Recap: Homogeneous eq'n $\vec{x}'(t) = A\vec{x}(t)$, $A \in M_{n\times n}(\mathbb{R})$.

- A matrix-valued fcn $X(t)$ is a <u>fundamental matrix</u> if its columns $\{\vec{x}_1(t), \ldots, \vec{x}_n(t)\}$ is a l.i. set of sol'ns.

- $\vec{x}(t)$ is a sol'n $\iff$ $\vec{x}(t) = X(t)\vec{c}$ for some $c \in \mathbb{R}^n$.

- $\forall$ two fundamental matrices $X(t)$ and $Y(t)$, $\exists!$ $M \in M_{n\times n}(\mathbb{R})$ invertible s.t. $X(t) \cdot M = Y(t)$.

- the <u>matrix exponential</u> $e^{tA} = \mathbb{I} + tA + \frac{t^2}{2}A^2 + \cdots \in M_{n\times n}(\mathbb{R})$ is a fundamental matrix.

<u>e.g.</u> when $A = PDP^{-1}$, where $P = [\vec{v}_1 \cdots \vec{v}_n]$ invertible, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

we proved that $\{e^{\lambda_1 t}\vec{v}_1, \ldots, e^{\lambda_n t}\vec{v}_n\}$ is a l.i. set of sol'ns.

Hence $\begin{bmatrix} e^{\lambda_1 t}\vec{v}_1 & \cdots & e^{\lambda_n t}\vec{v}_n \end{bmatrix}$ is a fundamental matrix.

<u>Rmk:</u> Since e^{tA} is a fundamental matrix of $\vec{x}' = A\vec{x}$, In principle, we can get all the sol'ns from the columns of e^{tA}.

<u>Issue:</u> e^{tA} is <u>hard to compute in general</u>.

<u>Fact:</u> If <u>$A\vec{v} = \lambda\vec{v}$</u>, then $\boxed{e^{tA}\vec{v}} = \boxed{e^{\lambda t}\vec{v}}$ easy.

When $A = PDP^{-1}$, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ eigvectors

$\boxed{e^{tA}\underbrace{\boxed{\vec{v}_1 \cdots \vec{v}_n}}_{P}} = \left[e^{\lambda_1 t}\vec{v}_1 \cdots e^{\lambda_n t}\vec{v}_n\right]$

also a fundamental matrix.

<u>pf: 1)</u>
$$e^{tA}\vec{v} = \left(\mathbb{I} + tA + \frac{t^2}{2}A^2 + \cdots\right)\vec{v}$$
$$= \vec{v} + t\underbrace{A\vec{v}}_{\lambda\vec{v}} + \frac{t^2}{2}\underbrace{A^2\vec{v}}_{\lambda^2\vec{v}} + \cdots$$
$$= \vec{v} + t\lambda\vec{v} + \frac{t^2}{2}\lambda^2\vec{v} + \cdots$$
$$= e^{\lambda t}\vec{v}$$

pf 2) $e^{tA}\vec{v} = e^{t\lambda}\boxed{e^{t(A-\lambda I)}}\vec{v}$

$(A-\lambda I)\vec{v}=0$

$\| $

$e^{t\lambda}\left(I + t(A-\lambda I) + \frac{t^2}{2}(A-\lambda I)^2 + \cdots\right)\vec{v}$

$\|$

$e^{t\lambda}\vec{v}.$

$A = \lambda I + (A-\lambda I)$

$e^{tA} = \boxed{e^{t\lambda I}}\, e^{t(A-\lambda I)}$

$\downarrow$

$e^{t\lambda}\cdot I$

$e^{t\lambda I}\cdot A = e^{t\lambda} A$

$\|$

$1 + t\lambda + \frac{1}{2}(t\lambda)^2 + \cdots$

Recall

Def (generalized eigenvector). If λ is an eigenvalue of A, say $\vec{v} \neq \vec{0}$ is a generalized eigenvector if

$$(A-\lambda I)^k \vec{v} = \vec{0} \quad \text{for some } k \geq 1.$$

We proved: for any A, $\exists$ generalized eigenbasis.

$$\{\vec{v}_1, \ldots, \vec{v}_n\}.$$

Idea: $e^{tA}\vec{v}$ is not hard to compute for any generalized eigenvector $\vec{v}$.

If $(A-\lambda I)^k \vec{v} = \vec{0}$. then

$$e^{tA}\vec{v} = e^{t\lambda} e^{t(A-\lambda I)}\vec{v}$$

$$= e^{tA}\Big(I + t(A-\lambda I) + \frac{t^2}{2}(A-\lambda I)^2 + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{k-1} + \frac{t^k}{k!}(A-\lambda I)^k + \cdots\Big)\vec{v}$$

$$= e^{tA}\Big(I + t(A-\lambda I) + \frac{t^2}{2}(A-\lambda I)^2 + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{k-1}\Big)\vec{v}.$$

$$\underbrace{e^{tA}[\underbrace{\vec{v}_1 \cdots \vec{v}_n}]}_{\text{fund. matrix.}} = [\underbrace{e^{tA}\vec{v}_1 \cdots e^{tA}\vec{v}_n}_{\text{not hard to compute}}]$$

Strategy for finding a fundamental matrix of $\vec{x}' = A\vec{x}$:

① Find all the eigenvalues $\{\lambda_i\}$ and their multiplicities $\{m_i\}$

$$\prod_i (\lambda_i - \lambda)^{m_i} = \det(A - \lambda I)$$

② Generalized eigenspace of λ_i = $\mathrm{Nul}\left((A - \lambda_i I)^{m_i}\right)$

(Reason: $A = P \begin{pmatrix} \boxed{\;} & & \\ & \boxed{\;} & \\ & & \ddots \end{pmatrix} P^{-1}$

$$\left(\underbrace{\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}}_{l \times l} - \lambda I \right)^{l} = 0$$)

③ Find a generalized eigenbasis $\{\vec{v}_1, -, \vec{v}_n\}$
(pick a basis for each generalized eigenspace.)

④ Compute $e^{tA}\vec{v}_i$ for each i.

⑤ Get a fund matrix $\underline{\begin{bmatrix} e^{tA}\vec{v}_1 & \cdots & e^{tA}\vec{v}_n \end{bmatrix}}$

Rmk To get the matrix exponential, $\underline{e^{tA}}$,

$$\begin{bmatrix} e^{tA}\vec{v}_1 & \cdots & e^{tA}\vec{v}_n \end{bmatrix}\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}^{-1}$$

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e.g. Find a fundamental matrix of $\vec{x}' = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{A} \vec{x}$

① $\lambda = 1$ or 3.

$\uparrow$ mult 2, $\uparrow$ mult 1.

② eigenvectors of 3: $\text{Nul}\begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$.

eigenvectors of 1: $\text{Nul}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

A not diagonalizable.

generalized eigenspace of 1:

$\text{Nul}\left((A - 1I)^2\right) = \text{Nul}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}\right\}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

eigvectors

$\rightsquigarrow$ gen. eigenbasis $\left\{ \underset{\substack{v_1 \\ \downarrow \\ 1}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{\substack{v_2 \\ \downarrow \\ 1}}{\begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}}, \underset{\substack{v_3 \\ \downarrow \\ 3}}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}} \right\}$

$$e^{tA}\left[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\right]$$

$$\boxed{e^{tA}\vec{v}_1} = e^{1t}\vec{v}_1 = e^{t}\vec{v}_1$$

$$e^{tA}\vec{v}_3 = e^{3t}\vec{v}_3$$

$$(A-I)^2\vec{v}_2 = 0$$

$$e^{tA}\vec{v}_2 = e^{t}e^{t(A-I)}\vec{v}_2$$

$$= e^{t}\left(I + t(A-I) + \frac{t^2}{2}(A-I)^2 + \cdots\right)\vec{v}_2$$

$$= e^{t}\left(\vec{v}_2 + t(A-I)\vec{v}_2\right)$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= e^{t}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$\Longrightarrow$ a fund. matrix

$$\begin{bmatrix} 0 & -2e^{t} & 0 \\ 0 & e^{t} & 2e^{3t} \\ e^{t} & te^{t} & e^{3t} \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \vec{x}$$

$$\begin{cases} x_1(t) = -2e^t \\ x_2(t) = e^t \\ x_3(t) = te^t \end{cases}$$ is a sol'n

$x_1'(t) = -2e^t = x_1(t)$

$x_2'(t) = e^t = \underbrace{x_1(t)}_{-2e^t} + \underbrace{3x_2(t)}_{3e^t}$

$x_3'(t) = e^t + te^t = x_2 + x_3$

Rmk: We saw "te^t" when we consider $y'' + by' + cy = 0$ when $r^2 + br + c = 0$ has a double root.

e.g. $\boxed{y'' - 4y' + 4y = 0}$ $\quad r^2 - 4r + 4 = 0$, double root $= 2$

Define $x_1(t) = y(t)$, $\quad x_1' = y' = x_2$

$x_2(t) = y'(t)$, $\quad x_2' = y'' = 4y' - 4y$

$= -4x_1 + 4x_2$

$$\boxed{\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \vec{x}(t).}$$

Char poly $= \det\begin{pmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 4$

$\lambda = 2$ eigenvalue w/ mult = 2.

$$\mathrm{Nul}\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

generalized eig space

$$(A - 2I)^2 = 0$$

$$\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$$e^{tA}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$e^{tA}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$e^{tA}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{2t} e^{t(A-2I)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= e^{2t}\left(I + t(A-2I) \cancel{+\cdots}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= e^{2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$\Rightarrow$ fund. matrix

$$\begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{bmatrix}$$

sol$^{\text{ns}}$ to $y'' - 4y' + 4y = 0$

$$\vec{x}\,' = \underset{\text{real}}{A}\,\vec{x}$$

has complex eigenvalues

$$\begin{cases} \bar{\lambda} = \alpha - i\beta \\ \bar{v} = \vec{x} - i\vec{y} \end{cases} \quad A\bar{v} = \bar{\lambda}\bar{v}$$

$\beta \neq 0.$

$$\begin{cases} \lambda = \alpha + i\beta \\ \vec{v} = \vec{x} + i\vec{y} \end{cases} \quad \text{s.t. } A\vec{v} = \lambda\vec{v}$$

$\Rightarrow$ We know $\boxed{e^{t\lambda}\vec{v}}$ is a sol^{n}

$\Leftarrow$ Complex conjugate $e^{t\bar{\lambda}}\bar{\vec{v}}$

$\parallel$

$$e^{t(\alpha+i\beta)}(\vec{x}+i\vec{y})$$

$$= e^{t\alpha}\left(\cos(\beta t)+i\sin(\beta t)\right)(\vec{x}+i\vec{y})$$

$$= \left(e^{t\alpha}\cos(\beta t)\vec{x} - e^{t\alpha}\sin(\beta t)\vec{y}\right) \quad \vec{u}$$

$$+ i\left(e^{t\alpha}\sin(\beta t)\vec{x} + e^{t\alpha}\cos(\beta t)\vec{y}\right) \quad \vec{v}$$

& its cpx conjugate are both sol^{n}s to $\vec{x}' = A\vec{x}$.

$u+iv$, $u-iv$ sol^{n}s $\Downarrow$ u, v sol^{n}

$$\Rightarrow \boxed{\begin{cases} e^{t\alpha}\cos(\beta t)\vec{x} - e^{t\alpha}\sin(\beta t)\vec{y} \\ e^{t\alpha}\sin(\beta t)\vec{x} + e^{t\alpha}\cos(\beta t)\vec{y} \end{cases} \text{ are both sol}^{n}\text{s}}$$

e.g.: $\vec{x}' = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}\vec{x}$.

$(\lambda+1)(\lambda+3)+2=0 \Rightarrow \lambda^2+4\lambda+5=0 \Rightarrow \lambda = \frac{-4\pm\sqrt{-4}}{2}$

$-2\pm i$

$\lambda = -2+i$

$\alpha = -2$ $\quad \vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\beta = 1$ $\quad \vec{y} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$\text{Nul}\begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -2 \\ 1-i \end{bmatrix}\right\}$$

$$\Rightarrow \begin{cases} e^{-2t}\cos t\begin{bmatrix} -2 \\ 1 \end{bmatrix} - e^{-2t}\sin t\begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ e^{-2t}\sin t\begin{bmatrix} -2 \\ 1 \end{bmatrix} + e^{-2t}\cos t\begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{cases} \text{ are sol}^{\text{s}}.$$

Non-homogeneous case: $\boxed{\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t)}$ $\quad \vec{f}: \mathbb{R} \to \mathbb{R}^n$, $A \in M_{n\times n}(\mathbb{R})$

Variation of parameters:

- general sol$^{\text{n}}$ of homog eq'n $\boxed{\vec{x}' = A\vec{x}}$.

  $X(t)$ fund matrix

  $\rightsquigarrow$ general sol$^{\text{s}}$ $\boxed{X(t)\vec{c}}$ const. vector

- Let $\boxed{\vec{x}(t) = X(t)\vec{c}(t)}$

  Goal: Find $\vec{c}(t)$ s.t. $\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t)$

$$\vec{x}'(t) = \boxed{X'(t)}\vec{c}(t) + X(t)\vec{c}'(t)$$
$$= \boxed{A\boxed{X(t)}\vec{c}(t)} + X(t)\vec{c}'(t)$$
$$= A\vec{x}(t) + X(t)\vec{c}'(t)$$

i.e. want to find $\vec{c}(t)$

s.t. $\underline{X(t)}\, \vec{c}\,'(t) = \vec{f}(t)$

$\Leftrightarrow \quad \vec{c}\,'(t) = X(t)^{-1} \vec{f}(t)$

So we can find $\vec{c}(t)$
by integrating each components of
the vector-valued fcn $X(t)^{-1} \vec{f}(t)$.

e.g. $\boxed{\vec{x}\,'(t) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}}$

① find a fund. matrix $\begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix} = X(t)$

② $X(t)^{-1} \vec{f}(t) = \begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}^{-1} \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{t} & 3e^{t} \end{bmatrix} \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{3t} + \frac{3}{2}e^{t} \end{bmatrix}$

③ get $\boxed{\vec{c}(t)}$ by $\int$

④ get a particular sol$^{\underline{n}}$ $\vec{x}_0(t) = X(t)\,\vec{c}(t)$

⑤ general sol$^{\underline{n}}$ $\boxed{X(t)\,\vec{d}} + \boxed{\vec{x}_0(t)}$

any const. vector

⑥ Plug in to $\vec{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ & solve for $\vec{d}$.