#1.

(a) Write
$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

Then
$$\det(cA) = \det \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & & \vdots \\ ca_{n1} & \cdots & ca_{nn} \end{bmatrix}.$$

det is linear in each column (and each row)

$$= c \cdot \det \begin{bmatrix} a_{11} & ca_{12} & \cdots & ca_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & ca_{n2} & \cdots & ca_{nn} \end{bmatrix}$$

$$= \cdots = c^n \det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = c^n \det(A). \square$$

(b) $A = -A^T \Rightarrow \det(A) = \det(-A^T)$
$$= (-1)^n \det(A^T)$$
$$= -\det(A). \text{ since } n \text{ is odd.}$$
$\Rightarrow \det(A) = 0.$

$\Rightarrow$ A is not invertible. $\square$

#2:

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

Then $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$.

$$(\operatorname{tr} A)^2 - \operatorname{tr}(A^2) = (a+d)^2 - (a^2+bc+bc+d^2)$$
$$= 2(ad-bc)$$
$$= 2\det(A). \square$$

#3: $(\Rightarrow)$ $A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & & \vdots \\ \vdots & & \\ C_{1n} & \cdots & \end{pmatrix}$,

where C_{ij} is the (i,j)-th cofactor of A.

Since A has integral entries, it's clear that

$C_{ij} \in \mathbb{Z} \quad \forall i,j.$

Hence A^{-1} also has integral entries since $\det A = \pm 1$. $\square$

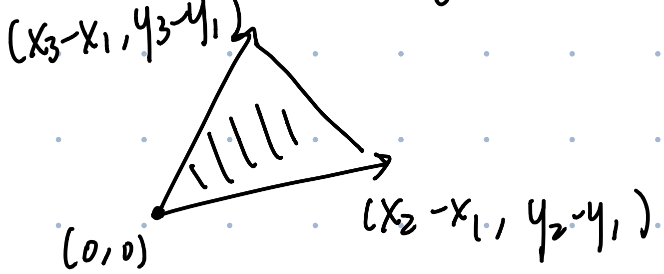
$(\Leftarrow)$. $AA^{-1} = I \Rightarrow \det(A)\det(A^{-1}) = \det(I) = 1$.

Since A, A^{-1} both have integral entries,

we have $\det(A), \det(A^{-1}) \in \mathbb{Z}$.

$\Rightarrow \det(A) = \pm 1$. $\square$

#4: Translate the triangle to:



Area of the triangle $= \frac{1}{2} \det \begin{bmatrix} x_2-x_1 & x_3-x_1 \\ y_2-y_1 & y_3-y_1 \end{bmatrix}$.

Observe that

$$\det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} x_1 & x_2-x_1 & x_3-x_1 \\ y_1 & y_2-y_1 & y_3-y_1 \\ 1 & 0 & 0 \end{bmatrix}$$

compute the cofactor exp.

$$= \det \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}. \quad \square$$

#5:

(a) not closed under scalar multiplication by $\mathbb{R}$.

(b) not closed under addition: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\det \neq 0 \quad \det \neq 0 \quad \det = 0.$

(c) not closed under addition: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det = 0 \quad \det = 0 \quad \det \neq 0.$

(d) not closed under addition: $(x-1) + (x-2) = (2x-3)$

$\downarrow \quad \downarrow \quad \downarrow$

$f(1) = 0 \quad f(2) = 0 \quad f(1) f(2) \neq 0.$

#6:

(a)(b) it's easy to check that $H_1 \cap H_2$ and $H_1 + H_2$ contain $\vec{0}$, and are closed under addition & scalar multi.

(c) 1) $H_1 + H_2 \subseteq \operatorname{Span}\{\vec{u}_1, \cdots, \vec{u}_k, \vec{w}_1, \cdots, \vec{w}_\ell\}$.

pf: $\forall \vec{v} \in H_1 + H_2$, $\exists\, \vec{v}_1 \in H_1, \vec{v}_2 \in H_2$ s.t. $\vec{v} = \vec{v}_1 + \vec{v}_2$.

Since $\vec{v}_1 \in H_1 = \operatorname{Span}\{\vec{u}_1, \cdots, \vec{u}_k\}$,

$\exists\, a_1, \cdots, a_k \in \mathbb{R}$ s.t. $\vec{v}_1 = a_1 \vec{u}_1 + \cdots + a_k \vec{u}_k$.

Similarly, $\exists\, b_1, \cdots, b_\ell \in \mathbb{R}$ s.t. $\vec{v}_2 = b_1 \vec{w}_1 + \cdots + b_\ell \vec{w}_\ell$.

$\Rightarrow \vec{v} = \vec{v}_1 + \vec{v}_2 = a_1 \vec{u}_1 + \cdots + a_k \vec{u}_k + b_1 \vec{w}_1 + \cdots + b_\ell \vec{w}_\ell \in \operatorname{Span}\{\vec{u}_1 \cdots \vec{u}_k, \vec{w}_1 \cdots \vec{w}_\ell\}$ $\square$

2) $\mathrm{Span}\{\vec{u}_1,\cdots,\vec{u}_k,\vec{w}_1,\cdots,\vec{w}_\ell\} \subseteq H_1+H_2$.

pf $\forall \vec{v} \in \mathrm{Span}\{\vec{u}_1,\cdots,\vec{u}_k,\vec{w}_1,\cdots,\vec{w}_\ell\}$,

$\exists\, a_1,\cdots,a_k,b_1,\cdots,b_\ell$ s.t.

$$\vec{v} = \underbrace{a_1\vec{u}_1+\cdots+a_k\vec{u}_k}_{\in H_1} + \underbrace{b_1\vec{w}_1+\cdots+b_\ell\vec{w}_\ell}_{\in H_2} \in H_1+H_2. \qquad \square$$

#7:

(a) • $\vec{0} \in T(U)$: since $\vec{0} \in U$ and $T(\vec{0}) = \vec{0}$.

• Suppose $\vec{w}_1, \vec{w}_2 \in T(U)$. Want to show: $\vec{w}_1+\vec{w}_2 \in T(U)$

$\vec{w}_1 = T(\vec{u}_1)$, $\vec{w}_2 = T(\vec{u}_2)$ for some $\vec{u}_1, \vec{u}_2 \in U$,

Since U is a subspace, $\vec{u}_1+\vec{u}_2 \in U$,

Hence $\vec{w}_1+\vec{w}_2 = T(\vec{u}_1)+T(\vec{u}_2) = T(\vec{u}_1+\vec{u}_2) \in T(U)$. $\square$

• Similarly, one can prove $c\vec{w} \in T(U)$ if $w \in T(U)$. $\square$

(b) • $\vec{0} \in T^{-1}(X)$: since $\vec{0} \in X$ and $T(\vec{0}) = \vec{0}$.

• Suppose $\vec{v}_1, \vec{v}_2 \in T^{-1}(X)$, Want to show: $\vec{v}_1+\vec{v}_2 \in T^{-1}(X)$.

$T(\vec{v}_1), T(\vec{v}_2) \in X$,

Since X is a subspace of W,

$T(\vec{v}_1+\vec{v}_2) = T(\vec{v}_1)+T(\vec{v}_2) \in X$,

Hence $\vec{v}_1+\vec{v}_2 \in T^{-1}(X)$. $\square$

• $c\vec{v}_1 \in T^{-1}(X)$ can be proved similarly. $\square$

#8:

(a) $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}.$

#9:

(a) It's not hard to prove by induction in m.

(b)

- $\det \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \det(A^2 - B^2)$ in general NOT true:

e.g.

$$\det \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}}^{A} & \boxed{\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}}^{B} \\ \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \end{bmatrix} \neq 0$$

But

$$\det\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^2 \right) = \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = 0.$$

- $\det((A+B)(A-B)) = \det(A^2 - B^2)$ in general NOT true:

e.g. Same example:

$$\det\left(\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A+B} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A-B} \right) = \det \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0.$$

- $\det\big((A+B)(A-B)\big) = \det(A+B)\det(A-B)$ true.

(c)

$$\det\left(\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}\begin{bmatrix} I & B \\ 0 & A-B \end{bmatrix}\begin{bmatrix} A+B & 0 \\ -I & I \end{bmatrix}\right) = \det\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}\det\begin{bmatrix} I & B \\ 0 & A-B \end{bmatrix}\det\begin{bmatrix} A+B & 0 \\ -I & I \end{bmatrix}$$

$$= \det\left(\begin{bmatrix} I & B \\ I & A \end{bmatrix}\begin{bmatrix} A+B & 0 \\ -I & I \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right)$$

$$\det\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}\det\begin{bmatrix} I & B \\ 0 & A-B \end{bmatrix}\det\begin{bmatrix} A+B & 0 \\ -I & I \end{bmatrix} = \det(A-B)\det(A+B).$$

$\square$