Today: • 3 equivalent definitions of conti. fcns.
• 2 important properties of conti. fcns. (cpt → cpt, conn. → conn.)

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examples of conti. fn. / $\mathbb{R}$:

- $f(x)=x$, const., x^2
- HW: addition, product, ... of conti fcns are still conti.
- $f(x)=\sin x$, $\cos x$, e^x ...

Thm: $\boxed{f: X \to Y \text{ continuous at } x_0 \in X}$ (I)

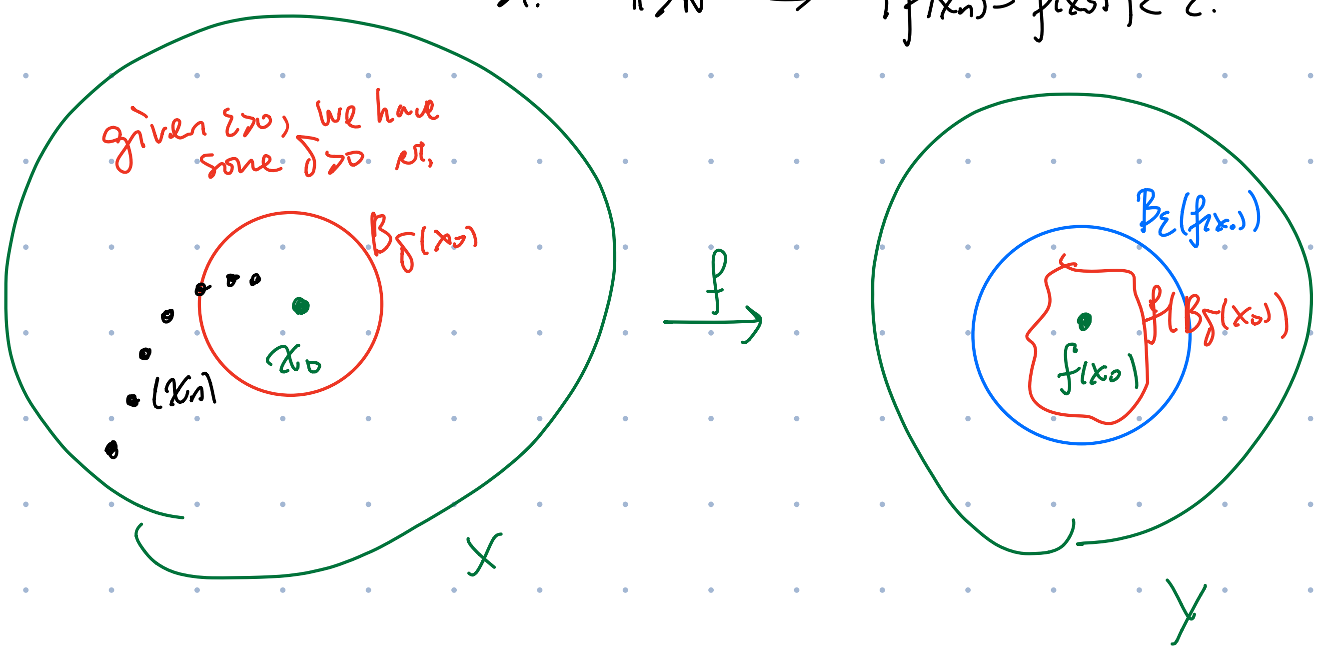
$\Longleftrightarrow$ $\boxed{\forall \varepsilon>0,\ \exists \delta>0 \text{ s.t. } d_X(x,x_0)<\delta \Rightarrow d_Y(f(x),f(x_0))<\varepsilon}$ (II)

pf:

($\Leftarrow$) need to show: If $\lim x_n = x_0$, then $\lim f(x_n)=f(x_0)$,

i.e. $\forall \varepsilon>0$, $\exists N>0$

s.t. $n>N \Rightarrow |f(x_n)-f(x_0)|<\varepsilon$.



- By (II), $\forall \varepsilon > 0$,

  $\exists \delta > 0$ s.t. $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$.

- Since $\lim x_n = x_0$, so, $\exists N > 0$

  s.t. $n > N \Rightarrow d(x_n, x_0) < \delta$

  $\Rightarrow d(f(x_n), f(x_0)) < \varepsilon$.

Hence $\lim f(x_n) = f(x_0)$. $\square$

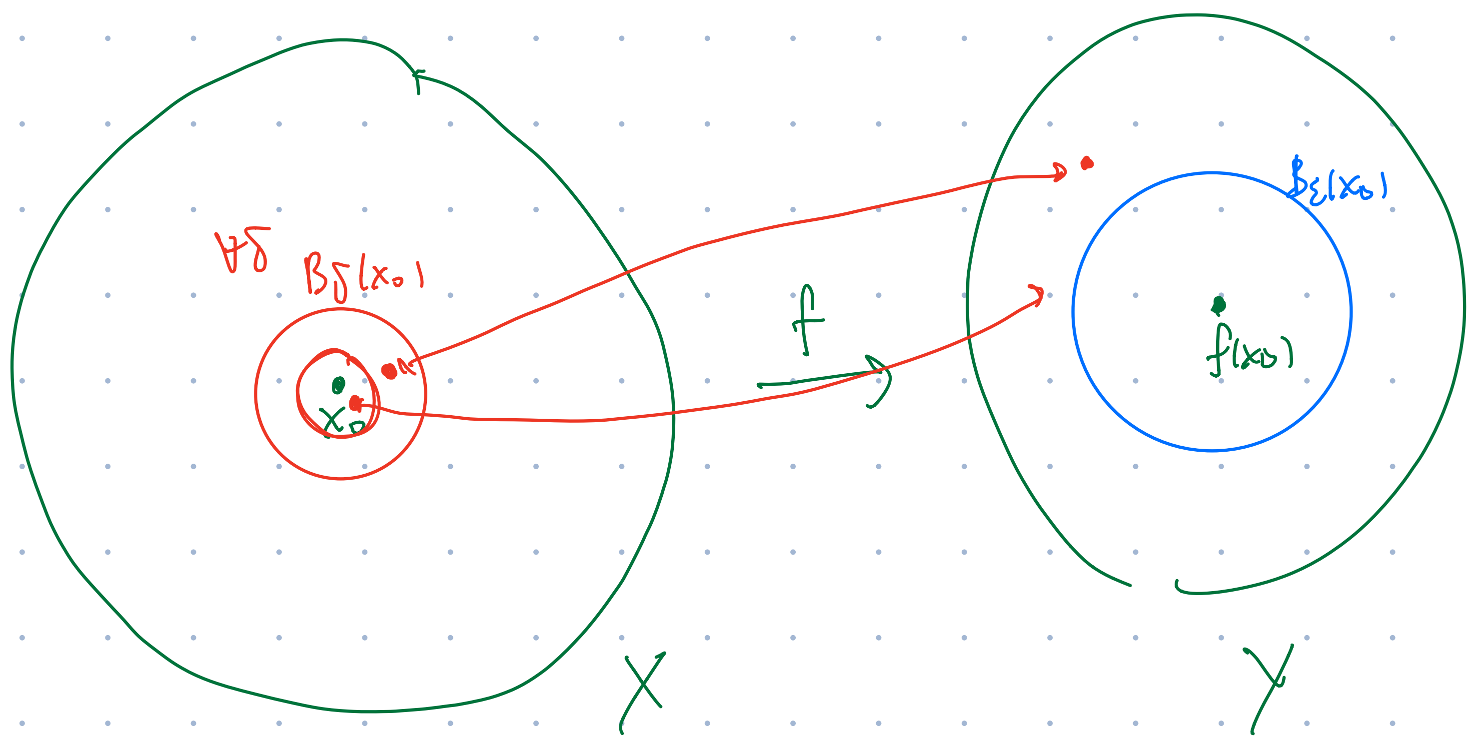
$(I) \Rightarrow (II)$:

Prove by contradiction:

Suppose (II) is not true, i.e. $\exists \varepsilon > 0$

s.t. $\forall \delta > 0$, $\exists x$ s.t. $d(x, x_0) < \delta$,

$d(f(x), f(x_0)) \geq \varepsilon$.



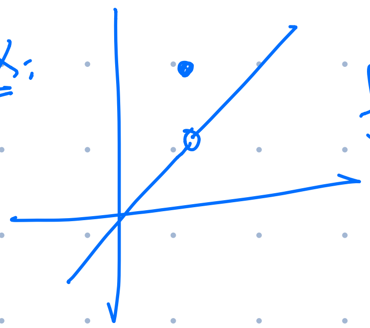
For each $n \in \mathbb{N}$, consider $\delta = 1/n > 0$.

$\exists\ x_n \in X$ s.t. $d(x_n, x_0) < 1/n$, $d(f(x_n), f(x_0)) \geq \varepsilon$

Then $\lim x_n = x_0$, But $\lim f(x_n) \neq f(x_0)$

Contradicts with (I). $\square$

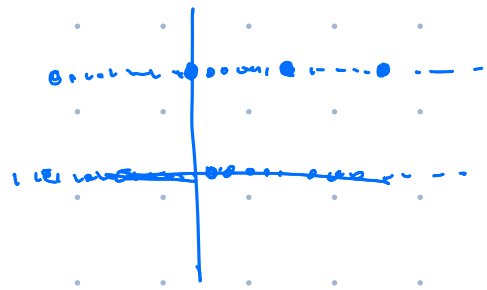
Ex:



$f(x) = \begin{cases} x & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$

Ex: $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

Determine whether f has any conti. pt



Thm: "$f: X \to Y$ is conti." $\iff$

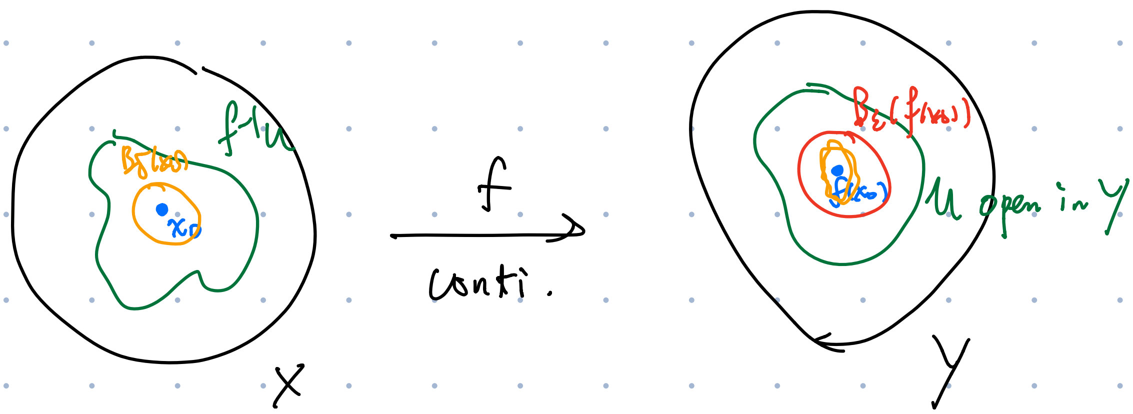
"$\forall\ \mathcal{U} \underset{\text{open}}{\subseteq} Y$, we have: $\boxed{f^{-1}(\mathcal{U})} \subseteq X$ is open"

$\shortparallel$

$\{x \in X \mid f(x) \in \mathcal{U}\}$.

(f may not have an "inverse fcn")

pf ($\Longrightarrow$)



- Let $U \subseteq Y$ open. Want to show: $f^{-1}(U) \subseteq X$ open.

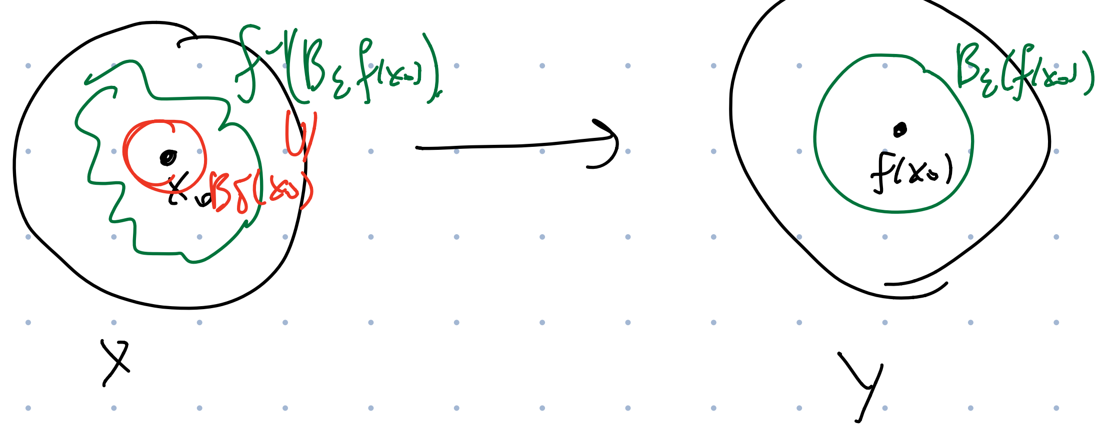
  i.e. $\boxed{\forall x_0 \in f^{-1}(U),\ \exists \delta > 0 \text{ s.t. } B_\delta(x_0) \subseteq f^{-1}(U).}$

- Let $x_0 \in f^{-1}(U)$, then $f(x_0) \in U$.
- Since U is open, $\exists \varepsilon > 0$, s.t. $B_\varepsilon(f(x_0)) \subseteq U$.
- Since f is conti, $\exists \delta > 0$ s.t. $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$

$$\Rightarrow f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0)) \subseteq U$$

$$\Rightarrow B_\delta(x_0) \subseteq f^{-1}(U). \quad \square$$

($\Leftarrow$)



- $\forall \varepsilon > 0$, consider $B_\varepsilon(f(x_0)) \underset{\text{open}}{\subseteq} Y$
- So, $f^{-1}(B_\varepsilon(f(x_0))) \underset{\text{open}}{\subseteq} X$
$$\underset{x_0}{\cup\!\!\!\!\!\!\;\mid}$$
- $\exists \delta > 0$ s.t. $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$.

$\Rightarrow \quad f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$. $\square$

Rmk: If $f: X \to Y$ conti, $\underset{\text{open}}{U \subseteq X} \not\Rightarrow \underset{\text{open}}{f(U) \subseteq Y}$

$f(x) = |x|$, $\quad \underset{\text{open}}{(-1,1) \subseteq \mathbb{R}}$, $\quad \underset{\text{not open in } \mathbb{R}.}{f(-1,1) = [0,1) \subseteq \mathbb{R}}$

Rmk: The definition of continuity using "the preimage of open subset is open" works in a more general setting of "topological space"

Thm $f: X \to Y$ conti.,

If $E \subseteq X$ compact, then $f(E) \subseteq Y$ is compact.

Q: $f: A \to B$, $D \subseteq A$, $C \subseteq B$

$$f(f^{-1}(C)) \subseteq C$$

$$f^{-1}(f(D)) \supseteq D$$

pf:

- Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(E)$
- Since f is conti, $\underline{f^{-1} U_\alpha} \subseteq X$ open.

$\{f^{-1} U_\alpha\}_{\alpha \in I}$ is an open cover of E.

$$\left(\begin{array}{c} f(E) \subseteq \cup U_\alpha \\ E \subseteq f^{-1} f(E) \subseteq \cup f^{-1} U_\alpha \end{array} \right)$$

- Since E cpt, $\exists$ finite subcover, i.e.
  $\exists\ \alpha_1, \dots, \alpha_n$ s.t.
  $$E \subseteq (f^{-1} U_{\alpha_1} \cup \dots \cup f^{-1} U_{\alpha_n})$$

- $$f(E) \subseteq f(f^{-1} U_{\alpha_1}) \cup \dots \cup f(f^{-1}(U_{\alpha_n}))$$
  $$\subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}. \quad \square$$

Specialize the thm. to the case $Y = \mathbb{R}$:

Corollary: (extreme value thm for compact sets):

$f: X \longrightarrow \mathbb{R}$ conti. $E \subseteq X$ cpt. Then

1) f is bounded on E. (i.e. $\exists M > 0$ s.t. $|f(x)| < M$ $\forall x \in E$)

2) f attains its maximum & minimum on E.
(i.e. $\exists\ x_1, x_2 \in E$ s.t. $f(x_1) \leq f(x) \leq f(x_2)$ $\forall x \in E$)

pf: By thm, $f(E) \subseteq \mathbb{R}$ compact. $\overset{\text{Heine-Borel}}{\iff}$ closed & bdd $\Downarrow$ 1)

HW $f(E) \subseteq \mathbb{R}$ closed & bdd

$\Rightarrow$ $\begin{matrix} \sup f(E) \\ \inf f(E) \end{matrix} \in f(E)$

i.e. $\exists x_1, x_2 \in E$

s.t. $\sup f(E) = f(x_2)$
$\inf f(E) = f(x_1)$

$\rightarrow f(x_1) \le f(x) \le f(x_2) \quad \forall x \in E.$ $\square$

Corollary (§18.1) $f : [a,b] \longrightarrow \mathbb{R}$ conti. Then

1) f is bounded.

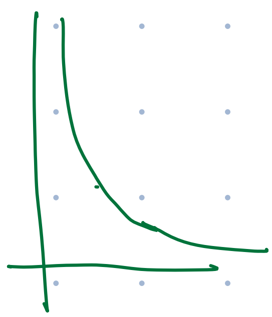
2) f attains its max. & min. on $[a,b]$

pf: $[a,b] \subseteq \mathbb{R}$ is compact. (Heine-Borel). $\square$

e.g. $f(x) = \frac{1}{x}$

$(0,1)$

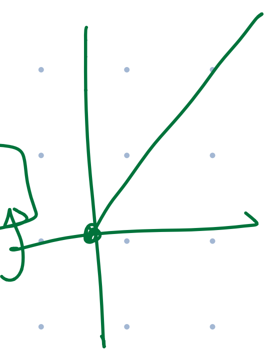
not bdd



e.g. $f(x) = x$

$[0, \infty)$

not bdd



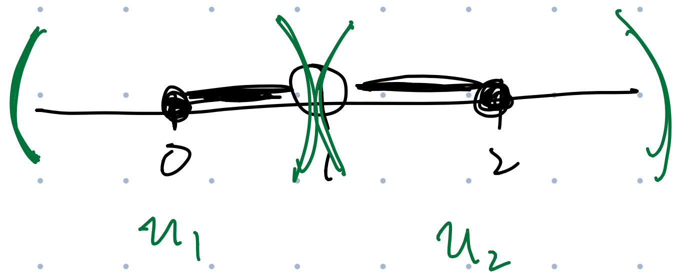
Ex: The proof of (§18.1, Ross) is more complicated than ours.
What's the hidden difficulties in our proof ??

Def $E \subseteq (X,d)$ is <u>disconnected</u> if $\exists\ \mathcal{U}_1, \mathcal{U}_2 \underset{\text{open}}{\subseteq} X$,
that "separates E", i.e.

1) $E \cap \mathcal{U}_1 \neq \phi$, $E \cap \mathcal{U}_2 \neq \phi$
2) $E \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$
3) $E \cap \mathcal{U}_1 \cap \mathcal{U}_2 = \phi$.

Otherwise, $E \subseteq X$ is called <u>connected</u>

e.g. $E = [0,1) \cup (1,2]$ is disconnected



$\mathcal{U}_1 = (-1,1)$, $\mathcal{U}_2 = (1,3)$

e.g. $E \subseteq \mathbb{R}$ is connected $\Longleftrightarrow$ E is an interval
(see §22 Ross)

Thm $f: X \longrightarrow Y$ conti.

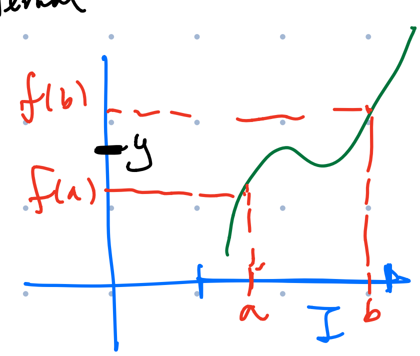
If $E \subseteq X$ connected, then $f(E) \subseteq Y$ is connected.

Coro: (<u>Intermediate value thm</u>). $f: \underset{\underset{\mathbb{R}}{\cap|\ \text{interval}}}{I} \longrightarrow \mathbb{R}$ conti.

Then $\forall a, b \in I$, $a < b$,

$\forall$ y between $f(a)$ & $f(b)$,

$\exists\ x \in [a,b]$ s.t. $f(x) = y$



pf: $[a,b] \subseteq I \xRightarrow{\text{Thm}} f([a,b]) \subseteq \mathbb{R}$ connected

connected.

$\Downarrow$

$f([a,b])$ is an interval.

$f(a), f(b) \in \underbrace{f([a,b])}_{\text{Interval in } \mathbb{R}}$

y

$\Rightarrow y \in f([a,b])$

$\Rightarrow \exists\, x \in [a,b]$ s.t. $f(x) = y$. $\square$