4/2/2020

Taylor series

Def $f: \underset{\hat{c}}{I} \to \mathbb{R}$, If f has derivatives of all orders at c, i.e.

- $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists & finite,

  & $f'(x)$ is defined on some open interval containing c.

- $f''(c) = \lim_{x \to c} \frac{f'(x)-f'(c)}{x-c}$ exists & finite,

  & $f''(x)$ is defined on some open interval containing c.

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Define the Taylor series for f about c:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

(power series centered at c)

Q: • For which x does the series conv.?

• When $\sum \frac{f^{(k)}(c)}{k!}(x-c)^k$ conv., is it the same as $f(x)$?

(No, in general: HW#2)

Def Remainder.

$$R_n(x) := f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Then $\lim_{n\to\infty} R_n(x) = 0$ for some $x \in \mathbb{R}$,

$$\Updownarrow$$

$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ conv. at x

$= f(x)$

Taylor's thm Suppose f has derivatives of all orders on I, $c \in I$. → open interval

$\forall x_0 \in I$, $x_0 \neq c$, $\exists y$ between c & x_0

s.t. $R_n(x_0) = \frac{f^{(n)}(y)}{n!}(x_0-c)^n$,

i.e. $f(x_0) = \underline{f(c) + \frac{f'(c)}{1!}(x_0-c) + \cdots + \frac{f^{(n-1)}(c)}{(n-1)!}(x_0-c)^{n-1}} + \frac{f^{(n)}(y)}{n!}(x_0-c)^n$

Rmk When $n=1$, $\forall x_0 \neq c$, $\exists y$ b/w x_0 & c s.t.

$f(x_0) = f(c) + \frac{f'(y)}{1!}(x_0-c)$ (MVT)

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pf Let M be the unique number s.t.

$f(x_0) = f(c) + \frac{f'(c)}{1!}(x_0-c) + \cdots + \frac{f^{(n-1)}(c)}{(n-1)!}(x_0-c)^{n-1} + \frac{M}{n!}(x_0-c)^n$

$\boxed{\text{WTS: } M = f^{(n)}(y) \text{ for some } y \text{ b/w } c \text{ \& } x_0}$

$$\Updownarrow$$

$\boxed{g^{(n)}(x) = f^{(n)}(x) - M}$ $g^{(n)}(y) = 0$ for some y b/w c & x_0

Define $$g(x) = f(x) - \left(f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f^{(2)}(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1} + \frac{M}{n!}(x-c)^n\right)$$

- $g(x_0) = 0$
- $g(c) = 0$

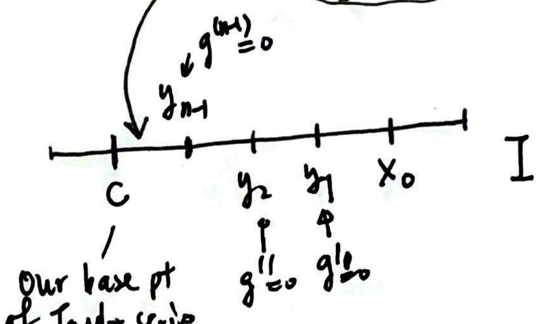
$g'(c) = 0$

$\vdots$

$g^{(n-1)}(c) = 0$

$$g'(x) = f'(x) - \left(f'(c) + \frac{f^{(2)}(c)}{1!}(x-c) + \cdots + \frac{f^{(n-1)}(c)}{(n-2)!}(x-c)^{n-2} + \frac{M}{(n-1)!}(x-c)^{n-1}\right)$$

WTS $\exists\, y$ btw c & x_0 s.t. $g^{(n)}(y) = 0$



$g(c) = g(x_0) = 0$

Rolle's thm $\Rightarrow$ $\exists\, y_1$ btw c & x_0 s.t. $g'(y_1) = 0$

$g'(c) = g'(y_1) = 0$

Rolles thm $\Rightarrow$ $\exists\, y_2$ btw c & y_1 s.t. $g''(y_2) = 0$

<u>Coro</u> $f: (a,b) \to \mathbb{R}$ has derivatives of all order on (a,b) ($c \in (a,b)$)

& $|f^{(n)}(x)| < D$ $\forall n$ $\forall x \in (a,b)$

Then $\lim_{n\to\infty} R_n(x) = 0$ $\forall x \in (a,b)$

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

pf $|R_n(x)| = \left|\frac{f^{(n)}(y)}{n!}(x-c)^n\right| < \frac{D}{n!}|x-c|^n$ as $n\to\infty$ $\to 0$

for some y btw x & c $\left(\lim_{n\to\infty} \frac{r^n}{n!} = 0\right)$

$\square$

<u>e.g.</u> $f(x) = e^x$, $f^{(n)}(x) = e^x$ $\forall n$

Taylor series for f about 0:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

radius of conv. of this power series is $+\infty$.

Use Taylor thm to prove that.

$$\left(f(x) = e^x = \right) \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \forall x \in \mathbb{R}$$

Say we want to prove for $x_0 \in \mathbb{R}$,

$$\boxed{\sum_{k=0}^{\infty} \frac{x_0^k}{k!} = e^{x_0} \quad \forall x_0 \in \mathbb{R}}$$

$$f(x_0) = \sum_{k=0}^{\infty} \frac{x_0^k}{k!}$$

$\Uparrow$

$$\boxed{\text{Claim: } \sum_{k=0}^{\infty} \frac{x^k}{k!} \to e^x \text{ unif. on } [-R,R]}$$

We can find some $R > 0$

s.t. $x_0 \in [-R, R]$.

$$|R_n(x_0)| = \left| \frac{f^{(n)}(y)}{n!} (x_0 - 0)^n \right| \le e^R \cdot \frac{x_0^n}{n!} \le \boxed{e^R \cdot \frac{R^n}{n!}} \to 0 \text{ as } n \to \infty$$

$< \varepsilon$ $\forall n > N$

for some y b/w 0 & x_0

$y \in [-R, R]$

$$|f^{(n)}(y)| = |e^y| \le e^R$$

$\forall \varepsilon > 0$, $\exists N > 0$

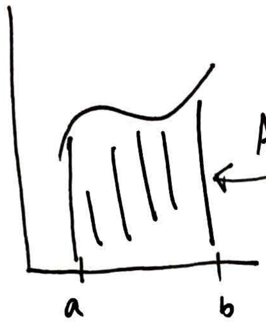
s.t. $\left| \sum_{k=0}^{n-1} \frac{x_0^k}{k!} - e^{x_0} \right| < \varepsilon$ ($R_n(x_0)$)

$\forall n > N$

& $\forall x_0 \in [-R, R]$

Integration

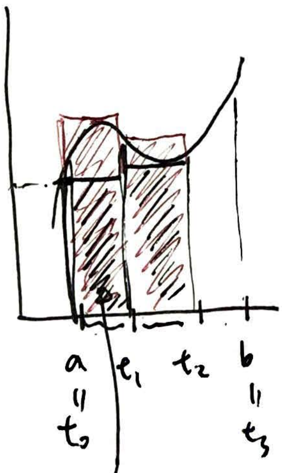
Setting $f : [a,b] \to \mathbb{R}$ bounded.



" $\int_a^b f(x)\,dx$ "

Def A partition of $[a,b]$ is a set of numbers

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$



$$\boxed{(t_1 - t_0)\cdot \inf_{x\in[t_0,t_1]} f(x)}$$

Def Upper Lower sum U(f,P)

$$L(f,P) := \sum_{k=1}^{n} (t_k - t_{k-1}) \cdot \underset{x\in[t_{k-1},t_k]}{\overset{\sup}{\inf}} f(x)$$

Clearly, $L(f,P) \le U(f,P)$

e.g. $f(x) = x$ on $[0,1]$

$$P_n = \left\{0 < \tfrac{1}{n} < \tfrac{2}{n} < \cdots < 1\right\}$$

$$L(f,P_n) = \sum_{k=1}^{n} \frac{1}{n}\cdot\frac{k-1}{n} = \frac{n-1}{2n}$$

$$U(f,P_n) = \sum_{k=1}^{n} \frac{1}{n}\cdot\frac{k}{n} = \frac{n+1}{2n}$$

Expect $U(f,P) \ge$ " $\int_a^b f(x)\,dx$ " $\forall P$

$$\Rightarrow \underbrace{\boxed{\inf_P U(f,P)}}_{U(f)} \ge \text{" } \int_a^b f(x)\,dx \text{ "}$$