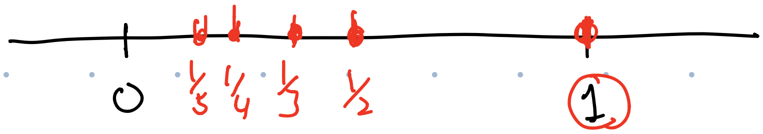
e.g. $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$.

$$= \{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\}$$

Is it compact or not?



$$\begin{cases} U_1 = (\frac{1}{2}, 2), & \ni 1 \\ U_2 = (\frac{1}{3}, 1) & \ni \frac{1}{2} \\ U_3 = (\frac{1}{4}, \frac{1}{2}) & \ni \frac{1}{3} \\ U_4 = (\frac{1}{5}, \frac{1}{3}) & \ni \frac{1}{4} \\ \vdots & \vdots \\ U_n = (\frac{1}{n+1}, \frac{1}{n-1}) & \vdots \end{cases}$$

is an open cover of E

Claim: There is no finite subcover of $\{U_n\}$.

pf $\{U_{k_1}, \dots, U_{k_n}\}$.

say $k_1 < k_2 < \dots < k_n$

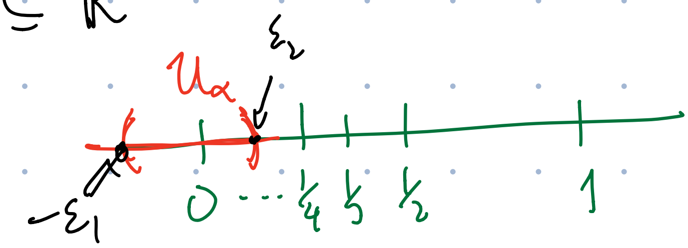
Then $\forall x \in (0, \frac{1}{k_n+1})$,

$x \notin U_{k_1} \cup \dots \cup U_{k_n}$ $\square$

$\Longrightarrow$ E is not compact

e.g. $\tilde{E} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$

Is $\tilde{E}$ compact or not??



Claim: $\tilde{E}$ is compact.

pf $\forall \{U_\alpha\}_{\alpha \in I}$ open cover of $\tilde{E}$

- $\exists \alpha \in I$ s.t. $0 \in U_\alpha = (-\varepsilon_1, \varepsilon_2)$
- $\exists N > 0$ s.t. $\forall n > N$, $0 < \frac{1}{n} < \varepsilon_2$

$\Longrightarrow$ $\frac{1}{n} \in U_\alpha$ $\forall n > N$.

- For $1, \frac{1}{2}, \cdots, \frac{1}{N}$, we know
$\exists\, \alpha_1, \cdots, \alpha_N \in I$
s.t. $1 \in U_{\alpha_1}, \frac{1}{2} \in U_{\alpha_2}, \cdots, \frac{1}{N} \in U_{\alpha_N}$

$\Longrightarrow$ $U_\alpha \cup U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_N}$
is a finite subcover of $\tilde{E}$. $\square$

e.g. $[a,b] \subseteq \mathbb{R}$ is compact.

Def. $(a_n) \subseteq (S,d)$ seq. of points in a metric space
Say $\underline{\lim a_n = a \in S}$ if
$\forall \varepsilon > 0, \exists N > 0$
s.t. $n > N \Rightarrow d(a_n, a) < \varepsilon$.
In this case, (a_n) is called convergent.

Def. $K \subseteq (S,d)$ is called sequentially compact if
$\forall$ sequence of points in K, there is a subsequence which converges to a point in K.

e.g. $\underline{(0,1)} \subseteq \mathbb{R}$ is NOT seq. cpt.
the seq. $(1, \frac{1}{2}, \frac{1}{3}, \cdots) \subseteq (0,1)$
but $\lim \frac{1}{n} = 0 \notin (0,1)$

Def $K \subseteq (S,d)$ is bounded if
$\exists\, x \in K, r > 0$
s.t. $K \subseteq B_r(x)$

Big theorems that we're working toward:

Compact $\Longleftrightarrow$ Sequentially cpt. $\Longrightarrow$ closed and bounded

True for $(S, d) = (\mathbb{R}^n, d_{std})$.

(Heine-Borel thm.) will prove later.

Rmk: "closed & bdd $\Rightarrow$ cpt" is NOT true in general:

e.g. $S = \{s_1, s_2, \dots\}$
set w/ ∞'ly many pts.

$$d : S \times S \longrightarrow \mathbb{R} \quad \text{where} \quad d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Ex: any subset of S is ~~both~~ open & closed. & bounded

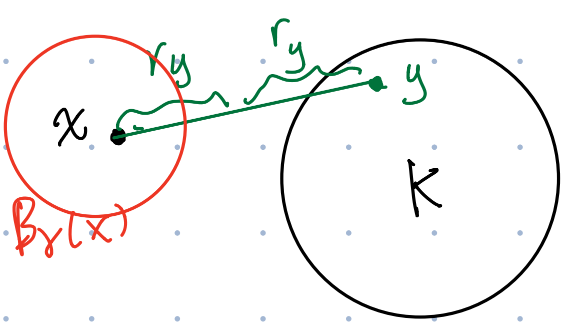
S is closed & bounded, but it's not compact

(since $\{s_1\}, \{s_2\}, \{s_3\}, \dots$ gives an open cover of S, but there is no finite subcover.)

Thm: $K \subseteq (S, d)$ If K is compact, then K is closed.

pf: need to show: K^c is open,

Say $x \in K^c$;



Need: find $r > 0$ s.t. $B_r(x) \subseteq K^c$.

Say $z \in B_r(x)$, $d(x,z) < r$.

If $z \in K$, then $\exists y_i$ s.t. $z \in B_{r_{y_i}}(y_i)$,

$d(y_i, z) < r_{y_i}$

$d(x, y_i) \leq d(x,z) + d(z, y_i) < r + r_{y_i} \leq 2 r_{y_i}$

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\textstyle d(x,y_i)}{\|}$

$\forall y \in K$, $\quad r_y := \frac{1}{2} d(x,y) > 0.$

*: $\square$

Consider the open ball $\quad B_{r_y}(y)$

Consider the collection of open sets $\quad \{B_{r_y}(y)\}_{y \in K}.$

an open cover of K.

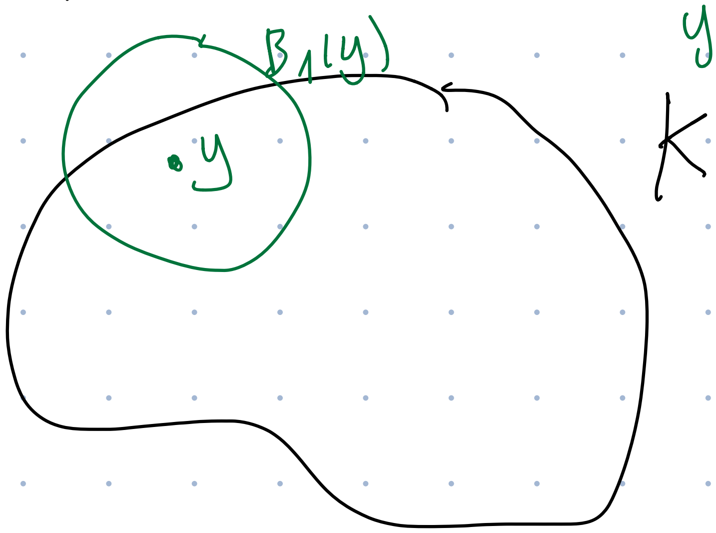
By cptness of K, $\quad \exists\, y_1, \dots, y_n \in K$

s.t. $\quad K \subseteq \left(B_{r_{y_1}}(y_1) \cup \dots \cup B_{r_{y_n}}(y_n)\right)$

Define $r := \min\{r_{y_1}, \dots, r_{y_n}\} > 0$

Claim: $\quad B_r(x) \cap K = \emptyset.$ $\quad$ (i.e. $B_r(x) \subseteq K^c$)

Hint of cpt $\Rightarrow$ bdd:



$\bigcup_{y \in K} B_1(y)$ is an open cover of K

$\Downarrow$ cptness

$\exists$ finite subcover.

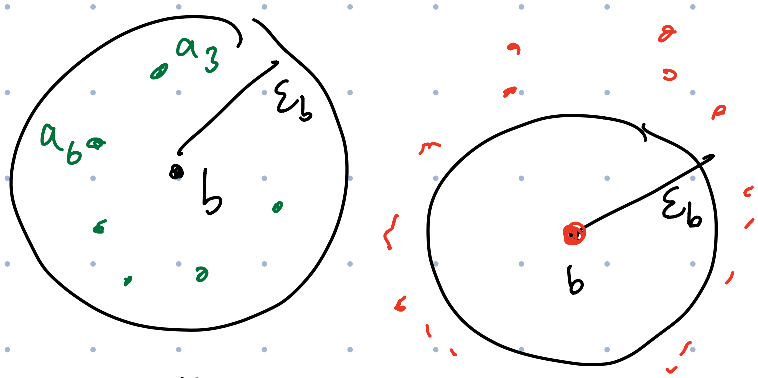
Thm (cpt $\Rightarrow$ seq. cpt):

Assume $K \subseteq (S, d)$ cpt., <u>not</u> seq. cpt.

i.e. $\exists\ (a_n) \subseteq K$ s.t. any subseq. of (a_n) doesn't conv. to a pt. in K.

(**Recall:** "$b \in K$ is a limit of a subseq. of (a_n)" $\Longleftrightarrow$ "$\forall \varepsilon > 0$, $\{n \in \mathbb{N} : d(a_n, b) < \varepsilon\}$ is infinite".)

So, $\forall\ b \in K$, $\exists\ \varepsilon_b > 0$ s.t. $B_{\varepsilon_b}(b)$ contains only finitely many pts in $\{a_1, a_2, \ldots\}$.



In fact, $\forall\ b \in K$, $\exists\ \varepsilon_b > 0$ s.t. $B_{\varepsilon_b}(b)$ contains no points from $\{a_1, a_2, \ldots\}$, except possibly b itself.

Consider the open cover $\{B_{\varepsilon_b}(b)\}_{b \in K}$ of K

By cptness of K, $\exists\ b_1, \ldots, b_n \in K$ s.t.

$$K \subseteq B_{\varepsilon_{b_1}}(b_1) \cup \cdots \cup B_{\varepsilon_{b_n}}(b_n)$$

Now consider each a_m, $a_m \in K$

$\Rightarrow \exists\, 1 \le i \le n$ s.t. $a_m \in B_{\varepsilon_{b_i}}(b_i)$

$\Rightarrow \exists\, 1 \le i \le n$ s.t. $a_m = b_i$

$\Rightarrow a_m \in \{b_1, \dots, b_n\} \quad \forall m$

$\Rightarrow \exists\, 1 \le j \le n$, s.t. $a_m = b_j$ for infinitely many m.

$\Rightarrow \exists$ conv. subseq. of a_n that converges to $b_j \in K$. $\rightarrow\!\times\!\leftarrow$.

$\square$

Thm (seq. cpt. $\Rightarrow$ cpt).

Lemma 1: If K is seq. cpt, and if $\{U_\alpha\}_{\alpha \in I}$ is an open cover of K, then

$\exists\, \delta > 0$ s.t. $\forall x \in K, \exists\, \alpha \in I$, s.t. $B_\delta(x) \subseteq U_\alpha$.

Rmk: Crucial pt: $\delta > 0$ is indep. of $x \in K$.

Lemma 2: If K is seq. cpt, then

$\forall \varepsilon > 0$, $\exists$ finite set $F \subseteq K$

s.t. $$K \subseteq \bigcup_{x \in F} B_\varepsilon(x)$$

---

pf of Thm assuming Lemma 1 & 2:

$\{U_\alpha\}$ open cover of K.

①: $\exists \delta > 0$ s.t.

$\forall x \in K$, $\exists \alpha \in I$, s.t. $B_\delta(x) \subseteq U_\alpha$.

②: $\exists$ finite set $F = \{x_1, \dots, x_n\} \subseteq K$,

s.t. $K \subseteq B_\delta(x_1) \cup \dots \cup B_\delta(x_n)$

By ①, for each x_i, $\exists \alpha_i \in I$ s.t. $B_\delta(x_i) \subseteq U_{\alpha_i}$

By ②,
$$K \subseteq B_\delta(x_1) \cup \dots \cup B_\delta(x_n)$$
$$\subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$$

$\Rightarrow$ K is cpt. $\square$