#1:

Suppose λ is an eigenvalue of A, i.e. $\exists \vec{v} \neq \vec{0}$ s.t. $A\vec{v} = \lambda\vec{v}$.

$\Rightarrow \quad \theta = 0\vec{v} = A^k\vec{v} = A^{k-1}(\lambda\vec{v}) = \lambda A^{k-1}\vec{v} = \cdots = \lambda^k\vec{v}$.

$\Rightarrow \quad \lambda = 0$ since $\vec{v} \neq \vec{0}$. $\square$

#2:

λ is an eigenvalue of $A \iff \det(A - \lambda I) = 0$.

$$\Updownarrow$$

λ is an eigenvalue of A^T. $\iff \det(A^T - \lambda I) = 0$.

#3:

Observe that $A\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. $\square$

#4: We have $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$.

(a) Plug in $\lambda = 0 \Longrightarrow \det(A) = \lambda_1\lambda_2\cdots\lambda_n$. $\square$

(i.e. consider the constant term of the char. poly).

(b) Consider the coefficient of λ^{n-1} of the char. poly:

$$\det\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn}-\lambda \end{bmatrix}$$

The λ^{n-1}-coeff. is given by: $(-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})$
$= (-1)^{n-1}\,tr(A)$.

On the other hand, the λ^{n-1}-coeff. of $(\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$ is given by: $(-1)^{n-1}(\lambda_1+\cdots+\lambda_n)$.

$\Rightarrow \quad tr(A) = \lambda_1 + \cdots + \lambda_n$. $\square$

#5: This follows directly from HW4(3)(d). $\square$

#6:
$$A = P \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{=:D} P^{-1} \quad \text{for some invertible } P.$$

Observe that

$$(\lambda_1 I - A)\cdots(\lambda_n I - A) = (\lambda_1 I - PDP^{-1})\cdots(\lambda_n I - PDP^{-1})$$
$$= P(\lambda_1 I - D)\cdots(\lambda_n I - D)P^{-1}$$

It's not hard to show that

$$(\lambda_1 I - D)\cdots(\lambda_n I - D) = 0. \quad \square$$

#7:

(a) $B\vec{v} = \lambda\vec{v} \Rightarrow BA\vec{v} = AB\vec{v} = A(\lambda\vec{v}) = \lambda A\vec{v}$. $\square$

(b) Since B has n distinct eigenvalues, B is diagonalizable and each eigenspace is of 1-dim'l.
Suppose $\vec{v} \neq \vec{0}$ is an eigenvector of B w/ $B\vec{v} = \lambda\vec{v}$.
By (a), $A\vec{v}$ also satisfies $B(A\vec{v}) = \lambda(A\vec{v})$.
Hence $A\vec{v} \in \mathrm{Span}\{\vec{v}\}$ since $\dim \mathrm{Nul}(B-\lambda I) = 1$.
$\Rightarrow \vec{v}$ is also an eigenvector of A.

(c) The eigenbasis of B also gives an eigenbasis of A.

AB also shares the same eigenvectors as A and B. by Part (b). $\square$

#8:

(a) (i) Let μ be an eigenvalue of A. $A\vec{v} = \mu\vec{v}$, $\vec{v} \neq \vec{0}$.

$\Rightarrow$ $A^2\vec{v} = \mu^2\vec{v}$. $\Rightarrow$ $\mu^2 \in \{\lambda_1, \cdots, \lambda_k\}$

$\Rightarrow$ $\mu \in \{\pm\sqrt{\lambda_1}, \cdots, \pm\sqrt{\lambda_k}\}$. $\square$

(ii) For each $1 \leq i \leq k$, $0 = \det(A^2 - \lambda_i I) = \det(A - \sqrt{\lambda_i} I)(A + \sqrt{\lambda_i} I)$.

$\Rightarrow$ at least one of $\pm\sqrt{\lambda_i}$ is an eigenvalue of A. $\square$

(b)

$$A^2 = \underbrace{\begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 20 \end{bmatrix}}_{P :=} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 20 \end{bmatrix}^{-1}$$

$\vec{w}_1$ $\vec{w}_2$ $\vec{w}_3$

Claim: $A = P \begin{bmatrix} \pm 1 & & \\ & \pm 2 & \\ & & \pm 3 \end{bmatrix} P^{-1}$, hence there are 8 possible such matrices.

pf:

By Part (a), A has eigenvalues $\lambda_1 = \pm 1$, $\lambda_2 = \pm 2$, $\lambda_3 = \pm 3$.

$\Rightarrow$ each of the eigenspaces of A is of 1-dim'l.

$\mathrm{Nul}(A - \lambda_i I) = \mathrm{Span}\{\vec{v}_i\}$.

$\Rightarrow$ $A\vec{v}_i = \lambda_i \vec{v}_i$

$\Rightarrow$ $A^2\vec{v}_i = \lambda_i^2 \vec{v}_i$

$\Rightarrow$ $\vec{v}_i$ is an eigenvector of A^2 w.r.t. eigenvalue λ_i^2.

Notice that each of the eigenspaces of A^2 is also of 1-dim'l.

Hence $\mathrm{Span}\{\vec{v}_i\} = \mathrm{Nul}(A^2 - \lambda^2 I) = \mathrm{Span}\{\vec{w}_i\}$.

In particular, we have $A\vec{w}_i = \lambda_i \vec{w}_i$.

$$\Rightarrow \quad AP = P\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

$$\Rightarrow \quad A = P\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}P^{-1}. \quad \square$$

#9:

(a) "$\mathrm{Col}(P) \subseteq \mathrm{Nul}(P-I)$": If $\vec{v} = P\vec{w}$, then

$$(P-I)\vec{v} = (P-I)P\vec{w} = (P^2-P)\vec{w} = 0\vec{w} = \vec{0}.$$

"$\mathrm{Nul}(P-I) \subseteq \mathrm{Col}(P)$": If $(P-I)\vec{v} = \vec{0}$,

then $\vec{v} = P\vec{v} \in \mathrm{Col}(P)$. $\square$

(b) $\mathrm{Nul}(P) =$ eigenspace of P w.r.t. eigenvalue 0.

By Part (a), $\mathrm{Col}(P) =$ eigenspace of P w.r.t. eigenvalue 1.

By rank-nullity thm, $\dim \mathrm{Nul}(P) + \dim \mathrm{Col}(P) = n$.

$\Rightarrow$ P is diagonalizable. $\square$

#10.

(a) $$\overline{\bar{x}^T A x} = x^T \bar{A}\,\bar{x} \underset{\substack{\uparrow \\ (\text{it's a } 1\times 1 \text{ matrix}) \\ (\bar{A} = A \text{ since } A \text{ real})}}{=} (x^T A \bar{x})^T = \bar{x}^T A^T x \underset{\substack{\uparrow \\ (A^T = A)}}{=} \bar{x}^T A x. \quad \square$$

(b) $$Ax = \lambda x \Rightarrow \underbrace{\bar{x}^T A x}_{\in \mathbb{R}} = \bar{x}^T \lambda x = \lambda \underbrace{\bar{x}^T x}_{\in \mathbb{R}}.$$

$$\Rightarrow \lambda \in \mathbb{R}. \quad \square$$

(c) Write $x = y + iz$, where $y, z \in \mathbb{R}^n$. ($y = \mathrm{Re}\, x$, $z = \mathrm{Im}\, x$).

Then

$$\begin{array}{ccc} A(y+iz) & = & \lambda(y+iz) \\ \| & & \| \\ Ay + iAz & & \lambda y + i\lambda z. \end{array}$$

Since $Ay, Az, \lambda y, \lambda z \in \mathbb{R}^n$ (A, λ: real),

we have $Ay = \lambda y$ and $Az = \lambda z$. $\square$