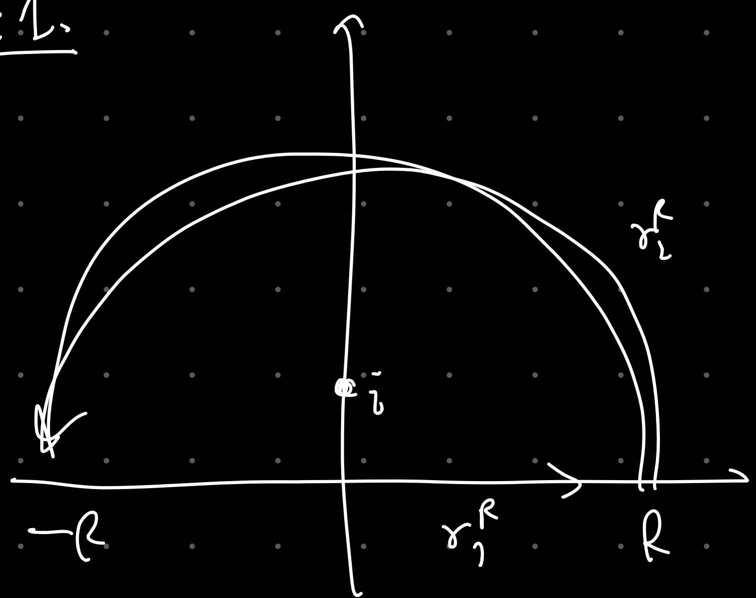
#1.



$\frac{\pi}{4}$

$$\int_{\gamma_2^R} \frac{1}{(1+z^2)^2}\, dz = \int_0^{\pi} \frac{iRe^{i\theta}}{(1+R^2e^{i2\theta})^2}\, d\theta$$

$\longrightarrow 0$ as $R\to\infty$

$$\lim_{R\to\infty} \int_{\gamma_1^R} \frac{1}{(1+z^2)^2}\, dz = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(1+z^2)^2} = \frac{\pi}{2}$$

$$\frac{1}{(z+i)^2(z-i)^2}$$

$f(z)$ has a double at $\underline{z=a}$.

$$f(z) = \frac{h(z)}{(z-a)^2} \quad \text{nonvanishing holo.} \qquad \boxed{\frac{1}{(z+i)^2}} = h(z)$$

$$= \frac{h(a) + h'(a)(z-a) + \dots}{(z-a)^2} \qquad \underline{h'(i)}$$

$$\frac{-2}{(z+i)^3}$$

$$\frac{-2}{(2i)^3} = \frac{-2}{-8i}$$

$$= \boxed{\frac{1}{4i}}$$

$$\pi i \left(\frac{i^2}{(2i)(i^2+9)} + \frac{(3i)^2}{((3i)^2+1)(6i)} \right)$$

$$= \pi i \left(\frac{-1}{2i \cdot 8} + \frac{-9}{(-8)(6i)} \right)$$

$$= \pi i \left(\frac{-1}{16 i} + \frac{\cancel{9}\,3}{\cancel{48}_{16}\, i} \right)$$

$$= \pi i \cdot \frac{i}{8} = \frac{\pi}{8}$$

$$\int_{|z|=1} z^4 e^{2/z^2} \, dz = 0$$

sing at $z=0$

Laurent series at $z=0$:

$$z^4 \left(1 + \frac{2}{z^2} + \frac{\left(\frac{2}{z^2}\right)^2}{2} + \cdots \right)$$

no $\frac{1}{z}$ term

$$\boxed{w = \frac{1}{z}}$$

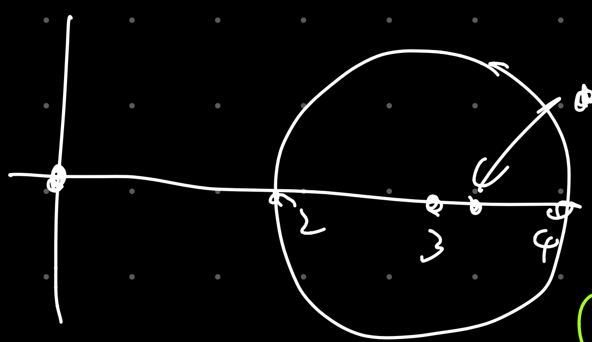
$$-\int_{|w|=1} \underbrace{\frac{1}{w^4} e^{2w^2} \frac{-1}{w^2}}_{\text{sig at } w=0 \text{ pole}} dw = 0$$

no $\frac{1}{w}$ term

$$\frac{1}{w^6}\left(1 + 2w^2 + \frac{(2w^2)^2}{2!} + \dots\right)$$

$$\frac{h(w)}{w^6} = \dots + \boxed{\frac{h^{(5)}(w)}{5!}} w^5 + \dots$$

$$\int_{|z-3|=1} \boxed{\frac{\cos z}{z(z-\pi)^2}} dz = 2\pi i \operatorname*{Res}_{z=\pi} \boxed{} = \boxed{\frac{2i}{\pi}}$$



$$\left(\frac{\cos z}{z}\right)'\Big|_{\pi} = \frac{(-\sin z)z - \cos z}{z^2}\Big|_{\pi} = \frac{1}{\pi^2}$$

$$\frac{}{z} + \frac{}{(z-\pi)^2}$$

#2 $\int_{|z|=1} \frac{1}{z} dz = 2\pi i$ $\int_{|z|=1} p(z)\,dz = 0$

$$\left|\int_{|z|=1} \left(\frac{1}{z} - p(z)\right) dz\right| = |2\pi i| = 2\pi$$

$$\wedge\!|$$

$$\sup_{|z|=1} \left|\frac{1}{z} - p(z)\right| \cdot \underbrace{\text{length}(|z|=1)}_{\substack{\| \\ 2\pi}}$$

#3 $\limsup |a_n b_n| \leq \overset{A}{\overset{\|}{\boxed{\limsup |a_n|}}}\ \overset{B}{\overset{\|}{\boxed{\limsup |b_n|}}}$

$\forall \varepsilon > 0$, $\limsup |a_n| = A$

$\exists N > 0$ s.t. $|a_n| < A + \varepsilon$ $\forall n > N$.

$|b_n| < B + \varepsilon$

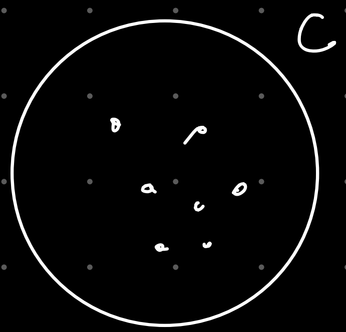
$$|a_n b_n| < (A+\varepsilon)(B+\varepsilon)$$

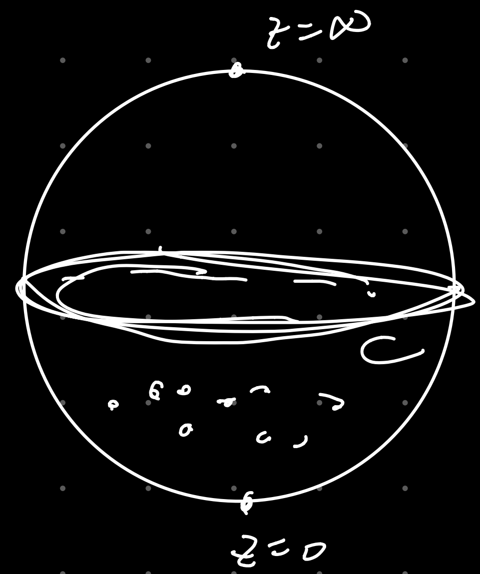
$$= AB + \varepsilon(A + B + \varepsilon)$$

$\Rightarrow \limsup |a_n b_n| \leq AB + \varepsilon(A+B+\varepsilon)$ $\forall \varepsilon > 0$

Take $\varepsilon \to 0$ $\Rightarrow$ $\limsup |a_n b_n| \leq AB$.

5 (Lecture 17)





14:

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$f_A : z \longmapsto \frac{a_1 z + b_1}{c_1 z + d_1} \qquad f_B : z \longmapsto \frac{a_2 z + b_2}{c_2 z + d_2}$$

$$AB = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix} \qquad f_{AB}$$

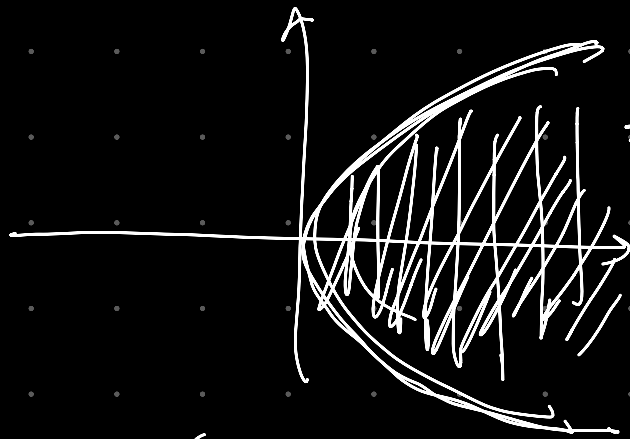
$$z \longmapsto \boxed{\frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}}$$

$$f_A(f_B(z)) = f_{AB}(z)$$

$$= \frac{a_1 (a_2 z + b_2) + b_1 (c_2 z + d_2)}{c_1 (a_2 z + b_2) + d_2 (c_2 z + d_2)}$$

$$f_A\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) = \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_2}$$

#9:

$f: \mathbb{C} \longrightarrow$  $= \{x > y^2\}$ in $\mathbb{C}$

open, connected, simply connected

$\wr$ biholo.

$\mathbb{D}$

e^z omits 0 as a value

Montel's thm.

#15: $\mathcal{F}$ normal on Ω $\Longleftrightarrow$ $\mathcal{F}$ unif bdd. on Ω

$$\boxed{\text{whether } \exists M > 0 \text{ st. } |f(x)| < M \quad \forall x \in \Omega, \ \forall f \in \mathcal{F}}$$

$$|f(z)| = |(z-a_1)\cdots(z-a_n)| = \underbrace{|z-a_1|}_{<1}\cdots\underbrace{|z-a_n|}_{<1} < 1$$

$|a_i| < \frac{1}{2}$

$\Omega = \{z \mid |z| < \frac{1}{2}\}$.

#16.

① write down the principal part of $\wp' \wp'''$ at the pole $z=0$

② try to find certain poly in $\wp$ w/ the same principal part.

①:

$$\wp = \frac{1}{z^2} + 3E_4 z^2 + 5E_6 z^4 + 7E_8 z^6$$

$$\wp' = \frac{-2}{z^3} + 6E_4 z + 20E_6 z^3 + 42E_8 z^5$$

$$\wp'' = \frac{6}{z^4} + 6E_4 + 60E_6 z^2 + 210E_8 z^4$$

$$\wp''' = \frac{-24}{z^5} + 120E_6 z + 840E_8 z^3$$

$$\boxed{\wp' \wp'''} = \boxed{\frac{48}{z^8} + \frac{-144E_4}{z^4} + \frac{-240E_6 - 480E_6}{z^2}} + \begin{matrix} -1680E_8 \\ -1008E_8 \end{matrix} = -2688E_8$$

②

$$\wp^2 = \frac{1}{z^4} + 6E_4 + 10E_6 z^2 + (9E_4^2 + 14E_8)z^4$$

$$\wp^3 = \frac{1}{z^6} + \frac{9E_4}{z^2} + 15E_6 + \dots$$

$$\wp^4 = \frac{1}{z^8} + \frac{12E_4}{z^4} + \frac{20E_6}{z^2} + (36E_4^2 + 18E_4^2 + 28E_8)$$

$$\boxed{\begin{array}{l} 48\wp^4 = \\ -720 E_4 \wp^2 = \\ -1680 E_6 \wp = \end{array}}$$

$$\frac{48}{z^8} + \frac{576 E_4}{z^4} + \frac{960 E_6}{z^2} + \cdots$$

$$- \frac{720 E_4}{z^4} + \cdots$$

$$\frac{-1680 E_6}{z^2} +$$

$$\Rightarrow \wp' \wp''' - \left(48 \wp^4 - 720 E_4 \wp^2 - 1680 E_6 \wp \right)$$

= holo ell. fn

= $\boxed{\text{const. fn}}$ $\equiv 0$

$$\wp'^2 = 4\wp^3 - 60 E_4 \wp - 140 E_6$$

$$2\wp' \wp'' = 12 \wp^2 \cancel{\wp'} - 60 E_4 \cancel{\wp'}$$

$$\wp'' = 6\wp^2 - 30 E_4$$

$$\wp' \wp''' = 12 \wp \wp' \wp'$$

$$= 12 \wp (4\wp^3 - 60 G_4 \wp - 140 G_6)$$

$$= \boxed{48 \wp^4 - 720 G_4 \wp^2 - 1680 G_6 \wp}$$

#17:

Key thm. $\Rightarrow$ any modular form of wt 0 ~~that is~~ nonvanishing on F.D. & $i\infty$ ($\mathbb{H}$)

Suppose f is a nonconst. modular form of wt 0.

$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

$f(z) = f(z+1) = f\left(\frac{-1}{z}\right)$

Pick any $a \in \mathbb{H}$.

Define $g(z) := f(z) - f(a)$ on $\mathbb{H}$.

$\Rightarrow g$ is also a modular form of wt 0

$g(a) = f(a) - f(a) = 0$.

$\Rightarrow g \equiv$ zero fn.