HOMEWORK 9
MATH 104, SECTION 6

Office Hours: Tuesday and Wednesday 9:30-11am at 735 Evans.

Nima's Office Hours: Monday, Tuesday and Thursday 9:30am-1pm at 1010 Evans.

Reading

There will be reading assigned for each lecture. You should come to the class having read the assigned sections of the textbook.

Due March 19: Ross, Section 23, 25

Problem set (8 problems; due March 19)

Submit your homework at the beginning of the lecture on Thursday. *Late homework will not be accepted under any circumstances.*

You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any.

Write clearly! Mastering mathematical writing is one of the goals of this course.

You have to staple your work if it is more than one page.

(1) Let X be a set, and (f_n) be a sequence of functions $f_n\colon X \to \mathbb{R}$.
   (a) Suppose that (f_n) converges to $f\colon X \to \mathbb{R}$ uniformly and each (f_n) is bounded. Prove that f is also bounded.
   (b) Find an example of (f_n) converges to $f\colon X \to \mathbb{R}$ pointwisely and each (f_n) is bounded, but f is unbounded.

(2) Let X be a set, and (f_n) be a sequence of functions $f_n\colon X \to \mathbb{R}$. Prove that if (f_n) converges to some function $f\colon X \to \mathbb{R}$ uniformly, then (f_n) is uniformly Cauchy.

(3) Consider the sequence of functions (f_n) defined by $f_n(x) = \frac{nx}{1+nx}$ for $x \geq 0$.
   (a) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \geq 0$.
   (b) Let $a > 0$. Prove or disprove: (f_n) converges uniformly to f on $[a, \infty)$.
   (c) Prove or disprove: (f_n) converges uniformly to f on $[0, \infty)$.

(4) Consider the sequence of functions (f_n) defined by $f_n(x) = \frac{1}{1+x^n}$ for $x \geq 0$.
   (a) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \geq 0$.
   (b) Let $0 < a < 1$. Prove or disprove: (f_n) converges uniformly to f on $[0, a]$.
   (c) Prove or disprove: (f_n) converges uniformly to f on $[0, 1]$.

(5) Prove that the series

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!} \right)^2$$

is continuous on $\mathbb{R}$.

(Hint: First show that the series converges uniformly on $[-T, T]$ using Weierstrass M-test for any $T > 0$.)

(6) Let X be a compact metric space, and (f_n) be a sequence of continuous functions $f_n \colon X \to \mathbb{R}$. Suppose that

- (f_n) converges pointwisely to a continuous function $f \colon X \to \mathbb{R}$.
- $f_{n+1}(x) \leq f_n(x)$ for any $x \in X$ and $n \in \mathbb{N}$.

Prove that (f_n) converges uniformly to f on X.

(Hint: Define $g_n \coloneqq f_n - f$. Consider the set

$$E_n \coloneqq \{x \in X \colon g_n(x) < \epsilon\}.$$

Show that $E_1 \subset E_2 \subset E_3 \subset \cdots$ and that $X = \cup E_n$.)

(7) A collection of functions (f_n) on X is called *uniformly equicontinuous* on X if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $n \in \mathbb{N}$, we have

$$|x - y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon.$$

(a) Prove that any finite set of continuous functions on a compact metric space X is uniformly equicontinuous.

(b) Let (f_n) be a sequence of uniformly convergent continuous functions on a compact metric space X. Prove that (f_n) is uniformly equicontinuous.

(The notion of uniformly equicontinuous is important in the study of ordinary differential equations, in particular for proving the existence of solutions to certain initial value problems.)

(8) Consider the metric space $(\mathcal{C}([0,1], \mathbb{R}), d_\infty)$, where

- $\mathcal{C}([0,1], \mathbb{R})$ is the set of all real-valued continuous on $[0,1]$.
- $d_\infty(f, g) \coloneqq \sup\{|f(x) - g(x)| \colon x \in [0,1]\}$ for $f, g \in \mathcal{C}([0,1], \mathbb{R})$.

Let $\mathfrak{o} \in \mathcal{C}([0,1], \mathbb{R})$ denotes the zero function on $[0,1]$. Consider the following subset in $\mathcal{C}([0,1], \mathbb{R})$:

$$\mathcal{S} \coloneqq \{f \in \mathcal{C}([0,1], \mathbb{R}) \colon d_\infty(f, \mathfrak{o}) \leq 1\}.$$

Prove that $\mathcal{S}$ is closed and bounded, but not compact.