1. A holomorphic mapping $f : U \to V$ is a **local bijection** on U if for every $z \in U$ there exists an open disc $D \subset U$ centered at z, so that $f : D \to f(D)$ is a bijection.

Prove that a holomorphic map $f : U \to V$ is a local bijection on U if and only if $f'(z) \neq 0$ for all $z \in U$.

[Hint: Use Rouché's theorem as in the proof of Proposition 1.1.]

① "local bijection" $\Longrightarrow$ "$f'(z) \neq 0 \quad \forall z \in U$".

- Assume the contrary that $\exists z_0 \in U$ s.t. $f'(z_0) = 0$.
- Then, near z_0, we have:

$$f(z) = f(z_0) + a_k (z - z_0)^k + (z - z_0)^{k+1} g(z),$$

where $k \geq 2$, $a_k \neq 0$, g: holo. near z_0.

- Choose $\varepsilon > 0$ small enough s.t.
  1) z_0 is the only zero of f' in $\mathbb{D}_\varepsilon(z_0)$.
  2) on $\partial D_\varepsilon(z_0)$, we have:

$$|a_k (z - z_0)^k| > |(z - z_0)^{k+1} g(z)|.$$

- $\exists\, w \neq 0$, $|w|$ small enough s.t.

$$|a_k (z - z_0)^k| > |(z - z_0)^{k+1} g(z) - w| \quad \forall z \in \partial \mathbb{D}_\varepsilon(z_0).$$

- By Rouché's thm,

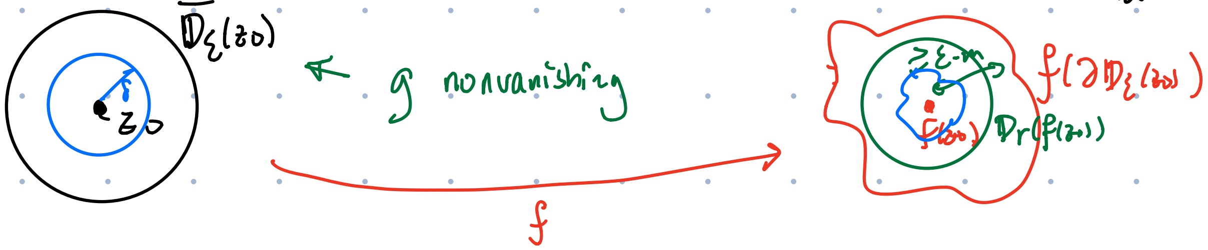
$$f(z) - f(z_0) - w \text{ has at least } k \geq 2 \text{ zeros in } \mathbb{D}_\varepsilon(z_0).$$

- If there is a zero $\tilde{z}$ of $f(z) - f(z_0) - w$ with order ≥ 2, then $f'(\tilde{z}) = 0$. $\Longrightarrow$ $\tilde{z} = z_0$ since z_0 is the only zero of f' in $\mathbb{D}_\varepsilon(z_0)$.
  But z_0 is not a zero of $f(z) - f(z_0) - w$ since $w \neq 0$.
- Therefore, $f(z) - f(z_0) - w$ has at least 2 distinct zeros in $\mathbb{D}_\varepsilon(z_0)$.
  This argument works for any $\varepsilon > 0$ sufficiently small, which contradicts with the local bijectivity of f. $\square$

②: "$f'(z) \neq 0 \quad \forall z \in U$" $\Longrightarrow$ "local bijection".

- For each $z_0 \in \mathcal{U}$, f can be written locally near z_0 as:

$$f(z) = f(z_0) + (z - z_0)\, g(z), \quad \text{where } g \text{ is nonvanishing near } z_0, \text{ holo.}$$



- Let $m := \inf\limits_{z \in \partial \mathbb{D}_\varepsilon(z_0)} |g(z)| > 0$. ($\because$ g nonvanishing & conti., $\partial \mathbb{D}_\varepsilon(z_0)$ compact)

- Then $|f(z) - f(z_0)| = |(z - z_0) g(z)| \geq \varepsilon \cdot m \quad \forall z \in \partial \mathbb{D}_\varepsilon(z_0)$

- Since f is conti., $\forall r > 0$ s.t. $r < \varepsilon \cdot m$.
  $\exists \delta > 0$ s.t. $f(\mathbb{D}_\delta(z_0)) \subseteq \mathbb{D}_r(f(z_0))$

- <u>Claim</u>: f is injective in $\mathbb{D}_\delta(z_0)$.

  <u>pf</u>: Let $w \in f(\mathbb{D}_\delta(z_0)) \subseteq \mathbb{D}_r(f(z_0))$.

  Then $|f(z) - f(z_0)| \geq \varepsilon \cdot m > r > |f(z_0) - w| \quad \forall z \in \partial \mathbb{D}_\varepsilon(z_0)$.

  Hence # of zeros of $f(z) - f(z_0)$ in $\mathbb{D}_\varepsilon(z_0)$ $= 1$
  $=$ # of zeros of $f(z) - w$ in $\mathbb{D}_\varepsilon(z_0)$. $\square$

3. Suppose U and V are conformally equivalent. Prove that if U is simply connected, then so is V. Note that this conclusion remains valid if we merely assume that there exists a continuous bijection between U and V.

- Suppose γ_0, γ_1 are two curves in V with the same end points.
  $\gamma_0, \gamma_1 : [a,b] \longrightarrow V$, $\gamma_0(a) = \gamma_1(a) = p$, $\gamma_0(b) = \gamma_1(b) = q$

- Let $f : \mathcal{U} \to V$, $f^{-1} : V \to \mathcal{U}$ be continuous bijections.

- $f^{-1}\circ\gamma_0$, $f^{-1}\circ\gamma_1$ are two curves in $\mathcal{U}$ w/ the same endpts $f^{-1}(p), f^{-1}(q)$
- Since $\mathcal{U}$ is simply connected, there exists homotopy:

$$F: [a,b]\times[0,1] \longrightarrow \mathcal{U} \quad \text{conti.}$$
$$(t, s) \longmapsto F(t,s)$$

s.t.
- $F(0,s) = f^{-1}(p)$, $F(1,s) = f^{-1}(q)$ $\forall s \in [0,1]$.
- $F(t,0) = f^{-1}(\gamma_0(t))$, $F(t,1) = f^{-1}(\gamma_1(t))$ $\forall t \in [a,b]$.

- Then $f\circ F: [a,b]\times[0,1] \xrightarrow{F} \mathcal{U} \xrightarrow{f} V$

is a homotopy between γ_0, γ_1. $\square$

4. Does there exist a holomorphic surjection from the unit disc to $\mathbb{C}$?
[Hint: Move the upper half-plane "down" and then square it to get $\mathbb{C}$.]

- $\mathbb{D}$ and $\mathbb{H}$ are conformally equivalent.
Hence it suffices to construct a holo. surj. $\mathbb{H} \to \mathbb{C}$
- e.g. $\mathbb{H} \longrightarrow \mathbb{C}$

$$z \longmapsto (z-i)^2$$

5. Prove that $f(z) = -\frac{1}{2}(z+1/z)$ is a conformal map from the half-disc $\{z = x+iy : |z| < 1,\ y > 0\}$ to the upper half-plane.
[Hint: The equation $f(z) = w$ reduces to the quadratic equation $z^2 + 2wz + 1 = 0$, which has two distinct roots in $\mathbb{C}$ whenever $w \neq \pm 1$. This is certainly the case if $w \in \mathbb{H}$.]

- The fn is holo. in the half-disc.
- Claim: $\forall w \in \mathbb{H}$, $-\frac{1}{2}\left(z+\frac{1}{z}\right) = w$ has a unique sol^{n} in the half-disc.

$\Updownarrow$

$z^2 + 2wz + 1 = 0$, easy to check!!

8. Find a harmonic function u in the open first quadrant that extends continuously up to the boundary except at the points 0 and 1, and that takes on the following boundary values: $u(x, y) = 1$ on the half-lines $\{y = 0,\ x > 1\}$ and $\{x = 0,\ y > 0\}$, and $u(x, y) = 0$ on the segment $\{0 < x < 1, y = 0\}$.

[Hint: Find conformal maps $F_1, F_2, \ldots, F_5$ indicated in Figure 11. Note that $\frac{1}{\pi}\arg(z)$ is harmonic on the upper half-plane, equals 0 on the positive real axis, and 1 on the negative real axis.]

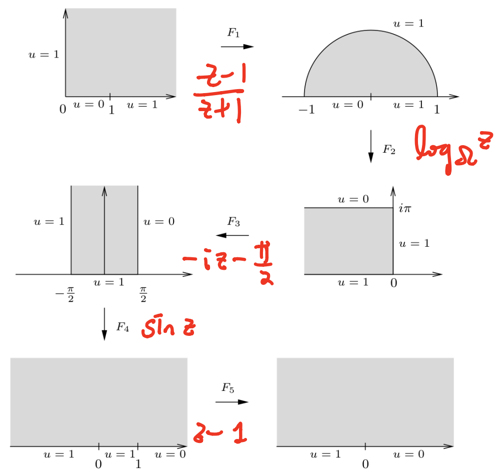
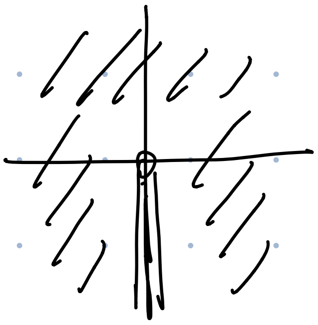


Figure 11. Successive conformal maps in Exercise 8

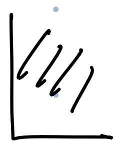
Consider  Ω, simply connected,

and $\log_\Omega(z)$ define on Ω s.t. $\log_\Omega(1) = 0$.

- Then $\operatorname{Im}\log_\Omega(z) = 0$ for $z \in \mathbb{R}_{>0}$, and
  $\operatorname{Im}\log_\Omega(z) = \pi$ for $z \in \mathbb{R}_{<0}$.

- Therefore $\log_\Omega(z)/\pi$ is holo. on Ω, so it's imaginary part is harmonic, and $\begin{cases} = 0 & \text{for } z \in \mathbb{R}_{>0}, \\ = 1 & \text{for } z \in \mathbb{R}_{<0}. \end{cases}$

- So we can take u to be:

$$\left(\frac{1}{\pi}\operatorname{Im}\log_\Omega\right) \circ F_5 \circ F_4 \circ F_3 \circ F_2 \circ F_1(z) :$$  $\longrightarrow \mathbb{R}$.

$$= \frac{1}{\pi}\operatorname{Im}\log_\Omega\left(\sin\left(-i\log_\Omega\left(\frac{z-1}{z+1}\right) - \frac{\pi}{2}\right) - 1\right).$$

$\square$