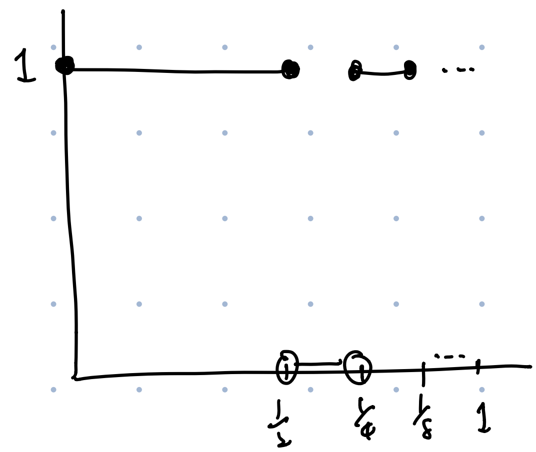
(1) Define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } 1-2^{-2k} \le x \le 1-2^{-(2k+1)} \quad \text{for } k = 0,1,2,\dots \\ 0 & \text{if } 1-2^{-(2k+1)} < x < 1-2^{-(2k+2)} \quad \text{for } k = 0,1,2,\dots \\ 0 & \text{if } x = 1 \end{cases}$$

Prove that f is integrable on $[0,1]$, and compute $\int_0^1 f(x)dx$.

The fcn looks like:



$\forall \varepsilon > 0$, take $N > 0$ large s.t. $\frac{2N+1}{2^{2N}} < \varepsilon$.

Consider the partition: $P = \{0 = t_0 < t_1 < \cdots < t_{2^{2N}} = 1\}$, $t_i = \frac{i}{2^{2N}}$.

Then

$$\sup_{x\in[t_{k-1},t_k]} f(x) - \inf_{x\in[t_{k-1},t_k]} f(x) = \begin{cases} 1 & \text{if } \begin{cases} \frac{k-1}{2^{2N}} = 1 - \frac{1}{2^{2l+1}} \text{ for } l = 0,\cdots,N-1, \ \underline{\text{OR}} \\ \frac{k}{2^{2N}} = 1 - \frac{1}{2^{2l}} \text{ for } l = 1,\cdots,N, \ \underline{\text{OR}} \\ k = 2^{2N} \end{cases} \\ 0 & \end{cases}$$

$$\Longrightarrow \quad U(f,P) - L(f,P) = \frac{2N+1}{2^{2N}} < \varepsilon.$$

Hence f is integrable, and

$$\int_0^1 f(x)\,dx = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots$$

$$= \frac{2}{3}. \quad \square$$

(2) Suppose that $f\colon [a,b] \to \mathbb{R}$ is integrable. Prove that the function $|f|\colon [a,b] \to \mathbb{R}$ which sends x to $|f(x)|$ is also integrable, and

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

See Ross, Thm 33.5.

(3) Let f be a positive and continuous function on $[0,1]$. Compute

$$\int_0^1 \frac{f(x)}{f(x)+f(1-x)}dx.$$

$$\int_0^1 \frac{f(1-x)}{f(x)+f(1-x)}\,dx = \int_0^1 \frac{f(x)}{f(x)+f(1-x)}\,dx$$

and $$\int_0^1 \frac{f(x)}{f(x)+f(1-x)}\,dx + \int_0^1 \frac{f(1-x)}{f(x)+f(1-x)}\,dx = 1.$$

$$\Rightarrow \int_0^1 \frac{f(x)}{f(x)+f(1-x)}\,dx = \frac{1}{2}. \quad \square$$

(4) Let $(\mathcal{C}[0,1], d_\infty)$ be the metric space of continuous functions on $[0,1]$, where the distance function is defined by

$$d_\infty(f,g) = \sup_{x\in[0,1]} |f(x)-g(x)|.$$

Consider the function $T\colon (\mathcal{C}[0,1], d_\infty) \to (\mathcal{C}[0,1], d_\infty)$ defined by

$$(Tf)(x) := \int_0^x f(t)dt.$$

Prove that:

(a) T is not a contraction, i.e. there does not exist $0 < K < 1$ such that

$$d_\infty(Tf, Tg) \leq K \cdot d_\infty(f,g)$$

holds for any $f, g \in \mathcal{C}[0,1]$.

(b) T has a unique fixed point, i.e. there is a unique $f \in \mathcal{C}[0,1]$ satisfies $Tf = f$.

(c) T^2 is a contraction.

(a) Consider $f \equiv 0$ and $g \equiv 1$ on $[0,1]$.

Then:
- $d_\infty(f,g) = 1$
- $Tf \equiv 0$, $(Tg)(x) = x \Rightarrow d_\infty(Tf, Tg) = 1$

Hence T is not a contraction. $\square$

(b) Observe that $f \equiv 0$ satisfies $Tf = f$.

Suppose $f \in \mathcal{C}[0,1]$ s.t. $Tf = f$.

By FTC, Tf is differentiable and $(Tf)' = f$.

$\Rightarrow$ f is differentiable and $f(x) = f'(x)$.

Define $g(x) := f(x) \cdot e^{-x}$.

Then $g'(x) = f'(x)e^{-x} - f(x)e^{-x} = 0$.

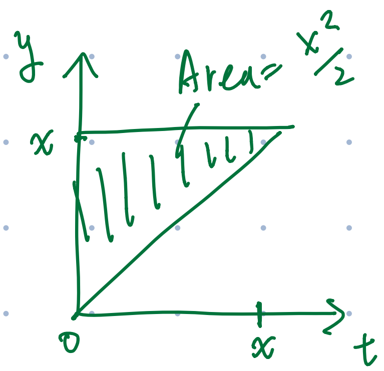
$\Rightarrow$ $g \equiv$ const.

$\Rightarrow$ $f(x) = C \cdot e^{-x}$ for some const $C \in \mathbb{R}$.

$$C = f(0) = (Tf)(0) = \int_0^0 f(t)\,dt = 0.$$

$\Rightarrow$ $f \equiv 0$. $\square$

(c)
$$d_\infty(T^2 f, T^2 g) = \sup_{x \in [0,1]} \left| (T^2 f)(x) - (T^2 g)(x) \right|$$
$$= \sup_{x \in [0,1]} \left| \int_0^x (Tf)(y)\,dy - \int_0^x (Tg)(y)\,dy \right|$$
$$= \sup_{x \in [0,1]} \left| \int_0^x \int_0^y (f(t) - g(t))\,dt\,dy \right|$$
$$\leq \sup_{x \in [0,1]} \int_0^x \int_0^y d_\infty(f,g)\,dt\,dy$$
$$= \sup_{x \in [0,1]} \frac{x^2}{2} \cdot d_\infty(f,g)$$
$$= \frac{1}{2} d_\infty(f,g). \quad \square$$



(5) Let f, g be integrable functions on $[a, b]$. Prove that

$$\left(\int_a^b f(x)g(x)\right)^2 \leq \left(\int_a^b f(x)^2 dx\right)\left(\int_a^b g(x)^2 dx\right).$$

(Hint: Consider $\int_a^b(\int_a^b(f(x)g(y) - f(y)g(x))^2 dx)dy$.)

$$0 \leq \int_a^b \int_a^b (f(x)g(y) - f(y)g(x))^2 \, dx \, dy$$

$$= \int_a^b \int_a^b f(x)^2 g(y)^2 + f(y)^2 g(x)^2 - 2f(x)f(y)g(x)g(y) \, dx \, dy$$

$$= \left(\int_a^b f(x)^2 dx\right)\left(\int_a^b g(y)^2 dy\right) + \left(\int_a^b f(y)^2 dy\right)\left(\int_a^b g(x)^2 dx\right)$$

$$- 2\left(\int_a^b f(x)g(x)\,dx\right)\left(\int_a^b f(y)g(y)\,dy\right)$$

$$= 2\left(\int_a^b f(x)^2 dx\right)\left(\int_a^b g(x)^2 dx\right) - 2\left(\int_a^b f(x)g(x)\right)^2. \quad \square$$