Upcoming topics: (Much more computations, much less proofs)

- second order ordinary differential equation.

  (e.g. find $y(t)$ s.t. $y''(t) - 3y'(t) + 2y(t) = t$)

- systems of first order ordinary differential equations.

  (e.g. find $x_1(t)$ and $x_2(t)$ s.t. $\begin{cases} x_1'(t) = 2x_1(t) - 3x_2(t) \\ x_2'(t) = x_1(t) + 4x_2(t) \end{cases}$)

- Fourier series, Some classical partial differential equation.

  (e.g. heat equation, wave equation, ...)

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e.g. Find $y(t)$ s.t. $\boxed{y'(t) - 2y(t) = 0.}$ (homogeneous differential eq'n)

Take $y(t) = e^{2t}$. (then $y'(t) = 2e^{2t}$)

Any $y(t) = Ce^{2t}$, $C \in \mathbb{R}$ is a solution.

If we let $y(t) = c_1 \cos(2t) + c_2 \sin(2t)$

$y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$

$-\quad 2y(t) = 2c_1 \cos(2t) + 2c_2 \sin(2t)$

$(-2c_1 - 2c_2) \sin(2t) + (2c_2 - 2c_1) \cos(2t) = 0$

$(-c_1 - c_2) \sin(2t) + (c_2 - c_1) \cos(2t) = 0$

$\frac{y'(t)}{y(t)} = 2. \Rightarrow \int \frac{y'(t)}{y(t)} dt = \int 2\, dt$

$\Rightarrow \log|y(t)| = 2t + C$

$\Rightarrow |y(t)| = e^{(2t+C)} = e^{2t} \cdot e^{C} = e^{2t} \cdot (\text{const.})$

e.g. Find $y(t)$ s.t. $y'(t) - 2y(t) = 0$, $y(0) = 1$

($\Rightarrow$ $C = 1$, so $y(t) = e^{2t}$ is the sol^{n}.)

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want to find y satisfies this eq^{n}.

e.g. $y'(t) - by(t) = f(t)$ (non-homogeneous diff^{l} eq^{n})

given

"method of variation of parameter"

1) consider the homog. eq^{n} $y'(t) - by(t) = 0$.
we know general sol^{n} of it is given by $y(t) = Ce^{bt}$.

2) Consider $y(t) = \underline{C(t)e^{bt}}$

(try to find function $C(t)$ s.t. $C(t)e^{bt}$ satisfies the non-homog. eq^{n})

$$y'(t) = C'(t)e^{bt} + bC(t)e^{bt}.$$

$$-by(t) = -bC(t)e^{bt}.$$

$$y'(t) - by(t) = C'(t)e^{bt}$$

$$\| \quad f(t)$$

(we want to find $C(t)$ that satisfies)

$\Rightarrow$ we want $C(t)$ to satisfy $C'(t) = \dfrac{f(t)}{e^{bt}}$.

$$\Rightarrow C(t) = \int_0^t \frac{f(s)}{e^{bs}}\,ds + (\text{const.})$$

(Fundamental thm of Calculus)

$\Rightarrow$ $y(t) = \left(\int_0^t \dfrac{f(s)}{e^{bs}}\,ds + \text{const}\right)e^{bt}$ are sol^{n}. to the non-homog eq^{n}.

$$= \underbrace{\left(\int_0^t \frac{f(s)}{e^{bs}}\,ds\right)e^{bt}}_{\text{a sol}^{\text{n}}\text{ of the non-homog eq}^{\text{n}}} + \underbrace{(\text{const.})\,e^{bt}}_{\text{any sol}^{\text{n}}\text{ of the homog. eq}^{\text{n}}}.$$

<u>Fact</u>.

$$y' - by = f \quad (*),$$

$$y' - by = 0 \quad (**)$$

If y_0 is a sol$^{\text{n}}$ to $(*)$, then

"y_1 is a sol$^{\text{n}}$ to $(*)$"

$\iff$ "$y_1 - y_0$ is a sol$^{\text{n}}$ to $(**)$".

(If $y_0' - by = f$, then

"$y_1' - by = f$" $\iff$ "$(y_1 - y_0)' - b(y_1 - y_0) = 0$")

(Similar to: If $\vec{x}_0$ is a sol$^{\text{n}}$ to $A\vec{x} = \vec{b}$,

then "$\vec{x}_1$ is a sol$^{\text{n}}$ to $A\vec{x} = \vec{b}$"

$\iff$ "$\vec{x}_1 - \vec{x}_0$ is a sol$^{\text{n}}$ to $A\vec{x} = \vec{0}$")

Find $y(t)$ s.t. $\boxed{y''(t) + b\,y'(t) + c\,y(t) = 0.}$ $\quad b, c \in \mathbb{R}$

e.g. $\boxed{y'' + 5y' - 6y = 0}$ $\left(y(0) = 0, \quad y'(0) = 1 \right)$

Guess $y(t) = e^{rt}$ $\quad r \in \mathbb{R}$

$y'(t) = r e^{rt}$

$y''(t) = r^2 e^{rt}$

$$y'' + 5y' - 6y = (r^2 + 5r - 6) e^{rt}$$

$\overset{\text{want}}{\|}$

0

$\Rightarrow$ If $r^2 + 5r - 6 = 0$, then e^{rt} is a sol$^{\underline{n}}$.

$\left(\overset{\Updownarrow}{r = 1 \text{ or } -6} \right)$

$\Rightarrow$ $y(t) = e^{t}$ or $y(t) = e^{-6t}$ is sol$^{\underline{n}}$.

$\Rightarrow$ $y(t) = c_1 e^{t} + c_2 e^{-6t}$ is a sol$^{\underline{n}}$ $\forall\, c_1, c_2 \in \mathbb{R}$.

$$\left(\begin{array}{ll} 0 = y(0) = c_1 + c_2 & c_1 = \frac{1}{7}, \\ 1 = y'(0) = c_1 - 6c_2. & c_2 = \frac{-1}{7} \\ \Rightarrow y(t) = \frac{1}{7} e^{t} - \frac{1}{7} e^{-6t} \text{ is the sol}^{\underline{n}} & \end{array} \right)$$

Thm (existence & uniqueness thm.)

Let I be any open interval of $\mathbb{R}$. (e.g. $(0,1)$, $(0,\infty)$, $\mathbb{R}$)

Let $p_0(t), p_1(t), \cdots, p_{n-1}(t), f(t)$ continuous fun's on I.

Let $t_0 \in I$ (where we impose initial conditions)

Let $Y_0, Y_1, \cdots, Y_{n-1} \in \mathbb{R}$

There exists a unique function $y(t)$ on I that satisfies:

$$\begin{cases} y^{(n)}(t) + p_{n-1}(t)\, y^{(n-1)}(t) + \cdots + p_1(t)\, y'(t) + p_0(t)\, y(t) = f(t) \\ y(t_0) = Y_0,\quad y'(t_0) = Y_1,\ \cdots,\ y^{(n-1)}(t_0) = Y_{n-1}. \end{cases}$$

Def: Say $y_1(t)$, $y_2(t)$ are linearly dependent if

$\exists\, c \in \mathbb{R}$ s.t. $\quad y_1(t) = c\, y_2(t) \quad \forall t \in \mathbb{R}$.

Thm: If y_1, y_2 are l.i. sol^{n} to $y'' + by' + cy = 0$.

then for any initial condition $y(t_0) = Y_0$, $y'(t_0) = Y_1$,

$\exists!\ c_1, c_2 \in \mathbb{R}$ s.t. $\quad y(t) = c_1 y_1(t) + c_2 y_2(t)$.

is the unique sol^{n} to $\begin{cases} y'' + by' + cy = 0 \\ y(t_0) = Y_0,\ y'(t_0) = Y_1 \end{cases}$

$$\left(\begin{aligned} Y_0 &= y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) \\ Y_1 &= y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0), \end{aligned} \quad \begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix} = \underbrace{\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}}_{\text{invertible } \forall t_0} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right)$$

y_1, y_2 are l.d. $\Leftrightarrow$ $\underline{y_1(t) = c\, y_2(t) \ \forall t.}$ (for some c)

$\Downarrow$

$y_1'(t) = c\, y_2'(t) \ \forall t$

$\Downarrow$

$\det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = 0$ for some t_0, $\Leftarrow$ $\det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = 0 \ \forall t$

Prop

If $\det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \neq 0$ for some t

then y_1, y_2 are l.i.

relies on the existence & uniqueness thm.

$\boxed{\text{Prop.}}$ If y_1, y_2 are sol^{n} to $y'' + by' + cy = 0$.

and suppose $\det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = 0$ for some $t_0 \in \mathbb{R}$.

Then $\{y_1, y_2\}$ is l.d.