FIRST MIDTERM SOLUTION
MATH H54

- There are 4 problems in the exam. You'll need to upload two separate PDF files: one of them has your answers to the first two problems, the other has your answers to the last two problems. No late submissions will be accepted.
- You may use only the textbook, notes from the lectures, homework, quizzes and their solutions. In particular, you're NOT allowed to receive/give any assistance in any form during the exam.
- Make sure your argument is as clear as possible. In case you wish to use a theorem, you should write down the name of the theorem or state the precise result. You can use any statement that is proved in the lectures or appeared in homework assignments.
- Please write clearly. Answers that are not legible cannot be given credit.
- Good luck!

(1) Consider the following basis of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

(a) (10 points) Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Compute $[\vec{v}]_{\mathcal{B}}$, the coordinate vector of $\vec{v}$ relative to the basis $\mathcal{B}$.

(b) (10 points) Suppose $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, T\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, T\left(\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Compute

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = ?$$

Solution. (a)

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 5 & -2 & -7 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 3 \end{bmatrix}.$$

(b) By Part (a), we have

$$T\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) - 6\cdot T\left(\begin{bmatrix}2\\1\\1\end{bmatrix}\right) + 3\cdot T\left(\begin{bmatrix}4\\3\\2\end{bmatrix}\right) = \begin{bmatrix}1\\-5\\-2\end{bmatrix}.$$

(2) Let V be the subspace of $\mathbb{R}^4$ defined by

$$V := \{\vec{x} \in \mathbb{R}^4 \colon x_1 - x_2 + 2x_3 - x_4 = 0\}.$$

(a) (20 points) Find a matrix A such that the linear transformation $T_A \colon \mathbb{R}^3 \to \mathbb{R}^4$ satisfies

$$\ker(T_A) = \{0\} \text{ and } \operatorname{Im}(T_A) = V.$$

(b) (20 points) Suppose A and A' are two matrices that both satisfy the conditions in the previous part. Then one of the following two statements must be true. Find out which one it is, and prove your answer.

(i) There exists an invertible matrix B such that $AB = A'$.

(ii) There exists an invertible matrix B such that $BA = A'$.

Solution. (a) Observe that A must be a 4×3 matrix. Suppose that we have $\operatorname{Im}(T_A) = V \subseteq \mathbb{R}^4$. Then since V is 3-dimensional, the rank-nullity theorem implies that $\ker(T_A) = \{0\}$. Therefore, it suffices to find a 4×3 matrix A such that

$$\operatorname{Col}(A) = \operatorname{Im}(T_A) = V \subseteq \mathbb{R}^4.$$

It is not hard to see that

$$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}.$$

form a basis of V. Hence we can choose A to be

$$A = \begin{bmatrix}1 & 2 & 1\\1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix}.$$

(The choice of A is not unique.)

(b) **Answer: (i).** Since A and A' both satisfy condition in Part (a), we have

$$\operatorname{Col}(A) = \operatorname{Col}(A') \subseteq \mathbb{R}^4.$$

In particular, each column of A' is a linear combination of the columns of A. Therefore, there exists a 3×3 matrix B such that

$$AB = A'.$$

Note that B must be invertible: otherwise, we have $\operatorname{rank}(B) < 3$. Then

$$3 = \operatorname{rank}(A') = \operatorname{rank}(AB) \le \operatorname{rank}(B) < 3,$$

contradiction.

(3) (20 points) Let A and B be $m \times n$ matrices. Then $A+B$ also is an $m \times n$ matrix. Prove that

$$\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

(Hint: First show that $\operatorname{Col}(A+B) \subseteq \operatorname{Col}(A) + \operatorname{Col}(B)$.)

Solution. Since each column of $A+B$ lies in $\operatorname{Col}(A) + \operatorname{Col}(B)$, we have

$$\operatorname{Col}(A+B) \subseteq \operatorname{Col}(A) + \operatorname{Col}(B) \subseteq \mathbb{R}^m.$$

Hence

$$\begin{aligned}
\operatorname{rank}(A+B) &= \dim \operatorname{Col}(A+B) \\
&\leq \dim(\operatorname{Col}(A) + \operatorname{Col}(B)) \\
&\leq \dim \operatorname{Col}(A) + \dim \operatorname{Col}(B) \\
&= \operatorname{rank}(A) + \operatorname{rank}(B).
\end{aligned}$$

(4) (20 points) Let A be a square matrix. Suppose there exists a positive integer k such that $A^k = 0$ (here 0 denotes the zero matrix). Prove that the matrix $\mathbb{I} - A$ is invertible. (Hint: For a real number x, we have $1 - x^k = (1-x)(1 + x + x^2 + \cdots + x^{k-1})$. Does a similar identity hold for square matrices?) (Hint: Consider determinants.)

Solution. For any square matrix A and positive integer k, it's easy to show that

$$\mathbb{I} - A^k = (\mathbb{I} - A)(\mathbb{I} + A + A^2 + \cdots + A^{k-1}).$$

Suppose now that $A^k = 0$, then the left hand side is the identity matrix, which has nonzero determinant. Therefore $\det(\mathbb{I} - A) \neq 0$, hence $\mathbb{I} - A$ is an invertible matrix. (Actually, the inverse of $\mathbb{I} - A$ is given by $\mathbb{I} + A + A^2 + \cdots + A^{k-1}$.)