Finite subgroups of derived automorphisms of generic K3 surfaces

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Joint work with Kuan-Wen Lai (Bonn)

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Slogan of main results

Let X be a Calabi–Yau manifold. It is expected that

$$D := D^b\mathrm{Coh}(X) \cong D^\pi\mathrm{Fuk}(X^\vee).$$

This triangulated category has autoequivalences arising from both complex geometry of X and symplectic geometry of $X^\vee$, for instance:

$$\mathrm{Aut}(X), \qquad L \otimes -; \qquad \mathrm{Symp}(X^\vee).$$

Slogan: When X is a K3 surface of Picard number one,

- there is no finite order autoequivalence arising from either $\mathrm{Aut}(X)$, $L \otimes -$, or $\mathrm{Symp}(X^\vee)$;
- there are interesting finite order autoequivalences given by mixings of complex and symplectic geometric autoequivalences;
- we give full classification and counting of finite subgroups of $\mathrm{Aut}(D)$ and $\mathrm{Aut}(D)/[2]$ up to conjugations.

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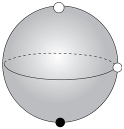
Finite order autoequivalences and Gepner points

Consider an one-parameter family of Calabi–Yau manifolds:

$$\left\{x_0^{n+1} + x_1^{n+1} + \cdots + x_n^{n+1} + t \cdot x_0 x_1 \cdots x_n = 0\right\} \subseteq \mathbb{CP}^n.$$



The monodromies correspond to autoequivalences of $D^b\mathrm{Coh}(X)$:

- large complex structure limit point: $\mathcal{O}(1) \otimes -$;
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In particular, when X is a K3 surface, we have $\Phi^4 = [2]$, therefore $\Phi^2[-1]$ is of order two in $\mathrm{Aut}(D^b(X))$.

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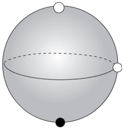

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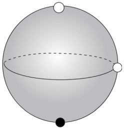

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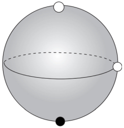

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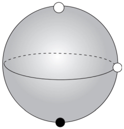

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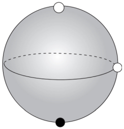

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Automorphisms of K3 surfaces

As a warm-up, let us sketch the proof of the following result.

Theorem (Nikulin)

Let X be a complex projective K3 surface with $NS(X) \cong \langle H \rangle$. Then

$$\mathrm{Aut}(X) = \begin{cases} \{id\} & \text{if } H^2 > 2 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } H^2 = 2 \end{cases}$$

- There is an injective map $\mathrm{Aut}(X) \hookrightarrow \mathrm{O}\left(H^2(X, \mathbb{Z})\right)$.
- $f \in \mathrm{Aut}(X)$ acts trivially on $\mathrm{NS}(X)$: pullback of H is still ample.
- f acts as $\pm\mathrm{id}$ on $T(X) := \mathrm{NS}(X)^{\perp}$: true for any odd Picard number.
- Its induced actions on the discriminant groups $T(X)^*/T(X)$ and $\mathrm{NS}(X)^*/\mathrm{NS}(X) \cong \mathbb{Z}/(H^2)\mathbb{Z}$ coincide $\implies f = \mathrm{id}$ unless $H^2 = 2$.
- When $H^2 = 2$, X is a double cover over $\mathbb{P}^2$ and the covering involution gives the nontrivial automorphism.

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Some difficulties of generalizing to $\mathrm{Aut}(D^b(X))$

- If $\Phi \in \mathrm{Aut}(D^b(X))$ is of finite order, then its induced actions on

$$T(X) \quad \text{and} \quad N(X) = H^0(X) \oplus \mathrm{NS}(X) \oplus H^4(X) \cong \mathbb{Z}^3$$

  are of finite order.
- Φ still acts as $\pm\mathrm{id}$ on $T(X)$; but its action on $N(X)$ can be more complicated.

Strategy: We show that any finite order Φ fixes a Bridgeland stability condition on $D^b(X)$. In particular, it fixes a 2-plane in $N(X)_{\mathbb{R}}$ pointwisely, therefore the action of Φ on $N(X)$ is either id or a reflection.

- In order to fully classify the finite subgroups of $\mathrm{Aut}(D^b(X))$, one needs to determine the conditions on finite subgroups of $T(X)$ and $N(X)$ of which there exists finite order lifts in $\mathrm{Aut}(D^b(X))$.

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Bridgeland stability conditions

A Bridgeland stability condition on D is a pair $\sigma = (Z, P)$:

- $Z\colon N(D) \to \mathbb{C}$ group homomorphism (central charge)
- $P = \{P(\phi)\}_{\phi \in \mathbb{R}}$ additive subcategories (semistable of phase ϕ)

satisfying several axioms, including the Harder–Narasimhan property:

- for any $E \in D$, there exists a unique sequence of exact triangles

$$\begin{tikzcd}
0 \arrow[rr] & & \star \arrow[rr] \arrow[dl] & & \star \arrow[r] \arrow[dl] & \dots \arrow[r] & \star \arrow[rr] & & E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
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where $A_i \in P(\phi_i)$ and $\phi_1 > \cdots > \phi_n$.

(Analogy in SG: $P \leftrightarrow$ special Lagrangian submanifolds, $Z \leftrightarrow \int_L \Omega$)
(Analogy in AG: $P \leftrightarrow$ slope semistable sheaves, $Z \leftrightarrow -\text{deg}+\sqrt{-1}\cdot\text{rank}$)
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Autoequivalences act on stability conditions

A natural way of studying a group is via considering its actions on various spaces. The group of autoequivalences $\mathrm{Aut}(D)$ of a triangulated category admits a natural action on the space of its Bridgeland stability conditions.

This action ($\mathrm{Aut}(D) \curvearrowright \mathrm{Stab}(D)$) can be used to:

- define complexity (e.g. categorical entropy) of autoequivalences;
- provide classifications of autoequivalences (e.g. finite order, "reducible", "pseudo-Anosov", etc.);
- provide analogy with Teichmüller theory –

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A dictionary of analogy

(after Gaiotto, Moore, Neitzke; Bridgeland, Smith; Dimitrov, Haiden, Katzarkov, Kontsevich, etc.)

Riemann surface Σ	Triangulated category $\mathcal{D}$
curve C	object E
$C_1 \cap C_2$	$\mathrm{Hom}(E_1, E_2)$
metric g	Bridgeland stability condition σ
geodesics	semistable objects
length $\ell_g(C)$	mass $m_\sigma(E)$
$\mathrm{MCG}(\Sigma)$	$\mathrm{Aut}(\mathcal{D})$
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Many analogous statements in Teichmüller theory can be proved in the categorical setting for $\mathcal{D} = D^b\mathrm{Coh}$(elliptic curve). An interesting general question is whether some of these can be generalized to dim ≥ 2.

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Nielsen realization problem

- Nielsen asked (1923): Let $G \subseteq \mathrm{MCG}(\Sigma)$ be a finite subgroup. Does there always exist a lifting $G \subseteq \mathrm{Diff}(\Sigma)$?
  (Recall that $\mathrm{MCG}(\Sigma) = \mathrm{Diff}(\Sigma)/\text{isotopy}$).
- Kerckhoff (1983): Yes! Moreover, there exists a metric g such that $G \subseteq \mathrm{Isom}(\Sigma, g)$. Or equivalently, G fixes a point in $\mathrm{Teich}(\Sigma)$.
  (There is a natural action of $\mathrm{MCG}(\Sigma)$ on $\mathrm{Teich}(\Sigma)$,
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- When $D = D^b(X)$, stability conditions on D are roughly Kähler structures on X; so this problem is similar to (but not quite the same) the mirror problem of Farb–Looijenga.

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Main theorems (F.–Lai, 2023)

- The answers to both problems are yes, for $D = D^b\mathrm{Coh}(X)$ where X is a curve, a (twisted) abelian surface, a generic twisted K3 surface, or a K3 surface of Picard number $\rho = 1$.

For K3 surfaces of $\rho = 1$, we obtain:

- every finite subgroup of $\mathrm{Aut}(D)$ is of order 2, and is generated by an anti-symplectic involution;
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which satisfies $T_Y^3 = [2]$. It is asked by Huybrechts whether the existence of such order 3 element of $\mathrm{Aut}(D^b(X))/[2]$ characterizes K3 surfaces with associated cubic fourfolds.

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Modulo certain conjectures regarding the (polynomial) entropy of the reducible autoequivalences, we expect the following trichotomy:

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For K3 or abelian surfaces, Bridgeland (2008) showed that there is an $\mathrm{Aut}(D)$-equivariant covering map

$$\mathrm{Stab}^{\dagger}_{\mathsf{red}}(D)/\mathbb{C} \xrightarrow{\pi} Q_0^+(D)$$

where $Q_0^+(D) = \{v \in \mathbb{P}(N(D) \otimes \mathbb{C}) \mid v^2 = 0, v\overline{v} > 0\} \setminus \bigcup_{\delta^2=-2} \delta^{\perp}$.

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  and it is not hard to show that finite subgroups of $\mathrm{Aut}(D)$ fix a point in $Q_0^+(D)$ using basic Lie theory.
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Avoiding $\delta^\perp$

Suppose X is a K3 surface of $\rho = 1$ and degree $2n$.

- We have $Q_0^+(D) \cong \mathbb{H} \backslash$ "(-2)-points".
- By Dolgachaev (1996) and Kawatani (2014), the action of $\mathrm{Aut}(D)$ on $Q_0^+(D)$ factors through $\mathrm{Im}(\mathrm{Aut}(D) \xrightarrow{f} \mathrm{PSL}(2,\mathbb{R})) = \Gamma_0^+(n)$ the Fricke modular group, where $\Gamma_0^+(n) = \left\langle \Gamma_0(n), \begin{bmatrix} & -1/\sqrt{n} \\ \sqrt{n} & \end{bmatrix} =: \omega_n \right\rangle$.

We showed that the following statements are equivalent:

- $f(\Phi)$ fixes a (-2)-point in $\mathbb{H}$
- $f(\Phi)$ is an involution, and $f(\Phi) = g_0\omega_n$ for some $g_0 \in \Gamma_0(n)$
- $\Phi = T_S\Psi$ for some spherical object S and some $\Psi \in \mathrm{Deck}(\pi)$

Moreover, we showed that autoequivalences of the form $T_S\Psi$ must be of infinite order in $\mathrm{Aut}(D)/[1]$. This resolves the first issue.

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- Let $\Phi \in \mathrm{Aut}(D)/[1]$ be of finite order, then the previous discussion shows that it fixes a point in $Q_0^+(D)$.
- Kawatani (2019): $\pi_1(Q_0^+(D)) \cong \star_{\mathsf{free}} T_S^2$.
- Bayer–Bridgeland (2017): $\mathrm{Stab}^\dagger_{\mathsf{red}}(D)/\mathbb{C}$ is contractible.
- Combining these two results, we have $\mathrm{Deck}(\pi) \cong \star_{\mathsf{free}} T_S^2$.
- We showed that this implies the fixed point of Φ in $Q_0^+(D)$ can be lifted to a fixed point in $\mathrm{Stab}^\dagger_{\mathsf{red}}(D)/\mathbb{C}$, which proves the realization problem for cyclic subgroups of $\mathrm{Aut}(D)/[1]$.
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A few further problems

- Find good representatives of 0-reducible autoequivalences.
- Do 0-reducible autoequivalences have zero entropy? $(h(\otimes\mathcal{O}(1)) = 0)$
- Generalize the realization results to:
  - general special cubic fourfolds $\mathrm{Ku}(Y)$
  - K3 surfaces of Picard number $\rho \geq 2$
  - $\cdots$?

Thank you for your attention!

Reference: F.–Lai, *Nielsen realization problem for derived automorphisms of generic K3 surfaces*, arXiv:2302.12663