Derived equivalent K3 surfaces & categories

Outline:
1) Review: Cubic 4-folds and K3 surfaces
2) Fourier-Mukai partners & Related Questions
3) Known Results.

- $X \subseteq \mathbb{CP}^5$ cubic 4-fold.
- $\mathcal{C}$ = moduli of smooth cubic 4-fold.

has $\dim = \dim \mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}(3)) - \dim PGL(6,\mathbb{C}) = 20.$

- Open question: Which cubic 4-folds are rational?

$$\begin{array}{ccccclcl} 0 & 1 & 21 & 1 & 0 & H^4(X) & \cong & (+1)^{\oplus 21} \oplus (-1)^{\oplus 2} \\ 0 & 1 & 20 & 1 & 0 & H^4_{prim}(X) = \langle H^2 \rangle^{\perp} & \cong & \underset{(20,2)}{A_2 \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}} \\ & 1 & 20 & 1 & & H^2(K3) & \cong & \underset{(3,19)}{U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}} \end{array}$$

<u>Def</u> (Hassett) A cubic 4-fold is <u>special</u> if $\exists\, T \in H^{2,2}(X, \mathbb{Z})$ not homologous to multiples of H^2.

<u>Def</u>: $\mathcal{C}_d = \{ \text{special cubic 4-fold w/ } T \text{ st. } \operatorname{disc}(H^2, T) = d \} \subseteq \mathcal{C}$

$$\operatorname{disc}(H^2,T) = \det \begin{pmatrix} H^2.H^2 & H^2.T \\ H^2.T & T^2 \end{pmatrix}$$

<u>Thm</u> (Hassett) $\mathcal{C}_d$ nonempty $\iff$ $d \geq 8$ and $d \equiv 0, 2 \ (6)$

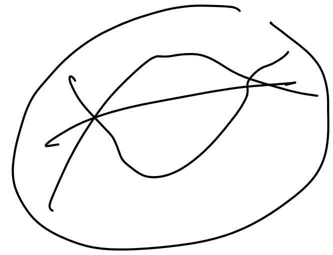
$\uparrow$ irreducible divisor of $\mathcal{C}$.

Furthermore,

$$\text{"}\ \underset{(19,2)}{\langle H^2, T\rangle^{\perp_{H^4(X,\mathbb{Z})}}} \cong \underset{(1,19)}{H^2_{prim}(K3,\mathbb{Z})[-1]}\ \text{"}$$

$\Longleftrightarrow$ d is not divisible by 4, 9, or any odd prime $\equiv 2\ (3)$ ↗ $(*)$

(e.g. $d = 14, 26, 38, 42, 62, \cdots$)



Conj: X is rational $\Longleftrightarrow$ $X \in \bigcup_{d \in (*)} \mathcal{C}_d$

The conj. holds for all known examples: $\mathcal{C}_{14}, \mathcal{C}_{26}, \mathcal{C}_{38}, \mathcal{C}_{42}$.

Katzarkov-Kontsevich claimed a proof via quantum cohomology a few years ago.
but the paper hasn't appeared yet.
(many talks available online).

$D^b(X) = \langle \mathcal{A}_X, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$ semiorthogonal decomposition,

where: $\mathcal{A}_X = \{ E \in D^b(X) \mid \mathrm{Hom}^\bullet(\mathcal{O}, E) = \mathrm{Hom}^\bullet(\mathcal{O}(1), E) = \mathrm{Hom}^\bullet(\mathcal{O}(2), E) = 0 \}$

Kuznetsov: $\mathcal{A}_X$ is a K3 cat, i.e. $S_{\mathcal{A}_X} \cong [2]$.

Conj (Kuznetsov). X is rat'l $\Longleftrightarrow$ $\mathcal{A}_X \cong D^b(S)$ for some K3 sf. S.

- Addington-Thomas, Bayer-Lahoz-Macri-Nuer-Perry-Stellari
  These two conjectures are equivalent.

(K3 cat. encodes rationality info. of the cubic)

Def: • X: K3 surface, Say a K3 surface Y is a <u>Fourier-Mukai partner</u> of X if $D^b(X) \cong D^b(Y)$

• X: cubic 4-fold, Say a cubic 4-fold Y is a <u>FM partner</u> if $\mathcal{A}_X \cong \mathcal{A}_Y$.

<u>Reasons to study FM partner:</u>

• <u>Conj</u> (Huybrechts) X, Y: cubic 4-folds   general $\to$ $X \cong Y$

$$\mathcal{A}_X \cong \mathcal{A}_Y \Longrightarrow X \overset{\sim}{\dashrightarrow} Y \text{ birational}$$

<u>Evidence</u>: <u>Thm</u> (F.-Lai) If $X \in \mathcal{C}_{20}$ general, then the conj. holds.

<u>Sketch</u>: $X \in \mathcal{C}_{20} \Leftrightarrow X \subseteq \mathbb{CP}^5$ containing a Veronese surface:

defined by 6 quadratic eq's $Q_0, \dots, Q_5$ $\leftarrow$ $V = \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$

$$[x,y,z] \longmapsto [x^2, xy, y^2, yz, z^2, xz]$$

$$\begin{tikzcd}
V \subseteq X \subseteq \mathbb{P}^5 \arrow[rr, dashed, "{(Q_0, \dots, Q_5)}", "F"'] & & (\mathbb{P}^5)' \supseteq F(X) \supseteq V' \\
& \mathbb{P} \cong Bl_V \mathbb{P}^5 \cong Bl_{V'}(\mathbb{P}^5)' \arrow[ul, "g"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

$F(X) = X'$

$\mathbb{P} \supseteq Y = Bl_V X$

• $X \not\cong X'$

• $\mathcal{A}_X \cong \mathcal{A}_{X'}$

• $|FM(X)| = 2$.

<u>In Pic$(\mathbb{P})$:</u>

$L := g^* \mathcal{O}_{\mathbb{P}^5}(1)$

$M := g'^* \mathcal{O}_{\mathbb{P}^{5'}}(1)$

$E := \text{excep}(g)$

$F := \text{excep}(g')$

f defined by quad. poly $\Rightarrow$ $M = 2L - E$, $L = 2M - F$

X cubic $\Rightarrow$ $Y = 3L - E$

$$3L - E = L + M = 3M - F \Rightarrow F(X) \text{ cubic}$$

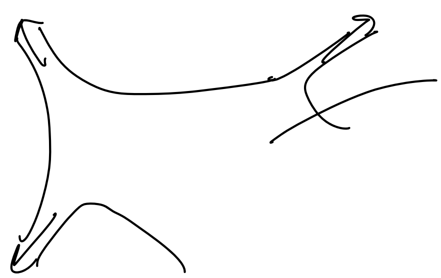
- In terms of HMS, $\underset{\text{(CY)}}{X}$ - K3 sf. $\longleftrightarrow$ $\underset{\text{(CY)}}{\check{X}}$ mirror K3

$$D^b(X) \cong \mathrm{Fuk}(\check{X})$$

On the complex-geometric side, derived equivalent varieties are undistinguishable. $D^b(X) \cong D^b(Y)$.

- One way to study all FM partners together is by considering the space of Bridgeland stab. cond^n. on $D^b(X) : (\cong D^b(Y))$

Hartmann, Ma:



$\mathrm{Aut}(D) \backslash \mathrm{Stab}^*(D) / \mathbb{C}$

$\cong$

$M_{cpx}(\check{X})$

each FM partner corresp. to a <u>cusp</u> (Large volume limit)

e.g. $\rho(X) = 1$, $H^2 = 2n$,

$$\mathrm{Aut}(D) \backslash \mathrm{Stab}^*(D) / \mathbb{C} \cong \mathbb{H} / \Gamma_0^+(n)$$

<u>defin</u> $\Gamma_0(n), \left\langle \begin{pmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix} \right\rangle$

cusp $\longleftrightarrow$ FM partner.

ell. pts $\longleftrightarrow$ finite order elt of $\mathrm{Aut}(D)/[1]$

Fr-Lai.

$|FM(K3)|$: • Oguiso: $\rho = 1$

• Hosono-Lian-Oguiso-Yau: all ρ. (not explicit)

$|FM(\text{general cubic } 4)| = 1$: Huybrechts.

$|FM(\text{special cubic } 4)|$: Fu-Lai ($\Leftrightarrow$ counting # of certain overlattices)

generic $X \in \mathcal{C}_d$

$$|(\mathbb{Z}_{2d}^{\times})_2| = \begin{cases} 4, & d = 2^{a+1} \\ 2^{k+1}, & d = 2p_1^{e_1}\cdots p_k^{e_k} \\ 2^{k+2}, & d = 2^{a+1}p_1^{e_1}\cdots p_k^{e_k} \end{cases}$$

$\mathcal{C}_{20}$ $d = 2^2 \cdot 5$

1) $d \equiv 2\ (6)$: $|FM(X)| = \frac{1}{4}|(\mathbb{Z}_{2d}^{\times})_2|$ $\quad |FM| = \frac{1}{4} \cdot 2^{1+2} = 2.$

2) $d \equiv 0\ (6)$, $9 \nmid d$: $|FM(X)| = \frac{1}{8}|(\mathbb{Z}_{2d}^{\times})_2|$

3) $9 \mid d$, $\frac{d}{18} \equiv 1\ (3)$: $|FM(X)| = \frac{1}{4}|(\mathbb{Z}_{2d}^{\times})_2|$

4) $9 \mid d$, $\frac{d}{18} \equiv 2\ (3)$: $|FM(X)| = \frac{1}{2}|(\mathbb{Z}_{2d}^{\times})_2|$

5) $27 \mid d$: $|FM(X)| = \frac{3}{4}|(\mathbb{Z}_{2d}^{\times})_2|$

$|FM| = \boxed{3} \cdot 2^N$

Q: Explanation in term of birational geometry?

Candidate "3-way flop" studied recently by Donovan of 4-folds

$$\begin{tikzcd} X_1 \arrow[rr, dashed, leftrightarrow] \arrow[dr, dashed, leftrightarrow] & & X_2 \arrow[dl, dashed, leftrightarrow] \\ & X_3 & \end{tikzcd}$$

Q: special Cubic 4-fold w/ 3-way flop

$\Updownarrow$??

$27 \mid d$

Kawatani, Huybrechts-Macri-Stellari: $(\rho = 1)$

preserve orientation of +ve def 2-plane

$$H^0 \oplus NS(S) \oplus H^4 \cong U \oplus \langle 2n \rangle$$

$$\begin{tikzcd}
\operatorname{Aut} D^b(S) \arrow[r] \arrow[dr] & O^+(N(D^b(S))) \\
& \langle \tilde{O}^+(N(D^b(S))), \underline{-id} \rangle \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

$\operatorname{Aut} D^b(S) \ni F$ acts as $\pm id$ on $T(S)$

$\tilde{O}^+$: if acts trivially on the disc. gp. N^*/N

$$N(D^b(S)) \overset{\text{canonical}}{\cong} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2n \end{pmatrix}$$

FM partner

$$D^b(S') \xrightarrow{\cong} D^b(S) \rightsquigarrow$$

$$\begin{tikzcd}
N(D^b(S')) \arrow[r, "\cong"] \arrow[d, "\wr", "\text{Canonial}"'] & N(D^b(S)) \arrow[d, "\wr\ \text{Canonial}"] \\
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2n \end{pmatrix} \arrow[r, bend left] & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2n \end{pmatrix}
\end{tikzcd}$$

$$\rightsquigarrow O^+(U \oplus \langle 2n \rangle)$$

$$\{D^b(S') \xrightarrow{\cong} D^b(S)\} \twoheadrightarrow O^+(U \oplus \langle 2n \rangle)$$

$$|FM(S)| = [O^+(U \oplus \langle 2n \rangle) : \tilde{O}^+(U \oplus \langle 2n \rangle)]$$

$O^+(U \oplus \langle 2n \rangle) \cong AL_n$, $\tilde{O}^+(U \oplus \langle 2n \rangle) \cong Fr_n \subseteq PSL_2 \mathbb{R}$

where: $AL_n = \coprod_{s \| n} W_s$, $W_s = \left\{ \begin{pmatrix} a\sqrt{s} & \frac{b}{\sqrt{s}} \\ c\frac{n}{\sqrt{s}} & d\sqrt{s} \end{pmatrix} : a,b,c,d \in \mathbb{Z} \right\}$

$$Fr_n = W_1 \amalg W_n = \Gamma_0(n) \amalg \Gamma_0(n)\tau_n, \quad \tau_n = \begin{pmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}$$

F.-Lai Similar results for special cubic 4-folds

$\rightsquigarrow$ arithmetic Fuchsian gps live in $PSU(1,1)$

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{Z}[\omega].$$

with similar cond$^{\underline{n}}$ as AL gp.

Finite subgps of $\operatorname{Aut} D^b(X)$

Quick recap of $\operatorname{Stab}_2^{\operatorname{Aut}}$,

$\langle \Omega, \Omega \rangle = 0, \ \langle \Omega, \bar{\Omega} \rangle > 0.$

$-\mathrm{id}$ $\quad\quad$ id.

(Dolgachev)

acts trivially
on L_{2n}/L_{2n}

$\boxed{\dfrac{\text{FM counting}}{27 \mid d}}$

birat'l explanation??

- Huybrechts' conj: $A_X \simeq A_Y \overset{?}{\Rightarrow} X \sim Y$

- $d = 20$

Start w/ sthg really basic, like Zili's talk $\mathbb{C}^{\times}\curvearrowright\mathbb{C}^{r}$ vs $E\curvearrowright\mathbb{C}$

$\boxed{O^{T}\begin{pmatrix} & 2n \\ 1 & \end{pmatrix} ?}$

EM $\longrightarrow$ $O^+ / \tilde{O}^+$ $\qquad$ $\mathrm{Aut}(A_X) \longrightarrow \tilde{O}^+$

Write down precisely:

$A^{T}\begin{pmatrix} & 2n \\ 1 & \end{pmatrix} A = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$