HOMEWORK 4
MATH 104, SECTION 6

Office Hours: Tuesday and Wednesday 9:30-11am at 735 Evans. I'll be holding extra office hours on February 12th (Wednesday) for the coming midterm. More info will be updated on the course website.

Reading

There will be reading assigned for each lecture. You should come to the class having read the assigned sections of the textbook.

Due February 13: Ross, Section 12

Due February 18: Ross, Section 13

Problem set (9 problems; due February 13)

Submit your homework at the beginning of the lecture on Thursday. *Late homework will not be accepted under any circumstances.*

You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any.

Write clearly! Mastering mathematical writing is one of the goals of this course.

You have to staple your work if it is more than one page.

(1) Let $a_1 = 3$ and $a_{n+1} = \sqrt{3a_n + 10}$ for $n \geq 1$. Prove that (a_n) converges, and find the limit. (Hint: Try to show that it is monotone and bounded first.)

(2) Let (a_n) be a bounded sequence. Prove that

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n).$$

(3) Prove that $\limsup |a_n| = 0$ if and only if $\lim a_n = 0$.

(4) Let (a_n) and (b_n) be bounded sequences.

(a) Prove that $(a_n + b_n)$ is bounded.

(b) Prove that

$$(\liminf_{n\to\infty} a_n)+(\liminf_{n\to\infty} b_n) \leq \liminf_{n\to\infty}(a_n+b_n) \text{ and } (\limsup_{n\to\infty} a_n)+(\limsup_{n\to\infty} b_n) \geq \limsup_{n\to\infty}(a_n+b_n)$$

(c) Find an example of (a_n) and (b_n) such that

$$(\liminf_{n\to\infty} a_n) + (\liminf_{n\to\infty} b_n) < \liminf_{n\to\infty}(a_n + b_n).$$

(5) (a) Let (a_n) be a sequence such that $|a_{n+1}-a_n| < C^n$ for all n for some constant $0 < C < 1$. Prove that (a_n) is a Cauchy sequence, therefore is convergent.

(b) Let (a_n) be a sequence such that $|a_{n+1} - a_n| < \frac{1}{n}$ for all n. Is it true that such (a_n) is always convergent?

(6) Let (a_n) be a sequence of nonzero real numbers. Assume that $\limsup \left|\frac{a_{n+1}}{a_n}\right| = L$ is finite. You'll prove $\limsup(|a_n|^{1/n}) \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$ in this problem. Using similar argument, you can show that

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf_{n\to\infty}(|a_n|^{1/n}) \leq \limsup_{n\to\infty}(|a_n|^{1/n}) \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|,$$

which will be important for us later on in the course.

(a) Let L' be any number bigger than L. Prove that there exists $N > 0$ such that $\left|\frac{a_{n+1}}{a_n}\right| < L'$ for any $n > N$.

(b) Prove that for any $n > N$, we have $|a_n| < (L')^{n-N}|a_N|$. Define $B := (L')^{-N}|a_N|$. Then we have $|a_n| < B(L')^n$ for any $n > N$.

(c) Prove that $\limsup(|a_n|^{1/n}) \leq L'$.

(d) Finally, prove that $\limsup(|a_n|^{1/n}) \leq L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

Also, think about why do we need to choose such $L' > L$ in step (a).

(7) Consider the metric space $\mathbb{R}$ with the usual distance function $d(x,y) = |x-y|$. Is the subset $\mathbb{Q} \subset \mathbb{R}$ open? Is it closed?

(8) (de Morgan's Laws) Let $\{S_\alpha\}$ be a collection of (possibly infinitely many) subsets of a set S. Prove that

(a) The complement of union is the intersection of complements: $(\cup_\alpha S_\alpha)^c = \cap_\alpha(S_\alpha^c)$.

(b) The complement of intersection is the union of complements: $(\cap_\alpha S_\alpha)^c = \cup_\alpha(S_\alpha^c)$.

(9) (a) For any metric space (S,d), show that the union of any infinitely many open subsets is open.

(b) For any metric space (S,d), show that the intersection of any finitely many open subsets is open.

(c) For any metric space (S,d), show that the intersection of any infinitely many closed subsets is closed. (Recall that a subset is *closed* if its complement is open.)

(d) For any metric space (S,d), show that the union of any finitely many closed subsets is closed.

(e) Find a counterexample of (a) if 'open' is replaced by 'closed'; find a counterexample of (c) if 'closed' is replaced by 'open'.