(a)
$$\sum \frac{(-1)^n (n-1)}{n}$$
; (b) $\sum \frac{n^n}{(n+1)^{2n}}$; (c) $\sum \frac{(-1)^n}{n^{1/12}}$;

(d)
$$\sum \frac{1}{(2n-1)^2}$$
; (e) $\sum \frac{1}{n \log n}$; (f) $\sum ne^{-n^2}$.

(a)
$$\frac{diverge}{h}$$
: $\left(\frac{(-1)^n(n-1)}{h}\right)$ doesn't conv. to 0.

(b) Converge: voot test:
$$\lim_{n\to\infty} \left(\frac{n^n}{(n+1)^{2n}}\right) = \lim_{n\to\infty} \frac{n}{(n+1)^2} = 0 < 1$$
.

(d) Converge:
$$\left|\frac{1}{(2n-1)^2}\right| \leq \frac{1}{n^2}$$
 and $\left|\frac{1}{n^2}\right| < +\infty$.

(e) diverge: integral test:
$$\int_{2}^{\infty} \frac{dx}{x \log x} = \int_{\log 2}^{\infty} \frac{dt}{t} = +\infty$$

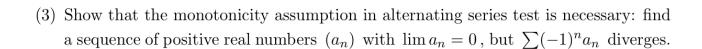
If): converge: not test:
$$\lim \sup_{n\to\infty} \left(\frac{n}{e^{n^2}}\right)^n = \lim \sup_{n\to\infty} \frac{n^n}{e^n} = 0 < 1$$
.

(2) Prove the triangle inequality for series: if $\sum a_n$ converges absolutely, then

$$\Big|\sum_{n=1}^{\infty} a_n\Big| \le \sum_{n=1}^{\infty} |a_n|.$$

By triangle inequality, we have:
$$\sum_{n=1}^{k} |a_n| \leq \sum_{n=1}^{k} |a_n| \leq \sum_{n=1}^{k} |a_n|$$
.

Take limit k > +00, we obtain:



Define (an) where:

•
$$(\alpha_{2n-1}) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots)$$

•
$$(\alpha_{2n}) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots)$$

Then ITM an =0.

But
$$\sum (-1)^n \alpha_n = -1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \frac{1}{6} - \cdots$$

= $-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots$ diverges.

(4) Let $(a_n^{(1)})_{n=1}^{\infty}, (a_n^{(2)})_{n=1}^{\infty}, \dots, (a_n^{(k)})_{n=1}^{\infty}$ denote k sequences of real numbers. (For instance, the first sequence is $(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}, \ldots)$.) Define another sequence $(b_n)_{n=1}^{\infty}$ where the *n*-th term is defined to be

$$b_n = a_n^{(1)} + a_n^{(2)} + \dots + a_n^{(k)}.$$

Suppose that the series $\sum_{n=1}^{\infty} a_n^{(i)}$ converges for each $i=1,2,\ldots,k$. Prove that (a) the series $\sum_{n=1}^{\infty} b_n$ also converges; moreover,

$$\sum_{n=1}^{\infty} b_n = \sum_{i=1}^{k} \left(\sum_{n=1}^{\infty} a_n^{(i)} \right).$$

This is a discrete version of Fubini's theore

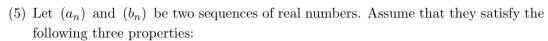
Let $\sum_{n=1}^{\infty} \alpha_n^{(i)} = A_i$ for each $1 \le i \le k$

4270, 416; sk, 3 N; >0

Let N := Max { N1, --, Nx} >0. Then Ymon, we have:

$$\left|\sum_{n=1}^{m}b_{n}-\sum_{i=1}^{k}A_{i}\right|=\left|\sum_{n=1}^{k}\left(\sum_{n=1}^{m}\alpha_{n}^{(i)}-A_{i}\right)\right|$$

$$\leq \sum_{i=1}^{k} \left| \sum_{n=1}^{m} a_n^{(i)} - A_i \right| < \epsilon$$
. Hence, $\sum_{n=1}^{\infty} b_n = \sum_{i=1}^{k} A_i$.



- (a) The partial sums of (b_n) is bounded: there exists L>0 such that $|s_k|=$ $|b_1 + \cdots + b_k| < L$ for any k;
- (b) $\lim a_n = 0$;
- (c) $\sum |a_{n+1} a_n|$ is convergent.

Prove that the series $\sum a_n b_n$ is convergent. This is known as *Abel's theorem*.

(Hint: Show that $\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} a_n (s_n - s_{n-1}) = \sum_{n=M}^{N-1} (a_n - a_{n+1}) s_n +$ $a_N s_N - a_M s_{M-1}$, then try to apply the assumptions.)

For any 270,

$$\left|\sum_{n=M}^{N} \alpha_n b_n\right| = \left|\sum_{n=M}^{N-1} (\alpha_n - \alpha_{n+1}) s_n + \alpha_N s_N - \alpha_M s_{M-1}\right|$$

$$\leq \sum_{n=M}^{N-1} |a_{n+1}-a_n| \cdot |b| + |a_N| \cdot |b| + |a_M| \cdot |b|$$

$$<\frac{\xi}{3}+\frac{\xi}{3}+\frac{\xi}{3}=\xi.$$

By Cauchy criterion, Early 3 therefore convergent.

(6) (optional; basic knowledge of complex numbers required) Show that the series

$$\sum \frac{\cos(n\theta)}{n}$$
 and $\sum \frac{\sin(n\theta)}{n}$

are convergent for any $0 < \theta < 2\pi$.

(Hint: Show that

$$\left(\sum_{n=1}^{N}\cos(n\theta)\right)+i\left(\sum_{n=1}^{N}\sin(n\theta)\right)=\sum_{n=1}^{N}e^{in\theta}=e^{i\theta}\frac{1-e^{iN\theta}}{1-e^{i\theta}}=e^{i(N+1)\theta/2}\frac{\sin(N\theta/2)}{\sin(\theta/2)}$$
 and use the previous problem.)

$$1 - e^{i\theta} = 1 - \cos\theta - i \sin\theta = 2 \sin^2(\frac{\theta}{5}) - i 2 \sin\frac{\theta}{5} \cos\frac{\theta}{5}$$

= $-2i \sin\frac{\theta}{5} \cdot e^{i\frac{\theta}{5}}$.

$$\begin{aligned} |\text{Hint} & \Rightarrow & \left| \sum_{n=1}^{N} \cos(n\theta) + i \sum_{n=1}^{N} \sin(n\theta) \right| \leq \frac{1}{\sin(\frac{\theta}{\delta})} \\ & \Rightarrow & \left| \sum_{n=1}^{N} \cos(n\theta) \right| \leq \frac{1}{\sin(\frac{\theta}{\delta})} \text{ and } \left| \sum_{n=1}^{N} \sin(n\theta) \right| \leq \frac{1}{\sin(\frac{\theta}{\delta})} \end{aligned}$$

$$|\text{Now, think of } \left(\cos(n\theta) \right) \text{ or } \left(\sin(n\theta) \right) \text{ as } \left(\sin(n\theta) \right) = \frac{1}{\sin(\frac{\theta}{\delta})}$$

Now, think of (astra)) or (sin(na)) as (bn) in #4; and let $(a_n = \frac{1}{n})$.

They satisfy the conditions in #4, hence $\sum \frac{cosn\theta}{n}$, $\sum \frac{stnn\theta}{n}$ conv.