HOMEWORK 9
MATH H54

Yu-Wei's Office Hours: Sunday 1-2:30pm and Friday 12-1:30pm (PST)
Michael's Office Hours: Monday 12-3pm (PST)

Some ground rules:

- You have to submit your solutions via **Gradescope**, to the assignment **HW9**.
- The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own.
- You're allowed to use any result that is proved in the lecture. But if you'd like to use other results, you have to prove it first before using it.

Problems: (mostly taken from the textbook)
You have to write down your computations, not just the final answers.

(1) Find general solutions to the following differential equations:
  (a) $6y'' + y' - 2y = 0$.
  (b) $4y'' + 20y' + 25y = 0$.
  (c) $y'' + 4y' + 8y = 0$.
  (d) $y''(t) + 4y(t) = \sin t - \cos t$.
  (e) $y''(t) - 2y'(t) + y(t) = t^{-1}e^t$.
  (f) $y''(t) + 16y(t) = \sec(4t)$.

(2) Solve the following initial value problems:
  (a) $y'' - 4y' - 5y = 0$; $y(-1) = 3$ and $y'(-1) = 9$.
  (b) $y'' + 2y' + 2y = 0$; $y(0) = 2$ and $y'(0) = 1$.
  (c) $y''(t) + y'(t) - 12y(t) = e^t + e^{2t} - 1$; $y(0) = 1$ and $y'(0) = 3$.

(3) When the values of a solution to a differential equation are specified at *two different points*, these conditions are called *boundary conditions.* The purpose of this problem is to show that for boundary value problems there is no existence-uniqueness theorem.
  (a) Find general solutions to the differential equation:

$$y'' + y = 0. \tag{*}$$

  (b) Show that there is a unique solution to (*) that satisfies the boundary conditions $y(0) = 2$ and $y(\pi/2) = 0$.

(c) Show that there is no solution to (*) that satisfies $y(0) = 2$ and $y(\pi) = 0$.

(d) Show that there are infinitely many solutions to (*) that satisfy $y(0) = 2$ and $y(\pi) = -2$.

(4) One way to define the *hyperbolic functions* is by means of differential equations. Consider the differential equation:

$$y'' - y = 0. \tag{**}$$

The *hyperbolic cosine*, denoted $\cosh t$, is defined as the solution of (**) subject to the initial values: $y(0) = 1$ and $y'(0) = 0$. The *hyperbolic sine*, denoted $\sinh t$, is defined as the solution of (**) subject to the initial values: $y(0) = 0$ and $y'(0) = 1$.

(a) Solve these two initial value problems to derive explicit formulas for $\cosh t$ and $\sinh t$. Also, show that $(\cosh t)' = \sinh t$ and $(\sinh t)' = \cosh t$.

(b) Prove that a general solution of (**) is given by $y(t) = c_1 \cosh t + c_2 \sinh t$.

(5) To see the effect of changing the coefficient b in the initial value problem

$$y'' + by' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 0,$$

solve the problem for $b = 5$, $b = 4$, and $b = 2$, and sketch the solutions.

(6) Prove the *sum of angles formula* for the sine function by following these steps. Fix $x \in \mathbb{R}$.

(a) Let $f(t) = \sin(x + t)$. Verify that $f''(t) + f(t) = 0$, $f(0) = \sin x$, and $f'(0) = \cos x$.

(b) Solve the initial value problem: $y'' + y = 0$; $y(0) = \sin x$ and $y'(0) = \cos x$.

(c) By uniqueness, the solution in Part (b) is the same as $f(t)$ from Part (a). Write this equality; this should be the standard sum of angles formula for $\sin(x+t)$.

(7) All that is known concerning a mysterious second-order constant-coefficient differential equation $y'' + by' + cy = f(t)$ is that $t^2 + 1 + e^t \cos t$, $t^2 + 1 + e^t \sin t$, and $t^2 + 1 + e^t \cos t + e^t \sin t$ are solutions.

(a) Determine two linearly independent solutions to the corresponding homogeneous equation $y'' + by' + cy = 0$.

(b) Find a suitable choice of $b, c \in \mathbb{R}$ and function $f(t)$ that enables these solutions.

(8) Use the method of variation of parameters to show that

$$y(t) = c_1 \cos t + c_2 \sin t + \int_0^t f(s) \sin(t - s) ds$$

is a general solution to the differential equation $y'' + y = f(t)$.

(9) Suppose the auxiliary equation $r^2 + br + c = 0$ of the differential equation $y'' + by' + cy = 0$ have two real roots. Prove that a nonzero solution to the differential equation can take the value 0 at most once, i.e. if $y(t)$ is a nonzero solution, then there is at most one point $t_0 \in \mathbb{R}$ such that $y(t_0) = 0$.