(4) Let $S$ be the set of nonempty compact subsets of $\mathbb{R}^2$. For any $r>0$ and $K \in S$, we define the $r$-*neighborhood* of $K$ to be

$$\boxed{B_r(K) := \{x \in \mathbb{R}^2 : d(x,a) < r \text{ for some } a \in K\} = \bigcup_{a \in K} B_r(a).}$$


For $K_1, K_2 \in S$, we define

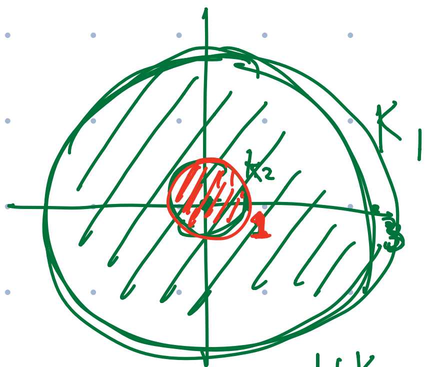
$$\boxed{d(K_1,K_2) := \inf\{\boxed{r>0} : \boxed{K_1 \subset B_r(K_2) \text{ and } K_2 \subset B_r(K_1)}\}}$$

(a) Prove that $\boxed{(S,d)}$ is a metric space, i.e. $d$ is a distance function on $S$.
(b) Let $F$ be the set of finite subsets of $\mathbb{R}^2$. Prove that $F$ is dense in $S$.

$S = \{$ nonempty cpt subsets in $\mathbb{R}^2 \}$

$\boxed{K_1}, \boxed{K_2} \in$

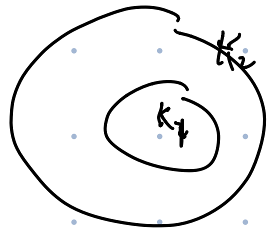
$d(K_1, K_2)$





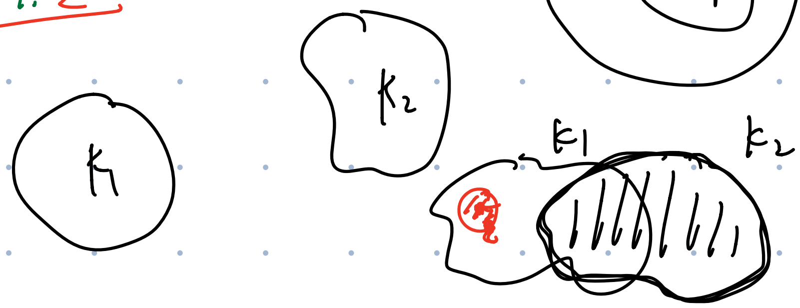
$$\left\{ \begin{array}{l} \boxed{K_1 \subseteq B_r(K_2)} \\ \boxed{K_2 \subseteq B_r(K_1)} \end{array} \right.$$

$d(K_1, K_2) = ??\ 2$




(a)

- $d(K_1, K_2) \in \mathbb{R}_{\geq 0}$.




- $d(K, K) = 0$
- ~~$d(K_1,K_2)$~~ if $K_1 \neq K_2$, then $\boxed{d(K_1,K_2) > 0}$
- $d(K_1, K_2) = d(K_2, K_1)$
- $d(K_1, K_2) + d(K_2, K_3) \geq \boxed{d(K_1, K_3)}$

pf:

$\inf\{r>0 \mid B_r(K_1) \supseteq K_2,\ B_r(K_2) \supseteq K_1\}$.

Choose any $r_1 > d(K_1, K_2)$, $\quad r_2 > d(K_2, K_3)$

$K_2 \subseteq B_{r_1}(K_1) \qquad K_2 \subseteq B_{r_2}(K_3)$

$K_1 \subseteq B_{r_1}(K_2) \qquad K_3 \subseteq B_{r_2}(K_2)$

$\Rightarrow K_1 \subseteq B_{r_1}(K_2) \subseteq B_{r_1+r_2}(K_3)$

$K_3 \subseteq B_{r_2}(K_2) \subseteq B_{r_1+r_2}(K_1)$

$\Rightarrow \quad d(K_1, K_3) \le r_1 + r_2.$

$\forall \varepsilon > 0, \quad d(K_1, K_3) \le (\underbrace{d(K_1, K_2)}_{r_1} + \varepsilon) + (\underbrace{d(K_2, K_3)}_{r_2} + \varepsilon)$

$= d(K_1, K_2) + d(K_2, K_3) + 2\varepsilon.$

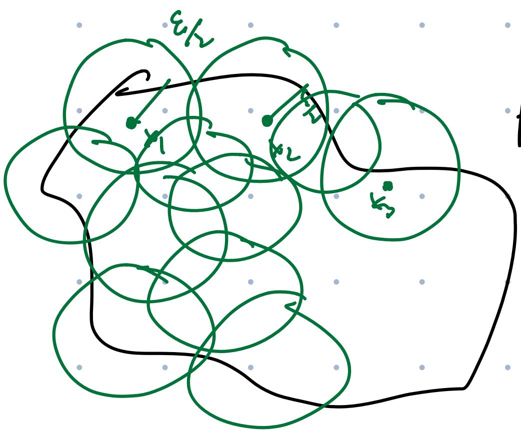
$\Rightarrow \quad d(K_1, K_3) \le d(K_1, K_2) + d(K_2, K_3).$

(b) • $F \subseteq S$.

• Need to show: $\forall K \in S, \ \varepsilon > 0.$

$\exists$ finite subset of $\mathbb{R}^2$ $E$

s.t. $d(E, K) < \varepsilon$.




$K \subseteq \mathbb{R}^2$

$\boxed{d(E, K) < \varepsilon}$

$E$

$\{x_1, \dots, x_n\} \subseteq K$

s.t. $K \subseteq B_{\frac{\varepsilon}{2}}(x_i)$

$d(E, K) = \inf\{r > 0 : E \subseteq B_r(K), \ K \subseteq B_r(E)\}$

$\le \frac{\varepsilon}{2} < \varepsilon$

$\square$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$

(1) Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and $s_k = a_1 + \cdots + a_k$ be the $k$-th partial sum.

(a) Suppose that $\boxed{\lim a_n = 0}$, and there exists a $m \in \mathbb{N}$ such that the sequence $(s_{mk})_{k=1}^{\infty} = (s_m, s_{2m}, s_{3m}, \ldots)$ converges. Prove that $\sum a_n$ converges.

(b) Find an example where $(s_{2k})_{k=1}^{\infty}$ converges and $(a_n)$ doesn't converge to $0$.

(c) Find an example where $\lim a_n = 0$, and there is a subsequence $(s_{k_n})$ of $(s_n)$ that converges, but $\sum a_n$ diverges.

(b) $(-1, 1, -1, 1, \cdots)$

(c) $(-1, 1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \cdots$

(a) Need $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $\left| \sum_{k=p}^{q} a_k \right| < \varepsilon \quad \forall q \geq p > N$

$$\underbrace{a_p + a_{p+1} + a_{p+2}}_{< m \text{ terms}} + \underbrace{a_{km+1} + \cdots + a_{lm}}_{\varepsilon/2} + \underbrace{\cdots + a_q}_{< m \text{ terms}} \qquad \frac{\varepsilon}{4m}$$

we have: $\underset{a_1+\cdots+a_m}{S_m}, \underset{a_1+\cdots+a_{2m}}{S_{2m}}, S_{3m}, \cdots$ conv.

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N$ $\quad k < l$

s.t. $\left| \boxed{S_{km}} - \boxed{S_{lm}} \right| < \varepsilon \quad \forall k, l > N$

$S_{km} = a_1 + \cdots + a_{km}$, $\quad S_{lm} = a_1 + \cdots + a_{lm}$

$\|$

$$\left| a_{km+1} + a_{km+2} + \cdots + a_{lm} \right| < \frac{\varepsilon}{2}$$

$km+1 \equiv 1 \pmod m$, $\quad lm \equiv 0 \pmod m$

pf: $\forall \varepsilon > 0$,

- $\lim a_n = 0 \Rightarrow \exists N_1 > 0$ st.

$|a_n| < \frac{\varepsilon}{4m} \quad \forall n > N_1$

• $(S_{km})_{k=1}^{\infty}$ conv. $\Rightarrow$ $\exists N_2 > 0$ s.t.

$$|a_{km+1} + \cdots + a_{lm}| < \frac{\varepsilon}{2} \quad \forall l > k > N_2,$$

• Choose $\underline{N = \max\{N_1, m(N_2+2)\}} > 0.$

Then $\forall q \geq p > N$,

$$|a_p + a_{p+1} + \cdots + a_q|$$

$$\leq |a_{km+1} + \cdots + a_{lm}| + 2m \cdot \frac{\varepsilon}{4m} < \varepsilon. \quad \square$$

where $k$ is the least integer s.t. $km+1 \geq p$.
$l$ is the largest integer s.t. $lm \leq q$

(2) (a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = 0$ for all $x \in \mathbb{Q}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

(b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then $f$ is linear, i.e. there exists $c$ so that $f(x) = cx$ for all $x$.

(a) directly from the continuity.

(b) • $f(0) = 0$ (plug in $x = y = 0$)

• let $f(1) = A$ (Claim: $f(x) = Ax \ \forall x \in \mathbb{R}$)

• 1) $f(x) = Ax \ \forall x \in \mathbb{Z}$ inductively.

2) $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$. $\underset{pA}{\underset{\|}{f(p)}} = q f(\frac{p}{q}) \Rightarrow f(\frac{p}{q}) = \frac{p}{q} A.$

$\Rightarrow f(x) = Ax \ \forall x \in \mathbb{Q}$

3) continuity & part (a). $\square$

(6) Let $(p_n)$ be a sequence of polynomials defined over real numbers, and let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Suppose that $(p_n)$ converges uniformly to $f$ on $\mathbb{R}$. Prove that $f$ is also a polynomial.

$$\boxed{p_n \to f \text{ unif. on } \mathbb{R}}$$

Weierstrass approx thm

$f$ conti. on $[0,1]$
Then $\exists\, p_n$
$p_n \to f$ unif. on $[0,1]$

$\forall \varepsilon > 0,\ \exists N \geq 0$
s.t. $\boxed{|p_n(x) - p_m(x)| < 1}$ $\forall\, n, m > N$ $\forall x \in \mathbb{R}$.

$\Downarrow$

$p_n(x)$ & $p_m(x)$ can only be different at the constant term! $\forall n, m > N$.

$\Downarrow$

$$\forall n \geq N,\quad p_n(x) = A_\ell x^\ell + A_{\ell-1} x^{\ell-1} + \cdots + A_1 x + C_n$$

$A_\ell, A_{\ell-1}, \ldots, A_1$ are indep. of $n \geq N$

$\forall \varepsilon > 0\ \exists N > 0,$

s.t. $|p_n(x) - p_m(x)| < \varepsilon \quad \forall n, m > N,\ \forall x \in \mathbb{R}$

$\| \, |C_n - C_m|$

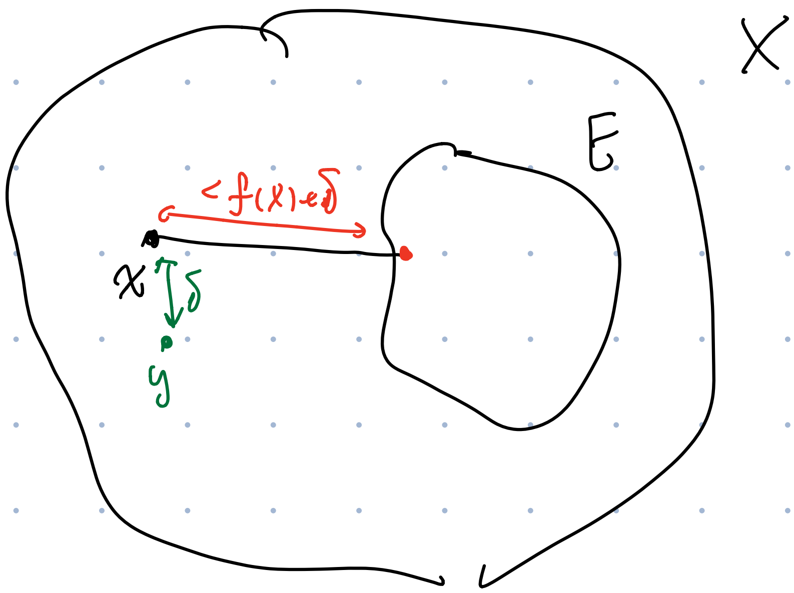
$\Rightarrow (C_n)$ converges $\lim C_n =: A_0$

$\Rightarrow f(x) = A_\ell x^\ell + \cdots + A_1 x + A_0.$ $\square$

(5) Let $(X, d)$ be a metric space and $E \subset X$ be a nonempty subset. Define a function $f\colon X \to [0, \infty)$ by:

$$f(x) := \inf\{d(x, y)\colon y \in E\}.$$

Prove that $f$ is uniformly continuous on $X$.




$\forall \varepsilon > 0,\ \exists \delta > 0$ $\varepsilon/2$

s.t.

if $d(x,y) < \delta$
$x, y \in X$

then $|f(x) - f(y)| < \varepsilon$

$f(y) < f(x) + 2\delta$

$f(x) < f(y) + 2\delta$

$\Rightarrow\ |f(x) - f(y)| < 2\delta$

(3) Let $X = (\mathbb{R}^n, d_{\text{std}})$ be the Euclidean space with the standard distance function

$$d_{\text{std}}(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Prove that any linear map $T\colon X \to X$ is uniformly continuous.

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{T} \underset{\uparrow \atop \mathbb{R}^n}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} = \underset{\uparrow \atop \mathbb{R}^n}{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ \vdots \end{bmatrix}} = T(\vec{x})$$

if $\|\vec{x}-\vec{y}\|$ small $\quad \|T(\vec{x})-T(\vec{y})\|$ small

$$\| T(\vec{x}-\vec{y})\|$$

$\|\vec{v}\| < \delta \qquad \|T(\vec{v})\| < \varepsilon$

$$\boxed{\text{Let } A := \max\{|a_{ij}|\} > 0}$$

$$\|T(\vec{v})\| = \sqrt{\left(\sum_k a_{1k} v_k\right)^2 + \cdots + \left(\sum a_{nk} v_k\right)^2}$$

$$\leq \sqrt{A^2 \left(\sum v_k\right)^2 + \cdots + A^2 \left(\sum v_k\right)^2}$$

$$= \sqrt{nA^2} \cdot \sqrt{\left(\sum v_k\right)^2} \leq \sqrt{nA^2} \cdot \sqrt{n\left(\sum v_k^2\right)} = nA \cdot \|\vec{v}\|$$

$$\left(\sum v_k\right)^2 \leq n\left(\sum v_k^2\right) \quad \text{(Cauchy ineq.)}$$