HOMEWORK 7
MATH H54

Yu-Wei's Office Hours: Sunday 1-2:30pm and Friday 12-1:30pm (PDT)
Michael's Office Hours: Monday 12-3pm (PDT)

Part I (no need to turn in)

This part of the homework provides some routine computational exercises. You don't have to turn in your solutions for this part, but being able to do the computations is vitally important for the learning process, so you definitely should do these practices before you start doing Part II of the homework.

The following exercises are from the corresponding sections of the UC Berkeley custom edition of Lay, Nagle, Saff, Snider, *Linear Algebra and Differential Equations*.

- **Exercise 6.3**: 11, 17
- **Exercise 6.4**: 11, 13, 15
- **Exercise 6.5**: 3, 11, 15

Part II (Due October 27, 8am PDT)

Some ground rules:

- You have to submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW7**.
- The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own.
- You're allowed to use any result that is proved in the lecture. But if you'd like to use other results, you have to prove it first before using it.

Problems:

(1) Let V be a finite dimensional inner product space, and let $W \subseteq V$ be a subspace. Prove that the projection map

$$\mathrm{proj}_W \colon V \to W, \quad \vec{v} \mapsto \mathrm{proj}_W \vec{v}$$

is linear.

(2) Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthonormal basis of an inner product space V. Let $\vec{x}$ be any vector in V, and let θ_i be the angle between $\vec{x}$ and $\vec{v}_i$ for each i. Prove that

$$\cos^2\theta_1 + \cdots + \cos^2\theta_n = 1.$$

(3) Prove the uniqueness of QR decomposition: Suppose A is an $m \times n$ matrix with linearly independent columns. Prove that there exists a *unique* pair of matrices Q and R such that
- $A = QR$,
- Q is an $m \times n$ matrix with orthonormal columns,
- R is an $n \times n$ upper triangular matrix with positive entries on the diagonal.

(4) Let A be an $m \times n$ matrix. Prove that its transpose A^T is the *unique* $n \times m$ matrix such that

$$\langle A\vec{x}, \vec{y}\rangle_{\mathbb{R}^m} = \left\langle \vec{x}, A^T\vec{y}\right\rangle_{\mathbb{R}^n} \quad \text{holds for any } \vec{x} \in \mathbb{R}^n \text{ and } \vec{y} \in \mathbb{R}^m.$$

(In fact, this is one of the main reasons that the transpose of a matrix is important.)

(5) Let A be a real $n \times n$ matrix. Prove that A is anti-symmetric (i.e. $A = -A^T$) *if and only if* $\langle A\vec{v}, \vec{v}\rangle = 0$ for all $\vec{v} \in \mathbb{R}^n$.

(6) Let $\vec{v}_1, \dots, \vec{v}_n$ be vectors in an inner product space V. Define the *Gram determinant* to be $G(\vec{v}_1, \dots, \vec{v}_n) \coloneqq \det(\langle \vec{v}_i, \vec{v}_j\rangle)$, i.e. the determinant of the matrix with entry $\langle \vec{v}_i, \vec{v}_j\rangle$ in the (i,j)-th component. Prove that $G(\vec{v}_1, \dots, \vec{v}_n) \neq 0$ *if and only if* $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a linearly independent set.

(7) Describe all least-square solutions of the linear system

$$\begin{aligned} x + y &= 1, \\ x + y &= -1. \end{aligned}$$

(8) Apply the Cauchy–Schwartz inequality (Problem 7 in the last homework) to show that for any continuous function $f\colon [a,b] \to \mathbb{R}$, we have

$$\left(\frac{1}{b-a}\int_a^b f(x)dx\right)^2 \leq \frac{1}{b-a}\int_a^b f(x)^2 dx.$$

In other words, the square of the average value of $f(x)$ on $[a,b]$ does not exceed the average value of $f(x)^2$. (Hint: Plug in f and 1 into the Cauchy–Schwartz inequality. Consider a suitable inner product on $\mathcal{C}[a,b]$.)

Part III (Extra credit problems, Due November 3, 8am PDT)

The following problem worths up to 2 extra points in total (out of 10), so you can potentially get 12/10 for this homework.

You have to submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW7_extra_credit**, which is separated from Part II.

(1) Let $V = \mathcal{C}[-1,1]$ be the vector space consisting of real-valued continuous functions on $[-1,1]$, with inner product given by $\langle f, g\rangle \coloneqq \int_{-1}^1 f(x)g(x)dx$. Consider the subspace

$$W = \Big\{f \in V\colon f(-x) = -f(x) \text{ for any } x \in [-1,1]\Big\} \subseteq V.$$

Find (with proof) the orthogonal complement of W in V.

(2) Continue the notations in Problem (6) of Part II. Suppose $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of W, where W is a subspace of V. Let $\vec{x}$ be any vector in V. Prove that

$$||\vec{x} - \mathrm{proj}_W \vec{x}||^2 = \frac{G(\vec{x}, \vec{v}_1, \ldots, \vec{v}_n)}{G(\vec{v}_1, \ldots, \vec{v}_n)}.$$