Math H54 First Midterm (Practice Exam) October 3, 2019

Name: Solution

- You have 80 minutes to complete the exam.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	20	
2	20	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	100	

1. (4 points each) Determine if each statement is TRUE or FALSE, and give a short justification.

FALSE (a) If A and B are both symmetric matrices, then AB must be symmetric as well. (A is called symmetric if $A = A^T$.)

e.g. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\Rightarrow$ $AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

not symmetric.

TRUE. (b) If the determinant of a 4×4 matrix A is 4, then $\text{rank}(A) = 4$.

$\det(A) = 4 \neq 0 \Rightarrow A$ invertible $\Rightarrow$ $\text{rank}(A) = 4$.

FALSE (c) If A is an $n \times n$ matrix, then the determinant of A is equals to the product of its diagonal entries.

e.g. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\det A = -1$,

but diagonals are 0.

FALSE (d) The set of all polynomials p satisfying $p(0) = 1$ forms a vector space.

It doesn't contain zero polynomial.,

and isn't closed under addition and scalar multiplication.

FALSE (e) If A has pivot in each row, then its associated linear transformation $\vec{x} \to A\vec{x}$ must be injective.

e.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. $T_A: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ not injective

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2. (10 points each) Let

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

(a) Find $\det(A)$.

Do cofactor expression along 1st column:

$$\det(A) = 2 \cdot \det\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} - \det\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= 4 \cdot \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= 3 \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \boxed{9.}$$

(b) Find A^{-1}.

$$\left[\begin{array}{cccc|cccc} 2 & 1 & 1 & 1 & 1 & & & \\ 1 & 2 & 1 & 1 & & 1 & & \\ 0 & 0 & 2 & 1 & & & 1 & \\ 0 & 0 & 1 & 2 & & & & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 1 & & 1 & & \\ 2 & 1 & 1 & 1 & 1 & & & \\ 0 & 0 & 2 & 1 & & & 1 & \\ 0 & 0 & 1 & 2 & & & & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 1 & & 1 & & \\ 0 & -3 & -1 & -1 & 1 & -2 & & \\ 0 & 0 & 2 & 1 & & & 1 & \\ 0 & 0 & 1 & 2 & & & & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 1 & & 1 & & \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & & \\ 0 & 0 & 2 & 1 & & & 1 & \\ 0 & 0 & 1 & 2 & & & & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & & \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & & \\ 0 & 0 & 2 & 1 & & & 1 & \\ 0 & 0 & 1 & 2 & & & & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & & \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & & \\ 0 & 0 & 1 & 0 & & & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & & & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{array}\right. \underbrace{\left.\begin{array}{cccc} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{9} & -\frac{1}{9} \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} \end{array}\right]}_{\boxed{A^{-1}}}$$

3. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}.$$

Find a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

The matrix A satisfies: $A\begin{bmatrix}1 & -1\\1 & 2\end{bmatrix} = \begin{bmatrix}3 & 0\\5 & 1\end{bmatrix}$.

$$\Rightarrow A = \begin{bmatrix}3 & 0\\5 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 2\end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix}3 & 0\\5 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\-1 & 1\end{bmatrix} = \boxed{\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}}. \quad \square$$

4. (10 points) Let A be a 3×3 matrix with the property that the linear transformation T defined by $\vec{x} \to A\vec{x}$ maps $\mathbb{R}^3$ *onto* $\mathbb{R}^3$. Explain why the transformation must be *injective*.

Since $\vec{x} \mapsto A\vec{x}$ is onto, A has pivot in each row, and since A is a square matrix, A also has pivot in each column, hence $\vec{x} \mapsto A\vec{x}$ is injective. $\square$

(There are lots of different ways to prove this.)

5. (10 points) Under what conditions on $a, b, c \in \mathbb{R}$ is the matrix $A = \begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ invertible?

Be as explicit as possible.

Since $\left\{ \begin{bmatrix} a \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} b \\ 2 \\ 2 \end{bmatrix} \right\}$ are linearly independent, so $\operatorname{rank}(A) \geq 2$,

So the matrix is <u>not</u> invertible $\Leftrightarrow$ $\operatorname{rank}(A)$ is exactly 2,

$\Leftrightarrow$ $\dim \operatorname{Nul}(A)$ is exactly 1. by the rank theorem.

Observe that $\operatorname{Nul}\left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}\right) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

So $\dim \operatorname{Nul}(A) = 1 \Leftrightarrow [a\ b\ c]\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0 \Leftrightarrow a - 2b + c = 0$.

So A is invertible $\Leftrightarrow$ $a - 2b + c \neq 0$.

Simpler way: A invertible $\Leftrightarrow$ $\det A \neq 0$ Then compute $\det(A)$

6. (10 points) Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{pmatrix} -7 \\ 9 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \end{pmatrix} \right\}$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} -7 & -5 \\ 9 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

$$= -\frac{1}{4} \begin{bmatrix} 7 & 5 \\ -9 & -7 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}}$$

$\square$

$A = [\vec{a}_1 \cdots \vec{a}_n]$

7. (10 points) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Show that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

$\operatorname{rank}(A) = \dim \operatorname{Col}(A) = \dim \operatorname{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$.

The columns of AB are all linear combinations of the columns of A, hence belong to $\operatorname{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$.

Hence, $\operatorname{Col}(AB) \subset \operatorname{Col}(A)$ is a subspace in $\operatorname{Col}(A)$.

$\Rightarrow \operatorname{rank}(AB) \leq \operatorname{rank}(A)$. $\square$

8. (10 points) Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces, and $H \subset V$ be a subspace. We know that $T(H)$ is a subspace of W. Prove that $\dim T(H) \leq \dim H$.

Let $\{\vec{v}_1, \cdots, \vec{v}_n\} \subset H$ be a basis of H, so $\dim H = n$.

Then $T(H) = \operatorname{Span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$.

By spanning set theorem, there is a subset of $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ that will give a basis of $T(H)$.

Therefore $\dim T(H) \leq n = \dim H$. $\square$