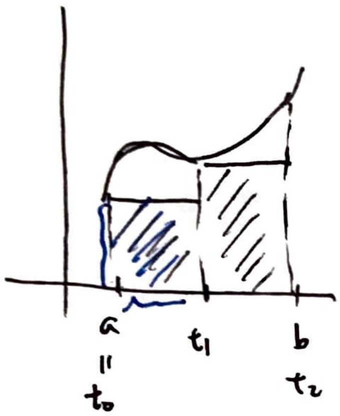
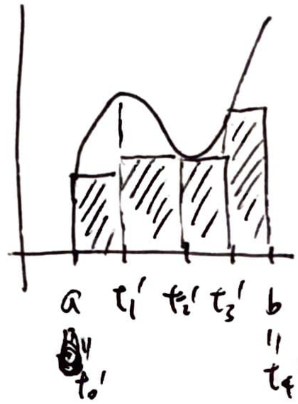
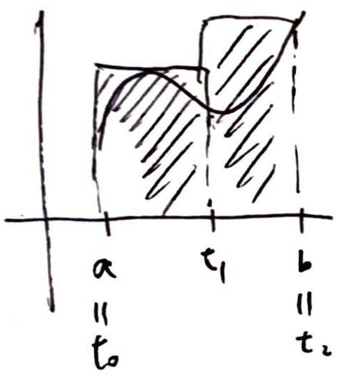


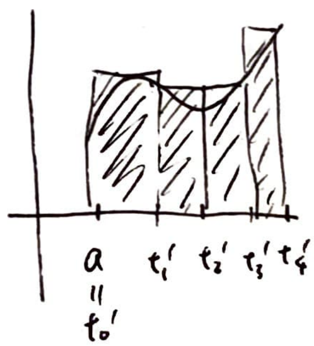
$L(f,P)$ →



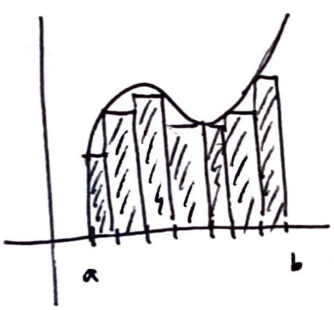
$L(f,P')$



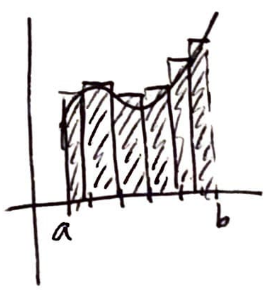
$U(f,P)$



$U(f,P')$



$L(f,P'')$



$U(f,P'')$

Last time:

- Partition of $[a,b]$: $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$
- $L(f,P) := \sum_{k=1}^{n} (t_k - t_{k-1}) \cdot \inf\{f(x) : x \in [t_{k-1}, t_k]\}$

  $U(f,P) := \sum_{k=1}^{n} (t_k - t_{k-1}) \sup\{f(x) : x \in [t_{k-1}, t_k]\}$
- $\boxed{L(f,P) \leq U(f,P)}$
- Expect: • $L(f,P) \leq$ "$\int_a^b f(x)\,dx$" $\leq U(f,P)$ if it's defined.
  - If we take finer and finer partitions, $L(f,P)$, $U(f,P)$ "should approach" $\int f(x)\,dx$

Def Lower integral

$$L(f) := \sup\{L(f,P) : P \text{ partition of } [a,b]\}$$

Upper integral

$$U(f) := \inf\{U(f,P) \ P \text{ partition}\}$$

Doesn't follow directly from $L(f,P) \le U(f,P)\ \forall P$.

We're going to show: $\boxed{L(f) \le U(f)}$

Def f is integrable on $[a,b]$ if

$$L(f) = U(f)$$

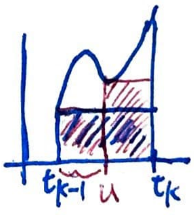
Def Say $Q = \{a = s_0 < s_1 < \cdots < s_m = b\}$ is a refinement of $P = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$ ("$P \subset Q$")

If $\{t_0, \dots, t_n\} \subset \{s_0, \dots, s_m\}$.

Lemma If $P \subset Q$, then

$$\boxed{L(f,P) \le L(f,Q)} \le U(f,Q) \le U(f,P)$$

suffices to show for Q consists of 1 more point than P.



$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$

$Q = \{a = t_0 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b\}$

③

$$L(f,Q) = \boxed{(t_1 - t_0)\inf_{x\in[t_0,t_1]} f(x) + \cdots + (t_{k-1} - t_{k-2})\inf_{x\in[t_{k-2},t_{k-1}]} f(x)}$$

$$+ (u - t_{k-1})\inf_{x\in[t_{k-1},u]} f(x) + (t_k - u)\inf_{x\in[u,t_k]} f(x)$$

$$+ \boxed{(t_{k+1} - t_k)\inf_{x\in[t_k,t_{k+1}]} f(x) + \cdots + (t_n - t_{n-1})\inf_{x\in[t_{n-1},t_n]} f(x)}$$

$$L(f,P) =$$

$$+ (t_k - t_{k-1})\inf_{x\in[t_{k-1},t_k]} f(x)$$

$$+$$

$$L(f,Q) - L(f,P)$$

$$= (u - t_{k-1})\inf_{x\in[t_{k-1},u]} f(x) + (t_k - u)\inf_{x\in[u,t_k]} f(x)$$

$$- (t_k - t_{k-1})\inf_{x\in[t_{k-1},t_k]} f(x)$$

$$(u - t_{k-1}) + (t_k - u)$$

$$= \overset{\overset{0}{\wedge}}{(u - t_{k-1})}\left(\inf_{x\in[t_{k-1},u]} f(x) - \inf_{x\in[t_{k-1},t_k]} f(x)\right) \geq 0$$

$$+ \overset{\overset{0}{\wedge}}{(t_k - u)}\left(\inf_{x\in[u,t_k]} f(x) - \inf_{x\in[t_{k-1},t_k]} f(x)\right) \geq 0$$

$\square$

Lemma

$$L(f,P) \le U(f,Q) \quad \forall P,Q$$

pf $P \cup Q :=$ common refinement of P, Q,

i.e. union of all pts in P & Q.

$$L(f,P) \le L(f, P\cup Q)$$
$$\le U(f, P\cup Q) \le U(f,Q). \quad \square$$

Prop: $L(f) \le U(f)$

pf $L(f,P) \le U(f,Q) \quad \forall P, Q$

$$\Rightarrow \underbrace{\sup_P L(f,P)}_{=L(f)} \le U(f,Q)$$

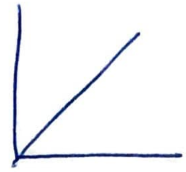
i.e. $L(f) \le U(f,Q) \quad \forall Q$

$$\Rightarrow L(f) \le \underbrace{\inf_Q U(f,Q)}_{=U(f)} \quad \square$$

Def A bounded fcn $f: [a,b] \to \mathbb{R}$ is integrable if $L(f) = U(f)$

In this case, $\int_a^b f(x)\,dx := L(f) = U(f)$

e.g. $f(x) = x$ on $[0,1]$



$P_n = \{0 < \frac{1}{n} < \frac{2}{n} < \cdots < 1\}$

$L(f, P_n) = \frac{n-1}{2n}$, $U(f,P_n) = \frac{n+1}{2n}$ $\forall n$

$$\underbrace{L(f,P_n)}_{=\frac{n-1}{2n}} \le L(f) \le U(f) \le \underbrace{U(f,P_n)}_{=\frac{n+1}{2n}}$$

By taking $n \to \infty$

$\Rightarrow L(f) = U(f) = \frac{1}{2}$

So f is integrable on $[0,1]$,

and $\int_0^1 f(x)\,dx = \frac{1}{2}$.

---

e.g. $f : [0,1] \longrightarrow \mathbb{R}$

$$x \longmapsto \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

. disconti. on every pt in $[0,1]$

$\Rightarrow U(f) = 1$, $L(f) = 0$ #

$\Rightarrow f$ is not integrable.

Claim: $U(f,P) = 1$, $L(f,P) = 0$ $\forall P$

$$\underbrace{U(f,P)}_{=\sum (t_k - t_{k-1}) \overbrace{\sup_{x\in[t_{k-1},t_k]} f(x)}^{=1}} = (t_1 - t_0) + (t_2 - t_1) + \cdots + (t_n - t_{n-1}) = 1$$

$[t_{k-1}, t_k]$ contains a rat'l number

Riemann-Lebesgue thm

$f : [a,b] \to \mathbb{R}$ bdd fcn is integrable

$\Leftrightarrow$ the set $\{x \in [a,b] : f \text{ is discont. at } x\}$

"has measure zero"

Later

Thm A bdd fcn $f : [a,b] \to \mathbb{R}$ integrable

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists P$ partition of $[a,b]$

s.t. $U(f,P) - L(f,P) < \varepsilon$.

---

pf ($\Rightarrow$) Assume $L(f) = U(f)$

$\forall \varepsilon > 0$,

Since $L(f) = \sup_P L(f,P)$

$$\begin{cases} \exists P_1 \text{ s.t. } 0 \le L(f) - L(f,P_1) < \frac{\varepsilon}{2} \\ \exists P_2 \text{ s.t. } 0 \le U(f,P_2) - U(f) < \frac{\varepsilon}{2} \end{cases}$$

$P := P_1 \cup P_2$

$U(f,P) - L(f,P)$

$\le U(f,P_2) - L(f,P_1)$

$< \left(\cancel{U(f)} + \frac{\varepsilon}{2}\right) + \left(-\cancel{L(f)} + \frac{\varepsilon}{2}\right)$

$= \varepsilon$ $\square$

$(\Leftarrow)$ $\forall \varepsilon>0, \exists P$ s.t. $U(f,P)-L(f,P)<\varepsilon$.

$$L(f) \le U(f) \le U(f,P) < L(f,P)+\varepsilon \le L(f)+\varepsilon$$

$$\Rightarrow L(f) \le \boxed{U(f) < L(f)+\varepsilon} \quad \forall \varepsilon>0$$

$$\Rightarrow L(f) = U(f). \quad \square$$

<u>Thm</u> Any conti. fcn. $f:[a,b]\to\mathbb{R}$ is integrable.

pf <u>WTS</u>: $\boxed{\forall \varepsilon>0, \exists P \text{ s.t. } U(f,P)-L(f,P)<\varepsilon}$

f is <u>unif. conti.</u> on $[a,b]$ (b/c compactness of $[a,b]$)

$\forall \varepsilon>0, \exists \delta>0$

s.t. $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon/(b-a)\cdot 2$

Choose P s.t. each subinterval has length $<\delta$.

$$\Rightarrow U(f,P)-L(f,P)$$

$$= \sum (t_k-t_{k-1})\cdot\Big(\underbrace{\sup_{x\in[t_{k-1},t_k]} f(x) - \inf_{x\in[t_{k-1},t_k]} f(x)}_{\le \frac{\varepsilon}{2(b-a)}}\Big)$$

$$\le \frac{\varepsilon}{2(b-a)}\cdot \boxed{\underbrace{\sum (t_k-t_{k-1})}_{=(b-a)}}$$

$$= \frac{\varepsilon}{2} < \varepsilon. \quad \square$$

Thm Any bounded monotone fcn $f: [a,b] \to \mathbb{R}$ is integrable

---

pf Say f is increasing, $f(a) \le f(b)$

① Choose any P s.t. each subinterval has length $< \dfrac{\varepsilon}{f(b)-f(a)}$

$$U(f,P) - L(f,P) = \sum \underbrace{(t_k - t_{k-1})}_{< \frac{\varepsilon}{f(b)-f(a)}} \cdot \Big(\underbrace{\sup_{x \in [t_{k-1}, t_k]} f(x)}_{= f(t_k)} - \underbrace{\inf_{x \in [t_{k-1}, t_k]} f(x)}_{= f(t_{k-1})} \Big)$$

$$< \frac{\varepsilon}{f(b)-f(a)} \underbrace{\sum \big(f(t_k) - f(t_{k-1}) \big)}_{= f(b)-f(a)}$$

$$= \varepsilon. \quad \square$$

RL thm + Thm ①

$\Rightarrow$ for bdd monotone fcn, the set of discont. pts has measure zero

In HW, you'll prove for bdd monotone fcn, the set of discont. pt is countable