathbb{Q}$ , then  $f(x) = 0$  for all  $x \in \mathbb{R}$ .
- (b) Show that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $f(x+y) = f(x) + f(y)$  for all  $x, y \in \mathbb{R}$ , then  $f$  is linear, i.e. there exists  $c$  so that  $f(x) = cx$  for all  $x$ .
- (3) Let  $X = (\mathbb{R}^n, d_{\text{std}})$  be the Euclidean space with the standard distance function

$$d_{\text{std}}(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Prove that any linear map  $T : X \rightarrow X$  is uniformly continuous.

- (4) Let  $S$  be the set of nonempty compact subsets of  $\mathbb{R}^2$ . For any  $r > 0$  and  $K \in S$ , we define the  $r$ -neighborhood of  $K$  to be

$$B_r(K) := \{x \in \mathbb{R}^2 : d(x, a) < r \text{ for some } a \in K\} = \bigcup_{a \in K} B_r(a).$$

For  $K_1, K_2 \in S$ , we define

$$d(K_1, K_2) := \inf\{r > 0 : K_1 \subset B_r(K_2) \text{ and } K_2 \subset B_r(K_1)\}.$$

- (a) Prove that  $(S, d)$  is a metric space, i.e.  $d$  is a distance function on  $S$ .
  - (b) Let  $F$  be the set of finite subsets of  $\mathbb{R}^2$ . Prove that  $F$  is dense in  $S$ .
- (5) Let  $(X, d)$  be a metric space and  $E \subset X$  be a nonempty subset. Define a function  $f : X \rightarrow [0, \infty)$  by:

$$f(x) := \inf\{d(x, y) : y \in E\}.$$

Prove that  $f$  is uniformly continuous on  $X$ .

- (6) Let  $(p_n)$  be a sequence of polynomials defined over real numbers, and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a real-valued function. Suppose that  $(p_n)$  converges uniformly to  $f$  on  $\mathbb{R}$ . Prove that  $f$  is also a polynomial.