HOMEWORK 3
MATH 104, SECTION 6

Office Hours: Tuesday and Wednesday 9:30-11am at 735 Evans.

Reading

There will be reading assigned for each lecture. You should come to the class having read the assigned sections of the textbook.

Due February 6: Ross, Section 10

Due February 11: Ross, Section 11

Problem set (10 problems; due February 6)

Submit your homework at the beginning of the lecture on Thursday. *Late homework will not be accepted under any circumstances.*

You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any.

Write clearly! Mastering mathematical writing is one of the goals of this course.

You have to staple your work if it is more than one page.

(1) Determine each of the following sequences is convergent or divergent. For convergent sequences, find the limit and prove it. For divergent sequences, prove that they are divergent.
   (a) $a_n = (\frac{2}{3})^n$.
   (b) $b_n = 2^n$.
   (c) $c_n = \frac{\sin(2n)}{\sqrt{n}}$.
   (d) $d_n = \sin(\frac{n\pi}{2})$.
   (e) $e_n = \sqrt{n^2 + 4n} - n$.
   (f) $f_n = \frac{2^n}{n!}$.

(2) Let (a_n) be a convergent sequence with $\lim_{n\to\infty} a_n = a$. Let (b_n) be another sequence such that $b_n = a_n$ for all but finitely many n . Prove that (b_n) is a convergent sequence and has the same limit as (a_n) .

(3) (Squeeze lemma) Let (a_n) , (b_n) , (c_n) be three sequences satisfying $a_n \leq b_n \leq c_n$ for all n . Suppose that (a_n) and (c_n) both converge with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a$. Prove that $\lim_{n\to\infty} b_n = a$.

(4) Let $a_n = \frac{n - \sin(n)}{n}$. Use the squeeze lemma to show that a_n converges and find the limit.

(5) Let (a_n) and (b_n) be two convergent sequences with limits a and b respectively. Suppose that $a_n \leq b_n$ for all but finitely many n . Prove that $a \leq b$.

(6) Show that if (a_n) converges to a , then the sequence of absolute values $(|a_n|)$ converges to $|a|$. What about the converse statement?

(7) Let $S \subset \mathbb{R}$ be a nonempty subset which is bounded above. Let $z = \sup S$. Prove that there exists a sequence (a_n) such that $a_n \in S$ for all n , and $\lim_{n\to\infty} a_n = z$.

(8) Let (a_n) be a sequence of nonzero real numbers. Suppose that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = b$ exists and is less than 1 . Prove that $\lim_{n\to\infty} a_n = 0$. (Hint: Choose any c so that $b < c < 1$ and show that there exists $N > 0$ such that $|a_{n+1}| < c|a_n|$ for all $n > N$.)

(9) (a) Suppose (a_n) is a bounded sequence and (b_n) is a sequence converging to 0 . Show that $(a_n b_n)$ converges to 0 .
   (b) Give an example where (a_n) is unbounded, (b_n) converges to 0 , and $(a_n b_n)$ is divergent.
   (c) Give an example where (a_n) is bounded, (b_n) converges to some $b \neq 0$, and $(a_n b_n)$ is divergent.

(10) Prove or find a counterexample of the following statements.
   (a) If (a_n) is a sequence such that (a_n^2) converges, then (a_n) converges.
   (b) If (a_n) is a sequence such that (a_n^3) converges, then (a_n) converges.