HOMEWORK 4
MATH 104, SECTION 2

Some ground rules:

- You have to submit your homework via **Gradescope** to the corresponding assignment. The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You're allowed to use any result that is proved in the lecture; but if you'd like to use other results, you have to prove them before using them.

Problem set (6 problems; due February 16 at 11am PT)

(1) A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *contractive* if there exists $0 < K < 1$ such that

$$|f(x) - f(y)| \leq K|x - y| \text{ holds for any } x, y \in \mathbb{R}.$$

You'll show that any contractive map on $\mathbb{R}$ has a unique fixed point.

  (a) Pick any $x_1 \in \mathbb{R}$. Construct a sequence (x_n) recursively via $x_{n+1} := f(x_n)$. Prove that such sequence (x_n) is a Cauchy sequence, therefore is convergent.

  (b) Moreover, prove that the limit x^* of (x_n) is a *fixed point* of f, i.e. $f(x^*) = x^*$. (Hint: First, show that $\lim f(x_n) = x^*$ by the construction of (x_n). On the other hand, show that $\lim f(x_n) = f(x^*)$ by the fact that f is contractive.)

  (c) Prove that f has a *unique* fixed point.

(2) Let $\{S_\alpha\}$ be a collection of (possibly infinitely many) subsets of a set S. Prove:

  (a) The complement of union is the intersection of complements: $(\cup_\alpha S_\alpha)^c = \cap_\alpha (S_\alpha^c)$.

  (b) The complement of intersection is the union of complements: $(\cap_\alpha S_\alpha)^c = \cup_\alpha (S_\alpha^c)$.

(3) Prove that in a metric space:

  (a) The union of (possibly infinitely many) open subsets is open.

  (b) The intersection of *finitely many* open subsets is open.

  (c) The intersection of (possibly infinitely many) closed subsets is closed.

  (d) The union of *finitely many* closed subsets is closed.

  (e) Find a counterexample of (a) if 'open' is replaced by 'closed'; find a counterexample of (c) if 'closed' is replaced by 'open'.

(4) Let (S, d) be a metric space, and $K \subseteq S$ be a compact subset. Prove that K is bounded (i.e. there exist $x \in K$ and $R > 0$ such that $K \subseteq B_R(x)$).

(5) Let (S, d) be a metric space, $K \subseteq S$ be a compact subset, and $C \subseteq S$ be a closed subset. Prove that $C \cap K$ is a compact subset of S. (Hint: First show that $C \cap K$ is a closed subset of S. Now let $\{U_\alpha : \alpha \in I\}$ be an open cover of $C \cap K$, then $\{U_\alpha : \alpha \in I\} \cup \{(C \cap K)^c\}$ is an open cover of the compact set K.)

(6) Let $E \subseteq (S, d)$ be a subset of a metric space. Define the *Cantor–Bendixson derivative* of E:

$$E' := \{x \in S \colon x \text{ is a limit point of } E\}.$$

(a) Show that E' is a closed subset of S.

(b) Show that if $E' \neq \emptyset$ then E contains infinitely many elements.

(Recall that $x \in S$ is a limit point of E if for any $r > 0$, the intersection $B_r(x) \cap E$ contains at least a point other than x.)