Math 104 (Section 4) Final Exam December 18, 2019

Name: Solution

- You have 170 minutes to complete the exam (3:10pm – 6:00pm).
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using. Exception: if you are asked to prove a theorem, you are not allowed to use it!
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	8	
2	8	
3	14	
4	14	
5	14	
6	14	
7	14	
8	14	
Total	100	

1. True/False questions. You don't need to justify your answers. No partial credit.

(a) (2 points) For any bounded sequence (a_n), $\limsup a_n$ always exists and is finite.

$\boxed{\text{True}}$

(b) (2 points) If a sequence (a_n) satisfies $\lim a_n = 0$ then the series $\sum a_n$ converges.

$\boxed{\text{False}}$ e.g. $a_n = \frac{1}{n}$

(c) (2 points) Every continuous function on $(0, 1)$ is bounded.

$\boxed{\text{False}}$ e.g. $f(x) = \frac{1}{x}$ on $(0,1)$

(d) (2 points) The function $f(x) = x^2$ is uniformly continuous on $\mathbb{R}$.

$\boxed{\text{False}}$ Take $\varepsilon = 1$,

Claim: $\forall \delta > 0$, $\exists x, y \in \mathbb{R}$ s.t. $|x-y| < \delta$ but $|x^2 - y^2| > 1$.

Let $x = \frac{1}{\delta} + \frac{\delta}{2}$ and $y = \frac{1}{\delta}$.

Then $|x-y| = \frac{\delta}{2} < \delta$,

and $|x^2 - y^2| = |x-y||x+y| = \frac{\delta}{2}\left(\frac{2}{\delta} + \frac{\delta}{2}\right) = 1 + \frac{\delta^2}{4} > 1$. $\square$

2. Find the following quantities. You don't need to justify your answers.

(a) (4 points) The radius of convergence of the power series

$$\sum_{n=0}^{\infty} 2^{-n} x^{2n}.$$

The power series can be written as: $\sum_{n=0}^{\infty} 2^{-n} x^{2n} = \sum_{n=0}^{\infty} a_n x^n$,

where $a_n = \begin{cases} 2^{-n/2}, & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd}. \end{cases}$

$$\limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(2^{-n/2}\right)^{1/n} = 2^{-1/2}.$$

$\Rightarrow$ Radius of convergence $= \left(2^{-1/2}\right)^{-1} = \boxed{\sqrt{2}}$ $\square$

(b) (4 points) The upper integral $U(f)$ for the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \cap [0,1]; \\ 0, & x \notin \mathbb{Q} \cap [0,1]. \end{cases}$$

For any partition P of $[0,1]$, $P = \{t_0 = 0 < t_1 < \cdots < t_n = 1\}$.

$U(f,P) = \sum_{k=1}^{n} (t_k - t_{k-1}) \sup_{x\in[t_{k-1},t_k]} f(x)$. Observe that $\sup_{x\in[t_{k-1},t_k]} f(x) = t_k^2$.

$$= \sum_{k=1}^{n} (t_k - t_{k-1}) \sup_{x\in[t_{k-1},t_k]} x^2$$

$$= U(x^2, P).$$

$\Rightarrow$ $U(f) = \inf_P U(f,P) = \inf_P U(x^2, P) = U(x^2) = \int_0^1 x^2\,dx = \boxed{\frac{1}{3}}$

Note: Using similar argument, one can show that $L(f) = 0$.

Hence f is NOT integrable.

3. (a) (2 points; no partial credit) Let (a_n) be a sequence of real numbers, and a be a real number. Write down the precise definition of "(a_n) converges to a".

$\forall \varepsilon > 0, \exists N > 0$

s.t. $n > N \Rightarrow |a_n - a| < \varepsilon$.

(b) (12 points) Prove that if (a_n) is a bounded increasing sequence then it converges. (You are not allowed to use any theorem for this problem.)
Hint: You will need to use the completeness axiom of $\mathbb{R}$.

(a_n) bounded $\Rightarrow$ $\sup\{a_n : n \in \mathbb{N}\}$ exists and is finite by completeness axiom of $\mathbb{R}$

Let $z = \sup\{a_n : n \in \mathbb{N}\}$.

We'll show that $\lim_{n\to\infty} a_n = z$.

- $\forall \varepsilon > 0$, Consider $z - \varepsilon$.
  Since $z-\varepsilon$ is not an upper bound of $\{a_n : n \in \mathbb{N}\}$,
  So there exists some $N \in \mathbb{N}$ s.t. $a_N > z - \varepsilon$.

  $\Rightarrow z - \varepsilon < a_N \leq z$.

- Since (a_n) is increasing, we have $z - \varepsilon < a_N \leq a_n \leq z$ $\forall n \geq N$.

$\Rightarrow |a_n - z| < \varepsilon$ $\forall n > N$. $\square$

4. (a) (2 points; no partial credit) Let $f\colon [0,1] \to \mathbb{R}$ be a real-valued function on $[0,1]$. Write down the precise definition of "f is continuous on $[0,1]$".

$\forall x_0 \in [0,1].\ \forall \varepsilon > 0,\ \exists \delta > 0$

s.t. $\begin{matrix} |x - x_0| < \delta \\ x \in [0,1] \end{matrix} \Rightarrow |f(x) - f(x_0)| < \varepsilon.$

(b) (12 points) Let f be a continuous real-valued function on $[0,1]$. Prove that f is a bounded function, i.e. there exists $M > 0$ such that $|f(x)| < M$ for any $x \in [0,1]$.
Hint: You will need to (and are only allowed to) use the Bolzano–Weierstrass theorem for this problem.
Hint: Try to prove by contradiction.

Assume by contradiction that f is unbounded.

Then $\forall n \in \mathbb{N}$, $\exists x_n \in [0,1]$ s.t. $|f(x_n)| > n$

By Bolzano-Weierstrass, the sequence $\{x_n\} \subset [0,1]$ has a convergent subsequence: $\{x_{k_n}\}$. Say $\lim_{n\to\infty} x_{k_n} = y \in [0,1]$.

Since f is continuous on $[0,1]$, we have $\lim_{n\to\infty} f(x_{k_n}) = f(y)$.

But the sequence $\{f(x_{k_n})\}$ doesn't converge since $|f(x_{k_n})| > k_n$ $\forall n$.

Contradiction. $\square$

5. (a) (2 points; no partial credit) Let (f_n) be a sequence of real-valued functions $[0,2]$, and f be a real-valued function on $[0,2]$. Write down the precise definition of "(f_n) converges uniformly to f on $[0,2]$".

$\forall \varepsilon > 0, \ \exists N > 0$

s.t. $|f_n(x) - f(x)| < \varepsilon \quad \forall n > N$
$\forall x \in [0,2]$.

(b) (12 points) Let (f_n) be a sequence of real-valued functions $[0,2]$, and f be a real-valued function on $[0,2]$. Suppose that (f_n) converges uniformly to f on $[0,2]$. Moreover, assume that each f_n is continuous at 1. Prove that f is continuous at 1. (You are not allowed to use any theorem for this problem.)

Hint: $|f(x) - f(1)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(1)| + |f_n(1) - f(1)|$.

$\forall \varepsilon > 0, \ \exists N > 0$

s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall n > N, \ x \in [0,2]$.

Let $n = N+1$.

Since f_n is continuous at 1, $\exists \delta > 0$ s.t.

$$\begin{matrix} |x-1| < \delta \\ x \in [0,2] \end{matrix} \Rightarrow |f_n(x) - f_n(1)| < \frac{\varepsilon}{3}.$$

Hence we have

$$|f(x) - f(1)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(1)| + |f_n(1) - f(1)|$$

$$< \varepsilon$$

for any $|x-1| < \delta$
$x \in [0,2]$.

Hence f is continuous at 1. $\square$

6. (a) (2 points; no partial credit) Let $f: \mathbb{R} \to \mathbb{R}$ be a real-valued function on $\mathbb{R}$. Write down the precise definition of "f is differentiable on $\mathbb{R}$".

$\forall x \in \mathbb{R}$. $\quad \lim_{y \to x} \frac{f(y)-f(x)}{y-x}$ exists and is finite.

(b) (12 points) Suppose f is a differentiable function on $\mathbb{R}$ such that f' is a bounded function on $\mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.
Hint: You will need to (and are only allowed to) use the Mean Value Theorem for this problem.

Suppose that $|f'(x)| < M \quad \forall x \in \mathbb{R}$.

$\forall \varepsilon > 0$, let $\delta = \frac{\varepsilon}{M}$.

Then for any $|x-y| < \delta = \frac{\varepsilon}{M}$, we have

$$|f(x)-f(y)| = |f'(z)||x-y| < M \cdot \frac{\varepsilon}{M} = \varepsilon. \quad \square$$

for some z in between x and y.
(mean value theorem)

7. (a) (2 points) Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 2^n}$$

converges.

$\left\{ \frac{1}{n \cdot 2^n} \right\}$ is a decreasing sequence that converges to 0.

By the alternating series thm, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 2^{n-1}}$ converges.

(b) (5 points) Define the function $f\colon (-1, \infty) \to \mathbb{R}$ by

$$f(x) = \log_e(1+x).$$

Find the Taylor series for $f(x)$ at the point 0. Also, find the exact interval of convergence of the Taylor series.

(Part (c) is on the next page.)

Same as the practice exam problem.

Get: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$.

$\limsup_{n\to\infty} \left| \frac{(-1)^{n-1}}{n} \right|^{1/n} = \limsup_{n\to\infty} \left| \frac{1}{n} \right|^{1/n} = 1.$

$\Rightarrow$ radius of convergence $= \frac{1}{1} = 1$.

- At $x = 1$, : $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges.

- At $x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-1)^n = \sum_{n=1}^{\infty} \frac{-1}{n}$ diverges

$\Rightarrow$ exact interval of convergence $= (-1, 1]$.

(c) (7 points) Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 2^n} = \log_e 3 - \log_e 2.$$

Hint: Taylor's theorem.

Let $R_n(\frac{1}{2}) := f(\frac{1}{2}) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} (\frac{1}{2})^k$.

By Taylor's thm, $\exists\, y_n \in (0, \frac{1}{2})$ s.t.

$$R_n(\tfrac{1}{2}) = \frac{f^{(n)}(y_n)}{n!} (\tfrac{1}{2})^n = \frac{(-1)^{n-1}}{n(1+y_n)^n} (\tfrac{1}{2})^n.$$

$$\Rightarrow |R_n(\tfrac{1}{2})| = \frac{1}{n \cdot 2^n \cdot (1+y_n)^n} < \frac{1}{n}.$$

$$\Rightarrow \lim_{n\to\infty} R_n(\tfrac{1}{2}) = 0.$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (\tfrac{1}{2})^k = f(\tfrac{1}{2})$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (\tfrac{1}{2})^k = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\tfrac{1}{2})^n$$

$$f(\tfrac{1}{2}) = \log(1+\tfrac{1}{2}) = \log 3 - \log 2. \quad \square$$

8. (a) (10 points) Let f and g be continuous functions on $[0,1]$. Assume that

$$\int_0^1 f = \int_0^1 g.$$

Prove that there exists $x \in [0,1]$ such that $f(x) - g(x) = 0$.
Hint: Prove by contradiction, and use Intermediate Value Theorem.

Suppose that $f(x) - g(x) \neq 0 \quad \forall x \in [0,1]$.

Then either $f(x) - g(x) > 0 \quad \forall x \in [0,1]$

or $f(x) - g(x) < 0 \quad \forall x \in [0,1]$

is true, by Intermediate value theorem.

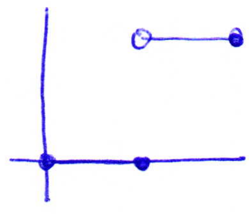
If $f(x) - g(x) > 0 \ \forall x \in [0,1]$, then $\int_0^1 f > \int_0^1 g$. Contradiction.

If $f(x) - g(x) < 0 \ \forall x \in [0,1]$, then $\int_0^1 f < \int_0^1 g$, contradiction.

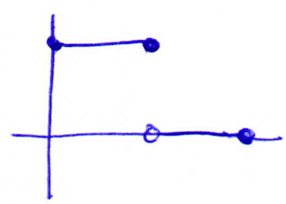
$\square$

(b) (4 points) Construct an example of functions f and g that are both integrable on $[0,1]$ such that $\int_0^1 f = \int_0^1 g$, but $f(x) \neq g(x)$ for any $x \in [0,1]$.

e.g. $f(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ 1 & \frac{1}{2} < x \le 1 \end{cases}$



$g(x) = \begin{cases} 1. & 0 \le x \le \frac{1}{2} \\ 0. & \frac{1}{2} < x \le 1. \end{cases}$



$\int_0^1 f = \int_0^1 g = \frac{1}{2}.$