Last time: $\begin{cases} u_t = \beta u_{xx} & t>0,\ 0<x<L \\ u_x(0,t) = u_x(L,t) = 0. & t>0 \\ u(x,0) = f(x) & 0<x<L \end{cases}$

TODAY:
- Heat eq^{n}
- Wave eq^{n}

Suppose $u(x,t) = X(x)T(t)$. Then

$$\underset{X \cdot T'}{\underset{\parallel}{u_t}} = \beta \underset{\beta X'' T}{\underset{\parallel}{u_{xx}}} \Rightarrow \begin{cases} \boxed{X''(x) + \lambda X(x) = 0} \text{ for some const. } \lambda. \\ T'(t) + \beta\lambda T(t) = 0 \end{cases}$$

$\frac{X''}{X} = \frac{T'}{\beta T} = -\lambda$

$\Rightarrow \sqrt{\lambda} = \frac{n\pi}{L}$, $X(x) = \cdot \cos\frac{n\pi x}{L}$

$T(t) =$

$$\underset{X'(0)T(t)}{u_x(0,t)} = \underset{X'(L)T(t)}{u_x(L,t)} = 0 \Rightarrow \boxed{X'(0) = X'(L) = 0.}$$

- If $\lambda < 0$, then $X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$.

  $X'(x) = c_1\sqrt{-\lambda}\, e^{\sqrt{-\lambda}x} - c_2\sqrt{-\lambda}\, e^{\sqrt{-\lambda}x}$,

  which can't take value 0 twice.

- If $\lambda = 0$, then $X(x) = c_1 + c_2 x$,

  $X'(x) = c_2$, so $c_2 = 0$ and $X(x) \equiv c_1$ const.

- If $\lambda > 0$, then: $X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

  $X'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$.

$$\begin{cases} 0 = X'(0) = c_2\sqrt{\lambda} \Rightarrow \boxed{c_2 = 0} \\ 0 = X'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) \end{cases}$$

$n = 1, 2, \dots$

To get a nontrivial $X(x)$, we should have $\sqrt{\lambda} = \frac{n\pi}{L}$

Summary: possible (nontrivial) sol^n for $X(x)$:

- $\lambda = 0$ and $X(x) \equiv \text{const}$.
- $\lambda = \left(\frac{n\pi}{L}\right)^2$ and $X(x) = \text{const} \cdot \cos\left(\frac{n\pi x}{L}\right)$, for $n = 1, 2, \cdots$

For each λ, $T'(t) + \beta\lambda T(t) = 0$

$$\Rightarrow \quad T(t) = \text{const.} \cdot e^{-\beta\lambda t}.$$

$\Longrightarrow$ For each $n \geq 0$,

$$u(x,t) = \cos\left(\frac{n\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \quad \text{is a sol}^n.$$

to $\begin{cases} u_t = \beta u_{xx} \\ u_x(0,t) = u_x(L,t) = 0. \end{cases}$

$\Longrightarrow$ For any $\{a_n\} \subseteq \mathbb{R}$,

$$u(x,t) := \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$

is a sol^n.

We need to find $\{a_n\}$ s.t. $u(x,0) = f(x)$,

i.e. $\frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$

In other words, we want to write $f(x)$ into the form

$f(x)$ on $(0,L)$

If $g(x)$ on $(-L,L)$,

$$\tilde{g} := \frac{a_0}{2} + \sum \left(a_k \cos\frac{k\pi x}{L} + \cancel{b_k} \sin\frac{k\pi x}{L}\right)$$

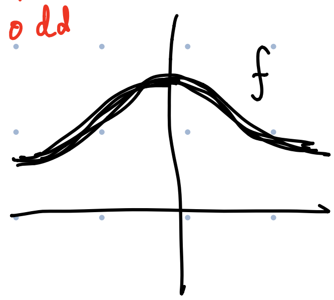
where

$$a_k = \frac{1}{L}\int_{-L}^{L} g(x) \cos\frac{k\pi x}{L}\, dx$$

If g is even fcn $(-L,L)$

$$b_k = \frac{1}{L}\int_{-L}^{L} \underset{\text{even}}{g(x)}\ \underset{\text{odd}}{\sin\frac{k\pi x}{L}}\, dx = 0$$

We do an "even extension" of f



$$\boxed{f_e(x)} = \begin{cases} f(x), & \text{where } 0<x<L \\ f(-x), & \text{where } -L<x<0. \end{cases}$$

Fourier coeff. of f_e:

$$a_k = \frac{1}{L}\int_{-L}^{L} \underset{\text{even}}{f_e(x)}\ \underset{\text{even}}{\cos\frac{k\pi x}{L}}\, dx$$

$$= \frac{2}{L}\int_{0}^{L} \underbrace{f_e(x)}_{f(x) \text{ on } (0,L)} \cos\frac{k\pi x}{L}\, dx$$

$$= \frac{2}{L}\int_{0}^{L} f(x) \cos\frac{k\pi x}{L}\, dx$$

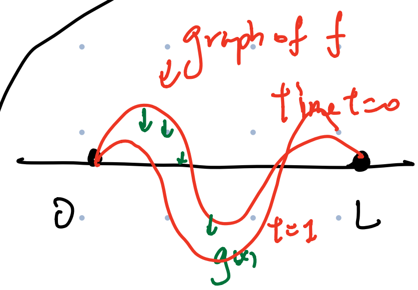
Summary: If f is conti. on $(0,L)$, then

$$a_k = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{k\pi x}{L}\, dx$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$ is the sol^{n}

<u>Wave eq^{n}</u>:

$$\begin{cases} u_{tt} = \alpha^2 u_{xx}, & 0<x<L,\ t>0. \\ u(0,t) = u(L,t) = 0, & t>0. \\ u(x,0) = f(x) & 0<x<L \\ u_t(x,0) = g(x) & 0<x<L \end{cases}$$



$u(x,t)$ = position of the string at $0<x<L$ at time $t>0$

Suppose $u(x,t) = X(x)\,T(t)$.

$$X \cdot T'' = \alpha^2 X'' \cdot T$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{T''}{\alpha^2 T} = -\lambda$$

$$\Rightarrow \begin{cases} \boxed{X''(x) + \lambda X(x) = 0 \qquad X(0) = X(L) = 0} \\ \boxed{T''(t) + \alpha^2 \lambda T(t) = 0} \end{cases}$$

$\boxed{\lambda = \left(\frac{n\pi}{L}\right)^2}$ and $X(x) = \text{const.}\ \sin\left(\frac{n\pi x}{L}\right)$.
$n = 1, 2, \ldots$

(aux. eq^{n}: $r^2 + \alpha^2\left(\frac{n\pi}{L}\right)^2 = 0 \Rightarrow r = \pm i \alpha \frac{n\pi}{L}$

$$T(t) = \text{const.}\ \cos\frac{n\pi\alpha t}{L} + \text{const.}\ \sin\frac{n\pi\alpha t}{L}$$

For each $n \geq 1$,

$$u(x,t) = \left(\text{const.}\ \cos\frac{n\pi\alpha t}{L} + \text{const.}\ \sin\frac{n\pi\alpha t}{L}\right) \cdot \sin\frac{n\pi x}{L}$$

$\Rightarrow$ ⊗

$$u(x,t) = \sum_{n\geq 1} \left(a_n \cos \frac{n\pi\alpha t}{L} + b_n \sin \frac{n\pi\alpha t}{L} \right) \sin \frac{n\pi x}{L}$$

is a sol^{n} of $\begin{cases} u_{tt} = \alpha^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \end{cases}$

for any $\{a_n\}, \{b_n\} \subseteq \mathbb{R}$.

Find $\{a_n\}, \{b_n\}$ s.t. $\begin{cases} u(x,0) = f(x) & \text{——— ①} \\ u_t(x,0) = g(x). & \text{——— ②} \end{cases}$

①: $u(x,0) = \sum_{n\geq 1} a_n \sin \frac{n\pi x}{L} = f(x)$

($\{a_n\}$ can be determined by the Fourier sine series of f).

② $u_t(x,t) = \sum_{n\geq 1} \left[\left(a_n \cdot \left(-\frac{n\pi\alpha}{L} \right) \sin \frac{n\pi\alpha t}{L} + b_n \left(\frac{n\pi\alpha}{L} \right) \cos \frac{n\pi\alpha t}{L} \right) \cdot \sin \frac{n\pi x}{L} \right]$

$$u_t(x,0) = \boxed{\sum_{n\geq 1} \boxed{b_n \cdot \boxed{\left(\frac{n\pi\alpha}{L} \right)}} \sin \frac{n\pi x}{L}} = g(x)$$

($\{b_n\}$ can be determined by the Fourier sine series of g)

Uniqueness of sol^{n} via energy method

$$\begin{cases} u_{tt} = \alpha^2 u_{xx}, & 0<x<L,\ t>0. \\ u(0,t) = u(L,t) = 0, & t>0. \\ u(x,0) = f(x) & 0<x<L \\ u_t(x,0) = g(x) & 0<x<L \end{cases}$$

Suppose u_1, u_2 are both sol^{ns}.

$W := u_1 - u_2$ Want: $\boxed{W \equiv 0}$

$$\boxed{\begin{cases} W_{tt} = \alpha^2 W_{xx} \\ \boxed{W(0,t)} = \boxed{W(L,t) = 0} \\ W(x,0) = 0 \\ W_t(x,0) = 0 \end{cases}}$$

??

$$E(0) = \frac{1}{2}\int_0^L \alpha^2 \underbrace{W_x(x,0)^2}_{=0} + \underbrace{W_t(x,0)^2}_{=0}\, dx = 0$$

$$E(t) := \frac{1}{2}\int_0^L \Big(\underbrace{\alpha^2 W_x^2}_{\text{potential energy}} + \underbrace{W_t^2}_{\text{kinetic energy}}\Big)\, dx.$$

$$\begin{aligned}(W_x^2)_t &= (W_x \cdot W_x)_t \\ &= W_{xt}W_x + W_x W_{xt} \\ &= 2W_{xt}W_x\end{aligned}$$

$$\frac{dE}{dt} = \frac{1}{2}\int_0^L \left(\alpha^2 \cdot 2W_{xt}W_x + 2W_tW_{tt}\right) dx.$$

$$\int_0^L W_{xt}W_x\, dx = \cancel{W_x W_t \Big|_0^L} - \int_0^L W_{xx}W_t\, dx.$$

$$W_x(L,t)\underbrace{\boxed{W_t(L,t)}}_{=0} - W_x(0,t)\underbrace{\boxed{W_t(0,t)}}_{=0}$$

$$= \cancel{\frac{1}{2}} \int_0^L \cancel{2}\,\alpha^2 \left(-w_{xx} w_t\right) + \cancel{2}\, w_t w_{tt} \quad dx$$

$$= 0 \qquad \left(w_{tt} = \alpha^2 w_{xx}\right)$$

$\Rightarrow$ $E(t)$ is a const. fn in t

$\Rightarrow$ $E(t) \equiv 0 \quad \forall t$

$$\| $$

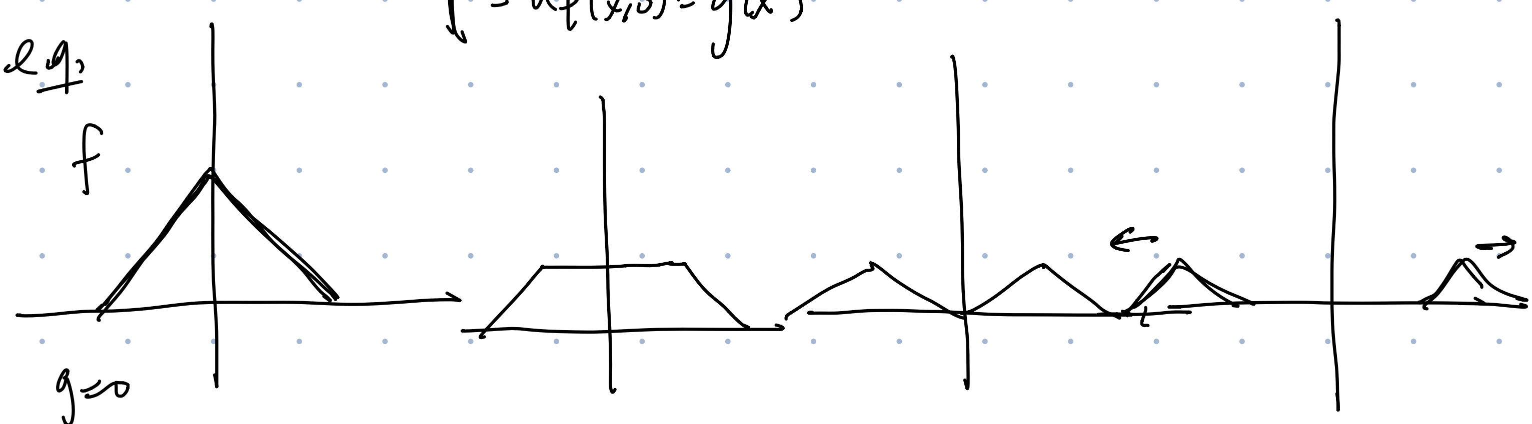
$$\boxed{\underbrace{\frac{1}{2}\int_0^L \left(\alpha^2 w_x^2 + w_t^2\right) dx}_{\geq 0}}$$

$\Rightarrow$ $w_x(x,t) = w_t(x,t) = 0.$

$\Rightarrow$ $w(x,t) \equiv \text{const.} \equiv 0.$ $\square$

Traveling wave: $\begin{cases} u_{tt} = \alpha^2 u_{xx} & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$

e.g.



d'Alembert sol^{n}:

$$\begin{cases} \psi = x + \alpha t \\ \eta = x - \alpha t \end{cases}$$

Write $\boxed{u_{tt} = \alpha^2 u_{xx}}$ in the variables ψ, η:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \psi}\frac{\partial \psi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \psi} + \frac{\partial u}{\partial \eta}$$

multivar. chain rule

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \psi}\frac{\partial \psi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = \alpha\left(\frac{\partial u}{\partial \psi} - \frac{\partial u}{\partial \eta}\right)$$

$$\boxed{u_{\psi\eta} = 0}$$

$$\|$$

$$\boxed{\frac{\partial}{\partial \psi}\left(\frac{\partial u}{\partial \eta}\right)} \Rightarrow \frac{\partial u}{\partial \eta} \text{ is const. in } \psi$$

$$\Rightarrow \frac{\partial u}{\partial \eta} = \underline{b(\eta)}$$

$$\Rightarrow u = B(\eta) + A(\psi)$$

$$\Rightarrow u(x,t) = A(x + \alpha t) + B(x - \alpha t)$$

for some fns A, B.

Plug this into $\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$

- $f(x) = u(x,0) = A(x) + B(x)$
- $g(x) = u_t(x,0) = \alpha\left(A'(x) - B'(x)\right)$

$$\Rightarrow \quad u(x,t) = \frac{1}{2}\left(f(x+\alpha t) + f(x-\alpha t)\right) + \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} g(s)\,ds$$