**Today:** Symmetric matrices, quadratic forms.
(end of material of $2^{nd}$ midterm). §5 - §7

**Thursday:** Preview of differential eq$^n$s; other topics on quad. forms.

**Next Tue:** Q & A for $2^{nd}$ midterm. + election day.

**Next Thu:** $2^{nd}$ midterm.

open: 11/4 (Wed.) 1pm PST.

close: 11/5 (Thu.) 1pm PST.

(OH: 11/~~6 (Fri.)~~ 12–1:30 pm → 11/3 (Tue.) 12–1:30 pm)

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**Symmetric matrices:** $A = A^T$

$\downarrow$

square, $a_{ij} = a_{ji}$

**Prop** $A^T = A \Rightarrow$ any 2 eigenvectors with distinct eigenvalues are orthogonal.

**pf:** $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$,

where $\vec{v}_1, \vec{v}_2 \neq \vec{0}$, $\lambda_1 \neq \lambda_2$.

$$\underbrace{\langle \underbrace{A\vec{v}_1}_{=\lambda_1 \vec{v}_1}, \vec{v}_2 \rangle}_{=\lambda_1 \langle \vec{v}_1, \vec{v}_2 \rangle} \underset{\text{(HW)}}{=} \langle \vec{v}_1, A^T \vec{v}_2 \rangle = \underbrace{\langle \vec{v}_1, \underbrace{A\vec{v}_2}_{=\lambda_2 \vec{v}_2} \rangle}_{=\lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle}$$

$$\Rightarrow \langle \vec{v}_1, \vec{v}_2 \rangle = 0 \qquad \square$$

**Def** Say a matrix $A$ is <u>orthogonally diagonalizable</u> if $\exists$ orthonormal eigenbasis of $A$. $\{\vec{v}_1, \dots, \vec{v}_n\}$
$A\vec{v}_i = \lambda_i \vec{v}_i$

i.e. $\exists$ orthogonal matrix $P$ and a diagonal matrix $D$

s.t. $A = PDP^T$. $\quad (P^T = P^{-1})$

$P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ $\quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

**Thm** $A$ is <u>orthogonally diagonalizable</u> $\iff$ $A^T = A$

(pf of "$\Rightarrow$"): $A = PDP^T$

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T$$
$$= P D^T P^T$$
$$= PDP^T = A.$$

(pf of "$\Leftarrow$").

① If $A^T = A$ and $A$ is diagonalizable, then $A$ is orthogonally diagonalizable.

② If $A^T = A$, then $A$ is diagonalizable.

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**pf of ①**: $\{\lambda_1, \dots, \lambda_k\}$ distinct eigenvalues of $A$. $\quad$ <u>$n \times n$</u>

$\mathrm{Nul}(A - \lambda_1 I), \dots, \mathrm{Nul}(A - \lambda_k I)$

$A$ diagonalizable $\sum \dim \mathrm{Nul}(A - \lambda_i I) = n$,

$$\mathbb{C}^n = \mathrm{Nul}(A-\lambda_1 I) \oplus \cdots \oplus \mathrm{Nul}(A-\lambda_k I)$$

If we pick a orthogonal basis $\{v_i^{(1)}, \dots, v_i^{(a_i)}\}$ of $\mathrm{Nul}(A-\lambda_i I)$ for each $i$.

then $\{v_1^{(1)}, \dots, v_1^{(a_1)}, v_2^{(1)}, \dots, v_2^{(a_2)}, \dots, v_k^{(1)}, \dots, v_k^{(a_k)}\}$ is a basis of $\mathbb{C}^n$.

By Prop., this is an orthogonal set

→ normalize each vector → orthonormal set.

eigenbasis. $\square$

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② "$A^T = A \implies$ diagonalizable"

Induction on the size of $A$.

$1\times 1$: ok.

Assume that any $(n-1)\times(n-1)$ sym. matrix is diag.

$\boxed{A: n\times n}$, $A^T = A$. $\overset{?}{\implies}$ diagonalizable.

Let $\vec{v}$ be an eigenvector of $A$, $\|\vec{v}\| = 1$

$\Rightarrow A\vec{v} = \lambda\vec{v}$, Choose $\{\vec{v}_2, \dots, \vec{v}_n\}$ s.t.

$\underbrace{\{\vec{v}, \vec{v}_2, \dots, \vec{v}_n\}}$ forms an orthonormal basis of $\mathbb{C}^n$.

$$\underline{Q} = \begin{bmatrix} | & | & & | \\ \vec{v} & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \quad \text{orthogonal}, \quad Q^T = Q^{-1}$$

$A\vec{v} = \lambda \vec{v}$.

$$AQ = \begin{bmatrix} \lambda\vec{v} & A\vec{v}_2 & \cdots & A\vec{v}_n \end{bmatrix}$$

$$\underline{Q^T A Q} = \begin{bmatrix} \vec{v} \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda\vec{v} & A\vec{v}_2 & \cdots & A\vec{v}_n \end{bmatrix}$$

$\parallel$

$$\left[\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array}\right] = \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & \\ \vdots & A' \\ 0 & \end{array}\right]$$

symmetric $(n-1)\times(n-1)$

$\Rightarrow$ $* = 0 \cdots 0$

$A' = (A')^T$

$$(Q^T A Q)^T = \left[\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline & & & \\ * & & (A')^T & \\ & & & \end{array}\right]$$

$\parallel$ $A^T = A$

$Q^T A Q$

By inductive hypothesis, $A'$ is diagonalisable,

By ①, $A'$ orthogonally diagonalisable.

$\exists\, P$ orthogonal & $D$ diag. s.t. $A' = PDP^T$.

$$Q^T A Q = \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & PDP^T \end{array}\right]$$

$$= \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array}\right]}_{R :=\ \text{orthogonal}} \cdot \underbrace{\left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & D \end{array}\right]}_{\text{diagonal}} \cdot \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^T \end{array}\right]$$

$$= R \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & D \end{array}\right] R^T$$

$$\Rightarrow A = \underbrace{Q}\,\underbrace{R} \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & D \end{array}\right] (QR)^T . \quad \square$$

orthogonal $\xRightarrow{\text{HW}}$ $QR$ orthogonal.

Rmk $A^T = A \Rightarrow$ all the eigenvalues of $A$ are real, (HW5) real & have real eigenvectors

$A\vec{x} = \lambda \vec{x}$, $\vec{x} \in \mathbb{C}^n$, $\vec{x} \neq \vec{0}$, $\lambda \in \mathbb{C}$

WTS: $\lambda \in \mathbb{R}$.

$\bar{x}^T A x \in \mathbb{C}$ ($\in \mathbb{R}$) actually is a real number.

b/c:

$$\overline{\bar{x}^T A x} = x^T \bar{A} \bar{x} = x^T A \bar{x} = \bar{x}^T A^T x = \bar{x}^T A x.$$

$$\mathbb{R}? \quad \bar{x}^T A x = \bar{x}^T \lambda x = \lambda \underbrace{\bar{x}^T x}_{\neq 0}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

$$\bar{x}^T x = \begin{bmatrix} \bar{x}_1 & \cdots & \bar{x}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= |x_1|^2 + \cdots + |x_n|^2 \in \mathbb{R}$$

Quadratic forms: homogeneous deg. 2 poly. in $x_1, \dots, x_n$ variables

$$Q(x_1, \dots, x_n) = \sum_i b_{ii} x_i^2 + \sum_{i<j} 2b_{ij} x_i x_j$$

$$\updownarrow$$

$$Q_A(\vec{x}) := \vec{x}^T A \vec{x} = [x_1\ x_2 \cdots x_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & \vdots \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \sum_{i=1}^n a_{ii} x_i^2 + \sum_{1 \le i < j \le n} 2a_{ij} x_i x_j$$

Symmetric matrix $A$ $n \times n$

e.g. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $Q_A(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$.

$$= [x_1\ x_2] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1\ x_2] \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = x_1^2 + x_1x_2 + x_1x_2 + x_2^2$$

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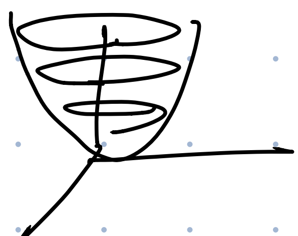
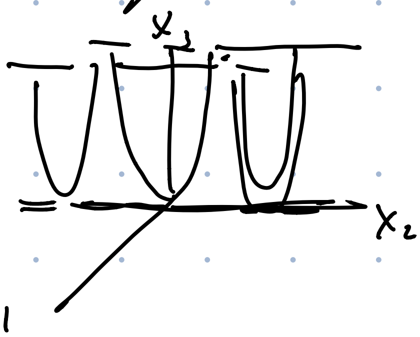
e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q_A(\vec{x}) = x_1^2 + x_2^2$

↑ positive definite



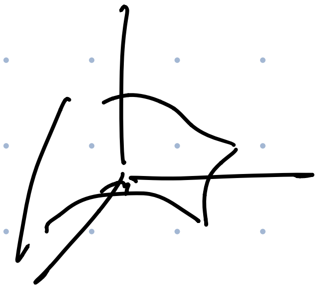
$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $Q_A(\vec{x}) = x_1^2$

↑ positive semidefinite



$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Q_A(\vec{x}) = x_1^2 - x_2^2$$

Indefinite.



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$$Q_A(\vec{x}) = \vec{x}^T \underbrace{A}_{PDP^T} \vec{x} = \vec{x}^T P D \underbrace{P^T \vec{x}}_{\vec{y}} = \vec{y}^T D \vec{y}$$

$$\vec{y}^T D \vec{y} = [y_1 \cdots y_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[d, "P^T"'] \arrow[r, "Q_A = \vec{x}^T A \vec{x}"] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, "\vec{y}^T D \vec{y}"'] & \end{tikzcd}$$

i.e. we can use $P^T$ to do a "change of variables" to repn the quadratic form by a diagonal matrix.

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e.g. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = PDP^T$

$\text{Nul}(A - 0I) = \text{Span}\left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$ $\qquad P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$\text{Nul}(A - 2I) = \text{Span}\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$ $\qquad P^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$$\vec{y} = P^T \vec{x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x_1 - x_2}{\sqrt{2}} \\ \frac{x_1 + x_2}{\sqrt{2}} \end{bmatrix}$$

$$Q_A(\vec{x}) = \vec{y}^T D \vec{y} = \vec{y}^T \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \vec{y}$$

$$= 2 y_2^2 = 2 \cdot \frac{1}{2} (x_1 + x_2)^2 = \underline{(x_1 + x_2)^2}$$

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<u>Def</u> A symmetric matrix, $Q_A(\vec{x}) = \vec{x}^T A \vec{x}$.

- Say $A$ or $Q_A$ is <u>positive definite</u> if $Q_A(\vec{x}) > 0 \quad \forall \vec{x} \neq \vec{0}$

  <u>positive semidefinite</u> if $Q_A(\vec{x}) \geq 0 \quad \forall \vec{x} \neq \vec{0}$.
- <u>negative definite</u>, <u>negative semidefinite</u>
- <u>indefinite</u> if it's not any of the above

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<u>Thm.</u> A symmetric,

- It's positive def. $\iff$ all <u>eigenvalues</u> $> 0$.
- It's positive semidef $\iff$ —— $\geq 0$

Similarly for negative (semi)def.

$$x^T A x \qquad x = \underbrace{v_1 + \cdots + v_n}$$

$$\|$$

$$\underbrace{(v_1^T + \cdots + v_n^T)\, A\, (v_1 + \cdots + v_n)}$$

$$\|$$

$$\sum_{i,j} v_i^T \underbrace{A v_j}_{\lambda_j v_j} = \sum_{i,j} \lambda_j v_i^T v_j$$

$$= \sum_i \lambda_i v_i^T v_i$$