16. Determine the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ when:

(a) $a_n = (\log n)^2$

(c) $a_n = \frac{n^2}{4^n+3n}$

(e) Find the radius of convergence of the **hypergeometric series**

$$F(\alpha,\beta,\gamma;z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n.$$

Here $\alpha, \beta \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, \ldots$.

Sol$^{\underline{n}}$:

(a) $\log\left(|a_n|^{1/n}\right) = \log\left((\log n)^{2/n}\right) = \frac{2}{n}\log(\log n)$

Hence $\lim_{n\to\infty} \log\left(|a_n|^{1/n}\right) = 0 \Rightarrow \lim_{n\to\infty} |a_n|^{1/n} = 1.$

$\Rightarrow$ Radius of conv. $R = 1/1 = 1.$ $\square$

(c) $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \frac{n^{2/n}}{(4^n+3n)^{1/n}} = \frac{1}{4}$

$\Rightarrow R = 4.$ $\square$

(e) Use #17:

Write $a_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\cdot\gamma(\gamma+1)\cdots(\gamma+n-1)}$

Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}\right| = \left|\frac{(1+\frac{\alpha}{n})(1+\frac{\beta}{n})}{\frac{n+1}{n}\cdot(1+\frac{\gamma}{n})}\right|$

$\longrightarrow 1$ as $n\to\infty$

By #17, $R = 1/1 = 1.$ $\square$

17. Show that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

then

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

In particular, this exercise shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

Sol^{n}: Since $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$,

$\forall \varepsilon > 0$, $\exists N > 0$ s.t.

$$L-\varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < L+\varepsilon \quad \forall n > N.$$

Then for any $n > N$, we have

$$(L-\varepsilon)^{n-N} < \left|\frac{a_n}{a_N}\right| = \left|\frac{a_n}{a_{n-1}}\right|\left|\frac{a_{n-1}}{a_{n-2}}\right|\cdots\left|\frac{a_{N+1}}{a_N}\right| < (L+\varepsilon)^{n-N}$$

$$\Rightarrow (L-\varepsilon)^{n-N}\cdot|a_N| < |a_n| < (L+\varepsilon)^{n-N}\cdot|a_N|$$

$$\Rightarrow \underbrace{(L-\varepsilon)^{\frac{n-N}{n}}}_{\downarrow\, L-\varepsilon}\ \underbrace{|a_N|^{\frac{1}{n}}}_{\downarrow\, 1} < |a_n|^{\frac{1}{n}} < \underbrace{(L+\varepsilon)^{\frac{n-N}{n}}}_{\downarrow\, L+\varepsilon}\ \underbrace{|a_N|^{1/n}}_{\downarrow\, 1}.$$

as $n\to\infty$

$$\Rightarrow L-\varepsilon \leq \liminf_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} |a_n|^{1/n} \leq L+\varepsilon.$$

Since this holds for any $\varepsilon > 0$, we have

$$L = \liminf |a_n|^{1/n} = \limsup |a_n|^{1/n} = L,$$

Hence $\lim |a_n|^{1/n} = L$. $\square$

19. Prove the following:

(a) The power series $\sum nz^n$ does not converge on any point of the unit circle.

(b) The power series $\sum z^n/n^2$ converges at every point of the unit circle.

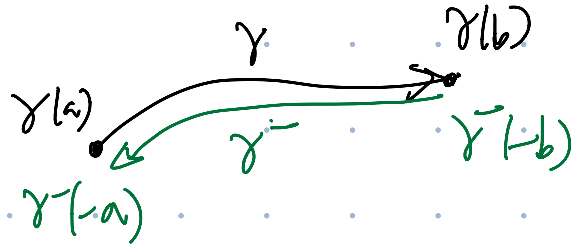
Sol$^{\underline{n}}$: (a) $\forall |z|=1$, we have $|nz^n| = n$.
hence $\sum nz^n$ doesn't converge on $|z|=1$. $\square$

(b) $\forall |z|=1$, $\sum \frac{|z|^n}{n^2} = \sum \frac{1}{n^2} < +\infty$

Hence $\sum \frac{z^n}{n^2}$ converges (absolutely) on $|z|=1$. $\square$

24. Let γ be a smooth curve in $\mathbb{C}$ parametrized by $z(t) : [a,b] \to \mathbb{C}$. Let γ^- denote the curve with the same image as γ but with the reverse orientation. Prove that for any continuous function f on γ

$$\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz.$$



Sol$^{\underline{n}}$: γ^- can be parametrized by $\gamma^- : [-b,-a] \longrightarrow \mathbb{C}$
$t \longmapsto \gamma^-(t) := \gamma(-t)$

$$\int_{\gamma^-} f(z)\,dz = \int_{-b}^{-a} f(\gamma^-(t))\,\frac{d}{dt}\left(\gamma^-(t)\right)dt$$

$$= \int_{-b}^{-a} f(\gamma(-t))\,\frac{d}{dt}\left(\gamma(-t)\right)dt$$

$$= \int_{-b}^{-a} f(\gamma(-t))\,\gamma'(-t)\cdot(-1)\,dt$$

$$= -\int_{-b}^{-a} f(\gamma(-t))\,\gamma'(-t)\,dt$$

$$= -\int_b^a f(\gamma(s))\gamma'(s)(-1)\,ds$$

$$= \int_b^a f(\gamma(s))\gamma'(s)\,ds$$

$$= -\int_a^b f(\gamma(s))\gamma'(s)\,ds = -\int_\gamma f(z)\,dz.$$

$\square$

25. The next three calculations provide some insight into Cauchy's theorem, which we treat in the next chapter.

(a) Evaluate the integrals

$$\int_\gamma z^n\,dz$$

for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

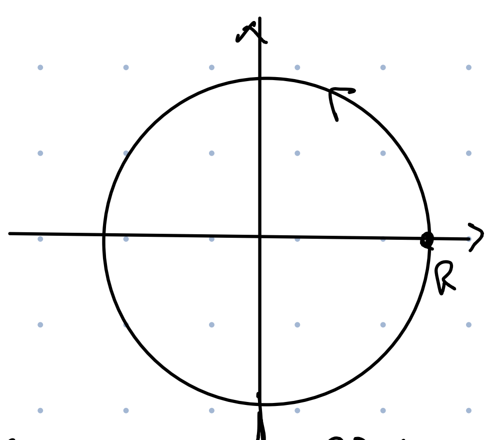
(b) Same question as before, but with γ any circle not containing the origin.

(c) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)}\,dz = \frac{2\pi i}{a-b},$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

(a)



$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$$
$$t \longmapsto \gamma(t) = Re^{it}.$$

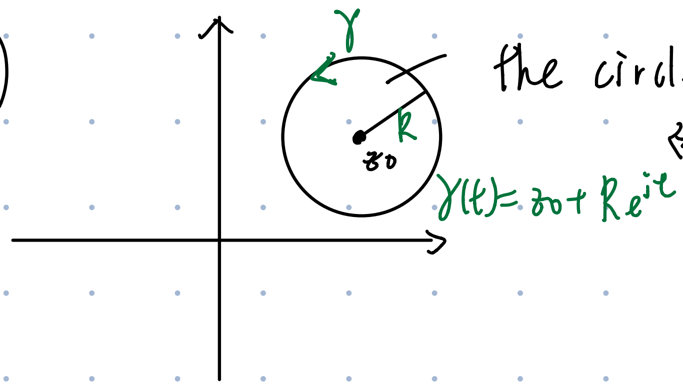
$$\int_\gamma z^n\,dz = \int_0^{2\pi} (Re^{it})^n \frac{d}{dt}(Re^{it})\,dt$$

$$= \int_0^{2\pi} R^n e^{int} \cdot R \cdot e^{it} \cdot i\,dt$$

$$= i R^{n+1} \int_0^{2\pi} e^{i(n+1)t}\, dt$$

$$= \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1. \end{cases} \qquad \square$$

(b)



the circle doesn't contain 0 $\iff R < |z_0|$

For $n \neq -1$, z^n has primitive $\frac{z^{n+1}}{n+1}$ defining on $\mathbb{C} \setminus \{0\}$,

hence $\int_\gamma z^n = 0$.

For $n = -1$,

$$\int_\gamma z^{-1} dz = \int_0^{2\pi} \frac{iRe^{it}}{z_0 + Re^{it}}\, dt = iR \int_0^{2\pi} \frac{e^{it}}{1 + \frac{R}{z_0} e^{it}}\, dt$$

Claim: $\int_0^{2\pi} \frac{e^{it}}{1 + \frac{R}{z_0} e^{it}}\, dt = 0$. (therefore $\int_\gamma z^{-1} dz = 0$).

pf: Since $R < |z_0|$, we have $\frac{1}{1 + \frac{R}{z_0} e^{it}} = \sum_{n=0}^{\infty} \left(\frac{-R}{z_0} e^{it}\right)^n$

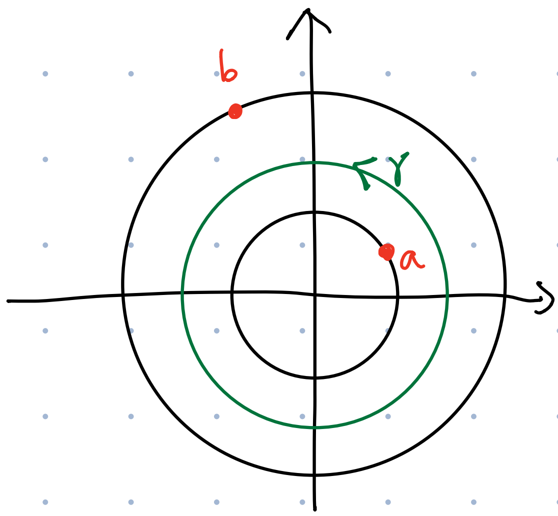
Hence $\int_0^{2\pi} \frac{e^{it}}{1 + \frac{R}{z_0} e^{it}}\, dt = \int_0^{2\pi} \sum_{n=0}^{\infty} \left(\frac{-R}{z_0}\right)^n e^{it(n+1)}\, dt.$

Note that $\displaystyle\int_0^{2\pi} \sum_{n=0}^{\infty} \left| \left(\frac{-R}{z_0}\right)^n e^{it(n+1)} \right| dt < +\infty.$

Hence one can exchange the integration & summation; so

$$\int_0^{2\pi} \frac{e^{it}}{1+\frac{R}{z_0}e^{it}}\,dt = \sum_{n=0}^{\infty} \int_0^{2\pi} \left(\frac{-R}{z_0}\right)^n e^{it(n+1)}\,dt = 0. \quad \square$$

(c)



$$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{1}{b-a}\left(\int_\gamma \frac{dz}{z-b} - \int_\gamma \frac{dz}{z-a}\right)$$

$$\int_\gamma \frac{dz}{z-b} = \int \frac{dz}{z}$$

Circle centered at b, with radius r

doesn't contain 0 (in its interior) therefore the integration is 0 by (b),

$$\int_\gamma \frac{dz}{z-a} = \int \frac{dz}{z}$$

Circle centered at a, with radius r.

contains 0. (in its interior). therefore the integration is $2\pi i$ by (a).

$$= \frac{1}{b-a}\left(0 - 2\pi i\right) = \frac{2\pi i}{a-b}. \quad \square$$