PRACTICE PROBLEMS FOR FINAL
MATH H54

(1) Let A and B be square matrices. Suppose that $A+B$ is invertible. Prove that $A(A+B)^{-1}B = B(A+B)^{-1}A$.

(2) Let A, B, C be rectangle matrices such that the product ABC is defined. Prove that
$$\operatorname{rank}(BC) - \operatorname{rank}(ABC) \leq \operatorname{rank}(B) - \operatorname{rank}(AB).$$

(3) Let A be an $n \times n$ matrix. Consider the linear transformation $T\colon \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ on the n^2-dimensional vector space $\mathrm{Mat}_{n\times n}(\mathbb{R})$ defined by $T(B) = AB$. Express $\det(T)$ in terms of $\det(A)$.

(4) Consider the function $f\colon \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathbb{R}$ given by $f(A) = \mathrm{tr}(A^2)$.
  (a) Prove that f is a quadratic form on the vector space $\mathrm{Mat}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$.
  (b) Find the *signature* of f.

  Signature of a quadratic form: Let Q be any quadratic form on $\mathbb{R}^k$. Recall from Lecture 18 that there exists a basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$ of $\mathbb{R}^k$ and real numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that for any $\vec{x} \in \mathbb{R}^k$ with $\vec{x} = x_1\vec{v}_1 + \cdots + x_k\vec{v}_k$, we have
$$Q(\vec{x}) = \lambda_1 x_1^2 + \cdots + \lambda_k x_k^2.$$
  The *signature* of Q is a triple of non-negative integers (n_+, n_0, n_-), where n_0 is the number of zeros in $\{\lambda_1, \ldots, \lambda_k\}$, and n_+ (resp. n_-) is the number of positive (resp. negative) numbers in $\{\lambda_1, \ldots, \lambda_k\}$. In particular, Q is positive (resp. negative) definite if and only if its signature is $(k, 0, 0)$ (resp. $(0, 0, k)$).

(5) Let A be a square matrix with columns given by unit vectors. Prove that $|\det(A)| \leq 1$. When does the equality hold?

(6) Let V be a finite-dimensional vector space, and let $T\colon V \to V$ be a diagonalizable linear transformation. Suppose $W \subseteq V$ is a subspace satisfying $T(W) \subseteq W$. Prove that the restriction $T|_W\colon W \to W$ also is diagonalizable.

(7) Consider a sequence of linear transformations between finite-dimensional vector spaces
$$\{0\} \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n \xrightarrow{T_n} \{0\}$$
  Assume that $\mathrm{Im}(T_{i-1}) = \mathrm{Ker}(T_i)$ for all $1 \leq i \leq n$. What is the value of
$$\dim(V_1) - \dim(V_2) + \dim(V_3) - \cdots + (-1)^n \dim(V_n)?$$

(8) Find all possible 5×5 real symmetric matrices A satisfying
$$A^3 - 2A = 4\mathbb{I}_5.$$

(9) Let $\mathrm{Mat}_{n\times n}(\mathbb{C})$ denote the n^2-dimensional complex vector space consisting of $n\times n$ complex matrices. Consider the linear transformation $f\colon \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ given by $f(A) = A^T$.

(a) Find all eigenvalues of f.
(b) Prove or disprove: f is diagonalizable.

(10) Let A be a real $n \times n$ matrix. Prove that the following two statements are equivalent:
(a) $A^2 = A$;
(b) $\operatorname{rank}(A) + \operatorname{rank}(\mathbb{I}_n - A) = n$.

(11) Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be an orthonormal set in a finite-dimensional inner product space V. Suppose that for any $\vec{v} \in V$ we have

$$||\vec{v}||^2 = \langle \vec{v}_1, \vec{v} \rangle^2 + \cdots + \langle \vec{v}_k, \vec{v} \rangle^2 .$$

Prove that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis of V.

(12) Let W_1 and W_2 be subspaces of a finite-dimensional inner product space V.
(a) Prove that $W_1^\perp \cap W_2^\perp = (W_1 + W_2)^\perp$.
(b) Prove that $\dim(W_1) - \dim(W_1 \cap W_2) = \dim(W_2^\perp) - \dim(W_1^\perp \cap W_2^\perp)$.

(13) Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Suppose that $\mathbb{I}_m - AB$ is invertible. Prove that $\mathbb{I}_n - BA$ also is invertible.

(14) Let W_1 and W_2 be subspaces of a vectors space V. Consider the union

$$W_1 \cup W_2 \coloneqq \{x \in V : x \in W_1 \text{ or } x \in W_2\}.$$

Prove that if $W_1 \cup W_2$ is a subspace of V, then we must have $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(15) Let A and B be two square matrices such that $AB = A^2 + A + \mathbb{I}$. Prove that $AB = BA$.

(16) Let $T\colon V \to V$ be a linear transformation on a (possibly infinite-dimensional) vector space V. Suppose that every subspace of V is invariant under V, i.e. $T(W) \subseteq W$ for any subspace $W \subseteq V$. Prove that T is a scalar multiple of the identity transformation.