RESEARCH STATEMENT

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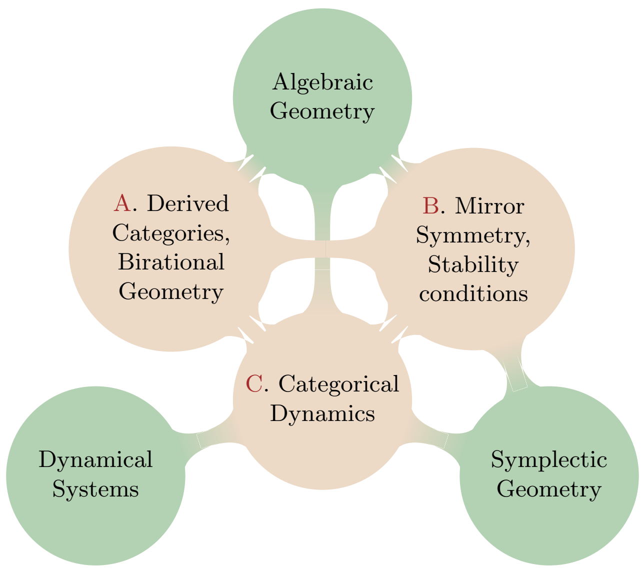
My research is primarily focused on *algebraic geometry*, and its intersections with *dynamical systems* and *symplectic geometry*. This document is divided into two parts. The first part provides an overview of my past work and contributions to the field, including those recognized by the ICCM award [1]. The second part delves into my ongoing research programs.

Past work



A. **Derived Categories and Birational Geometry.** This section provides a summary of my work on derived categories and birational geometry, with a particular focus on K3 surfaces and cubic fourfolds.

A cubic fourfold is a hypersurface $X \subseteq \mathbb{P}^5$ defined by a cubic polynomial. Its derived category admits a semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$, where $\mathcal{A}_X$ is a K3 category. A conjecture due to Huybrechts, which relates birational geometry of cubic fourfolds and their K3 categories, can be formulated as follows:

$$\mathcal{A}_X \cong \mathcal{A}_{X'} \overset{??}{\Longrightarrow} X \dashrightarrow X' \text{ birational.}$$

We prove that the conjecture holds on a divisor $\mathcal{C}_{20}$ of the moduli space of cubic fourfolds.

Theorem ([FL23a])**.** *Let X and X' be two cubic fourfolds. Suppose X is a general element in $\mathcal{C}_{20}$. Then $\mathcal{A}_X \cong \mathcal{A}_{X'}$ implies that X and X' are birational.*

The distinctiveness of $\mathcal{C}_{20}$ is that it parametrizes cubic fourfolds containing a Vernoese surface $\mathbb{P}^2 \cong V \subseteq X \subseteq \mathbb{P}^5$. The key tool in our proof is certain Cremona transformations on $\mathbb{P}^5$, defined as follows. Remember that a Veronese surface $V \subseteq \mathbb{P}^5$ is given by six quadratic defining polynomials $Q_0 = \cdots = Q_5 = 0$. One can use these polynomials to define a birational map

$$F := (Q_0, ..., Q_5) \colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^5.$$

A crucial observation in our work is that $X' := F(X)$ is again a cubic fourfold in $\mathcal{C}_{20}$; moreover, it is not isomorphic to X, but they share the same K3 category $\mathcal{A}_X \cong \mathcal{A}_{X'}$ and are birational to each other. Our construction also yields the rationality of certain cubic fourfolds that were previously unknown.

[1] I was awarded ICCM Silver Medal of Mathematics (formally the Morningside Awards) in 2022, which is awarded to exceptional mathematicians of Chinese descent under the age of 45 for their seminal achievements in mathematics and applied mathematics, "*for important contributions in the moduli space of Calabi–Yau manifolds and its dynamics*".

Theorem ([FL23a]). *There are at least three irreducible divisors in $\mathcal{C}_{20}$ which parametrize rational cubic fourfolds which were not known before.*

In the context of Huybrechts' conjecture, the *Fourier–Mukai number* of a given cubic fourfold X is of significant importance in the study of its birational geometry. This number represents the count of isomorphism classes of cubic fourfolds X' such that $\mathcal{A}_X \cong \mathcal{A}_{X'}$. For example, we established that the Fourier–Mukai number of a general element $X \in \mathcal{C}_{20}$ is two. This is a key step in finalizing the proof of the first theorem mentioned above. Subsequently, we extend this result to encompass $\mathcal{C}_d$ for all discriminants d.

Theorem ([FL24]). *Let $X \in \mathcal{C}_d$ be a general element. We derive an explicit formula of its Fourier–Mukai number. In particular, if $d \not\equiv 0 \pmod{27}$, then $|FM(X)|$ is a power of 2; if $d \equiv 0 \pmod{27}$, then $|FM(X)| = 3 \cdot 2^n$ for some integer $n \geq 0$.*

In a conjecture proposed by Kuznetsov, the geometric nature of $\mathcal{A}_X$, that is, whether $\mathcal{A}_X \cong \mathrm{D}^b(S)$ for a certain K3 surface S, is equivalent to the rationality of X. Conversely, it is crucial to ascertain whether a given K3 surface S has a corresponding cubic fourfold. Note that any $\mathcal{A}_X$ admits an autoequivalence F such that $F^3 = [2]$. Hence, for a K3 surface S to have an associated cubic fourfold, it is necessary that $\mathrm{Aut}(\mathrm{D}^b(S))/[2]$ contains an element of order 3. We establish that this condition is not only necessary but also sufficient for a generic K3 surface S.

Theorem ([FL23b]). *Let S be a K3 surface of Picard number one. We derive an explicit formula of the number of conjugacy classes of finite subgroups of $\mathrm{Aut}(D^b(S))/[2]$. In particular, S admits associated cubic fourfold if and only if $\mathrm{Aut}(D^b(S))/[2]$ contains an element of order 3.*

A crucial element in our proof is the categorical analogue of the Nielsen realization problem, which states that any finite subgroup of the mapping class group of a Riemann surface fixes a point on its Teichmüller space.

Theorem ([FL23b]). *Let S be a K3 surface of Picard number one, and $G \subseteq \mathrm{Aut}(D^b(S))/[2]$ be a subgroup.*

$$G \text{ is finite} \iff G \text{ fixes a point on } \mathrm{Stab}^\dagger(S)/\mathbb{C}.$$

Here $\mathrm{Stab}^\dagger(S)$ denotes the distinguished component of the space of Bridgeland stability conditions on $D^b(S)$.

B. **Mirror Symmetry and Bridgeland Stability Conditions.** Mirror symmetry is a conjectured duality between *complex* and *symplectic* geometry. My work explores the interaction of various "objects" such as operations, structures, and invariants within the realms of complex and symplectic geometry.

We constructed the mirror operation corresponding to one of the most fundamental operations in complex geometry, the Atiyah flop. Recall that an Atiyah flop $\hat{X} \to X \leftarrow \hat{X}^\dagger$ contracts a $(-1,-1)$-rational curve C in a complex threefold $\hat{X}$ and resolves the resulting conifold singularity with another $(-1,-1)$-curve $C^\dagger$.

Theorem ([FHLY18]). *Given a symplectic sixfold (Y, ω) and a Lagrangian three-sphere $S \subseteq Y$, we construct another symplectic sixfold $(Y^\dagger, \omega^\dagger)$ with a corresponding Lagrangian three-sphere $S^\dagger \subseteq Y^\dagger$, together with a symplectomorphism*

$$f^{(Y,S)} : (Y, \omega) \to (Y^\dagger, \omega^\dagger).$$

It has the property that $f^{(Y^\dagger,S^\dagger)} \circ f^{(Y,S)} = \tau_S^{-1}$, where τ_S is the Dehn twist along S. Moreover, we prove that the symplectomorphism is compatible with the Atiyah flop in terms of equivalence of certain triangulated categories, and wall-crossings on their space of stability conditions.

We constructed the mirror of the Weil–Petersson metric on the moduli of complex structures on Calabi–Yau manifolds. In the context of mirror symmetry, the complex moduli space is identified with the *stringy Kähler moduli space* $\mathcal{M}_{\mathrm{Kah}}(X)$ on the mirror manifold. This space is conjectured to be embedded in the space of stability conditions on $\mathrm{D}^b(X)$. Our approach involves defining the Weil–Petersson geometry on the space of stability conditions, then restrict it to the stringy Kähler moduli space.

Theorem ([FKY21]). *For any Calabi–Yau category D, we define the Weil–Petersson metric on $\mathrm{Stab}^\dagger(D)/\mathbb{C}$ for an appropriate subset $\mathrm{Stab}^\dagger(D) \subseteq \mathrm{Stab}(D)$. Below are some low-dimensional example:*

- *Let E be an elliptic curve. Then*
$$\mathcal{M}_{\mathrm{Kah}}(E) \cong \mathrm{Aut}(D^b(E))\backslash \mathrm{Stab}^\dagger(D^b(E))/\mathbb{C} \cong \mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H};$$
our Weil–Petersson metric coincides with the Poincaré metric *on $\mathbb{H}$.*
- *Let $A = E_\tau \times E_\tau$ be the self-product of a generic elliptic curve. Then*
$$\mathcal{M}_{\mathrm{Kah}}(A) \cong \overline{\mathrm{Aut}}_{\mathrm{CY}}(D^b(A))\backslash \mathrm{Stab}^\dagger(D^b(A))/\mathbb{C} \cong \mathrm{Sp}(4,\mathbb{Z})\backslash\mathfrak{H}_2$$
is the Siegel modular variety; our Weil–Petersson metric coincides with its Bergman metric.

While the Weil–Petersson metric on $\mathrm{Aut}(\mathrm{D})\backslash\mathrm{Stab}^\dagger(\mathrm{D})/\mathbb{C}$ is generally degenerate, the Weil–Petersson metric on the complex moduli space is non-degenerate. As a result, we derive a criterion for the embedding
$$\iota\colon \mathcal{M}_{\mathrm{Kah}}(X) \hookrightarrow \mathrm{Aut}(\mathrm{D}^b(X))\backslash\mathrm{Stab}^\dagger(\mathrm{D}^b(X))/\mathbb{C}.$$
which is that the pullback of the Weil–Petersson metric is non-degenerate on $\mathcal{M}_{\mathrm{Kah}}(X)$. This gives a more refined characterization of the stringy Kähler moduli space.

I introduced a broadening of the concept of *systole* in the context of Calabi–Yau geometry and stability conditions. For a Riemannian manifold (M,g), its systole is defined as the shortest length of a non-contractible loop in M. When $M = \mathbb{T}^2$ is a two-torus, Loewner proved that the inequality $\mathrm{sys}(\mathbb{T}^2,g)^2 \leq \frac{2}{\sqrt{3}}\mathrm{vol}(\mathbb{T}^2,g)$ holds for any metric g on $\mathbb{T}^2$. I introduced the notions of systole and volume of Calabi–Yau structures, as well as Bridgeland stability conditions [Fan22]. These notions are natural generalizations of $\mathrm{sys}(\mathbb{T}^2,g)$ and $\mathrm{vol}(\mathbb{T}^2,g)$. Moreover, I proved the following generalized systolic inequality for K3 surfaces.

Theorem ([Fan22])**.** *Let X be a complex projective K3 surface. There exists a constant $C>0$ such that*
$$\mathrm{sys}(\sigma)^2 \leq C\cdot\mathrm{vol}(\sigma) \quad \textit{holds for all } \sigma\in\mathrm{Stab}^\dagger(D^b(X)).$$

In our work [AFL23], we have also explored the generalization of certain counting results in flat surfaces. The growth rate of the number of saddle connections of length $\leq L$ has been a subject of intensive study. We have extended these results to K3 surfaces, focusing on the counting of special Lagrangian submanifolds and semistable objects.

C. **Categorical Dynamics.** Categorical dynamics, a relatively new research subject, studies the properties of endofunctors of triangulated categories under large iterations. My research has shed light on its connections with various other fields, such as holomorphic dynamics, Teichmüller theory, symplectic mapping class groups, and rotation theory.

In 2014, Dimitrov, Haiden, Katzarkov, and Kontsevich introduced a notion of *categorical entropy* of endofunctors $F\colon \mathrm{D}\to\mathrm{D}$ of a triangulated category D, which generalizes the classical notion of topological entropy. Kikuta and Takahashi proved that if $\mathrm{D}=\mathrm{D}^b(X)$ is the derived category of a smooth complex projective variety X and $f\colon X\to X$ is a holomorphic self-map, then $h_{\mathrm{cat}}(\mathbb{L}f^*) = h_{\mathrm{top}}(f) = \log\rho([\mathbb{L}f^*])$. As an analogue of a fundamental result of Gromov and Yomdin in holomorphic dynamics, it is conjectured that $h_{\mathrm{cat}}(F) = \log\rho([F])$ holds for any $F\in\mathrm{Aut}(\mathrm{D}^b(X))$. I found the first counterexample to this conjecture.

Theorem ([Fan18a])**.** *Let X be a Calabi–Yau hypersurface in $\mathbb{P}^{d+1}$ and $d\geq 4$ be an even integer. Consider the autoequivalence $F \coloneqq T_{\mathcal{O}_X}\circ(-\otimes\mathcal{O}(-1))$ of $D^b(X)$, where $T_{\mathcal{O}_X}$ is the spherical twist of $\mathcal{O}_X$. Then*
$$h_{\mathrm{cat}}(F) > 0 = \log\rho([F]).$$
In fact, $h_{\mathrm{cat}}(F)$ is the unique positive real number $\lambda>0$ satisfying $\sum_{k\geq 1}\frac{\chi(\mathcal{O}(k))}{e^{k\lambda}} = 1$.

In the construction of F, the spherical twist $T_{\mathcal{O}_X}$ is considered as the mirror of the Dehn twist along a Lagrangian sphere in a mirror Calabi–Yau manifold. Consequently, the theorem illustrates that categorical dynamical systems formed by compositions of both holomorphic and symplectic ones, can exhibit a larger entropy than what is expected from the theory of holomorphic dynamics.

In another study, we introduce new canonical invariants, the *shifting numbers*, that measure the asymptotic amount by which an endofunctor $F\colon\mathrm{D}\to\mathrm{D}$ *translates* within the triangulated category. These invariants are the complement of the categorical entropy, and are analogues of the notion of *Poincaré translation number* which we now recall. There is a central extension of the group of orientation-preserving homeomorphisms of the circle $S^1 \cong \mathbb{R}/\mathbb{Z}$: $0 \to \mathbb{Z} \to \mathrm{Homeo}^+_Z(\mathbb{R}) \to \mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z}) \to 1$. The Poincaré translation number of $f \in \mathrm{Homeo}^+_\mathbb{Z}(\mathbb{R})$ is defined by $\rho(f) := \lim_{n\to\infty}(f^{(n)}(x_0) - x_0)/n$ for some $x_0 \in \mathbb{R}$. It is a classical fact that the limit exists and is independent of the choice of x_0. Now, for a ($\mathbb{Z}$-graded) triangulated category D, we have a similar central extension

$$0 \to \mathbb{Z} \to \mathrm{Aut}(\mathrm{D}) \to \mathrm{Aut}(\mathrm{D})/[1] \to 1$$

in which the shift functor [1] plays the role of the integral shift by 1 in the classical setting. Instead of the basepoint $x_0 \in \mathbb{R}$, we consider a split generator $G \in \mathrm{D}$, and define

$$\epsilon^+(G, F^nG) := \max\{k \colon \mathrm{Hom}(G, F^nG[-k]) \neq 0\} \text{ and } \epsilon^-(G, F^nG) := \min\{k \colon \mathrm{Hom}(G, F^nG[-k]) \neq 0\},$$

which serve as categorical analogues of the difference "$f^{(n)}(x_0) - x_0$".

Theorem ([FF23])**.** *Let $F\colon D \to D$ be an endofunctor and G be a split generator of D. Then the following limits exist and are finite real numbers:*

$$\tau^+(F) := \lim_{n\to\infty} \frac{\epsilon^+(G, F^nG)}{n} \quad \text{and} \quad \tau^-(F) := \lim_{n\to-\infty} \frac{\epsilon^-(G, F^nG)}{n}.$$

Moreover, we prove that $\tau^\pm(F)$ can be computed via the categorical entropy function $h_t(F)$, or via the phase functions of any *Bridgeland stability condition on D.*

An important property of the Poincaré translation number is that it gives a nontrivial *quasimorphism* on $\mathrm{Homeo}^+_\mathbb{Z}(\mathbb{R})$. In light of it, I prove the following categorical analogue, which generalizes our result in [FF23].

Theorem ([Fan24])**.** *Let X be an abelian variety. Then the shifting numbers satisfy $\tau^+(F) = \tau^-(F)$ for any $F \in \mathrm{Aut}(D^b(X))$, and*

$$\tau = \tau^\pm \colon \mathrm{Aut}(D^b(X)) \to \mathbb{R} \quad \textit{is a quasimorphism.}$$

In conclusion, I would like to highlight another two contributions I have made in the field of categorical dynamics. Firstly, drawing parallels with Teichmüller theory, we introduced the concept of *pseudo-Anosov autoequivalences* [FFH+21]. This condition is excepted to be satisfied by generic autoequivalences. We further illustrated this concept through various examples of pseudo-Anosov autoequivalences in Calabi–Yau categories of dimensions greater than one.

Secondly, in [FFO21] we proposed a notion of *categorical polynomial entropy*. This measure can be employed to quantify the complexity of dynamical systems exhibiting slow growth rates, such as the functor $(-\otimes \mathcal{O}(1))$. This leads to a *categorical trichotomy*: autoequivalences are classified into three categories – those with positive entropy, those with zero entropy and positive polynomial entropy, and those with zero polynomial entropy. This trichotomy behaves similarly to the classification of elements of $\mathrm{SL}(2,\mathbb{R})$ into hyperbolic, parabolic, and elliptic elements.

Ongoing Research Programs

In what follows, I will briefly touch on one of my ongoing research programs. While the details are sparse, I hope to provide a quick glimpse into some of my current work.

This program is a natural extension of my previous work on K3 surfaces, stability conditions, and categorical dynamics. In a nutshell, the goal of this program is to classify the autoequivalences of the derived categories of coherent sheaves on K3 surfaces via their actions on the stability spaces, and then study the geometric properties of each type of autoequivalence.

An analogy can be drawn to the Nielsen–Thurston classification of mapping classes of Riemann surfaces, which classifies mapping classes into three types: finite order, reducible, and pseudo-Anosov. Each type of mapping class exhibits distinct geometric and dynamical behaviors. Note that such a classification has been partially obtained in our joint work [FL23b] for K3 surfaces of Picard number one. Here, we seek a more thorough understanding for all K3 surfaces. For each type of autoequivalence, we aim to study their different geometric and dynamical properties.

Question (a). *Let $F \in \mathrm{Aut}(D^b(X))$ be an autoequivalence of the derived category of a K3 surface X. Suppose F is of finite order. Does there always exist a stability condition $\sigma \in \mathrm{Stab}(D^b(X))$ such that $F \cdot \sigma = \sigma$?*

The answer to this question is crucial for understanding finite subgroups of $\mathrm{Aut}(D^b(X))$. The answer is affirmative for K3 surfaces of Picard number one [FL23b], and we used this to provide a full classification of finite subgroups of $\mathrm{Aut}(D^b(X))$ in this case. For K3 surfaces of larger Picard number, one would need a better understanding of the corresponding period domain and the covering map from the stability space to the period domain.

Question (b). *Let $F \in \mathrm{Aut}(D^b(X))$ be an autoequivalence of infinite order. Suppose it has zero entropy $h_{\mathrm{cat}}(F) = 0$. Does it always have positive polynomial entropy $h_{\mathrm{poly}}(F) > 0$? Does the polynomial entropy have some geometric meaning?*

When F is a spherical twist (i.e., the monodromy around a (-2)-hyperplane, or the *mirror* of the Dehn twist along a Lagrangian sphere), it has $h_{\mathrm{poly}}(F) = 1$. On the other hand, when $F = (- \otimes \mathcal{O}(1))$ is tensoring an ample line bundle (which preserves a semirigid object, as opposed to a spherical object for spherical twists), it has $h_{\mathrm{poly}}(F) = 2$. Therefore, it is plausible that the polynomial entropy encodes certain geometric properties of the autoequivalences.

Question (c). *Let $F \in \mathrm{Aut}(D^b(X))$ be an autoequivalence of infinite order. Suppose that $h_{\mathrm{cat}}(F) > 0$. Under what conditions is it pseudo-Anosov in the sense of [FFH+21]? If F is not pseudo-Anosov, does it always preserve a nontrivial subcategory?*

When $h_{\mathrm{cat}}(F) > 0$, we expect to obtain a filtration of $D^b(X)$ (depending on the autoequivalence F) via the exponential growth rates of the mass functions of objects, and use this filtration to characterize the properties of F (such as being pseudo-Anosov or not).

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