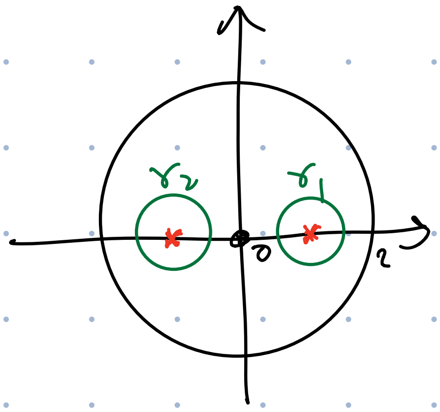
Today: Exam logistics, Practice problems.

(3) Compute

$$\int_{\gamma_2(0)} \boxed{\frac{\cos(\pi z)}{z^2-1}}\, dz$$

where $\gamma_2(0)$ is the circle of radius two centered at $0 \in \mathbb{C}$, oriented positively.



$$\int_{\gamma_2(0)} \frac{\cos(\pi z)}{z^2-1}\, dz$$

$$= \underbrace{\int_{\gamma_1} \frac{\cos(\pi z)}{z^2-1}\, dz}_{} + \underbrace{\int_{\gamma_2} \frac{\cos \pi z}{z^2-1}\, dz}_{\text{can be computed similarly}}$$

keyhole argument

$$\| \quad \int_{\gamma_1} \frac{1}{z-1} \cdot \boxed{\frac{\cos(\pi z)}{z+1}}\, dz = 2\pi i \cdot \frac{\cos(\pi \cdot 1)}{1+1}$$

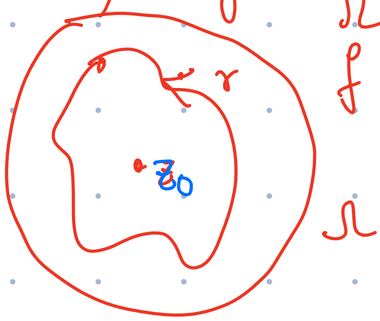
$$= 2\pi i \cdot \frac{-1}{2}$$

Cauchy integral formula.

Ω = open set contains γ & its interior

f: holo in Ω

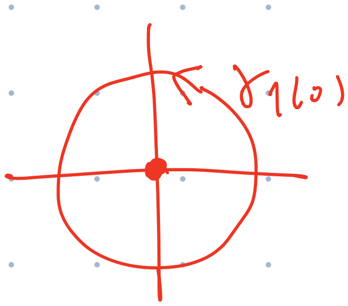
$$f(z_0) = \frac{1}{2\pi i} \int_{z \in \gamma} \boxed{f(z)} \frac{1}{z - z_0}\, dz$$



(4) Compute

$$\int_{\gamma_1(0)} \frac{e^z}{z}, = 2\pi i \cdot e^0 = 2\pi i$$

where $\gamma_1(0)$ is the circle of radius two centered at $0 \in \mathbb{C}$, oriented positively.



(6) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Assume that there exists a nonempty open subset $\Omega \subseteq \mathbb{C}$ such that $f(z) \notin \Omega$ for any $z \in \mathbb{C}$. Prove that f is a constant function.

$f : \mathbb{C} \to \mathbb{C}$

We have $f(z) \notin \mathbb{D}_R(z_0)$ $\forall z \in \mathbb{C}$

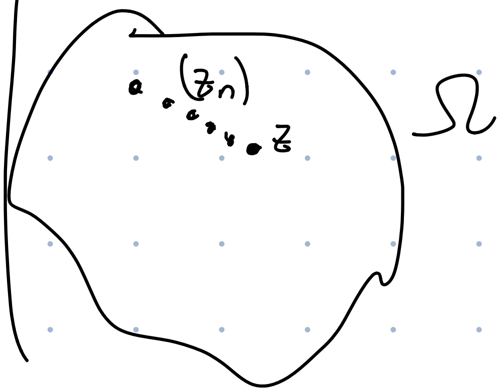
$$g(z) := \frac{1}{f(z) - z_0} : \mathbb{C} \xrightarrow[\text{holo.}]{} \mathbb{C}$$

(i.e. entire)

$$|g(z)| = \frac{1}{\underbrace{|f(z) - z_0|}_{\geq R}} \leq \frac{1}{R} \implies \underline{g \text{ is bounded}}$$

Liouville $\implies$ g const. $\implies$ f const. $\square$

(2) Let $\Omega \subseteq \mathbb{C}$ be an open and connected subset of $\mathbb{C}$, and let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function. Suppose that there is a curve $\gamma \subseteq \Omega$ such that f is constant on γ. Prove that f is constant in Ω.



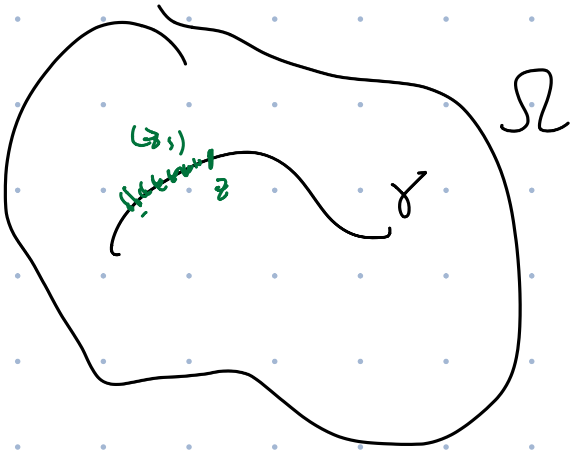
Thm

$\lim z_n = z$

$z_n \in \Omega,\ z \in \Omega$

$f(z_n) = 0 \quad \forall n$

$\Longrightarrow f \equiv 0$ in Ω



$f(z) = \underline{\underline{c}} \quad \forall z \in \gamma$

$g(z) = f(z) - c$ on Ω

By thm $\Rightarrow g \equiv 0$ on Ω

$\Rightarrow f \equiv c$ on Ω

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(7) Let $f(z) = z^2$.

(a) Calculate $\int_0^{2\pi} f(2+e^{it})dt$, and confirm that it is non-zero.

(b) Does Cauchy's theorem imply $\int_{\gamma_1(2)} f(z)dz = 0$? (Here $\gamma_1(2)$ is the circle of radius one centered at $2 \in \mathbb{C}$, oriented positively.) Explain the seeming discrepancy with part (a).

$\gamma_1(2)$ can be parametrized by $z\colon t \longmapsto 2+e^{it}$

$$\underbrace{\int_{\gamma_1(2)} f(z)\,dz}_{=\,0} = \int_0^{2\pi} f(2+e^{it}) \cdot \underbrace{(ie^{it})}_{z'(t)}\,dt$$

$\leftarrow$ (a) doesn't include this term.

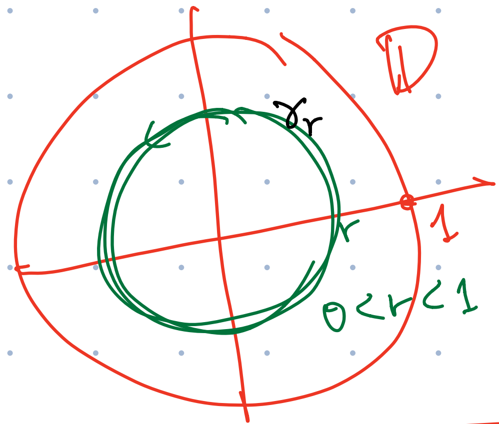
(8) Let $f\colon \mathbb{D} \to \mathbb{C}$ be a holomorphic function on the unit disk. Suppose that

$$|f(z)| \le \frac{1}{1-|z|} \quad \text{for any} \quad |z|<1.$$

Prove that

$$\left|f^{(n)}(0)\right| \le (n+1)!\left(1+\frac{1}{n}\right)^n \quad \text{for all} \quad n \ge 1.$$

$$|f^{(n)}(0)| \le \frac{n!}{r^n} \sup_{|z|=r} |f(z)| \le \frac{n!}{r^n} \frac{1}{1-r}$$



$$\left(\left|f^{(n)}(0)\right| = \left|\frac{n!}{2\pi i}\int_{\gamma_r} \frac{f(w)}{(w-0)^{n+1}}\,dw\right| \le \frac{n!}{\cancel{2\pi}} \cdot \cancel{2\pi} r \cdot \sup_{w\in\gamma_r} \frac{|f(w)|}{|w|^{n+1}} \right)$$

$$\Rightarrow \boxed{|f^{(n)}(0)| \le \frac{n!}{r^n(1-r)} \quad \forall\, 0<r<1}$$

When does $\boxed{r^n(1-r)}$ achieve its max. in $0<r<1$?

$$n\cancel{r^{n-1}}(1-r) - r^n = 0$$

$$\Rightarrow n(1-r) = r$$

$$\Rightarrow r = \frac{n}{n+1}$$

$$|f^{(n)}(0)| \le \frac{n!}{\left(\frac{n}{n+1}\right)^n \frac{1}{n+1}} = (n+1)!\left(\frac{n+1}{n}\right)^n.$$

$\square$

(9) Prove that if a power series $\sum a_n z^n$ converges to some function $f\colon \mathbb{C} \to \mathbb{C}$ uniformly in $\mathbb{C}$, then $a_n = 0$ for all but finitely many n, hence f must be a polynomial.

Fact (f_n) fcn on S

$\sum f_n \to f$ conv. unif. on S

$\iff$ $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $\left| \sum_{k=n}^{m} f_k(x) \right| < \varepsilon$

$\forall x \in S$

$\forall m \geq n > N$.

Our case. $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $\left| \sum_{k=n}^{m} a_k z^k \right| < \varepsilon$ $\quad \forall z \in \mathbb{C}$

$\forall m \geq n > N$.

$\implies$ $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $|a_n z^n| < \varepsilon$ $\forall z \in \mathbb{C}$, $\forall n > N$.

Now, if $\sum a_n z^n$ is not a poly.,

then $\forall N > 0$, $\exists n > N$ s.t. $a_n \neq 0$

But this implies: $|z|^n < \dfrac{\varepsilon}{|a_n|}$ $\forall z \in \mathbb{C}$

Contradiction. $\square$

(5) Prove that the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \frac{z}{1+|z|}$$

is not holomorphic at *any* point $z_0 \in \mathbb{C} \setminus \{0\}$

$z = x + iy$

$$f(z) = \frac{x+iy}{1+\sqrt{x^2+y^2}} = \underline{u} + i\underline{v}$$

$$u = \frac{x}{1+\sqrt{x^2+y^2}}, \quad v = \frac{y}{1+\sqrt{x^2+y^2}}$$

<u>Need</u>: Show that $\forall (x_0, y_0) \in \mathbb{C}$,

$$\begin{cases} u_x(x_0,y_0) = v_y(x_0,y_0) \\ u_y(x_0,y_0) = -v_x(x_0,y_0) \end{cases}$$

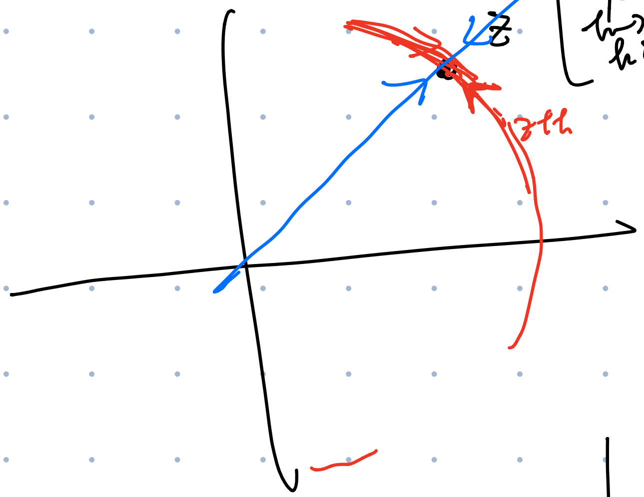
Cauchy-Riemann eq'n.

doesn't hold simultaneously.

$f = |z|$ is not holo. at <u>any</u> pt in $\mathbb{C}$

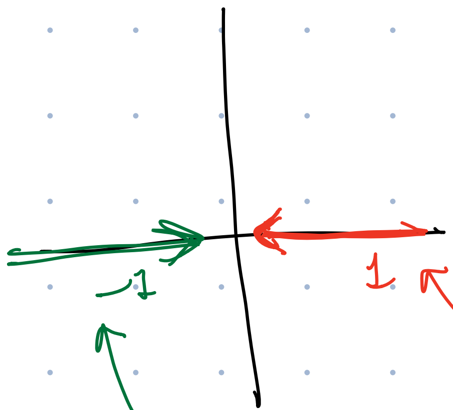
$$\boxed{\lim_{\substack{h \to 0 \\ \text{in } \mathbb{C}}} \frac{f(z+h) - f(z)}{h} \text{ doesn't exist } \forall z \in \mathbb{C}}$$

Case 1: $z \neq 0$



$\boxed{\lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{|z+h|-|z|}{h}}$ doesn't exist

Case 2: $z = 0$



$$\lim_{h\to 0} \frac{|h|-|0|}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

Assume

$$\boxed{f(z) = \frac{z}{1+|z|}} \text{ holo. at } z_0 \neq 0$$

$$\Rightarrow \frac{f(z)}{z} = \frac{1}{1+|z|}$$

holo. at z_0

$\Rightarrow 1+|z|$ holo. at z_0

$\Rightarrow |z|$ holo. at z_0 $\times$.

At $z_0 = 0$;

$$\boxed{\frac{f(z_0+h)-f(z_0)}{h}}$$

$$= \frac{\frac{h}{1+|h|} - 0}{h} = \frac{1}{1+|h|}$$