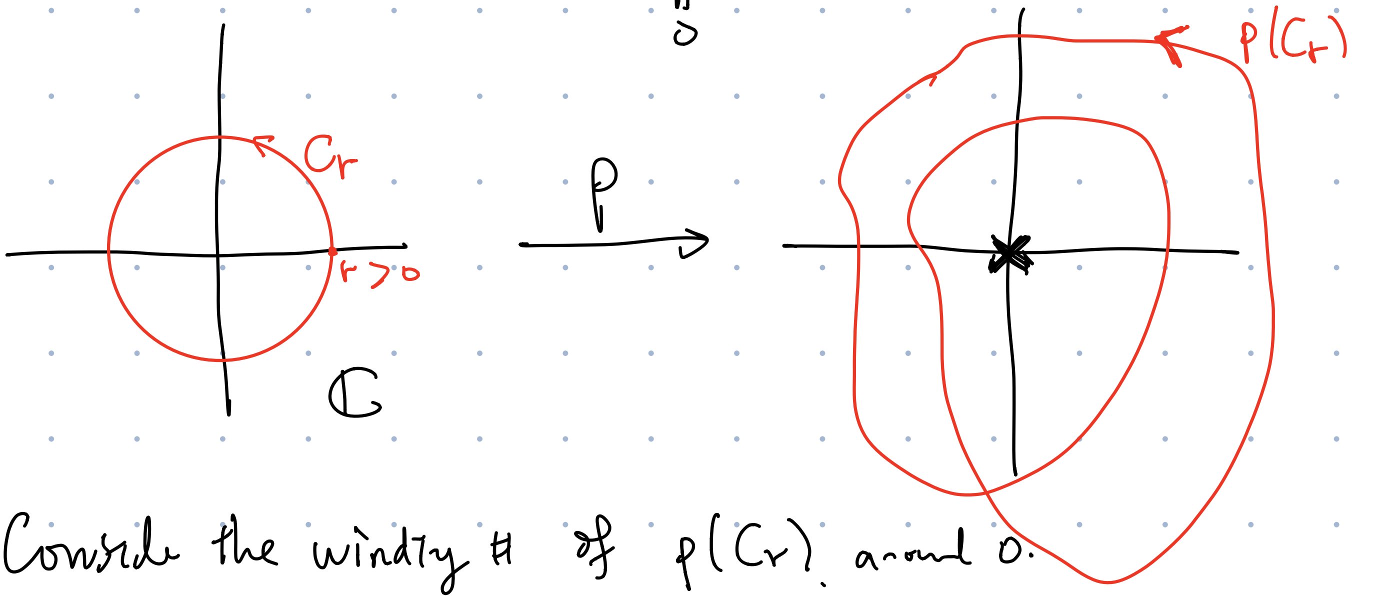
Idea (of using winding # to prove the fundamental thm. of alg.)

$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, $a_0 \neq 0$. Assume $p(z) \neq 0$ $\forall z \in \mathbb{C}$

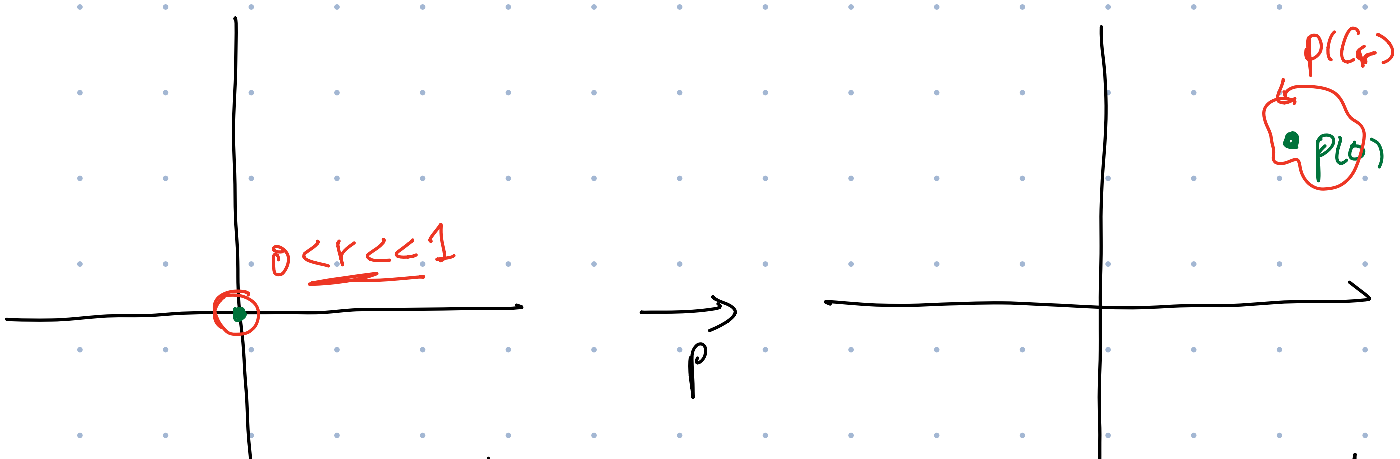


Consider the winding # of $p(C_r)$ around 0.

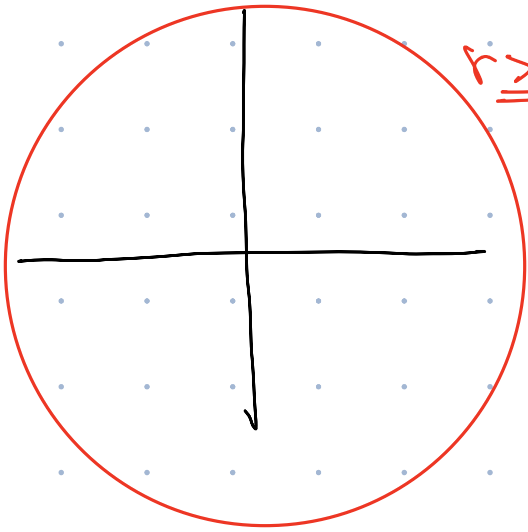
- $\frac{1}{2\pi i}\int_{p(C_r)} \frac{1}{z}\,dz \in \mathbb{Z}$.

- As we move $r>0$ continuously in $(0,\infty)$, the winding # of $p(C_r)$ should also vary continuously.

- Since winding # is a discrete set $(\mathbb{Z})$, so the winding # of $p(C_r)$ is a constant $\forall r>0$

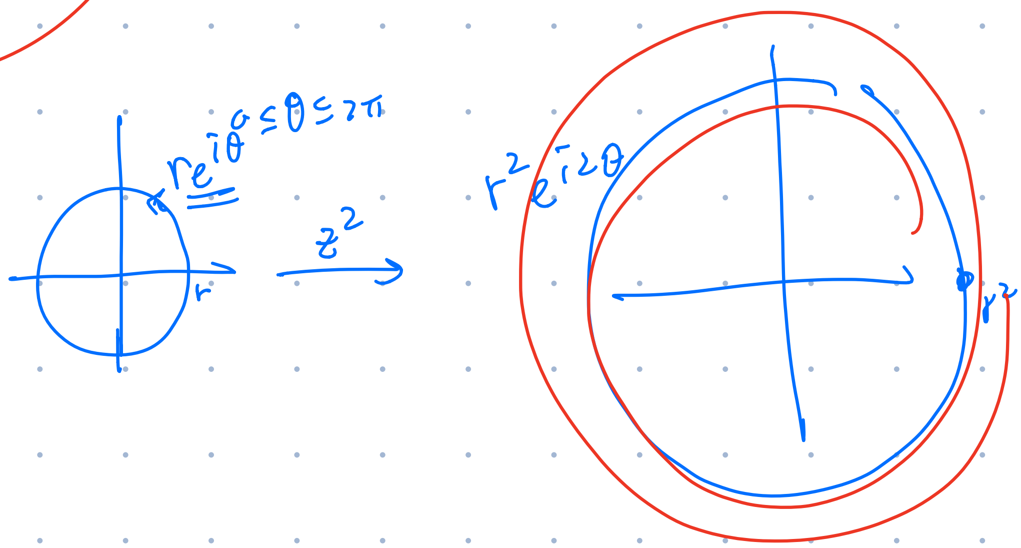


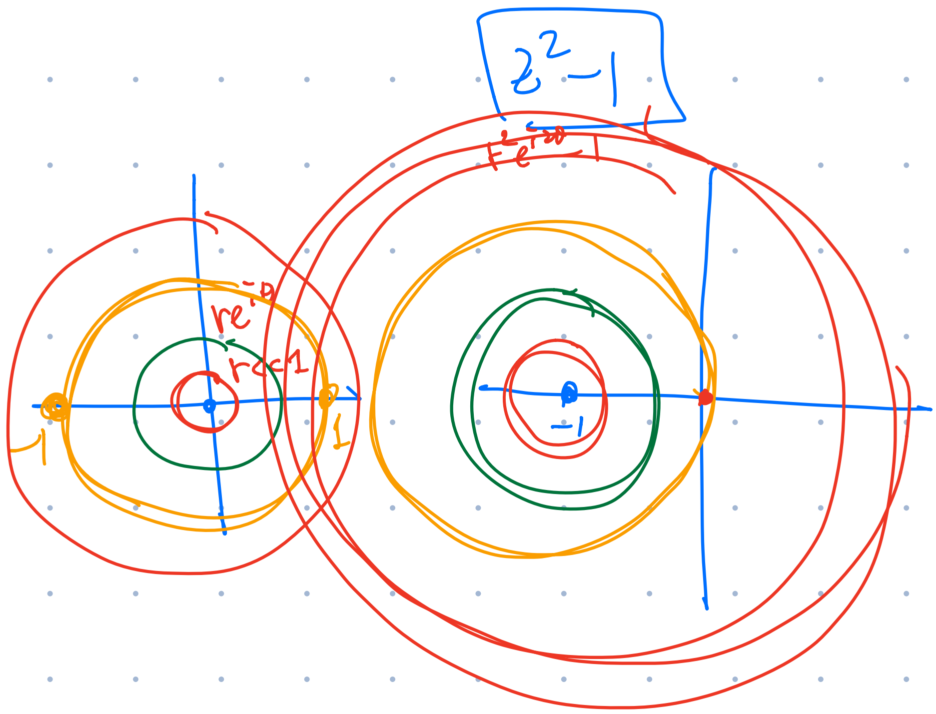
So, when r is very small, the winding # of $p(C_r) = 0$



$r \gg 1$ → p is essentially governed by the leading term z^n.,
so the winding # of $p(C_r)$ is n.

e.g.



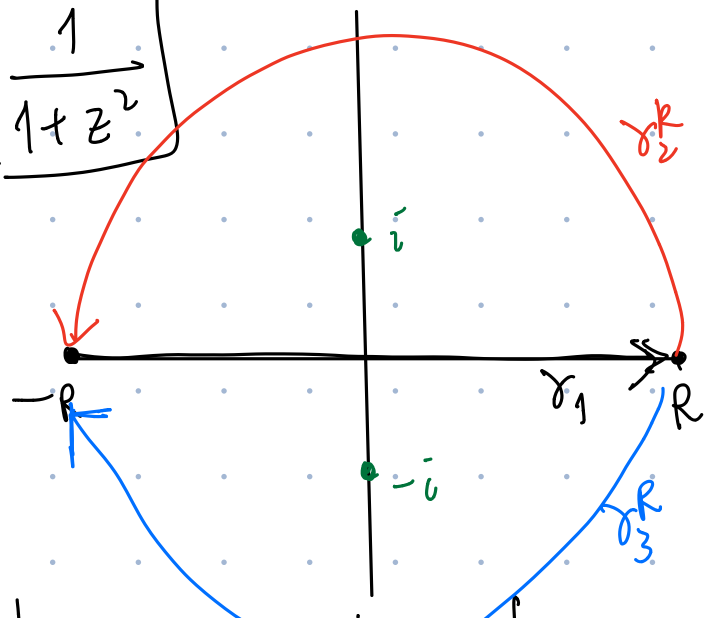


eg: $\displaystyle\int_{\mathbb{R}} \frac{1}{1+x^2}\,dx = \lim_{R\to+\infty} \int_{-R}^{R} \frac{1}{1+x^2}\,dx$

$$\| $$

$$\lim_{R\to\infty} \int_{\gamma_1} \frac{1}{1+z^2}\,dz$$

$$\boxed{\frac{1}{1+z^2}}$$



Rmk: $\displaystyle\int_{\gamma_1} + \int_{\gamma_3} = -2\pi i \operatorname*{Res}_{z=-i}$

$\overrightarrow{\gamma_1\gamma_3}$ is negatively oriented

In residue formula, the curve is positively oriented.

$$\left|\int_{\gamma_2^R} \frac{1}{1+z^2}\,dz\right| = \left|\int_0^{\pi} \frac{1}{1+(Re^{i\theta})^2}(iRe^{i\theta})\,d\theta\right|$$

$$\le \int_0^{\pi} \frac{R}{|1+R^2e^{i2\theta}|}\,d\theta \le \pi\cdot\frac{R}{R^2-1} \longrightarrow 0 \quad \text{as } R\to+\infty$$

By Residue Thm,

$$\int_{\mathbb{R}} \frac{1}{1+x^2}\,dx = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{1+z^2} = \pi. \qquad \square$$

$$\operatorname*{Res}_{z=i} \frac{1}{1+z^2} = \frac{1}{2i}$$

$f(z)$ has a pole at z_0, of order n,

$f(z) = \dfrac{h(z)}{(z-z_0)^n}$ h: nonvanishing holo. near z_0.

$$= \frac{* + (z-z_0)* + (z-z_0)^2 * + \cdots + (z-z_0)^{n-1} * + \cdots}{(z-z_0)^n}$$

$$f(z) = \frac{h(z)}{z-z_0} \quad \underline{\underline{n=1}}$$

$$= \frac{h(z_0) + h'(z_0)(z-z_0) + \cdots}{z-z_0}$$

res.

$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$$

has pole of order 1 at both $\pm i$

(simple pole)

$$= \frac{\left(\frac{1}{z+i}\right)}{z-i}$$

holo, nonvanishing near i.

$$\Rightarrow \operatorname*{Res}_{z=i} \frac{1}{1+z^2} = \frac{1}{2i}.$$

$$\operatorname*{Res}_{z=-i} \frac{1}{1+z^2} = \frac{1}{-2i}.$$

---

e.g. $\underbrace{\int_{\mathbb{R}} \frac{e^{ax}}{1+e^x}\,dx}$ for $\boxed{0 < a < 1.}$

- $$\int_{\mathbb{R}}^{+\infty} \frac{e^{ax}}{1+e^x}\,dx \sim \int_{\mathbb{R}}^{+\infty} \frac{e^{ax}}{e^x}\,dx = \int_{\mathbb{R}}^{+\infty} e^{(a-1)x}\,dx < +\infty$$

since $\underline{a<1}$

- $\displaystyle\int_{-\infty}^{-R} \frac{e^{ax}}{1+e^{x}}\,dx \sim \int_{-\infty}^{-R} e^{ax}\,dx = -\int_{R}^{\infty} e^{-ax}\,dx < +\infty$

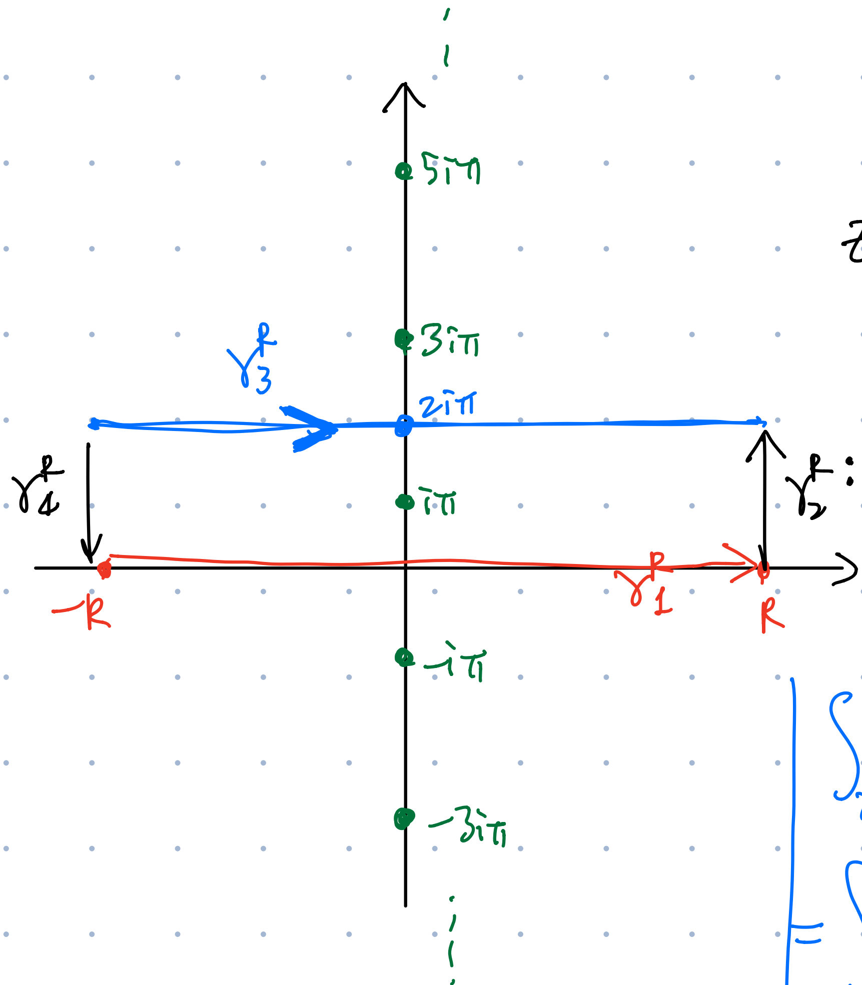
since $\underline{\underline{a>0}}$

$$\boxed{\frac{e^{az}}{1+e^{z}}}$$

Q: Is it entire? Are there any $\underline{\underline{\text{singular points}}}$?

where $e^{z}=-1$

$z = i\pi + 2i\pi\mathbb{Z}$



$\gamma_2^R : [0,1] \to \mathbb{C}$
$t \mapsto R + 2\pi i t$

$\gamma_3^R : [-R,R] \to \mathbb{C}$
$t \mapsto t + 2\pi i$

$$\int_{\gamma_3^R} \frac{e^{az}}{1+e^{z}}\,dz = \int_{-R}^{R} \frac{e^{a(t+2\pi i)}}{1+e^{(t+2\pi i)}}\,(1)\,dt = \int_{-R}^{R} \frac{e^{at}\cdot e^{a2\pi i}}{1+e^{t}}\,dt = e^{a\cdot 2\pi i}\cdot \int_{-R}^{R} \frac{e^{at}}{1+e^{t}}\,dt = e^{a\cdot 2\pi i} \int_{\gamma_1^R} \frac{e^{az}}{1+e^{z}}\,dz$$

$\underline{\text{Want:}}$ $\displaystyle\lim_{R\to\infty} \int_{\gamma_1^R} \frac{e^{az}}{1+e^{z}}\,dz.$

$\underline{\text{Claim:}}$ $\displaystyle\lim_{R\to\infty} \int_{\gamma_2^R} \frac{e^{az}}{1+e^{z}}\,dz = 0$

$\displaystyle\lim_{R\to\infty} \int_{\gamma_4^R} \frac{e^{az}}{1+e^{z}}\,dz = 0$

$$\left|\int_{\gamma_2^R} \frac{e^{az}}{1+e^z}\,dz\right| = \left|\int_0^1 \frac{e^{a(R+2\pi i t)}}{1+e^{(R+2\pi i t)}}\cdot(2\pi i)\,dt\right|$$

$$\leq \int_0^1 \frac{e^{aR}}{\left|1+e^{(R+2\pi i t)}\right|}\cdot 2\pi\,dt$$

$$\leq \frac{e^{aR}}{e^R-1}\cdot 2\pi. \longrightarrow 0 \quad \text{as } R\to\infty$$

since $\underline{a<1}$

By similar estimate, you can show

$$\left|\int_{\gamma_4^R} \frac{e^{az}}{1+e^z}\,dz\right| \longrightarrow 0 \quad \text{as } R\to\infty$$

by $0<a$.

By residue thm.,

$$2\pi i\cdot \operatorname*{Res}_{z=i\pi} \frac{e^{az}}{1+e^z} = \int_{\gamma_1^R} + \int_{\gamma_2^R} - \int_{\gamma_3^R} + \int_{\gamma_4^R}$$

$\forall R.$

As $R\to+\infty$, $\int_{\gamma_2^R}, \int_{\gamma_4^R} \longrightarrow 0$

$$\Rightarrow 2\pi i \boxed{\operatorname*{Res}_{z=i\pi} \frac{e^{az}}{1+e^z}} = (1-e^{a\cdot 2\pi i})\cdot \underbrace{\lim_{R\to\infty}\int_{\gamma_1^R} \frac{e^{az}}{1+e^z}dz}_{= \boxed{\int_{\mathbb{R}} \frac{e^{ax}}{1+e^x}dx}}$$

??

What's the order of pole of $\frac{e^{az}}{1+e^z}$ at $z=i\pi$??

- e^{az} doesn't vanish at $z=i\pi$. $(0<a<1)$

$\Rightarrow$ order of pole of $\frac{e^{az}}{1+e^z}$ at $z=i\pi$

$=$ order of zero of $\boxed{1+e^z}$ at $z=i\pi$.

$$\boxed{1+e^z} = \boxed{(z-i\pi)^n}\, g(z)$$

nonvanishing holo. near $i\pi$

$$\frac{e^{az}}{1+e^z} = \frac{e^{az}}{(z-i\pi)^n\, g(z)}$$

- $f(z)$ has zero at z_0.

the order $=$ the smallest $n\in\mathbb{N}$ s.t. $f^{(n)}(z_0)\neq 0$

- $(1+e^z)' = e^z\Big|_{z=i\pi} = -1 \neq 0$

$\Rightarrow$ $z=i\pi$ is a simple zero of $1+e^z$

$\Rightarrow$ $z = i\pi$ is a simple pole of $\dfrac{e^{az}}{1+e^{z}}$

$$\operatorname*{Res}_{z=i\pi} \frac{e^{az}}{1+e^{z}} = \lim_{z \to i\pi} (z - i\pi) \cdot \frac{e^{az}}{1+e^{z}}$$

$$= e^{ai\pi} \lim_{z \to i\pi} \frac{z - i\pi}{1+e^{z}}$$

$$= e^{ai\pi} \lim_{z \to i\pi} \frac{1}{e^{z}}$$

$$= e^{ai\pi} \frac{1}{e^{i\pi}} \qquad \square$$