#1: By Gram-Schmidt, $\exists$ orthonormal basis of W. $\{\vec{v}_1, \dots, \vec{v}_n\}$

Then $\forall \vec{v} \in V$,

$$\mathrm{proj}_W \vec{v} = \langle \vec{v}, \vec{w}_1 \rangle \vec{w}_1 + \cdots + \langle \vec{v}, \vec{w}_n \rangle \vec{w}_n .$$

The linearity of proj_W then follows from the linearity of inner product. $\square$

#2: We have

$$\vec{x} = \langle \vec{x}, \vec{v}_1 \rangle \vec{v}_1 + \cdots + \langle \vec{x}, \vec{v}_n \rangle \vec{v}_n$$

$$\Rightarrow \langle \vec{x}, \vec{x} \rangle = \langle \vec{x}, \vec{v}_1 \rangle \langle \vec{x}, \vec{v}_1 \rangle + \cdots + \langle \vec{x}, \vec{v}_n \rangle \langle \vec{x}, \vec{v}_n \rangle$$

$$= \langle \vec{x}, \vec{v}_1 \rangle^2 + \cdots + \langle \vec{x}, \vec{v}_n \rangle^2 .$$

$$\Rightarrow 1 = \frac{\langle \vec{x}, \vec{v}_1 \rangle^2}{\|\vec{x}\|^2} + \cdots + \frac{\langle \vec{x}, \vec{v}_n \rangle^2}{\|\vec{x}\|^2}$$

$$= \cos^2\theta_1 + \cdots + \cos^2\theta_n . \quad \square$$

#3: Existence of QR decomp. is proved in class.

Suppose $A = QR = Q'R'$, with $\{Q, R\}$, $\{Q', R'\}$ satisfy the conditions.

Note that R is invertible upper-triangular, and its inverse R^{-1} is also upper-triangular.

$$Q = Q' \underbrace{R' R^{-1}}_{\overset{!!}{=} \, R''} = Q' R''.$$

R'' also upper triangular

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \\ & \text{orthonormal} & \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} \vec{w}_1 & \cdots & \vec{w}_n \\ & \text{orthonormal} & \end{bmatrix}}_{Q'} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots \\ & r_{22} & \cdots \\ 0 & & \ddots & r_{nn} \end{bmatrix}}_{R''}$$

Compare the 1^{st}-column:

$$\vec{v}_1 = r_{11}\vec{w}_1$$

$r_{11} > 0$ and $\vec{v}_1, \vec{w}_1$ unit vectors $\Rightarrow$ $\vec{w}_1 = \vec{v}_1$ and $r_{11} = 1$.

Compare 2^{nd} column: $\vec{v}_2 = r_{12}\vec{w}_1 + r_{22}\vec{w}_2$

$$= r_{12}\vec{v}_1 + r_{22}\vec{w}_2$$

$0 = \langle \vec{v}_1, \vec{v}_2 \rangle = \langle \vec{w}_1, \vec{v}_2 \rangle = r_{12} + r_{22}\langle \vec{w}_1, \vec{w}_2 \rangle = r_{12}.$

$\Rightarrow$ $\vec{v}_2 = r_{22}\vec{w}_2$, $r_{22} > 0$ and $\vec{v}_2, \vec{w}_2$ unit vectors

$\Rightarrow$ $\vec{w}_2 = \vec{v}_2$ and $r_{22} = 1$.

Continue Inductively, suppose we have:

$$\begin{bmatrix} \vec{v}_1 \cdots \vec{v}_{k-1} \; \vec{v}_k \cdots \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_{k-1} \; \vec{w}_k \cdots \end{bmatrix} \begin{bmatrix} 1 & & 0 & r_{1k} & \\ & \ddots & & \vdots & * \\ 0 & & 1 & & \\ & & & r_{kk} & \\ & 0 & & 0 & \ddots \end{bmatrix}$$

Then $\vec{v}_k = r_{1k}\vec{v}_1 + \cdots + r_{k-1,k}\vec{v}_{k-1} + r_{kk}\vec{w}_k$.

<u>Claim</u>: $r_{1k} = \cdots = r_{k-1,k} = 0$, $r_{kk} = 1$, $\vec{v}_k = \vec{w}_k$.

pf: $0 = \langle \vec{v}_1, \vec{v}_k \rangle = r_{1k} \langle \vec{v}_1, \vec{v}_1 \rangle + \cdots + r_{k-1,k} \langle \vec{v}_{k-1}, \vec{v}_k \rangle + r_{kk} \langle \vec{v}_1, \vec{w}_k \rangle$

$= r_{1k} + 0 + \cdots + 0 + r_{kk} \cancel{\langle \vec{w}_1, \vec{w}_k \rangle}^{0}$

$= r_{1k}.$

Similarly, we also have $r_{2k} = \cdots = r_{k-1,k} = 0$.

Hence $\vec{v}_k = r_{kk} \vec{w}_k$, where $r_{kk} > 0$ and $\vec{v}_k, \vec{w}_k$ unit.

$\Rightarrow$ $r_{kk} = 1$ and $\vec{v}_k = \vec{w}_k$. $\square$

Therefore, we showed that $Q = Q'$ and $R' R^{-1} = R'' = I_n$. $\square$

#4: $\langle A\vec{x}, \vec{y} \rangle = (A\vec{x})^T \vec{y} = \vec{x}^T A^T y = \langle \vec{x}, A^T \vec{y} \rangle$.

It remains to show the uniqueness., i.e.

" if $\langle \vec{x}, A^T \vec{y} \rangle = \langle \vec{x}, B\vec{y} \rangle \quad \forall \vec{x}, \vec{y}$,

then $A^T = B$."

$\Leftrightarrow$ " If an $n \times m$ matrix C satisfies

$$\langle \vec{x}, C\vec{y} \rangle = 0 \quad \forall \vec{x} \in \mathbb{R}^n, \vec{y} \in \mathbb{R}^m,$$

then $C = 0$."

Observe that

$$\underbrace{[0 \cdots 0\, 1\, 0 \cdots 0]}_{i\text{-th entry}} \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \\ c_{n1} & & \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \xleftarrow{j\text{-th entry}} = c_{ij} \quad \forall i,j.$$

$\Rightarrow$ $c_{ij} = 0 \quad \forall i,j$

$\Rightarrow$ $C = 0$. $\square$

#5: If $A = -A^T$, then

$$\langle A\vec{v}, \vec{v}\rangle = \langle \vec{v}, A\vec{v}\rangle = \vec{v}^T A \vec{v} = -\vec{v}^T A^T \vec{v}$$
$$= -(A\vec{v})^T \vec{v} = -\langle A\vec{v}, \vec{v}\rangle$$

$\Rightarrow \langle A\vec{v}, \vec{v}\rangle = 0.$

Conversely, If $\langle A\vec{v}, \vec{v}\rangle = 0 \quad \forall \vec{v}$,

$\langle A\vec{e}_i, \vec{e}_i\rangle = 0 \Rightarrow$ the diagonal entries $a_{ii} = 0$.

$$A = \begin{bmatrix} 0 & a_{12} & \cdots & \cdots \\ a_{21} & 0 & & \\ \vdots & & \ddots & \\ \vdots & & & 0 \end{bmatrix}$$

$$0 = \left\langle A\begin{bmatrix}1\\1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\\vdots\\0\end{bmatrix}\right\rangle = \left\langle \begin{bmatrix}a_{12}\\a_{21}\\ *\\ *\\ \vdots\\ *\end{bmatrix}, \begin{bmatrix}1\\1\\0\\\vdots\\ \vdots\\0\end{bmatrix}\right\rangle = a_{12} + a_{21}.$$

Hence $a_{12} = -a_{21}$.

Similarly, one can show that $a_{ij} = -a_{ji} \quad \forall i, j$.

Hence $A = -A^T$. $\square$

#6: Let $W = \text{Span}\{\vec{v}_1, \cdots, \vec{v}_n\} \subseteq V$.

W is a finite dim'l inner product space.

Let $\{\vec{w}_1, \cdots, \vec{w}_k\}$ be an orthonormal basis of W.

and define
$$f: W \xrightarrow{\cong} \mathbb{R}^k$$
$$\vec{w}_i \longmapsto \vec{e}_i$$

It's not hard to check that $\forall \vec{x}, \vec{y} \in W$, we have

$$\langle \vec{x}, \vec{y} \rangle = \langle f(\vec{x}), f(\vec{y}) \rangle.$$

Hence we can compute the Gram determinant on $\mathbb{R}^k$:

$$G(\vec{v}_1, \dots, \vec{v}_n) = \det\left(\left[\langle \vec{v}_i, \vec{v}_j \rangle\right]\right) = \det\left(\left[\langle f(\vec{v}_i), f(\vec{v}_j) \rangle\right]\right).$$

Let
$$A := \begin{bmatrix} | & & | \\ f(\vec{v}_1) & \cdots & f(\vec{v}_n) \\ | & & | \end{bmatrix} \in M_{k\times n}(\mathbb{R})$$

Then
$$G = \det(A^T A),$$

It remains to show the following claim:

Claim: A has l.i. columns $\Leftrightarrow$ $\det(A^T A) \neq 0$.

pf. A has l.i. columns $\Leftrightarrow \mathrm{Nul}(A) = \{0\}$

$\Leftrightarrow \mathrm{Nul}(A^T A) = \{0\}$. (HW4 #7 sol^{n}).

$\Leftrightarrow \det(A^T A) \neq 0$. $\square$

#7: The normal eq'n of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is:

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\Rightarrow$ least square sol^{ns} are $\{x + y = 0\}$. $\square$

#8: Consider the inner product $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$ on $\mathcal{C}[a,b]$.

By HW6 Problem #7,

$$\|f\|^2 \cdot \|1\|^2 \geq |\langle f, 1\rangle|^2$$

$$\|f\|^2 \cdot \|1\|^2 = \left(\int_a^b f(x)^2\,dx\right)\cdot\left(\int_a^b 1\,dx\right) = (b-a)\int_a^b f(x)^2\,dx$$

$$|\langle f, 1\rangle|^2 = \left(\int_a^b f(x)\,dx\right)^2.$$

$\square$