HOMEWORK 4
MATH H54, FALL 2021

DUE SEPTEMBER 28, 11AM

Some ground rules:

- Please submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW4**.
- The submission should be a **single PDF file**.
- Late homework will not be accepted/graded under any circumstances.
- Make sure the writing in your submission is clear enough. Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own, and acknowledge the students with whom you worked.
- **For True/False questions**: You have to prove the statement if your answer is "True"; otherwise, you have to provide an explicit counterexample and justification.
- You are allowed to use any result that is proved in the lecture. But if you would like to use other results, you have to prove it first before using it.

Problems:

(1) Let $T\colon V \to W$ be a linear transformation between vector spaces (not necessarily finite dimensional). Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a subset of vectors in V.

 (a) Prove that if $\{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ is a linearly independent set, then $\{\vec{v}_1, \dots, \vec{v}_n\}$ also is linearly independent.

 (b) Prove that if T is injective, then the converse also is true, namely:

$$\text{“}\{\vec{v}_1, \dots, \vec{v}_n\} \text{ is linearly independent”} \implies \text{“}\{T(\vec{v}_1), \dots, T(\vec{v}_n)\} \text{ is linearly independent”}.$$

(2) Prove that the vector space Poly of all polynomials is an infinite-dimensional space, i.e. any finite set of polynomials does not form a basis of Poly.

(3) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. In this problem, you'll prove that $\operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

 (a) Prove that $\operatorname{rank}(AB) \le \operatorname{rank}(A)$. (Hint: Consider the column spaces.)

 (b) Prove that $\dim \operatorname{Nul}(AB) \ge \dim \operatorname{Nul}(B)$.

 (c) Use (b) to prove that $\operatorname{rank}(AB) \le \operatorname{rank}(B)$. (Hint: Rank-nullity theorem.)

 (d) Suppose that A is an invertible square matrix. Prove that $\operatorname{rank}(AB) = \operatorname{rank}(B)$. Similarly, suppose B is an invertible square matrix, show that $\operatorname{rank}(AB) = \operatorname{rank}(A)$.

(4) Let $T\colon V \to W$ be a linear transformation between finite dimensional vector spaces, and let $H \subseteq V$ be a subspace. We know from last homework that $T(H)$ is a subspace of W. Prove that $\dim T(H) \leq \dim H$. (Hint: Spanning set theorem.)

(5) Let $\mathcal{B} = \{-1+t, 1-2t\}$ and $\mathcal{C} = \{13-5t, 5-2t\}$ be two bases of the vector space $\text{Poly}_{\leq 1}$. Let $\vec{x} \in \text{Poly}_{\leq 1}$ be a polynomial with $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\vec{x}]_{\mathcal{C}}$, the coordinate vector of $\vec{x}$ with respect to the basis $\mathcal{C}$.

(6) Prove that $\text{rank}(A) = \text{rank}(A^T)$. (Hint: Recall that $\text{rank}(A)$ is the dimension of the column space of A, which we proved that is the same as the number of pivots of A. Hence it suffices to prove that the dimension of the row space of A coincides with the number of pivots. One can prove this by showing that the row operations do not change the row space.)

(7) Let A be a *real* $m \times n$ matrix. Prove that $\text{rank}(A^T A) = \text{rank}(A)$. (Hint: If $A^T A\vec{v} = \vec{0}$, then $\vec{0} = \vec{v}^T A^T A\vec{v} = (A\vec{v})^T(A\vec{v})$, and deduce that $A\vec{v} = \vec{0}$.)

(8) Let H_1 and H_2 be subspaces of a finite dimensional vector space of V. Recall the definition of the subspaces $H_1 \cap H_2$ and $H_1 + H_2$ of V from last homework. Prove that

$$\dim(H_1 + H_2) + \dim(H_1 \cap H_2) = \dim H_1 + \dim H_2.$$

(Hint: Start with a basis $\{x_1, \ldots, x_n\}$ of $H_1 \cap H_2$. Show that it can be complemented to a basis $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ of H_1 and to a basis $\{x_1, \ldots, x_n, z_1, \ldots, z_k\}$ of H_2. Then show that $\{x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_k\}$ is a basis of $H_1 + H_2$.)

(9) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that

$$\text{rank}(A) + \text{rank}(B) \leq \text{rank}(AB) + n.$$

(Hint: Consider the restriction of $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ to the subspace $\text{Col}(B) \subseteq \mathbb{R}^n$, i.e. consider

$$T_A|_{\text{Col}(B)}\colon \text{Col}(B) \to \mathbb{R}^m, \text{ which sends } \vec{x} \mapsto A\vec{x}.$$

Apply the rank-nullity theorem to the linear map $T_A|_{\text{Col}(B)}$. What's $\text{Im}(T_A|_{\text{Col}(B)})$?)