Important theorems:

$\boxed{\text{Cauchy's thm.}}$ $\Longrightarrow$ "keyhole argument"

$\Downarrow$

$\boxed{\begin{array}{c}\text{Cauchy integral formula} \\ \Downarrow \\ \text{Regularity of holo. fcns. \&} \\ \text{(Generalization of) Cauchy integral formula}\end{array}}$

Liouville's thm $\Longleftarrow$ Cauchy integral formula

Liouville's thm $\Longrightarrow$ Fundamental thm. of Algebra

Cauchy integral formula $\Longrightarrow$ existence of power series exp. of holo. fcns. $\Longrightarrow$ "local determine global" principle for holo. fcns.

Cauchy integral formula $\Longrightarrow$ Cauchy inequalities

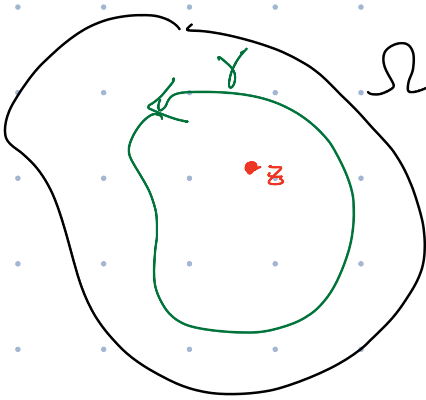
Cauchy integral formula $\Longrightarrow$ Morera's thm.

<u>Thm</u> (Cauchy integral formula).

γ: simple closed curve,

$\Omega \subseteq \mathbb{C}$ open, which contains γ & its interior

$f: \Omega \to \mathbb{C}$ holo.



Ω

$\forall z$ in the interior of γ, we have:

$$\boxed{f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw.}$$

Thm $\Omega \subseteq \mathbb{C}$ open.

- If f is holo. in Ω, then it has infinitely many complex derivatives in Ω.
- $\gamma \subseteq \Omega$ simple closed curve, and Ω also contains the interior of γ. then
$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,dw$$
for any z in the interior of γ.

pf: Induction on n: $n=0$: by Cauchy integral formula.
Suppose the statement is true for $n-1$, i.e.
$f^{(n-1)}$ exists, and $\underbrace{f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^n}\,dw}$

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i\, h}\int_\gamma f(w)\left(\frac{1}{(w-z-h)^n} - \frac{1}{(w-z)^n}\right)dw$$

$$= \frac{(n-1)!}{2\pi i \cancel{h}}\int_\gamma f(w)\cdot\frac{\cancel{h}\left((w-z)^{n-1} + (w-z)^{n-2}(w-z-h) + \cdots + (w-z-h)^{n-1}\right)}{(w-z-h)^n (w-z)^n}\,dw$$

$$\left(A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + \cdots + B^{n-1})\right)$$

$$\xrightarrow[h\to 0]{} \frac{(n-1)!}{2\pi i}\int_\gamma f(w)\frac{n\cdot(w-z)^{n-1}}{(w-z)^{2n}}\,dw$$

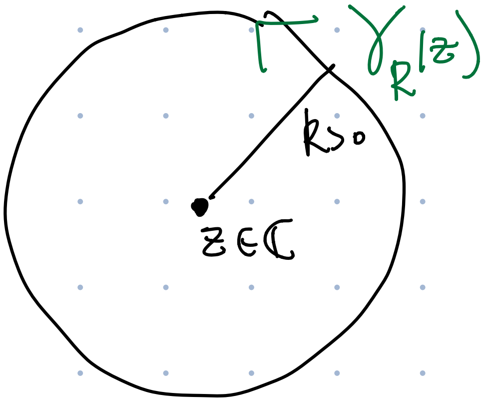
$$= \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw$$

$\Rightarrow f^{(n)}(z)$ exists and $=$

$\square$

<u>Thm (Liouville)</u> If f is <u>entire</u> (i.e. holo. on $\mathbb{C}$) and <u>bounded</u>, then f is a constant function.

<u>pf:</u> Suffices to prove: $f' = 0$



$$\boxed{\text{Say } |f(z)| < M \quad \forall z \in \mathbb{C}}$$

$$f'(z) = \frac{1}{2\pi i} \int_{\gamma_R(z)} \frac{f(w)}{(w-z)^2}\, dw$$

$$|f'(z)| = \frac{1}{2\pi} \left| \int_{\gamma_R(z)} \frac{f(w)}{(w-z)^2}\, dw \right|$$

$$\leq \frac{1}{2\pi} \left(\sup_{w \in \gamma_R(z)} \frac{\overbrace{|f(w)|}^{M}}{\underbrace{|(w-z)^2|}_{R^2 \text{ on } \gamma_R(z)}} \right) \cdot \text{length}(\gamma_R(z))$$

$$< \frac{1}{2\pi} \cdot \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R}$$

$$\Rightarrow |f'(z)| < \frac{M}{R} \quad \forall R > 0$$

$$\Rightarrow f'(z) = 0 \qquad \square$$

<u>Fundamental thm of algebra</u>: Any nonconstant polynomial in $\mathbb{C}$ always has a root (in $\mathbb{C}$).

(i.e. $\forall\, p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$, $a_i \in \mathbb{C}$
$\exists\, z_0 \in \mathbb{C}$ st. $p(z_0) = 0$)

<u>pf</u>: Assume the contrary that $p(z)$ has no root in $\mathbb{C}$.

Then $\frac{1}{p(z)}$ is a holo. fcn on $\mathbb{C}$. (i.e. entire)

<u>Claim</u>: $\frac{1}{p(z)}$ is <u>bounded</u> on $\mathbb{C}$.

Choose $R \gg \max\{|a_{n-1}|, \ldots, |a_0|\} \cdot 2n$.

Then $\forall |z| > R$,

$$|p(z)| = |z^n + a_{n-1} z^{n-1} + \cdots + a_0| \geq |z|^n - |a_{n-1} z^{n-1} + \cdots + a_0|$$

$$> R^n - (\underline{|a_{n-1}|} R^{n-1} + \cdots + |a_0|)$$

$$> R^n - \left(\frac{R}{2n} R^{n-1} + \cdots + \frac{R}{2n}\right)$$

$$> R^n - \frac{1}{2} R^n = \frac{1}{2} R^n.$$

$$\Longrightarrow \left|\frac{1}{p(z)}\right| < \frac{2}{R^n} \quad \forall\, |z| > R.$$

~~For $|z| \leq R$,~~ $\frac{1}{p(z)}$ is bounded on $\{|z| \leq R\}$

b/c $\{|z| \leq R\}$ compact.

$\Rightarrow$ $\frac{1}{p(z)}$ is bounded in $\mathbb{C}$.

By Liouville's thm, $\frac{1}{p(z)}$ is a const. fcn.

$\Longrightarrow$ $p(z)$ const. $\Leftrightarrow$ contradiction. $\square$

<u>Thm</u> (Cauchy ineq.)

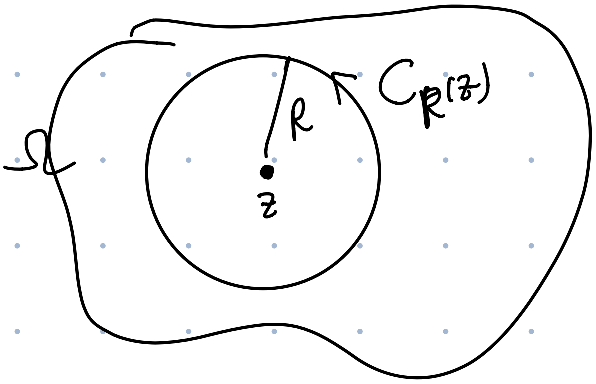
$f: \Omega \xrightarrow[\text{holo}]{} \mathbb{C}$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \sup_{w \in \gamma} \frac{|f(w)|}{|w-z|^{n+1}} \cdot \text{length}(\gamma)$$

$$\leq \frac{n!}{2\pi} \cdot \frac{\|f\|_\gamma}{d(z,\gamma)^{n+1}} \cdot \text{length}(\gamma).$$

$\|f\|_\gamma := \sup_{w \in \gamma} |f(w)|, \quad d(z,\gamma) := \inf_{w \in \gamma} |w-z|$



$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \cdot \frac{\|f\|_{C_R(z)}}{R^{n+1}} \cdot 2\pi R$$

$$= \frac{n!}{R^n} \cdot \|f\|_{C_R(z)}$$

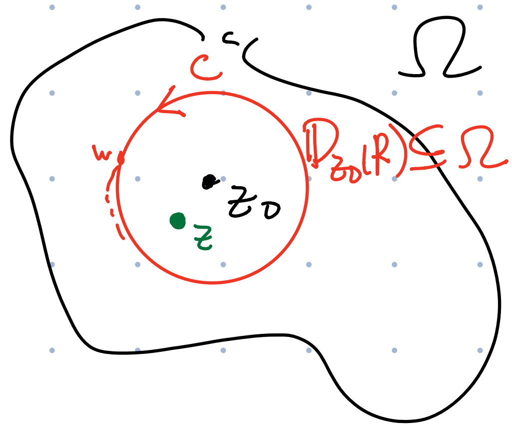
<u>Thm</u> (existence of power series expansion).

($f : \Omega \to \mathbb{C}$ holo.)

$\forall z \in \mathbb{D}_{z_0}(R)$,

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n.$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$



<u>pf</u>

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z} \, dw$$

$$= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0) - (z - z_0)} \, dw.$$

$$= \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}} \, dw.$$

$$\left| \frac{z - z_0}{w - w_0} \right| < 1$$

$$= \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0} \left(\sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n \right) dw.$$

$$= \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} \frac{f(w)}{(w-z_0)^{n+1}} \cdot (z-z_0)^n \, dw.$$

$\boxed{\text{Suppose we can exchange } \int_C \text{ \& } \sum_{n=0}^{\infty},}$ then

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \frac{f(w)}{(w-z_0)^{n+1}} (z-z_0)^n \, dw$$

$$= \sum_{n=0}^{\infty} (z-z_0)^n \underbrace{\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}} \, dw}_{= \frac{f^{(n)}(z_0)}{n!}}$$

this proves the existence of power series expansion.

"Fubini type theorem"

To show that we can exchange $\int_C$ & $\sum_{n=0}^{\infty}$, we need to show;

$$\int_C \sum_{n=0}^{\infty} \left| \frac{\overset{\rightarrow \text{bdd}}{f(w)}}{\underbrace{(w-z_0)^{n+1}}_{\to R^{n+1}}} \right| \underbrace{|z-z_0|^n}_{\to (<R)^n} \, dw < +\infty$$

$$\int \quad \sum_{n=0}^{\infty} \frac{(<R)^n}{R^{n+1}} \cdot 2\pi R \sim \sum_{n=0}^{\infty} \left(\frac{<R}{R}\right)^n < +\infty$$

Thm: $\Omega \subseteq \mathbb{C}$ open connected subset of $\mathbb{C}$.

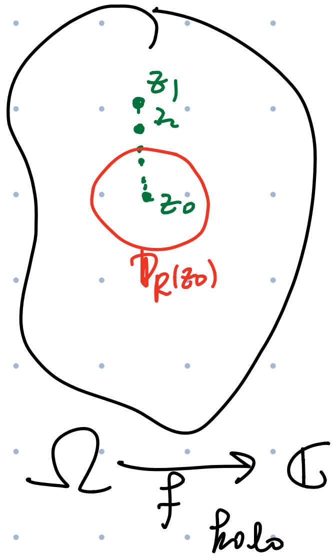
$f: \Omega \to \mathbb{C}$ holo.

If $\exists$ distinct points $z_1, z_2, \cdots$ in Ω

s.t. $\lim z_n = z_0 \in \Omega$,

and $f(z_n) = 0 \quad \forall n$.

Then $f \equiv 0$ in Ω



pf: Consider the power expansion of f at z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall z \in D_R(z_0) \subseteq \Omega.$$

Claim: $a_n = 0 \ \forall n$. (i.e. $f \equiv 0$ on $D_R(z_0)$)

Suppose a_m is the first nonzero coeff.

$$f(z) = \underset{\substack{\neq \\ 0}}{a_m} (z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots$$

$$= a_m (z-z_0)^m \underbrace{\left(1 + (z-z_0) \sum_{n=0}^{\infty} \frac{a_{m+n+1}}{a_m} (z-z_0)^n\right)}_{g(z)}$$

holo. fcn. which $= 1$ at z_0

$\exists \delta > 0$ s.t. $g(z) \neq 0 \quad \forall z \in D_\delta(z_0)$

$\Rightarrow$ $f(z)$ only vanishes at z_0 in $D_\delta(z_0)$

this contradicts with the assumption.

So far, we proved: $f\equiv 0$ on $\mathbb{D}_R(z_0)$. $\Rightarrow$ $\boxed{\mathbb{D}_{R_0}(z_0)\subseteq U}$

$\Downarrow$

$\mathbb{D}_R(z_0)\subseteq V$

Consider $U=\{z\in\Omega : f(z_0)=0\}\subseteq\Omega$

$V=\text{Interior}(U)\subseteq\Omega$

- V is open (clear)
- V is nonempty, since $z_0\in V$
- V is closed: Suppose $z_1,\dots,z_n\in V$, $\lim z_n=z_0$

  need to show: $z_0\in V$.

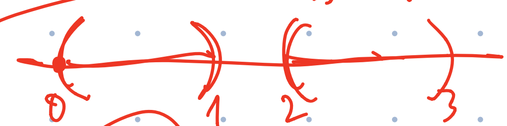
  By previous argument, $\exists\, r>0$ s.t. $\mathbb{D}_r(z_0)\subseteq U \Rightarrow z_0\in V$

Fact: The only nonempty open & closed subset of a connected metric space is the whole metric space.

$\Longrightarrow V=\Omega$

$\Longrightarrow f\equiv 0$ on Ω

$(0,1)\cup(2,3)=S$



$(0,1)$ open & closed subset in S.

$(2,3)$ open & closed

Coro: (local determine global). If $\Omega\subseteq\mathbb{C}$ open connected. f,g holo fcns in Ω.

If $\exists\, \underset{\text{open}}{U}\subseteq\Omega$ s.t. $f(z)=g(z)$ $\forall z\in U$.

then $f(z)=g(z)$ $\forall z\in\Omega$.

(If $\exists \overset{(z_n)}{z_1, \dots, \cancel{z_n}} \in \Omega$, s.t. $\lim z_n = z_0 \in \Omega$

and $f(z_i) = g(z_i) \ \forall i \in \mathbb{N}$,

then $f(z) = g(z)$ on Ω.)