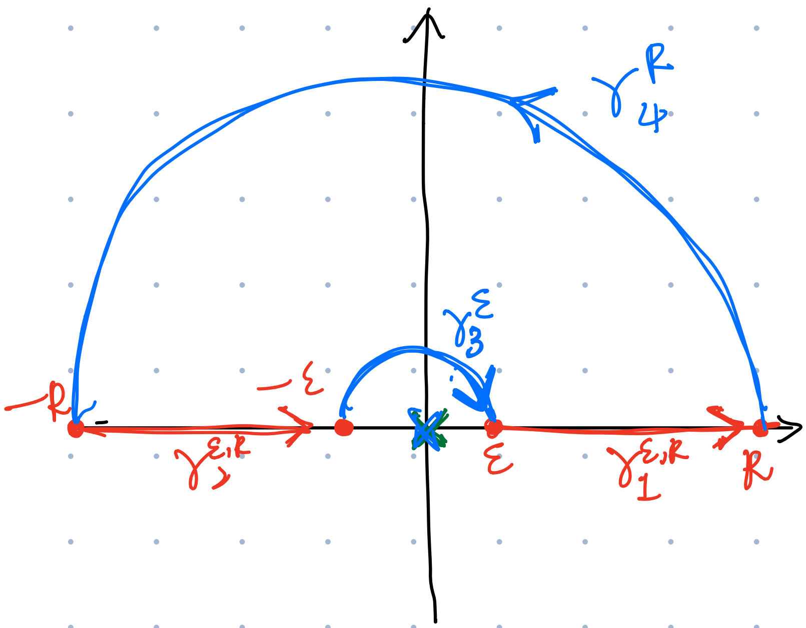


$$\boxed{f(z) = \frac{1-e^{iz}}{2z^2}}$$

holo. fun on $\mathbb{C}\setminus\{0\}$

Last time:

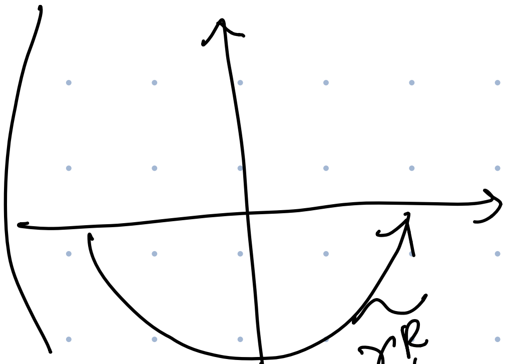
$$\lim_{R\to\infty}\left(\int_{\gamma_4^R} f(z)\,dz\right) = 0.$$

$$\lim_{\varepsilon\to 0}\left(\int_{\gamma_3^\varepsilon} f(z)\,dz\right) = \frac{-\pi}{2}.$$

By Cauchy's thm, we have:

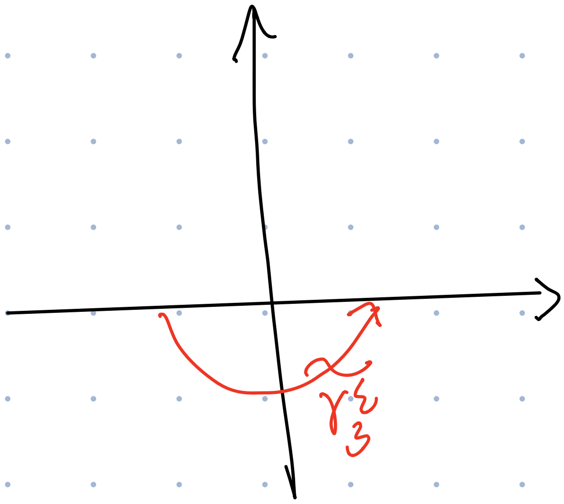
$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\left(\int_{\gamma_1^{\varepsilon,R}} f + \int_{\gamma_2^{\varepsilon,R}} f\right)$$

$$= \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\left(-\int_{\gamma_3^\varepsilon} f - \int_{\gamma_4^R} f\right) = \frac{\pi}{2}.$$



We saw last time that the estimate doesn't work for $\lim_{R\to\infty}\int_{\tilde{\gamma}_4^R} f$.

$\pi = \lim_{R\to\infty}\int_{\tilde{\gamma}_4^R} f \neq 0$



$$f(z) = \frac{1-e^{iz}}{2z^2}$$

$$\tilde{\gamma}_3^\varepsilon : [0,\pi] \longrightarrow \mathbb{C}$$

$$\theta \longmapsto \underline{-\varepsilon e^{i\theta}}$$

$$\int_{\tilde{\gamma}_3^\varepsilon} f(z)\,dz = \int_0^\pi \frac{1-e^{i(-\varepsilon e^{i\theta})}}{2(-\varepsilon e^{i\theta})^{\cancel{2}}} \left(\cancel{+}\, i \cancel{\varepsilon} \cancel{e^{i\theta}}\right) d\theta$$

$$= \frac{-i}{2\varepsilon} \int_0^\pi \frac{1-e^{-i\varepsilon e^{i\theta}}}{e^{i\theta}}\, d\theta$$

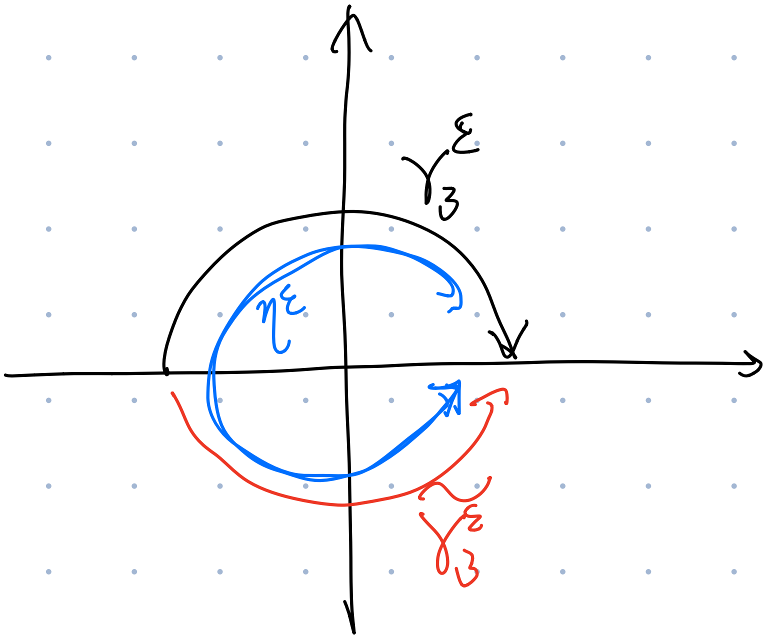
$$\left(e^{-i\varepsilon e^{i\theta}} = 1 + (-i\varepsilon e^{i\theta}) + O(\varepsilon^2) \right)$$

$$= \frac{-i}{2\varepsilon} \int_0^\pi \frac{i\varepsilon e^{i\theta} + O(\varepsilon^2)}{e^{i\theta}}\, d\theta$$

$$= \frac{-i}{2\varepsilon}(\varepsilon i \pi) + O(\varepsilon)$$

$$= \frac{\pi}{2} + O(\varepsilon)$$

$$\lim_{\varepsilon \to 0} \int_{\tilde{\gamma}_3^\varepsilon} f = \frac{\pi}{2}$$



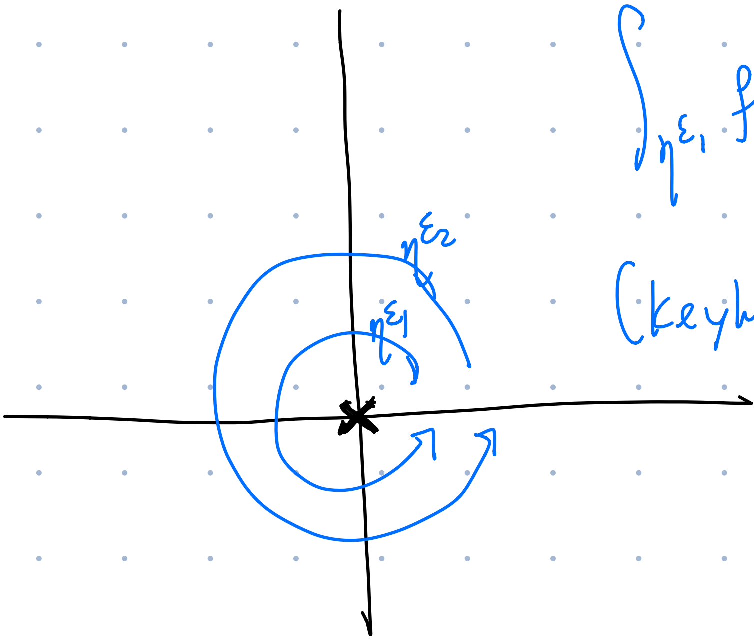
$$\lim_{\varepsilon\to 0}\int_{\gamma_3^\varepsilon} f = \frac{-\pi}{2}$$

$$\lim_{\varepsilon\to 0}\int_{\tilde{\gamma}_3^\varepsilon} f = \frac{\pi}{2}$$

$$\int_{\eta^\varepsilon} f(z)\,dz = -\int_{\gamma_3^\varepsilon} f(z)\,dz + \int_{\tilde{\gamma}_3^\varepsilon} f(z)\,dz$$

$$\lim_{\varepsilon\to 0}\int_{\eta^\varepsilon} f(z)\,dz = \pi \implies \int_{\eta^\varepsilon} f(z)\,dz = \pi \quad \forall \varepsilon>0$$

oriented positively (counterclockwise for this example)



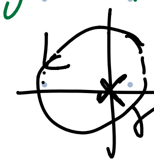
$$\int_{\eta^{\varepsilon_1}} f = \int_{\eta^{\varepsilon_2}} f$$

(keyhole argument)

Rmk: The integral of f along any simple closed curve which contains 0 in its interior is the same.

i.e. $\int_{\eta_\varepsilon} f$ depends only on the "local behavior" of the fn f at the "singular point" 0.

$2\pi i$ "$\boxed{\operatorname{Res}_{z=0} f}$"

e.g.  $f(z)=\frac{1}{z}$ $\quad \int_\gamma f = 2\pi i$

roughly, it's the "coeff. of "$\frac{1}{z}$"" in the power series exp. of f at $z=0$

In this e.g. $$f = \frac{1-e^{iz}}{2z^2} = \frac{1-\left(1+iz+\frac{1}{2}(iz)^2+\cdots\right)}{2z^2}$$

$$\operatorname{Res}_{z=0} f = \frac{-i}{2}$$

$$2\pi i \operatorname{Res} f = 2\pi i\cdot\frac{-i}{2} = \pi$$

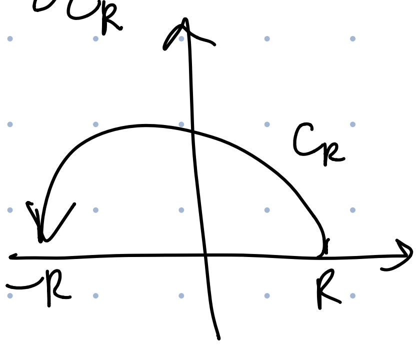
Lemma (Jordan) $Q:\mathbb{H}\longrightarrow\mathbb{C}$ s.t. $|Q(z)|\to 0$ uniformly as $|z|\to\infty$

$\mathbb{H} = \{(x,y)\in\mathbb{C} \mid y>0\}$

(i.e. $\forall \varepsilon>0$, $\exists M>0$ s.t. $|Q(z)|<\varepsilon$ $\forall |z|>M$ and $z\in\mathbb{H}$)

Then $$\lim_{R\to\infty}\int_{C_R} Q(z)\cdot e^{ipz}\,dz = 0 \quad \forall p>0$$

where



pf: $C_R:[0,\pi]\longrightarrow\mathbb{C}$

$\theta\longmapsto Re^{i\theta}$

$$\left|\int_{C_R} Q(z)\, e^{ipz}\,dz\right| = \left|\int_0^{\pi} Q(Re^{i\theta})\, e^{ipRe^{i\theta}}\,(iRe^{i\theta})\,d\theta\right|$$

$$\le \int_0^{\pi} |Q(Re^{i\theta})| \cdot |e^{ipRe^{i\theta}}|\, R\,d\theta$$

$\Big(\forall \boxed{\varepsilon>0}$, $\exists \boxed{M>0}$ s.t. $|Q(z)|<\varepsilon$ $\forall\, |z|>M$ and $z\in H$

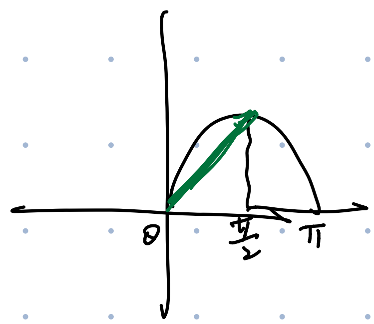
$\forall\, R>M \Big)$

$$\le \varepsilon R \int_0^{\pi} |e^{ipRe^{i\theta}}|\,d\theta$$

$$|e^{ipRe^{i\theta}}| = |e^{ipR(\cos\theta + i\sin\theta)}| = |e^{-pR\sin\theta + ipR\cos\theta}|$$

$$= e^{-pR\sin\theta}$$

$$= \varepsilon R \int_0^{\pi} e^{-pR\sin\theta}\,d\theta$$



$$= 2\varepsilon R \int_0^{\pi/2} e^{-pR\sin\theta}\,d\theta$$

Since $\sin\theta$ concave on $[0,\pi]$,

$\Rightarrow \sin\theta \ge \frac{2}{\pi}\theta$ $\forall\, \theta\in[0,\frac{\pi}{2}]$

$$\le 2\varepsilon R \int_0^{\pi/2} e^{-pR\cdot\frac{2}{\pi}\theta}\,d\theta$$

$$< 2\varepsilon R \int_0^{\infty} e^{-pR\frac{2}{\pi}\theta}\,d\theta = 2\varepsilon R\left(\frac{\pi/2}{pR}\right) = \frac{\varepsilon\pi}{p}$$

$\forall \varepsilon > 0$, $\exists M > 0$ s.t.

for any $R > M$, we have

$$\left| \int_{C_R} Q(z) e^{ipz} dz \right| < \frac{\varepsilon \pi}{p}$$

$$\Rightarrow \quad \limsup_{R \to \infty} \left| \int_{C_R} Q(z) e^{ipz} dz \right| \leq \frac{\varepsilon \pi}{p} \quad \forall \varepsilon > 0$$

$$\Rightarrow \quad \lim_{R \to \infty} \int_{C_R} Q(z) e^{ipz} dz = 0. \quad \square$$

$$e^{-\pi(x^2 + 2ix\xi)} = e^{-\pi(x+i\xi)^2} \cdot e^{\pi(i\xi)^2} = e^{-\pi(x+i\xi)^2} \cdot e^{-\pi\xi^2}$$

e.g.: $\xi \in \mathbb{R}$, Prove:

$$\int_{\mathbb{R}} \boxed{e^{-\pi x^2} e^{-2\pi i x \cdot \xi}} dx = e^{-\pi \xi^2} \quad \Leftrightarrow \quad \boxed{\int_{\mathbb{R}} e^{-\pi(x+i\xi)^2} dx = 1.}$$

(Rmk: the Fourier transf. of $e^{-\pi x^2}$ is itself)

e.g.: $\xi = 0$: $\boxed{\int_{\mathbb{R}} e^{-\pi x^2} dx = 1}$ (Gaussian dist. in prob. th.)

$$\left(\int_{\mathbb{R}} e^{-\pi x^2} dx\right)^2 = \left(\int_{\mathbb{R}} e^{-\pi x^2} dx\right) \cdot \left(\int_{\mathbb{R}} e^{-\pi y^2} dy\right)$$

$$= \iint_{\mathbb{R}^2} e^{-\pi(x^2+y^2)} dx dy$$

$(x, y) \to (r, \theta)$

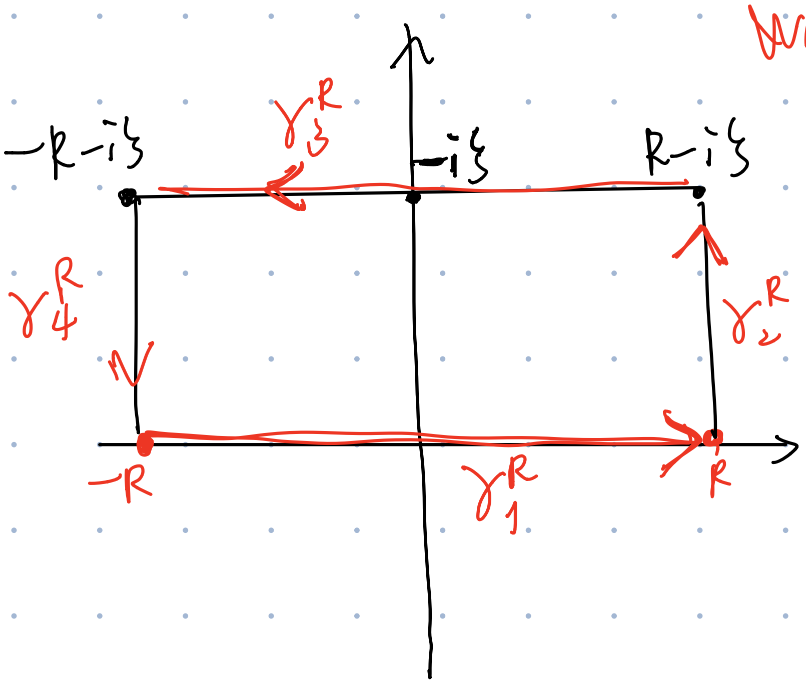
$$= \int_0^\infty \int_0^{2\pi} e^{-\pi r^2} r \, dr \, d\theta$$

$\boxed{s = r^2}$

$$= 2\pi \int_0^\infty r \cdot e^{-\pi r^2} dr$$

$$= 2\pi \cdot \frac{1}{2} \int_0^\infty e^{-\pi s}\, ds$$

$$= \pi \int_0^\infty e^{-\pi s}\, ds = 1. \quad \square$$



Want to prove:

$$\boxed{\int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\, dx = 1.} \qquad \boxed{\lim_{R\to\infty} \int_{\gamma_1^R} f(z)\, dz}$$

$$\| \qquad\qquad //$$

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-\pi(x+i\xi)^2}\, dx$$

$$\boxed{f(z) = e^{-\pi(z+i\xi)^2}}$$

holo. fcn in $\mathbb{C}$

$$\gamma_3^R : [-R, R] \longrightarrow \mathbb{C}$$

$$t \longmapsto -t - i\xi.$$

$$\int_{\gamma_3^R} f = \int_{-R}^{R} e^{-\pi t^2} (-1)\, dt = -\int_{-R}^{R} e^{-\pi t^2}\, dt$$

By what we computed earlier, we have:

$$\lim_{R\to\infty} \int_{\gamma_3^R} f(z)\, dz = -1$$

We can show that $\lim_{R\to\infty} \int_{\gamma_2^R} f = 0$, $\lim_{R\to\infty} \int_{\gamma_4^R} f = 0$.

Then by Cauchy's thm, we have: $\lim_{R\to\infty} \int_{\gamma_1^R} f = 1.$ $\square$

$$\gamma_2^R : [0,1] \to \mathbb{C}$$
$$t \mapsto R - it\xi.$$

$$\left|\int_{\gamma_2^R} f\right| = \left|\int_0^1 e^{-\pi(R-it\xi+i\xi)^2} (-i\xi)\, dt\right|$$

$$\leq \boxed{\int_0^1 \underbrace{\left|e^{-\pi(R-it\xi+i\xi)^2}\right|}\cdot |\underline{\xi}|\, dt}$$

$$\left|e^{-\pi(R-it\xi+i\xi)^2}\right| = \left|e^{-\pi\left(\boxed{R^2}+2R(-it\xi+i\xi)+\boxed{(-it\xi+i\xi)^2}\right)}\right|$$

$$= e^{-\pi(R^2-(t-1)^2\xi^2)}$$

$$\leq e^{-\pi(R^2-\xi^2)}$$

$$\leq \int_0^1 e^{-\pi(R^2-\xi^2)} |\xi|\, dt$$

$$= e^{-\pi(R^2-\xi^2)} |\xi| \to 0$$

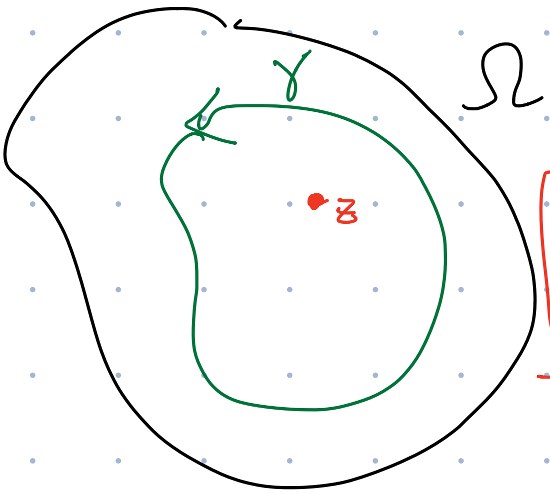
as $R \to \infty$

Thm (Cauchy integral formula).

γ: simple closed curve,

$\Omega \subseteq \mathbb{C}$ open, which contains γ & its interior

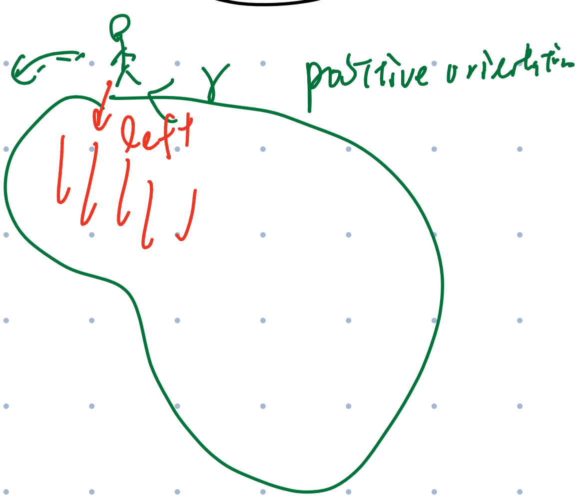
$f: \Omega \to \mathbb{C}$ holo.



$\forall z$ in the interior of γ, we have:

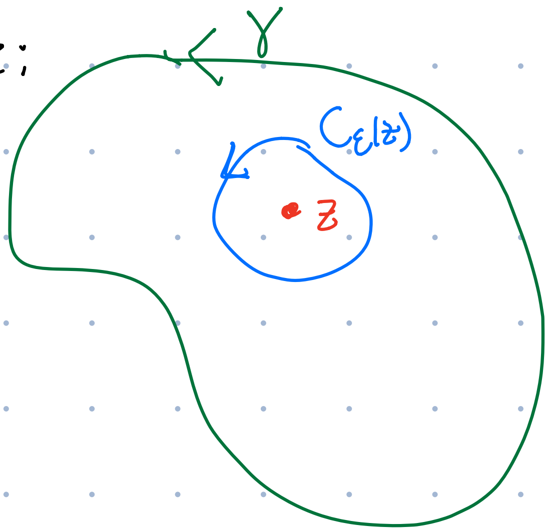
$$\boxed{f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w-z}\, dw.}$$

holo. fcn. on $\Omega \setminus \{z\}$ in w



Rmk: The behavior of a holo. fcn. on the boundary determines its behavior in the interior.

pf:



Claim:

① $$\int_{C_\varepsilon(z)} \frac{f(w)-f(z)}{w-z}\, dw = 0$$

② $$\int_{C_\varepsilon(z)} \frac{1}{w-z}\, dw = 2\pi i$$

By keyhole argument, $\displaystyle\int_\gamma \frac{f(w)}{w-z}\, dw = \int_{C_\varepsilon(z)} \frac{f(w)}{w-z}\, dw$

$$= \underbrace{\int_{C_\varepsilon(z)} \frac{f(w)-f(z)}{w-z}\,dw}_{① \; = \; 0} + \underbrace{\int_{C_\varepsilon(z)} \frac{f(z)}{w-z}\,dw}_{② \; = \; f(z)\cdot 2\pi i}$$

② : $\displaystyle\int_{C_\varepsilon(z)} \frac{1}{w-z}\,dw = \int_{C_\varepsilon(0)} \frac{1}{w}\,dw = 2\pi i$

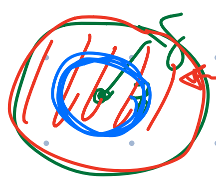
① : " $\displaystyle\int_{C_\varepsilon(z)} \frac{f(w)-f(z)}{w-z}\,dw = 0$ "

Claim: $\dfrac{f(w)-f(z)}{w-z}$ is a bounded fcn in w near z.

(since f is holo. at z,

$\forall \varepsilon>0,\ \exists \delta>0$ s.t.

$$0<|w-z|<\delta \Rightarrow \left|\frac{f(w)-f(z)}{w-z} - f'(z)\right| < \varepsilon$$



$$\Rightarrow \left|\frac{f(w)-f(z)}{w-z}\right| < \underbrace{|f'(z)|+\varepsilon}_{M} \quad)$$

$0<\boxed{\eta<\delta}$

$$\left|\int_{C_\eta(z)} \frac{f(w)-f(z)}{w-z}\,dw\right| \le \sup_{w\in C_\eta(z)} \left(\frac{f(w)-f(z)}{w-z}\right) \cdot \text{length}(C_\eta(z))$$

indep. of $0<\eta<\delta$ $\quad < M \cdot 2\eta\pi$

$$\Rightarrow \int_{C_\eta(z)} \frac{f(w)-f(z)}{w-z}\,dw = 0 \qquad \square$$