Plan:

- This week: 2^{nd}-order diff'l eq'ns $(y''(t) - 3y'(t) + 2y(t) = t)$
- Next week: System of 1^{st}-order diff'l eq'ns $\left(\begin{cases} x_1'(t) = 2x_1(t) - 3x_2(t) \\ x_2'(t) = x_1(t) - 2x_2(t) \end{cases} \right)$
- Fourier series, various partial diff'l eq'ns.

(more computations, less proofs).

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e.g.: $y'(t) - 2y(t) = 0.$ homogeneous diff'l eq'n,

- $y(t) = e^{2t}$ is a sol'n

$$(e^{2t})' = \frac{d}{dt}(e^{2t}) = 2e^{2t} = 2y(t). \checkmark$$

- general sol'n: Ce^{2t}, where $C \in \mathbb{R}$ is any const.

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e.g. $y'(t) - 2y(t) = 0$, $y(1) = 1$. (initial value problem)

$$y(t) = Ce^{2t}.$$

$$1 = y(1) = Ce^2 \Rightarrow C = e^{-2}$$

$\Rightarrow$ sol'n: $e^{-2} \cdot e^{2t} = e^{2t-2}$

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given $f(t)$ and b,

$\boxed{y'(t) - by(t) = f(t)}$ non-homogeneous diff'l eq'n

"method of variation of parameters"

- first, consider the homog. diff^{l} eq^{n}: $y' - by = 0$

  we know general sol^{n} is $y = C \cdot e^{bt}$.

- Let $y(t) = C(t) e^{bt}$.

  Want to find a fcn $C(t)$ s.t. $y' - by = f$.

$$(C(t)e^{bt})' - b\,C(t)e^{bt}$$
$$= C'(t)e^{bt} + \cancel{C(t) \cdot b e^{bt}} - \cancel{b\,C(t)e^{bt}} = f(t)$$

$$\Rightarrow \quad C'(t) = \frac{f(t)}{e^{bt}}.$$

$$\Rightarrow \text{ Choose } C(t) := \int_0^t \frac{f(s)}{e^{bs}}\,ds. + \text{const.}$$

Then $C'(t) = \dfrac{f(t)}{e^{bt}}$

$$\Rightarrow \quad y(t) := \left(\int_0^t \frac{f(s)}{e^{bs}}\,ds\right) \cdot e^{bt}$$

is a sol^{n} of $y' - by = f$.

$$\left(\int_0^t \frac{f(s)}{e^{bs}}\,ds + \underbrace{\text{const.}\Big)\, e^{bt}}$$

is a sol^{n} to the homog. eq^{n} $y' - by = 0$

Fact: $y'-by=f$ $(*)$, $y'-by=0$ $(**)$

If y_0 is a sol^n to $(*)$, then any other sol^n y_1 of $(*)$ satisfies:

y_1-y_0 is a sol^n of $(**)$.

$$\begin{cases} y_0'-by_0=f \\ y_1'-by_1=f \end{cases} \Rightarrow (y_0'-y_1')-b(y_0-y_1)=0$$

$$\Downarrow$$

y_0-y_1 is a sol^n of $(**)$

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e.g. $y'-y=e^t\cos t$. $(*)$

- $y'-y=0 \to$ general sol^n is Ce^t
- let $C(t):=\displaystyle\int_0^t \frac{e^s\cos s}{e^s}\,ds$

$$=\int_0^t \cos s\,ds$$

$$=\sin t$$

So $y(t)=(\sin t)\cdot e^t$ is a sol^n of $(*)$

$y' = (\cos t)e^t + (\sin t)e^t$

$y = (\sin t)e^t$

Today: $\boxed{y'' + by' + cy = 0}$ $b, c \in \mathbb{R}$.

(auxiliary eq'n $r^2 + br + c = 0$

e.g. $y'' + 5y' - 6y = 0$.

Guess $y = e^{rt}$ for some $r \in \mathbb{R}$.

$y' = re^{rt}$

$y'' = r^2 e^{rt}$

auxiliary eq'n of

$e^{rt}(r^2 + 5r - 6) = 0$

$\Downarrow$

$r = -6$ or 1.

$\Downarrow$

$y(t) = e^t$ or e^{-6t} is a sol'n to $y'' + 5y' - 6y = 0$

$\Downarrow$

$\boxed{y(t) = c_1 e^t + c_2 e^{-6t}}$ is a sol'n $\forall c_1, c_2 \in \mathbb{R}$.

Fact: The sol'ns of a homog diff'l eq'n form a vector space e.g. $y''+5y'-6y=0$, $\boxed{y(0)=0}$, $\boxed{y'(0)=1}$.

$y'(t)=c_1e^{t}-6c_2e^{-6t}$

$0=y(0)=c_1+c_2$ $\qquad$ $1=y'(0)$
$\qquad\qquad\qquad\qquad\qquad\qquad = c_1-6c_2$

$c_1=\frac{1}{7}$, $c_2=\frac{-1}{7}$

$y(t)=\frac{1}{7}e^{t}-\frac{1}{7}e^{-6t}$ is the sol. to the Initial value problem.

Thm (existence and uniqueness theorem).

Let I be an open interval $\mathbb{R}$. (e.g. $(0,1)$, $(0,\infty)$, $(-\infty,\infty)$).

Let $p_0(t),\dots,p_{n-1}(t)$, $f(t)$ continuous fns on I.

Let $t_0\in I$. (where we impose initial conditions).

Initial value problem
$$\begin{cases} y^{(n)}+p_{n-1}(t)y^{(n-1)}+\cdots+p_1(t)y'(t)+p_0(t)y(t)=f(t) \\ y(t_0)=Y_0,\ y'(t_0)=Y_1,\ \dots,\ y^{(n-1)}(t_0)=Y_{n-1} \end{cases}$$

where $Y_0,\dots,Y_{n-1}\in\mathbb{R}$.

$\Rightarrow$ $\exists!$ $y(t)$ on I satisfies both conditions.

Def $y_1(t)$ and $y_2(t)$ are functions on $\mathbb{R}$. Say they're linearly dependent if

$\exists\, c \in \mathbb{R}$ s.t. $y_1(t) = c\, y_2(t)$ $\forall t \in \mathbb{R}$.

$\Downarrow$

$y_1'(t) = c\, y_2'(t)$ $\forall t \in \mathbb{R}$

$\Downarrow$

$$\boxed{\det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = 0 \quad \forall t \in \mathbb{R}}$$

$\det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = 0$ for some $t_0 \in \mathbb{R}$

If y_1, y_2 are l.i. sol$^{\text{n}}$s, to $y'' + by' + cy = 0$, then $\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \neq 0$ $\forall t_0$

lemma.

Lemma: If y_1 and y_2 are sol$^{\text{n}}$s of

$$y'' + by' + cy = 0$$

and suppose that

$$\det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = 0 \text{ for some } t_0 \in \mathbb{R}$$

$\Longrightarrow$ $\{y_1, y_2\}$ is linearly dependent.

pf: We have:
- y_1, y_2 are sol$^{\text{n}}$ to $y'' + by' + cy = 0$.
- $y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0) = 0$ for some t_0

Case 1: $y_1(t_0) \neq 0$.

Define $k := \dfrac{y_2(t_0)}{y_1(t_0)}$, and consider $y_3(t) = k\, y_1(t)$

Claim: $y_3 = y_2$.

pf:
- $y_3(t_0) = k\, y_1(t_0) = y_2(t_0). = \alpha$
- $y_3'(t_0) = k\, y_1'(t_0) = \dfrac{y_2(t_0)}{y_1(t_0)}\, y_1'(t_0) = y_2'(t_0) = \beta$
- y_3 is a sol$^{\text{n}}$ to $y'' + by' + cy = 0$

Since $y_3 = k y_1$ and y_1 is a sol$^{\text{n}}$

$\Rightarrow$ both y_2 and y_3 satisfy:

- $y'' + by' + cy = 0$
- $y(t_0) = \alpha, \quad y'(t_0) = \beta.$

$\xRightarrow{\text{uniqueness thm}} \quad y_2 = y_3 = k y_1.$

Case 2: $y_1(t_0) = 0, \quad y_1'(t_0) \neq 0$

$$\underbrace{y_1(t_0)}_{=0} y_2'(t_0) = \underbrace{y_1'(t_0)}_{\neq 0} y_2(t_0) \Rightarrow y_2(t_0) = 0$$

$$k := \frac{y_2'(t_0)}{y_1'(t_0)}, \quad y_3(t) := k\, y_1(t)$$

$$\begin{cases} y_3(t_0) = k\, y_1(t_0) = 0 = y_2(t_0) \\ y_3'(t_0) = k\, y_1'(t_0) = y_2'(t_0) \end{cases}$$

$\xRightarrow{\text{uniqueness}} \quad y_3 = y_2.$

Case 3: $y_1(t_0) = y_1'(t_0) = 0$

Claim: $y_1 \equiv 0$

pf uniqueness thm: both y_1 and 0 (zero fn) satisfy:

$$\begin{cases} y'' + by' + cy = 0 \\ y(t_0) = 0, \quad y'(t_0) = 0 \end{cases} \quad \square$$

Thm: If y_1, y_2 are l.i. sol^{ns} to $y'' + by' + cy = 0$.
Then for any initial condition $y(t_0) = Y_0$, $y'(t_0) = Y_1$,
$\exists!$ $c_1, c_2 \in \mathbb{R}$ s.t. $y(t) = c_1 y_1(t) + c_2 y_2(t)$
is the unique sol^n to the initial value problem.

pf For any t_0, Y_0, Y_1,

Want to find c_1, c_2 s.t.

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = Y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = Y_1 \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix}$$

We know that $\det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \neq 0$

since y_1, y_2 are l.i. (By Lemma). $\square$

Strategy to solve $y'' + by' + cy = 0$, $y(t_0) = Y_0$, $y'(t_0) = Y_1$.

① find two l.i. sol^{ns} y_1, y_2 to the diff^{al} eq^n.

② let $y = c_1 y_1 + c_2 y_2$, use the initial conditions to determine c_1, c_2

$\boxed{y'' + by' + cy = 0}$ $\rightsquigarrow$ auxiliary eq'n.

$r^2 + br + c = 0$

① two distinct real roots r_1, r_2,

$\Rightarrow e^{r_1 t}, e^{r_2 t}$ are l.i. sol's.

② root r_0 with multiplicity 2. $\{ e^{r_0 t}, \boxed{te^{r_0 t}} \}$ l.i.

$r^2 + br + c = (r - r_0)^2$

$\Rightarrow b = -2r_0, \ c = r_0^2.$

$y(t) = te^{r_0 t}$

$y'(t) = e^{r_0 t} + r_0 t e^{r_0 t}$

$= (1 + r_0 t) e^{r_0 t}$

$y''(t) = r_0 e^{r_0 t} + (1 + r_0 t) r_0 e^{r_0 t}$

$= e^{r_0 t} (2r_0 + r_0^2 t)$

$y'' + by' + cy = e^{r_0 t} \left(2r_0 + r_0^2 t + b + br_0 t + ct \right) = 0$

$2r_0 + b = 0$

$r_0^2 + br_0 + c = 0$

③ complex roots. $\boxed{\alpha \pm i\beta}$

$e^{(\alpha \pm i\beta)t}$ is a sol^n to $y'' + by' + cy = 0$

$\left(e^{i\theta} = \cos\theta + i\sin\theta\right)$

$$\left.\begin{aligned} e^{(\alpha+i\beta)t} &= e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right) \\ e^{(\alpha-i\beta)t} &= e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right) \end{aligned}\right\} \text{sol}^{n}\text{s}$$

$$\Rightarrow \boxed{\left\{ \boxed{e^{\alpha t}\cos(\beta t)}, \boxed{e^{\alpha t}\sin(\beta t)} \right\} \text{ l.i.}}$$

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e.g. $y'' + 2y' + 4y = 0$, $y(0) = 0$, $y'(0) = 1$

- $r^2 + 2r + 4 = 0$ roots: $-1 \pm i\sqrt{3}$.
- general sol^n:

  $$y(t) = C_1 e^{-t}\cos(\sqrt{3}t) + C_2 e^{-t}\sin(\sqrt{3}t)$$

- $$\begin{cases} 0 = y(0) = C_1 \\ 1 = y'(0) = C_2 \end{cases} \Rightarrow y(t) = e^{-t}\sin(\sqrt{3}t). \quad \square$$