Uniform continuity

Recall: $f: X \to Y$ conti. if

$\forall x_0 \in X, \ \forall \varepsilon > 0, \ \exists \ \delta = \delta(x_0, \varepsilon) > 0$

s.t. $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$.

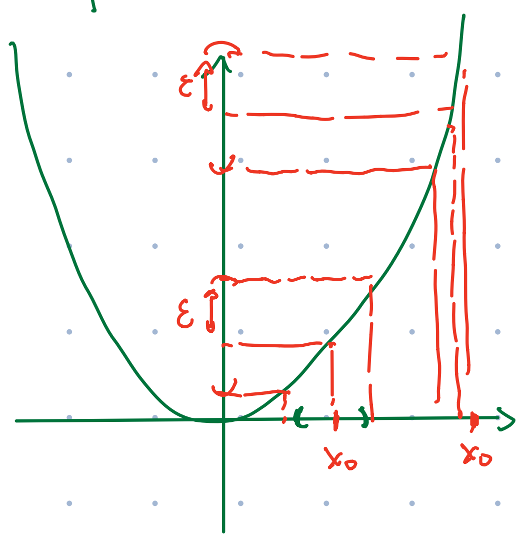
Def: $f: (X, d_X) \to (Y, d_Y)$ is uniformly conti. on $E \subseteq X$

if $\forall \varepsilon > 0, \ \exists \delta > 0$

s.t. $\forall x, y \in E$ s.t. $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \varepsilon$.

Say $f: X \to Y$ is uniformly continuous if it's unif conti. on X.

e.g. $f(x) = x^2$



Q: Is it uniformly continuous on $\mathbb{R}$?

$\boxed{\text{No}}$ Take $\varepsilon = 1$.

Need: $\forall \delta > 0$,

find $x, y \in \mathbb{R}$, s.t.

$|x - y| < \delta$, but $|x^2 - y^2| \geq 1$.

$\boxed{x} > 0$

$y = x + \frac{\delta}{2} > 0$

$|x^2 - (x + \frac{\delta}{2})^2| = \underset{\substack{\parallel \\ 1/\delta}}{x\delta} + \frac{\delta^2}{4} \quad \geq 1$

pf: Take $\varepsilon = 1$,

$\forall \delta > 0$

Consider $x = \frac{1}{\delta}$, $y = \frac{1}{\delta} + \frac{\delta}{2}$

Then $|x - y| = \frac{\delta}{2} < \delta$

$|x^2 - y^2| \geq 1$ $\square$

Rmk: Intuitively, the reason that $f(x) = x^2$ fails to be unif. conti. on $\mathbb{R}$ is b/c the slope blows up to $\pm\infty$ as $x \to \pm\infty$.

Q: Is it unif. conti. on $(0,1)$?

Yes $\forall \varepsilon>0,\ \exists \delta>0$ s.t.

If $x,y\in(0,1)$, $|x-y|<\delta$ $\Longrightarrow$ $|x^2-y^2|<\varepsilon$.

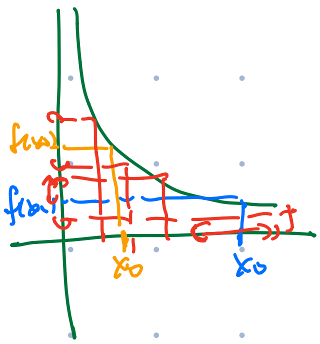
$|x+y||x-y| < \boxed{2|x-y|}!$

pf: $\forall \varepsilon>0$, Choose $\delta=\frac{\varepsilon}{2}>0$.

Then if $x,y\in(0,1)$, $|x-y|<\delta=\frac{\varepsilon}{2}$,

we have: $|x^2-y^2| = |x+y||x-y| < 2|x-y| = \varepsilon$.

e.g. $f(x)=\frac{1}{x}$ • $f:(0,\infty)\longrightarrow\mathbb{R}$.



Q: Is f unif. conti. on $(0,\infty)$?

No Take $\varepsilon=1$.

Need: $\forall\delta>0$, find $x,y>0$, where $|x-y|<\delta$,

$\left|\frac{1}{x}-\frac{1}{y}\right| \geq 1$.

$\frac{|x-y|}{xy} = \frac{\delta/2}{x(x+\delta/2)}$ ≥ 1

$x(x+\frac{\delta}{2}) \leq \frac{\delta}{2}$

X

$y = x+\frac{\delta}{2}$

$\boxed{x \leq \sqrt{\frac{\delta}{2}+\frac{\delta^2}{16}}-\frac{\delta}{4}}$

$(x+\frac{\delta}{4})^2 \leq \frac{\delta}{2}+\frac{\delta^2}{16}$

Pf: Take $\varepsilon = 1$, $\forall \delta > 0$

Consider $x = \sqrt{\frac{\delta}{2} + \frac{\delta^2}{16}} - \frac{\delta}{4} > 0$

$y = \sqrt{\frac{\delta}{2} + \frac{\delta^2}{16}} + \frac{\delta}{4} > 0.$

$|x - y| = \frac{\delta}{2} < \delta.$

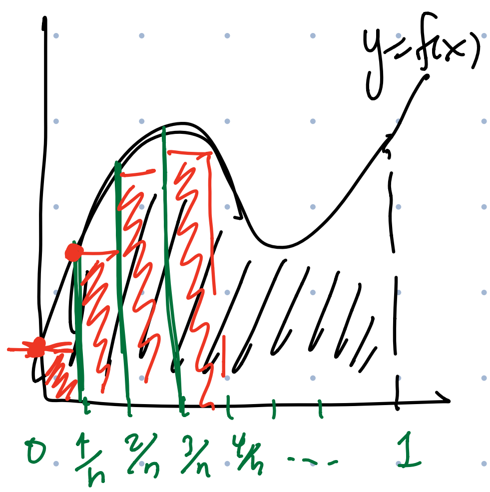
$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x-y}{xy}\right|$$

$$= \frac{\frac{\delta}{2}}{\frac{\delta}{2} + \cancel{\frac{\delta^2}{16}} - \cancel{\left(\frac{\delta}{4}\right)^2}} = 1. \quad \square$$

Why unif. conti. is important?

- It's important in the definition of integration

$f : [0,1] \longrightarrow \mathbb{R}$ unif. conti $\rightsquigarrow$ we can define $\int_0^1 f(x)\,dx$



$$L_n = \frac{1}{n} \sum_{k=1}^{n} \inf \left\{ f(x) : x \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \right\}.$$

$$U_n = \frac{1}{n} \sum_{k=1}^{n} \sup \left\{ f(x) : x \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \right\}$$

$$L_n \leq \text{"} \int_0^1 f(x)\,dx \text{"} \leq U_n.$$

Idea: Let $n \to +\infty$, then L_n, U_n should approach the actual area below the graph of f.

Want: $\boxed{\lim_{n\to\infty} (U_n - L_n) = 0} \Rightarrow f$ is "integrable"

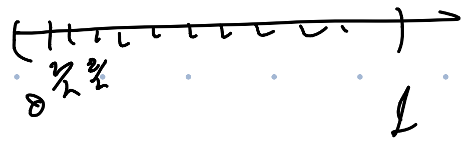
$$\parallel$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\sup\left\{f(x) : x \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right]\right\} - \inf\left\{f(x) : x \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right]\right\} \right)$$

Now, since f is $\boxed{\text{unif. conti}}$, so $\forall \varepsilon > 0, \exists \delta > 0$

s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

So if $\boxed{n > \frac{1}{\delta}}$, then



$x, y \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$, $|x-y| < \delta$

$\Rightarrow |f(x) - f(y)| < \varepsilon$.

$$\Rightarrow \quad (U_n - L_n) \leq \frac{1}{n}(n\varepsilon) = \varepsilon. \quad \forall n > \frac{1}{\delta}.$$

$$\Rightarrow \lim_{n\to\infty} (U_n - L_n) = 0$$

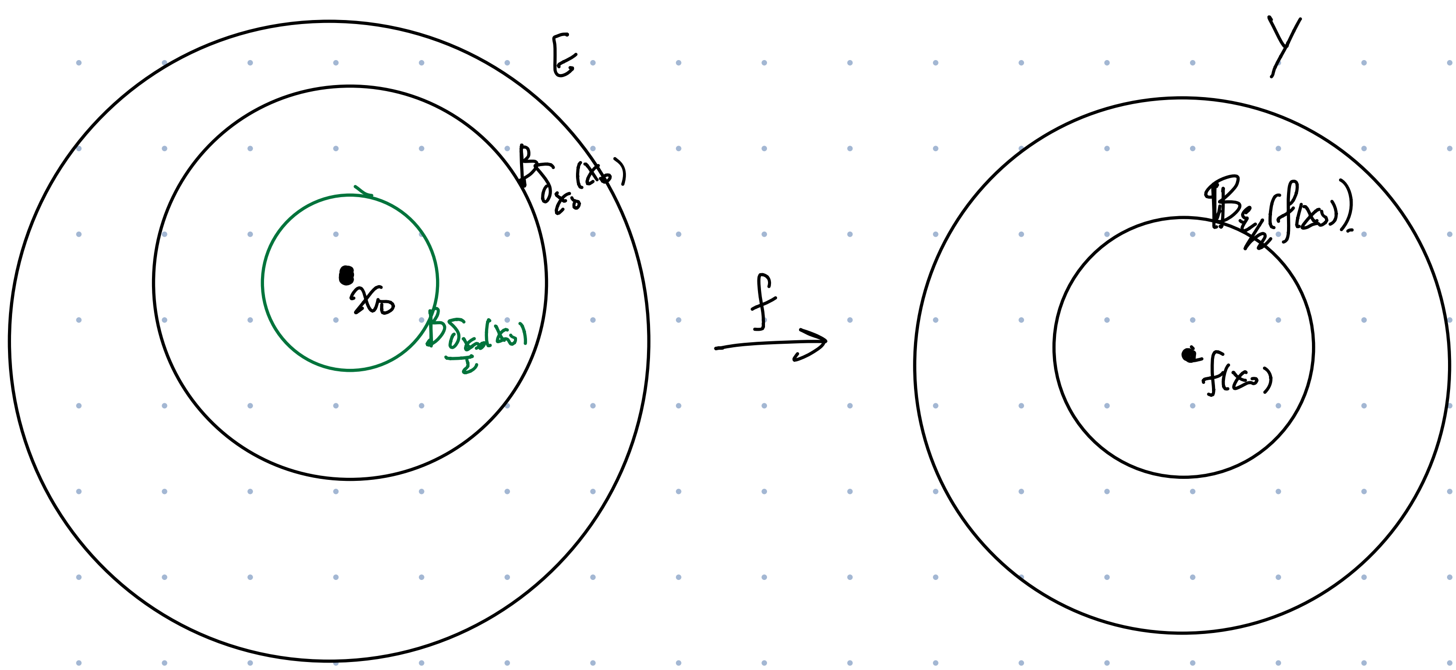
Thm $f : (X, d_X) \to (Y, d_Y)$ conti. $E \subseteq X$ $\boxed{\text{compact}}$ subset.

Then f is uniformly conti. on E.

pf:

- $\forall \varepsilon > 0$, $\forall x_0 \in E$,

$$\exists\, \delta_{x_0} > 0 \text{ s.t. } d(x, x_0) < \delta_{x_0} \Rightarrow d(f(x), f(x_0)) < \frac{\varepsilon}{2}$$



Consider open sets $\{B_{\frac{1}{2}\delta_{x_0}}(x_0)\}_{x_0 \in E}$. $\to$ open cover of E.

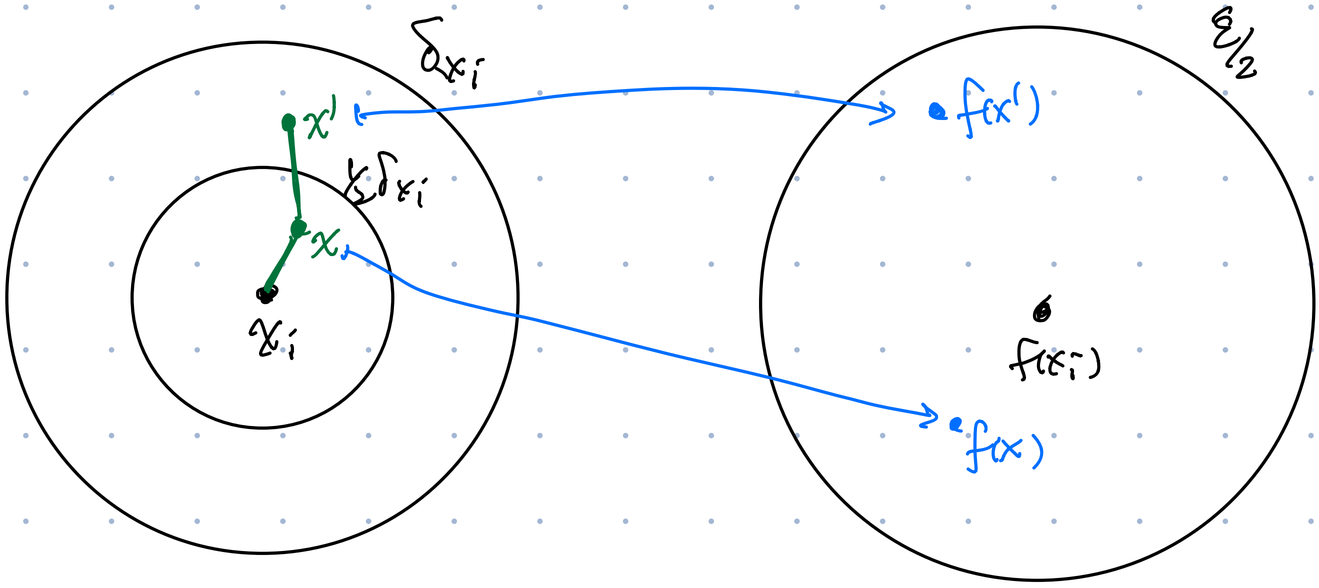
E cpt $\Rightarrow$ $\exists$ finite subcover.

$$E \subseteq \left(B_{\frac{1}{2}\delta_{x_1}}(x_1) \cup \cdots \cup B_{\frac{1}{2}\delta_{x_n}}(x_n) \right)$$

Define $\delta := \min\{ \frac{1}{2}\delta_{x_1}, \dots, \frac{1}{2}\delta_{x_n} \} > 0$

<u>Claim</u>: $\forall x, x' \in E$, $d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \varepsilon$.

<u>pf</u>:
- $x \in B_{\frac{1}{2}\delta_{x_i}}(x_i)$ for some i.
- $d(x, x') < \delta \leq \frac{1}{2}\delta_{x_i} \Rightarrow d(x_i, x') < \delta_{x_i}$
- Since $x, x' \in B_{\delta_{x_i}}(x_i) \Rightarrow f(x), f(x') \in B_{\frac{\varepsilon}{2}}(f(x_i))$
  $\Rightarrow d(f(x), f(x')) < \varepsilon$. $\square$



Def: $f : (X, d_X) \to (Y, d_Y)$ is Lipschitz conti. if $\exists k > 0$ s.t. $d_Y(f(x), f(y)) \leq k\, d_X(x, y)$. $\forall x, y \in X$.

Rmk Lip. conti. $\Longrightarrow$ unif conti. (Take $\delta = \varepsilon/k$.).

Rmk, unif. conti $\not\Longrightarrow$ Lip. conti. (HW).

Rmk: We'll see that if f is "differentiable" w/ bounded derivative then f is Lip. conti.