FIRST MIDTERM SOLUTION
MATH H54, FALL 2021

Problem 1: (20 points) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}.$$

Find a matrix C such that $T_B(T_C(\vec{v})) = T_A(\vec{v})$ for any $\vec{v} \in \mathbb{R}^3$. Please write down every step of your calculation.

Solution: By the definition of matrix product, we have $T_A(\vec{v}) = T_B(T_C(\vec{v})) = T_{BC}(\vec{v})$ for any $\vec{v} \in \mathbb{R}^3$. By the uniqueness of matrix representative of linear transformations we proved in class (end of Lecture 3), we have $A = BC$. Hence

$$C = B^{-1}A = \begin{bmatrix} -4 & -13 & -6 \\ 1 & 3 & 2 \\ 2 & 5 & 2 \end{bmatrix}.$$

Problem 2: (20 points) Let V be the subspace of $\mathbb{R}^5$ defined by

$$V := \{\vec{x} \in \mathbb{R}^5 \colon x_1 - x_2 + x_3 - x_4 + x_5 = 0 \text{ and } x_1 - 2x_2 + x_5 = 0\}.$$

Find a matrix A such that its associated linear transformation $T_A \colon \mathbb{R}^3 \to \mathbb{R}^5$ satisfies $\operatorname{Im}(T_A) = V$, and justify your answer.

Solution: The goal is to find a 5×3 matrix A such that its column space $\operatorname{Col}(A) = V$. It's not hard to show that

$$\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis of V. Therefore we can choose A to be

$$A = \begin{bmatrix} -2 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that the choice of A is not unique.

Problem 3: (20 points) Let V be the same subspace of $\mathbb{R}^5$ as in the previous problem. Suppose matrices A and A' both satisfy the condition in the previous problem, namely, T_A and $T_{A'}$ are both linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^5$ such that $\mathrm{Im}(T_A) = \mathrm{Im}(T_{A'}) = V$. Then at least one of the following two statements must be true. Find a correct statement and prove it.

(i) There exists an invertible matrix B such that $AB = A'$.
(ii) There exists an invertible matrix B such that $BA = A'$.

Solution: Both statements are true. (So points will be rewarded only if your proof makes sense.)

Proof of statement (i): Since A and A' both satisfy the condition in Part (a), we have $\mathrm{Col}(A) = V = \mathrm{Col}(A')$. In particular, each column of A' is a linear combination of the columns of A. Therefore, there exists a 3×3 matrix B such that $AB = A'$. The matrix B must be invertible: otherwise, we would have $\mathrm{rank}(B) < 3$, and

$$3 = \mathrm{rank}(A') = \mathrm{rank}(AB) \le \mathrm{rank}(B) < 3,$$

contradiction.

Proof of statement (ii): Since $\mathrm{Col}(A) = V = \mathrm{Col}(A')$, both A and A' have rank 3, therefore each has 3 pivots. So the reduced echelon forms of A and A' are both given by

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0. \end{bmatrix}$$

There exist sequences of 5×5 elementary matrices $E_1, \ldots, E_k$ and $E'_1, \ldots, E'_\ell$ such that

$$E_1 \cdots E_k A = R = E'_1 \cdots E'_\ell A'.$$

Then the invertible matrix $B = (E'_1 \cdots E'_\ell)^{-1}(E_1 \cdots E_k)$ satisfies the condition $BA = A'$.

Problem 4: (25 points) Let A be a 10×10 matrix with $\operatorname{rank}(A) = 6$. Find the integer N such that the following two statements hold, and prove your answer.

(i) There exists a 10×10 matrix B of rank N such that $AB = 0$ (the zero matrix).

(ii) There does not exist a 10×10 matrix B of rank $N+1$ such that $AB = 0$.

(You'll have to declare the integer N that you find, and prove that the two statements hold for any 10×10 matrix A of rank 6.)

Solution: $N = 4$**.**

By the rank-nullity theorem, we have $\dim \operatorname{Nul}(A) = 10 - 6 = 4$. Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} \subseteq \mathbb{R}^{10}$ be a basis of $\operatorname{Nul}(A)$. Consider the matrix B whose first four columns are $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$, and the remaining columns are the zero vector. Then it's clear that $\operatorname{rank}(B) = 4$ and $AB = 0$. This proves Part (i).

On the other hand, if $\operatorname{rank}(B) = 5$, then $\operatorname{Col}(B) \nsubseteq \operatorname{Nul}(A)$ since $\dim \operatorname{Col}(B) = 5 > 4 = \dim \operatorname{Nul}(A)$. So there exists a column vector $\vec{v}_i$ (say, the i-th column) of B such that $\vec{v}_i \notin \operatorname{Nul}(A)$. Then the i-th column of AB is not the zero vector, therefore $AB \neq 0$. This proves Part (ii).

Problem 5: (15 points) Recall that the cofactor matrix of an $n \times n$ matrix A is the matrix with entries given by

$$C_{ij} = (-1)^{i+j} \det(A_{ij}),$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and j-th column of A.

Let A be a 10×10 matrix with $\mathrm{rank}(A) = 6$. Compute the rank of its cofactor matrix, and prove your answer.

Solution: The rank is 0.

Since $\mathrm{rank}(A) = 6$, any collection of 9 vectors among the column vectors of A must be linearly dependent. Therefore, any 9×9 matrix A_{ij} obtained by removing the i-th row and j-th column of A must be non-invertible. Hence $C_{ij} = 0$ for any i, j, so the cofactor matrix of A is in fact the zero matrix, which has rank zero.