HOMEWORK 1
MATH 104, SECTION 2

Some ground rules:

- You have to submit your homework via **Gradescope** to the corresponding assignment. The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You're allowed to use any result that is proved in the lecture; but if you'd like to use other results, you have to prove them before using them.

Problem set (9 problems; due January 26 at 11am PT)

(1) For any ordered field F and any $a \in F$, one can define the notion of *absolute value* $|a| \in F$ of a (cf. Ross, Definition 3.3). Prove that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| \quad \text{for any} \quad a_1, a_2, \ldots, a_n \in F.$$

(Hint: First prove that the triangle inequality $|a+b| \leq |a|+|b|$ holds for any ordered field, then use induction to prove the desired statement.)

(2) Let F be an ordered field and $S \subseteq F$ be a subset. Complete the following definitions:

(a) We say $a \in F$ is a *lower bound* of S if ...?

(b) We say $S \subseteq F$ is *bounded below* in F if ...?

(c) We say $a \in F$ is the *greatest lower bound* of S if ...?

(d) We say F satisfies the *greatest lower bound property* if ...?

(Hint: These are essentially the opposites of *upper bound*, *bounded above*, *least upper bound*, and *least upper bound property* that we discussed in class; cf. Ross §4 if you're not sure what to do.)

(3) (continue on the previous problem) Prove that an ordered field F satisfies the least upper bound property *if and only if* F satisfies the greatest lower bound property. (Hint: Follow the same idea of the proof of Ross, Corollary 4.5.)

(4) Let $S \subseteq \mathbb{R}$ be a nonempty subset which is bounded above, and let $z = \sup S$. Prove that for any $\epsilon > 0$, there exists $a \in S$ such that $z - \epsilon < a \leq z$. Can $a \in S$ always be found so that $z - \epsilon < a < z$?

(5) Let $x, y \in \mathbb{R}$. Suppose that $x < y + \epsilon$ for any $\epsilon > 0$. Prove that $x \leq y$.

(6) Prove that $\sup\{1-\frac{1}{n} : n \in \mathbb{N}\} = 1$.

(7) Let (a_n) be a convergent sequence with $\lim_{n\to\infty} a_n = a$. Let (b_n) be another sequence such that $b_n = a_n$ for all but finitely many n. Prove that (b_n) is a convergent sequence and has the same limit as (a_n).

(8) Let (a_n) and (b_n) be two convergent sequences with limits a and b respectively. Suppose that $a_n \leq b_n$ for all but finitely many n. Prove that $a \leq b$.

(9) Let $S \subset \mathbb{R}$ be a nonempty subset which is bounded above. Let $z = \sup S$. Prove that there exists a sequence (a_n) such that $a_n \in S$ for all n, and $\lim_{n\to\infty} a_n = z$.