(1) (15 points) Compute

$$\int_{|z|=1} \frac{(2\bar{z}-1)(3\bar{z}-1)(4\bar{z}-1)\cdots(185\bar{z}-1)}{(2z-1)(3z-1)(4z-1)\cdots(2021z-1)}dz,$$

where the curve $|z|=1$ is oriented positively.

(Note 1: Your answer should be an explicit number.)

(Note 2: The integrand is not a meromorphic function on $\mathbb{C}$.)

- On $|z|=1$ we have $\bar{z} = 1/z$. Hence the integral is:

$$\int_{|z|=1} \frac{(2/z-1)(3/z-1)\cdots(185/z-1)}{(2z-1)(3z-1)\cdots(2021z-1)}dz$$

- The integrand is now meromorphic, with lots of poles in $\mathbb{D}$. Recall that in this situation, it's easier to evaluate the integral by changing the variable $w = 1/z$.

- The integral

orientation changes as we changed to the w-plane.

$$= - \int_{|w|=1} \frac{(2w-1)(3w-1)\cdots(185w-1)}{(2/w-1)(3/w-1)\cdots(2021/w-1)} \frac{-1}{w^2} dw.$$

$$= \int_{|w|=1} \boxed{\frac{(2w-1)(3w-1)\cdots(185w-1)}{(2-w)(3-w)\cdots(2021-w)} w^{2018}} dw.$$

holo. in $\overline{\mathbb{D}}$

$= 0$ by Cauchy's thm. □

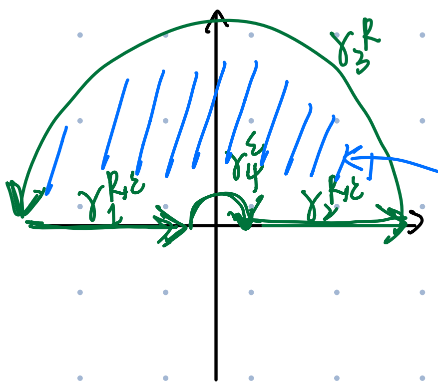
(2) (25 points) Compute
$$\int_0^\infty \left(\frac{\sin(x)}{x}\right)^3 dx.$$

Note: the computation below follows the same idea as in HW3 #2.

- $\left(\frac{\sin x}{x}\right)^3 = \left(\frac{e^{ix}-e^{-ix}}{2ix}\right)^3 = \frac{e^{3ix}-e^{-3ix}-3e^{ix}+3e^{-ix}}{-8ix^3}$

- $\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx = \lim_{\substack{R\to+\infty\\ \varepsilon\to 0}} \int_\varepsilon^R \frac{e^{3ix}-e^{-3ix}-3e^{ix}+3e^{-ix}}{-8ix^3}\,dx$

$= \lim_{\substack{R\to+\infty\\ \varepsilon\to 0}} \left(\int_\varepsilon^R \frac{e^{3ix}-3e^{ix}}{-8ix^3}\,dx + \int_{-R}^{-\varepsilon} \frac{e^{3ix}-3e^{ix}}{-8ix^3}\,dx \right)$



$= \lim_{\substack{R\to+\infty\\ \varepsilon\to 0}} \left(\int_{\gamma_1^{R,\varepsilon}} + \int_{\gamma_2^{R,\varepsilon}} \frac{e^{3iz}-3e^{iz}}{-8iz^3}\,dz \right)$

f is holo. in the interior $\quad f(z)$

- By Jordan's lemma, $\lim_{R\to 0} \int_{\gamma_3^R} f = 0.$

- $e^{3iz} - 3e^{iz} = -2 - 3z^2 + z^3 \cdot (\text{something holo.})$

- $\int_{\gamma_4^\varepsilon} f = \int_{\gamma_4^\varepsilon} \frac{-2}{-8iz^3}\,dz + \int_{\gamma_4^\varepsilon} \frac{-3z^2 + z^3(\text{something holo.})}{-8iz^3}$

- $\frac{-2}{-8iz^3}$ has primitive $\frac{1}{-8iz^2}$ on $\mathbb{C}\setminus\{0\}$.

Hence $\int_{\gamma_4^\varepsilon} \frac{-2}{-8iz^3} = \frac{1}{-8i\varepsilon^2} - \frac{1}{-8i(-\varepsilon)^2} = 0. \ \forall \varepsilon > 0.$

- By what we discussed in the begining of Lecture 16,

$$\lim_{\varepsilon\to 0}\int_{\gamma_4^\varepsilon} \boxed{\frac{-3z^2+z^3(\text{something hol.})}{-8iz^3}} = -\pi i\cdot\frac{-3}{-8i} = \frac{-3\pi}{8}$$

has simple pole at 0.

- Therefore,

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx = \lim_{\substack{R\to+\infty\\ \varepsilon\to 0}}\left(\int_{\gamma_1^{R,\varepsilon}} f + \int_{\gamma_2^{R,\varepsilon}} f\right)$$

$$= -\lim_{\substack{R\to+\infty\\ \varepsilon\to 0}}\left(\int_{\gamma_3^R} f + \int_{\gamma_4^\varepsilon} f\right) = \frac{3\pi}{8}. \quad \square$$

Cauchy's thm.

(3) (15 points) Let f and g be two entire functions with $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant $c \in \mathbb{C}$ such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$.
(Note: If your proof is very short, then you probably are missing something.)

- If g is the zero fcn $g(z) \equiv 0$, then f also is the zero fcn.
- Assume that g is not identically zero, then the zeros of g are isolated. by local-determine-global principle.
- Define $F(z) := f(z)/g(z)$. Then F is a meromorphic fcn on $\mathbb{C}$, with isolated singularities at the zeros of g.
- Let z_0 be an isolated singularity of F, and let $\varepsilon > 0$ small enough s.t. F is holo. in the punctured disk $\mathbb{D}_\varepsilon^\times(z_0)$.
- Observe that F is bounded on $\mathbb{D}_\varepsilon^\times(z_0)$ since

  $$|F(z)| = \frac{|f(z)|}{|g(z)|} \leq 1. \quad \text{for any } z \text{ with } g(z) \neq 0.$$

$\Rightarrow$ F can be extended to a holo. fcn $\tilde{F}$ on $\mathbb{D}_\varepsilon(z_0)$.
where $|\tilde{F}(z_0)| = |\lim_{z \to z_0} F(z)| \leq 1$.

- Therefore, F can be extended to an entire fcn $\tilde{F}$, and $|\tilde{F}(z)| \leq 1 \quad \forall z \in \mathbb{C}$.
- By Liouville thm, $\tilde{F}$ is a const. fcn. $\tilde{F}(z) \equiv c \quad \forall z \in \mathbb{C}$.

$\Rightarrow$
- $\frac{f(z)}{g(z)} = F(z) = \tilde{F}(z) = c$ if $g(z) \neq 0$.
- $f(z) = g(z) = 0$ if $g(z) = 0$.

$\Rightarrow$ $f(z) = c\,g(z) \quad \forall z \in \mathbb{C}$. $\square$

(4) (a) (20 points) Let (f_n) be a sequence of entire functions, and let p be a polynomial of degree $m > 0$. Suppose that (f_n) converges uniformly on every compact subset $K \subseteq \mathbb{C}$ to the polynomial p. Prove that there exists $N > 0$ such that f_n has at least m zeros (counting multiplicities) for all $n > N$.

(b) (5 points) Give an example of a sequence of entire functions (f_n), each with at least two zeros, which converges uniformly on every compact subset $K \subseteq \mathbb{C}$ to a polynomial p of degree one.

(a)

- By fundamental thm of algebra, p has m zeros in $\mathbb{C}$.
- Choose $R>0$ large enough s.t. $\mathbb{D}_R(0)$ contains all the zeros of p in its interior.
- Let $M := \inf\{|p(z)| : z \in \boxed{\partial\mathbb{D}_R(0)}\}$. We have $M > 0$ since $\partial\mathbb{D}_R(0)$ is compact so the infimum is achieved.

  the boundary circle of $\mathbb{D}_R(0)$;
- Since $f_n \to p$ uniformly on $\partial\mathbb{D}_R(0)$, $\exists N>0$ s.t. $|f_n(z) - p(z)| < M \quad \forall n>N, z \in \partial\mathbb{D}_R(0)$.
- Then we have $|f_n(z) - p(z)| < M \le |p(z)| \quad \forall z \in \partial\mathbb{D}_R(0) \; \forall n>N$.
- By Rouché's thm, for any $n>N$, we have

  $\#$ zeros of $f_n(z)$ in $\mathbb{D}_R(0)$

  $= \#$ zeros of $p(z)$ in $\mathbb{D}_R(0)$ $= m$.

$\Rightarrow$ $f_n(z)$ has at least m zeros $\forall n>N$. $\square$

(b)

Consider $f_n(z) = z\left(1 - \frac{z}{n}\right)$ and $p(z) = z$.

- $f_n(z)$ has two zeros at 0 and n.
- Let $K \subseteq \mathbb{C}$ be a compact subset. $\exists R>0$ s.t. $K \subseteq B_R(0)$.

  $\forall \varepsilon > 0$, let $N = R^2/\varepsilon > 0$.

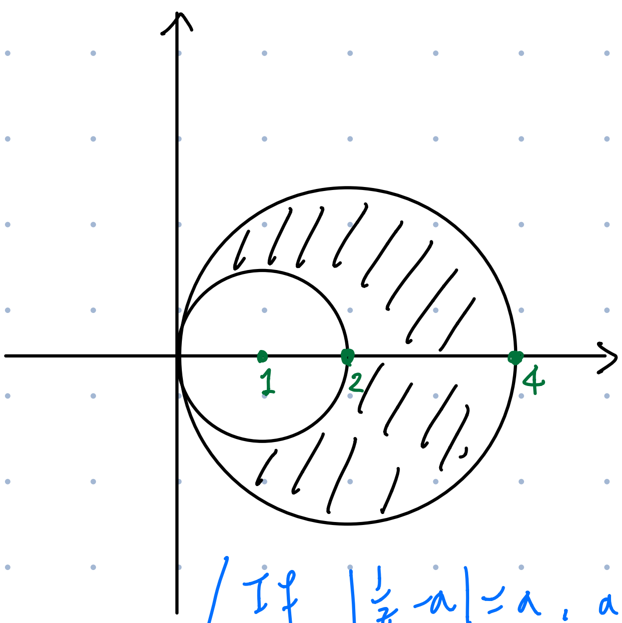
  Then $|f_n(z) - p(z)| = \left|\frac{z^2}{n}\right| < \frac{R^2}{n} < \varepsilon \quad \forall n>N, z \in K$.

  Hence $f_n \to p$ uniformly on K. $\square$

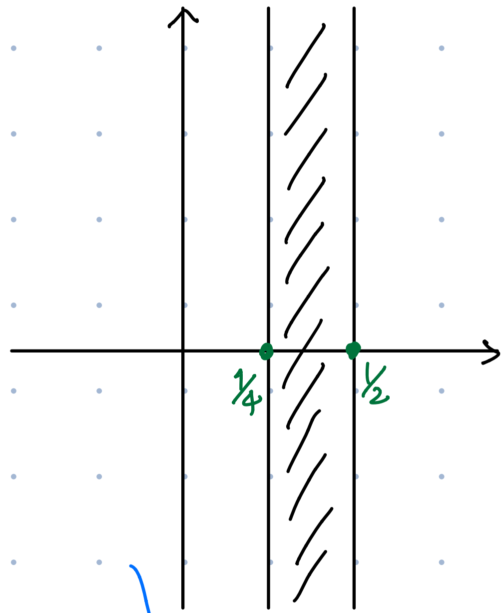
(5) (15 points) Write down (and provide justification) an explicit biholomorphic map from the set

$$\{z \in \mathbb{C} \colon |z-1| > 1 \ \text{ and } \ |z-2| < 2\}$$

to the open unit disk $\mathbb{D}$. (Hint: First, consider $z \mapsto \frac{1}{z}$.)



$\xrightarrow{\frac{1}{z}}$



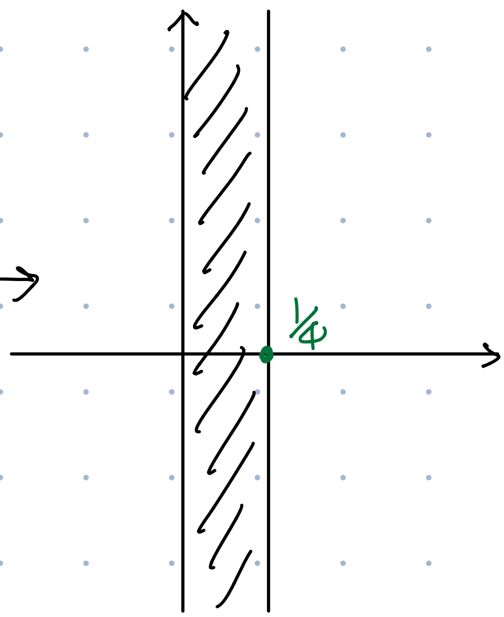
If $\left|\frac{1}{z} - a\right| = a$, $a \in \mathbb{R}$, then

$a^2 = \left(\frac{1}{z} - a\right)\left(\frac{1}{\bar{z}} - a\right) = \frac{1}{|z|^2} + a^2 - a\left(\frac{1}{z} + \frac{1}{\bar{z}}\right)$

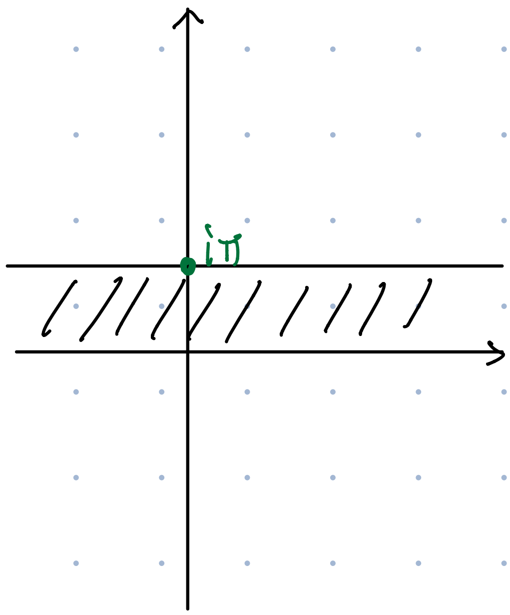
$= a^2 + \frac{1}{|z|^2} - a \cdot \frac{2\,\mathrm{Re}\, z}{|z|^2}$

$\Rightarrow \mathrm{Re}\, z = \frac{1}{2a}.$

$\xrightarrow{z - \frac{1}{4}}$



$\xrightarrow{4\pi i z}$



$\xrightarrow{e^z} \mathbb{H} \xrightarrow{\frac{i-z}{i+z}} \mathbb{D}.$

$$\boxed{\frac{i - e^{4\pi i\left(\frac{1}{z} - \frac{1}{4}\right)}}{i + e^{4\pi i\left(\frac{1}{z} - \frac{1}{4}\right)}}}$$

(6) (20 points) Let f be a non-constant holomorphic function defined on the upper-half plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. Assume that $\operatorname{Im}(f(z)) \geq 0$ for any $z \in \mathbb{H}$. Prove that for any $z_1, z_2 \in \mathbb{H}$, we have

$$\frac{|f(z_1) - f(z_2)|}{|f(z_1) - \overline{f(z_2)}|} \leq \frac{|z_1 - z_2|}{|z_1 - \overline{z_2}|}.$$

(Hint: First, show that $z \mapsto \frac{z - z_0}{z - \overline{z_0}}$ is a biholomorphic map from $\mathbb{H}$ to $\mathbb{D}$ for any $z_0 \in \mathbb{H}$.)

- First, we claim that $\operatorname{Im} f(z) > 0$ $\forall z \in \mathbb{H}$.:

  We have

$$\begin{tikzcd} \mathbb{H} \arrow[r, "f"] & \mathbb{H} \cup \mathbb{R} \arrow[d, "\frac{i-z}{i+z} =: g_2"] \\ \mathbb{D} \arrow[u, "g_1 := i\frac{1-z}{1+z}", "\wr"'] & \overline{\mathbb{D}} \end{tikzcd}$$

  If there exists $z_0 \in \mathbb{H}$ s.t. $f(z_0) \in \mathbb{R}$, then $|g_2(f(z_0))| = 1$.

  $\Rightarrow$ the holo. map $g_2 \circ f \circ g_1$ attends max. modulus at an interior point $g_1^{-1}(z_0)$. Contradiction. $\square$

- Second, we claim that $\forall z_0 \in \mathbb{H}$, $\mathbb{H} \longrightarrow \mathbb{D}$, $z \longmapsto \frac{z - z_0}{z - \overline{z_0}}$ is biholo.
  - the map is holo. on $\mathbb{H}$ since $\overline{z_0} \notin \mathbb{H}$.
  - the image lies in $\mathbb{D}$ since $|z - z_0| < |z - \overline{z_0}|$ $\forall z, z_0 \in \mathbb{H}$
  - the map is injective since $w = \frac{z - z_0}{z - \overline{z_0}}$

    $\Leftrightarrow$ $z - z_0 = w(z - \overline{z_0})$

    $\Leftrightarrow$ $z = \frac{z_0 - w\overline{z_0}}{1 - w}$ $\forall w$.
  - the map is surjective since $\forall |w| < 1$,

    $\operatorname{Im}\left(\frac{z_0 - w\overline{z_0}}{1 - w}\right) = \frac{\operatorname{Im}\left((z_0 - w\overline{z_0})(1 - \overline{w})\right)}{1 - |w|^2} = \operatorname{Im}(z_0) > 0.$

- $\forall z_2 \in \mathbb{H}$, Consider

$$\begin{tikzcd} \mathbb{H} \arrow[r, "f"] \arrow[d, "f_1 := \frac{z - z_2}{z - \overline{z_2}}"', "\wr"] \arrow[dr, bend left, "F := f_2 \circ f \circ f_1^{-1}"] & \mathbb{H} \arrow[d, "\frac{z - f(z_2)}{z - \overline{f(z_2)}} =: f_2", "\wr"'] \\ \mathbb{D} & \mathbb{D} \end{tikzcd}$$

Then $F(0) = f_2(f(f_1^{-1}(0))) = f_2(f(z_2)) = 0$.

By Schwarz lemma, $|F(f_1(z_1))| \leq |f_1(z_1)| = \left|\frac{z_1 - z_2}{z_1 - \overline{z_2}}\right|$. $\forall z_1 \in \mathbb{H}$.

$$\left|\frac{f(z_1) - f(z_2)}{f(z_1) - \overline{f(z_2)}}\right| = |F(f_1(z_1))|$$

$\square$

(7) Let $\Lambda \subseteq \mathbb{C}$ be a lattice, and let $z_1, \dots, z_n$ be points in its fundamental domain. For each z_i, consider a function

$$p_i(z) = \frac{a^{(i)}_{-k_i}}{(z-z_i)^{k_i}} + \frac{a^{(i)}_{-(k_i-1)}}{(z-z_i)^{k_i-1}} + \cdots + \frac{a^{(i)}_{-1}}{z-z_i},$$

where $a^{(i)}_{-1}, a^{(i)}_{-2}, \dots, a^{(i)}_{-k_i}$ are complex numbers.

(a) (5 points) Prove that if there exists a meromorphic elliptic function f (with respect to Λ) such that the poles of f in the fundamental domain are given by $z_1, \dots, z_n$, with principal parts given by $p_1(z), \dots, p_n(z)$, then

$$a^{(1)}_{-1} + \cdots + a^{(n)}_{-1} = 0.$$

(b) (20 points) Conversely, given any $p_1(z), \dots, p_n(z)$ with the property that $a^{(1)}_{-1} + \cdots + a^{(n)}_{-1} = 0$. Prove that there exists a meromorphic elliptic function f (with respect to Λ) such that the poles of f in the fundamental domain are given by $z_1, \dots, z_n$, with principal parts given by $p_1(z), \dots, p_n(z)$.

(Hint: Use $\wp(z - z_i)$ and its derivatives to reduce to the case of simple poles; then consider $\zeta(z) - \zeta(z - z_i)$.)

(a) By residue formula, $\displaystyle\sum_{i=1}^{n} a^{(i)}_{-1} = \frac{1}{2\pi i}\int_{\partial\,\text{F.D.}} f(z)\,dz = 0$ since f is elliptic. $\square$

(b).

- Observe that $\wp(z-z_i)$ has a double pole at $z=z_i$ and no other poles in the F.D.
  (its Laurent series exp. is $\frac{1}{(z-z_i)^2}$ + (something holo.)).
- Similarly, $\left(\frac{d}{dz}\right)^n \wp(z-z_i)$ has a pole of order $n+2$ at $z=z_i$, and no other poles in the F.D.
- Therefore, there exists an ell. fcn f_i given by some linear combination of $\wp(z-z_i)$ and its derivatives, so that z_i is the only pole in the F.D., and its principal part is given by:
  $$\frac{a^{(i)}_{-k_i}}{(z-z_i)^{k_i}} + \cdots + \frac{a^{(i)}_{-2}}{(z-z_i)^2}.$$
- Consider $g(z) := \displaystyle\sum_{i=1}^{n} a^{(i)}_{-1}\left(\zeta(z) - \zeta(z-z_i)\right)$, where ζ is the Weierstrass ζ-fcn.
  Recall that $\zeta(z-a) - \zeta(z-b)$ is an ell. fcn. $\forall a, b.$, hence so is $g(z)$.
- Laurent series exp. of g at z_i: $\dfrac{-a^{(i)}_{-1}}{z-z_i} + \cdots$

  —— " —— z: $\dfrac{\overbrace{a^{(1)}_{-1} + \cdots + a^{(n)}_{-1}}^{=0 \text{ by assumption.}}}{z} + \cdots$

Hence g only has poles (in the F.D.) at $z_1, \dots, z_n$. Each pole is simple, with residue $-a_{-1}^{(i)}$.

- Then $f := f_1 + \dots + f_n - g$ has the desired properties. $\square$