4/21/2020

System of ODEs:

$p \in U \overset{\text{open}}{\subset} \mathbb{R}^m$

$f_1, \dots, f_m : U \to \mathbb{R}$

interval in $\mathbb{R}$ that contains 0

Find $x_1(t), \dots, x_m(t) : (a,b) \to U$

s.t.

$$(1) \quad \begin{cases} x_1'(t) = f_1(x_1(t), \dots, x_m(t)) \\ x_2'(t) = f_2(x_1(t), \dots, x_m(t)) \\ \vdots \\ x_m'(t) = f_m(x_1(t), \dots, x_m(t)) \end{cases}$$

$$\left(\text{e.g. } \vec{x}'(t) = \underbrace{A}_{m \times m \text{ matrix}} \vec{x}(t) \right)$$

(2) $(x_1(0), \dots, x_m(0)) = p$

---

e.g. $x'(t) = 2x(t), \quad x(0) = 3$

$\downarrow$

sol'ns: $\boxed{3e^{2t}}$

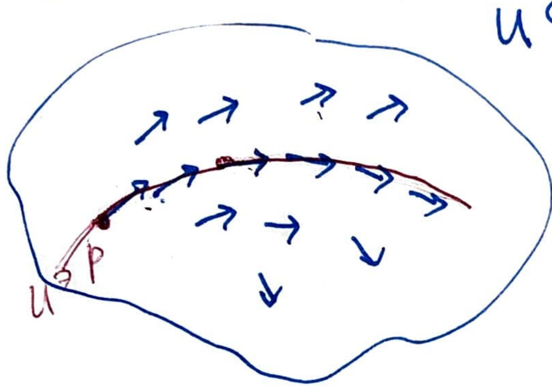
Define $F: \underset{\subset \mathbb{R}^m}{U} \longrightarrow \mathbb{R}^m$

$$(x_1, \dots, x_m) \longmapsto (f_1(x_1,\dots,x_m), \dots, f_m(x_1,\dots,x_m))$$

"vector field on U".

e.g. $m = 2$

$U \subset \mathbb{R}^2$



Sol'n of the ODEs

$\Updownarrow$

a "curve" γ: in U, ...

$$\boxed{\begin{aligned} \gamma: (a,b) &\longrightarrow U \\ t &\longmapsto (x_1(t), \dots, x_m(t)) \end{aligned}}$$

$\gamma'(t) := (x_1'(t), \dots, x_m'(t))$

satisfies:

$$\begin{cases} \gamma'(t) = F(\gamma(t)) \\ \gamma(0) = p \end{cases}$$

Tangent direction of the curve γ

Thm Suppose $F: U \longrightarrow \mathbb{R}^m$ is Lipschitz conti., i.e. $\exists\, L > 0$

s.t. $|F(x) - F(y)| < L\,|x-y|$ $\quad \forall x, y \in U$

Then such γ exists and is unique "locally", i.e.

• $\exists\, \varepsilon > 0$ s.t. $\exists!\, \gamma : (-\varepsilon, \varepsilon) \longrightarrow U$

s.t. $\boxed{\begin{cases} \gamma'(t) = F(\gamma(t)) \\ \gamma(0) = p \in U \end{cases}}$

~~exists & unique for~~

~~$\exists!\, \gamma : (-\varepsilon, \varepsilon) \longrightarrow U$~~

$\Updownarrow$ FTC

$$\boxed{\gamma(t) = p + \int_0^t F(\gamma(t))\,dt}$$

Idea: • Consider the metric space $\mathcal{C}$ of all curves $\gamma : (-\varepsilon, \varepsilon) \longrightarrow U$

• $\Phi : \mathcal{C} \longrightarrow \mathcal{C}$

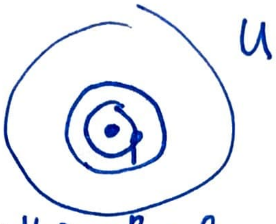
$$\gamma_0(t) \longmapsto p + \int_0^t F(\gamma_0(t))\,dt$$

Then ~~the eq'n~~ γ satisfies the eq'n

$\Longleftrightarrow \quad \Phi\gamma = \gamma$

i.e. γ is a fixed pt.

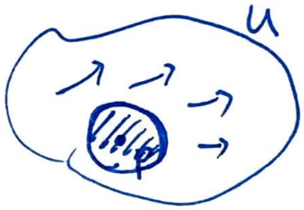
pf: $p \in U$



- $\exists\, r>0$ cpt by Heine-Borel.
  s.t. $p \in \overline{B_r(p)} \subset U$
- $\exists\, M>0$ s.t. $|F(x)|<M \quad \forall x \in \overline{B_r(p)}$

Choose $\tau>0$ s.t.

- $\tau M < r$
- $\tau L < 1$



$\mathcal{C} := \{\text{conti. fcn. } [-\tau,\tau] \longrightarrow \overline{B_r(p)}\}$

$d_\infty(f,g) := \sup\{|f(x)-g(x)| : x \in [-\tau,\tau]\}$

↳ complete metric space (HW12 #1)

$\Phi: \mathcal{C} \longrightarrow \mathcal{C}$

$\gamma \longmapsto \overset{\Phi(\gamma)}{p + \int_0^t F(\gamma(s))\,ds}$ → in $\mathcal{C}$??

$(\gamma: [-\tau,\tau] \longrightarrow \overline{B_r(p)})$

<u>Check</u>: $|\Phi(\gamma)(t) - p|$

$$= \left|\int_0^t F(\gamma(s))\,ds\right|$$

$$\le |t|\,M \qquad t \in [-\tau,\tau]$$

$$\le \tau M < r$$

<u>Fact</u> Any "contraction" on a complete metric space has a unique fixed point.

$$\left(\begin{array}{l} F: X \longrightarrow Y,\ \exists\, 0 < M < 1 \\ \text{s.t.} \\ |F(x_1) - F(x_2)| \le M\, |x_1 - x_2| \end{array} \right)$$

(cf. the proof of HW 8 #8)

It suffices to show Φ is a contraction:

$$d_\infty(\Phi(\gamma_1), \Phi(\gamma_2))$$

$$= \sup_{t \in [-\varepsilon, \varepsilon]} \left| \Phi(\gamma_1)(t) - \Phi(\gamma_2)(t) \right|$$

$$\Phi(\gamma_1)(t) = p + \int_0^t F(\gamma_1(s))\,ds, \quad \Phi(\gamma_2)(t) = p + \int_0^t F(\gamma_2(s))\,ds$$

$$= \sup_{t \in [-\varepsilon, \varepsilon]} \left| \int_0^t \big(F(\gamma_1(s)) - F(\gamma_2(s))\big)\, ds \right|$$

$$\le \varepsilon \sup_s \left| F(\gamma_1(s)) - F(\gamma_2(s)) \right|$$

$$\le \underbrace{\varepsilon L}_{\text{(Lip. const. of } F)} \underbrace{\sup_s \left| \gamma_1(s) - \gamma_2(s) \right|}_{= d_\infty(\gamma_1, \gamma_2)}$$

$\boxed{\varepsilon L < 1}$

$\Rightarrow \Phi$ is a contraction. □

Construction of Real numbers

$\mathbb{Z} \subset \mathbb{Q}$

Def A (Dedekind) <u>cut</u> is a subset

$A \subset \mathbb{Q}$ s.t.

- $A \neq \phi$, $A \neq \mathbb{Q}$.
- "leftward-closed", i.e. If $a \in A$ and $b < a$ then $b \in A$
- A has no largest element, i.e. $\forall a \in A$, $\exists b \in A$ st $b > a$

$\mathbb{R} := \{\text{cuts } A \subset \mathbb{Q}\} \ (\subset P(\mathbb{Q}))$

a set

$$\mathbb{Q} \longrightarrow \mathbb{R}$$

$$r \longmapsto r^* := \{x \in \mathbb{Q} \mid x < r\}$$

injective.

r_1 r_1^* ← doesn't contain r_1

$\wedge$

r_2 r_2^* ← contains r_1

① Order "$\leq$"

Def Write "$A \leq B$" if $A \subseteq B$

$A, B \in \mathbb{R} = \{\text{cuts } A \subset \mathbb{Q}\}$

Properties:

① If $A \leq B, B \leq C$ then $A \leq C$ ✓

② if $A \leq B, B \leq A$, then $A = B$ ✓

③ $\forall A, B \in \mathbb{R}$ either $A \leq B$ or $B \leq A$

---

pf ③: $A, B \in \mathbb{R}$

Suppose $A \leq B$ is not true.

$\Updownarrow$

$A \nsubseteq B$

$\Updownarrow$

$\boxed{\exists\, a \in A \text{ s.t. } a \notin B.}$

Claim $b < a \quad \forall b \in B$

$(\Rightarrow B \subseteq A \Leftrightarrow B \leq A)$

Otherwise,

- $b > a \Rightarrow a \in B$ $\ast$
- $\underset{\substack{\Uparrow \\ B}}{b = a}$ $\ast$

2 Least upper bound property

Suppose $\{A_i\}_{i\in I} \subset \mathbb{R}$ bounded above.

i.e. $\exists\, B \in \mathbb{R}$, s.t. $A_i \subseteq B \ \forall i \in I$.

Define: $C = \bigcup_{i\in I} A_i \subset \mathbb{Q}$

Check: ✓ $\boxed{C \text{ is a cut.}}$

✓. C is ~~the~~ least bound of $\{A_i\}$

C is an upper bound. b/c

$A_i \subseteq C$

C is the least upper bound:

C' is another upper bound,

then $A_i \subseteq C' \ \forall i$

$$C = \bigcup_{i\in I} A_i \subseteq C'$$

$$C \leq C'$$

For "+" "·", ... on $\mathbb{R} = \{\text{cuts } A \subset \mathbb{Q}\}$,

see Supplementary Reading on

Course website.