Today:

- (finish the proof of "compact $\Leftrightarrow$ sequentially compact")
- Gradescope & exam logistics
- discuss practice problems. (or any other questions)

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(1) $a_n = \sqrt{n^2+1} - n$. Prove (a_n) conv. by def$^{\underline{n}}$.

$$\parallel$$

$$\frac{1}{\sqrt{n^2+1}+n}$$

$$\underbrace{\frac{1}{\sqrt{n^2+1}+n}}_{\parallel \atop |a_n - 0|} < \frac{1}{2n} < \varepsilon$$

$N = \frac{1}{2\varepsilon}$

$|a_n - 0| < \varepsilon$

Claim: $\lim a_n = 0$.

pf: $\forall \varepsilon > 0$, Pick $N = \frac{1}{2\varepsilon} > 0$.

Then $n > N$, we have:

$$|a_n - 0| = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{2n} < \frac{1}{2N} = \varepsilon. \quad \square$$

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(2) $a_1 = 1$, $a_{n+1} = \frac{n}{n+3} a_n^2$, Prove: (a_n) conv. & find limits.

Claim: (a_n) decreasing & bdd. $\boxed{0 \le a_{n+1} \le a_n \le 1}$ $\forall n$

pf

By induction, true for $n=1$.

$$a_{n+2} = \frac{n+1}{n+4} a_{n+1}^2 = \underbrace{\frac{n+1}{n+4}}_{< 1} \underbrace{a_{n+1}}_{\le 1} a_{n+1} < a_{n+1} \le 1$$

$\Rightarrow$ (a_n) conv., call $\underline{\lim a_n = \boxed{a}}$

$$a_{n+1} = \frac{n}{n+3} a_n^2$$

$$\boxed{\lim a_{n+1} = a}$$

$$\| $$

$$\lim\left(\frac{n}{n+3} a_n^2\right) = \lim\left(\underbrace{\frac{n}{n+3}}_{\downarrow\, 1} \cdot \underbrace{a_n}_{\downarrow\, a} \cdot \underbrace{a_n}_{\downarrow\, a}\right) = a^2$$

by limit thm.

$\Rightarrow$ $a = a^2$

$\Rightarrow$ $a = \cancel{1}$ or 0 $\square$

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(3) $a_n = (n!)^{1/n}$ $\underline{Q}$: conv. or not ??

$$= (1 \cdot 2 \cdot 3 \cdot \dots \cdot n)^{1/n}$$

n even

$$> \Big(\underbrace{\left(\tfrac{n}{2}+1\right)\cdot\left(\tfrac{n}{2}+2\right)\cdots n}_{n/2\text{-many terms}}\Big)^{1/n}$$

$$> \left[\left(\frac{n}{2}\right)^{n/2}\right]^{1/n}$$

$$= \left(\frac{n}{2}\right)^{1/2} \to +\infty \text{ as } n \to \infty$$

$\Rightarrow$ (a_n) is not bdd. $\Rightarrow$ (a_n) divergent.

(4) Pf: Suppose (a_n^3) is conv. $\overset{??}{\Longrightarrow}$ (a_n) is conv.

Q: (a_n^2) is conv. $\overset{??}{\Longrightarrow}$ (a_n) is conv.

$(a_n) = (-1, 1, -1, 1, \ldots)$ div.

$(a_n^2) = (1, 1, 1, \ldots)$ conv.

~~Idea~~ sketch. $\lim a_n^3 = A$ guess: then $\lim a_n = A^{1/3}$

$|a_n - A^{1/3}|$ small

want

we have: $|a_n^3 - A| = |a_n - A^{1/3}|\,|a_n^2 + a_n A^{1/3} + A^{2/3}|$

small as n large

$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

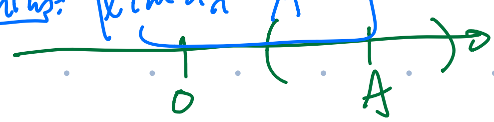
$$|a_n - A^{1/3}| = \frac{|a_n^3 - A|}{|a_n^2 + a_n A^{1/3} + A^{2/3}|} \le \frac{|a_n^3 - A|}{A^{2/3}} < \varepsilon$$

$|a_n^3 - A| < \varepsilon \cdot A^{2/3}$

$|a_n^2 + a_n A^{1/3} + A^{2/3}| \ge A^{2/3}$

Claim: $\boxed{\lim a_n = A^{1/3}}$

Pf Let $\lim a_n^3 = A$



Case 1: $A > 0$ (the case $A < 0$ can be proved in the same way)

- $\exists N_1 > 0$ s.t. $a_n > 0$ $\forall n > N_1$
- $\forall \varepsilon > 0$, $\exists N_2 > 0$ s.t. $n > N_2 \Rightarrow |a_n^3 - A| < \varepsilon \cdot A^{2/3}$

- let $N = \max\{N_1, N_2\} > 0$.

Then $\forall n > N$, we have: $\begin{cases} a_n > 0 \\ |a_n^3 - A| < \varepsilon \cdot A^{2/3} \end{cases}$

$$\Rightarrow |a_n - A^{1/3}| = \frac{|a_n^3 - A|}{|a_n^2 + a_n A^{1/3} + A^{2/3}|} < \frac{\varepsilon \cdot A^{2/3}}{A^{2/3}} = \varepsilon. \quad \square$$

positive

Case 2: If $\underline{\lim a_n^3 = 0}$, then $\lim a_n = 0$.

pf

$\forall \varepsilon > 0$, $\exists N > 0$ s.t. $n > N \Rightarrow |a_n^3| < \varepsilon^3$

$\Rightarrow |a_n| < \varepsilon$. $\square$

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(5) $\underline{0 \le a_{n+m} \le a_n + a_m}$. $\quad b_n = \frac{a_n}{n}$ $\quad$ Prove: $\underline{(b_n) \text{ conv.}}$

- (b_n) bounded:

$0 \le b_1 = \frac{a_1}{1} = a_1$

$0 \le b_2 = \frac{a_2}{2} \le \frac{a_1 + a_1}{2} = a_1$

$0 \le b_3 = \frac{a_3}{3} \le \frac{3a_1}{3} = a_1$

$0 \le b_n = \frac{a_n}{n} \le \frac{na_1}{n} = a_1$

- So, $\liminf b_n$ & $\limsup b_n$ exist. $\in \mathbb{R}$.

(recall: (b_n) is conv. $\Leftrightarrow \liminf b_n = \limsup b_n$)

- $\underline{z := \limsup_{n \to \infty} b_n.}$
- We proved in class that $\exists$ subseq. (b_{k_n}) of (b_n) s.t. $\lim b_{k_n} = z$
- $\underline{\text{Claim}}$: $z \leq b_m$ $\forall m.$

  $\underline{\text{pf}}$: $\underline{\forall m \in \mathbb{N},}$

  Look at the indices of the subseq. (b_{k_n})

  For each k_n, we can write $k_n = l_n \cdot m + r_n$

  where $l_n \in \mathbb{Z}_{\geq 0}$, $0 \leq r_n < m$

  By the subadditive cond'n $(a_{m+n} \leq a_m + a_n)$,

  we have: $\left(k_n = \underbrace{m+m+\cdots+m}_{l_n\text{-many } m\text{'s}} + r_n\right)$

$$a_{k_n} \leq l_n \cdot a_m + a_{r_n}.$$

$$b_{k_n} = \frac{a_{k_n}}{k_n} \leq \frac{l_n \cdot a_m}{k_n} + \frac{a_{r_n}}{k_n}$$

$$= \frac{l_n \cdot b_m \cdot m}{k_n} + \frac{a_{r_n}}{k_n}$$

$\boxed{* \; 0 \leq r_n \leq m-1}$

$$= \frac{(k_n - r_n) b_m}{k_n} + \frac{a_{r_n}}{k_n}.$$

$$= \underbrace{\left(1 - \frac{r_n}{k_n}\right)}_{\to 1 \text{ as } n \to \infty} b_m + \underbrace{\frac{a_{r_n}}{k_n}}_{\to 0 \text{ as } n \to \infty}$$

$\Rightarrow \quad z = \lim b_{k_n} \leq b_m. \quad \square$

- $\Rightarrow \quad z \leq I_N := \inf\{b_n : n > N\} \quad \forall N$

$\Rightarrow \quad z \leq \liminf_{n\to\infty} b_n = \lim_{N\to\infty} I_N$

$\quad\quad\; \parallel$

$\limsup_{n\to\infty} b_n.$

$\Rightarrow \liminf b_n = \limsup b_n \quad \Rightarrow (b_n) \text{ conv.} \quad \square$

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(6) $\mathbb{Q} \subseteq \mathbb{R}$ is open?

is closed?

(If $\mathbb{Q} \subseteq \mathbb{R}$ is open, then $\forall x \in \mathbb{Q}, \exists r > 0$ s.t. $(x-r, x+r) \subseteq \mathbb{Q}$.)

Claim: $\mathbb{Q} \subseteq \mathbb{R}$ is NOT open.

pf $0 \in \mathbb{Q}$, $\forall r > 0$, $(-r, r) \not\subseteq \mathbb{Q}$. $\square$

Claim: $\mathbb{Q}^c \subseteq \mathbb{R}$ is NOT open ($\Leftrightarrow$ $\mathbb{Q}$ is NOT closed).

pf: $\sqrt{2} \in \mathbb{Q}^c$, $\forall r > 0$, $(\sqrt{2}-r, \sqrt{2}+r) \not\subseteq \mathbb{Q}^c$. $\square$