Math H54 First Midterm October 3, 2019

Name: Solution

- You have 80 minutes to complete the exam.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	20	
2	20	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	100	

1. (4 points each) Determine if each statement is TRUE or FALSE, and give a short justification.

(a) Let A and B be two $n \times n$ square matrices. If AB is an invertible matrix, then A and B must be invertible as well.

TRUE.

$\exists C$ s.t. $(AB)C = C(AB) = I_n$

$\Rightarrow A(BC) = I_n$, so A is invertible

$(CA)B = I_n$, so B is invertible.

(b) If the determinant of a 4×4 matrix A is 0, then $\dim \mathrm{Nul}(A) = 0$.

FALSE

$\det(A) = 0 \Leftrightarrow A$ is NOT invertible

$\Leftrightarrow \dim \mathrm{Null}(A) > 0$.

(c) Let A and B be two $n \times n$ square matrices. We have $(AB)^T = A^T B^T$.

FALSE

$$\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)^T \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) The set $\{p(t) : p(0) = 0 \text{ or } p(1) = 0\}$ of polynomials p satisfying $p(0) = 0$ or $p(1) = 0$ with the usual addition and scalar multiplication forms a vector space.

FALSE

x and $x-1$ are both in the set.

But $x + (x-1) = 2x - 1$ is not in the set.

(e) A set of n linearly independent vectors in $\mathbb{R}^n$ must form a basis of R^n.

TRUE

If the set doesn't span $\mathbb{R}^n$, then one can add another vector, get a linearly independent set of $n+1$ vectors in $\mathbb{R}^n$, which is impossible.

2. (20 points) Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & -1 \\ 1 & -1 & 2 & 4 & -1 \\ 2 & -2 & 1 & 5 & -2 \end{pmatrix}.$$

row reductions

$$\sim \begin{pmatrix} 1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccc} \uparrow & & \uparrow & & \\ \text{pivot} & & \text{pivot.} & & \end{array}$$

(a) Find a basis of the column space Col(A).

$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\} = \text{Col}(A).$$

(b) Find a basis of the null space Nul(A).

$$\text{Nul}(A) = \left\{ \begin{bmatrix} x_2 - 2x_4 + x_5 \\ x_2 \\ -x_4 \\ x_4 \\ x_5 \end{bmatrix} : x_2, x_4, x_5 \in \mathbb{R} \right\}$$

$$= \left\{ x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

3. (10 points) Recall that for any $m \times n$ matrix A, one can associate a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ that sends $\vec{x}$ to $A\vec{x}$. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{pmatrix}.$$

Find a matrix C such that $T_B \circ T_C = T_A$ (i.e. $T_B(T_C(\vec{x})) = T_A(\vec{x})$ for any $\vec{x} \in \mathbb{R}^3$).

$\Leftrightarrow BC = A \Leftrightarrow C = B^{-1}A$ if B is invertible.

$$\left(\underbrace{\begin{matrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{matrix}}_{B} \left| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{matrix}}_{A} \right.\right) \sim \left(\begin{matrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} 1 & 0 & 0 \\ -1 & -2 & 0 \\ 2 & 5 & 2 \end{matrix} \right.\right)$$

$$\sim \left(\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} -3 & -10 & -4 \\ 1 & 3 & 2 \\ 2 & 5 & 2 \end{matrix} \right.\right) \sim \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \left| \underbrace{\boxed{\begin{matrix} -4 & -13 & -6 \\ 1 & 3 & 2 \\ 2 & 5 & 2 \end{matrix}}}_{=\,C.} \right.\right)$$

4. (10 points) Let

$$A = \begin{pmatrix} a & 1 & 0 \\ 1 & b & 1 \\ 0 & 1 & c \end{pmatrix},$$

where a, b, c are real numbers. Suppose that the transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ sends the unit ball $\mathbb{B} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ to certain three dimensional region $T_A(\mathbb{B})$ with volume $\mathrm{Vol}(T_A(\mathbb{B})) = \frac{32}{3}\pi$. What can you say about the numbers a, b, c? Be as explicit as possible.
(Recall that $\mathrm{Vol}(\mathbb{B}) = \frac{4}{3}\pi$.)

$$\mathrm{Vol}\, T_A(\mathbb{B}) = |\det A|\, \mathrm{Vol}(\mathbb{B})$$

$$\Rightarrow |\det A| = 8.$$

$$\det A = abc - a - c$$

$$\Rightarrow \boxed{|abc - a - c| = 8.}$$

5. (10 points) Let $\mathcal{B} = \{-1+t, 1-2t\}$ and $\mathcal{C} = \{13-5t, 5-2t\}$ be two bases of the vector space P_1 of polynomials of degree ≤ 1. Suppose the coordinate vector of an element $\vec{x} \in P_1$ with respect to $\mathcal{B}$ is $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Find its coordinate vector $[\vec{x}]_{\mathcal{C}}$ with respect to $\mathcal{C}$.

$$\vec{x} = 3(-1+t) + (1-2t) = -2+t$$

Solve $\begin{bmatrix} 13 & 5 \\ -5 & -2 \end{bmatrix} [\vec{x}]_C = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$$[\vec{x}]_C = \begin{bmatrix} 13 & 5 \\ -5 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= -\begin{bmatrix} -2 & -5 \\ 5 & 13 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \square$$

6. (5 points each)

(a) Let A be an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be its associated linear transformation. Suppose that T_A is injective. What can you say about the relation between m and n? What can you say about the dimensions of the column space $\mathrm{Col}(A)$ and the null space $\mathrm{Nul}(A)$? ~~You have to provide proofs for your answers.~~

T_A inj $\Leftrightarrow$ pivot in each column

$\Rightarrow \boxed{m \geq n.}$

$\boxed{\dim \mathrm{Nul}(A) = 0}$ ($\Leftrightarrow$ T_A inj.)

$\boxed{\dim \mathrm{Col}(A) = n}$ By rank thm.

(b) Same problems as in part (a), but replace injective by surjective.

T_A surj. $\Rightarrow$ pivot in each row

$\Updownarrow$ $\Rightarrow \boxed{m \leq n}$

$\mathrm{Col}(A) = \mathbb{R}^m$ $\Leftrightarrow$ $\boxed{\dim \mathrm{Col}(A) = m.}$

$\boxed{\dim \mathrm{Nul}(A) = n - m}$ by rank thm.

7. (10 points) Let V be a vector space and $\{v_1, \ldots, v_n\}$ be a linearly independent set in V. Suppose that $w \in V$ and $w \notin \text{Span}\{v_1, \ldots, v_n\}$. Prove that $\{v_1, \ldots, v_n, w\}$ is a linearly independent set.

~~(Hint: Prove by contradiction.)~~

Suppose $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + bw = 0$ for some $a_1, \cdots, a_n, b \in \mathbb{R}$.

Case 1: $b \neq 0$, then $w = -\frac{a_1}{b} v_1 - \frac{a_2}{b} v_2 - \cdots - \frac{a_n}{b} v_n$.

contradicts with $w \notin \text{Span}\{v_1, \cdots, v_n\}$.

Case 2: $b = 0$, then $a_1 v_1 + \cdots + a_n v_n = 0$.

Since $\{v_1, \cdots, v_n\}$ is linearly independent,

So $a_1 = \cdots = a_n = 0$.

$\square$

8. (10 points) Let A be an $m \times n$ matrix and B be an $n \times n$ square matrix. Show that $\dim \mathrm{Nul}(AB) \geq \dim \mathrm{Nul}(A)$. What can you say about the relation between $\dim \mathrm{Nul}(AB)$ and $\dim \mathrm{Nul}(A)$ if B is invertible?

Since AB and A are both $m \times n$ matrices.

By rank thm, It suffices to show that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.

This follows from the fact that the columns of AB are in $\mathrm{Col}(A)$,

therefore $\mathrm{Col}(AB) \subset \mathrm{Col}(A)$. $\square$

If B is invertible, then we have

$$\dim \mathrm{Nul}(A) = \dim \mathrm{Nul}(ABB^{-1}) \geq \dim \mathrm{Nul}(AB) \geq \dim \mathrm{Nul}(A)$$

$$\Rightarrow \quad \dim \mathrm{Nul}(AB) = \dim \mathrm{Nul}(A). \quad \square$$