4/30/2020

$f: \mathbb{R} \to \mathbb{R}$ satisfies the conclusion of IVT,

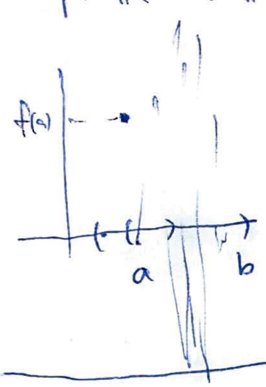
(i.e. $\forall (a,b)$, $\forall y$ btw $f(a)$ & $f(b)$,

$\exists c \in (a,b)$ s.t. $y = f(c)$)

but f is discontinuous at any point in $\mathbb{R}$.

Conway's "base 13 function"

$f: \mathbb{R} \to \mathbb{R}$ s.t. $\boxed{\forall (a,b) \text{ and } \forall y \in \mathbb{R}\ \ \exists c \in (a,b) \text{ s.t. } y = f(c).}$



Ex $\Rightarrow$ f is discontinuous at any pt in $\mathbb{R}$.

- Write any $x \in \mathbb{R}$ in base 13:

digits: $0, 1, 2, \cdots, 9, A, B, C$

$12_{10} = C_{13}$

$13_{10} = 10_{13}$

$$\pm \underbrace{\overset{\times 13^2}{0}\ \overset{\times 13}{0}\ \overset{\times 1}{0}}_{\text{integral part}} \, . \, \underbrace{\overset{\frac{1}{13}}{0}\ \overset{\frac{1}{13^2}}{0} \cdots}_{\text{fractional part.}}$$

$\underline{\qquad\quad}$ 2 C C C C C $\cdots$

$\to$ $\underline{\qquad\quad}$ 3 0 0 0 0 $\cdots$

(in base 10, $0.9999\cdots$
$\to 1.000\cdots$)

Define f: For $x \in \mathbb{R}$.

Consider the base 13 expansion of x,

and forget $\pm$, .

$\to$ a string of digits in $\{0, \dots, 9, A, B, C\}$

- If the string ends with

$$A\, \boxed{x_1 \cdots x_n}\, C\, \boxed{y_1 y_2 \cdots}$$

where $x_i, y_i \in \{0, \dots, 9\}$,

then define

$$f(x) = \underbrace{x_1 \cdots x_n}_{\text{integral}} . \underbrace{y_1 y_2 \cdots}_{\text{fractional}} \text{ in base 10}$$

- If the string ends with

$$B\, x_1 \cdots x_n\, C\, y_1 y_2 \cdots$$

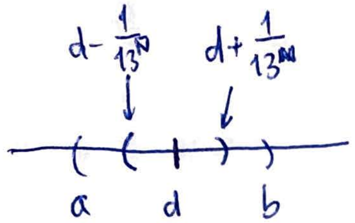
then $f(x) = -x_1 \cdots x_n . y_1 y_2 \cdots$ in base 10

- Otherwise, $f(x) = 0$.

$$f(A1C1234\cdots_{13}) = 1.1234\cdots_{10}$$

$$\parallel$$

$$f(123A1C1234\cdots_{13})$$



$\forall\, r \in \mathbb{R}.$

Want to find $x \in (a,b)$

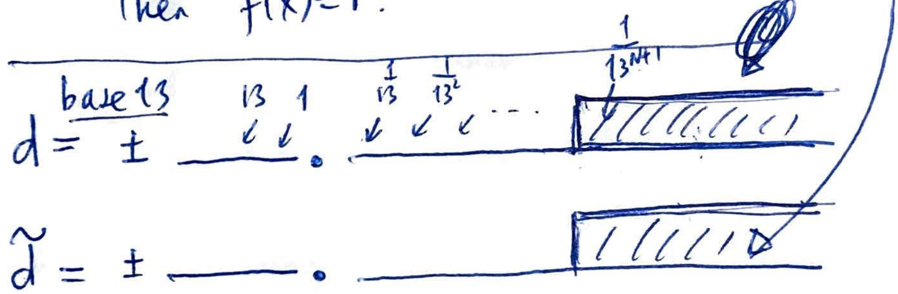
s.t. $f(x) = r$.

$$r \overset{\text{base 10}}{=} \pm\, \underline{x_1 \cdots x_n} \,.\, \underline{y_1 y_2 \cdots}$$

$$\text{``}\tilde{x}\text{''} = \begin{matrix} A \\ B \end{matrix}\ \underline{x_1 \cdots x_n}\ C\ \underline{y_1 y_2 \cdots}$$

If $x \in \mathbb{R}$ ends with the string $\tilde{x}$ (in base 13),

then $f(x) = r$.



$$|d - \tilde{d}| < \frac{1}{13^N}$$

$\Rightarrow \tilde{d} \in (a,b)$

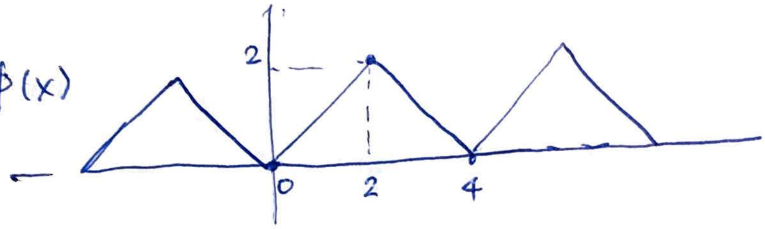
and $f(\tilde{d}) = r$. $\square$

$f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous.

but nowhere differentiable

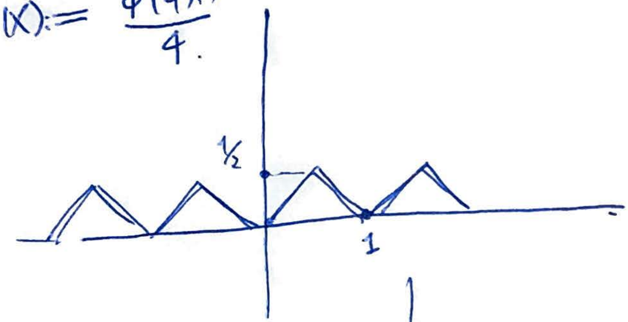
Var der Waerden fcn: Ross §??

$\phi(x)$

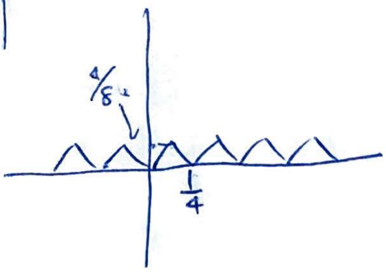


$\phi(x+4) = \phi(x) \quad \forall x$

$f_1(x) := \dfrac{\phi(4x)}{4}.$



$f_2(x) := \dfrac{\phi(4^2 x)}{4^2}$



$\vdots$

$f_n(x) = \dfrac{\phi(4^n x)}{4^n}$

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

- $|f_n(x)| \le \frac{2}{4^n} \quad \forall x \in \mathbb{R}$

$\sum_{n=1}^{\infty} \frac{2}{4^n}$ conv.

By Weierstrass M-test,

$\sum f_n$ conv. unif.

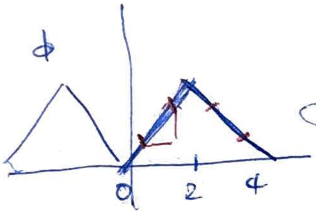
$\Rightarrow f$ conti.

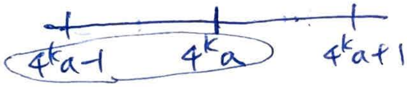
$\forall a \in \mathbb{R}$, we'll find a seq. $(h_k) \to 0$

s.t. $\lim_{k\to\infty} \frac{f(a+h_k)-f(a)}{h_k}$ doesn't exist.

$\forall k \in \mathbb{N}$, consider $4^k \cdot a \in \mathbb{R}$

$\exists\, \varepsilon_k = 1$ or -1, s.t. there is no even integer btw $4^k a$ and $4^k a + \varepsilon_k$





$h_k = \frac{\varepsilon_k}{4^k}$

$\Rightarrow \left|\phi(4^k a+\varepsilon_k) - \phi(4^k a)\right| = 1$

$\forall n \le k$, no even integer btw $4^n a$ & $4^n a + \frac{\varepsilon_k}{4^{k-n}}$

$$\left|\phi\left(4^n a + \frac{\varepsilon_k}{4^{k-n}}\right) - \phi(4^n a)\right| = \frac{1}{4^{k-n}} \quad \forall 1 \le k \le n$$

$\forall n > k$, $\phi(4^n a + 4^{n-k}\varepsilon_k) = \phi(4^n a)$

$$\left|f_n\left(a+\frac{\varepsilon_k}{4^k}\right) - f_n(a)\right| = \frac{1}{4^n}\left|\phi(4^n a + 4^{n-k}\varepsilon_k) - \phi(4^n a)\right|$$

h_k

$$= \begin{cases} \frac{1}{4^k} & \text{if } 1 \le n \le k \\ 0 & \text{if } n > k. \end{cases}$$

$$\frac{f(a+h_k)-f(a)}{h_k} = \frac{\sum_{n=1}^{\infty}(f_n(a+h_k)-f_n(a))}{h_k} = \frac{\sum_{n=1}^{k} \pm \frac{1}{4^k}}{\frac{\varepsilon_k \pm 1}{4^k}} =$$

$$= \sum_{n=1}^{k} (\pm 1)$$

$$= \begin{cases} \text{even integer if } k \text{ is even} \\ \text{odd integer if } k \text{ is odd.} \end{cases}$$

$\Rightarrow \lim_{k\to\infty} \frac{f(a+h_k)-f(a)}{h_k}$ doesn't exist. $\square$

"Space-filling curve" Peano, Hilbert

$I = [0,1]$, $I \times I = [0,1]^2$

$I \subset \mathbb{R}$, $I \times I \subset \mathbb{R}^2$

Ex: $|I| = |I \times I|$ same cardinality

But $I \xrightarrow[\text{homeom.}]{\not\cong} I \times I$.

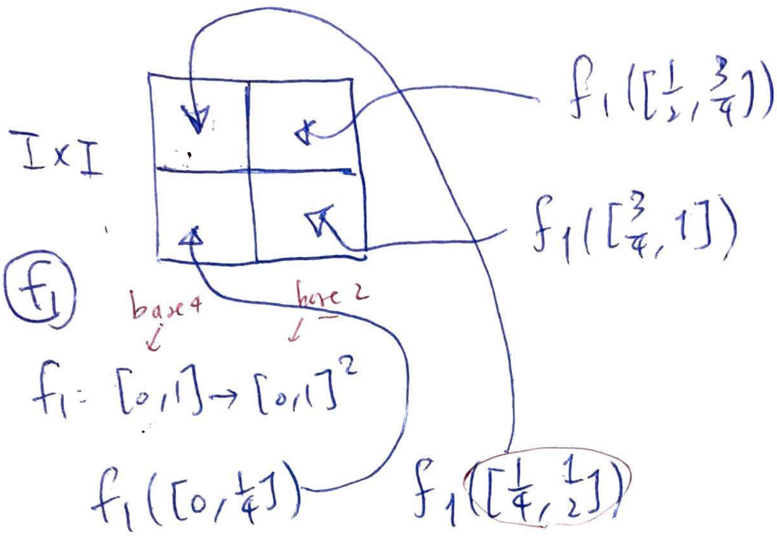
b/c If we remove 1 pt in I^0,

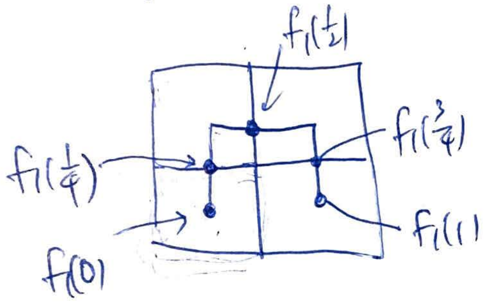
then $I \setminus pt$ is disconnected set.

but $I \times I \setminus pt$ is connected.

$\exists\ f: I \longrightarrow I \times I$ conti.

surjective.





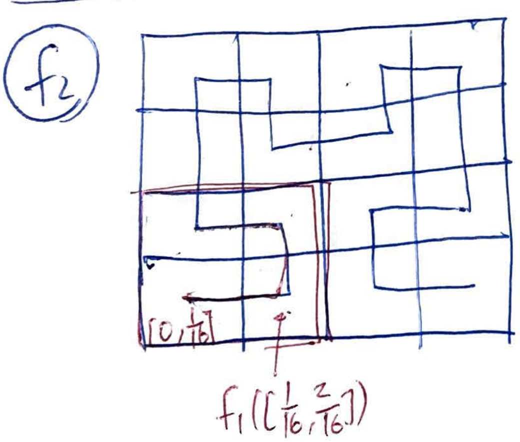
base 4 $\qquad$ base 2

$0.00\ldots \mapsto (0.0\ldots, 0.0\ldots)$

$0.1\ldots \mapsto (0.0\ldots, 0.1\ldots)$

$0.2\ldots \mapsto (0.1\ldots, 0.1,\ldots)$

$0.3\ldots \mapsto (0.1\ldots, 0.0\ldots)$



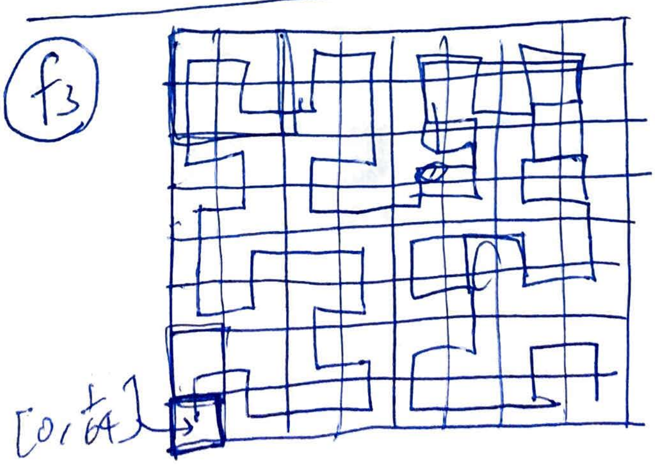
base 4

$0.00\ldots \mapsto (0.00\ldots, 0.00\ldots)$

$0.01\ldots \mapsto (0.01\ldots, 0.00\ldots)$

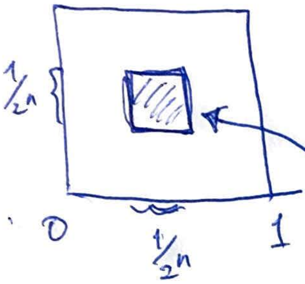
$\vdots$

$[0, \frac{1}{4}]$



<u>Claim</u>: (f_n) conv. unif. to f

$\Rightarrow f$ is conti.



$\forall x \in [0,1]$

$\forall m \geq n$, $f_m(x)$ is required to lies in a square of size $1/2^n$.

$$|f_{m_1}(x) - f_{m_2}(x)| \leq \frac{\sqrt{2}}{2^n} \quad \forall m_1, m_2 \geq n \quad \forall x$$

By <u>Cauchy criterion</u>

$\Rightarrow (f_n(x))$ conv. unif. $\to f$

<u>f is surjective</u>:

$(x, y) \in I^2$

$\downarrow \ \downarrow$

base 2 expansion of x and y.

The definition of (f_n) gives a way to find a preimage of (x,y) in base 4.

<u>Q</u>: Why is it not injective?