HOMEWORK 7
MATH 104, SECTION 6

Office Hours: Tuesday and Wednesday 9:30-11am at 735 Evans.

Nima's Office Hours: Monday, Tuesday and Thursday 9:30am-1pm at 1010 Evans.

Reading

There will be reading assigned for each lecture. You should come to the class having read the assigned sections of the textbook.

Due March 5: Ross, Section 18, 21

Due March 10: Ross, Section 19

Problem set (10 problems; due March 5)

Submit your homework at the beginning of the lecture on Thursday. *Late homework will not be accepted under any circumstances.*

You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any.

Write clearly! Mastering mathematical writing is one of the goals of this course.

You have to staple your work if it is more than one page.

(1) Let E be a nonempty compact subset of $\mathbb{R}$. Prove that $\sup E$ and $\inf E$ belong to E.

(2) Explain why the following sets are compact.
   (a) The Sierpiński triangle in $\mathbb{R}^2$. (You may want to read more about the construction of the Sierpiński triangle on Wikipedia.)
   (b) The set $X = \{A \in M_n(\mathbb{R}) : A^t A = I\} \subseteq M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ of orthogonal matrices.

(3) Let $E \subseteq (X, d)$ be a subset of a metric space. Define the *Cantor–Bendixson derivative* of E:

$$E' \coloneqq \{x \in X : x \text{ is a limit point of } E\}.$$

Show that E' is closed, and if $E' \neq \emptyset$ then E contains infinitely many elements. (Recall that $x \in X$ is a limit point of E if for any $r > 0$, the intersection $B_r(x) \cap E$ contains at least a point other than x.)

(4) Consider the following two functions on $\mathbb{R}$:

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

For each of the functions, prove or disprove that it is continuous at the point $x = 0$.

(5) In each case, find $\delta > 0$ such that $|f(x) - \ell| < \epsilon$ for all satisfying $|x - x_0| < \delta$.
  (a) $f(x) = \frac{1}{x}$; $x_0 = 1$, $\ell = 1$.
  (b) $f(x) = \sqrt{|x|}$; $x_0 = 0$, $\ell = 0$.
  (c) $f(x) = \sqrt{x}$; $x_0 = 1$, $\ell = 1$.

As we discussed in class, δ typically depends on both ϵ and x_0. Note that x_0 is given in these problems, so your δ should be depending on ϵ.

(6) Prove the following generalization of Ross, Theorem 17.4: Let (X, d) be any metric space, and let $f, g : X \to \mathbb{R}$ be two real-valued functions that are continuous at $x_0 \in X$. Prove that the functions $f + g$ and fg are both continuous at x_0. Moreover, if $g(x_0) \neq 0$, then f/g is also continuous at x_0. (The proofs are very similar, so you can pick one of $f + g, fg, f/g$ and prove it.)

(7) Prove the following generalization of Ross, Theorem 17.5: Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be three metric spaces and let $f : X \to Y$ and $g : Y \to Z$ be two maps among them. Define the composite function $g \circ f : X \to Z$ via $(g \circ f)(x) := g(f(x))$. Prove that if f is continuous at $x_0 \in X$ and g is continuous at $f(x_0) \in Y$, then the composition $g \circ f$ is continuous at x_0.

(8) Prove that any polynomial function of odd degree has at least one real root. (Hint: First show that polynomial functions are continuous on $\mathbb{R}$. Then try to apply intermediate value theorem.)

(9) Suppose f, g are real-valued continuous functions on the closed interval $[a, b]$, and that $f(a) < g(a)$ and $f(b) > g(b)$. Prove that $f(c) = g(c)$ for some $c \in (a, b)$. (Hint: if your proof is not very short, then it's probably not the right one.)

(10) (a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = 0$ for all $x \in \mathbb{Q}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.
  (b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then f is linear, i.e. there exists c so that $f(x) = cx$ for all x.