(1) Determine each of the following sequences is convergent or divergent. For convergent sequences, find the limit and prove it. For divergent sequences, prove that they are divergent.

(a) $a_n = (\frac{2}{3})^n$.
(b) $b_n = 2^n$.
(c) $c_n = \frac{\sin(2n)}{\sqrt{n}}$.
(d) $d_n = \sin(\frac{n\pi}{2})$.
(e) $e_n = \sqrt{n^2 + 4n} - n$.
(f) $f_n = \frac{2^n}{n!}$.

Sol^{n}:

(a) $\lim a_n = 0$. (similar to the proof $\lim \frac{1}{2^n} = 0$ we did in class).

(b) divergent, since (b_n) is not bounded.

(c) $\lim c_n = 0$.

$\forall \varepsilon > 0$, let $N = 1/\varepsilon^2$.

then $n > N \Rightarrow |c_n - 0| \le \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \varepsilon$. $\square$

(d) divergent. (similar to the proof $(-1, 1, -1, 1, \cdots)$ is div. we did in class.)

(e) $\lim e_n = 2$,

$\forall \varepsilon > 0$, let $N = \frac{2}{\varepsilon}$.

then $n > N \Rightarrow |e_n - 2| = |\sqrt{n^2+4n} - n - 2|$

$$= \frac{4}{\sqrt{n^2+4n} + (n+2)}$$

$$< \frac{4}{2n} < \frac{2}{N} = \varepsilon. \quad \square$$

(f). $\lim f_n = 0$.

$\forall \varepsilon > 0$, Let $N = 3 + \dfrac{\log\left(\frac{4}{3\varepsilon}\right)}{\log 2}$.

Then for any $n > N$, we have:

$$\frac{2^n}{n!} = \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\frac{2}{4}\cdot \cdots\cdots \frac{2}{n}$$

$$< 2\cdot 1\cdot \frac{2}{3}\cdot \underbrace{\frac{1}{2}\cdot\frac{1}{2}\cdots\frac{1}{2}}_{(n-3)\text{-copies}}$$ since $\frac{2}{n} < \frac{2}{4}$ $\forall n > 4$.

$$= \frac{4}{3}\cdot\frac{1}{2^{n-3}} < \varepsilon. \quad \square$$

(2) (Squeeze lemma, **very useful**) Let (a_n), (b_n), (c_n) be three sequences satisfying $a_n \le b_n \le c_n$ for all n. Suppose that (a_n) and (c_n) both converge with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a$. Prove that $\lim_{n\to\infty} b_n = a$.

pf: $\forall \varepsilon > 0$,

$\exists N_1 > 0$ s.t. $n > N_1 \Rightarrow |a_n - a| < \varepsilon$,

$\exists N_2 > 0$ s.t. $n > N_2 \Rightarrow |c_n - a| < \varepsilon$.

Let $N = \max\{N_1, N_2\} > 0$,

Then $n > N \Rightarrow a - \varepsilon < a_n \le b_n \le c_n < a + \varepsilon$

$\Rightarrow |b_n - a| < \varepsilon. \quad \square$

(3) Let $a_n = \frac{n-\sin(n)}{n}$. Use the squeeze lemma to show that a_n converges and find the limit.

Claim: $\lim a_n = 1$.

pf: We have $1-\frac{1}{n} \le a_n \le 1+\frac{1}{n}$ $\forall n$.

$\lim(1-\frac{1}{n}) = \lim(1+\frac{1}{n}) = 1$.

By squeeze lemma, we have $\lim a_n = 1$. $\square$

(4) Let $a_1 = 3$ and $a_{n+1} = \sqrt{3a_n + 10}$ for $n \geq 1$. Prove that (a_n) converges, and find the limit.

We'll show that $\lim a_n = 5$.

Claim: $a_n \le a_{n+1} \le 5$ $\forall n$.

pf: Induction on n: $a_1 = 3 < a_2 = \sqrt{19} < 5$.

Suppose $a_{n-1} \le a_n \le 5$. Then:

"$a_n \le a_{n+1}$" $\iff$ "$a_n \le \sqrt{3a_n+10}$"

$\iff$ "$a_n^2 \le 3a_n + 10$" (Since $a_n > 0$).

$\iff$ "$(a_n-5)(a_n+2) \le 0$"

$\iff$ "$-2 \le a_n \le 5$" $\iff$ "$a_n \le 5$" (Since $a_n > 0$)

True, by inductive assumption

"$a_{n+1} \le 5$" $\iff$ "$\sqrt{3a_n+10} \le 5$"

$\iff$ "$a_n \le 5$"

Therefore $a_n \le a_{n+1} \le 5$. $\square$

$\Rightarrow$ (a_n) is a bounded increasing seq.
therefore convergent.

Let $a = \lim a_n$.

We have $a_{n+1}^2 = 3a_n + 10$.

$$\lim a_{n+1}^2 = a^2$$
$$\|$$
$$\lim (3a_n+10) = 3a+10$$

by limit theorems.

$\Rightarrow$ $a^2 = 3a+10$.

$\Rightarrow$ $a = 5$ or $\cancel{-2}$

not possible since $a_n > 0$ $\forall n$.

$\square$

(5) Show that if (a_n) converges to a, then the sequence of absolute values $(|a_n|)$ converges to $|a|$. Is the converse statement true?

pf: By triangle ineq.,

$$-|a_n - a| \leq |a_n| - |a| \leq |a_n - a|.$$

$$\Rightarrow \quad \big||a_n| - |a|\big| \leq |a_n - a|.$$

Since $\lim a_n = a$,

$\forall \varepsilon > 0$, $\exists N > 0$

s.t. $n > N \Rightarrow |a_n - a| < \varepsilon.$

$\Rightarrow \big||a_n| - |a|\big| \leq |a_n - a| < \varepsilon.$ $\square$

The converse statement is NOT true.

e.g. $(a_n) = (-1, 1, -1, 1, \ldots)$ div.

but $(|a_n|) = (1, 1, 1, 1, \ldots)$ conv.

(6) Let (a_n) be a sequence of nonzero real numbers. Suppose that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = b$ exists and is less than 1. Prove that $\lim_{n\to\infty} a_n = 0$. (Hint: Choose any c so that $b < c < 1$ and show that there exists $N > 0$ such that $|a_{n+1}| < c|a_n|$ for all $n > N$.)

Choose any $c \in \mathbb{R}$ s.t. $b < c < 1$.

Let $\varepsilon = c - b > 0$.

Since $\lim \left|\frac{a_{n+1}}{a_n}\right| = b$,

$\exists N > 0$

s.t. $n > N \implies \left| \left|\frac{a_{n+1}}{a_n}\right| - b \right| < \varepsilon = c - b.$

$\implies \left|\frac{a_{n+1}}{a_n}\right| < b + \varepsilon = c.$

$\implies |a_{n+1}| < c|a_n|.$

Now fix any $n_0 > N$. Then we have

$$0 < |a_{n_0+k}| < c\,|a_{n_0+(k-1)}| < \cdots < c^k |a_{n_0}| \quad \forall k > 0.$$

Since $0 < c < 1$, we have $\lim (c^k |a_{n_0}|) = 0$.

By squeeze lemma, we have $\lim a_n = 0$. $\square$