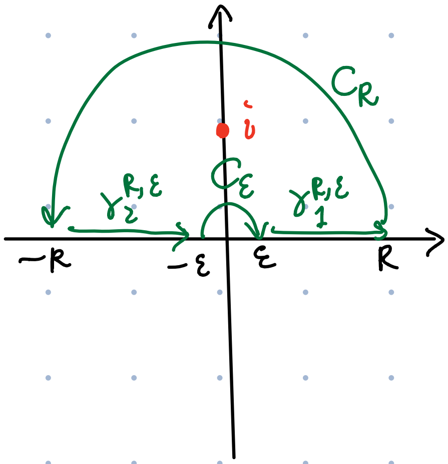
(1) (25 points) Compute (with detail calculations) the following integral
$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}dx.$$



- $$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}dx = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_\varepsilon^R \underbrace{\boxed{\frac{\sin(x)}{x(x^2+1)}}}_{\text{even fcn.}} dx.$$

$$= \frac{1}{2} \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \left(\int_\varepsilon^R + \int_{-R}^{-\varepsilon} \frac{\sin(x)}{x(x^2+1)} dx \right)$$

$$= \frac{1}{2} \operatorname{Im}\left(\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{\gamma_1^{R,\varepsilon}} + \int_{\gamma_2^{R,\varepsilon}} \frac{e^{iz}}{z(z^2+1)} dz \right)$$

- $$\left| \int_{C_R} \frac{e^{iz}}{z(z^2+1)} dz \right| = \left| \int_0^\pi \frac{e^{iRe^{i\theta}}}{\cancel{Re^{i\theta}}(R^2e^{2i\theta}+1)} i\cancel{Re^{i\theta}} \, d\theta \right|$$

$$\leq \int_0^\pi \frac{e^{-R\sin\theta}}{R^2-1} d\theta \leq \frac{\pi}{R^2-1} \longrightarrow 0 \quad \text{as } R\to\infty.$$

- $\dfrac{e^{iz}}{z(z^2+1)}$ has a simple pole at $z=0$, hence:

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{e^{iz}}{z(z^2+1)} dz = -\pi i \cdot \operatorname*{Res}_{z=0} \frac{e^{iz}}{z(z^2+1)} = -\pi i.$$

[see beginning of Lecture 16]

- Hence: $$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \left(\int_{\gamma_1^{R,\varepsilon}} + \int_{\gamma_2^{R,\varepsilon}} \frac{e^{iz}}{z(z^2+1)} dz \right)$$

$$= 2\pi i \cdot \operatorname*{Res}_{z=i} \frac{e^{iz}}{z(z^2+1)} - 0 - (-\pi i)$$

$$= 2\pi i \cdot \frac{e^{-1}}{i\cdot 2i} + \pi i = \pi i\left(1-\frac{1}{e}\right).$$

- So $$\int_0^\infty \frac{\sin x}{x(x^2+1)} dx = \frac{\pi}{2}\left(1-\frac{1}{e}\right). \quad \square$$

Note: "$|\sin(z)| \le 1$" is not true for all $z \in \mathbb{C}$!!

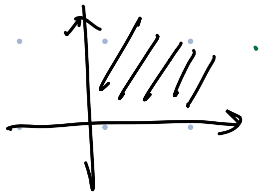
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{-y+ix} - e^{y-ix}}{2i}$$

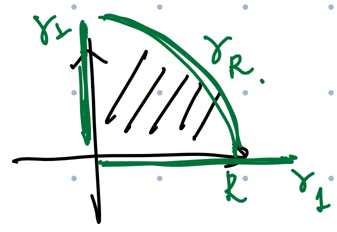
Fix any $x \in \mathbb{R}$, $|\sin(x+iy)| \nearrow +\infty$ exponentially as $|y| \to +\infty$.

Therefore, "$\left| \int_{C_R} \frac{\sin(z)}{z(z^2+1)} dz \right| \to 0$ as $R \to +\infty$." is not true.

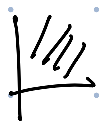
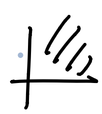
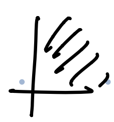
(2) (20 points) Determine the number of zeros (counting multiplicities) of the polynomial $z^8 - 5z^4 + 16$ inside the quadrant $\{x + iy \colon x, y > 0\} \subseteq \mathbb{C}$, and give a proof.

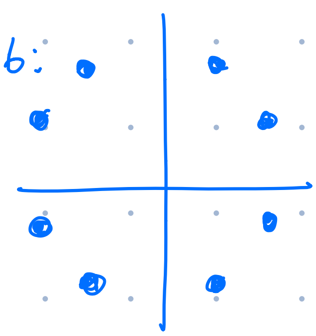
Claim: # of zeros (counting multiplicities) of $z^8 - 5z^4 + 16$

in:  is $\underline{\underline{2}}$.



pf: Let $f(z) = z^8 + 16$, $g(z) = -5z^4$.

- On γ_1 (positive x-axis),
$$|f(z)| = |z^8 + 16| > |5z^4| = |g(z)|.$$
- On γ_2 (positive y-axis), $z = iy$ where $y \geq 0$,
$$|f(z)| = |z^8 + 16| = |y^8 + 16| > |5y^4| = |g(z)|.$$
- For any $R > 0$ large enough s.t. $R^8 - 16 > 5R^4$, we have:
$$|f(z)| \geq |z^8| - 16 = R^8 - 16 > 5R^4 = |g(z)| \quad \text{for } z \in \gamma_R.$$
- By Rouché's thm, # of zeros of f in 
= # of zeros of $f+g$ in 
- There are exactly 2 zeros of f in . $\square$

(zeros of $z^8 + 16$: )

(3) (20 points) Consider the function $f\colon \mathbb{R} \to \mathbb{C}$ given by

$$f(x) = a_0 + a_1 e^{ix} + a_2 e^{2ix} + a_3 e^{3ix} + a_4 e^{4ix},$$

where each a_i is a nonzero complex number. Prove that there exists $x \in \mathbb{R}$ such that $|f(x)| > |a_0|$.

Consider $\tilde{f} : \mathbb{C} \longrightarrow \mathbb{C}$

$$z \longmapsto a_0 + a_1 z + \cdots + a_4 z^4.$$

Then $\tilde{f}$ is an entire function, and $f(x) = \tilde{f}(\underbrace{e^{ix}}_{\text{on the unit circle.}})$ $\forall x \in \mathbb{R}$.

Since $\tilde{f}$ is not a constant fcn, by max. modulus principle,

$$|a_0| = |\tilde{f}(0)| < \max_{z \in \underbrace{\partial D_1(0)}_{\text{unit circle.}}} |\tilde{f}(z)| = \max_{x \in \mathbb{R}} |f(x)|. \quad \square$$

(4) (20 points) Prove that there does not exist a holomorphic function $f\colon \mathbb{D} \to \mathbb{C}$ on the unit open disk $\mathbb{D}$ such that

$$f\left(\frac{1}{n^3}\right) = \frac{1}{n^5} \text{ holds for all positive integer } n.$$

Assume the contrary that $\exists\, f : \mathbb{D} \longrightarrow \mathbb{C}$ holo. s.t. $f(1/n^3) = 1/n^5$ $\forall n \in \mathbb{N}$.

- Since $\lim_{n\to\infty} \frac{1}{n^3} = 0$ and f is conti., we have:

$$f(0) = \lim_{n\to\infty} \frac{1}{n^5} = 0.$$

- Define $g(z) := \begin{cases} f(z)/z, & z \neq 0. \\ f'(0), & z = 0. \end{cases}$

Then g is holo. on $\mathbb{D}$, and

$$g(1/n^3) = \frac{1/n^5}{1/n^3} = 1/n^2. \quad \forall n \in \mathbb{N}.$$

- Again, we have $g(0)=0$, therefore

$$h(z) := \begin{cases} g(z)/z, & z \neq 0 \\ g'(0), & z = 0 \end{cases} \quad \text{is holo. on } \mathbb{D}.$$

- $h(1/n^3) = g(1/n^3) \big/ \frac{1}{n^3} = n. \quad \forall n \in \mathbb{N}.$

$\lim_{n\to\infty} \frac{1}{n^3} = 0$ but $\lim_{n\to\infty} h(\frac{1}{n^3}) = \infty$. Contradiction. $\square$

(5) (15 points) Let $\Omega \subseteq \mathbb{C}$ be an open subset (not necessarily simply connected), and let $f\colon \Omega \to \mathbb{C}\backslash\{0\}$ be a non-vanishing holomorphic function. Let $n \geq 2$ be a positive integer. Prove that if there exists a non-vanishing holomorphic function $g\colon \Omega \to \mathbb{C}\backslash\{0\}$ such that $f(z) = g(z)^n$ for all $z \in \Omega$, then we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \in n\mathbb{Z} = \{\cdots, -2n, -n, 0, n, 2n, \cdots\}$$

for any closed curve γ in Ω. (Note that Ω may not contain the interior of γ, so the argument principle does not apply.)

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_\gamma \frac{n\,g(z)^{n-1}g'(z)}{g(z)^n}dz$$

$$= \underbrace{\frac{1}{2\pi i}\cdot\int_\gamma \frac{g'(z)}{g(z)}dz}_{} \cdot n.$$

is an integer, since it's the winding # of the curve η:

$$\begin{tikzcd} {[a,b]} \arrow[r, "\gamma"] \arrow[rr, bend right, "\eta"'] & \Omega \arrow[r, "g"] & \mathbb{C}\backslash\{0\} \end{tikzcd}$$

around the point $0 \in \mathbb{C}$. $\square$