(1) <u>Claim:</u> If $\vec{u}, \vec{v}$ are solutions of the linear system, then so is $t\vec{u} + (1-t)\vec{v}$, $\forall t \in \mathbb{R}$.

pf: Write $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$. Then $t\vec{u} + (1-t)\vec{v} = \begin{bmatrix} tu_1 + (1-t)v_1 \\ \vdots \\ tu_n - (1-t)v_n \end{bmatrix}$.

We need to check that $t\vec{u} + (1-t)\vec{v}$ satisfies each of the eq^n:

For any $1 \le k \le m$, we have:

$$a_{k1}(tu_1 + (1-t)v_1) + a_{k2}(tu_2 + (1-t)v_2) + \cdots + a_{kn}(tu_n + (1-t)v_n)$$

$$= t(a_{k1}u_1 + \cdots + a_{kn}u_n) + (1-t)(a_{k1}v_1 + \cdots + a_{kn}v_n)$$

since $\vec{u}, \vec{v}$ are sol^n to the eq^{ns}.

$$= tb_k + (1-t)b_k$$

$$= b_k.$$

Hence $t\vec{u} + (1-t)\vec{v}$ is a sol^n to the linear system $\forall t \in \mathbb{R}$. □

The claim shows that if the system has at least 2 sol^{ns}, then it must have infinitely many sol^{ns}. This proves the desired statement. □

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(2) It suffices to show that each elementary row operation doesn't change the sol^n set.

(i) $\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{k1} + ca_{\ell 1} & \cdots & a_{kn} + ca_{\ell n} & b_k + cb_\ell \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_n \end{array}\right]$

Call this system L_1 Call this system L_2.

$\vec{x} \in \mathbb{R}^n$ is a sol^n to L_2

$$\Leftrightarrow \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ (a_{k1} + ca_{\ell 1})x_1 + \cdots + (a_{kn} + ca_{\ell n})x_n = b_k + cb_\ell \\ \vdots \\ a_{\ell 1}x_1 + \cdots + a_{\ell n}x_n = b_\ell \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$\Updownarrow$

(Observe that

$$\begin{cases} (a_{k1} + ca_{\ell 1})x_1 + \cdots + (a_{kn} + ca_{\ell n})x_n = b_k + cb_\ell \\ a_{\ell 1}x_1 + \cdots + a_{\ell n}x_n = b_\ell \end{cases}$$

$$\Leftrightarrow \begin{cases} a_{k1}x_1 + \cdots + a_{kn}x_n = b_k. \\ a_{\ell 1}x_1 + \cdots + a_{\ell n}x_n = b_\ell. \end{cases}$$
)

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{k1}x_1 + \cdots + a_{kn}x_n = b_k \\ \vdots \\ a_{\ell 1}x_1 + \cdots + a_{\ell n}x_n = b_\ell \\ \vdots \end{cases} \Longleftrightarrow \vec{x} \text{ is a sol}^n \text{ to } L_1$$

The fact that the remaining two types of row operations don't change the sol^n set can be proved similarly. so I'll omit here. $\square$

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(3) <u>Claim</u>: " $\{\vec{v}_1, \ldots, \vec{v}_k\}$ are linearly independent" (<u>l.i.</u> for short)

$\Leftrightarrow$ " $\{\vec{v}_2, \ldots, \vec{v}_k\}$ are linearly independent and

$\vec{v}_1 \notin \text{Span}\{\vec{v}_2, \ldots, \vec{v}_k\}$ ".

<u>pf</u>: $(\Rightarrow)$: It's easy to see that if $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is l.i.

then a subcollection of vectors $\{\vec{v}_2, \ldots, \vec{v}_k\}$ is l.i. (why?)

Also, assume the contrary that $\vec{v}_1 \in \text{Span}\{\vec{v}_2, \ldots, \vec{v}_k\}$,

then $\exists\, c_2, \ldots, c_k \in \mathbb{R}$ s.t. $\vec{v}_1 = c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$.

Hence $\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_k\vec{v}_k = 0$, which implies

that $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is NOT l.i. Contradiction. $\square$

$(\Leftarrow)$ Assume the contrary that $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = 0$

for some $c_1, \ldots, c_k$ not all zero.

(i) Suppose $c_1 = 0$.

Then $c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = 0$ and $c_2, \ldots, c_k$ not all 0.

This contradicts with $\{\vec{v}_2, \ldots, \vec{v}_k\}$ is l.i.

(ii) Suppose $c_1 \neq 0$.

Then $\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2 - \cdots - \frac{c_k}{c_1}\vec{v}_k \in \text{Span}\{\vec{v}_2, \ldots, \vec{v}_k\}$.

Also a contradiction. $\square$

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(4) <u>False</u>: $\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 2 \\ 3x_1 + 3x_3 = 3. \end{cases}$ is a counterexample

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(5) True:

Assume the contrary that for some $m<n$, $\exists$ m vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$ s.t. $\text{Span}\{\vec{v}_1, \dots, \vec{v}_m\} = \mathbb{R}^n$.

Let
$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}.$$
an $n \times m$ matrix.

By a theorem we proved in class,
$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_m\} = \mathbb{R}^n \Leftrightarrow A \text{ has pivot in each row.}$$
But this is impossible since there are at most m pivots in A, and the # of rows $= n > m$. $\square$

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(6) Claim: $A\vec{u} = \vec{0}$ has no non-trivial sol^n $\vec{u} \neq \vec{0}$.

pf: Assume the contrary that $\exists$ $\vec{u} \neq \vec{0}$ s.t. $A\vec{u} = \vec{0}$.
Then we have $A\vec{x} = \vec{b}$ and $A(\vec{x}+\vec{u}) = A\vec{x} + A\vec{u} = \vec{b}$,
which contradicts with the assumption. $\square$

Then by a theorem we proved in class,
$$A\vec{u} = \vec{0} \text{ has no non-trivial sol'n} \Leftrightarrow A \text{ has pivots in each column.}$$

Since A is a square matrix (# of columns $=$ # of rows),
we have
$$A \text{ has pivots in each column} \Leftrightarrow A \text{ has pivots in each row},$$
Finally, we use the theorem that
$$A \text{ has pivots in each row} \Leftrightarrow \text{the columns of } A \text{ spans } \mathbb{R}^n.$$
to conclude the proof. $\square$

(7) Claim: "$T(\vec{x}) = \vec{0} \ \forall \vec{x} \in \mathbb{R}^n$" $\Leftrightarrow$ "$T(\vec{v_i}) = \vec{0} \ \forall i$".

pf ($\Rightarrow$) obviously true.

($\Leftarrow$). Since $\text{span}\{\vec{v}_1, \cdots, \vec{v}_k\} = \mathbb{R}^n$, for any $\vec{x} \in \mathbb{R}^n$, $\exists c_1, \cdots, c_k \in \mathbb{R}$ s.t. $\vec{x} = c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k$.

Hence

$$\begin{aligned} T(\vec{x}) &= T(c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k) \\ &= c_1 T(\vec{v}_1) + \cdots + c_k T(\vec{v}_k) \\ &= \vec{0}. \ \square \end{aligned}$$

Since T is linear.

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(8) Since $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is linearly dependent, $\exists c_1, \cdots, c_k$ not all 0 s.t. $c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k = \vec{0}$.

Then $\vec{0} = T(\vec{0}) = T(c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k)$

$= c_1 T(\vec{v}_1) + \cdots + c_k T(\vec{v}_k)$ since T is linear.

$\Rightarrow \{T(\vec{v}_1), \cdots, T(\vec{v}_k)\}$ is linearly dependent. $\square$

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(9) $\exists c_1, \cdots, c_k$ not all 0 s.t. $c_1 T(\vec{v}_1) + \cdots + c_k T(\vec{v}_k) = \vec{0}$

$\| \ (T \text{ is linear})$

$T(c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k)$

Let $\vec{x} := c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k$. Then $T(\vec{x}) = \vec{0}$.

Since $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is l.i. and $c_1, \cdots, c_k$ not all 0,

we have $\vec{x} \neq 0$. $\square$

(10) • $T_2(T_1(\vec{u}_1+\vec{u}_2)) = T_2(T_1(\vec{u}_1)+T_1(\vec{u}_2))$ since T_1 linear

$= T_2(T_1(\vec{u}_1))+T_2(T_1(\vec{u}_2))$ since T_2 linear.

• Similarly, one can show that $T_2(T_1(c\vec{u})) = cT_2(T_1(\vec{u}))$.

Hence $T_2 \circ T_1$ is a linear transform^n. $\square$

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(11). (For (a)(b)(c), there are infinitely many possible T. Below are some examples)

(a) Take T to be the identity transformation:

$$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\vec{x} \longmapsto \vec{x}.$$

Then the image of P under T is P itself.

(b) Consider $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix}.$$

(What's the standard matrix of T?)

(What's this map geometrically?)

One can check that T is a linear transform^n, which sends P to the line $\left\{ \begin{bmatrix} x_1 \\ 0 \\ 1 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$.

(c) Consider $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}.$$

One can check that T is linear, and sends P to the point $(0,0,1) \in \mathbb{R}^3$.

(d) Assume the contrary that $\exists$ linear transf^{s} $T: \mathbb{R}^3 \to \mathbb{R}^3$ whose image of P is the whole $\mathbb{R}^3$.

Then $\exists\ \vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3 \in P$ s.t.

$$T(\vec{v}_0) = \begin{bmatrix}0\\0\\0\end{bmatrix},\quad T(\vec{v}_1) = \begin{bmatrix}1\\0\\0\end{bmatrix},\quad T(\vec{v}_2) = \begin{bmatrix}0\\1\\0\end{bmatrix},\quad T(\vec{v}_3) = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Hence $T(\vec{v}_1 - \vec{v}_0) = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $T(\vec{v}_2 - \vec{v}_0) = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $T(\vec{v}_3 - \vec{v}_0) = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

By Problem (8) and the fact that $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is l.i., we have:

$$\{\vec{v}_1 - \vec{v}_0,\ \vec{v}_2 - \vec{v}_0,\ \vec{v}_3 - \vec{v}_0\} \subseteq \mathbb{R}^3 \text{ is l.i.}$$

Observe that for each $i = 1, 2, 3$, $\vec{v}_i - \vec{v}_0$ lies in the plane $P' = \{x_3 = 0\} \subseteq \mathbb{R}^3$. So they can be written as:

$$\vec{v}_1 - \vec{v}_0 = \begin{bmatrix}a_{11}\\a_{21}\\0\end{bmatrix},\quad \vec{v}_2 - \vec{v}_0 = \begin{bmatrix}a_{12}\\a_{22}\\0\end{bmatrix},\quad \vec{v}_3 - \vec{v}_0 = \begin{bmatrix}a_{13}\\a_{23}\\0\end{bmatrix}.$$

Then $\{\vec{v}_1 - \vec{v}_0,\ \vec{v}_2 - \vec{v}_0,\ \vec{v}_3 - \vec{v}_0\} \subseteq \mathbb{R}^3$ is l.i.

$\Leftrightarrow$ $\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ 0 & 0 & 0\end{bmatrix}$ has pivot in each column,

which is impossible since the matrix clearly has at most 2 pivots. $\square$

(e) Follow the same idea as in (d). We'd like to pick 4 points $\vec{w}_0, \cdots, \vec{w}_3$ on the hyperboloid s.t.

$$\{\vec{w}_1 - \vec{w}_0,\ \vec{w}_2 - \vec{w}_0,\ \vec{w}_3 - \vec{w}_0\} \subseteq \mathbb{R}^3 \text{ is l.i.}$$

For instance: $\vec{w}_0 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{w}_1 = \begin{bmatrix}-1\\0\\0\end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{w}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

Then $\vec{w}_1 - \vec{w}_0 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$, $\vec{w}_2 - \vec{w}_0 = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$, $\vec{w}_3 - \vec{w}_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Then we can proceed with the same argument as (d). $\square$