Math 104 (Section 4) Second Midterm (Practice Exam) November 5, 2019

Name: Solution

- You have 70 minutes to complete the exam.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	25	
2	25	
3	20	
4	30	
Total	100	

1. (a) (20 points) Consider the power series $\sum_{n=1}^{\infty} a_n x^n$ where

$$a_n = \begin{cases} \frac{2^k}{k} & \text{if } n = 2k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Find the exact interval of convergence of this power series. You need to justify your answer.

$$|a_n|^{1/n} = \begin{cases} \frac{\sqrt{2}}{\sqrt[n]{n/2}} & \text{if } n\text{: even} \\ 0 & \text{if } n\text{: odd.} \end{cases}$$

Since $\lim_{n\to\infty} \frac{\sqrt{2}}{\sqrt[n]{n/2}} = \sqrt{2}$, we get $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \sqrt{2}$,

so the radius of convergence is $1/\sqrt{2}$.

- If $x = 1/\sqrt{2}$, $\sum_{n=1}^{\infty} a_n \left(\frac{1}{\sqrt{2}}\right)^n = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty$.
- If $x = -1/\sqrt{2}$, $\sum_{n=1}^{\infty} a_n \left(\frac{-1}{\sqrt{2}}\right)^n = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty$.

Hence the exact interval of convergence is $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. $\square$

(b) (5 points) Give an explicit example of the power series in the form

$$\sum_{n=0}^{\infty} b_n (x-c)^n$$

whose exact interval of convergence is $(2, 6]$. You don't need to justify your answer. My power series is defined by

$b_n = \frac{(-1)^n}{2^n \cdot (n+1)}$, and $c = 4$

2. (25 points) Let $a < b$ be two real numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous function satisfying $f(a) < f(b)$. You are going to prove the intermediate value theorem for $f(x)$. Namely, prove that for any $y \in \mathbb{R}$ with $f(a) < y < f(b)$, there exists $c \in (a, b)$ such that $f(c) = y$.

(a) (5 points) Let $S = \{x \in [a, b] : f(x) < y\}$. Prove that $\sup S$ exists and is a real number.

- S is nonempty, since $a \in S$.
- S is bounded above (by b).

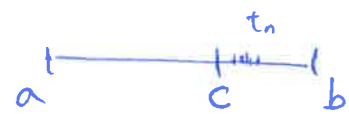
$\Rightarrow$ $\sup S$ exists & is a real number by completeness axiom.

(b) (10 points) Let $c = \sup S$. Prove that $f(c) \leq y$.
Hint: First show that for each $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ satisfying $c - \frac{1}{n} \leq x_n \leq c$ and $f(x_n) < y$. Then use the continuity of f.

- $\forall n \in \mathbb{N}$, since $c - \frac{1}{n}$ is not an upper bound of S,
  so $\exists \underline{x_n \in S}$ s.t. $\underline{c - \frac{1}{n} < x_n \leq c}$.
  $\hookrightarrow$ $x_n \in [a, b]$ and $f(x_n) < y$.
- By squeeze lemma, $\lim\limits_{n\to\infty} x_n = c$.
- By continuity of f, $\lim\limits_{n\to\infty} f(x_n) = f(c)$.
  Since $f(x_n) < y \ \forall n$, so $f(c) \leq y$. $\square$

(c) (10 points) Prove $f(c) \geq y$. Combine with (b), we conclude that $f(c) = y$.
Hint: First explain that $c < b$. Then consider the sequence (t_n) in $[a, b]$ defined by

$$t_n = \min\{c + \frac{1}{n}, b\}.$$



- $c < b$ since $f(c) \leq y < f(b)$.
- Let $t_n = \min\{c + \frac{1}{n}, b\}$.
  Then $t_n > c = \sup S$, hence $t_n \notin S$.
  Since $t_n \in [a, b]$, so $f(t_n) \geq y$. $\forall n$.
- We have $c \leq t_n \leq c + \frac{1}{n}$.
  By squeeze lemma, $\lim_{n \to \infty} t_n = c$.
- By continuity of f, $\lim_{n \to \infty} f(t_n) = f(c)$.
  Since $f(t_n) \geq y$ $\forall n$, so $f(c) \geq y$. $\square$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(x+1) = f(x)$ for every $x \in \mathbb{R}$.

(a) (5 points) Explain that f is uniformly continuous on $[-1, 2]$.

f is continuous on a closed and bounded interval, $[-1,2]$,
so it's uniformly continuous on $[-1,2]$.

(b) (15 points) Prove that f is uniformly continuous on $\mathbb{R}$.

$\forall \varepsilon > 0$, since f is uniformly continuous on $[-1,2]$,
$\exists \delta' > 0$ s.t. $|f(z) - f(w)| < \varepsilon$ for any $z, w \in [-1,2]$, and $|z-w| < \delta'$

Define $\delta := \min\{\delta', 1\} > 0$.

Claim: $\forall |x-y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

pf of Claim: $\exists n \in \mathbb{Z}$ s.t. $n \le x < n+1$.

Define $z = x - n$ and $w = y - n$.

- By periodicity condition of f, we have $f(x) = f(z)$ & $f(y) = f(w)$.
- By definition, $z \in [0,1]$.
- Since $|z-w| = |x-y| \underset{\text{assumption}}{<} \delta = \min\{\delta', 1\} \le 1$,

  so $w \in [-1,2]$.
- Since $|z-w| = |x-y| < \delta = \min\{\delta', 1\} \le \delta'$ and $z, w \in [-1,2]$,

  we have $|f(z) - f(w)| < \varepsilon$.

$\Rightarrow |f(x) - f(y)| = |f(z) - f(w)| < \varepsilon$. $\square$

4. There are four statements below:

(I) For $n \in \mathbb{N}$, define the continuous function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{3n - \sin x}{2n^2 + \cos^2 x}.$$

Then the sequence (f_n) converges uniformly to a continuous function on $\mathbb{R}$.

(II) For any continuous function $f : (0,2) \to \mathbb{R}$ and any Cauchy sequence (x_n) in $(0,2)$, the sequence $(f(x_n))$ is also a Cauchy sequence.

(III) Any bounded and continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

(IV) Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}, \\ x^2 & x \in \mathbb{R} \backslash \mathbb{Q}. \end{cases}$$

Then $f(x)$ is continuous at $x = 0$.

(a) (15 points) Choose a statement that is true and prove it. *You are not allowed to choose more than one statement.*

My statement is (I) or (IV)

(I): Let f be the zero function on $\mathbb{R}$. i.e. $f(x)=0 \ \forall x \in \mathbb{R}$.

Claim: $f_n \longrightarrow f$ uniformly.

$\forall \varepsilon > 0$, take $N = \frac{2}{\varepsilon}$, then for any $x \in \mathbb{R}$ and $n > N$, we have

$$|f_n(x) - f(x)| = \left|\frac{3n - \sin x}{2n^2 + \cos^2 x}\right| \le \frac{3n+1}{2n^2} \le \frac{4n}{2n^2} = \frac{2}{n} < \frac{2}{N} = \varepsilon. \quad \square$$

(IV): $\forall \varepsilon > 0$, take $\delta = \sqrt{\varepsilon} > 0$, then for any $x \in \mathbb{R}$ satisfying $|x - 0| < \delta$,

we have $|f(x) - f(0)| = |f(x)| \le x^2 < \delta^2 = \varepsilon. \quad \square$

(b) (15 points) Choose a statement that is false. Give an explicit counterexample and justify it. *You are not allowed to choose more than one statement.*
My statement is (I) or (II).

(I): Consider $f(x) = \frac{1}{x}$ on $(0, 2)$.
$(x_n = \frac{1}{n})$ is a Cauchy sequence, but $(f(x_n) = n)$ is not. $\square$

(II): Consider $f(x) = \cos(x^2)$ on $\mathbb{R}$. It's bounded & continuous.
Claim: f is NOT uniformly continuous.
Set $\varepsilon = 1$. For any $\delta > 0$, we want to find $x, y \in \mathbb{R}$ s.t.
$|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon = 1$.

Take any $n \in \mathbb{N}$, s.t. $n > \frac{\pi}{2\delta^2}$.
Set $x = \sqrt{(2n+1)\pi}$ and $y = \sqrt{2n\pi}$. Then

$$|x - y| = \sqrt{(2n+1)\pi} - \sqrt{2n\pi} = \frac{\pi}{\sqrt{(2n+1)\pi} + \sqrt{2n\pi}} < \frac{\pi}{\sqrt{2n\pi}} < \delta.$$

And

$$|f(x) - f(y)| = |\cos((2n+1)\pi) - \cos(2n\pi)| = 2 > 1 = \varepsilon. \quad \square.$$