* One problem (#10) has been added to HW 6.

Def: An <u>inner product space</u> is a pair $(V, \langle -,- \rangle)$, where:

- V is a real vector space.
- $\langle -,- \rangle : V \times V \longrightarrow \mathbb{R}$ (called <u>inner product</u>) s.t.:

1) $\langle \vec{v}_1, \vec{v}_2 \rangle = \langle \vec{v}_2, \vec{v}_1 \rangle \quad \forall \vec{v}_1, \vec{v}_2 \in V.$
2) $\langle \vec{v}_1 + \vec{v}_2, \vec{v}_3 \rangle = \langle \vec{v}_1, \vec{v}_3 \rangle + \langle \vec{v}_2, \vec{v}_3 \rangle \quad \forall \vec{v}_1, \vec{v}_2, \vec{v}_3 \in V.$
3) $\langle c\vec{v}_1, \vec{v}_2 \rangle = c \langle \vec{v}_1, \vec{v}_2 \rangle \quad \forall \vec{v}_1, \vec{v}_2 \in V,\ c \in \mathbb{R}$
4) $\langle \vec{v}, \vec{v} \rangle \geq 0$
5) $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$.

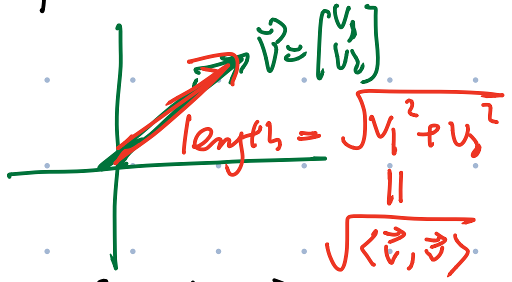
Rmk: Importance of inner product: one can define length of a vector, angle between two vectors, orthogonality, <u>w.r.t. an inner product.</u>

E.g. $V = \mathbb{R}^n$.

$$\langle \vec{v}, \vec{w} \rangle := v_1 w_1 + v_2 w_2 + \cdots + v_n w_n,$$

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

(Standard inner product on $\mathbb{R}^n$)



$$\langle \vec{v}, \vec{v} \rangle = \boxed{v_1^2 + \cdots + v_n^2} \geq 0$$

and $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$.

Rmk There are other inner products on $\mathbb{R}^n$. (HW)

Def $(V, \langle -,- \rangle)$ inner product space

Define length of $\vec{v} \in V$ (w.r.t. $\langle -,- \rangle$) to be:

$$\|\vec{v}\| := \sqrt{\langle \vec{v}, \vec{v} \rangle} \geq 0$$

Ex: $V = \mathcal{C}[a,b]$ = vector space of $\left\{ \begin{array}{l} \text{conti. fcns. on } [a,b] \\ f : [a,b] \to \mathbb{R} \end{array} \right\}$

For $f, g \in V$,

$$\langle f, g \rangle := \int_a^b f(x) g(x)\, dx.$$

$$\langle f, f \rangle = \int_a^b f(x)^2\, dx \geq 0$$

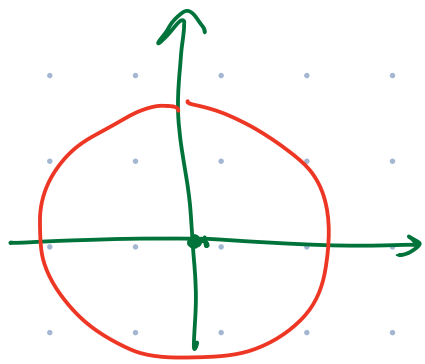
and $\langle f, f \rangle = 0$ if and only if $f \equiv 0$ (f is conti.)

Rmk

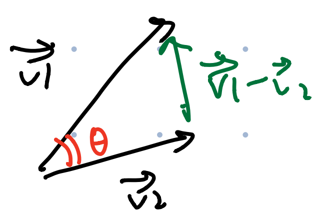
- $c \in \mathbb{R}$, $\|c\vec{v}\|$ $\quad |c|\,\|\vec{v}\|$

$$\|c\vec{v}\| = \sqrt{\langle c\vec{v}, c\vec{v} \rangle} = \sqrt{c^2 \langle \vec{v}, \vec{v} \rangle} = |c| \sqrt{\langle \vec{v}, \vec{v} \rangle} = |c|\,\|\vec{v}\|$$

- We say a vector $\vec{v} \in V$ is a <u>unit vector</u> if $\|\vec{v}\| = 1$.



- <u>distance</u> btw $\vec{v}_1$ and $\vec{v}_2$: $\|\vec{v}_1 - \vec{v}_2\|$



In $\mathbb{R}^n$: $\|\vec{v}_1-\vec{v}_2\|^2 = \|\vec{v}_1\|^2+\|\vec{v}_2\|^2 - \underline{2\|\vec{v}_1\|\|\vec{v}_2\|}\cos\theta$

$$2\|\vec{v}_1\|\|\vec{v}_2\|\cos\theta = \|\vec{v}_1\|^2+\|\vec{v}_2\|^2-\|\vec{v}_1-\vec{v}_2\|^2$$

$$= \cancel{\langle\vec{v}_1,\vec{v}_1\rangle} + \cancel{\langle\vec{v}_2,\vec{v}_2\rangle} - \langle\vec{v}_1-\vec{v}_2,\vec{v}_1-\vec{v}_2\rangle$$

$$\langle\vec{v}_1,\vec{v}_1-\vec{v}_2\rangle - \langle\vec{v}_2,\vec{v}_1-\vec{v}_2\rangle$$

$$\langle\vec{v}_1,\vec{v}_1\rangle - \langle\vec{v}_1,\vec{v}_2\rangle - \langle\vec{v}_2,\vec{v}_1\rangle + \langle\vec{v}_2,\vec{v}_2\rangle$$

$$\cancel{\langle\vec{v}_1,\vec{v}_1\rangle} - 2\langle\vec{v}_1,\vec{v}_2\rangle + \cancel{\langle\vec{v}_2,\vec{v}_2\rangle}$$

$$= 2\langle\vec{v}_1,\vec{v}_2\rangle$$

$$\Rightarrow \cos\theta = \frac{\langle\vec{v}_1,\vec{v}_2\rangle}{\|\vec{v}_1\|\|\vec{v}_2\|}$$

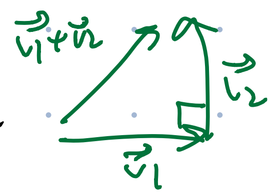
$$\Rightarrow \theta = \arccos\left(\frac{\langle\vec{v}_1,\vec{v}_2\rangle}{\|\vec{v}_1\|\|\vec{v}_2\|}\right)$$

In particular, $\theta = \frac{\pi}{2} \Leftrightarrow \cos\theta = 0 \Leftrightarrow \langle\vec{v}_1,\vec{v}_2\rangle = 0$

Def $(V,\langle -,-\rangle)$ inner product space.
We say $\vec{v}_1,\vec{v}_2\in V$ are <u>orthogonal</u> if $\langle\vec{v}_1,\vec{v}_2\rangle = 0$

Pythagorean thm If $\vec{v}_1,\vec{v}_2$ are orthogonal, then

$$\|\vec{v}_1\|^2+\|\vec{v}_2\|^2 = \|\vec{v}_1+\vec{v}_2\|^2$$



$$\langle \vec{v}_1+\vec{v}_2, \vec{v}_1+\vec{v}_2\rangle$$
$$\|$$
$$\langle \vec{v}_1, \vec{v}_1\rangle + \cancel{2\langle \vec{v}_1, \vec{v}_2\rangle}^{\,0} + \langle \vec{v}_2, \vec{v}_2\rangle$$

Since orthogonal

Def $(V, \langle -,-\rangle)$ inner product space

$W \subseteq V$ subspace

$(\Rightarrow (W, \langle -,-\rangle)$ is an inner product space$)$

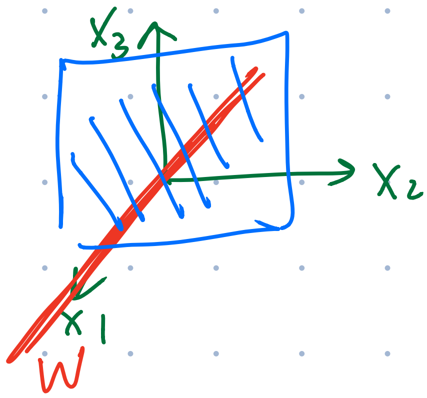
The orthogonal complement $W^\perp \subseteq V$ of W:

$$W^\perp := \{\vec{x} \in V \mid \langle \vec{x}, \vec{w}\rangle = 0 \text{ for all } \vec{w} \in W\}$$

e.g. $V = \mathbb{R}^3$

$W = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

$W^\perp = ??\quad \text{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$



$\begin{bmatrix}x\\y\\z\end{bmatrix} \in$

Rmk: HW: $W^\perp$ is a subspace of V,

$$W \cap W^\perp = \{\vec{0}\}$$

e.g. A: $m \times n$

$$(\text{Col}(A))^\perp = \text{Nul}(A^T)$$

$$(\text{Row}(A))^\perp = \text{Nul}(A)$$

$$\begin{bmatrix} \text{---} \; v_1 \; \text{---} \\ \text{---} \; v_2 \; \text{---} \\ \vdots \\ \text{---} \; v_m \; \text{---} \end{bmatrix} \begin{bmatrix} \\ \vec{w} \\ \\ \end{bmatrix} = \begin{bmatrix} \langle \vec{v}_1, \vec{w} \rangle \\ \langle \vec{v}_2, \vec{w} \rangle \\ \vdots \\ \langle \vec{v}_m, \vec{w} \rangle \end{bmatrix}$$

$$\vec{w} \in (\text{Row } A)^{\perp} \iff \langle \vec{v}_1, \vec{w} \rangle = \cdots = \langle \vec{v}_n, \vec{w} \rangle = 0$$

$$\iff A\vec{w} = \vec{0}$$

$$\iff \vec{w} \in \text{Nul}(A)$$

Def $(V, \langle \cdot, - \rangle)$

Say $\{\vec{v}_1, \dots, \vec{v}_n\}$ $\boxed{\text{nonzero}}$ is an orthogonal set if

$$\langle \vec{v}_i, \vec{v}_j \rangle = 0 \quad \forall\, 1 \le i < j \le n.$$

Thm Any orthogonal set is l.i.

$\{\vec{v}_1, \dots, \vec{v}_n\}$ orthogonal set.

pf

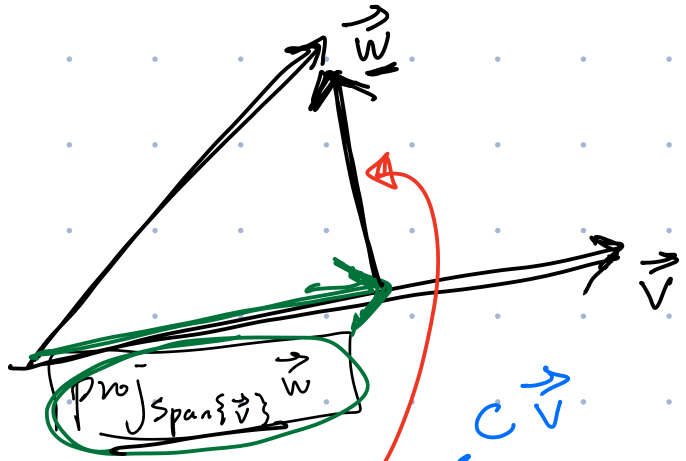
$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$$

$$0 = \langle \vec{0}, \vec{v}_1 \rangle = \langle a_1\vec{v}_1 + \cdots + a_n\vec{v}_n, \vec{v}_1 \rangle$$

$$= a_1 \langle \vec{v}_1, \vec{v}_1 \rangle + a_2 \underbrace{\cancel{\langle \vec{v}_2, \vec{v}_1 \rangle}}_{0} + \cdots + a_n \underbrace{\cancel{\langle \vec{v}_n, \vec{v}_1 \rangle}}_{0}$$

$$= a_1 \underbrace{\langle \vec{v}_1, \vec{v}_1 \rangle}_{\substack{\vee \\ 0}}$$

$$\Rightarrow a_1 = 0. \quad \text{Similarly, } a_2 = \cdots = a_n = 0. \quad \square$$



$c\,\vec{v}$

$$\left\langle \vec{w} - \text{proj}_{\text{span}\{\vec{v}\}} \vec{w},\ \vec{v} \right\rangle = 0$$

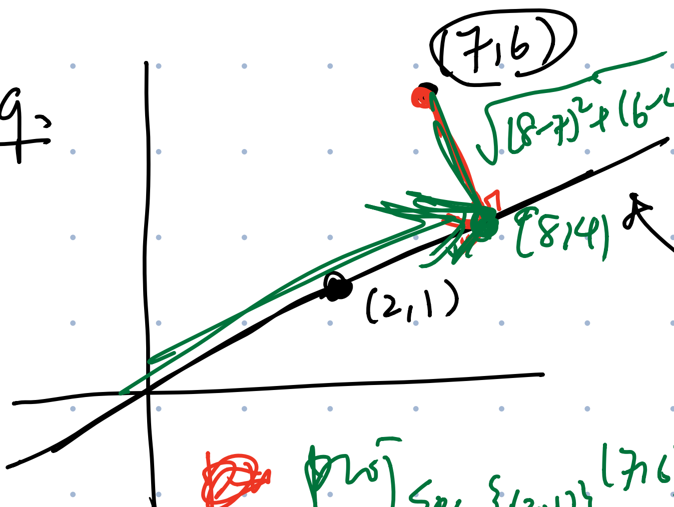
$$\| $$

$$\langle \vec{w}, \vec{v} \rangle - c \langle \vec{v}, \vec{v} \rangle$$

$$\Rightarrow \quad c = \frac{\langle \vec{w}, \vec{v} \rangle}{\|\vec{v}\|^2}$$

$$\text{proj}_{\text{span}\{\vec{v}\}} \vec{w} = \frac{\langle \vec{w}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}$$

e.g.



Compute distance from (7,6) to the line

$$\text{proj}_{\text{span}\{(2,1)\}}(7,6) = \frac{\left\langle \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}7\\6\end{bmatrix} \right\rangle}{2^2+1^2} \begin{bmatrix}2\\1\end{bmatrix}$$

$$= \frac{20}{5} \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}8\\4\end{bmatrix}$$

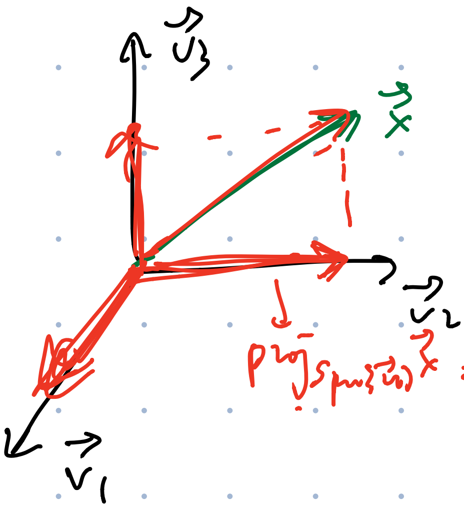
Thm $(V, \langle -,- \rangle)$, $\{\vec v_1, \dots, \vec v_n\}$ orthogonal basis

(i.e. it's an orthogonal set, and a basis of V)

Then $\forall \vec x \in V$, we have

$$\vec x = a_1 \vec v_1 + \dots + a_n \vec v_n,$$

where $a_i = \dfrac{\langle \vec v_i, \vec x \rangle}{\|\vec v_i\|^2}$



$$\text{proj}_{\text{span}\{\vec v_2\}} \vec x = \frac{\langle \vec v_2, \vec x \rangle}{\|\vec v_2\|^2} \vec v_2$$

Rmk This is not true if $\{\vec v_1, \dots, \vec v_n\}$ is not orthogonal

pf: $\vec x - (a_1 \vec v_1 + \dots + a_n \vec v_n) = \vec y$

Want to prove: $\vec y = \vec 0$

$$\langle \vec y, \vec v_1 \rangle = \langle \vec x - (a_1 \vec v_1 + \dots + a_n \vec v_n), \vec v_1 \rangle$$

$$= \langle \vec x, \vec v_1 \rangle - a_1 \langle \vec v_1, \vec v_1 \rangle - a_2 \underbrace{\langle \vec v_2, \vec v_1 \rangle}_{0} - \dots$$

$$\frac{\langle \vec v_1, \vec x \rangle}{\|\vec v_1\|^2}$$

$$= 0$$

Similarly, $\boxed{\langle \vec{y}, \vec{v}_i \rangle = 0}$ $\forall i$.

$$\vec{y} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$$

$$\langle \vec{y}, \underbrace{b_1\vec{v}_1 + \cdots + b_n\vec{v}_n}_{\vec{y}} \rangle = 0 \Rightarrow \vec{y} = \vec{0}. \quad \square$$

Def Say $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal set if it's an orthogonal set and $\|\vec{v}_i\| = 1$ $\forall i$.

Rmk If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthogonal set, then $\left\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \ldots, \frac{\vec{v}_n}{\|\vec{v}_n\|}\right\}$ is orthonormal.

Thm U: $m \times n$ has orthonormal columns $\Leftrightarrow$ $U^T U = I_n$

pf

$$\underbrace{\begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix}}_{U^T} \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}}_{U} = \begin{bmatrix} \overset{1}{\langle \vec{v}_1, \vec{v}_1 \rangle} & 0 \\ \langle \vec{v}_2, \vec{v}_1 \rangle & \\ \vdots & \\ 0 & \underset{}{\overset{1}{\langle \vec{v}_n, \vec{v}_n \rangle}} \end{bmatrix} = I_n$$

Thm $U : m\times n$, $U^T U = I_n$. Then

$\forall \vec{x}, \vec{y} \in \mathbb{R}^n$, we have:

$$T_U : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\langle U\vec{x}, U\vec{y}\rangle_{\mathbb{R}^m} = \langle \vec{x}, \vec{y}\rangle_{\mathbb{R}^n}$$

$\|U\vec{x}\| = \|\vec{x}\|$

$\langle \vec{x}, \vec{y}\rangle = 0 \Rightarrow \langle U\vec{x}, U\vec{y}\rangle = 0$

pf $\boxed{\langle \vec{x}, \vec{y}\rangle = \vec{x}^T \vec{y}}$

$$[x_1 \cdots x_n]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + \cdots + x_ny_n$$

$$\langle U\vec{x}, U\vec{y}\rangle = (U\vec{x})^T (U\vec{y})$$

$$= \vec{x}^T \underbrace{U^T U}_{I_n} \vec{y} = \vec{x}^T \vec{y} = \langle \vec{x}, \vec{y}\rangle. \quad \square$$

Def A square matrix A ($n\times n$) is called orthogonal if

$A^T A = I_n$, i.e. the columns of A is an orthonormal set

$\Updownarrow$

$A^{-1} = A^T$

$\Rightarrow$ $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ respects the inner product on $\mathbb{R}^n$

$$\langle \vec{x}, \vec{y}\rangle = \langle A\vec{x}, A\vec{y}\rangle$$