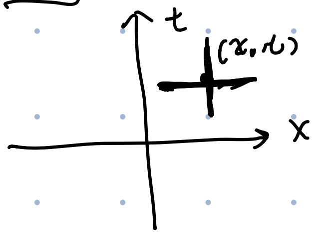
Today: • heat eq^n (a partial differential eq^n).
• Fourier series.

$u(x,t)$: $\mathbb{R}^2 \to \mathbb{R}$

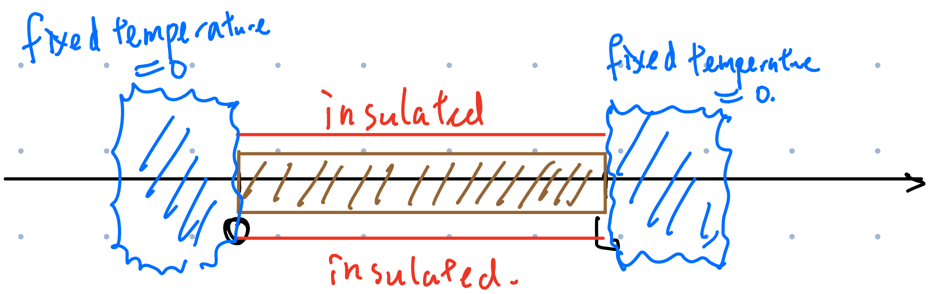
$$\boxed{\frac{\partial u}{\partial x}}(x_0,t_0) := \lim_{h\to 0} \frac{u(x_0+h,t_0)-u(x_0,t_0)}{h}$$



much more complicated than ODEs, e.g. Calabi conjecture, proved by Yau:
$\forall$ M compact Kähler manifold, $\forall f \in C^\infty(M)$,
there exists $\varphi \in C^\infty(M)$ s.t.

$$\partial\bar\partial\left(\log \frac{\det(g_{j\bar k}+\varphi_{j\bar k})}{\det(g_{j\bar k})} + f\right) = 0$$

1-D heat eq^n:



Initial condition: temperature at $x \in [0,L]$, at time $t=0$: given by $f(x)$. ; $f(0)=f(L)=0$

Let $\boxed{u(x,t)}$ = temperature at $x \in \underset{[0,L]}{\mathbb{R}}$, at time $t \geq 0$.

- $u(x,0) = f(x)$

$\frac{\partial u}{\partial t}$

- $u(0,t) = u(L,t) = 0 \quad \forall t \geq 0.$

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

Claim: $u_t(x,t) = \beta\, u_{xx}(x,t) \quad \forall x \in (0,L),\ t>0.$
for some $\beta > 0$

• $c :=$ specific heat of the material, $\rho :=$ density

- the total amount of heat contained in $[0,L]$, is:

$$H(t) = \int_0^L \underbrace{c\rho}_{>0}\, u(x,t)\, dx.$$

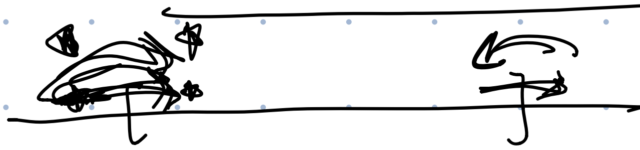
$\Rightarrow$ change of heat

$$\frac{dH}{dt} = \int_0^L c\rho\, u_t(x,t)\, dx.$$

- Fourier's law: heat flows from hot to cold regions at a rate $\kappa > 0$ proportional to the temperature gradient.

In our case,

$$\frac{dH}{dt} = \boxed{\kappa\left(-u_x(0,t) + u_x(L,t)\right)}$$



$$= \kappa \int_0^L u_{xx}(x,t)\, dx.$$

$$\Longrightarrow \quad c\rho\, u_t = \kappa\, u_{xx}.$$

$$\Longrightarrow \quad u_t = \beta u_{xx}, \text{ where } \beta = \frac{\kappa}{c\rho} > 0.$$

Question: $\beta > 0$. Find $u(x,t)$ s.t.

$$\begin{cases} u_t(x,t) = \beta\, u_{xx}(x,t), & \forall\, x \in (0,L),\ t>0. \\ u(0,t) = u(L,t) = 0 & \forall\, t>0 \\ u(x,0) = f(x) & \forall\, x \in (0,L). \end{cases}$$

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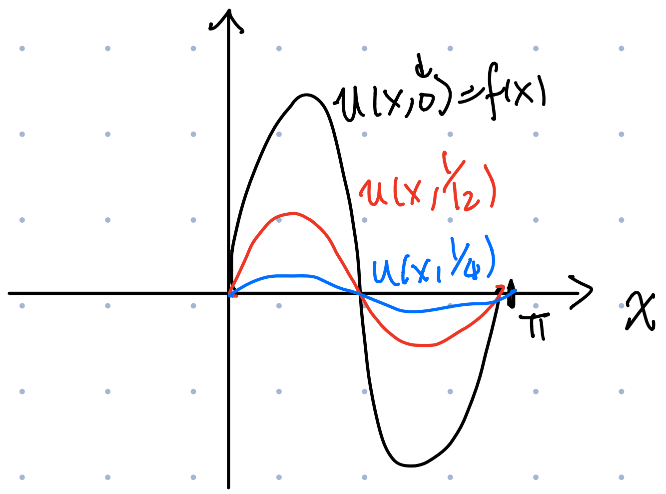
e.g. $L = \pi$, $f(x) = \sin(2x)$, $\beta = 3$.



$$\boxed{u(x,t) = \sin(2x)\cdot e^{-3\cdot 2^2 t}}$$

$u(x,t)$ converges (uniformly) to the zero fcn as $t \to +\infty$.

Uniqueness of sol$^{\text{n}}$ of heat eq$^{\text{n}}$: Energy method:

Suppose u_1, u_2 are both sol$^{\text{s}}$ to

$$\begin{cases} u_t(x,t) = \beta\, u_{xx}(x,t), & \forall\, x \in (0,L),\ t>0. \\ u(0,t) = u(L,t) = 0 & \forall\, t>0 \\ u(x,0) = f(x) & \forall\, x \in (0,L). \end{cases}$$

Define $w := u_1 - u_2$ (the goal: $w \equiv 0$).

$$\begin{cases} w_t(x,t) = \beta\, w_{xx}(x,t) \\ w(0,t) = w(L,t) = 0. \\ w(x,0) = 0 \end{cases}$$

$$E(t) := \boxed{\frac{1}{2}\int_0^L w(x,t)^2\,dx}$$

$$\frac{dE}{dt} = \cancel{\frac{1}{2}}\int_0^L \cancel{2}\, w w_t \,dx$$

heat eq'n

$$= \int_0^L w \cdot \beta w_{xx}\,dx.$$

Integrat'n by parts

$$= \beta\Big(\underbrace{w w_x \Big|_0^L}_{0} - \int_0^L w_x^2\,dx \Big)$$

because $w(0,t) = w(L,t) = 0$

$$= \beta\left(-\int_0^L w_x^2\,dx\right) \leq 0$$

$\Rightarrow$ $E(t)$ is decreasing.

Also, $E(0) = \frac{1}{2}\int_0^L \underbrace{w(x,0)}_{0}{}^2\,dx = 0$

$$\Rightarrow \quad E(t) \leq 0 \quad \forall t \geq 0.$$

$$\boxed{\frac{1}{2}\int_0^L w^2\,dx} \geq 0$$

(w is conti.)

$\Rightarrow$ $E(t) = 0$, $\Rightarrow$ $w(x,t) \equiv 0$ $\forall t, \forall x$.

$$\begin{cases} u_t(x,t) = \beta\, u_{xx}(x,t), & \forall\, x \in (0,L),\ t>0. \\ u(0,t) = u(L,t) = 0 & \forall\, t>0 \\ u(x,0) = f(x) & \forall\, x \in (0,L). \end{cases}$$

not the zero fcn

Method of separation of variables:

Assuming: $u(x,t) = \boxed{X(x)} \cdot \boxed{T(t)}$

$$u_t = \beta u_{xx}$$

$$\| \qquad\qquad \|$$

$$X \cdot T' \qquad \beta X'' \cdot T$$

$$\Rightarrow \boxed{\frac{T'(t)}{\beta T(t)} = \frac{X''(x)}{X(x)}} = \text{const.} = -\lambda$$

When can this hold?

$$\Rightarrow \begin{cases} X''(x) = -\lambda X(x) \\ T'(t) = -\beta\lambda T(t). \end{cases}$$

① $\boxed{u(0,t) = u(L,t) = 0}$

$$\| \qquad\qquad \|$$

$$\boxed{X(0)}\,\boxed{T(t)} \qquad \boxed{X(L)}\,T(t)$$

② $u(x,0) = f(x)$

$$\|$$

$X(x)\,T(0)$ Later

Suppose $X(0) \neq 0$ or $X(L) \neq 0$.

then $T(t) \equiv 0$, $\Rightarrow u(x,t) \equiv 0$ ✗

$\boxed{3.\ X(0) = X(L) = 0}$

$$\begin{cases} X'' = -\lambda X \\ \boxed{X(0) = X(L) = 0.} \end{cases}$$

HW: If the roots of the auxiliary eq'n $\boxed{r^2 = -\lambda}$ are real, then any sol'n can't take value 0 more than once.

$\Rightarrow \boxed{\lambda > 0}$

$\Rightarrow X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$$\begin{cases} 0 = X(0) = c_1 \\ 0 = X(L) = \underset{\neq 0}{c_2} \sin(\sqrt{\lambda}\, L) \end{cases}$$

$\Rightarrow \sqrt{\lambda} = \frac{n\pi}{L}$ for some $n = 1, 2, 3, \dots$

- $\lambda = \left(\frac{n\pi}{L}\right)^2$ for some $n \in \mathbb{N}$.
- in this case, $X(x) = C \sin\left(\frac{n\pi x}{L}\right)$

Still need to find $T(t)$ s.t.

$$\begin{cases} T'(t) = -\beta\lambda T(t) \\ X(x)T(0) = f(x). \end{cases}$$

$\lambda = \left(\frac{n\pi}{L}\right)^2$

$u(x,0) = f(x)$

$$T(t) = c \cdot e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$

$\cancel{T(0)c} \sin\left(\frac{n\pi x}{L}\right) \doteq f(x)$

const.

<u>Summarise</u>: If $f(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$,

then

$$u(x,t) = C_n \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$

is the sol^{n}.

(so far, we only solved the problem in a very particular case, i.e. $f = \text{const.} \sin\left(\frac{n\pi x}{L}\right)$)

$$\begin{cases} u_t(x,t) = \beta u_{xx}(x,t), & \forall x \in (0,L),\ t > 0. \\ u(0,t) = u(L,t) = 0 & \forall t > 0 \\ u(x,0) = f(x) & \forall x \in (0,L). \end{cases}$$

$[0,L]$

If $f(x) = \sum_n C_n \sin\left(\frac{n\pi x}{L}\right)$, Then $$u(x,t) = \sum_n c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$

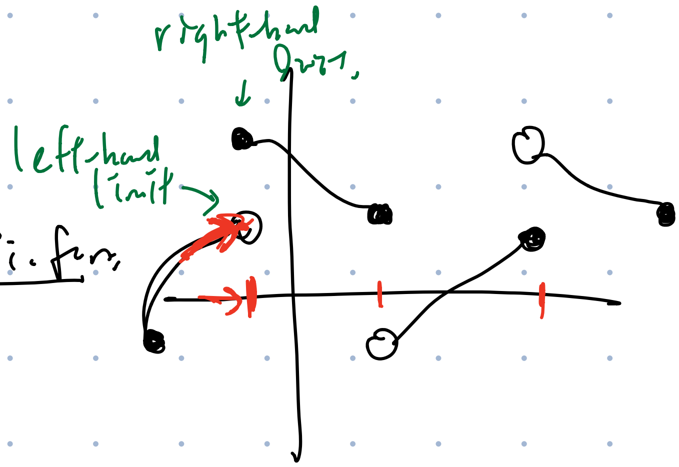
is the sol^n

<u>Question</u>: Can we write (or express as a limit) any fcn as $\sum_n c_n \sin\frac{n\pi x}{L}$?

→ <u>Fourier series</u>

$[-L, L]$

<u>Def</u>: piecewise <u>conti. fcn</u>



Fourier expansion ≡ orthogonal projection of fcn to a particular orthonormal basis in $\boxed{\mathcal{C}[-L,L]}$

<u>Inner product space</u>: $$\langle f,g\rangle := \frac{1}{L}\int_{-L}^{L} f(x)\,g(x)\,dx$$

There is an orthonormal basis:

$$\left\{\frac{1}{\sqrt{2}},\ \cos\frac{\pi x}{L},\ \sin\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \sin\frac{2\pi x}{L},\ \dots\right\}$$

If we do projection of f to these basis vectors,

$$\langle f, \tfrac{1}{\sqrt{2}} \rangle \cdot \tfrac{1}{\sqrt{2}} = \frac{1}{L}\left(\int_{-L}^{L} f(x) \cdot \frac{1}{\sqrt{2}}\,dx\right) \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx.$$

$$\langle f, \underset{\text{sin}}{\cos}\tfrac{n\pi x}{L} \rangle \underset{\text{sin}}{\cos}\frac{n\pi x}{L} = \frac{1}{L}\left(\int_{-L}^{L} f(x) \underset{\text{sin}}{\cos}\frac{n\pi x}{L}\,dx\right) \underset{\text{sin}}{\cos}\frac{n\pi x}{L}$$

<u>Def:</u> f piecewise conti. fun. on $[-L, L]$, the <u>Fourier series</u> of f is:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}\right),$$

where

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}\,dx, \qquad k \geq 0$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx. \qquad k \geq 1$$