#1: Let $A = \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 2 & 4 & 3 & -2 & 11 \\ -1 & -2 & 2 & 1 & 5 \end{bmatrix}$.

(a) Find a basis of the column space of A.

(b) ______ null ______.

Sol^n: row reductions on A:

$$\longrightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 3 & 0 & 9 \\ 0 & 0 & 2 & 0 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

(a) $\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \right\}$

(b). $\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\square$

#2. Let $\mathcal{B} = \{-1+x, 1-2x\}$ and $\mathcal{C} = \{13-5x, 5-2x\}$ be two bases of $\mathrm{Poly}_{\le 1}$. Suppose the coordinate vector of $p(x) \in \mathrm{Poly}_{\le 1}$ w.r.t. $\mathcal{B}$ is $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$,

Find its coord. vector $[p(x)]_{\mathcal{C}}$ w.r.t. $\mathcal{C}$.

Sol^n: $p(x) = 3(-1+x) + (1-2x) = -2+x$.

Then $\begin{bmatrix} 13 & 5 \\ -5 & -2 \end{bmatrix} [p(x)]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$$\Rightarrow [p(x)]_{\mathcal{E}} = \begin{bmatrix} 13 & 5 \\ -5 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= -\begin{bmatrix} -2 & -5 \\ 5 & 13 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \quad \square$$

#3. Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Consider the linear map

$$T : \mathrm{Mat}_{3\times 3} \longrightarrow \mathrm{Mat}_{3\times 2}$$
$$M \longmapsto MA.$$

Find $\dim \ker(T)$ and $\dim \mathrm{Im}(T)$, and prove your answer.

Sol^n: $\dim \ker(T) = 3$ and $\dim \mathrm{Im}(T) = 6$.

Since $\mathrm{rank}(A) = 2$, $\exists$ invertible matrix P s.t.

$$PA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

For any $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \in \mathrm{Mat}_{3\times 2}$, we have

$$\begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & 0 \end{bmatrix} PA = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = B.$$

$\Rightarrow \mathrm{Im}(T) = \mathrm{Mat}_{3\times 2}$ is of 6-dim'l.

By rank-nullity thm, we have $\dim \ker(T) = 3$. $\square$