Today: Convergence of sequence of functions.

Weakest convergence: pointwise convergence.

Def: X: a set

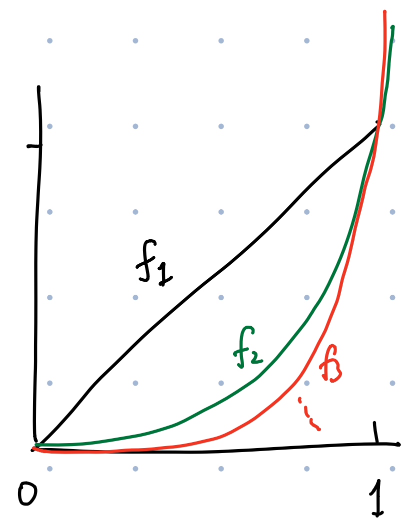
(f_n) sequence of functions $f_n : X \longrightarrow \mathbb{R}$.

Say $(f_n) \longrightarrow f$ converges pointwisely to a fun $f : X \longrightarrow \mathbb{R}$

If $\forall x_0 \in X$, we have $\lim_{n \to \infty} f_n(x_0) = f(x_0)$.

e.g.: $f_n : [0,1] \longrightarrow \mathbb{R}$

$x \mapsto x^n$.

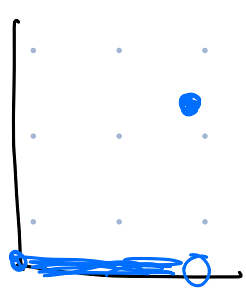


Q: Does (f_n) conv. to some fun $f : [0,1] \longrightarrow \mathbb{R}$?

- For $0 \le x < 1$, $\lim x^n = 0$.
- For $x = 1$, $\lim x^n = 1$

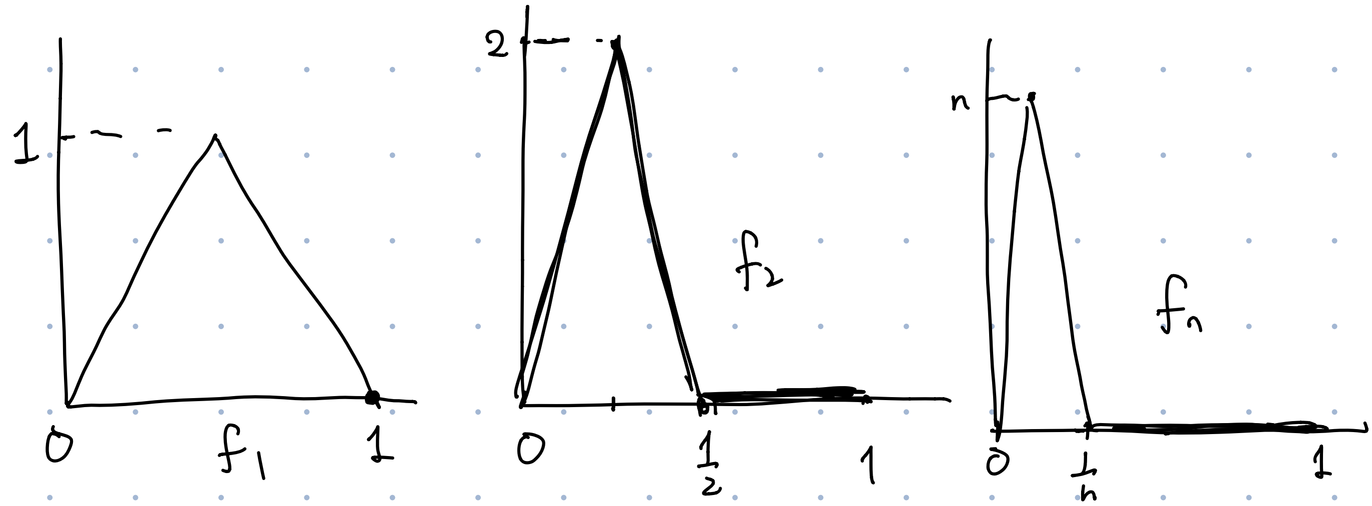
So, the seq. $(f_n = x^n)$ on $[0,1]$ has a pointwise limit f:

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$



Rmk: the pointwise limit of a seq. of conti. funs may not be conti.

e.g.



Q: Does (f_n) have a pointwise limit $f: [0,1] \to \mathbb{R}$??

- For $0 < x \leq 1$, $\exists N > 0$ s.t. $\frac{1}{N} < x$.

  Then $f_n(x) = 0 \quad \forall n > N$

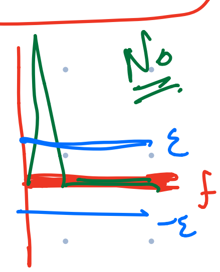
  $\Rightarrow \quad \lim f_n(x) = 0 \quad \forall\, 0 < x \leq 1.$

- For $x = 0$, $\quad f_n(0) = 0 \qquad \lim f_n(0) = 0$

$\Longrightarrow$ (f_n) has a pointwise limit $f: [0,1] \longrightarrow \mathbb{R}$
$x \longmapsto 0.$

Q: Is $(f_n) \to f$ uniform??

No



Rmk: In this example, we have a seq. of conti. fns on $[0,1]$, which converges pointwise to another conti. fn. on $[0,1]$,

But $$\underbrace{\int_0^1 f_n(x)\,dx}_{= \frac{1}{2}} \not\longrightarrow \underbrace{\int_0^1 f(x)\,dx}_{= 0}$$

(Need a stronger notion of conv.)

Recall: $(f_n) \to f$ pointwise $\iff \forall x_0 \in X,\ \boxed{\lim f_n(x_0) = f(x_0)}$
$X \to \mathbb{R}$

$\Updownarrow$

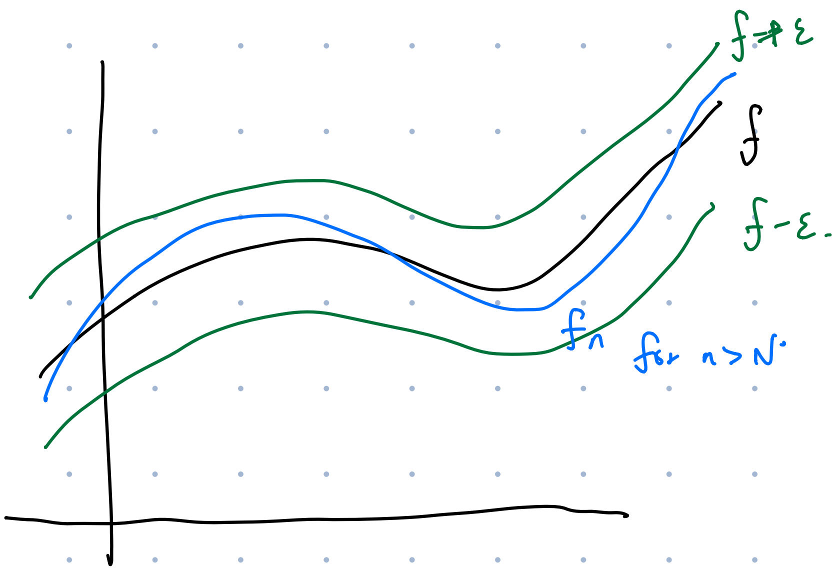
$\forall x_0 \in X,\ \boxed{\forall \varepsilon > 0,\ \exists N > 0 \text{ s.t. } |f_n(x_0) - f(x_0)| < \varepsilon \quad \forall n > N}$

depends on ε, x_0

Def: (unif. convergence). Say $(f_n) \to f$ uniformly if

$\forall \varepsilon > 0,\ \exists N > 0$

s.t. $|f_n(x_0) - f(x_0)| < \varepsilon \quad \forall n > N,\ \forall x_0 \in X.$



$\boxed{\varepsilon = \quad} > 0$

① $\boxed{x_0 = \frac{1}{2}}$

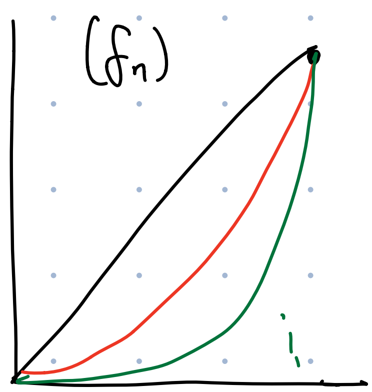
$\lim \boxed{\left(\frac{1}{2}\right)^n} = 0$

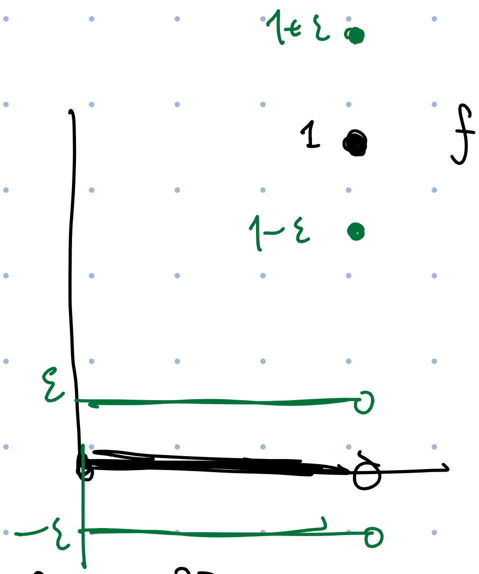
$N = \log_2\left(\frac{1}{\varepsilon}\right)$

② $x_0 = \frac{2}{3}$

When x_0 gets closer to 1, we need larger and larger N s.t. $x_0^N < \varepsilon$.

e.g. $\boxed{f_n(x) = x^n}$





Q: Does $(f_n) \to f$ uniformly??

Claim: $(f_n) \to f$ is not uniform convergence

pf: Pick $\varepsilon = \frac{1}{3}$.

Need to show: $\forall N > 0$,

we can find $n \geq N$ and $x_0 \in [0,1]$

s.t. $|f_n(x_0) - f(x_0)| \geq \frac{1}{3}$.

We actually have: $\forall n > 0$,

$\exists\ x_0 \in (0,1)$ s.t. $f_n(x_0) = \frac{1}{2}$. (by Intermediate value thm)

$\Rightarrow |f_n(x_0) - f(x_0)| > \frac{1}{3}$. $\square$

Thm (X,d) metric space, $f_n : X \to \mathbb{R}$ seq. of conti. fcns.

Suppose $(f_n) \to f$ uniformly convergence

Then f is also continuous.

Idea:

~~Pf~~ Need: $\forall x_0 \in X$, $\forall \varepsilon > 0$, $\boxed{\exists \delta > 0}$

s.t. if $d(x, x_0) < \delta$, then $\boxed{|f(x) - f(x_0)|} < \varepsilon$.

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$

continuity of f_n.

Control these 2 terms by $(f_n) \to f$ unif.

Proof: $\forall x_0 \in X$, $\forall \varepsilon > 0$,

- Since $(f_n) \to f$ unif, $\exists N > 0$ s.t.

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall n > N, \ \forall x \in X.$$

- Pick any $n > N$,

  Since f_n is conti., $\exists \underline{\underline{\delta > 0}}$

  s.t. $d(x, x_0) < \delta \implies |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$.

So, $\forall x \in X$ s.t. $d(x, x_0) < \delta$, we have:

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$\square$

Thm: $f_n : [a, b] \to \mathbb{R}$ conti.

Suppose $(f_n) \to f$ unif. conv.

Then $$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

pf: $$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| = \left| \int_a^b (f_n(x) - f(x))\,dx \right|$$

$$\leq \int_a^b |f_n(x) - f(x)|\,dx \overset{\forall n > N}{\leq} \int_a^b \frac{\varepsilon}{b-a}\,dx = \varepsilon. \quad \square$$

$(f_n) \to f$ unif: $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$ $\forall n > N$, $\forall x \in [a, b]$

Alternative description of unif. conv:

X: a set.

Define a metric space $\mathcal{B}(X)$ as follows:

- $\mathcal{B}(X) = \{ \text{bounded functions } f: X \to \mathbb{R} \}$.
- $d_{\mathcal{B}(X)}(f_1, f_2) := \sup_{x \in X} |f_1(x) - f_2(x)|$

(In HW, you'll show that it's a metric space).

Prop: (f_n) seq. of bdd fcns on X. $f_n: X \to \mathbb{R}$

Then $(f_n) \to f$ unif. $\iff$ (f_n) converges to f in $\mathcal{B}(X)$

Pf: ① the uniform limit f of bounded fcns f_n is also bounded, so $f \in \mathcal{B}(X)$. (HW)

② $f_n \to f$ conv. in $\mathcal{B}(X)$

$\iff$ $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $\underbrace{d_{\mathcal{B}(X)}(f_n, f)}_{\sup_{x \in X} |f_n(x) - f(x)|} < \varepsilon \quad \forall n > N$.

$\iff$ $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $|f_n(x) - f(x)| < \varepsilon \quad \forall n > N, x \in X$.

$\iff$ $f_n \to f$ uniformly $\square$