4/28/2020

Brouwer fixed pt thm

Any conti. fcn. $f: D^n \longrightarrow D^n$ has a fixed point.

$D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$

It suffices to show that there is no conti. fcn. g s.t.

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "i"] \arrow[rr, bend right, "id"'] & D^n \arrow[r, "g"] & S^{n-1} \end{tikzcd}$$

$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$

Homology gps

$\boxed{\mathbb{Z} \oplus \mathbb{Z}}$ (n,m)

$$H_k : \left\{ \begin{matrix} \text{top. spaces} \\ \text{conti. fcns} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{abel gps} \\ \text{gp homomorphisms} \end{matrix} \right\}$$

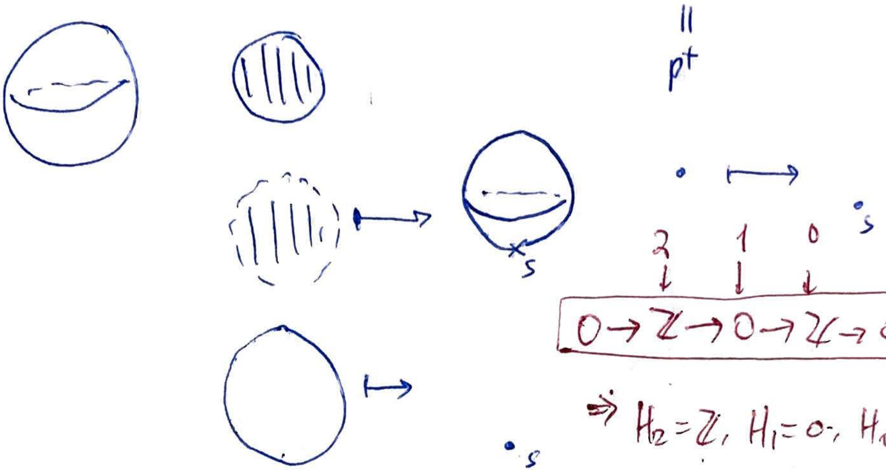
Def. A CW cpx str. on a top. space X is:

a collection of conti. fcns. $\{ f_\alpha : D^{n_\alpha} \to X \}$

s.t.
- $X = \bigcup_\alpha f_\alpha(D^{n_\alpha})$.
- $f_\alpha|_{(D^{n_\alpha})^\circ} : (D^{n_\alpha})^\circ \longrightarrow X$ is homeom. onto its image.
- $f_\alpha(\partial D^{n_\alpha})$ is the union of $\{ f_{\beta_1}(D^{n_{\beta_1}}), \ldots, f_{\beta_k}(D^{n_{\beta_k}}) \}$ for some $n_{\beta_1}, \ldots, n_{\beta_k} < n_\alpha$.

e.g. S^2 $\qquad D^2 \xrightarrow{f_2} S^2 \qquad D^0 \longrightarrow S^2$

$D^0 = pt$

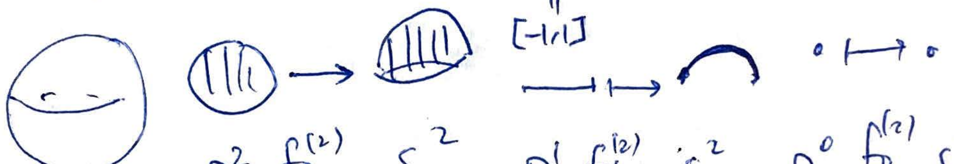


$$\boxed{0 \to \underset{2}{\mathbb{Z}} \to \underset{1}{0} \to \underset{0}{\mathbb{Z}} \to 0}$$

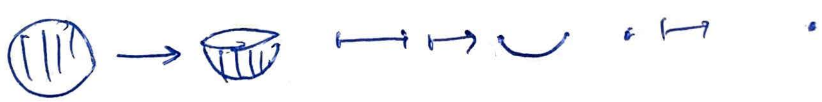
$\Rightarrow H_2 = \mathbb{Z},\ H_1 = 0,\ H_0 = \mathbb{Z}$

e.g. S^2 $\qquad D^2 \xrightarrow{f_2^{(1)}} S^2 \qquad D^1 \xrightarrow{f_1^{(1)}} S^2 \qquad D^0 \xrightarrow{f_0^{(1)}} S^2$

$D^1 = [-1,1]$



$D^2 \xrightarrow{f_2^{(2)}} S^2 \qquad D^1 \xrightarrow{f_1^{(2)}} S^2 \qquad D^0 \xrightarrow{f_0^{(2)}} S^2$



Rmk The # of cells in a CW cpx str. is <u>not</u> a topological invariant.

Consider $0 \to \underset{\#\,2\text{-cells}}{\mathbb{Z}^2} \xrightarrow{\partial_2} \underset{\#\,1\text{-cells}}{\mathbb{Z}^2} \xrightarrow{\partial_1} \underset{\#\,0\text{-cells}}{\mathbb{Z}^2} \to 0$ $\qquad \underline{\partial_1 \circ \partial_2 = 0}$

$f_2^{(1)} \longmapsto f_1^{(1)} + f_1^{(2)} \longmapsto 0$ $\qquad H_0 = \dfrac{\mathbb{Z}^2}{\langle f_0^{(1)} + f_0^{(2)} \rangle}$

$f_2^{(2)} \longmapsto -(f_1^{(1)} + f_1^{(2)}) \longmapsto 0$ $\qquad = \mathbb{Z}$

$H_2 = \ker \partial_2$ $\qquad f_1^{(1)} \longmapsto f_0^{(1)} + f_0^{(2)}$

$= \langle f_2^{(1)} + f_2^{(2)} \rangle$ $\qquad f_1^{(2)} \longmapsto -(f_0^{(1)} + f_0^{(2)})$

$= \mathbb{Z}$ $\qquad H_1 = \dfrac{\ker \partial_1}{\operatorname{Im} \partial_2} = \dfrac{\langle f_1^{(1)} + f_1^{(2)} \rangle}{\langle f_1^{(1)} + f_1^{(2)} \rangle} = 0$

In general,

$$0 \to \mathcal{C}_n \xrightarrow{\partial_n} \mathcal{C}_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \mathcal{C}_0 \to 0$$

where $\mathcal{C}_k = \mathbb{Z}^{\oplus\{k\text{-cells}\}}$

Fact: $\partial_k \circ \partial_{k+1} = 0$

$$\begin{tikzcd} \mathcal{C}_{k+1} \arrow[r, "\partial_{k+1}"] \arrow[rr, bend right, "0"'] & \mathcal{C}_k \arrow[r, "\partial_k"] & \mathcal{C}_{k-1} \end{tikzcd}$$

$\ker \partial_k = \{ g \in \mathcal{C}_k : \partial_k(g) = 0 \}$

$\operatorname{Im} \partial_{k+1} = \{ g \in \mathcal{C}_k : g = \partial_{k+1}(h)$ for some $h \in \mathcal{C}_{k+1} \}$

Fact $\Rightarrow \operatorname{Im} \partial_{k+1} \subseteq \ker \partial_k$

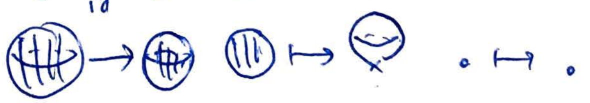
$$H_k(X) := \frac{\ker \partial_k}{\operatorname{Im} \partial_{k+1}} \leftarrow \text{top. invar.}$$

In general,

$$H_k(S^n) = \begin{cases} \mathbb{Z}, & k = 0, n \\ 0 \end{cases}$$

$$H_k(D^n) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 \end{cases}$$

D^3 $\quad D^3 \xrightarrow[\text{id}]{f_3} D^3 \quad D^2 \xrightarrow{f_2} D^3 \quad D^0 \xrightarrow{f_0} D^3$



$$\mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \to 0 \to \mathbb{Z}$$

$f_3 \mapsto f_2$

pf of Brouwer fixed pt thm

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "i"] \arrow[rr, bend right, "id"'] & D^n \arrow[r, "g"] & S^{n-1} \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} H_{n-1}(S^{n-1}) \arrow[r, "i_*"] \arrow[rr, bend right, "id_*"'] \arrow[d, equal] & H_{n-1}(D^n) \arrow[r, "g_*"] \arrow[d, equal] & H_{n-1}(S^{n-1}) \arrow[d, equal] \\ \mathbb{Z} \arrow[r] \arrow[rr, bend right, "id"'] & 0 \arrow[r] & \mathbb{Z} \\ 1 \arrow[rr, mapsto] & & 1 \end{tikzcd}$$

$\square$

Fundamental thm of algebra

$$f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0 \in \mathbb{C}[z]$$

has a root in $\mathbb{C}$

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pf Let $R = 2 + |a_{n-1}| + \cdots + |a_0| > 0$

Define

$$g(z) := \begin{cases} z - \dfrac{f(z)}{R \cdot e^{i(n-1)\theta} r}, & |z| \le 1 \\[2ex] z - \dfrac{f(z)}{R \cdot z^{n-1}}, & |z| \ge 1 \end{cases}$$

$z = re^{i\theta}$

$r \ge 0$

$0 \le \theta < 2\pi$

Claim For $|z| \le R$,

we have $|g(z)| \le R$.

① $|z| \le 1$,

$$|g(z)| = \left| z - \frac{f(z)}{Re^{i(n-1)\theta r}} \right|$$

$$\le |z| + \frac{|f(z)|}{R} \le 1 + \frac{1 + |a_{n-1}| + \dots + |a_0|}{R}$$

$$< 1 + 1 \le R.$$

② $1 \le |z| \le R$,

$$|g(z)| \le R.$$

By fixed pt thm, g has a fixed pt in $z \in B_R(0)$

$$g(z) = z$$

$$\Rightarrow f(z) = 0. \quad \square$$

<u>Perron-Frobenius thm</u>

$A \in M_{n \times n}(\mathbb{R})$ entries are all positive.

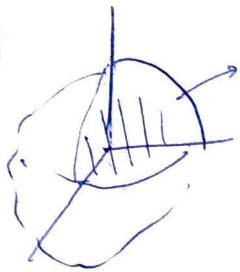
$\Rightarrow$ $\exists!$ largest eigenvalue

and ~~its~~ ~~o~~ has an eigenvector with all entries positive.

<u>Applications</u>:

Markov chain, Dynamical systems, ...

Let's prove $\exists$ eigenvector with all entries ≥ 0



$K = \{x \in \mathbb{R}^n \mid x_i \geq 0,\ \|x_i\| = 1\}$

$x = (x_1, \dots, x_n)$

A : $n \times n$ positive matrix

$f_A : K \longrightarrow K$ conti.

$x \longmapsto \dfrac{Ax}{\|Ax\|}$

$$\begin{tikzcd} K \arrow[r, "\sim", "h"'] & D^2 \\ K \arrow[u, "f_A"] & D^2 \arrow[l, "\sim", "h^{-1}"'] \end{tikzcd}$$

$x \longmapsto h^{-1}(x)$

$D^2 \xrightarrow{h \circ f_A \circ h^{-1}} D^2$ conti.

Brouwer fixed pt thm.

$\exists\, x \in D^2$ s.t.

$h f_A h^{-1}(x) = x$

$f_A(h^{-1}(x)) = h^{-1}(x)$

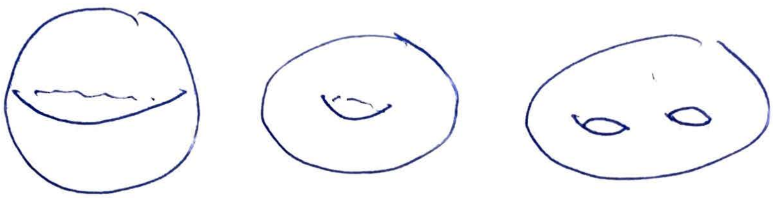
If x is a fixed pt. of f_A

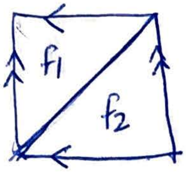
then $x = \dfrac{Ax}{\|Ax\|}$

$\Rightarrow Ax = \|Ax\|\, x$

$\Rightarrow x$ is an eigenvect.

$\square$

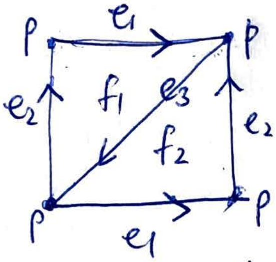




2-cells: f_1, f_2

1-cells: e_1, e_2, e_3

0-cell: p



$$0 \to \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}^1 \to 0$$

$$f_1 \longmapsto e_1+e_2+e_3$$

$$f_2 \longmapsto e_1+e_2+e_3$$

$$e_1 \longmapsto p-p=0$$

$$e_2 \longmapsto$$

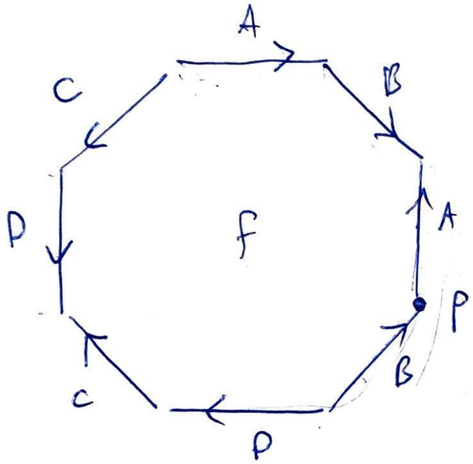
$$e_3 \longmapsto$$

$$H_2 = \frac{\langle f_1-f_2 \rangle}{0} = \boxed{\mathbb{Z}}$$

$$H_1 = \frac{\mathbb{Z}^3}{\langle e_1+e_2+e_3 \rangle} = \boxed{\mathbb{Z}^2}$$

$$H_0 = \frac{\mathbb{Z}^1}{0} = \boxed{\mathbb{Z}}$$

e.g.



2-cell: f

1-cells: A, B, C, D.

0-cell: p

$$0 \to \mathbb{Z} \longrightarrow \mathbb{Z}^4 \longrightarrow \mathbb{Z} \to 0$$

$$f \mapsto 0$$

$$A, B, C, D \to 0$$

$H_2 = \mathbb{Z}$	$H_2(\Sigma_g) = \mathbb{Z}$
$H_1 = \mathbb{Z}^4$	$H_1(\Sigma_g) = \mathbb{Z}^{2g}$
$H_0 = \mathbb{Z}$	$H_0(\Sigma_g) = \mathbb{Z}$