$A, B: n \times n$.

AB invertible $\Rightarrow$ A, B invertible.

① $\exists\, C$ st $(AB)C = C(AB) = I$

$\Rightarrow A(BC) = I \Rightarrow A$ invertible

$(CA)B = I \Rightarrow B$ invertible

② linear transfmt

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T_B"] \arrow[rr, bend right, "T_A \circ T_B = T_{AB}"'] & \mathbb{R}^n \arrow[r, "T_A"] & \mathbb{R}^n \end{tikzcd}$$

bijective

$\Rightarrow$ T_B is inj. T_A is surj.

Digression on inj/surj:

- $$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Suppose $g \circ f$ is injective,

Then what can we conclude about f, g??

If $g \circ f$ surjective then what about f, g?? g surj!

$g(f(X)) \subseteq g(Y) \subseteq Z$

$f(X) \subseteq Y$

f is inj,

b/c if not, $\exists\, x_1 \neq x_2$ in X st $f(x_1) = f(x_2) \in Y$

$\Rightarrow g(f(x_1)) = g(f(x_2))$ in Z ✗

$g \circ f$ inj, f surj $\Rightarrow g$ inj

g is inj ???? ✗

$y_1 \neq y_2$ in Y $\quad g(y_1) = g(y_2)$ in Z ✗

$f(x_1) \quad f(x_2) \quad g(f(x_1)) = g(f(x_2))$.

b/c not every elt. in Y is in the range of f.

e.g. $\mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}$

$x \longmapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$

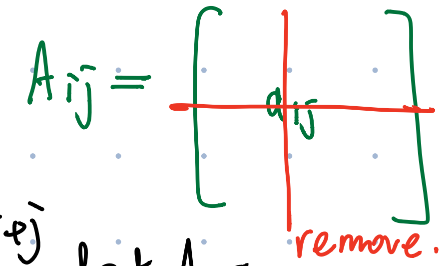
$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto x$

Recap:

Def Determinant $\det(A)$ is defined inductively as follows:

Suppose we've defined $\det$ for any $(n-1)\times(n-1)$ matrix

Now suppose $A: n\times n$



Define the (i,j)-cofactor of A as:

$$C_{ij} := (-1)^{i+j} \det A_{ij}$$

Pick any row $a_{i1}, \dots, a_{in}$ or any column $a_{1j}, \dots, a_{nj}$ of A

Define

$$\det(A) := a_{i1}C_{i1} + \dots + a_{in}C_{in}$$

or

$$\det(A) = a_{1j}C_{1j} + \dots + a_{nj}C_{nj}$$

(cofactor expression of the ith row or jth column)

The definition of determinant doesn't make much sense at the first glance.

Why is det. useful?

Today:

- $\det(A) \neq 0 \iff A$ invertible. & other basic properties of det.
- $\det(AB) = \det(A)\det(B)$
- Compute area/volume of: 

---

e.g.

$$\det \begin{bmatrix} a_{11} & & * \\ \vdots & a_{22} & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} = a_{11} a_{22} \cdots a_{nn}$$

$$a_{11} \underline{\underline{C_{11}}} + \cancel{a_{21}} C_{21} + \cdots \cancel{a_{n1}} C_{n1}$$

$$\| $$

$$\underbrace{(-1)^{1+1}}_{1} \det \begin{bmatrix} a_{22} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}_{(n-1)\times(n-1)}$$

$$\| $$

$$a_{11} \det \begin{bmatrix} a_{22} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}_{(n-1)\times(n-1)}$$

e.g. elementary matrices

$$\det \begin{bmatrix} 1 & & a \\ & \ddots & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = 1 \cdot 1 \cdots \cdot 1 = 1$$

$$\det\begin{bmatrix} c & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = c \neq 0$$

$$\det\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & 1 \ddots 1 & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} = ?\ -1$$

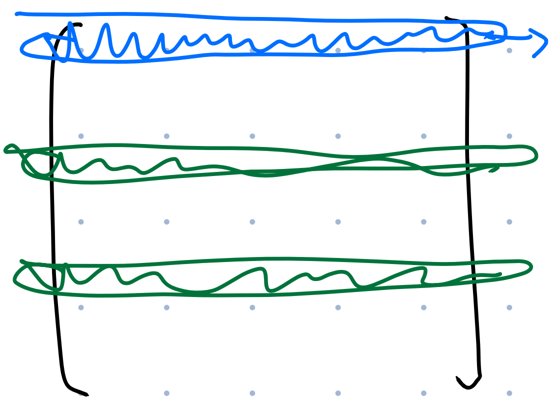
$$= 1 \cdot \det\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & 1 \\ & & & 1 \ddots 1 & \\ & & 1 & & 0 \ddots 1 \end{bmatrix}$$

$(-1)^{i+i}$

$$\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$$

Lemma: If A has 2 identical rows or columns, then $\det(A) = 0$.

pf.



$a_{11}C_{11} + \cdots + a_{1n}C_{1n} = 0.$ $\square$

$(-1)^{1+1}\det\begin{bmatrix} \sim\sim\sim \\ \sim\sim\sim \end{bmatrix}$ $(n-1)\times(n-1)$

e.g.
$$\det\begin{bmatrix} a & b \\ a & b \end{bmatrix} = ab - ab = 0$$

Prove by induction. 2×2 ✓

Suppose the statement is true for any $(n-1)\times(n-1)$ matrix

Lemma: det is linear in each column and each row, i.e.

$$\det\begin{bmatrix} a_{11}+a_{11}' & a_{12}+a_{12}' & a_{1n}+a_{1n}' \\ a_{21} & a_{22} & \vdots \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{2n} & a_{nn}\end{bmatrix} = (a_{11}+a_{11}')(-1)^{1+1}\det\begin{bmatrix} a_{22} & & \\ & \ddots & \\ & & a_{nn}\end{bmatrix} + (a_{12}+a_{12}')(-1)^{1+2}\det\begin{bmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \end{bmatrix} + \cdots$$

$$= \det\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \end{bmatrix} + \det\begin{bmatrix} a_{11}' & \cdots & a_{1n}' \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \end{bmatrix}$$

$$\det\begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \end{bmatrix} = c\det\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\det\begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ ca_{21} & \cdots & ca_{2n} \\ a_{31} & \cdots & a_{3n} \end{bmatrix}$$

$$(a_{11}C_{11}+a_{12}C_{12}+\cdots+a_{1n}C_{1n}) + (a_{11}'C_{11}+a_{12}'C_{12}+\cdots+a_{1n}'C_{1n})$$

$$= \det\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \end{bmatrix} + \det\begin{bmatrix} a_{11}' & \cdots & a_{1n}' \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \end{bmatrix}$$

Prop $\forall$ elementary matrix E, $\det(EA) = \det(E)\det(A)$

pf ① $$\det\Bigg(\underbrace{\begin{bmatrix} 1 & c & \\ & \ddots & \\ & & 1 \end{bmatrix}}_{E}\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \vdots & & \end{bmatrix}}_{A}\Bigg) = \det\begin{bmatrix} a_{11}+ca_{21} & a_{12}+ca_{22} & \cdots \\ a_{21} & a_{22} & \\ \vdots & \vdots & \end{bmatrix}$$

$[a_{11}\ a_{12} \cdots a_{1n}]$ $[ca_{21}\ ca_{22} \cdots ca_{2n}]$

$$\det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots \end{bmatrix} + \det\begin{bmatrix} ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots \end{bmatrix}$$

$$\underset{A}{\det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ & \vdots & & \end{bmatrix}} + c\,\underbrace{\det\begin{bmatrix} a_{21} & \cdots & a_{2n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & & a_{3n} \\ \vdots & & \vdots \end{bmatrix}}_{=0}$$

②
$$\det\begin{bmatrix} c & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ & \vdots & \end{bmatrix} = \det\begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \end{bmatrix}$$
$$= c\det\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ & \vdots & \end{bmatrix}$$

③
$$\det\begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}\begin{bmatrix} \\ A \\ \\ \end{bmatrix} = \det\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & & & \vdots \\ \vdots & & & \vdots \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \\ a_{41} & a_{42} & \cdots & \\ a_{31} & a_{32} & \cdots & \\ a_{21} & a_{22} & \cdots & \\ a_{51} & a_{52} & \cdots & \end{bmatrix}$$

2×2

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-bc$$

$$\det\begin{bmatrix} c & d \\ a & b \end{bmatrix} = bc-ad$$

$$\begin{bmatrix} a_{11} & \cdots & \\ a_{21} & & \\ a_{31} & & \\ a_{41} & & \\ \vdots & & \\ a_{j1} & \cdots & a_{jn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & \\ a_{41} & \\ a_{31} & \\ a_{21} & \\ \vdots & \\ & \end{bmatrix}$$

$$(a_{j1}c_{j1}+\cdots a_{jn}c_{jn}) = -(a_{j1}c'_{j1}+\cdots+a_{jn}c'_{jn})$$

differ by a sign

by Induction

Thm A is invertible $\Leftrightarrow$ $\det(A)\neq 0$.

pf $(\Rightarrow)$ $\exists$ elementary matrices s.t. $E_1\cdots E_k A = I$

$$1 = \det(\mathbb{I}) = \det(E_1 E_2 \cdots E_k A)$$

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$= \det(E_1)\det(E_2 \cdots E_k A)$$

$$\vdots$$

$$= \underbrace{\det(E_1) \cdots \det(E_k)}_{\neq 0} \det(A)$$

$$\Rightarrow \det(A) \neq 0.$$

A IS NOT invertible $\Rightarrow$ $\det(A) = 0$

$$E_1 \cdots E_k A = \begin{bmatrix} 1 & 0 & * & 0 & & \\ & & 1 & * & 0 & \\ & & 0 & & 1 & \\ & & & & 0 & \\ & & & & & 0 \\ & & \varphi & & & 0 \end{bmatrix}$$

$\det$ ... $= 0$

reduced echelon form of A

$$0 = \det(E_1 \cdots E_k A)$$

$$= \det(E_1)\det(E_2 \cdots E_k A)$$

$$\vdots$$

$$= \underbrace{\det(E_1) \cdots \det(E_k)}_{\neq 0} \det(A)$$

$$\Rightarrow \det(A) = 0.$$

$\square$

Thm $\det(AB) = \det(A)\det(B)$.

Coro: $\det(AB) = \det(BA)$ although $AB \neq BA$.

pf

Case 1:

- $\underline{\det(A) \neq 0}$

$$\underbrace{E_1 \cdots E_k}\, A = I$$

$$A = \underbrace{E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1}}_{\text{these are still elementary matrices}}$$

$$\begin{bmatrix} 1 & & a \\ & \ddots & \\ & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & -a \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} c & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/c & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \ddots & 1 \end{bmatrix}^2 = I$$

$$A = \underbrace{F_1 \cdots F_k}_{\text{elementary matrices}}$$

$$\det(AB) = \det(F_1 \cdots F_k B)$$

$$= \det(F_1)\det(F_2 \cdots F_k B)$$

$$\vdots$$

$$= \underline{\det(F_1) \cdots \det(F_k)}\, \det(B)$$

$$\det(A) = \det(F_1 \cdots F_k)$$

$$= \det(F_1) \cdots \det(F_k)$$

- Case 2. A is not invertible $\Rightarrow$ AB not invertible

$\Rightarrow \det(A) = 0$ $\qquad$ $\not\Leftarrow$ $\qquad \Updownarrow$

We should prove: $\boxed{\det(AB)=0}$

Thm $\det(A) = \det(A^T)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & & & \\ \vdots & & & \\ a_{1n} & & & \end{bmatrix}$$

$$a_{11}C_{11} + \cdots + a_{1n}C_{1n} \qquad a_{11}D_{11} + \cdots + a_{1n}D_{1n}$$

$$(-1)^{1+1}\det\begin{bmatrix} a_{22} & a_{23} & \cdots \\ a_{32} & & \\ \vdots & & \end{bmatrix} \qquad (-1)^{1+1}\det\begin{bmatrix} a_{22} & a_{32} & \cdots \\ a_{23} & & \\ \vdots & & \end{bmatrix}$$

Rmk: (Characterization of det).

If $F: M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$, satisfies:

1) $F(I_n) = 1$

2) "linear" in each column:

For each $\vec{v}_1, \cdots, \vec{v}_{i-1}, \vec{v}_{i+1}, \cdots, \vec{v}_n \in \mathbb{R}^n$

$F_i : \mathbb{R}^n \longrightarrow \mathbb{R}$

$\vec{w} \longmapsto F\left(\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_{i-1} & \vec{w} & \cdots \end{bmatrix}\right)$

$$F\left(\begin{bmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_i + \vec{v}_i' & \cdots & \vec{v}_n \\ | & & | & & | \end{bmatrix}\right) = F\left(\begin{bmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_i & \cdots & \vec{v}_n \\ | & & | & & | \end{bmatrix}\right) + F\left(\begin{bmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_i' & \cdots & \vec{v}_n \\ | & & | & & | \end{bmatrix}\right)$$

$$F\left(\left[\vec{v}_1 \cdots c\vec{v}_r \cdots \vec{v}_n\right]\right) = cF\left(\left[\vec{v}_1 \cdots \vec{v}_n\right]\right)$$

3) alternating in columns

$$F\left(\left[\vec{v}_1 \cdots \vec{v}_r \ \vec{v}_j \cdots\right]\right) = -F\left(\left[\cdots \vec{v}_j \cdots \vec{v}_r \cdots\right]\right)$$

$\Rightarrow$ $\underline{F = \det.}$