FIRST MIDTERM PRACTICE PROBLEMS
MATH 185, SECTION 3

Suggestions for preparation:

- Understand thoroughly the proofs of all the important theorems.
- Review all the homework problems. Rethink the problems before you read the solutions: understanding the thought process is much more important than memorizing the solutions, since the exam problems are not likely to be exactly the same as something you've seen before.
- Do as many practice problems as possible.

(1) Compute the following integrals:

$$\int_0^\infty \frac{1}{(1+x^2)^2}dx; \quad \int_0^\infty \frac{x^2}{(x^2+1)(x^2+9)}dx; \quad \int_{|z|=1} z^4 e^{2/z^2} dz; \quad \int_{|z-3|=1} \frac{\cos(z)}{z(z-\pi)^2}dz.$$

(2) Prove that if f is a holomorphic function on $\mathbb{D}$ and $|f|$ is constant, then f is a constant function.

(3) If the power series $\sum a_n z^n$ has radius of convergence $R_1 > 0$ and the power series $\sum b_n z^n$ has radius of convergence $R_2 > 0$. Prove that the radius of convergence of the power series $\sum a_n b_n z^n$ is at least $R_1 R_2$.

(4) Find the power series expansion of $f(z) = 1/z$ at the point $z_0 = 1+i$. What's its radius of convergence?

(5) Let $p(z)$ be a polynomial of degree at least two, and C be a simple closed curve which contains all zeros of $p(z)$ in its interior. Prove that

$$\int_C \frac{1}{p(z)}dz = 0.$$

(6) If f is holomorphic in a disc $D_R(z_0)$, then for any $0 < r < R$ we have

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta.$$

(7) Prove that there exists $\epsilon > 0$ such that for every polynomial $p(z)$,

$$\max_{|z|=1}\left|\frac{1}{z} - p(z)\right| > \epsilon.$$

(8) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a function such that $f(1/n) = 1$ for all positive integer n and $f(2i) = 2$. Prove that f is not an entire function.

(9) Find all the entire functions f such that $\mathrm{Re}(f(z)) > (\mathrm{Im}(f(z)))^2$ holds for any $z \in \mathbb{C}$.

(10) Prove that there is no entire function $f(z)$ with $\mathrm{Re}(f(z)) = |z|^2$.

(11) How many roots does the polynomial $z^5 + 2z^2 + 1$ have in the annulus $A_{\frac{1}{2},2}(0)$?

(12) If f is a nonconstant entire function, then

$$\max_{|z|\leq 1}|f(z)| < \max_{|z|\leq 2}|f(z)|.$$

(13) Review the biholomorphic maps among some basic simply connected domains, e.g. page 213 of the textbook.

(14) Let f_A be the Möbius transformation associated to $A \in \mathrm{SL}_2(\mathbb{R})$. Prove that $f_A \circ f_B = f_{AB}$ for any $A, B \in \mathrm{SL}_2(\mathbb{R})$.

(15) Let $\Omega = \{z \in \mathbb{C} \colon |z| < 1/2\}$, and let $\mathcal{F}$ be the family of holomorphic functions on Ω consisting of polynomials of the form

$$f(z) = (z - a_1)(z - a_2)\cdots(z - a_n), \text{ where } |a_i| < 1/2 \text{ for all } 1 \leq i \leq n.$$

Is $\mathcal{F}$ a normal family on Ω? Justify your answer.

(16) Let $\wp(z)$ be the Weierstrass $\wp$-function with respect to a lattice $\Lambda \subseteq \mathbb{C}$. Express $\wp'(z)\wp'''(z)$ as a polynomial in $\wp(z)$.

(17) Prove that the only modular forms of weight zero are the constant functions.