4/14/2020

Recall:
- Countable/uncountable sets.
- Measure zero sets in $\mathbb{R}$
- Cantor set.
- Riemann-Lebesgue thm.

Recall: E countable: $|E| = |\mathbb{N}|$

i.e. $\exists$ bijection $f: \mathbb{N} \to E$

$|\mathbb{N}| < |P(\mathbb{N})|$ → set of all subsets of $\mathbb{N}$.

$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$

Cantor: $|\mathbb{R}| > |\mathbb{N}|$ (diagonal argument)

$\mathbb{R}$ ↑ uncountable

① $\exists f: \mathbb{N} \to \mathbb{R}$ injective

② $\nexists g: \mathbb{N} \to \mathbb{R}$ bijective.

It suffices to show:

any $g: \mathbb{N} \to \mathbb{R}$ is not surjective.

$1 \mapsto [\cdots . a_{11} a_{12} a_{13} \cdots]$ e.g. $\pi = 3.1415\cdots$

$2 \mapsto [\cdots . a_{21} a_{22} a_{23} \cdots]$ $\mathbb{Q}$.

$\vdots$

$a_{ij} \in \{0, \ldots, 9\}$

~~Define~~ Choose

$r := 0.b_1 b_2 b_3 \cdots$

where $b_i \neq a_{ii}$

$b_i \in \{0, \dots, 9\}$

$r \neq g(i) \quad \forall i \in \mathbb{N}$

since their i-th digits in the expansion are not the same. $\square$

<u>Thm</u> $|P(\mathbb{N})| \overset{?}{=} |\mathbb{R}|$ $\quad (= |P(\mathbb{Q})|)$

pf <u>Schröder-Berstein thm</u>: A, B sets

$\exists\, f: A \to B$ inj $(|A| \le |B|)$

$\exists\, g: B \to A$ inj $(|B| \le |A|)$

$\Rightarrow \exists\, h: A \to B$ bij. $(|A| = |B|)$

<u>Ex</u> $|A| = |B| \Rightarrow |P(A)| = |P(B)|$

- $f: \mathbb{R} \longrightarrow P(\mathbb{Q}) \xrightarrow{\text{bij}} P(\mathbb{N})$ injective

$x \longmapsto \{r \in \mathbb{Q} \mid r \le x\}$

$x < r < y$ denseness of $\mathbb{Q}$

$r \in f(y)$

$r \notin f(x)$

- $g: P(\mathbb{N}) \longrightarrow \mathbb{R}$ injective.

$$\begin{array}{ccc} \psi & & \\ S = \{n_1, n_2, \ldots\} \subset \mathbb{N} & \longmapsto & \displaystyle\sum_{i=1}^{\infty} \frac{a_i}{3^i} = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots \\ \updownarrow & & \Uparrow \\ (a_i) & & [0,1] \end{array}$$

seq. $(0, 0, 2, 2, 0, \ldots)$

where $\begin{cases} a_i = 2 & \text{if } i \in S \\ a_i = 0 & \text{if } i \notin S \end{cases}$

"Cantor set"

$\square$

In 2-adic expansion:

$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \leftarrow 0.01111\cdots$

$\frac{1}{2} \leftarrow = 0.10000\cdots$

<u>Rmk</u> $|\mathbb{N}| < |\mathbb{R}| = |P(\mathbb{N})|$

<u>Q</u>: Is there any set S

s.t. $|\mathbb{N}| < |S| < |\mathbb{R}|$?

<u>A</u>: <u>Gödel'1940</u>, <u>Cohen'1963</u>

the existence of S can <u>not</u> be proved using "standard" set theory axioms.

Def A subset $E \subset \mathbb{R}$ has <u>measure zero</u>

if $\forall \varepsilon > 0$

$\exists$ finite or countably many

open intervals $\{I_1, I_2, \cdots\}$

(possibly overlapped)

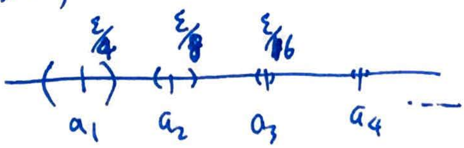
s.t.
- $E \subset \bigcup_{i=1}^{\infty} I_i$
- $\sum_{i=1}^{\infty} \text{length}(I_i) < \varepsilon$

(In HW12: measure zero in $\mathbb{R}^n$)

Lemma: A countable subset in $\mathbb{R}$ has measure zero.

pf $E = \{a_1, a_2, a_3, \cdots\} \subset \mathbb{R}$

$\forall \varepsilon > 0$,



$$I_n = \left(a_n - \frac{\varepsilon}{2^{n+2}},\ a_n + \frac{\varepsilon}{2^{n+2}}\right)$$

$$\ell(I_n) = \frac{\varepsilon}{2^{n+1}}$$

$$\sum \ell(I_n) = \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \cdots = \frac{\varepsilon}{2} < \varepsilon.$$

$\square$

Ex:

- Any subset of a measure zero set has measure zero.
- Any countable union of measure zero sets has measure zero.

$A_1, A_2, A_3, \cdots$ measure zero sets

$A = \bigcup_{i=1}^{\infty} A_i$ also has measure zero

Cantor set (uncountable, measure zero in $\mathbb{R}$)

$F_1 = [\text{———}]$ $[0,1]$

$[1,0,0,0,\cdots]$
$[0,2,2,2,\cdots]$

their 3-adic expansion start with 1.

$F_2 = [\text{—}] \quad [\text{—}]$

$[2,0,0,\cdots]$

$r = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots$

$[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$[0,1,\cdots]$ $\qquad$ $[2,1,\cdots]$

$F_3 =$ $[\,]\ [\,]$ $\qquad$ $[\,]\ [\,]$
00 $\quad$ 02 $\qquad$ 20 $\quad$ 22

$[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

$\vdots$

$F_1 \supset F_2 \supset F_3 \supset \cdots$ closed subsets in $[0,1]$

$$\mathcal{C} := \bigcap_{n=1}^{\infty} F_n$$

$$= \left\{ x \in [0,1] \,\middle|\, x = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots \right\}$$

where $a_i \in \{0, 2\}$

$\mathcal{C}$ is measure zero :

F_{n+1} = union of 2^n closed intervals,
each with length $1/3^n$

⋮

We can cover $F_{n+1} \supset \mathcal{C}$ with
2^n open intervals,
w/ total length $\boxed{\frac{2^n}{3^n} + \frac{\varepsilon}{2}}$

$\left(\frac{2}{3}\right)^n \to 0$

$\mathcal{C}$ is uncontable :

any $f: \mathbb{N} \longrightarrow \mathcal{C}$ is not surjective.

$1 \longmapsto \frac{a_{11}}{3} + \frac{a_{12}}{3^2} + \frac{a_{13}}{3^3} + \cdots$

$2 \longmapsto \frac{a_{21}}{3} + \frac{a_{22}}{3^2} + \cdots$

⋮

$a_{ij} \in \{0, 2\}$

Choose
$b := \frac{b_1}{3} + \frac{b_2}{3^2} + \frac{b_3}{3^3} + \cdots \in \mathcal{C}$

st $b_i \in \{0, 2\}$

and $\underline{b_i \neq a_{ii}}$

~~:0002222222~~
00100000
0012222...
0020000...

$b \neq f(i)$ since their i-th digits in the 3-adic exp. are different. $\square$

Riemann-Lebesgue thm

$f : [a,b] \to \mathbb{R}$ bdd.

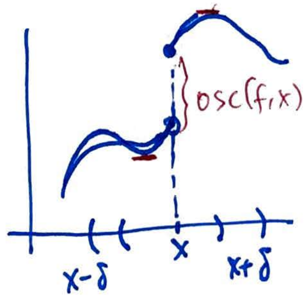
~~f is integrable ⟺~~

$A = \{ x \in [a,b] \mid f$ is not continuous at $x \}$

f is integrable $\Leftrightarrow$ A has measure zero

Def (Oscillation of f at x)

$$\operatorname{osc}(f,x) := \lim_{\delta \to 0} \left(\sup \left\{ |f(x_1) - f(x_2)| : x_1, x_2 \in (x-\delta, x+\delta) \cap [a,b] \right\} \right)$$



Ex: Why $\lim_{\delta \to 0}$ exists?

- f is conti. at $x \Leftrightarrow \operatorname{osc}(f;x) = 0$

($\Rightarrow$) $\forall \varepsilon > 0, \exists \delta > 0$

s.t. $|y - x| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$

Rmk: $\operatorname{osc}(f,x)$ measures the discontinuity of f at x.