HOMEWORK 8
MATH H54, FALL 2021

Due November 2, 11am

Some ground rules:

- Please submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW8**.
- The submission should be a **single PDF file**.
- Late homework will not be accepted/graded under any circumstances.
- Make sure the writing in your submission is clear enough. Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own, and acknowledge the students with whom you worked.
- **For True/False questions**: You have to prove the statement if your answer is "True"; otherwise, you have to provide an explicit counterexample and justification.
- You are allowed to use any result that is proved in the lecture. But if you would like to use other results, you have to prove it first before using it.

Problems:

(1) Let A be a real $n \times n$ matrix, and consider the function

$$\langle -, - \rangle_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

defined by $\langle \vec{x}, \vec{y} \rangle_A := \vec{x}^T A \vec{y}$. Prove that " $\langle -, - \rangle_A$ defines an inner product on $\mathbb{R}^n$ " if and only if " A is a symmetric positive definite matrix".

(2) Let A be a real symmetric matrix.
  (a) Prove that A is positive definite if and only if all the eigenvalues of A are positive.
  (b) Prove that A is positive semidefinite if and only if all the eigenvalues of A are non-negative.

(3) Let A be a real anti-symmetric $n \times n$ matrix (i.e. $A = -A^T$).
  (a) Prove that $\vec{x}^T A \vec{x} = 0$ for any $\vec{x} \in \mathbb{R}^n$.
  (b) Prove that A^2 is a symmetric negative semidefinite matrix.
  (c) Prove that $\mathbb{I} - A$ is invertible.

(4) Let A be a real anti-symmetric $n \times n$ matrix (i.e. $A = -A^T$), and let B be a real symmetric positive definite $n \times n$ matrix. Prove that $A + B$ is invertible.

(5) Let S_n^+ be the set of all $n \times n$ real symmetric matrices whose eigenvalues are all positive.

(a) Prove that if $A \in S$, then A is invertible and $A^{-1} \in S$.

(b) Prove that if $A, B \in S$, then $A + B \in S$.

(6) (a) Let A be an $m \times n$ real matrix (not necessarily a square matrix). Prove that both $A^T A$ and AA^T are symmetric positive semidefinite matrices.

(b) Prove that "both $A^T A$ and AA^T are symmetric positive definite matrices" if and only if " A is a square matrix and is invertible".

(7) Consider the function $f\colon M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ given by $f(A) = \operatorname{tr}(A^2)$.

(a) Prove that f is a quadratic form on the vector space $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

(b) Find the *signature* of f.

Signature of a quadratic form: Let Q be any quadratic form on $\mathbb{R}^k$. Recall from the lecture that there exists a basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$ of $\mathbb{R}^k$ and real numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that for any $\vec{x} \in \mathbb{R}^k$ with $\vec{x} = x_1\vec{v}_1 + \cdots + x_k\vec{v}_k$, we have

$$Q(\vec{x}) = \lambda_1 x_1^2 + \cdots + \lambda_k x_k^2.$$

The *signature* of Q is a triple of non-negative integers (n_+, n_0, n_-), where n_0 is the number of zeros in $\{\lambda_1, \ldots, \lambda_k\}$, and n_+ (resp. n_-) is the number of positive (resp. negative) numbers in $\{\lambda_1, \ldots, \lambda_k\}$. In particular, Q is positive (resp. negative) definite if and only if its signature is $(k, 0, 0)$ (resp. $(0, 0, k)$).