Reminder:

- 1^{st} quiz: Tuesday noon - Wednesday noon. PDT. Has computational & proof-based problems.

  Time limit: 45 min. (may vary for each quiz).

<u>Make sure to upload your answer within the time limit!!</u>

<u>Please write as clear as possible!!</u>

Recap:

Any problem on HW 1 that you'd like to go over?

- linear combinations, Span of vectors.

  (e.g. $\vec{b} \in \mathrm{Span}\{\vec{a}_1, \dots, \vec{a}_n\} \Leftrightarrow \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \vec{x} = \vec{b}$ has a sol$^{\underline{n}}$)
- linearly (in)dependence.

  (e.g. $\{\vec{v}_1, \dots, \vec{v}_n\}$ is l.i. $\Leftrightarrow \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \vec{x} = \vec{0}$ has no nontrivial sol$^{\underline{n}}$)
- A: $m \times n$ matrix $\rightsquigarrow$ $T_A: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation

  $$\vec{x} \mapsto A\vec{x}.$$

  $T: \mathbb{R}^n \to \mathbb{R}^m$ linear transf$^{\underline{n}}$ $\rightsquigarrow$ $\exists!\ A: m \times n$ s.t. $T_A = T$.
- Criterions for T_A being surjective or injective via pivots of A.

Today: Matrix product, invertible matrices.

In HW1, you showed that the composition of linear transform$^{\underline{n}}$'s is still linear.

$A: m \times n$, $B: n \times p$

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "T_B"] \arrow[rr, bend right, swap, "T_A \circ T_B"] & \mathbb{R}^n \arrow[r, "T_A"] & \mathbb{R}^m \end{tikzcd}$$

$T_A \circ T_B$ is linear.

$\exists!$ $m\times p$ matrix "AB" s.t. $T_A \circ T_B = T_{AB} : \mathbb{R}^p \to \mathbb{R}^m$

$\uparrow$

the product of A, B.

<u>Let's write AB explicitly:</u>

$$AB = \begin{array}{c} \quad\quad i\text{th} \\ \begin{bmatrix} | & | & | & | & | & | \\ | & | & | & | & | & | \end{bmatrix} \end{array}$$

The i-th column of AB

$$= T_{AB}(\vec{e}_i)$$

$$= T_A \circ T_B(\vec{e}_i)$$

$$= T_A(B\vec{e}_i)$$

$$= A(\underbrace{B\vec{e}_i}_{i\text{th column of } B})$$

$$\vec{e}_i = \begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} \\ \\ \leftarrow i\text{th} \\ \\ \\ \end{array} \end{array}$$

$$AB = A\underbrace{\begin{bmatrix} \vec{b}_1 \cdots \vec{b}_p \end{bmatrix}}_{B} = \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \\ | & | & & | \end{bmatrix}$$

in particular, each column of AB is a linear comb. of columns of A.

More explicitly,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}$$

i-th column of AB

$$= A\vec{b}_i = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{1i} \\ \vdots \\ b_{ni} \end{bmatrix}$$

$$= b_{1i}\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + b_{ni}\begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{n} a_{1k} b_{ki} \\ \vdots \\ \sum_{k=1}^{n} a_{mk} b_{ki} \end{bmatrix}$$

$\sum_{k=1}^{n} a_{1k} b_{ki} = a_{11}b_{1i} + a_{12}b_{2i} + \cdots + a_{1n}b_{ni}$

$$AB = \begin{bmatrix} \sum_{k=1}^{n} a_{1k} b_{k1} & \cdots & \sum_{k=1}^{n} a_{1k} b_{kp} \\ \vdots & & \vdots \\ \sum_{k=1}^{n} a_{mk} b_{k1} & \cdots & \sum_{k=1}^{n} a_{mk} b_{kp} \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

i.e. standard inner product btw
ith row of A & jth column of B.

e.g.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 1 & 7 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 1\times1+3\times1+5\times1 & 1\times7+3\times7+5\times7 \\ 2\times1+4\times1+6\times1 & 2\times7+4\times7+6\times7 \end{bmatrix}$$

e.g.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$ doesn't make sense

Def (sum, scalar multiple)

$A, B : m \times n$ $\quad A + B = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & \cdot \end{bmatrix}$

$r \in \mathbb{R},$ $\quad rA = \begin{bmatrix} ra_{11} & \cdots & ra_{1n} \\ \vdots & & \vdots \\ ra_{m1} & \cdots & \end{bmatrix}$

Prop. A, B, C matrices of size such that the following make sense.

1) • $A(BC) = (AB)C$

• $A(B+C) = AB + AC$, $(A+B)C = AC + BC$

• $r(AB) = (rA).B = A.(rB)$

4) • $\mathbb{I}_m A_{m\times n} = A_{m\times n} = A_{m\times n} \mathbb{I}_n$,

where $I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{n\times n}$ identity matrix

pf 1) $C = [\vec{c}_1 \cdots \vec{c}_p]$.

$BC = [B\vec{c}_1 \cdots B\vec{c}_p]$

$A(BC) = [A(B\vec{c}_1) \cdots A(B\vec{c}_p)]$

$$(AB)C = (AB)[\vec{c}_1 \cdots \vec{c}_p] = [(AB)\vec{c}_1 \cdots (AB)\vec{c}_p]$$

Need: $\underbrace{A(B\vec{v})}= (AB)\vec{v}$.

$$A(B\vec{v}) = A(T_B(\vec{v})) = T_A(T_B(\vec{v})) = T_{AB}(\vec{v}) = (AB)\vec{v}$$

$(AB)C \overset{??}{=} A(BC)$

$\Updownarrow$

$T_{(AB)C} \overset{??}{=} T_{A(BC)}$

$T_{(AB)C} = T_{AB} \circ T_C = (T_A \circ T_B) \circ T_C$

$T_{A(BC)} = T_A \circ T_{BC} = T_A \circ (T_B \circ T_C)$

4) Claim: $I_n = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

$T_{I_n} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is identity

$\vec{x} \longmapsto \vec{x}$.

$$T_{I_n}(\vec{x}) = I_n(\vec{x}) = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}$$

$$\underline{I_m}\,\underline{A_{m\times n}} = A_{m\times n} = A_{m\times n} I_n$$

$$T_{I_m A} = T_A = T_{A I_n}$$

$$T_{I_m A} = T_{I_m} \circ T_A \qquad T_{A I_n} = T_A \circ T_{I_n}$$

$T_{I_m} \circ T_A = T_A = T_A \circ T_{I_n}$

$\mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m \xrightarrow{T_{I_m}} \mathbb{R}^m \qquad \mathbb{R}^n \xrightarrow{T_{I_n}} \mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m$

Q: Are the following true for any matrices??

- $\boxed{AB = BA}$ A, B: $n \times n$ FALSE in general
- $AB = AC \Rightarrow B = C$
- $AB = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & & & \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 \end{bmatrix}$ or $B = \begin{bmatrix} 0 \end{bmatrix}$

e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \neq BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

e.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Notations:

- A: square matrix ($n \times n$), $A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k \text{ copies of } A}$
- A^T transpose of A $(A^T)_{ji} = A_{ij}$

$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Prop
- $(A^T)^T = A$,
- $(A+B)^T = A^T + B^T$
- $(AB)^T \overset{??}{=} A^T \cdot B^T$ ✗

$m \times n \quad n \times p \qquad n \times m \quad p \times n$

- $(AB)^T = B^T A^T$

Pf: $(AB)_{ij}$ = std inner product b/w $\boxed{i\text{th row of } A}$ & jth column of B.

$$\begin{array}{ccc} i\text{th row of } A & & j\text{th column of } B \\ \parallel & & \parallel \\ i\text{th column of } A^T & & j\text{th row of } B^T \end{array}$$

$= (j\text{th row of } B^T) \cdot (i\text{th column of } A^T)$

inner product.

$= (B^T A^T)_{ji}$

$\Rightarrow (AB)^T = B^T A^T$ $\square$

Def A: $n \times n$ is invertible (non-singular) if

$\exists$ $n \times n$ matrices B and C

s.t. $AB = I_n = CA$

Rmk:
- If A invertible, $B = C$:

  $B = I_n \cdot B = (CA)B = C(AB) = C I_n = C.$
- If A invertible, $AB = I_n = BA$ and $AB' = I_n = B'A$

  then $B = B'$:

  $B = I_n \cdot B = (B'A)B = B'(AB) = B' I_n = B'.$

- If A invertible, such B is called the <u>inverse</u> of A, denoted by A^{-1}.

$$AA^{-1} = A^{-1}A = I.$$

Thm A: $n \times n$ invertible,
Then $A\vec{x} = \vec{b}$ has a unique sol^{n} $\forall \vec{b} \in \mathbb{R}^n$.
In fact, $\vec{x} = A^{-1}\vec{b}$

Pf
- Check $\vec{x} = A^{-1}\vec{b}$ is indeed a sol^{n}:

$$A\vec{x} = A(A^{-1}\vec{b}) = (\underline{A}\,\underline{A^{-1}})\vec{b} = I_n\vec{b} = \vec{b}.$$

- <u>uniqueness</u>: Suppose $A\vec{y} = \vec{b}$.

$$\Rightarrow A^{-1}(A\vec{y}) = A^{-1}\vec{b}$$
$$\| $$
$$(A^{-1}A)\vec{y}$$
$$\| $$
$$\vec{y}$$

$\square$

A invertible $\Rightarrow$ T_A is <u>injective & surjective</u> (bijective)

{invertible matrices} form a "group"

Thm
- A is invertible, then A^{-1} is invertible, and $(A^{-1})^{-1} = A$
- If A, B invertible, then so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1}$$

- If A is invertible, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$

Pf 2) $\boxed{AB}$

Take $C = \boxed{B^{-1}A^{-1}}$

$$ABC = (AB)(B^{-1}A^{-1}) = AA^{-1} = I_n$$

$$CAB = (B^{-1}A^{-1})(AB) = I_n$$

$$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}. \quad \square$$

How to characterize invertible matrices?

e.g. 1×1 $\quad [a_{11}]$ invertible $\Leftrightarrow a_{11} \neq 0$.

$$\left[\frac{1}{a_{11}}\right] = [a_{11}]^{-1}.$$

We can realize row operations as left multiplication of certain matrices.

e.g.
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+2a_{21} & a_{12}+2a_{22} \\ \boxed{a_{21} \quad a_{22}} \\ a_{31} & a_{32} \end{bmatrix}$$

e.g.
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

e.g.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 2a_{11} & 2a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Def <u>elementary matrices</u>:

$$\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & a_{ij} & & \ddots & \\ & & & & 1 \end{bmatrix}$$

→ left multiplⁿ:

↓

replace the ith row by (ith row) + a(jth row)

$$\begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array} \begin{bmatrix} 1 & & & & & \\ & \ddots\ 1 & & & & \\ & & 0 & & 1 & \\ & & & 1 \ddots 1 & & \\ & & 1 & & 0 & \\ & & & & & 1 \ddots 1 \end{bmatrix}$$

with column i and column j marked: swap (ith row) & (jth row)

$$\begin{array}{c} \\ \\ i \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & & \\ & \ddots\ 1 & & & \\ & & a & & \\ & & & 1 \ddots & \\ & & & & 1 \end{bmatrix}$$

multiply the (ith row) by a

Fact these matrices are all invertible