#1: Find the solution set of:

$$\begin{cases} 2x_1 + 2x_2 + 2x_3 + 8x_4 = 2 \\ 2x_1 + x_2 + x_3 + 8x_4 = 2. \\ 2x_1 + 6x_2 + 3x_3 + 11x_4 = 1 \end{cases}$$

Sol^n:

$$\left[\begin{array}{cccc|c} 2 & 2 & 2 & 8 & 2 \\ 2 & 1 & 1 & 8 & 2 \\ 2 & 6 & 3 & 11 & 1 \end{array}\right] \xrightarrow{\text{row operations}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 1 \\ 2 & 1 & 1 & 8 & 2 \\ 2 & 6 & 3 & 11 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 4 & 1 & 3 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & 3 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & 1 & -1/3 \\ 0 & 0 & 1 & -1 & 1/3 \end{array}\right]$$

$$\text{Sol}^n \text{ set} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-4t \\ -1/3 - t \\ 1/3 + t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} \quad \square$$

#2: Find all possible values of $a \in \mathbb{R}$ s.t. the following vectors form a linearly dependent set:

$$\begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ 2 \\ 10 \end{bmatrix}.$$

Sol^n:

They're linearly dependent $\iff$ the matrix

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 2 & 2 \\ a & 1 & 10 \end{bmatrix}$$ doesn't have pivot in each column.

$$\xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 1 \\ 0 & 1 & 10-a^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 1 \\ 0 & 0 & 9-a^2 \end{bmatrix}$$

The matrix doesn't have pivot in each column

$\iff$ $9-a^2=0$.

$\iff$ $a=\pm 3$. $\square$

#3: Prove that $\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is linearly independent if and only if the following statement holds:

" for any $1 \le i \le k$, $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \neq \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$ "

pf: ($\Rightarrow$): Since $\{\vec{v}_1, \dots, \vec{v}_k\}$ is l.i., we have

$\vec{v}_i \notin \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$ $\quad \forall 1 \le i \le k$.

(Otherwise, suppose $\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$,

then $\vec{v}_i = a_1\vec{v}_1 + \dots + a_{i-1}\vec{v}_{i-1} + a_{i+1}\vec{v}_{i+1} + \dots + a_k\vec{v}_k$

for some $a_j \in \mathbb{R}$.

Then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is l.d., contradiction.)

On the other hand, it's clear that $\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.

Hence $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \neq \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$ $\forall i$.

($\Leftarrow$) Assume the contrary that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is l.d.

i.e., $\exists$ $c_1, \dots, c_k$ not all zero s.t.

$$c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}.$$

Choose i s.t. $c_i \neq 0$. Then

$$\vec{v}_i = \frac{-1}{c_i}\left(c_1 \vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \dots + c_k \vec{v}_k\right)$$

$$\in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}.$$

It's then clear that

$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}.$$

Contradiction. $\square$