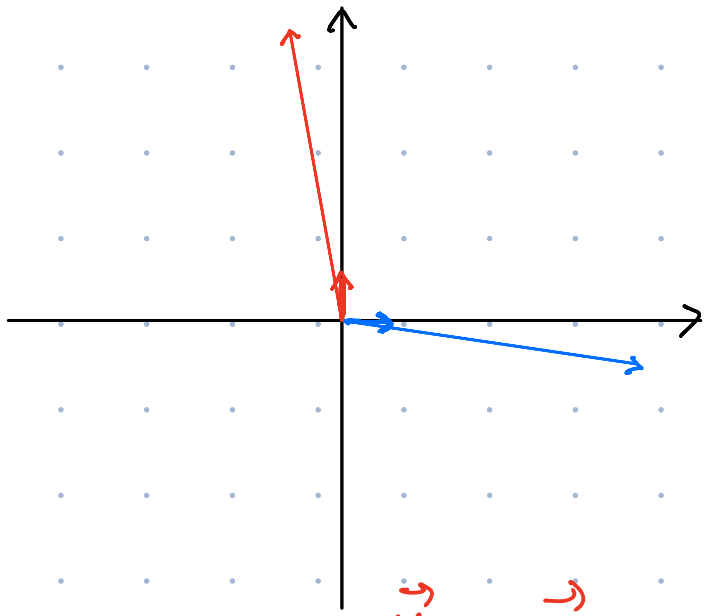
Recall

Rmk: Why do we want to consider different basis?

e.g. Consider $A = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}$, and $T_A : \mathbb{R}^2 \to \mathbb{R}^2$

Say we want to better understand T_A geometrically.

$$T_A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 11 \\ -2 \end{bmatrix}, \quad T_A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 14 \end{bmatrix}.$$



If we consider the basis $\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ -2 \end{bmatrix}} \right\}$.

$$T_A\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix} = 10\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

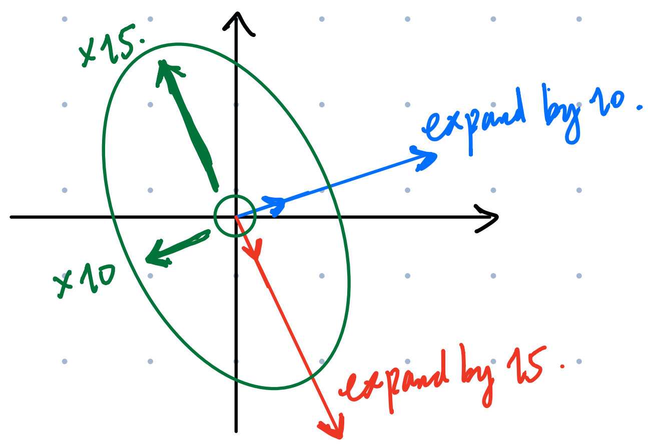
$$T_A\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 15 \\ -30 \end{bmatrix} = 15\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$\vec{x} \in \mathbb{R}^2$

$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$

$\Downarrow$

$T_A(\vec{x}) = 10c_1\vec{v}_1 + 15c_2\vec{v}_1$



It's easier to understand T using this basis!!

$\boxed{\text{Recall}}$

e.g.

$B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "T_A"] \arrow[d, "{[\ ]_B}"'] & \mathbb{R}^2 \arrow[d, "{[\ ]_B}"] \\
\mathbb{R}^2 & \mathbb{R}^2
\end{tikzcd}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 11 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} -2 \\ 14 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "{[\ ]_B^{-1}}"] & \mathbb{R}^2 \arrow[r, "T_A"] & \mathbb{R}^2 \arrow[r, "{[\ ]_B}"] & \mathbb{R}^2
\end{tikzcd}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longmapsto 10\begin{bmatrix} 2 \\ 1 \end{bmatrix} \longmapsto 10\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 \\ -2 \end{bmatrix} \longmapsto 15\begin{bmatrix} 1 \\ -2 \end{bmatrix} \longmapsto 15\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T = \begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix} = [\ ]_B \circ T_A \circ [\ ]_B^{-1}$$

$$= T_{P_B^{-1}} \circ T_A \circ T_{P_B}$$

$$= T_{P_B^{-1} A P_B}.$$

What we've done is:

$A = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}$,

We found $P_B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, s.t. diagonalization

$$P_B^{-1} A P_B = \begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix}$$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "T_A"] \arrow[d, "{[\ ]_B}"', "\cong"] & \mathbb{R}^n \arrow[d, "\cong"', "{[\ ]_B}"] \\
\mathbb{R}^n \arrow[r, dashed] & \mathbb{R}^n
\end{tikzcd}$$

$\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$

$P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix}$

$T_{P_{\mathcal{B}}^{-1}} = [\ ]_{\mathcal{B}}$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "{[\ ]_{\mathcal{B}}^{-1}}", "{= T_{P_{\mathcal{B}}}}"'] \arrow[rrr, bend right] & \mathbb{R}^n \arrow[r, "T_A"] & \mathbb{R}^n \arrow[r, "{[\ ]_{\mathcal{B}}}"] & \mathbb{R}^n
\end{tikzcd}$$

$$T_{P_{\mathcal{B}}^{-1}} \circ T_A \circ T_{P_{\mathcal{B}}} = T_{P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}}$$

Def Say A, B (square matrices) are similar if $\exists$ an invertible matrix P

s.t. $A = PBP^{-1} \iff P^{-1}AP = B$

Rmk: "similar" is an equivalence relation:

- $A \sim A$
- $A \sim B \Rightarrow B \sim A$
- $A \sim B,\ B \sim C \Rightarrow A \sim C$

Def Say a square matrix A is <u>diagonalizable</u> if it's similar to a diagonal matrix $D = \begin{bmatrix} * & & & 0 \\ & * & & \\ & & * & \\ 0 & & & \ddots * \end{bmatrix}$

i.e. $\exists$ P invertible s.t. $\underline{P^{-1}AP = D}$.

Rmk: If A diagonalizable then it's easy to compute A^k.

$$A = PDP^{-1}$$

$$A^k = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cancel{\cdot}\cdots\cancel{(}\cancel{P}DP^{-1})$$

$$= PD^kP^{-1} \qquad D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\Rightarrow \quad AP = P \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\Rightarrow \quad A\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix}$$

$\|$

$\{\vec{v}_1 \cdots \vec{v}_n\}$ is a basis of $\mathbb{R}^n$.

$$\begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \end{bmatrix} \quad \Rightarrow \quad \underline{A\vec{v}_i = \lambda_i\vec{v}_i} \text{ for each column } \vec{v}_i \text{ of } P$$

Def: • say $\vec{v} \neq \vec{0}$ is an eigenvector if $\boxed{A\vec{v} = \lambda\vec{v}}$ for some $\lambda \in \mathbb{C}$. (eigenvalue)

• λ is an eigenvalue $\iff (A - \lambda I)\vec{v} = \vec{0}$

$\iff \mathrm{Nul}(A - \lambda I) \neq \{0\}$

$\iff \boxed{\det(A - \lambda I) = 0}$

e.g. $A = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{bmatrix} 11-\lambda & -2 \\ -2 & 14-\lambda \end{bmatrix}$$

$$= (11-\lambda)(14-\lambda) - (-2)(-2)$$

$$= \lambda^2 - 25\lambda + 150$$

$$= (\lambda - 10)(\lambda - 15)$$

$\Rightarrow$ eigenvalues are 10, 15.

Plug in $\lambda = 10$: $A - 10I = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

$$\mathrm{Nul}(A - 10I) = \mathrm{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

Plug in $\lambda = 15$: $A - 15I = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}$

$$\mathrm{Nul}(A - 15I) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}.$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}^{-1} A \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix}.$$

Def.

- λ is an eigenvalue $\Leftrightarrow$ $\underline{\mathrm{Nul}(A-\lambda I)} \neq \{0\}$

  $\mathrm{Nul}(A-\lambda I)$ is called the eigenspace of λ

- characteristic poly. of A: $\boxed{\det(A-\lambda I)}$

$$\det \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{bmatrix}$$

  Is a poly. of deg. n in λ.

- eigenvalues are the roots of the char. poly

$$\underline{\det(A-\lambda I)} = \prod_{i=1}^{k} (\lambda_i - \lambda)^{\mathrm{mult}(\lambda_i)}$$

has n roots (counting multiplicity)

$\{\lambda_1, \dots, \lambda_k\}$ distinct eigenvalues of A

e.g. $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad \det \begin{bmatrix} 2-\lambda & 1 & 3 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 3-\lambda \end{bmatrix}$

$= (2-\lambda)^2 (3-\lambda)$

mult.(2) = 2.
mult.(3) = 1

e.g. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1$

eigenvalues are $\pm i \notin \mathbb{R}$

e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

diagonalizable $\qquad$ $\boxed{\text{not diagonalizable}}$

same char. poly. $(2-\lambda)^2$

$\mathrm{Nul}(A-2I) = \mathbb{R}^2$ $\qquad$ $\mathrm{Nul}(A-2I)$

$= \mathrm{Nul}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$

$= \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

Thm a $n \times n$ matrix A is diagonalizable

$\Leftrightarrow$ $\exists$ an eigenbasis of A in $\mathbb{R}^n$,

i.e. $\exists \{\vec{v}_1, \dots, \vec{v}_n\}$ basis of $\mathbb{R}^n$

where each $\vec{v}_i$ is an eigenvector of A.

pf ($\Rightarrow$) $P^{-1}AP = D$, $\quad AP = PD$

$$A\underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}}_{P} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}\underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{D} \Rightarrow A\vec{v}_i = \lambda_i \vec{v}_i \quad \forall i$$

$\Rightarrow \{\vec{v}_1, \dots, \vec{v}_n\}$ is an eigenbasis.

($\Leftarrow$) Define $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$, $A\vec{v}_i = \lambda_i \vec{v}_i$ $\forall i$ for some

~~Check~~ $AP = A\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$

$$= \begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}}_{P} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \quad \square$$

Rmk $\boxed{A, B \text{ similar}} \Rightarrow$ they have the same char. poly.

$\Rightarrow$ they have same eigenvalues (w/ same mult.)

$\exists P$ invertible

$$A = PBP^{-1}$$

$$\det(A - \lambda I) = \det\left(\boxed{PBP^{-1} - \lambda I}\right) = \det\left(\boxed{P(B - \lambda I)P^{-1}}\right) = \det(P)\det(B - \lambda I)\det(P^{-1}) = \det(B - \lambda I).$$

$P(\lambda I)P^{-1}$

Rmk: 0 is an eigenvalue of A $\Leftrightarrow$ A is not invertible

$$\mathrm{Nul}(A - 0I) \neq \{0\}$$

Thm Suppose $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of A

Suppose $\vec{v}_1, \dots, \vec{v}_k$ are eigenvectors assoc. to $\lambda_1 \dots \lambda_k$

$$(A\vec{v}_i = \lambda_i \vec{v}_i)$$

$$\Rightarrow \{\vec{v}_1, \dots, \vec{v}_k\} \text{ is l.i.}$$

pf Assume $\underbrace{a_1 \vec{v}_1 + \dots + a_k \vec{v}_k}_{\text{not all } 0} = \vec{0}$

Remove the terms w/ $a_i = 0$

$$\boxed{a_1 \vec{v}_1 + \dots + a_\ell \vec{v}_\ell = \vec{0} \text{ where } a_1, \dots, a_\ell \neq 0}$$

$$A(a_1 \vec{v}_1 + \dots + a_\ell \vec{v}_\ell) = A(\vec{0}) = \vec{0}$$

$$\|$$

$$\boxed{a_1 \lambda_1 \vec{v}_1 + \dots + a_\ell \lambda_\ell \vec{v}_\ell = \vec{0}} \quad (\lambda_1, \dots, \lambda_\ell \text{ distinct})$$

$\times \lambda_1$

$$a_1 \lambda_1 \vec{v}_1 + a_2 \lambda_1 \vec{v}_2 + \dots + a_\ell \lambda_1 \vec{v}_\ell = \vec{0}$$

$$\Rightarrow a_2 \lambda_2 \vec{v}_2 + a_3 \lambda_3 \vec{v}_3 + \cdots + a_\ell \lambda_\ell \vec{v}_\ell - a_2 \lambda_1 \vec{v}_2 - a_3 \lambda_1 \vec{v}_3 - \cdots - a_\ell \lambda_1 \vec{v}_\ell = \vec{0}$$

$$\Rightarrow \underset{\neq 0}{a_2(\lambda_2-\lambda_1)} \vec{v}_2 + \overset{\neq 0}{a_3(\lambda_3-\lambda_1)} \vec{v}_3 + \cdots + \underset{\neq 0}{a_\ell(\lambda_\ell-\lambda_1)} \vec{v}_\ell = \vec{0}$$

$\Rightarrow$ Do this inductively

$$\Rightarrow \underset{\neq 0}{b} \vec{v}_\ell = \vec{0}$$

$\Rightarrow \vec{v}_\ell = \vec{0}$ Contradiction. $\square$

$$\ell := \min\left\{ \ell' \leq k \;\middle|\; \exists \underline{\{\vec{v}_{i_1}, \ldots, \vec{v}_{i_{\ell'}}\}} \subseteq \{\vec{v}_1, \ldots, \vec{v}_k\},\ \exists a_1, \ldots, a_{\ell'} \neq 0 \text{ s.t. } a_1 \vec{v}_{i_1} + \cdots + a_{\ell'} \vec{v}_{i_{\ell'}} = \vec{0} \right\}$$

Thm If A is $\boxed{\text{diagonalizable}}$ ($n\times n$), then any $\vec{v}\in\mathbb{R}^n$ can be uniquely written as

$$\vec{v}=\vec{v}_1+\cdots+\vec{v}_k,$$

where each $\vec{v}_i$ is an eigenvector with different eigenvalues

pf $\exists$ $\boxed{\{\vec{w}_1,\cdots,\vec{w}_n\}\text{ eigenbasis.}}$

$$\Big\{\underbrace{\vec{w}_{\lambda_1}^{(1)},\cdots,\vec{w}_{\lambda_1}^{(m_1)}}_{\text{e-value}=\lambda_1},\ \underbrace{\vec{w}_{\lambda_2}^{(1)},\cdots,\vec{w}_{\lambda_2}^{(m_2)}}_{\lambda_2},\ \cdots\cdots\Big\}$$

- $\vec{v}\in\mathbb{R}^n$, $\vec{v}=c_1\vec{w}_1+\cdots+c_n\vec{w}_n$

$$=\boxed{\underbrace{c_{\lambda_1}^{(1)}\vec{w}_{\lambda_1}^{(1)}+\cdots+c_{\lambda_1}^{(m_1)}\vec{w}_{\lambda_1}^{(m_1)}}_{\text{e-value}=\lambda_1}}+\boxed{(\cdots\cdots)}+\cdots$$

$\boxed{\mathrm{Nul}(A-\lambda_1 I)}$

eigenvector of λ_1

$\Rightarrow$ each $\vec{v}\in\mathbb{R}^n$ can be written as sum of eigenvectors which corresp. to distinct eigenvalues

- $\vec{v}=\vec{v}_1+\cdots+\vec{v}_k$ $\qquad$ $\lambda_1,\cdots,\lambda_k$ eigenvalues of A

$=\vec{w}_1+\cdots+\vec{w}_k$ $\qquad$ $\vec{v}_i,\vec{w}_i\in\mathrm{Nul}(A-\lambda_i I)$

$$\Rightarrow \vec{0} = \underbrace{(\vec{v}_1 - \vec{w}_1)}_{\substack{\Uparrow \\ Nul(A-\lambda_1 I)}} + \underbrace{(\vec{v}_2 - \vec{w}_2)}_{\substack{\Uparrow \\ Nul(A-\lambda_2 I)}} + \cdots$$

$\Rightarrow$ By thm we just proved,

$$\Rightarrow \vec{v}_i = \vec{w}_i \quad \forall i.$$