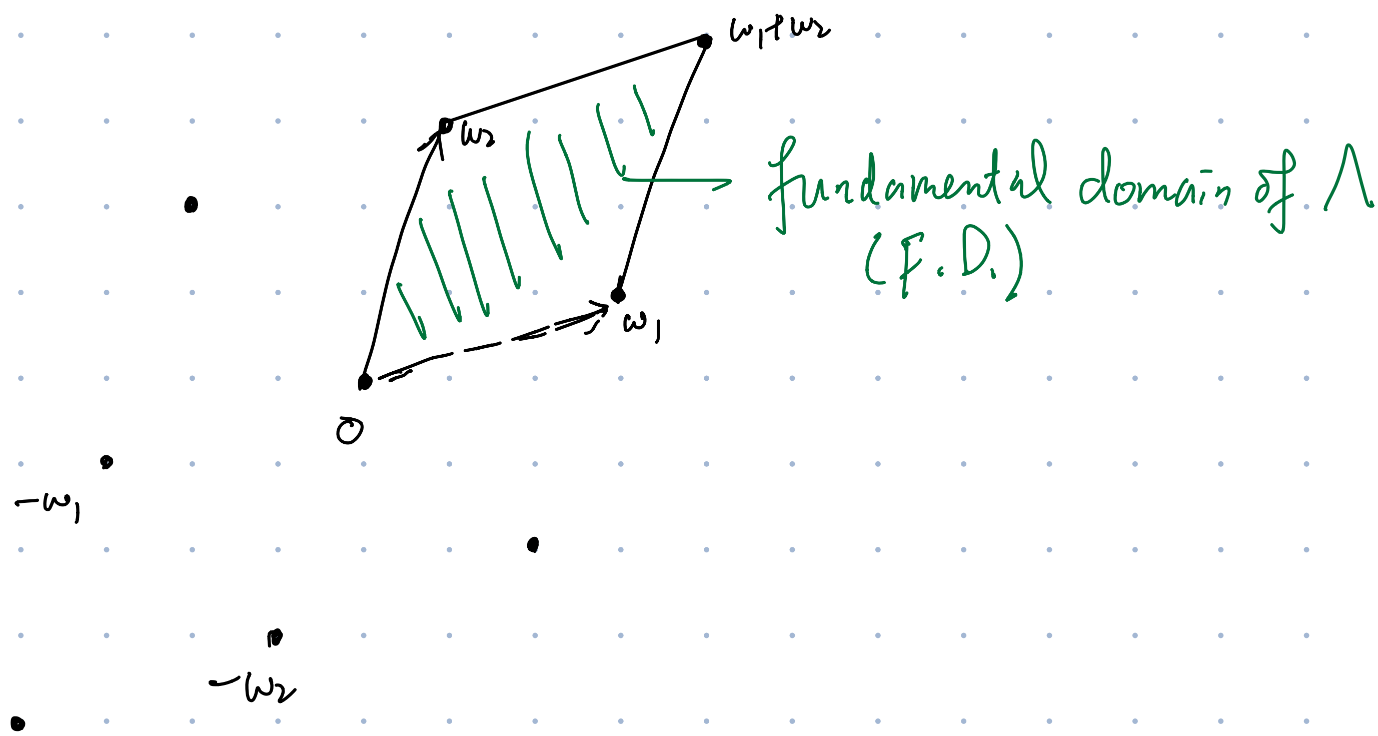
Today: elliptic functions.

Def. Say $f: \mathbb{C} \to \mathbb{C}$ is elliptic if it's doubly periodic.

i.e. $\exists\ \omega_1, \omega_2 \in \mathbb{C}$ s.t. $f(z) = f(z+\omega_1) = f(z+\omega_2)$



- $\Lambda := \{ m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z} \} \subseteq \mathbb{C}$ lattice
- Doubly periodic w.r.t. $\Lambda \iff f(z) = f(z+\lambda)\ \forall \lambda \in \Lambda$.
- elliptic functions w.r.t. Λ is determined by its value in the fundamental domain of Λ

Q: What are all possible holomorphic elliptic fns $f: \mathbb{C} \to \mathbb{C}$?

(f is bdd on the F.D. $\xrightarrow[\text{ell.}]{}$ f is bdd. entire

$\Downarrow$ Liouville

f is constant.)

Q: Meromorphic elliptic functions?

Idea: g: mero. on $\mathbb{C}$

$$f(z) = \boxed{\sum_{m,n\in\mathbb{Z}} g(z+m\omega_1+n\omega_2)}$$ — may not be convergent.

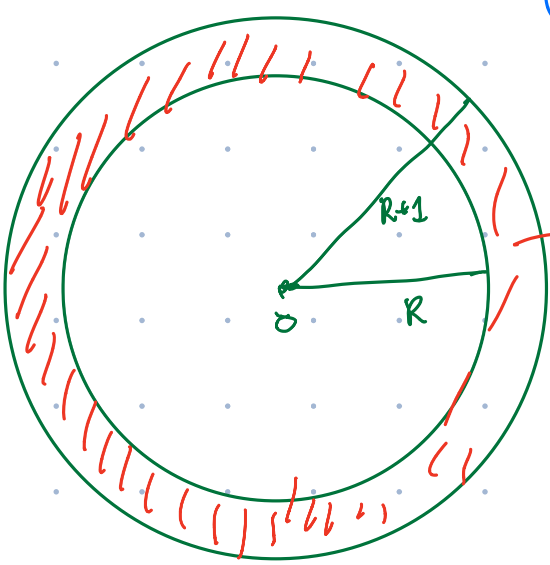
is obviously elliptic

$$f(z+\omega_1) = \sum_{m,n\in\mathbb{Z}} g(z+\omega_1+m\omega_1+n\omega_2)$$

$$= \sum_{m,n\in\mathbb{Z}} g(z+m\omega_1+n\omega_2)$$

Suppose $|g(z)| < \frac{C}{|z|^\alpha}$ for $|z| >> 0$.

(We'll consider $\sum_{m,n\in\mathbb{Z}} g(z+m\omega_1+n\omega_2)$)



of $(m,n)\in\mathbb{Z}^2$ s.t. $z+m\omega_1+m\omega_2 \in$

$\approx$ (Area of) $\cdot$ const.

$\approx$ R $\cdot$ Const.

$$|f(z)| = \Big|\sum_{m,n\in\mathbb{Z}} g(z+m\omega_1+n\omega_2)\Big|$$

$$\leq \sum_{R=0}^{\infty} \sum_{\substack{m,n\in\mathbb{Z},\\ R\leq|z+m\omega_1+n\omega_2|<R+1}} |g(z+m\omega_1+n\omega_2)|$$

$\wedge$

$\frac{c}{R^\alpha}$ when R lage.

$$\approx \sum_{R=0}^{\infty} \frac{c}{R^\alpha}\cdot R$$

$$\approx c\sum_{R} \frac{1}{R^{\alpha-1}}$$

So:, if $\alpha > 2$, then $\sum_{m,n\in\mathbb{Z}} g(z+m\omega_1+n\omega_2)$

is absolutely convergent

Fact: If $|g(z)| < \frac{c}{|z|^\alpha}$ for all $|z| \gg 0$ and some $\alpha > 2$
$c > 0$ const.

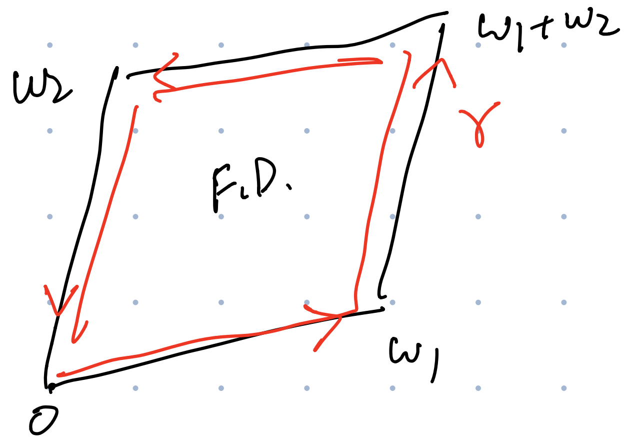
then $f(z) := \sum_{\lambda\in\Lambda} g(z+\lambda)$ is a meromorphic elliptic function.

e.g. $g(z) = \frac{1}{(z-\alpha)(z-\beta)(z-\gamma)}$ for some $\alpha,\beta,\gamma \in \mathbb{C}$

$\Rightarrow$ $f(z) := \sum_{\lambda\in\Lambda} g(z+\lambda)$ mero. ell. fcn.

(with 3 poles in the F.D.)

Q: $\exists$ ell. fun w/ exactly 1 or 2 poles in the F.D. ??



f: mero. ell. fun.

Fact 1: # zeros of f in F.D. = # poles in the F.D.

pf: # zeros of f in F.D. − # poles in the F.D.

$$= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0. \qquad \square$$

(Since f is doubly periodic w.r.t. Λ,

so $\displaystyle\int_{\omega_1}^{\omega_1+\omega_2} \frac{f'(z)}{f(z)}\,dz = \int_0^{\omega_2} \frac{f'(z)}{f(z)}\,dz$

(b/c $f(z) = f(z+\omega_1)$ $\forall z$))

Fact 2: $\left(\sum \text{zeros in the F.D.}\right) - \left(\sum \text{poles in the F.D.}\right) \in \Lambda$

Claim: $\displaystyle \frac{1}{2\pi i}\int_\gamma z\cdot\frac{f'(z)}{f(z)}\,dz = \sum \text{zeros inside } \gamma - \sum \text{poles}_{\text{inside } \gamma}$

pf: Say z_0 is a zero of order n of f, so near z_0,

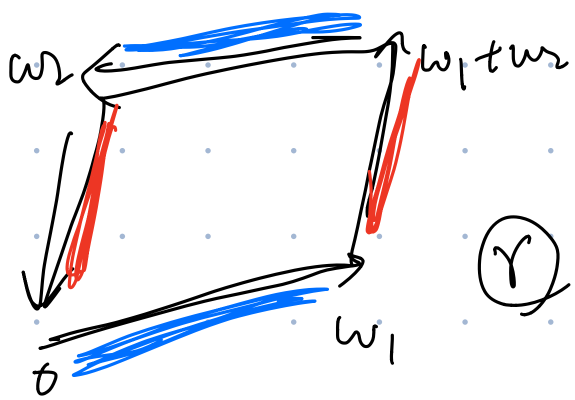
$$f(z) = (z-z_0)^n h(z),$$

$$\frac{1}{2\pi i}\oint z \cdot \frac{n(z-z_0)^{n-1}h(z) + (z-z_0)^n h'(z)}{(z-z_0)^n h(z)}\, dz.$$

$$= \frac{1}{2\pi i}\oint z \cdot \frac{n}{z-z_0}\, dz$$

$$= \frac{1}{2\pi i}\oint (z-z_0)\frac{n}{z-z_0} + z_0 \cdot \frac{n}{z-z_0}\, dz$$

$$= n z_0.$$

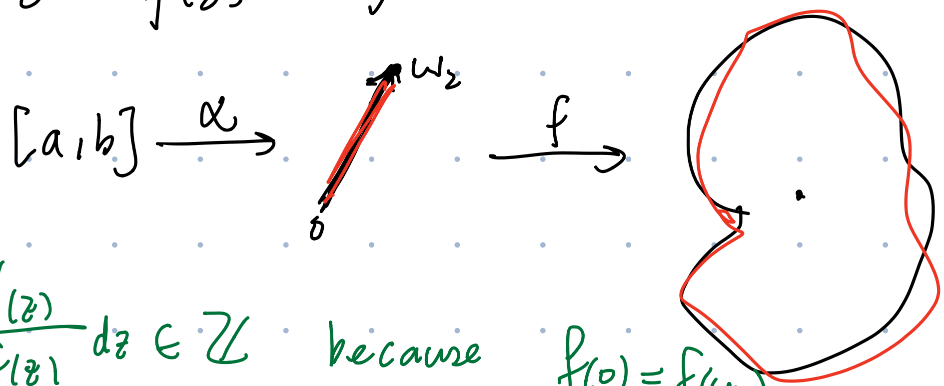


$$\frac{1}{2\pi i}\int_\gamma \frac{z \cdot f'(z)}{f(z)}\, dz.$$

$$\frac{1}{2\pi i}\left(\int_{w_1}^{w_1+w_2} \frac{z f'(z)}{f(z)}\, dz - \int_0^{w_2} \frac{z f'(z)}{f(z)}\, dz\right)$$

$$= \frac{1}{2\pi i}\left(\int_0^{w_2} \frac{(z+w_1)\overbrace{f'(z+w_1)}^{f'(z)}}{\underbrace{f(z+w_1)}_{f(z)}}\, dz - \int_0^{w_2} \frac{z f'(z)}{f(z)}\, dz\right)$$

$$= \frac{1}{2\pi i}\left(\omega_1 \int_{\gamma}^{\omega_2} \frac{f'(z)}{f(z)}\,dz\right) \in \omega_1 \mathbb{Z}$$



$$\frac{1}{2\pi i}\int_0^{\omega_2} \frac{f'(z)}{f(z)}\,dz \in \mathbb{Z} \quad \text{because} \quad f(0) = f(\omega_2)$$

$$\frac{1}{2\pi i}\left(\int_0^{\omega_1} \frac{z f'(z)}{f(z)}\,dz - \int_{\omega_2}^{\omega_1+\omega_2} \frac{z f'(z)}{f(z)}\,dz\right) \in \omega_2 \mathbb{Z}.$$

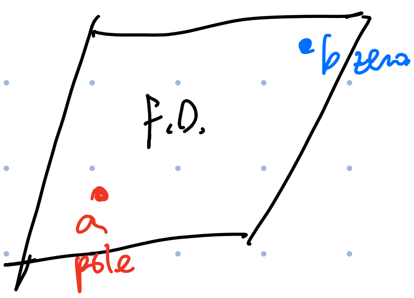
$\square$

__Thm__ f: mero. ell. fn w.r.t. Λ. Then:

$$\begin{cases} \#\text{ zeros in F.D.} = \#\text{ poles in F.D.} \\ \sum \text{zeros in F.D.} - \sum \text{poles in F.D.} \in \Lambda. \end{cases}$$

__Fact__: There is no ell. fn. with exactly 1 pole (counted w/ multiplicity) in the F.D.

__pf__:



But $b - a \notin \Lambda$ for any distinct points a, b in F.D. $\square$

__pf 2__:

$$\underbrace{\frac{1}{2\pi i}\int_\gamma f}_{\substack{\| \\ 0 \text{ (b/c } f \text{ is doubly periodic)}}} = \operatorname*{Res}_{z=a} f(z) \Rightarrow \text{contradiction} \quad \square$$

Laurent exp. at $z=a$:

$$\left(\cdots + \frac{0}{z^3} + \frac{0}{z^2} + \frac{*\neq 0}{z} + \underbrace{* + *z + *z^2 + \cdots}_{\text{holo.}} \right)$$

Q: $\exists$ ell. fcn w/ exactly 2 poles in the F.D.?

Idea: $\displaystyle\sum_{m,n\in\mathbb{Z}} \frac{1}{(z+m\omega_1+n\omega_2)^3}$ conv.

but $\displaystyle\sum_{m,n\in\mathbb{Z}} \frac{1}{(z+m\omega_1+n\omega_2)^2}$ divergent.

$$\parallel$$

$$\sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^2}$$

$$\frac{\lambda^2-(z+\lambda)^2}{(z+\lambda)^2\lambda^2} = \boxed{\frac{-z^2-2z\lambda}{(z+\lambda)^2\lambda^2}}$$

of deg -3 in λ.

Consider $\displaystyle\frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\boxed{\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}} \right)$ Convergent

$$\left(= \frac{1}{z^2} + \sum_{\substack{m,n\in\mathbb{Z} \\ \text{where} \\ (m,n)\neq(0,0)}} \left(\frac{1}{(z+m\omega_1+n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2} \right) \right)$$

Weierstrass $\wp$-fcn: $\displaystyle\wp(z) := \frac{1}{z^2} + \sum_{\substack{\lambda\neq 0 \\ \lambda\in\Lambda.}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)$

Prop: $\wp(z)$ is elliptic w.r.t. Λ.

pf:

- $$\wp'(z) = \frac{-2}{z^3} + \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{-2}{(z+\lambda)^3}$$
$$= -2 \sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^3} \quad \text{elliptic.}$$
  so $\wp'(z) = \wp'(z+\omega_1) = \wp'(z+\omega_2)$.

- $\wp(z) - \wp(z+\omega_1) = \text{const.} \equiv C$

- $\wp(z)$: is an even fcn. $(\wp(z) = \wp(-z))$

  so $$\wp\left(-\frac{\omega_1}{2}\right) - \wp\Big(\underbrace{-\frac{\omega_1}{2}+\omega_1}_{\frac{\omega_1}{2}}\Big) = C$$

  $\Rightarrow C = 0.$

  $\Rightarrow \wp(z) = \wp(z+\omega_1)$

- Similarly, $\wp(z) = \wp(z+\omega_2)$. $\square$

Fact: $\wp(z)$ has a double pole in the F.D. at $z=0$, and no other poles in the F.D.

$\Longrightarrow$ there are 2 zeros of $\wp(z)$ in the F.D.

But they're hard to write down explicitly.

$\Lambda = \{m + n\tau \mid m, n \in \mathbb{Z}\}$, $\tau \in \mathbb{H}$.

Theorem. *The zeros of* $\wp(z, \tau)$ $(\tau \in \mathfrak{H},\ z \in \mathbb{C})$ *are given by*

Eichler-Zagier 1982
$$z = m + \frac{1}{2} + n\tau \pm \left(\frac{\log(5+2\sqrt{6})}{2\pi i} + 144\pi i \sqrt{6} \int_{\tau}^{i\infty} (t-\tau) \frac{\Delta(t)}{E_6(t)^{3/2}} dt \right)$$

$(m, n \in \mathbb{Z})$, *where* $E_6(t)$ *and* $\Delta(t)$ $(t \in \mathfrak{H})$ *denote the normalized Eisenstein series of weight 6 and unique normalized cusp form of weight 12 on* $SL_2(\mathbb{Z})$, *respectively, and the integral is to be taken over the vertical line* $t = \tau + i\mathbb{R}_+$ *in* $\mathfrak{H}$.

Here $\Delta(t) = q \cdot \prod_{n \geq 1}^{\infty} (1-q^n)^{24}$, $q = e^{2\pi i t}$,

$E_6(t) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$, where $\sigma_5(n) = \sum_{d|n} d^5$

Idea: using modular forms. which we'll discuss next week.

Eichler: There are 5 elementary arithmetical operations: addition, subtraction, multiplication, division, and modular forms.