# HOMEWORK 5
# MATH H54

**Yu-Wei's Office Hours**: Sunday 1-2:30pm and Thursday 12-1:30pm (PDT)
**Michael's Office Hours**: Monday 12-3pm (PDT)

## Part I (no need to turn in)

This part of the homework provides some routine computational exercises. You don't have to turn in your solutions for this part, but being able to do the computations is vitally important for the learning process, so you definitely should do these practices before you start doing Part II of the homework.

The following exercises are from the corresponding sections of the UC Berkeley custom edition of Lay, Nagle, Saff, Snider, *Linear Algebra and Differential Equations*.

- **Exercise 5.1**: 13, 19, 21
- **Exercise 5.2**: 13
- **Exercise 5.3**: 15, 21, 23, 27, 31
- **Exercise 5.4**: 7, 11, 15
- **Exercise 5.5**: 3, 9, 15

## Part II (Due October 13, 8am PDT)

Some ground rules:

- You have to submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW5**.
- The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own.
- You're allowed to use any result that is proved in the lecture. But if you'd like to use other results, you have to prove it first before using it.

Problems:

(1) Let $A$ be a square matrix. Suppose that there exists a positive integer $k$ such that $A^k = 0$, then the only eigenvalue of $A$ is $0$.
(2) Prove that $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is an eigenvalue of $A^T$.
(3) Let $A$ be an $n \times n$ matrix. Suppose that the sum of entries in each row of $A$ equals to $1$ (i.e. $a_{i1} + a_{i2} + \cdots + a_{in} = 1$ for any $1 \leq i \leq n$). Prove that $1$ is an

eigenvalue of $A$. (Hint: Find an eigenvector.) (Remark: These matrices arise in the study of Markov chains.)

(4) Let $A$ be an $n \times n$ matrix (not necessarily diagonalizable) with eigenvalues $\lambda_1, \dots, \lambda_n$ (possibly with multiplicities).
  (a) Prove that $\det(A) = \lambda_1 \cdot \dots \cdot \lambda_n$.
  (b) Prove that $\operatorname{tr}(A) = \lambda_1 + \dots + \lambda_n$. (Recall that $\operatorname{tr}(A) = a_{11} + a_{22} + \dots + a_{nn}$.)
  (Hint: Consider the characteristic polynomial of $A$.)

(5) Prove that if two square matrices $A$ and $B$ are similar, then $\operatorname{rank}(A) = \operatorname{rank}(B)$.

(6) Suppose that $A$ is a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ (possibly with multiplicities). Prove that

$$(\lambda_1 \mathbb{I} - A)(\lambda_2 \mathbb{I} - A) \cdots (\lambda_n \mathbb{I} - A) = 0 \text{ (the zero matrix).}$$

  (In fact, the statement holds true without the diagonalizable assumption; this is the Cayley–Hamilton theorem.)

(7) Suppose that $A$ and $B$ are two $n \times n$ matrices that commute, i.e. $AB = BA$. Also, suppose that $B$ has $n$ distinct eigenvalues.
  (a) Prove that if $B\vec{v} = \lambda\vec{v}$, then $BA\vec{v} = \lambda A\vec{v}$.
  (b) Prove that any eigenvector of $B$ is also an eigenvector of $A$.
  (c) Prove that $A, B, AB$ are all diagonalizable.

(8) (a) Let $A$ be a real $n \times n$ matrix. Let $\{\lambda_1, \dots, \lambda_k\}$ be the set of distinct eigenvalues of $A^2$. Assume that $\lambda_i$ is real and positive for each $1 \le i \le k$. Let $\mathcal{S}$ be the set of distinct eigenvalues of $A$.
    (i) Prove that $\mathcal{S}$ is a subset of $\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, \dots, \sqrt{\lambda_k}, -\sqrt{\lambda_k}\}$.
    (ii) Prove that for each $1 \le i \le k$, $\mathcal{S}$ contains at least one of $\pm\sqrt{\lambda_i}$. (Hint: Consider $(A^2 - \lambda\mathbb{I}) = (A - \sqrt{\lambda}\mathbb{I})(A + \sqrt{\lambda}\mathbb{I})$.)
  (b) Find a $3 \times 3$ matrix $A$ such that

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 9 \end{bmatrix}.$$

  (It's enough to write $A = PDP^{-1}$ with explicit $P$ and $D$.)
  How many such matrices are there? Justify your answer.

(9) Let $P$ be a square matrix such that $P^2 = P$.
  (a) Prove that $\operatorname{Col}(P) = \operatorname{Nul}(P - \mathbb{I})$.
  (b) Prove that $P$ is diagonalizable via the rank-nullity theorem.

(10) Let $A$ be an $n \times n$ *real* symmetric matrix, i.e. $A^T = A$.
  (a) Let $\mathbf{x}$ be any vector in $\mathbb{C}^n$. Denote its conjugate by $\bar{\mathbf{x}} \in \mathbb{C}^n$. Prove that $\bar{\mathbf{x}}^T A\mathbf{x}$ is a *real* number. (Hint: Prove that $\bar{\mathbf{x}}^T A\mathbf{x} = \overline{\bar{\mathbf{x}}^T A\mathbf{x}}$.)
  (b) Prove that the eigenvalues of any real symmetric matrix are real, i.e. if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x} \in \mathbb{C}^n$, then $\lambda \in \mathbb{R}$. (Hint: Compute $\bar{\mathbf{x}}^T A\mathbf{x}$ and use the Part (a).)
  (c) Same notations as is Part (b). Prove that the real and imaginary parts of $\mathbf{x}$ satisfy $A(\mathrm{Re}\mathbf{x}) = \lambda(\mathrm{Re}\mathbf{x})$ and $A(\mathrm{Im}\mathbf{x}) = \lambda(\mathrm{Im}\mathbf{x})$.

## Part III (Extra credit problems, Due October 20, 8am PDT)

The following problem worths up to 2 extra points in total (out of 10), so you can potentially get 12/10 for this homework.

You have to submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW5_extra_credit**, which is separated from Part II.

(1) Let $T\colon V \to V$ be a linear transformation of a finite dimensional vector space $V$. Consider the series of subspaces of $V$

$$\{0\} \subseteq \ker(T) \subseteq \ker(T^2) \subseteq \cdots \quad \text{and} \quad V \supseteq \operatorname{Im}(T) \supseteq \operatorname{Im}(T^2) \supseteq \cdots .$$

(a) Prove that there exists $k > 0$ sufficiently large such that

$$\ker(T^k) = \ker(T^{k+1}) = \cdots \quad \text{and} \quad \operatorname{Im}(T^k) = \operatorname{Im}(T^{k+1}) = \cdots .$$

We'll call them the *stable kernel* and the *stable image* of $T$ and denoted by

$$\ker^s(T) \coloneqq \ker(T^k) \quad \text{and} \quad \operatorname{Im}^s(T) \coloneqq \operatorname{Im}(T^k).$$

(b) Prove the following equivalent descriptions of stable kernel and stable image:

$$\ker^s(T) = \{\vec{v} \in V : T^\ell \vec{v} = \vec{0} \ \text{ for some } \ell\},$$

$$\operatorname{Im}^s(T) = \{\vec{v} \in V : \text{for any } \ell, \text{ there exists } \vec{w} \in V \text{ s.t. } \vec{v} = T^\ell(\vec{w})\}.$$

(c) Prove that $\ker^s(T) \cap \operatorname{Im}^s(T) = \{0\}$.

(d) Prove that $V = \ker^s(T) + \operatorname{Im}^s(T)$.

(e) The statement without stabilization "$V = \ker(T) + \operatorname{Im}(T)$" is not true in general. Find a counterexample.

(2) Same notations as previous problem.

(a) Prove that $T(\ker^s(T)) \subseteq \ker^s(T)$ and $T(\operatorname{Im}^s(T)) \subseteq \operatorname{Im}^s(T)$. (In other words, the linear transformation $T$ respects the decomposition $V = \ker^s(T) \oplus \operatorname{Im}^s(T)$.) Hence one can define the restriction linear maps $T|_{\ker^s(T)}\colon \ker^s(T) \to \ker^s(T)$ and $T|_{\operatorname{Im}^s(T)}\colon \operatorname{Im}^s(T) \to \operatorname{Im}^s(T)$.

(b) Prove that there exists $L > 0$ such that $(T|_{\ker^s(T)})^L = 0$.

(c) Prove that $T|_{\operatorname{Im}^s(T)}$ is invertible.