Digression / Motivation

Linear algebra from "categorical perspective":

- Objects: $(\mathbb{R}^n, +, \cdot)$ (generally: vector spaces)
  $+$: addition, $\cdot$: scalar multiple.
- "Morphisms": Linear transform^n $\mathbb{R}^n \longrightarrow \mathbb{R}^m$
  (a function which is compatible w/ the additions & scalar mult. on $\mathbb{R}^n, \mathbb{R}^m$)

1) composition of linear transform^{n}s.
   you proved in HW1 that if
   $\mathbb{R}^p \xrightarrow{g} \mathbb{R}^n$, $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ linear transform^{n}s
   $\Rightarrow$ $\mathbb{R}^p \xrightarrow{f \circ g} \mathbb{R}^m$ also a linear transform^n.

2) identity transform^n: $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$
   $\vec{x} \longmapsto \vec{x}$.

Q: What's the corresponding matrix of the identity transf. $T: \mathbb{R}^n \to \mathbb{R}^n$?
$\vec{x} \mapsto \vec{x}$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 1 \end{bmatrix} = I_n$$

identity matrix

$A: m \times n$, $B: n \times p$

$T_A: \mathbb{R}^n \to \mathbb{R}^m$, $T_B: \mathbb{R}^p \to \mathbb{R}^n$

Consider the composition:

$$T_A \circ T_B : \mathbb{R}^p \longrightarrow \mathbb{R}^m$$

$$\vec{x} \longmapsto T_A(T_B(\vec{x}))$$

linear (proved in HW1)

By Thm we proved last time, $\exists!$ $m \times p$ matrix "AB"

s.t. $T_{AB} = T_A \circ T_B$.

definition of the product of A, B.

Let's write down AB explicitly:

$\vec{e}_1, \dots, \vec{e}_p$

The columns of AB are given by $\boxed{T_A(T_B(\vec{e}_i))}$

i-th column of AB

$$T_A(T_B(\vec{e}_i)) = T_A(B\vec{e}_i) = A(\boxed{B\vec{e}_i})$$

$$\underbrace{\begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix}}_{n \times p} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{e}_i} \begin{matrix} \\ \\ \\ \leftarrow i\text{th} \\ \\ \\ \\ \end{matrix} = i\text{-th column of } B.$$

$$B = \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_p \\ | & & | \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \\ | & | & & | \end{bmatrix}$$

Rmk: each column of AB is a linear combination of the columns of A.

Even more explicit:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}$$

i-th column of $AB = A\vec{b}_i$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix}$$

$$= b_{1i} \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + b_{2i} \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + b_{ni} \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{1i} + a_{12}b_{2i} + \cdots + a_{1n}b_{ni} \\ \vdots \\ a_{m1}b_{1i} + a_{m2}b_{2i} + \cdots + a_{mn}b_{ni} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{n} a_{1k} b_{ki} \\ \vdots \\ \sum_{k=1}^{n} a_{mk} b_{ki} \end{bmatrix} \leftarrow i\text{th column of } AB$$

$$\forall\, 1 \le i \le p$$

$$AB = \begin{bmatrix} \sum_{k=1}^{n} a_{1k} b_{k1} & \cdots & \sum_{k=1}^{n} a_{1k} b_{kp} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk} b_{k1} & \cdots & \sum_{k=1}^{n} a_{mk} b_{kp} \end{bmatrix}$$

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ \boxed{a_{r1} \quad \cdots \quad a_{rn}} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \end{array} \leftarrow r\text{-th row}, \qquad B = \begin{bmatrix} b_{11} & \cdots & \boxed{b_{1s}} & \cdots & b_{1p} \\ \vdots & & \vdots & \ddots & \vdots \\ b_{n1} & \cdots & \cdots & \cdots & b_{np} \end{bmatrix}$$

s-th column

$$\begin{array}{cc} & \overset{s}{\longleftrightarrow} \\ r \updownarrow & \begin{bmatrix} & & \\ & (r,s) & \\ & & \end{bmatrix} \end{array}$$

the (r,s)-entry of AB: $\displaystyle\sum_{k=1}^{n} a_{rk} b_{ks}$

$$= a_{r1} b_{1s} + a_{r2} b_{2s} + \cdots + a_{rn} b_{ns}$$

= standard inner product between the r-th row of A & the s-th column of B.

e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\neq$$

$$BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

e.g. $AB = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & & \vdots \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \underline{0} \not\Rightarrow A = \underline{0}$ or $B = 0$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Prop: $A: m \times n$, $B: n \times p$, $C: p \times q$

Then $A(BC) = (AB)C$ (associativity)

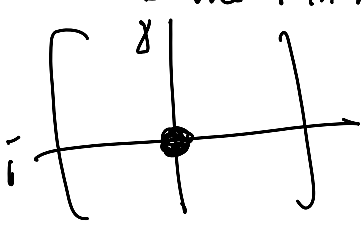
pf. (entry-wise computation)

$$[A(BC)]_{ij} = \sum_{k=1}^{n} a_{ik} (BC)_{kj}$$

$$= \sum_{k=1}^{n} a_{ik} \left(\sum_{\ell=1}^{p} b_{k\ell} c_{\ell j} \right)$$

$$= \sum_{k=1}^{n} \sum_{\ell=1}^{p} a_{ik} b_{k\ell} c_{\ell j}$$

the entry of $A(BC)$ at the i-th row & j-th colm



$$[(AB)C]_{ij} = \sum_{l=1}^{p} (AB)_{il}\, c_{lj}.$$

$$= \sum_{l=1}^{p} \left(\sum_{k=1}^{n} a_{ik} b_{kl} \right) c_{lj}$$

$$= \sum_{l=1}^{p} \sum_{k=1}^{n} a_{ik} b_{kl} c_{lj}.$$

pf (composition of linear transform:)

$$\underline{\underline{A(BC)}} \rightsquigarrow T_{A(BC)} = T_A \circ T_{BC} = T_A \circ (T_B \circ T_C)$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\underline{\underline{(AB)C}} \rightsquigarrow T_{(AB)C} = T_{AB} \circ T_C = (T_A \circ T_B) \circ T_C$$

any linear transf. has a unique corresponding matrix. $\square$

Prop: $A_{m\times n} = I_m \cdot A_{m\times n} = A_{m\times n} \cdot I_n$

pf:

$$\mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m$$

$$T_{I_m \cdot A} = T_{I_m} \circ T_A$$

$$\mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m \xrightarrow{T_{I_m}} \mathbb{R}^m$$

$$\vec{x} \longmapsto T_A(\vec{x}) \longmapsto T_A(\vec{x})$$

Notations:

- $A, B: m \times n$, $\quad A+B = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$

sum entry-wise.

$r \in \mathbb{R}$, $\quad rA = \begin{bmatrix} ra_{11} & \cdots & ra_{1n} \\ \vdots & & \\ ra_{m1} & \cdots & ra_{mn} \end{bmatrix}$

- $A: n \times n$ square matrix $\qquad A^2 = A \cdot A$

$$A^k = \underbrace{A \cdot A \cdots\cdots A}_{k \text{ times}}$$

- A^T: transpose of A. $\qquad (A^T)_{ij} = A_{ji}$

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Prop:
- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$, $\quad (rA)^T = rA^T$
- $\cancel{(AB)^T \overset{??}{=} A^T B^T}$ $\quad \underset{m \times n}{A} \; \underset{n \times p}{B}$, $\quad \underset{n \times m}{A^T} \; \underset{p \times n}{B^T}$

$$(AB)^T = B^T A^T$$

$\underbrace{m \times n \quad n \times p}_{m \times p}$ $\qquad \underbrace{p \times n \quad n \times m}_{p \times m}$

$((AB)^T)_{ji}$

$\underline{(AB)_{ij}}$ = Inner product btw i-th row of A & j-th column of B.

= inner product btw i-th column of A^T & j-th row of B^T

$= (B^T A^T)_{ji}$

$\Rightarrow (AB)^T = B^T A^T$. $\square$

Def: $A: n\times n$ is invertible (non-singular) if $\exists$ $n\times n$ matrices B and C s.t.

$$AB = I_n = CA.$$

Rmk:
- If A is invertible, then $B = C$:
  $B = I_n \cdot B = (CA)B = C(AB) = C \cdot I_n = C$
- If A is invertible, then such B is unique:
  Suppose $AB = I_n = BA$, and $AB' = I_n = B'A$.
  then $B = I_n . B = (B'A)B = B'(AB) = B' \cdot I_n = B'$
- If A invertible, such B is called the inverse of A and is denoted by A^{-1}: $AA^{-1} = A^{-1}A = I_n$

Thm A: $n\times n$ invertible.

Then $A\vec{x}=\vec{b}$ has a unique sol$^{\underline{\text{n}}}$ $\forall\, \vec{b}\in\mathbb{R}^n$.

In fact, $\vec{x}=A^{-1}\vec{b}$.

pf: 1) check $\vec{x}=A^{-1}\vec{b}$ is indeed a sol$^{\underline{\text{n}}}$:

$$A\vec{x}=A(A^{-1}\vec{b})=(AA^{-1})\vec{b}=I_n\cdot\vec{b}=\vec{b}.$$

2) uniqueness: Suppose $A\vec{y}=\vec{b}$,

then

$$\begin{aligned} A^{-1}(A\vec{y}) &= A^{-1}\vec{b}. \\ \| \\ (A^{-1}A)\vec{y} \\ \| \\ I_n\cdot\vec{y} \\ \| \\ \vec{y}. \end{aligned}$$

$\square$