Today: subseq., intro. to metric spaces
Next week: More on metric spaces
The week after next week: Review of 1st midterm & 1st midterm.

Bolzano-Weierstrass thm. Any bdd seq. in $\mathbb{R}$ has a convergent subseq.

1st proof): It suffices to show the following:

Lemma: Any bdd seq. in $\mathbb{R}$ has a monotone subseq.
($\Rightarrow$ B-W thm) (a_n) (so a_n is not "big" $\Leftrightarrow$ $\exists\, m > n$ s.t. $a_n < a_m$)

Def: Say a_n is "big" if $\boxed{a_n \geq a_m \;\; \forall\, n < m.}$

pf of Lemma: given any bdd seq. (a_n), there are 2 cases:

Case 1: There are infinitely many "big" terms in the seq.

Let's say the "big" terms are $a_{k_1}, a_{k_2}, a_{k_3}, \ldots$ where $1 \leq k_1 < k_2 < k_3 < \ldots$

this is a decreasing subseq. of (a_n).

Case 2: There is only finitely many "big" terms in (a_n)

$\exists\, N > 0$ s.t. a_n is NOT "big" $\forall\, n > N$.

Start with any $k_1 > N$, so a_{k_1} is NOT "big"

So, $\exists\, k_2 > k_1$ s.t. $a_{k_1} < a_{k_2}$

Since $k_2 > k_1 > N$, so a_{k_2} is NOT "big"

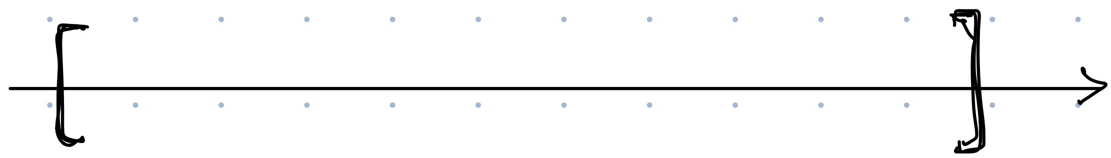
So, $\exists\, k_3 > k_2$ s.t. $a_{k_2} < a_{k_3}$.

$\vdots$

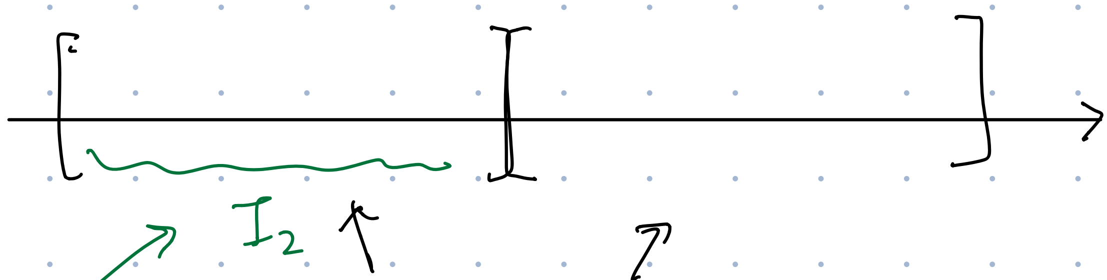
We can continue this process to construct an increasing subseq. of (a_n). $\square$

<u>2nd proof)</u> (a_n) bdd seq.

$\text{length}(I_1) = L$



$\{a_n : n \in \mathbb{N}\} \subseteq I_1$



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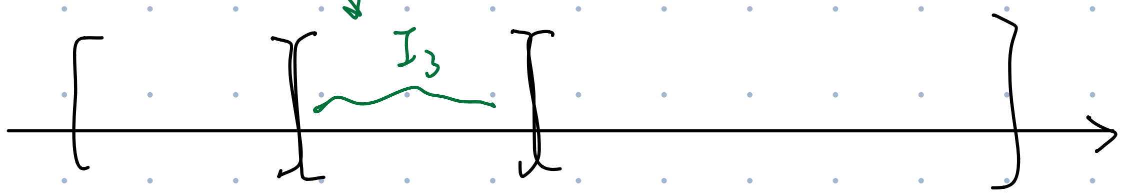
at least one of these two subintervals contains infinitely many $a_1, a_2, \cdots$

it contains ∞'ly many terms in (a_n)



$\Rightarrow$ We can construct a "nested intervals"

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \quad \text{s.t.}$$

- each I_n contains ∞'ly many terms of (a_n)
- $\text{length}(I_n) = \dfrac{L}{2^{n-1}}$

Claim: $\exists$ subseq. (a_{k_n}) of (a_n)

s.t. $a_{k_n} \in I_n \quad \forall n$

(i.e. $a_{k_1} \in I_1$, $a_{k_2} \in I_2$, ...)

pf: • Pick any k_1 s.t. $a_{k_1} \in I_1$

• $\exists k_2 > k_1$ s.t. $a_{k_2} \in I_2$

(since I_2 contains ∞ 'ly many terms in ~~the~~ (a_n)).

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$\square$

Claim: Such (a_{k_n}) is convergent:

pf: $\forall N$, $n > N$, we know:

$$a_{k_n} \in I_n \subseteq I_N$$

$\Rightarrow \forall n, m > N$,

Since $a_{k_n}, a_{k_m} \in I_N$, and $\text{length}(I_N) = \frac{L}{2^{N-1}}$

So, $|a_{k_n} - a_{k_m}| < \frac{L}{2^{N-1}}$

$\forall \varepsilon > 0$, Choose $N > 0$ large s.t. $\frac{L}{2^{N-1}} < \varepsilon$.

Then $\forall n, m > N$,

we have: $|a_{k_n} - a_{k_m}| < \frac{L}{2^{N-1}} < \varepsilon$.

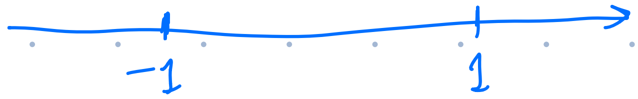
So, (a_{k_n}) Cauchy $\Rightarrow$ conv. $\square$

Thm: (a_n) bdd in $\mathbb{R}$,

$\exists$ subseq. (a_{k_n}) of (a_n) s.t. it converges to $\liminf a_n$

$\exists$ — — — — — — — — — — — $\limsup a_n$

(e.g. $(a_n) = (-1, 1, -1, 1, -1, 1, \dots)$)



Lemma: (a_n) bdd in $\mathbb{R}$.

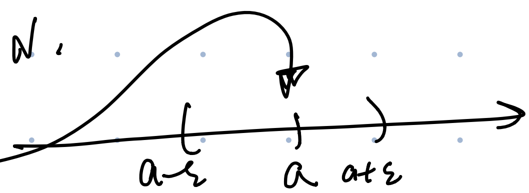
"$\exists$ subseq. (a_{k_n}) conv. to $a \in \mathbb{R}$" $\iff$

"$\forall \varepsilon > 0$, the set $\{n \in \mathbb{N} : |a_n - a| < \varepsilon\}$ is infinite".

pf. ($\Rightarrow$) $\forall \varepsilon > 0$, $\exists N > 0$ s.t.

$$|a_{k_n} - a| < \varepsilon \quad \forall n > N.$$

$$\boxed{a_{k_{N+1}},\ a_{k_{N+2}},\ \dots}$$



$$\{k_{N+1}, k_{N+2}, \dots\} \subseteq \{n \in \mathbb{N} : |a_n - a| < \varepsilon\}$$

($\Leftarrow$) Apply the cond^{le} to $\varepsilon = 1$:

$\Rightarrow \{n \in \mathbb{N} : |a_n - a| < 1\}$ is infinite

pick any $k_1 \in$

so, $|a_{k_1} - a| < 1$.

Apply for $\varepsilon = \frac{1}{2}$: $\{n \in \mathbb{N} : |a_n - a| < \frac{1}{2}\}$ is infinite

$\Downarrow$

$\exists\ k_2 > k_1$ s.t. k_2

$$|a_{k_2} - a| < \frac{1}{2}$$

Continue this process, we can find

$$1 \le k_1 < k_2 < k_3 < \cdots$$

s.t. $\quad |a_{k_n} - a| < 1/n.$

$\Rightarrow \quad \lim a_{k_n} = a. \quad \square$

~~pf of thm~~: Let's prove for $\boxed{\limsup_{n\to\infty} a_n = z}$

By lemma, it suffices to prove the following:

(*) "$\forall \varepsilon > 0$, the set $\{n \in \mathbb{N} : |a_n - z| < \varepsilon\}$ is infinite".

Recall: $\boxed{z = \limsup a_n} = \lim_{N\to\infty} S_N$

$$= \lim_{N\to\infty} \left(\sup \{a_n : n > N\} \right)$$

So $\boxed{\forall \varepsilon > 0, \ \exists N_1 > 0}$ s.t.

(**)

$$z \le \underline{S_{N_1}} < z + \varepsilon.$$

$$S_{N_1} = \sup\{a_n : n > N_1\}.$$

So, $\boxed{a_n < z + \varepsilon \quad \forall n > N_1.}$

We'll prove (*) by contradiction.

Suppose (*) is NOT true, i.e.

$\exists \varepsilon > 0$ s.t. $\{n \in \mathbb{N} : |a_n - z| < \varepsilon\}$ is finite

$\Rightarrow \exists N_2 > 0$ s.t.

"$|a_n - z| < \varepsilon$" is not true $\forall n > N_2$

$\Rightarrow$ $\boxed{\text{"} a_n \geq z + \varepsilon \text{ or } a_n \leq z - \varepsilon \text{"} \quad \forall n > N_2}$

By the previous argument (**), $\exists N_1 > 0$

s.t. $\boxed{a_n < z + \varepsilon \quad \forall n > N_1}$

Let's ~~pick~~ $N := \max\{N_1, N_2\}$. Then

$\forall n > N$, we have:

- "$\cancel{a_n \geq z + \varepsilon}$ or $a_n \leq z - \varepsilon$"
- "$a_n < z + \varepsilon$"

$\Rightarrow \quad a_n \leq z - \varepsilon \quad \forall n > N.$

$\Longrightarrow \qquad S_N = \sup\{a_n : n > N\} \le z - \varepsilon$

$//$

$\limsup a_n$

$//$

z

Contradiction. $\square$

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§ Metric spaces: (simplest mathematical structure that allows us to talk about "distance" ⇝ "ε-nbd", "conv.", ...)

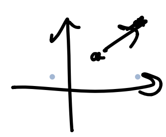
Def: A metric space (S, d) is a set S with a distance fcn $d : S \times S \longrightarrow \mathbb{R}_{\ge 0}$ s.t.

1) $d(x,x) = 0 \quad \forall x \in S.$
2) $d(x,y) > 0 \quad \forall x, y \in S$ and $x \ne y.$
3) $d(x,y) = d(y,x) \quad \forall x, y \in S$
4) $d(x,z) \le d(x,y) + d(y,z) \quad \forall x, y, z \in S$

Q: Any examples of metric spaces??

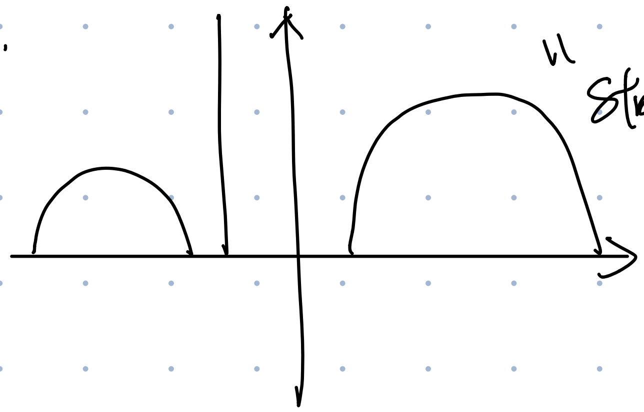
e.g. $(\mathbb{R}, d)$, where $d(x,y) = |x-y|$ "d_{std}" standard dist. fcn. on $\mathbb{R}$.

e.g. $(\mathbb{R}^2, d_{std})$, where $d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$



e.g. $(\mathbb{R}^n, d_{Euc})$, where $d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n))$
$= \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$

e.g. (non-Euclidean metric) $S = \mathbb{H} = \{(x,y) \in \mathbb{R}^2, y > 0\}$
$\exists$ a dist. fn d_{hyp} (hyperbolic metric on $\mathbb{H}$)
s.t.



"straight lines" in $(\mathbb{H}, d_{hyp})$

e.g. On $\mathbb{R}$, there are lots of possible dist. fns beside the standard one.

e.g. $\tilde{d}(x,y) = 2|x-y|$
$\tilde{\tilde{d}}(x,y) = \sqrt{|x-y|}$ (for x,y,z distinct pts in $\mathbb{R}$
$\tilde{\tilde{d}}(x,y) + \tilde{\tilde{d}}(y,z) > \tilde{\tilde{d}}(x,z)$)