#1: Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A. Then

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 1 & — ① \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 5 & — ② \\ \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 7 & — ③ \end{cases}$$

$$①, ② \Rightarrow \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = -2$$

$$\lambda_1^3 + \lambda_2^3 + \lambda_3^3 - 3\lambda_1\lambda_2\lambda_3 = (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_1\lambda_3)$$

$$// \qquad = 1 \cdot (5 - (-2))$$

$$7 - 3\lambda_1\lambda_2\lambda_3 \qquad = 7$$

$$\Rightarrow \det(A) = \lambda_1\lambda_2\lambda_3 = 0. \quad \square$$

#2: • It's not hard to show that the eigenvalues of A are $1, 2, 3$.

• hence $A = P \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} P^{-1}$ for some P invertible

$$\Rightarrow \begin{bmatrix} 1 & & \\ & 8 & \\ & & 27 \end{bmatrix} = A^3 = P \begin{bmatrix} 1 & & \\ & 8 & \\ & & 27 \end{bmatrix} P^{-1}.$$

$$\Rightarrow \begin{bmatrix} 1 & & \\ & 8 & \\ & & 27 \end{bmatrix} P = P \begin{bmatrix} 1 & & \\ & 8 & \\ & & 27 \end{bmatrix}.$$

Write $P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$. Then

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ 8P_{21} & 8P_{22} & 8P_{23} \\ 27P_{31} & 27P_{32} & 27P_{33} \end{bmatrix} = \begin{bmatrix} P_{11} & 8P_{12} & 27P_{13} \\ P_{21} & 8P_{22} & 27P_{23} \\ P_{31} & 8P_{32} & 27P_{33} \end{bmatrix} \Rightarrow P_{ij} = 0 \ \forall i \neq j.$$

Hence
$$A=\begin{bmatrix} p_{11} & & \\ & p_{22} & \\ & & p_{33}\end{bmatrix}\begin{bmatrix} 1 & & \\ & 2 & \\ & & 3\end{bmatrix}\begin{bmatrix} p_{11}^{-1} & & \\ & p_{22}^{-1} & \\ & & p_{33}^{-1}\end{bmatrix}$$
$$=\begin{bmatrix} 1 & & \\ & 2 & \\ & & 3\end{bmatrix}. \quad \square$$

#3: Max $=3$, Min $=1$. (Use HW6 #8).

#4: Use HW8 #7.

#5: (a) By HW8 #4, $\exists$ B positive def. s.t. $A=B^TB$.
B is invertible, $\exists!$ Q orthogonal, R: upper triangular w/ positive diagonal.
s.t. $B=QR$.

$$\Rightarrow A=(QR)^TQR=R^TR. \quad \square$$

(b)
$$R=\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & \vdots \\ & & \ddots & \vdots \\ 0 & & & r_{nn}\end{bmatrix} \qquad r_{11}, r_{22}, \dots, r_{nn}>0.$$

$$\det(A)=\det(R^TR)=(\det R)^2=r_{11}^2\cdot\dots\cdot r_{nn}^2.$$

$$A=R^TR=\begin{bmatrix} r_{11} & & & 0 \\ r_{12} & r_{22} & & \\ \vdots & \vdots & \ddots & \\ r_{1n} & & & r_{nn}\end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & \vdots \\ & & \ddots & \vdots \\ 0 & & & r_{nn}\end{bmatrix}$$

Hence $a_{11}=r_{11}^2$

$$a_{22} = r_{12}^2 + r_{22}^2 \geq r_{22}^2$$

$$\vdots$$

$$a_{nn} = r_{1n}^2 + \cdots + r_{nn}^2 \geq r_{nn}^2.$$

$$\Rightarrow \quad a_{11} a_{22} \cdots a_{nn} \geq r_{11}^2 \cdots r_{nn}^2 = \det(A). \quad \square$$

#6: $\exists$ orthogonal diagonalization $A = PDP^T$

$$\Rightarrow \quad I = A^k = PD^kP^T$$

$$\Rightarrow \quad D^k = I.$$

Since A is real symmetric, its eigenvalues are real, hence the eigenvalues are either 1 or -1.

$$\Rightarrow \quad D^2 = I.$$

$$\Rightarrow \quad A^2 = PD^2P^T = I. \quad \square$$

#7: $A^2B - BA^2 = A(AB - BA) + (AB - BA)A$

$$= 2A^2.$$

$$A^3B - BA^3 = A(A^2B - BA^2) + (AB - BA)A^2$$

$$= 3A^3.$$

one can prove inductively that $A^kB - BA^k = kA^k \quad \forall k \geq 1.$

Consider the linear map:

$$F: M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R})$$
$$X \longmapsto XB - BX.$$

Assume the contrary that $A^k \neq 0$ $\forall k \geq 1$.

Then A^k is an eigenvector of F w/ eigenvalue k.

$\Rightarrow$ F has infinitely many eigenvalues.

But $\dim M_{n\times n}(\mathbb{R}) = n^2 < +\infty$.

Contradiction. $\square$