Today: generalized eigenspace; preview of differential eq'n

$A \in M_{n\times n}(\mathbb{C})$, λ is an eigenvalue of A,

generalized eigenspace:

$$V_\lambda^{gen} := \{ \vec{v} \in \mathbb{C}^n \mid (A-\lambda I)^k \vec{v} = \vec{0} \text{ for some } k \geq 1 \}.$$

Rmk: $0 \subseteq \underbrace{Nul(A-\lambda I)}_{\text{eigenspace of } \lambda} \subseteq Nul(A-\lambda I)^2 \subseteq \cdots$

Slightly more generally, $T: V \to V$, ← finite dim'l

$$0 \subseteq \ker(T) \subseteq \ker(T^2) \subseteq \cdots \subseteq \underbrace{\ker(T^s)}_{\text{stablized kernel of } T.} = \ker(T^{s+1}) = \cdots$$

Thm: $A: n\times n$ $\{\lambda_1, \dots, \lambda_k\}$ distinct eigenvalues of A.

$$\mathbb{C}^n = V_{\lambda_1}^{gen} \oplus \cdots \oplus V_{\lambda_k}^{gen}$$

i.e. any vector $\vec{v} \in \mathbb{C}^n$ can be uniquely written as

$$\vec{v} = \vec{v}_1 + \cdots + \vec{v}_k,$$

where each $\vec{v}_i \in V_{\lambda_i}^{gen}$.

Rmk: When $k=1$, we proved it using the Cayley-Hamilton thm.

Let's prove such decomposition is unique first:

Suppose $\vec{v} = \vec{v}_1 + \cdots + \vec{v}_k$ $\quad \vec{v}_i, \vec{w}_i \in V_{\lambda_i}^{gen}$.

$= \vec{w}_1 + \cdots + \vec{w}_k$,

$$\vec{0} = \underset{\underset{V^{gen}_{\lambda_1}}{\cap}}{(\vec{v}_1 - \vec{w}_1)} + \cdots + \underset{\underset{V^{gen}_{\lambda_k}}{\cap}}{(\vec{v}_k - \vec{w}_k)}$$

$(A-\lambda_1 I)^{\ell}\vec{v}_1 = \vec{0}$ for some $\ell \geq 1$

We need to prove: If $\vec{v}_1 \in V^{gen}_{\lambda_1}, \cdots, \vec{v}_k \in V^{gen}_{\lambda_k}$,

$$\underset{\neq 0}{\vec{v}_1} + \cdots + \underset{\neq 0}{\vec{v}_k} = \vec{0},$$

then $\vec{v}_1 = \cdots = \vec{v}_k = \vec{0}$.

Recall: If $\vec{v}_i \in Nul(A-\lambda_i I)$, $\boxed{\underset{\neq \vec{0}}{\vec{v}_1} + \cdots + \vec{v}_k = \vec{0}}$

$$A\vec{v}_1 + \cdots + A\vec{v}_k = \vec{0}$$

$$\parallel$$

$$\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 + \cdots + \lambda_k\vec{v}_k$$

$$\Rightarrow \underbrace{(\lambda_2-\lambda_1)\vec{v}_2}_{\substack{\neq \\ \vec{0} \\ \text{in } Nul(A-\lambda_2 I)}} + \cdots + \underbrace{(\lambda_k-\lambda_1)\vec{v}_k}_{\substack{\neq \\ \vec{0}}} = \vec{0}$$

$$(A-\lambda_1 I)^{\ell}\vec{v}_2 + \cdots + (A-\lambda_1 I)^{\ell}\vec{v}_k = \vec{0}$$

Claim: $(A-\lambda_1 I)^{\ell}\vec{v}_2 \neq \vec{0}$ and

$\underline{(A-\lambda_1 I)^{\ell}\vec{v}_2} \in V^{gen}_{\lambda_2}$. easy

$\underline{(A-\lambda_2 I)}^m \vec{v}_2 = \vec{0}$ for some $m \geq 1$.

$$\Rightarrow (A-\lambda_2 I)^m (A-\lambda_1 I)^{\ell}\vec{v}_2 = \vec{0}$$

Claim: $V^{gen}_{\lambda_1} \cap V^{gen}_{\lambda_2} = \{\vec{0}\}$.

(i.e. given any $\vec{v} \in \mathbb{C}^n$. Suppose $\exists\, k \geq 1$

s.t. $(A - \lambda_1 I)^k \vec{v} = \vec{0}$,

$(A - \lambda_2 I)^k \vec{v} = \vec{0}$.

$\Longrightarrow \vec{v} = \vec{0}$.)

looks quite hard to prove.

pf Suppose $V^{gen}_{\lambda_1} \cap V^{gen}_{\lambda_2} = W \neq \{\vec{0}\}$.

Choose some $\vec{v} \in W \setminus \{\vec{0}\}$,

$\exists\, k \geq 1$ s.t. $\underline{(A - \lambda_1 I)^k \vec{v} = \vec{0}}$, and $\boxed{(A-\lambda_1 I)^{k-1} \vec{v}} \neq \vec{0}$ in W.

$\boxed{(A-\lambda_1 I)^{k-1} \vec{v}} = \vec{v}'$

- $\vec{v}' \neq \vec{0}$
- $(A - \lambda_1 I)\vec{v}' = \vec{0} \Rightarrow \boxed{A\vec{v}' = \lambda_1 \vec{v}'}$
- $\underbrace{(A - \lambda_2 I)^{\ell} \vec{v}'}_{= (\lambda_1 - \lambda_2)^{\ell} \vec{v}'} = \vec{0}$ for some $\ell \geq 1$.

$(\lambda_1 - \lambda_2)^{\ell} \vec{v}'$ contradiction. $\square$

Recall from HW5 extra credit problem: $T: V \to V$ finite dim'l

$0 \subseteq \ker(T) \subseteq \ker(T^2) \subseteq \cdots \subseteq \ker(T^k) = \ker(T^{k+1}) = \cdots = \ker^s(T)$

$V \supseteq \operatorname{Im}(T) \supseteq \operatorname{Im}(T^2) \supseteq \cdots \supseteq \operatorname{Im}(T^k) = \operatorname{Im}(T^{k+1}) = \cdots = \operatorname{Im}^s(T)$

- $V = \ker^s(T) \oplus \operatorname{Im}^s(T)$
- T preserves this decomposition: $T(\ker^s(T)) \subseteq \ker^s(T)$,
  $T(\operatorname{Im}^s(T)) \subseteq \operatorname{Im}^s(T)$

- $T|_{\ker^s(T)}$: nilpotent; $T|_{\mathrm{Im}^s(T)}$: invertible.

Prove: $A = n \times n$, has $\{\lambda_1, \ldots, \lambda_k\}$ distinct eigenvalues
any $\vec{v} \in \mathbb{C}^n$ can be written $\vec{v} = \vec{v}_1 + \cdots + \vec{v}_k$, where $\vec{v}_i \in V_{\lambda_i}^{gen}$

Prove by induction on k.

- $k = 1$. (we proved it before using Cayley-Hamilton).
- Suppose it's true for operators w/ at most $k-1$ eigenvalues.

Consider $T = T_{A - \lambda_k I} : \mathbb{C}^n \to \mathbb{C}^n$

- $\mathbb{C}^n = \underbrace{\ker^s(T_{A-\lambda_k I})}_{\text{generalized eigenspace of } \lambda_k} \oplus \mathrm{Im}^s(T_{A-\lambda_k I})$

- $T_{A-\lambda_k I}$ preserves this decomp.

$\Rightarrow T_A$ preserves this decomp.

So, it makes sense to consider $T_A|_{\mathrm{Im}^s(T_{A-\lambda_k I})}$

- Claim: $T_A|_{\mathrm{Im}^s(T_{A-\lambda_k I})}$ only has eigenvalues $\lambda_1, \ldots, \lambda_{k-1}$

pf: $T_{A-\lambda_k I}|_{\mathrm{Im}^s(A-\lambda_k I)}$ is invertible.

so $\forall \vec{v} \in \mathrm{Im}^s(A - \lambda_k I) \setminus \{\vec{0}\}$, $T_{A-\lambda_k I}\vec{v} \neq \vec{0}$

$\Rightarrow (A - \lambda_k I)\vec{v} \neq \vec{0} \Rightarrow A\vec{v} \neq \lambda_k \vec{v}$. $\square$

By inductive hypotheses, $T_A|_{Im^s(T_{A-\lambda_k I})} : Im^s(T_{A-\lambda_k I}) \circlearrowleft$,

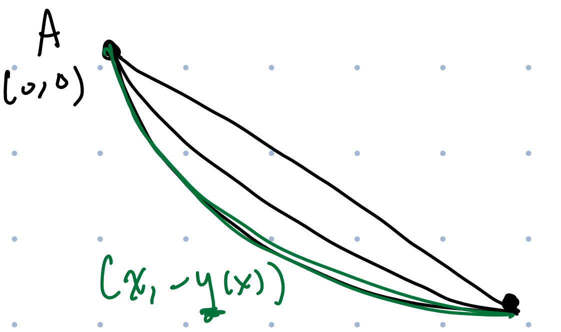
so $\forall \vec{v} \in Im^s(T_{A-\lambda_k I})$,

$\exists \vec{v}_1, \dots, \vec{v}_{k-1}$

s.t. $\vec{v} = \vec{v}_1 + \dots + \vec{v}_{k-1}$, where $\vec{v}_i \in V^{gen}_{\lambda_i}$

Together with $\mathbb{C}^n = V^{gen}_{\lambda_k} \oplus Im^s(T_{A-\lambda_k I})$,

proves the existence part of the thm. $\square$



Q: Find the path along which an object slide (without friction), in the shortest possible $\underline{\underline{\text{time}}}$

(Brachistochrone problem),

$y: [0, a] \longrightarrow [0, b]$

s.t. $y(0) = 0$, $y(a) = b$.

<u>Write down the traveling time in terms of y</u>

$$T = \frac{1}{\sqrt{2g}} \int_0^a \sqrt{\frac{1 + y'(x)^2}{y(x)}}\, dx.$$

(length of the path in terms of y: $L = \int_0^a \sqrt{1 + y'(x)^2}\, dx$)

$T: \{\text{space of functions}\} \longrightarrow \mathbb{R}$

$$y \longmapsto \frac{1}{\sqrt{2g}} \int_0^a \sqrt{\frac{1+y'(x)^2}{y(x)}}\, dx.$$

Goal: Find ~~the~~ y that minimize T.

Euler-Lagrange eq'n: If y is a minimizer of $F(y) = \int_0^a f(x,y,y')\,dx$ then y satisfies:

$$\frac{\partial}{\partial y} f(x,y,y') = \frac{d}{dx} \frac{\partial}{\partial y'} f(x,y,y')$$

Result:

length: E-L $\Rightarrow y'' = 0 \Rightarrow y$ is linear

time: E-L $\Rightarrow 2yy'' + (y')^2 + 1 = 0$

$\Rightarrow$ cycloid