Math 104 (Section 4) First Midterm (Practice Exam) October 1, 2019

Name: Solution

- You have 70 minutes to complete the exam.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	25	
2	20	
3	25	
4	30	
Total	100	

1. (a) (5 points) Write down the definition of a sequence (a_n) converging to a real number a.

$\forall \varepsilon > 0, \exists N > 0$ s.t.

$|a_n - a| < \varepsilon \quad \forall n > N.$

(b) (20 points) Prove the following statement based on the definition: Let (a_n) be a sequence converging to a and (b_n) be a sequence converging to b. Then the sequence $(a_n - b_n)$ converges to $a - b$.

Since $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$

$\forall \varepsilon > 0, \exists N_1 > 0$ s.t. $|a_n - a| < \frac{\varepsilon}{2} \quad \forall n > N_1.$

$\exists N_2 > 0$ s.t. $|b_n - b| < \frac{\varepsilon}{2} \quad \forall n > N_2.$

Take $N = \max\{N_1, N_2\}$.

Then for any $n > N$, we have

$$|(a_n - b_n) - (a - b)| = |(a_n - a) - (b_n - b)|$$
$$\leq |a_n - a| + |b_n - b| < \varepsilon.$$

Hence $\lim_{n\to\infty}(a_n - b_n) = a - b.$ $\square$

2. (20 points) Let $a_n = \sqrt{n^2+1} - n$. Prove that (a_n) converges to a real number based on the definition.

Claim: $\lim_{n\to\infty} a_n = 0$.

$\forall \varepsilon > 0$, Choose $N = \frac{1}{2\varepsilon}$. Then for any $n > N = \frac{1}{2\varepsilon}$,

we have

$$\cancel{|a_n - 0| = |\sqrt{n^2+1} - n|}$$

$$2n\varepsilon > 1$$

$$\Rightarrow n^2 + 1 < n^2 + 2n\varepsilon + \varepsilon^2 = (n+\varepsilon)^2$$

$$\Rightarrow n < \sqrt{n^2+1} < n + \varepsilon$$

$$\Rightarrow |\sqrt{n^2+1} - n| < \varepsilon$$

$$\Rightarrow |a_n - 0| = |\sqrt{n^2+1} - n| < \varepsilon.$$

This proves the claim. $\square$

3. Let $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 6}$ for $n \geq 1$.

(a) (20 points) Prove that (a_n) converges to a real number and find the limit.

Claim: $a_n < a_{n+1} < 3 \quad \forall n \geq 1.$ (true for $n=1$)

pf: Prove by induction. $a_1 = 1$, $a_2 = \sqrt{7}$, $1 < \sqrt{7} < 3$.

Suppose $a_n < a_{n+1} < 3$, we want to prove $a_{n+1} < a_{n+2} < 3$.

$a_{n+2} = \sqrt{a_{n+1} + 6} < \sqrt{3+6} = 3.$

$a_{n+2} = \sqrt{a_{n+1} + 6} > a_{n+1}$ because $\sqrt{a_{n+1}+6} > a_{n+1} \Leftrightarrow -2 < a_{n+1} < 3$ which is true by induction hypothesis $\square$.

By the claim, we know (a_n) is a monotone bounded sequence, therefore $\lim_{n\to\infty} a_n = a$ exists.

Now we take limit on both sides of $a_{n+1} = \sqrt{a_n + 6}$; $\Leftrightarrow \underline{a_{n+1}^2 = a_n + 6}$

$$a^2 = \left(\lim_{n\to\infty} a_{n+1}\right)\left(\lim_{n\to\infty} a_{n+1}\right) = \lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty}(a_n + 6) = \left(\lim_{n\to\infty} a_n\right) + 6 = a + 6$$

limit theorem limit theorem

Hence $\lim_{n\to\infty} a_n = a = 3$. $\square$

(b) (5 points) Is (a_n) a Cauchy sequence? Give a brief reason for your answer.

It's Cauchy because it converges.

4. There are four statements below:

(I) For every nonempty subset S of $\mathbb{R}$ that is bounded above, the set $\{x^2 : x \in S\}$ has a supremum that is a real number.

(II) Let (a_n) and (b_n) be bounded sequences. If $a_n < b_n$ for every $n \in \mathbb{N}$, then $\limsup a_n < \limsup b_n$.

(III) Let $M > 0$ and let (a_n) be any sequence satisfying $-M \leq a_n \leq M$ for every $n \in \mathbb{N}$. Then (a_n) admits a subsequence that converges to a real number in $[-M, M]$.

(IV) For all $a \in \mathbb{R}$, $\lim_{n\to\infty} \frac{a^n}{n!} = 0$. (Recall that $n! = 1 \cdot 2 \cdot \cdots \cdot n$.)

(a) (15 points) Choose a statement that is true and prove it. *You are not allowed to choose more than one statement.*

My statement is ________.

(III). (a_n) is a bounded sequence. So $\limsup_{n\to\infty} a_n \in \mathbb{R}$ and there exists a subsequence of (a_n) that converges to $\limsup_{n\to\infty} a_n$.

Since each $-M \leq a_n \leq M$, we have $-M \leq \limsup_{n\to\infty} a_n \leq M$. $\square$

(IV) $\exists N \in \mathbb{N}$ s.t. $N > a$. Write $\delta = \frac{a}{N} < 1$

Then for any $n > N$, we have

$$\frac{a^n}{n!} = \frac{\overbrace{a \cdot a \cdots a}^{n}}{1 \cdot 2 \cdot 3 \cdots n} = \underbrace{\frac{a \cdot a \cdots a}{1 \cdot 2 \cdots N}}_{A} \frac{a}{N+1} \frac{a}{N+2} \cdots \frac{a}{n}$$

$$= A\left(\frac{a}{N+1}\right)\left(\frac{a}{N+2}\right)\cdots\left(\frac{a}{n}\right) < A \cdot \delta^{N-n}.$$

$$\Rightarrow 0 < \frac{a^n}{n!} < A \cdot \delta^{N-n} \quad \forall n > N$$

Since $\lim_{n\to\infty} \delta^{N-n} = 0$ ($\because \delta < 1$), by squeeze lemma, we have $\lim_{n\to\infty} \frac{a^n}{n!} = 0$. $\square$

(b) (15 points) Choose a statement that is false. Give an explicit counterexample and justify it. *You are not allowed to choose more than one statement.*

My statement is ________.

(I) $S = \{-1, -2, -3, \cdots\}$ is bounded above, but $\{x^2 : x \in S\} = \{1, 2^2, 3^2, \cdots\}$ ~~[illegible]~~ isn't bounded above.

(II) $(a_n) = (0, 0, 0, \cdots)$

$(b_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$

$a_n < b_n \quad \forall n \in \mathbb{N}$,

but $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = 0$

$\|$

$\limsup_{n\to\infty} b_n = \lim_{n\to\infty} b_n = 0$