$(f_n), f : \underset{\text{set.}}{X} \longrightarrow \mathbb{R}$

$f_n \longrightarrow f$ pointwise.

$$\boxed{\lim_{n\to\infty} \boxed{f_n(x)} = f(x)} \quad \forall x \in X$$

$\forall x \in X, \ \forall \varepsilon > 0, \ \exists N > 0$

s.t. $|f_n(x) - f(x)| < \varepsilon \quad \forall n > N.$

$f_n \to f$ uniformly if

$\forall \varepsilon > 0, \ \exists N > 0$

s.t. $|f_n(x) - f(x)| < \varepsilon \quad \forall n > N, \ \forall x \in X.$

"$\sum f_n$" conv. unf. if $\exists F$

s.t. $\displaystyle\sum_{k=1}^{n} f_k \longrightarrow F$ unif.

#12 $f_n(x) \geq 0$, $\quad X \longrightarrow \mathbb{R}$

$f_n(x) \geq f_{n+1}(x)$, $\quad \underset{x}{\Psi}$

$$\boxed{\lim_{n\to\infty} \sup\{ f_n(x) : x \in X \}^{<\varepsilon} = 0}$$

$\forall \varepsilon > 0$

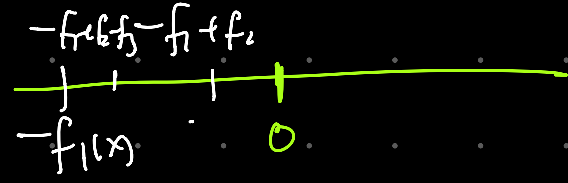
$\exists N > 0$

$\forall n$

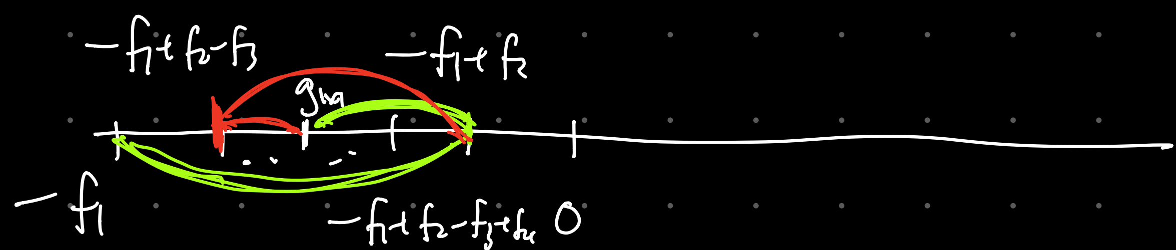
$$\boxed{f_1(x) + f_2(x) - f_3(x) + \cdots \text{ conv. unif.}}$$

- $\sum (-1)^n f_n(x)$ is an alternating series.

If we fix any $x \in X$ $\quad g(x)$



- $\left| \sum_{n=1}^{M} (-1)^n f_n(x) - g(x) \right| \le |f_M(x)| < \varepsilon \quad \forall M > N.$



#13 $\quad f : [0,1] \longrightarrow \mathbb{R}$ conti.

cpt $\qquad$ unif conti.

$\forall \varepsilon > 0, \ \exists \delta > 0$

st. $\quad |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$

$$\lim_{n\to\infty} \frac{1}{n}\left(\underbrace{-f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right)}_{} - f\left(\frac{3}{n}\right) + \cdots \cdots \right) - f(1)$$

$\wedge$

ε

$< \boxed{\frac{n}{2}\varepsilon + |f(1)|}$

$$\boxed{\frac{1}{2}\varepsilon + \frac{|f(1)|}{n}}$$

#16:

$$f(x) = \begin{cases} 1, & x = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{11}}, \cdots \\ 0, & o/w. \end{cases}$$

$$\boxed{\lim_{n \to \infty} f\left(\frac{r}{n}\right) = 0}$$

$\forall r \in \mathbb{R}$

Claim: $\frac{1}{\sqrt{p}}$ can only appear at most once,

in the seq. $\frac{r}{1}, \frac{r}{2}, \frac{r}{3}, \frac{r}{4}, \cdots$

$$\frac{1}{\sqrt{p}} = \frac{r}{n} \quad \frac{1}{\sqrt{q}} = \frac{r}{m}$$

$$\mathbb{Q} \not\ni \sqrt{\frac{q}{p}} = \frac{m}{n} \in \mathbb{Q}.$$

#17:

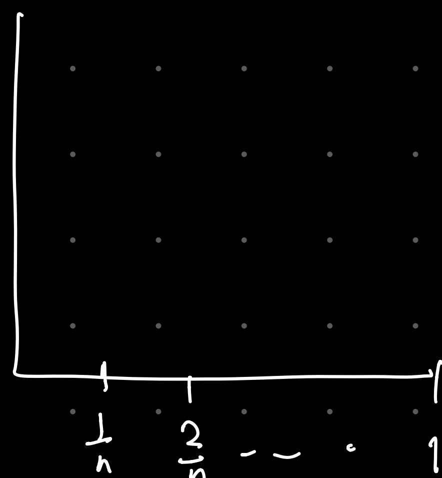
$$g' f + f' = 0$$

$F(x) = \boxed{f(x)\, e^{g(x)}}$ $\qquad F(a) = F(b) = 0$

$$F'(x) = f' \cdot e^{g} + f \cdot g' \cdot e^{g}$$

$$= e^{g} \underbrace{\left(f' + f g' \right)}_{=\,0}$$

#18



$$R_n = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

(b) f not integrable, $\lim_{n\to\infty} R_n$ exists.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

(a) If f integrable, then $\lim R_n = \int_0^1 f$

$\forall \varepsilon > 0$, $\exists P = \{0 = t_0 < t_1 < \cdots < t_\ell = 1\}$

s.t. $U(f,P) - L(f,P) < \varepsilon$.

Compare R_n w/ $U(f,P)$, $L(f,P)$

$$R_n = \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right)$$

For each $\frac{k}{n}$, $\exists$ $1 \le i \le \ell$

s.t. $\frac{k}{n} \in [t_{i-1}, t_i]$

$$\inf_{x \in [t_{i-1}, t_i]} f(x) \le f\left(\frac{k}{n}\right) \le \sup_{x \in [t_{i-1}, t_i]} f(x)$$

$$R_n = \boxed{\frac{1}{n} \sum f\left(\frac{k}{n}\right)} \le \frac{1}{n} \cdot \sum_{i=1}^{\ell} \#\left\{1 \le k \le n \,\middle|\, \frac{k}{n} \in [t_{i-1}, t_i]\right\} \cdot \sup_{x \in [t_{i-1}, t_i]} f(x)$$

$$n(t_i - t_{i-1}) - 1 \le \#\left\{1 \le k \le n \,\middle|\, \frac{k}{n} \in (t_{i-1}, t_i]\right\} \le n \cdot (t_i - t_{i-1}) + 1$$

$$R_n \le \frac{1}{n} \cdot \sum_{i=1}^{l} \left(n(\underline{t_i - t_{i-1}}) + 1 \right) \cdot \sup_{x \in [t_{i-1}, t_i]} f(x)$$

$$= \underbrace{\sum_{i=1}^{l} (t_i - t_{i-1}) \sup_{x \in [t_{i-1}, t_i]} f(x)}_{U(f,P)} + \underbrace{\frac{1}{n} \sum \sup_{x \in [t_{i-1}, t_i]} f(x)}_{\le \frac{l}{n} \cdot \sup_{x \in [0,1]} f(x)}$$

Let $n \to \infty$, $\quad \limsup R_n \le U(f,P)$.

Similarly, $\quad \liminf R_n \ge L(f,P)$

$\Rightarrow$

$$\limsup R_n \le U(f,P) < L(f,P) + \varepsilon$$

$$\le \liminf R_n + \varepsilon. \quad \forall \varepsilon > 0$$

$\Rightarrow \quad \limsup R_n \le \liminf R_n$

$\Rightarrow \quad \lim R_n$ exist.

$\lim R_n \le U(f,P) \quad \forall P$

$\lim R_n \geq L(f, P) \quad \forall P$

$\overline{\lim} R_n \leq U(f)$ $\quad \lim R_n \geq L(f)$

$L(f) \leq \lim R_n \leq U(f)$

$\Rightarrow \lim R_n = \int_0^1 f(x)\,dx.$

#14.

$g(x) = f(x) - f(x + \frac{T}{2})$

$f(0) = f(T).$

$g(0) = \boxed{f(0) - f(\frac{T}{2})} \neq 0$
> 0

$g(\frac{T}{2}) = f(\frac{T}{2}) - f(T) = \boxed{f(\frac{T}{2}) - f(0)} < 0$

$\boxed{g(c) = 0}$ IVT.

#3(c)

$$F(x) = \det\begin{pmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{pmatrix}$$

$F'(x_0) = 0$ for some $a < x_0 < b$

$\|$

$f'(x_0)\left(g(a)\cancel{h(b)} - g(b)\cancel{h(a)}\right)$

$+ g'(x_0)\left(\cancel{h(a)}f(b) - \cancel{h(b)}f(a)\right)$

$\cancel{+ h'(x_0)\left(f(a)g(b) - f(b)g(a)\right)}$

$\boxed{h(x) \equiv 1}$

#9(a)

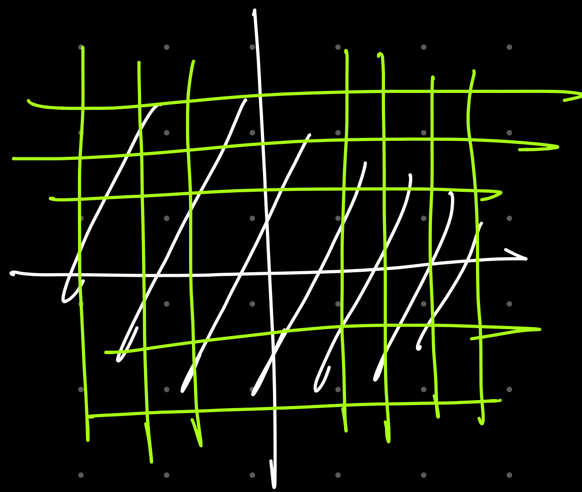
Fix $y \in \mathbb{R}$

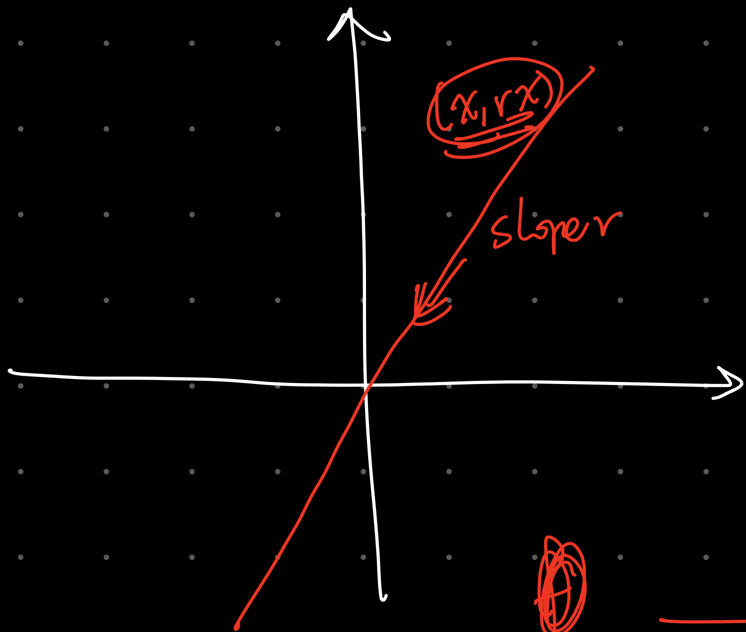
$$f_y : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto f(x, y)$$

- $y \neq 0$, $\quad f_y(x) = \dfrac{xy}{x^2+y^2}$

- $y = 0$, $\quad \boxed{f_y(x) = \begin{cases} \dfrac{xy}{x^2+y^2} = 0, & x \neq 0 \\ 0, & x = 0 \end{cases}}$

(b) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$





$$\frac{x(rx)}{x^2+(rx)^2} = \frac{r\,x^2}{x^2+r^2x^2} = \boxed{\frac{r}{1+r^2}}$$