HOMEWORK 7
MATH 104, SECTION 2

Some ground rules:

- You have to submit your homework via **Gradescope** to the corresponding assignment. The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You're allowed to use any result that is proved in the lecture; but if you'd like to use other results, you have to prove them before using them.

Problem set (5 problems; due March 16 at 11am PT)

(1) Explain why the Sierpiński triangle in $\mathbb{R}^2$ is compact. (You may want to read more about the construction of the Sierpiński triangle on Wikipedia.)

(2) Prove that any polynomial function of odd degree has at least one real root.

(3) (a) Let $f\colon (X, d_X) \to (Y, d_Y)$ be a uniformly continuous function (on the whole domain X). Suppose that (x_n) is a Cauchy sequence in X. Prove that $(f(x_n))$ is a Cauchy sequence in Y. (See Ross, Definition 13.2 for the definition of Cauchy sequences in metric spaces.)

(b) Find an example of a continuous function $f\colon (X, d_X) \to (Y, d_Y)$ and a Cauchy sequence (x_n) in X such that $(f(x_n))$ is not Cauchy in Y.

(4) Determine whether the following functions are uniformly continuous, and give proofs:

(a) $A(x) = \log x$ on $(0, 1)$.

(b) $B(x) = \frac{1}{x^2+1}$ on $\mathbb{R}$.

(c) $C(x) = \sin(\frac{1}{x})$ on $(0, \infty)$.

(Hint: Problem 3(a) could be useful for proving non-uniform continuity.)

(5) Consider the function $f\colon [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$.

(a) Prove that f is not Lipschitz continuous on $[0, \infty)$, i.e. there does not exist $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y| \text{ holds for any } x, y \geq 0$$

(b) Prove that f is uniformly continuous on $[0, \infty)$. (Hint: Show that $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$ for any $x, y > 0$.)