HW 10 sol'n

#1:

- For $x \neq 0$,

$$f'(x) = 2x\sin\left(\tfrac{1}{x}\right) + \cancel{x^2}\cos\left(\tfrac{1}{x}\right)\cdot\frac{-1}{\cancel{x^2}}$$
$$= 2x\sin\left(\tfrac{1}{x}\right) - \cos\left(\tfrac{1}{x}\right).$$

- For $x=0$,

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{x}$$
$$= \lim_{x\to 0} x\sin\left(\tfrac{1}{x}\right) = 0 \quad \text{since } \left|\sin\left(\tfrac{1}{x}\right)\right| \le 1 \text{ bounded.}$$

- $f' : \mathbb{R} \longrightarrow \mathbb{R}$ is not continuous at 0. :

Consider $\left(x_n = \frac{1}{2n\pi}\right)_{n\in\mathbb{N}}$.

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{1}{2n\pi} = 0,$$

but
$$\lim_{n\to\infty} f'(x_n) = \lim_{n\to\infty}\left(\frac{2}{2n\pi}\underbrace{\sin(2n\pi)}_{0} - \underbrace{\cos(2n\pi)}_{1}\right) = -1 \neq f'(0)$$

$\square$

(In fact, $\lim_{x\to 0} f'(x)$ doesn't exist.)

#2: See Ross, §31, Example 3.

#3: (a) Claim: $f_x : \mathbb{R} \longrightarrow \mathbb{R}$, $y \longmapsto f(x,y)$ is conti. $\forall x\in\mathbb{R}$.

① $x \neq 0$: $f_x : \mathbb{R} \longrightarrow \mathbb{R}$, $y \longmapsto f(x,y) = \dfrac{xy}{\underbrace{x^2+y^2}_{\neq 0}}$ is a rational function, where the denominator is always nonzero. Hence is conti. (by Ross, Thm 17.4)

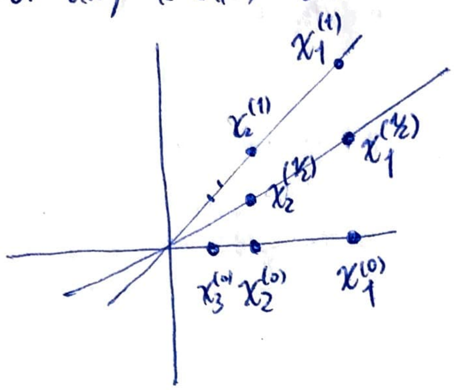
② $x=0$: $f_0 : \mathbb{R} \longrightarrow \mathbb{R}$,

$$y \longmapsto f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} \overset{\text{since } x=0}{=} 0, & y\neq 0 \\ 0 & y=0 \end{cases} \equiv 0.$$

obviously conti. $\square$

#3 (b):

For any $k \in \mathbb{R}$, consider a seq. of pts in $\mathbb{R}^2$: $\{\underbrace{(\frac{1}{n}, \frac{k}{n})}_{} \}_{n \in \mathbb{N}}$ where $x_n^{(k)} = (\frac{1}{n}, \frac{k}{n})$



Then:
- $\lim_{n\to\infty} (\frac{1}{n}, \frac{k}{n}) = (0,0)$ in $\mathbb{R}^2$ $\forall k \in \mathbb{R}$.
- $\lim_{n\to\infty} f(\frac{1}{n}, \frac{k}{n}) = \dfrac{\frac{k}{n^2}}{\frac{1}{n^2} + \frac{k^2}{n^2}} = \dfrac{k}{1+k^2}$ $\forall k \in \mathbb{R}$.

i.e. If we take a seq. of pts in $\mathbb{R}^2$ approaching $(0,0)$ on the line of slope k, then the value of these pts approaches $\frac{k}{1+k^2}$.

Hence f is not conti. at $(0,0)$. $\square$

#4 (a) False. Counterexample: $f(x) = x^3$, $f'(0) = 0$.

(b) True. $\forall$ $a < x < y < b$, by MVT,

$\exists$ $x < c < y$ st. $\underset{\underset{0}{\vee}}{f'(c)} = \dfrac{f(y)-f(x)}{y-x}$ by assumption.

$\Rightarrow f(y) > f(x)$

$\Rightarrow f$ is strictly increasing. $\square$

#5. See Ross $\S 31$, Example 2.

#6 Let $f(x) = e^x + x - 1$. $f(0) = 0$.

Claim: $f(x) \neq 0 \quad \forall x \neq 0$.

If not, i.e., $\exists\, x \neq 0$ s.t. $f(x) = 0$.

Then by Rolle's thm, $\exists\, \underset{\substack{\downarrow \\ (\text{btw } 0 \text{ and } x)}}{y \neq 0}$ s.t. $f'(y) = 0$.

$\Rightarrow \quad 0 = f'(y) = e^y + 1.$

$\Rightarrow \quad e^y = -1$, which is impossible. □

#7: Claim: $f'(x) = 0 \quad \forall x \in \mathbb{R}$.

(We proved in class that this will imply f is a constant function.)

$$\left|\frac{f(y)-f(x)}{y-x}\right| \leq \frac{|y-x|^2}{|y-x|} = \underbrace{|y-x|}_{\searrow 0 \text{ as } y \to x}$$

Hence $\displaystyle\lim_{y \to x} \left|\frac{f(y)-f(x)}{y-x}\right| = 0$

$\Rightarrow \quad \displaystyle\lim_{y \to x} \frac{f(y)-f(x)}{y-x} = 0.$

$\Rightarrow \quad f'(x)$ exists $\forall x \in \mathbb{R}$, and $f'(x) \equiv 0$. □

#8: Fix a point $c \in (a,b)$. Since f is unbounded on (a,b), $\forall M > 0$, $\exists\, d \in (a,b)$ s.t. $|f(d)| > (b-a)M + |f(c)|$

Then $\exists\, e$ between c and d s.t.

$$|f'(e)| = \left|\frac{f(d)-f(c)}{d-c}\right| > \frac{|f(d)|-|f(c)|}{b-a} > M. \ \square$$

#9 Let $S := \{x \in [0,1] \mid f(y) = 0 \ \forall\, 0 \le y \le x\}$. WTS: $1 \in S$

- S is nonempty since $0 \in S$. Let $z := \sup S$.
- By the continuity of f, it suffices to show that $z = 1$.

Suppose $z < 1$. Define $\varepsilon := \min\{\frac{1-z}{2}, \frac{1}{2M}\} > 0$.

Then $z + \varepsilon < 1$ and $M\varepsilon \le \frac{1}{2}$.

Since $z = \sup S$, $\exists\, y \in (z-\varepsilon, z]$ s.t. $y \in S$, i.e. $f(b) = 0 \ \forall\, 0 \le b \le y$.

Consider the interval $[y, z+\varepsilon] \subset [0,1]$.

Since $|f|$ is conti. on $[0,1]$, it achieves max. on $[y, z+\varepsilon]$.

i.e. $\exists\, x \in [y, z+\varepsilon]$ s.t. $|f(x)| \ge |f(a)| \ \forall a \in [y, z+\varepsilon]$.

- Suppose that $f(x) = 0$, then $f(a) = 0 \ \forall a \in [y, z+\varepsilon]$,

  $\Rightarrow f(a) = 0 \ \forall\, 0 \le a \le z+\varepsilon$. $\Rightarrow z+\varepsilon \in S$,

  This contradicts with $z = \sup S$.

- Hence $|f(x)| > 0$.

  By MVT, $\exists\, w \in (y, x)$ s.t.

$$\left|\frac{f(x) - \overset{0}{f(y)}}{x-y}\right| = |f'(w)| \le M|f(w)|$$

Since $0 < x - y < 2\varepsilon \le \frac{1}{M}$,

$$|f(x)| \le M|x-y|\,|f(w)| < |f(w)|.$$

This contradicts with the assumption that $|f(x)| \ge |f(a)| \ \forall a \in [y, z+\varepsilon]$.

This proves $z = 1$. $\square$

#10. Assume $|f'(l)| > 1$.

- Since f is conti. and $x_{n+1} = f(x_n)$. $\Rightarrow$ $f(l) = l$.
- $f'(l) = \lim_{x\to l} \frac{f(x) - f(l)}{x - l} = \lim_{x\to l} \frac{f(x) - l}{x - l}$

$$\Rightarrow \lim_{x\to l} \frac{|f(x) - l|}{|x - l|} = |f'(l)| > 1.$$

Let $\varepsilon = \frac{|f'(l)| - 1}{2} > 0$.

Then $\exists\ \delta > 0$ s.t. $\forall\ |x - l| < \delta$, we have

$$\frac{|f(x) - l|}{|x - l|} > \frac{|f'(l)| + 1}{2}.$$

$$\Rightarrow |x - l| < |x - l| \cdot \frac{|f'(l)| + 1}{2} < |f(x) - l| \quad \forall\ |x - l| < \delta.$$

- Since $\lim_{n\to\infty} x_n = l$, $\exists\ N > 0$ s.t. $|x_n - l| < \delta$ $\forall n > N$.

$$\Rightarrow |x_n - l| < |x_{n+1} - l| \quad \forall n > N.$$

$$\Rightarrow |x_{N+1} - l| < |x_M - l| \quad \forall M > N+1$$

Let $M \to \infty$, then $\lim_{M\to\infty} |x_M - l| = 0$.

$$\Rightarrow |x_{N+1} - l| < 0 \quad \text{contradiction. } \square$$