Today:
- residues
- classification of singularities (local behavior)

Reminder: Lecture this Thursday will be asynchronous.

$$\operatorname*{Res}_{z=i\pi} \frac{1}{1+e^z}$$

has an isolated sing at $z=i\pi$

$m \geq n$

$$\frac{(z-z_0)^{m-n}\tilde{f}(z)}{\tilde{g}(z)}$$

Locally, $\dfrac{f(z)}{g(z)} = \dfrac{(z-z_0)^m \tilde{f}(z)}{(z-z_0)^n \tilde{g}(z)}$

When $m<n$, $= \dfrac{\tilde{f}(z)}{(z-z_0)^{n-m}\tilde{g}(z)}$

$$\boxed{\frac{z^3}{z}} = z^2$$

f, g holo. in a nbd. of z_0

Suppose f has zero of order m at z_0
g has zero of order n at z_0.

Locally, $f(z) = (z-z_0)^m \tilde{f}(z)$, $\tilde{f}$ nonvanishing near z_0
$g(z) = (z-z_0)^n \tilde{g}(z)$, $\tilde{g}$ ——

$\dfrac{1}{1+e^z}$ has a simple pole at $z=i\pi$

(Since $1+e^z$ has a simple zero at $z=i\pi$).

How to compute the residue? (of a pole).

z_0 is a pole of f nonvanishing holo. near z_0

$$f(z) = \frac{h(z)}{(z-z_0)^n}$$

$$= \frac{h(z_0) + h'(z_0)(z-z_0) + \frac{h''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{h^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1}}{(z-z_0)^n}$$

$$\operatorname*{Res}_{z=z_0} f = \frac{h^{(n-1)}(z_0)}{(n-1)!}$$

$$\operatorname*{Res}_{z=i\pi} \underbrace{\frac{1}{1+e^z}}_{f(z)} = h(i\pi)$$

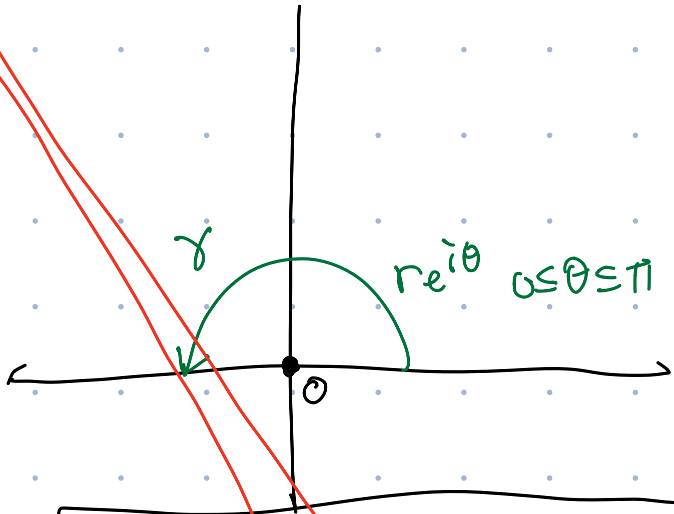
$$= \lim_{z \to i\pi} h(z)$$

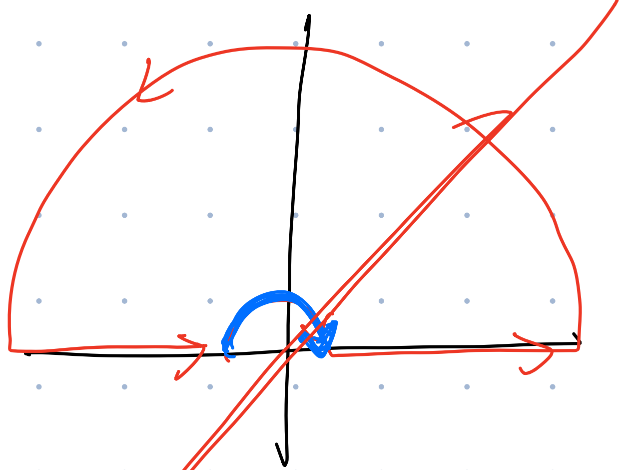
$$= \lim_{z \to i\pi} \frac{1}{1+e^z} \cdot (z - i\pi)$$

$$= \lim_{z \to i\pi} \frac{z - i\pi}{1+e^z}$$

$$= \lim_{z \to i\pi} \frac{1}{e^z}$$

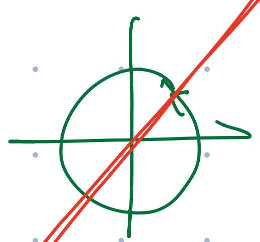
$$= \frac{1}{e^{i\pi}} = -1$$





f has simple pole at 0.

$$\frac{\operatorname{Res}_{z_0} f}{z - z_0} + (\text{holo.})$$



$\to 2\pi i \operatorname{Res} f$

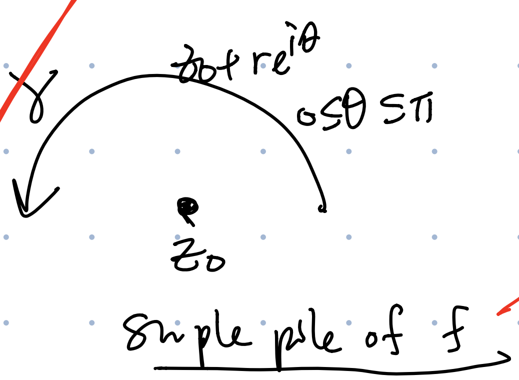
only works for simple pole

$$\int_\gamma \boxed{\frac{1}{z}}\, dz = \int_0^\pi \frac{1}{\cancel{re^{i\theta}}} \cdot (i\cancel{re^{i\theta}})\, d\theta$$

See next lecture for correction!!!

$$= \int_0^\pi i\, d\theta = i\pi = 2\pi i \cdot \frac{1}{2}$$

$$\left(\int_\gamma \frac{1}{z^2}\, dz = \int_0^\pi \frac{1}{(re^{i\theta})^{\cancel{2}}} (i\cancel{re^{i\theta}})\, d\theta \right)$$



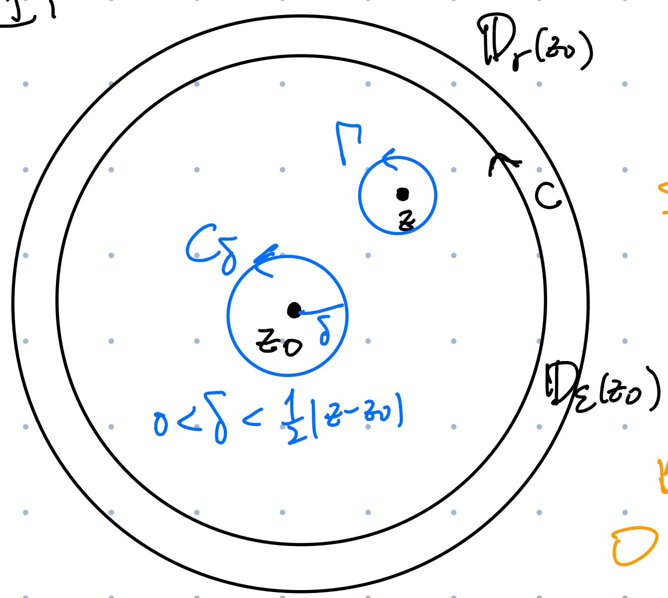
$$\int_\gamma f(z)\, dz = 2\pi i \operatorname{Res}_{z=2} f \cdot \frac{1}{2}$$

???

Thm. Suppose f is holo. on $\mathbb{D}_r^\times(z_0) = \mathbb{D}_r(z_0)\setminus\{z_0\}$, and suppose f is **bounded**.

Then f has a removable sing. at z_0.

Pf.



$$\left|\int_{C_\delta}\frac{f(w)}{w-z}\,dw\right| \le \sup_{w\in C_\delta}\frac{|f(w)|}{|w-z|}\cdot 2\pi\delta \le \frac{M}{\frac{1}{2}|z-z_0|}\cdot 2\pi\delta \quad \forall\,\delta>0$$

$|f(w)|$: bounded near z_0 by a fixed const.

$|w-z| \ge \frac{1}{2}|z-z_0|$

$\frac{1}{2}|z-z_0| > \delta$

$\Downarrow$

$$0 \Leftarrow \frac{1}{2\pi i}\left(\boxed{\int_{C_\delta}\frac{f(w)}{w-z}\,dw} + \int_{\Gamma}\frac{f(w)}{w-z}\,dw\right)$$

$f(z)$ $\Uparrow$ Cauchy integral fmla

indep. of $0<\delta<\frac{1}{2}|z-z_0|$.

For any $z\in\mathbb{D}_\varepsilon(z_0)$, define

$$g(z) := \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw.$$

Claim: g is holo. in $\mathbb{D}_\varepsilon(z_0)$, and $g(z)=f(z)$ $\forall z\in\mathbb{D}_\varepsilon^\times(z_0)$

($\Rightarrow$ f has removable sing. at z_0)

- $$\lim_{h\to0}\frac{g(z+h)-g(z)}{h} = \lim_{h\to0}\frac{1}{h}\frac{1}{2\pi i}\int_C\left(\frac{f(w)}{w-(z+h)}-\frac{f(w)}{w-z}\right)dw$$

$$= \lim_{h\to0}\frac{1}{\cancel{h}}\frac{1}{2\pi i}\int_C\frac{\cancel{h}\,f(w)}{(w-z-h)(w-z)}\,dw$$

$$= \frac{1}{2\pi i}\int_C\frac{f(w)}{(w-z)^2}\,dw. \quad\Rightarrow\quad g \text{ is holo.}$$

Thm f has isolated sing at z_0.

"$|f(z)| \to +\infty$ as $z \to z_0$" $\iff$ $\boxed{\text{"} f \text{ has a pole at } z_0 \text{"}}$

clear.

$\Updownarrow$

$\exists R > 0$

$\exists F : \mathbb{D}_R(z_0) \to \mathbb{C}$ holo st.

$F(z) = \begin{cases} 1/f(z) & \forall z \in \mathbb{D}_R^\times(z_0) \\ 0 & z = z_0. \end{cases}$

pf ($\Rightarrow$)

$$\frac{1}{|f(z)|} \to 0 \quad \text{as} \quad z \to z_0$$

$\Rightarrow$ $\frac{1}{f(z)}$ is bounded near z_0.

$\Rightarrow$ $\frac{1}{f(z)}$ has a removable sing at z_0

$\Rightarrow$ $\exists F : \mathbb{D}_R(z_0) \to \mathbb{C}$ holo.

st $F(z) = \begin{cases} 1/f(z) & z \in \mathbb{D}_R^\times(z_0) \\ L = 0 & z = z_0 \end{cases}$

And L must be 0 since $\frac{1}{|f(z)|} \to 0$ as $z \to z_0$.

$\Rightarrow$ f has pole at z_0.

$\square$

Thm (Casorati-Weierstrauss) (not removable, not pole)

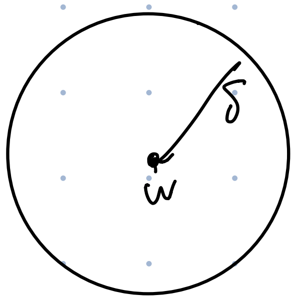
If f is holo. on $\mathbb{D}_R^\times(z_0)$, has essential sing at z_0.

Then the image $f(\mathbb{D}_R^\times(z_0)) \subseteq \mathbb{C}$ is dense in $\mathbb{C}$.

pf: Prove by contradiction. Suppose $f(\mathbb{D}_R^\times(z_0))$ is not dense in $\mathbb{C}$.

$\Rightarrow \exists\, w \in \mathbb{C},\ \delta > 0$ st. $\mathbb{D}_\delta(w) \cap f(\mathbb{D}_R^\times(z_0)) = \phi$.

$\Downarrow$



$$|f(z) - w| \geq \delta \quad \forall z \in \mathbb{D}_R^\times(z_0)$$

$\Downarrow$

Define $g(z) = \dfrac{1}{f(z) - w}$, holo. on $\mathbb{D}_R^\times(z_0)$

then $|g(z)| \leq 1/\delta \quad \forall z \in \mathbb{D}_R^\times(z_0)$.

$\Rightarrow g(z) = \dfrac{1}{f(z) - w}$ has a removable sing at z_0

i.e.

$$G(z) = \begin{cases} \dfrac{1}{f(z) - w}, & z \in \mathbb{D}_R^\times(z_0) \\ G(z_0), & z = z_0 \end{cases}$$ holo. on $\mathbb{D}_R(z_0)$

Case 1: $G(z_0) \neq 0$, then

$$F(z) := \frac{1}{G(z)} + w \quad \text{holo. in } \mathbb{D}_R(z_0)$$

and $F(z) = f(z) \quad \forall z \in \mathbb{D}_r^\times(z_0)$.

$\Rightarrow$ f has removable sing at z_0. $\rightarrow\!\!\!\!\times$

<u>Case 2:</u> $G(z_0) = 0$

$$\Rightarrow \frac{1}{|f(z)-w|} \longrightarrow 0 \quad \text{as} \quad z \to z_0$$

$$\Rightarrow |f(z)-w| \longrightarrow +\infty \quad \text{as} \quad z \to z_0$$

$$\Rightarrow |f(z)| \longrightarrow +\infty \quad \text{as} \quad z \to z_0.$$

$\Rightarrow$ z_0 is a pole of f. $\rightarrow\!\!\!\!\times$ $\square$

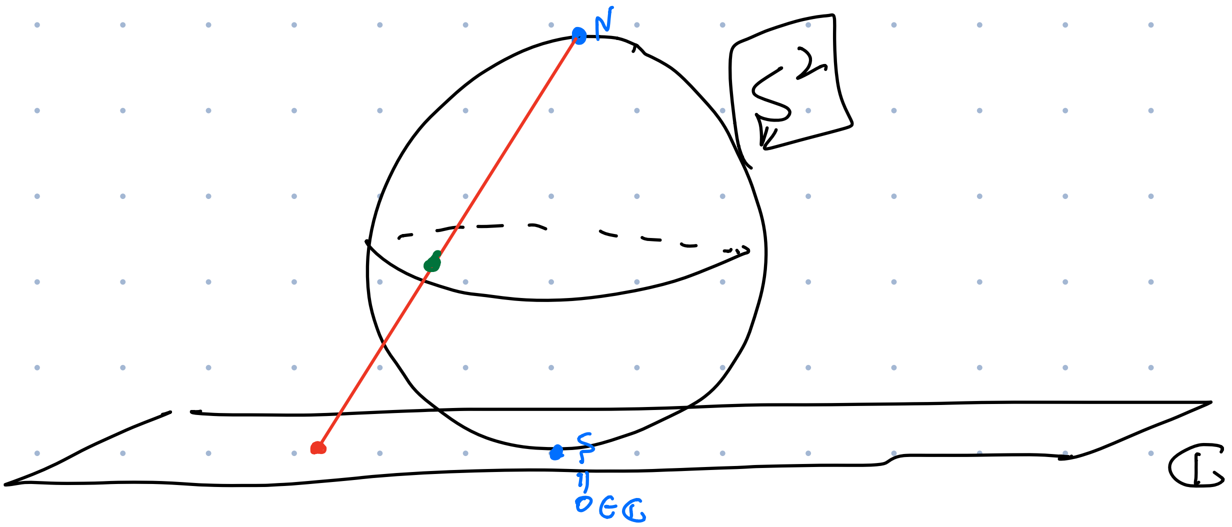
<u>Def</u> $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \simeq S^2$

<u>Def</u>: $f : \Omega \to \hat{\mathbb{C}}$ is <u>meromorphic</u> if

f is holo. on Ω except at $\{z_1, z_2, \cdots\} = S$

where • S is isolated (no accumulating pt.)

• each point in S is a pole of f.



1-1 map between $\mathbb{C}$ & $S^2 \setminus \{N\}$

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong S^2$$