HOMEWORK 3
MATH H54

Yu-Wei's Office Hours: Sunday 1-2:30pm and Thursday 12-1:30pm (PDT)
Michael's Office Hours: Monday 12-3pm (PDT)

Part I (no need to turn in)

This part of the homework provides some routine computational exercises. You don't have to turn in your solutions for this part, but being able to do the computations is vitally important for the learning process, so you definitely should do these practices before you start doing Part II of the homework.

The following exercises are from the corresponding sections of the UC Berkeley custom edition of Lay, Nagle, Saff, Snider, *Linear Algebra and Differential Equations*.

- **Exercise 3.2**: 7, 11, 19, 25, 27, 34, 44
- **Exercise 3.3**: 23, 31
- **Exercise 4.1**: 15, 22
- **Exercise 4.2**: 3, 15, 27

Part II (Due September 22, 8am PDT)

Some ground rules:

- You have to submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW3**.
- The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own.
- You're allowed to use any result that is proved in the lecture. But if you'd like to use other results, you have to prove it first before using it.

Problems:

(1) (a) Let A be an $n \times n$ matrix, and $c \in \mathbb{R}$. Prove that $\det(cA) = c^n \det(A)$.
    (b) Let A be an $n \times n$ matrix where n is an odd number. Suppose that A is anti-symmetric, i.e. $A + A^T = 0$. Prove that A is not invertible.

(2) Let A be a 2×2 matrix. Prove that

$$(\mathrm{tr}(A))^2 - \mathrm{tr}(A^2) = 2\det(A).$$

(3) Let A be an $n \times n$ invertible matrix whose entries are all integers. Prove that "$\det(A) = \pm 1$" if and only if "the entries of A^{-1} are all integers".

(4) Let P be the triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3) in $\mathbb{R}^2$. Prove that the area of the triangle is the same as

$$\frac{1}{2} \det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix}.$$

(Hint: Translate the triangle so that one of the vertices becomes the origin $(0,0)$.)

(5) Each of the following sets is not a subspace of the specified vector space. For each set, give a reason why it is not a subspace.
   (a) $\{f(x) \in \mathrm{Poly}_{\leq 3} : f(1) \in \mathbb{Z}\}$ in the vector space $\mathrm{Poly}_{\leq 3}$ of real polynomials of degree 3 or less.
   (b) $\{A \in M_{2\times 2} \colon \det(A) \neq 0\}$ in the vector space $M_{2\times 2}$ of 2×2 matrices.
   (c) $\{A \in M_{2\times 2} \colon \det(A) = 0\}$ in the vector space $M_{2\times 2}$ of 2×2 matrices.
   (d) $\{f \in \mathcal{C}([0,3]) \colon f(1)f(2) = 0\}$ in the vector space $\mathcal{C}([0,3])$ of continuous functions on $[0,3]$.

(6) Let H_1 and H_2 be subspaces of a vector space V.
   (a) Define the intersection of H_1 and H_2, denoted by $H_1 \cap H_2$, to be the set

$$H_1 \cap H_2 \coloneqq \{\vec{v} \in V \colon \vec{v} \in H_1 \text{ and } \vec{v} \in H_2\} \subseteq V.$$

   Prove that $H_1 \cap H_2$ is a subspace of V.
   (b) Define the sum of H_1 and H_2, denoted by $H_1 + H_2$, to be the set

$$H_1 + H_2 \coloneqq \{\vec{v} \in V \colon \vec{v} = \vec{v}_1 + \vec{v}_2 \text{ for some } v_1 \in H_1 \text{ and } v_2 \in H_2\} \subseteq V.$$

   Prove that $H_1 + H_2$ is a subspace of V.
   (c) Suppose that $H_1 = \mathrm{Span}\{\vec{u}_1, \ldots, \vec{u}_k\}$ and $H_2 = \mathrm{Span}\{\vec{w}_1 \ldots, \vec{w}_\ell\}$. Prove that

$$H_1 + H_2 = \mathrm{Span}\{\vec{u}_1, \ldots, \vec{u}_k, \vec{w}_1 \ldots, \vec{w}_\ell\}.$$

   (i.e. prove $H_1 + H_2 \subseteq \mathrm{Span}\{\vec{u}_1, \ldots, \vec{u}_k, \vec{w}_1 \ldots, \vec{w}_\ell\}$ and $\mathrm{Span}\{\vec{u}_1, \ldots, \vec{u}_k, \vec{w}_1 \ldots, \vec{w}_\ell\} \subseteq H_1 + H_2$.)

(7) Let $T \colon V \to W$ be a linear transformation between two vector spaces.
   (a) Let $U \subseteq V$ be a subspace. Prove that the set

$$T(U) \coloneqq \{\vec{w} \in W \colon \vec{w} = T(\vec{u}) \text{ for some } \vec{u} \in U\} \subseteq W$$

   is a subspace of W.
   (b) Let $X \subseteq W$ be a subspace. Prove that the set

$$T^{-1}(X) \coloneqq \{\vec{v} \in V \colon T(\vec{v}) \in X\} \subseteq V$$

   is a subspace of V.

(8) Let $T \colon M_{3\times 3} \to M_{3\times 3}$ be the linear transformation on the vector space of 3×3 matrices given by

$$T(A) \coloneqq A + A^T.$$

   (a) Find a set of matrices in the range of T, such that they span the range of T, and are linearly independent in $M_{3\times 3}$.

(b) Find a set of matrices in the kernel of T, such that they span the kernel of T, and are linearly independent in $M_{3\times 3}$.

(9) (a) Let A be an $n\times n$ matrix and B be an $n\times m$ matrix. Prove that

$$\det(A) = \det \begin{bmatrix} A & B \\ 0_{m\times n} & \mathbb{I}_{m\times m} \end{bmatrix}$$

Here $0_{m\times n}$ denotes the $m\times n$ matrix with all entries 0 and $\mathbb{I}_{m\times m}$ denotes the $m\times m$ matrix with all entries 1. The right hand side is the determinant of an $(n+m)\times(n+m)$ matrix.

(b) Let A and B be two $n\times n$ matrices. Your good friend Yu-Wei believes that

$$\det \begin{bmatrix} A & B \\ B & A \end{bmatrix}_{2n\times 2n} = \det(A+B)\det(A-B).$$

He offers the following argument to support his claim:

$$\begin{aligned} \det \begin{bmatrix} A & B \\ B & A \end{bmatrix} &= \det(A^2 - B^2) \\ &= \det\Big((A+B)(A-B)\Big) \\ &= \det(A+B)\det(A-B). \end{aligned}$$

Explain carefully why each of the three steps in his argument is correct or incorrect.

(c) Is the result Yu-Wei is trying to prove actually true?
(Hint: Consider $\begin{bmatrix} \mathbb{I} & 0 \\ \mathbb{I} & \mathbb{I} \end{bmatrix} \begin{bmatrix} \mathbb{I} & B \\ 0 & A-B \end{bmatrix} \begin{bmatrix} A+B & 0 \\ -\mathbb{I} & \mathbb{I} \end{bmatrix}$.)

Part III (Extra credit problems, Due September 29, 8am PDT)

The following problem worths up to 2 extra points in total (out of 10), so you can potentially get 12/10 for this homework.

You have to submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW3_extra_credit**, which is separated from Part II.

Prove that for any $x_1, \dots, x_n \in \mathbb{R}$,

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix} = \prod_{1\le i<j\le n} (x_j - x_i).$$

Note that "$\prod$" is the product symbol; the right hand side means:

$$\prod_{1\le i<j\le n} (x_j - x_i) = (x_2-x_1)(x_3-x_1)(x_3-x_2)\cdots(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1}).$$

(Hint: Consider both sides of the equation as polynomials in $x_1, \dots, x_n$.)