10. Let $F : \mathbb{H} \to \mathbb{C}$ be a holomorphic function that satisfies

$$|F(z)| \le 1 \quad \text{and} \quad F(i) = 0.$$

Prove that

$$|F(z)| \le \left|\frac{z-i}{z+i}\right| \quad \text{for all } z \in \mathbb{H}.$$

- Consider

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\cong"] \arrow[rr, bend left, "G"] & \mathbb{H} \arrow[r, "F"] & \mathbb{C} \end{tikzcd}$$

$$w \longmapsto i\,\frac{1-w}{1+w}$$

$$\frac{i-z}{i+z} \longleftarrow\!\!\shortmid\; z$$

Then $|G(w)| \le 1$, $G(0) = 0$.

- $\forall\, 0 < r < 1$, Consider

$$\begin{tikzcd} \mathbb{D} \arrow[r, "G"] \arrow[rr, bend left, "G_r"] & \mathbb{C} \arrow[r, "\times r"] & \mathbb{C} \end{tikzcd}$$

Then $|G_r(w)| < 1$, $G_r(0) = 0$.

Therefore $G_r : \mathbb{D} \to \mathbb{D}$ holo., $G_r(0) = 0$.

- By Schwarz lemma, $|G_r(w)| \le |w| \quad \forall w \in \mathbb{D}$,

$$|G_r(w)| = r\,|G(w)| = r\left|F\left(i\,\frac{1-w}{1+w}\right)\right|$$

$$\iff |F(z)| \le \frac{1}{r}\left|\frac{i-z}{i+z}\right| \quad \forall z \in \mathbb{H}.$$

- Take $r \to 1$, we have: $|F(z)| \le \left|\frac{z-i}{z+i}\right| \quad \forall z \in \mathbb{H}.$ $\square$

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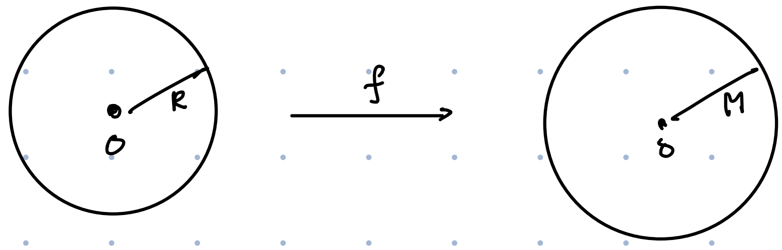
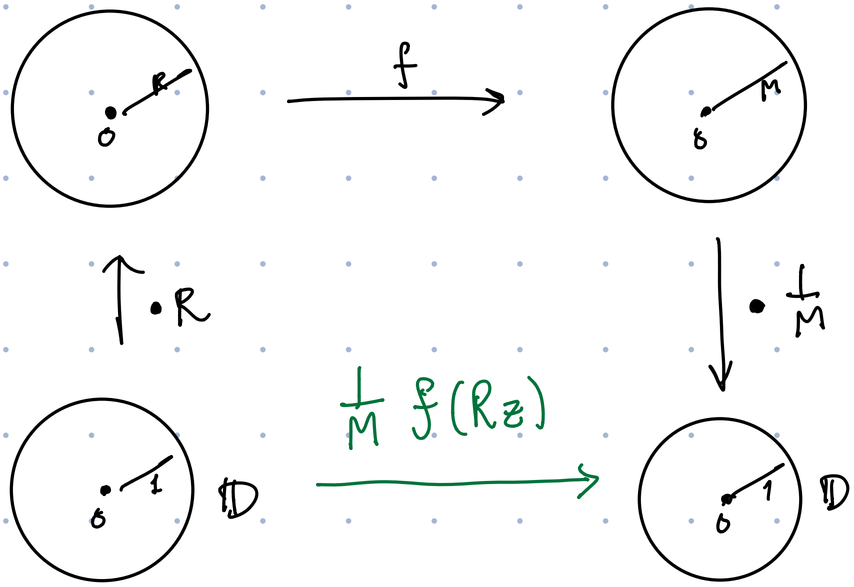
11. Show that if $f : D(0, R) \to \mathbb{C}$ is holomorphic, with $|f(z)| \le M$ for some $M > 0$, then

$$\left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)}\right| \le \frac{|z|}{MR}.$$

[Hint: Use the Schwarz lemma.]

- 

Note that $|f(z)| < M \quad \forall z \in \mathbb{D}_R(0)$ by max. modulus principle

- 

- Then $F(z) := \psi_{\frac{f(0)}{M}}\left(\frac{f(Rz)}{M}\right) : \mathbb{D} \longrightarrow \mathbb{D}$

and $F(0) = 0$.

- Schwarz lemma $\Longrightarrow$ $|F(z)| \leq |z| \quad \forall z \in \mathbb{D}$

$$|F(z)| = \left| \frac{\frac{f(0)}{M} - \frac{f(Rz)}{M}}{1 - \frac{\overline{f(0)}}{M} \cdot \frac{f(Rz)}{M}} \right| = M \cdot \left| \frac{f(Rz) - f(0)}{M^2 - \overline{f(0)}\, f(Rz)} \right|$$

$$\Longrightarrow \left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)}\, f(z)} \right| \leq \frac{|z|}{MR} \quad \forall z \in \mathbb{D}_R(0). \quad \square$$

12. A complex number $w \in \mathbb{D}$ is a **fixed point** for the map $f : \mathbb{D} \to \mathbb{D}$ if $f(w) = w$.

(a) Prove that if $f : \mathbb{D} \to \mathbb{D}$ is analytic and has two distinct fixed points, then f is the identity, that is, $f(z) = z$ for all $z \in \mathbb{D}$.

(b) Must every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ have a fixed point? [Hint: Consider the upper half-plane.]

(a) We may assume one of the fixed points is 0 (using φ_α).
Then, by Schwarz lemma, $|f(z)| = |z|$ at some $z \in \mathbb{D}$ only if f is a rotation.
Since there is another fixed point $f(z_0) = z_0$, $z_0 \neq 0$.
f is a rotation; and $f(z_0) = z_0$ makes f the identity map.

(b) No. It suffices to construct an example for $\mathbb{H}$. Since $\mathbb{H} \cong \mathbb{D}$.

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{H} \\ z &\longmapsto z + a \end{aligned} \qquad \text{has no fixed pt.} \quad a \in \mathbb{R} \setminus \{0\}$$

$\square$

(A) Let $\mathcal{F}$ be a normal family of holomorphic functions on Ω. Prove that $\mathcal{F}$ is uniformly bounded on every compact subset of Ω.

Let $K \subseteq \Omega$ compact subset. Suppose $\mathcal{F}$ is not unif. bdd. on K.
Then $\forall n \in \mathbb{N}$, $\exists f_n \in \mathcal{F}$ s.t. $|f_n(x_n)| > n$ for some $x_n \in K$.

Since $\mathcal{F}$ is a normal family, $\exists$ subseq. (f_{k_n}) of (f_n) s.t.
$f_{k_n} \to f$ unif. on K.

holo.

Since f is conti, K cpt $\Rightarrow$ f is bdd. on K,
say $|f(z)| < M$ $\forall z \in K$.

Since $f_{k_n} \to f$ uniformly on K, $\exists N > 0$ s.t.

$$|f_{k_n}(x) - f(x)| < 1 \quad \forall n > N, \ \forall x \in K.$$

$$\Rightarrow \ |f_{k_n}(x)| < M + 1 \quad \forall n > N, \ \forall x \in K.$$ Contradiction. $\square$

(B) Let $\mathcal{F}$ be the family of holomorphic functions on the open unit disc $\mathbb{D}$, consisting of the functions f_a for all $|a| > 1$, where $f_a(z) = \frac{1}{z-a}$ holomorphic on $\mathbb{D}$. Determine whether $\mathcal{F}$ is a normal family, and give a proof.

Yes.

By Montel's thm, it suffices to show that $\mathcal{F}$ is unif. bdd. on every cpt. subset, of $\mathbb{D}$.

Note that $\forall$ cpt subset $K \subseteq \mathbb{D}$, $\exists$ $0 < r < 1$ s.t. $K \subseteq \overline{\mathbb{D}_r}$

Then $$|f_a(z)| \leq \frac{1}{1-r} \quad \forall |a| > 1, \ \forall z \in \overline{\mathbb{D}_r}.$$

Hence $\mathcal{F}$ is unif. bdd. on K. $\square$