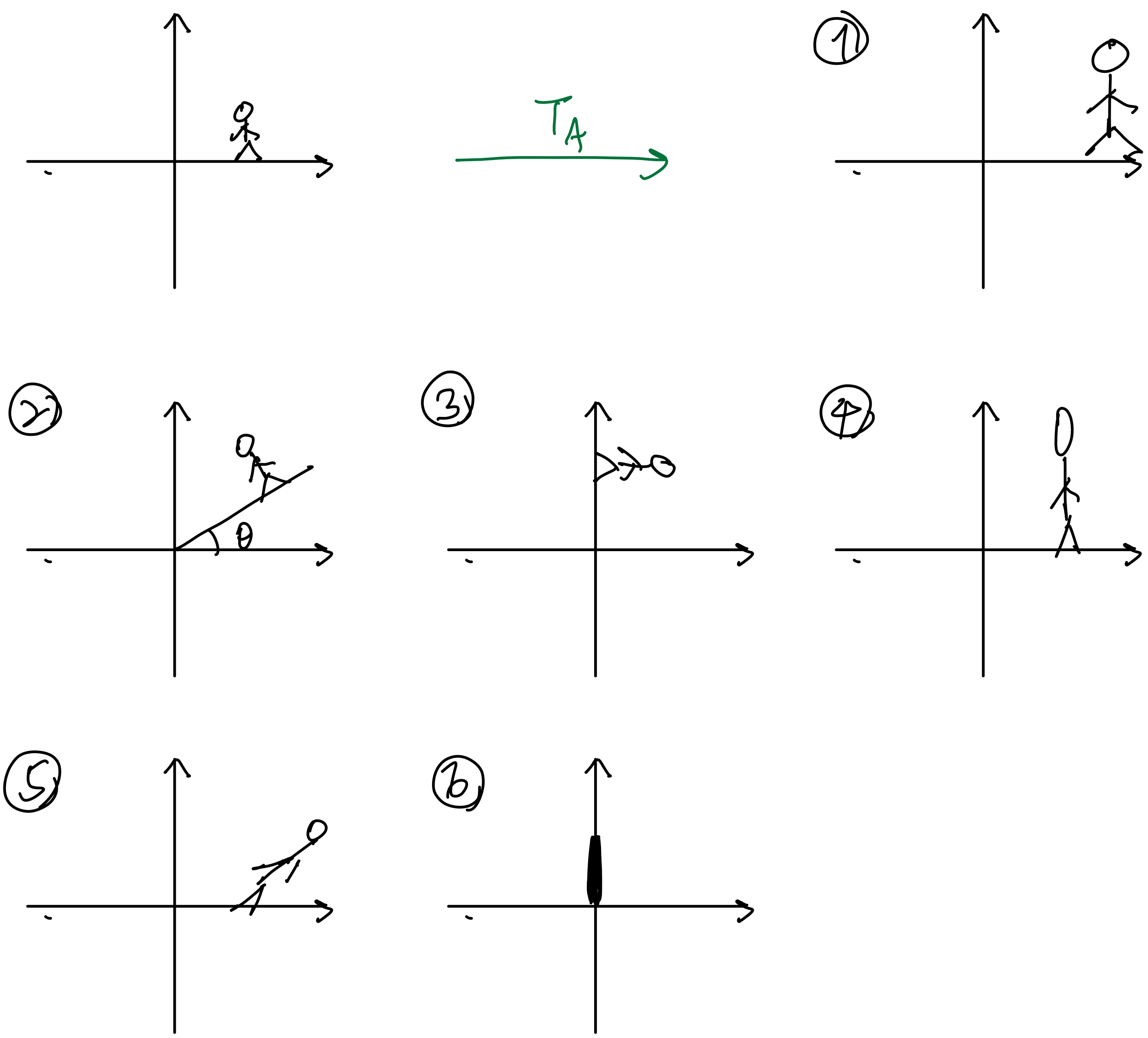
Find a 2×2 matrix A s.t. T_A looks like:



Last time:

- $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}_{m \times n} \rightsquigarrow T_A : \mathbb{R}^n \to \mathbb{R}^m$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto A\vec{x}$$

$$\| $$

$$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

- T_A surjective $\Leftrightarrow$ A has pivots in each row.

$\Leftrightarrow$ $[A \mid \vec{b}]$ has sol'n $\forall \vec{b} \in \mathbb{R}^m$.

$\Leftrightarrow$ $\text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} = \mathbb{R}^m$

Q: When is T_A injective?

i.e. $\forall\ \vec{x}_1 \neq \vec{x}_2$ in $\mathbb{R}^n$, we have $A\vec{x}_1 \neq A\vec{x}_2$ in $\mathbb{R}^m$.

Idea: Suppose T_A is NOT injective.

Then $\exists\ \vec{x}_1 \neq \vec{x}_2$ in $\mathbb{R}^n$, s.t. $A\vec{x}_1 = A\vec{x}_2$.

$$\Rightarrow \quad \vec{0} = A\vec{x}_1 - A\vec{x}_2 = A(\vec{x}_1 - \vec{x}_2)$$

$$\Rightarrow \quad \exists\ \vec{y} := \underbrace{\vec{x}_1 - \vec{x}_2}_{\neq \vec{0}} \text{ in } \mathbb{R}^n \quad \text{s.t.} \quad A\vec{y} = \vec{0}$$

i.e. the linear system $A\vec{x} = \vec{0}$ has a nontrivial sol^{n}.

$([A \mid \vec{0}])$ $\qquad x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$

i.e. $\exists\ \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \neq \vec{0}$ s.t. $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{0}$

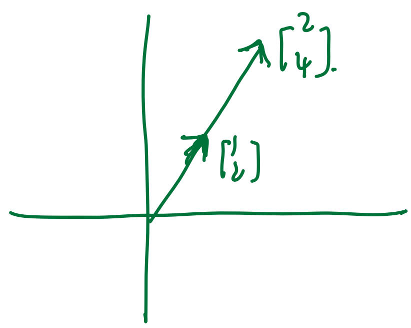
($x_1, \cdots, x_n$ not all 0)

Def: $\{\vec{v}_1, \cdots, \vec{v}_k\}$ vectors in $\mathbb{R}^n$. say this set of vectors is linearly dependent (l.d.) if $\exists\ c_1, \cdots, c_k \in \mathbb{R}$ not all 0, s.t. $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$.

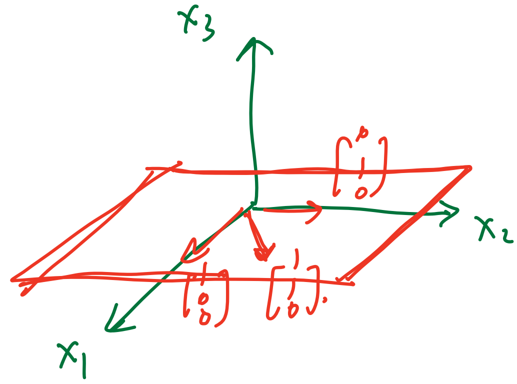
Otherwise, it's called linearly independent. (l.i.)

e.g. $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ in $\mathbb{R}^2$

l.d. $\begin{bmatrix} 2 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.



e.g. $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\}$ in $\mathbb{R}^3$ $\quad \begin{bmatrix}1\\1\\0\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$



l.d. $\quad 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} - 1\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Thm $\{\vec{v}_1, \dots, \vec{v}_k\}$ is l.d.

$\Leftrightarrow \exists\ 1 \le i \le k$ s.t. $\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$.

pf: $(\Leftarrow)$ $\vec{v}_i = a_1\vec{v}_1 + \dots + a_{i-1}\vec{v}_{i-1} + a_{i+1}\vec{v}_{i+1} + \dots + a_k\vec{v}_k$.

for some $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k \in \mathbb{R}$

$\Rightarrow a_1\vec{v}_1 + \dots + a_{i-1}\vec{v}_{i-1} - \vec{v}_i + a_{i+1}\vec{v}_{i+1} + \dots + a_k\vec{v}_k = \vec{0}$

$\Rightarrow \{\vec{v}_1, \dots, \vec{v}_k\}$ is l.d. $\square$

$(\Rightarrow)$ $\exists\ a_1, \dots, a_k \in \mathbb{R}$ not all 0, s.t.

$$a_1\vec{v}_1 + \dots + a_k\vec{v}_k = \vec{0}.$$

Say $a_i \neq 0$.

$$-a_i\vec{v}_i = a_1\vec{v}_1 + \dots + a_{i-1}\vec{v}_{i-1} + a_{i+1}\vec{v}_{i+1} + \dots + a_k\vec{v}_k.$$

$$\Rightarrow \vec{v}_i = \frac{-a_1}{a_i}\vec{v}_1 + \dots + \frac{-a_{i-1}}{a_i}\vec{v}_{i-1} + \frac{-a_{i+1}}{a_i}\vec{v}_{i+1} + \dots + \frac{-a_k}{a_i}\vec{v}_k.$$

$$\in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}. \quad \square$$

Ex: $\vec{v}_1 \in \text{Span}\{\vec{v}_2, \dots, \vec{v}_k\}$.

Then $\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_2, \dots, \vec{v}_k\}$

Thm $A: m\times n$, The following are equivalent:

1) $T_A: \mathbb{R}^n \to \mathbb{R}^m$ is injective
$\vec{x} \mapsto A\vec{x}$

2) $A\vec{x} = \vec{b}$ has at most 1 sol$^{\underline{n}}$ $\forall \vec{b} \in \mathbb{R}^m$.

3) $\underbrace{A\vec{x}}_{= x_1\vec{a}_1 + \cdots + x_n\vec{a}_n} = \vec{0}$ has no nontrivial sol$^{\underline{n}}$ (i.e. the only sol$^{\underline{n}}$ is $\vec{x} = \vec{0}$)

4) The columns of A are l.i.

5) A has pivots in each column.

Pf: • 1) $\Leftrightarrow$ 2): by definition of injectivity.

• 2) $\Rightarrow$ 3): clear (set $\vec{b} = \vec{0}$)

• 3) $\Rightarrow$ 2): Suppose 2) is not true,

$\exists\, \vec{b} \in \mathbb{R}^m$ s.t. $A\vec{x} = \vec{b}$ has > 1 sol$^{\underline{n}}$.

say $A\vec{x}_1 = A\vec{x}_2 = \vec{b}$ and $\vec{x}_1 \neq \vec{x}_2$

Consider $\vec{y} := \vec{x}_1 - \vec{x}_2 \neq \vec{0}$

$$A\vec{y} = A(\vec{x}_1 - \vec{x}_2) \underset{\uparrow}{=} A\vec{x}_1 - A\vec{x}_2 = \vec{0}$$

Contradiction. $\square$ Since T_A is linear!

• 3) $\Leftrightarrow$ 4): by definition of l.i.

• 3) $\Rightarrow$ 5):

$A\vec{x} = \vec{0}$ has no nontrivial sol$^{\underline{n}}$

A has pivot in each column.

$$A \sim \begin{bmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}$$

Suppose $\exists$ column w/ no pivots.

$$\left[\begin{matrix}A\end{matrix}\,\middle|\,\begin{matrix}0\\0\\\vdots\\0\end{matrix}\right] \rightsquigarrow \left[\begin{matrix}1 & * & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots\end{matrix}\,\middle|\,\begin{matrix}0\\\vdots\\\vdots\\0\end{matrix}\right]$$

free variable $\Rightarrow$ sol^n is not unique.

- 5) $\Rightarrow$ 3):

$$\left[\begin{matrix}A\end{matrix}\,\middle|\,\begin{matrix}0\\\vdots\\0\end{matrix}\right] \rightsquigarrow \left[\begin{matrix}1 & 0 & 0 & \cdots & \\ 0 & 1 & & & \\ 0 & & \ddots & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & 0 & 0\end{matrix}\,\middle|\,\begin{matrix}0\\\vdots\\\vdots\\\vdots\\\vdots\\\vdots\\0\end{matrix}\right] \Rightarrow$$

the only sol^n is $x_1 = x_2 = \cdots = x_n = 0$.

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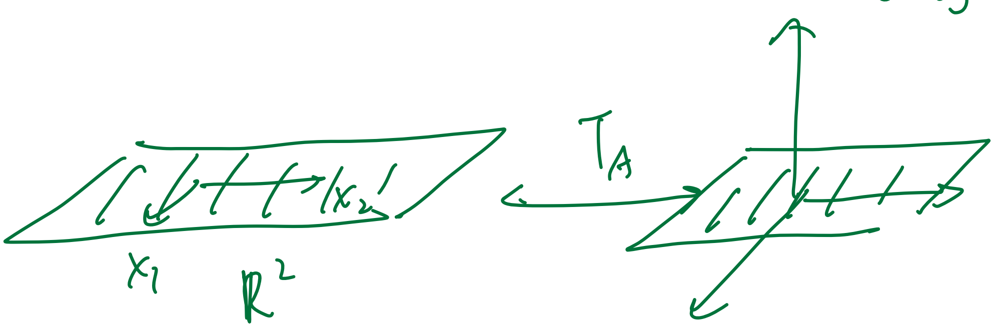
e.g. $A = \begin{bmatrix}1 & 0\\0 & 1\\ \boxed{0 \quad 0}\end{bmatrix}$

$T_A : \mathbb{R}^2 \to \mathbb{R}^3$

$\begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix}$

$= \begin{bmatrix}x_1\\x_2\\0\end{bmatrix}$



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§ Linear transformation between Euclidean spaces

Def: A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is called a <u>linear transformation</u> if: $\forall \vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, $c \in \mathbb{R}$

- $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$
- $T(c\vec{v}_1) = cT(\vec{v}_1)$

e.g. A: $m\times n$ matrix. $T_A: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transf.
$\vec{x} \mapsto A\vec{x}$

Rmk: T is linear $\Rightarrow$
- $T(\vec{0}) = \vec{0}$
- $T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = c_1 T(\vec{v}_1) + \cdots + c_k T(\vec{v}_k)$

Thm: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ linear transf.
Then $\exists!$ A: $m\times n$ s.t. $T_A(\vec{v}) = T(\vec{v})$ $\forall \vec{v} \in \mathbb{R}^n$.

there exists a unique ...

In fact, the i-th column of A is given by: $T(\vec{e}_i)$,
where

$$\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i-th entry.

pf: Consider

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

We'll check that $T_A(\vec{v}) \overset{??}{=} T(\vec{v})$ $\forall \vec{v} \in \mathbb{R}^n$.

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \vec{v}$$

$$T_A(\vec{v}) = A\vec{v} = v_1 \underbrace{T(\vec{e}_1)}_{} + v_2 \underbrace{T(\vec{e}_2)}_{} + \cdots + v_n \underbrace{T(\vec{e}_n)}_{} = T(v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n)$$

columns of A.

T is linear.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$$

$$T(\vec{v}) = T(v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n)$$

<u>Uniqueness</u>: Suppose $\exists$ A, B : $m \times n$ s.t.

$$T_A(\vec{v}) = T_B(\vec{v}) = T(\vec{v}) \quad \forall \vec{v} \in \mathbb{R}^n$$

$$A\vec{v} = B\vec{v}.$$

plug in $\vec{v} = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

$$\Rightarrow \quad \underbrace{A\vec{e}_1}_{= A\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \text{the } 1^{st} \text{ column vector of } A} = B\vec{e}_1 = \text{the } 1^{st} \text{ column of } B$$

By the same argument (i.e. plug in $\vec{v} = \vec{e}_i$ for $i = 1, \dots, n$) we know that

the i^{th} column of A = the i^{th} column of B $\forall i$.

$\Rightarrow$ $A = B$. $\square$