What's linear algebra?

- In the most concrete form, it studies systems of <u>linear</u> eq'ns. like:

$$\begin{cases} x_1 + x_2 = 2 \\ -2x_1 + 3x_2 = 1. \end{cases}$$

- More abstractly, it studies "transformations" of "spaces" which carries "lines" to "lines".

<u>e.g.</u> Schrödinger eq'n:

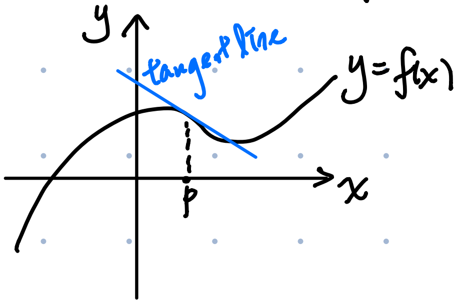
$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - V(x,t)\right) F(x,t) = 0.$$

where: "space" $\mathcal{F}$:= space of functions $F(x,t)$.

"transformation": $\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - V(x,t)\right) : \mathcal{F} \longrightarrow \mathcal{F}$

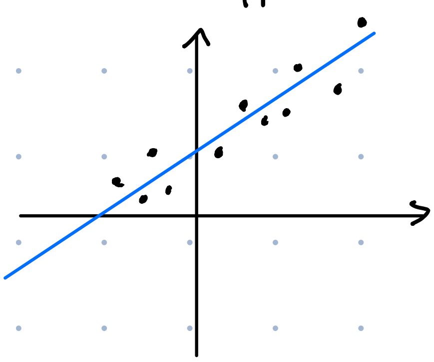
Some applications of linear algebra:

- To understand a map $f: X \longrightarrow Y$ at a point $p \in X$, we usually start with studying its "first order approximation", i.e. the "tangent map" $f_p: T_pX \longrightarrow T_{f(p)}Y$, which is a <u>linear</u> transformation.



This is the most fundamental thing in multivariable calculus, differential geometry, ...

- Linear approximation



You'll leave this class equipped with a powerful conceptual framework on which the vast majority of math, science, engineering, ... depend.

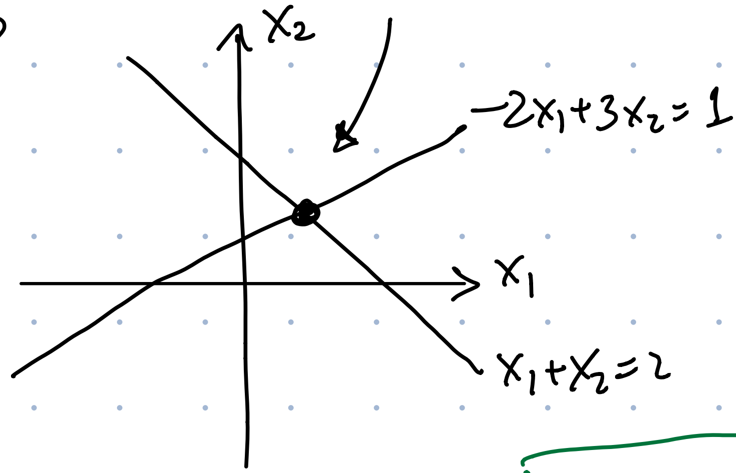
Warm-up:

e.g. $\begin{cases} x_1 + x_2 = 2 \\ -2x_1 + 3x_2 = 1. \end{cases}$

Any solutions?

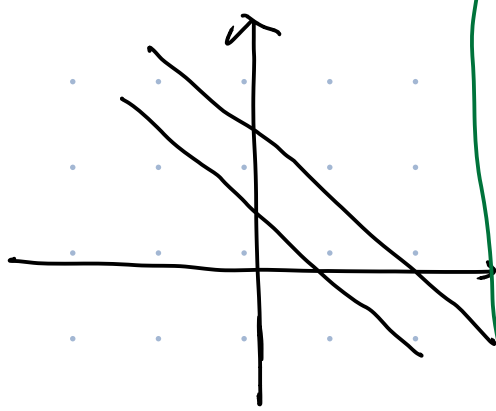
$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ solves the linear system

Geometrically?



e.g. $\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}$

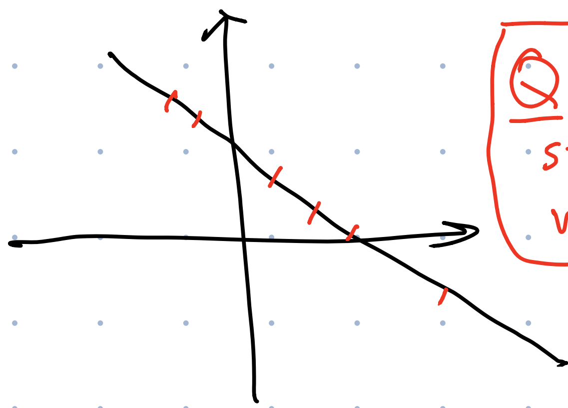
No sol'n



For a system of linear eq'ns in 2 variables, the sol'n set is either empty, a pt, a line, whole $\mathbb{R}^2$

e.g. $\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 2 \\ 3x_1 + 3x_2 = 3 \\ -x_1 - x_2 = -1 \end{cases}$

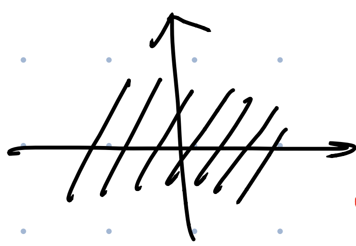
∞ many sol'ns

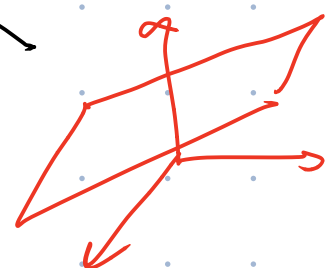


Q: What about system of linear eq'ns w/ 3 variables?

e.g. "trivial" linear eq'n:

$\{ \underline{0} x_1 + \underline{0} x_2 = 0$





$\mathbb{R}^3$, plane, line, pt, ϕ

In HW you'll show that $\forall$ linear system, it can have either
(for any)
no sol^{n}s, a unique sol^n, or ∞ many sol^{n}s.

<u>Def</u> <u>System of linear eq^{n}s</u> (<u>linear system</u>) in variables $x_1, \dots, x_n$:

$$(*) \quad \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

where each a_{ij}, b_i are all real numbers ($\mathbb{R}$)

<u>Def</u> A <u>vector</u> in $\mathbb{R}^n$ is an ordered list of n real numbers, written as

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

<u>Def</u> We say a vector $\vec{v}$ is a <u>sol^n of $(*)$</u> if

$$\begin{cases} a_{11}v_1 + \cdots + a_{1n}v_n = b_1 \\ \quad \vdots \\ a_{m1}v_1 + \cdots + a_{mn}v_n = b_m. \end{cases}$$

e.g. $\begin{cases} x_1 - 2x_2 + x_3 = 0 & \text{--- ①} \\ 3x_1 - 4x_2 - 5x_3 = 8 & \text{--- ②} \\ 5x_1 - 2x_2 - 13x_3 = 18 & \text{--- ③} \end{cases}$

It's more convenient to write down the system without the variables:

$$\underbrace{\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 3 & -4 & -5 & 8 \\ 5 & -2 & -13 & 18 \end{array}\right]}_{\text{augmented matrix}}$$

coefficient matrix.

augmented matrix

Replace ② with ② − 3 × ①:

$$\begin{array}{rl} ②: & 3x_1 - 4x_2 - 5x_3 = 8 \\ -\quad 3\times ①: & 3x_1 - 6x_2 + 3x_3 = 0 \\ \hline & 2x_2 - 8x_3 = 8 \end{array}$$

sol'n set of

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & -2 & -13 & 18 \end{array}\right]$$

is the same as the sol'n set of

Replace ③ with ③ − 5 × ①:

$$\begin{array}{rl} ③: & 5x_1 - 2x_2 - 13x_3 = 18 \\ 5\times ①: & 5x_1 - 10x_2 + 5x_3 = 0 \\ \hline & 8x_2 - 18x_3 = 18 \end{array}$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 8 & -18 & 18 \end{array}\right] \end{array} \begin{array}{l} \\ — ②' \\ \\ \end{array}$$

Replace ②' by ②'/2:

$2x_2 - 8x_3 = 8$

$\Updownarrow$

$x_2 - 4x_3 = 4$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & \boxed{1} & -4 & 4 \\ 0 & 8 & -18 & 18 \end{array}\right] \begin{array}{l} — ①'' \\ — ②'' \\ — ③'' \end{array}$$

Replace ①'' by ①'' + 2 × ②'':

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 8 & -18 & 18 \end{array}\right]$$

Replace ③'' by ③'' − 8 × ②''

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 14 & -14 \end{array}\right]$$

$-18 - 4 \times (-8)$

$18 - 8 \times 4$

$8 + 7(-1)$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$4 + (-1) \times 4$

$$\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$$

Row reduction algorithm:

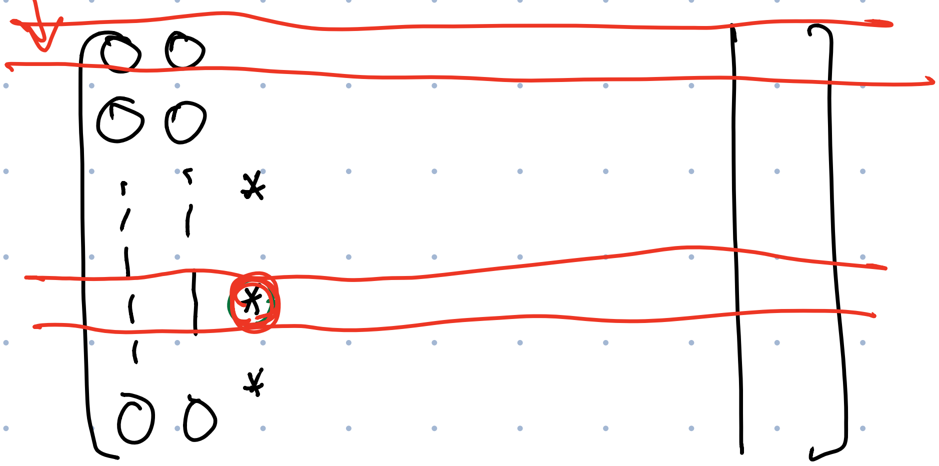
(Start w/ a linear system, we do some "row operations" to get a "simpler" linear systems, while the solⁿ set is unchanged)

Elementary row operations

① Replace a row by the sum of itself and a multiple of another row.

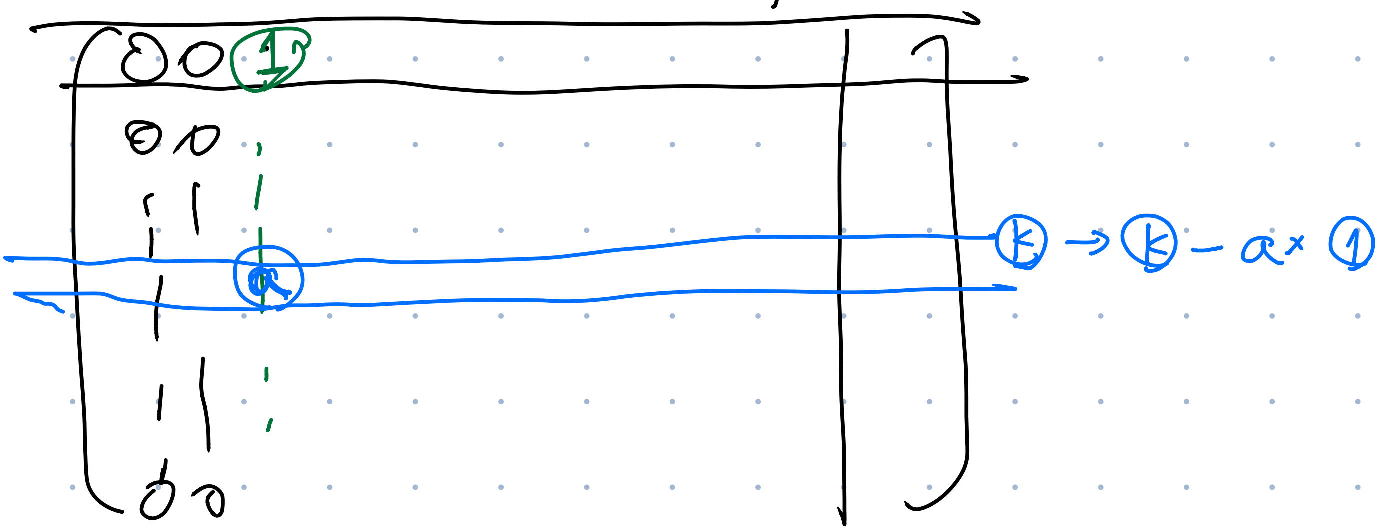
② multiply all entries of a row by a nonzero constant.

③ Interchange 2 rows



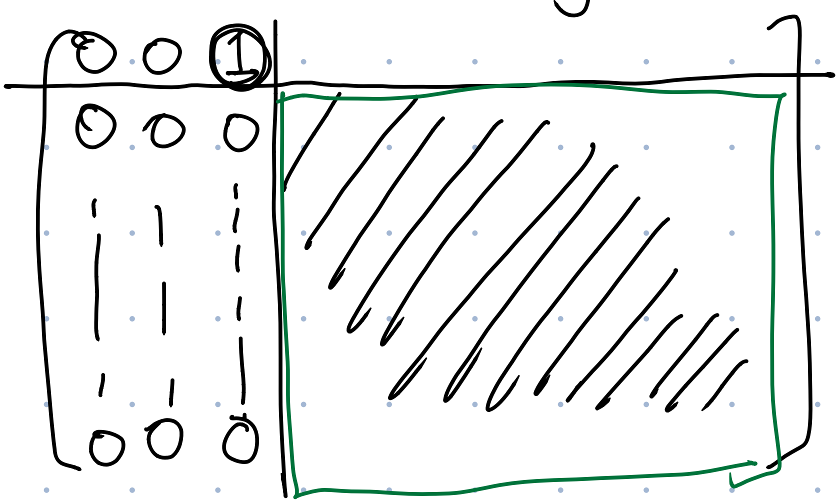
① find a left-most nonzero entry

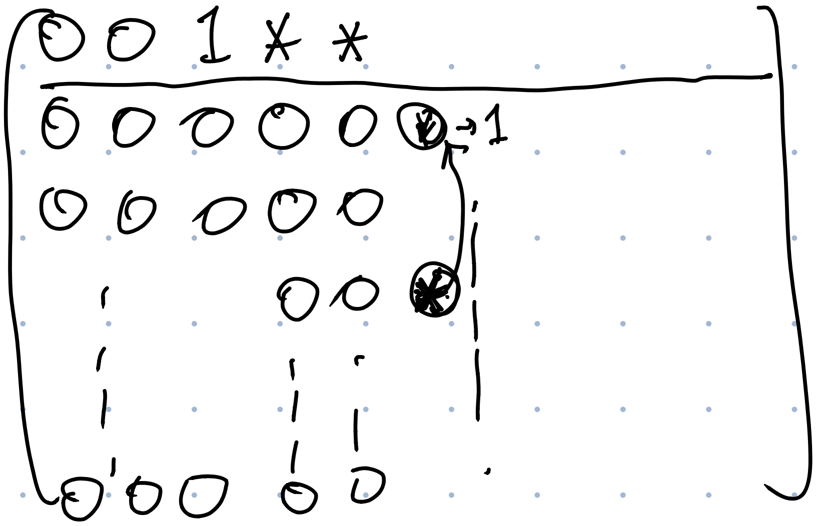
② Do ③ to make the nonzero entry to be in the 1st row.

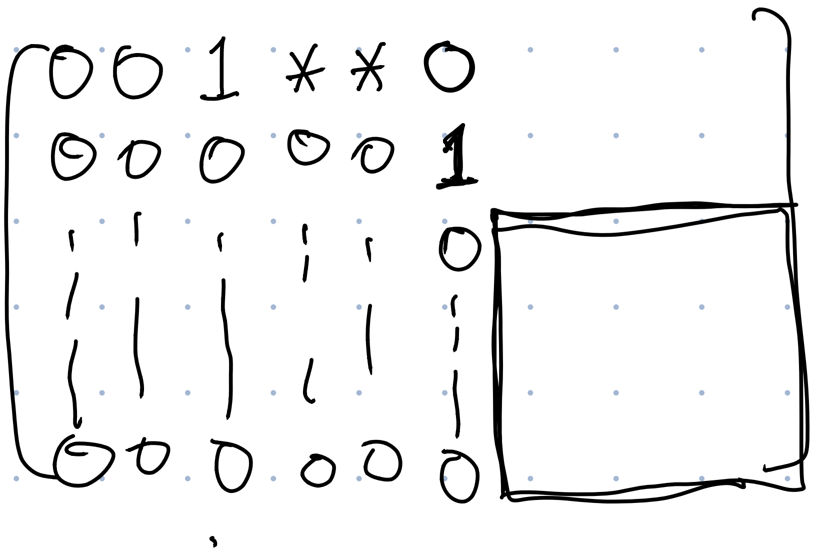


③ Do ② to make it = 1.

④ Do ① to make the remaining column = 0

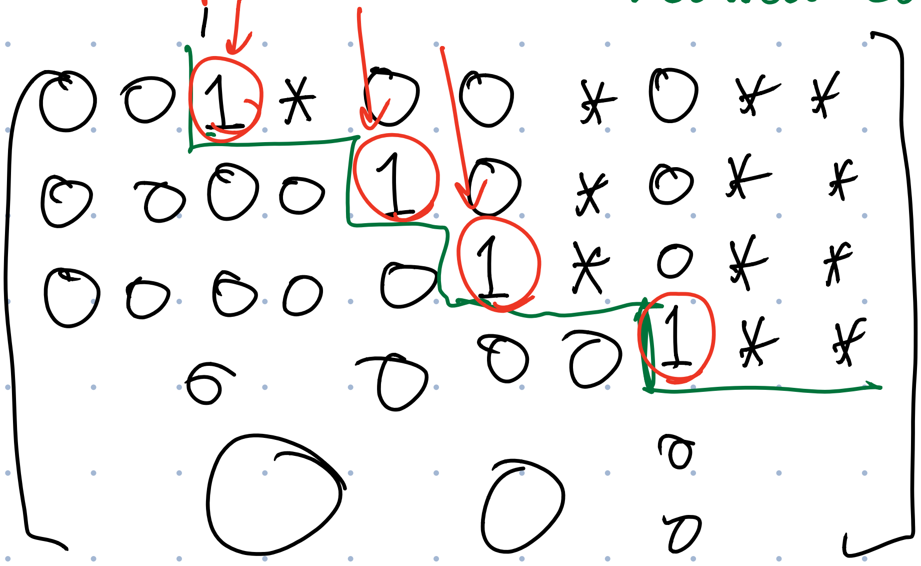




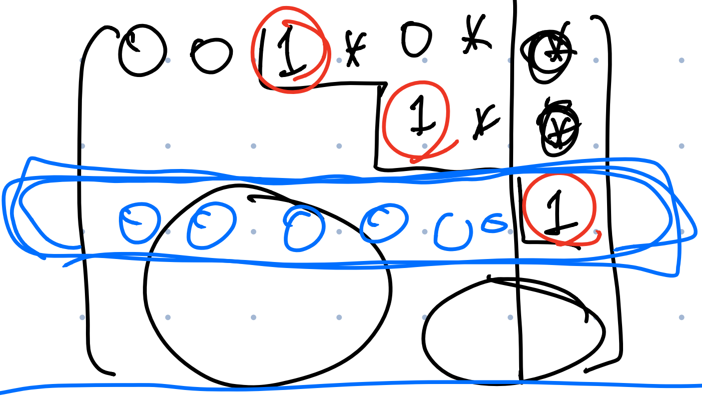


pivot positions.

Reduced echelon form.



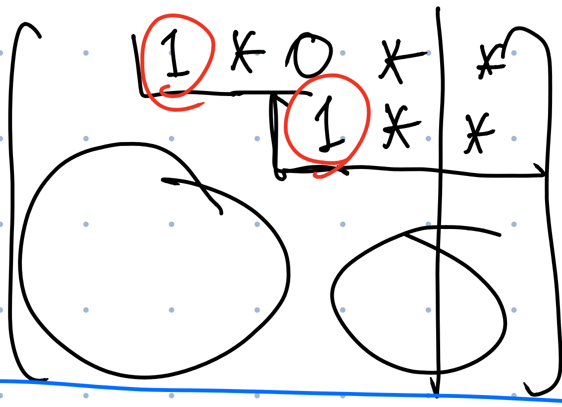
Case 1: The last column of / is pivot



$0x_1 + 0x_2 + \cdots + 0x_n = 1$

NO sol'n !!

Case 2: The last column is not ~~pivot~~ pivot.



Q: In which case does the linear system have sol'n ??

Q: In case 2, how to find sol's ?

$$
\begin{array}{c}
\left[\begin{array}{cccccc|c}
0 & 1 & 0 & 3 & 0 & 5 & 8 \\
0 & 0 & 1 & 4 & 0 & 6 & 9 \\
0 & 0 & 0 & 0 & 1 & 7 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \\
\begin{array}{cccccc}
\uparrow & \uparrow & \uparrow & \uparrow & & \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6
\end{array}
\end{array}
$$

$$
\begin{cases}
x_2 + 3x_4 + 5x_6 = 8 \\
x_3 + 4x_4 + 6x_6 = 9 \\
x_5 + 7x_6 = 10
\end{cases}
$$

$x_2 = 8 - 3x_4 - 5x_6$

$x_3 = 9 - 4x_4 - 6x_6$

$x_5 = 10 - 7x_6$