**14.** Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$, and $a \neq 0$.

[Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.]

- consider the fun $g(z) := f(1/z)$, which is holo. on $\mathbb{C} \setminus \{0\}$.
- $g$ has an isolated singularity at $z=0$.

<u>Claim 1</u>: The sing. at $z=0$ is not essential.

pf: Suppose that $0$ is an essential sing. of $g(z)$.

- By Casorati-Weierstrass, the image of any neighborhood of $0$ under $g$ is dense in $\mathbb{C}$.
- Take $\mathbb{D}_1^\times(0) = \{z \in \mathbb{C} : 0 < |z| < 1\}$.
  Then $g(\mathbb{D}_1^\times(0)) \subseteq \mathbb{C}$ is dense.
- Let $R := \{z \in \mathbb{C} : |z| > 1\} \subseteq \mathbb{C}$.
  Then $f(R) \subseteq \mathbb{C}$ is dense since $g(z) = f(1/z)$.
- Now consider $\mathbb{D}_1(0) = \{z \in \mathbb{C} : |z| < 1\}$,
  we have $\mathbb{D}_1(0) \cap R = \phi$.
- By open mapping thm., $f(\mathbb{D}_1(0)) \subseteq \mathbb{C}$ is open.
- Since $f(R) \subseteq \mathbb{C}$ is dense, we have:

$$f(R) \cap f(\mathbb{D}_1(0)) \neq \phi.$$

This contradicts with the assumption that $f$ is injective. $\square$

<u>Claim 2</u>: The sing. at $z=0$ of $g$ is not removable.

pf: Assume it's a removable singularity.

Then $g$ is bounded on $\overline{\mathbb{D}_1(0)} = \{z \in \mathbb{C} : |z| \leq 1\}$.

Hence $f$ is bounded on $\overline{R} = \{z \in \mathbb{C} : |z| \geq 1\}$.

Therefore $f$ is bounded on $\mathbb{C}$.

By Liouville thm, $f$ is constant, contradict with injectivity $\square$

<u>Therefore</u>: $g(z) = f(1/z)$ has a <u>pole</u> at $z=0$.

- Near $z=0$, $g(z)$ can be written as:

$$g(z) = \underbrace{\frac{a_{-n}}{z^n} + \cdots + \frac{a_{-1}}{z}}_{g_{prin}(z)} + \underbrace{a_0 + a_1 z + \cdots}_{\text{holo. in } \overline{\mathbb{D}_\delta(0)} \text{ for some } \delta>0.}$$

- Consider $\tilde{f}(z) := f(z) - g_{prin}(1/z)$.

$$= f(z) - \underbrace{\left( a_{-n} z^n + a_{-n+1} z^{n-1} + \cdots + a_{-1} z \right)}_{\text{entire}}$$

<u>Claim</u>: $\tilde{f}$ is bounded on $\mathbb{C}$

<u>pf</u>: It suffices to show that $\tilde{f}$ is bounded on $\{z: |z| \geq 1/\delta\}$,

or equivalently $\tilde{f}(1/z)$ is bounded on $\{z: |z| \leq \delta\}$.

$$\tilde{f}(1/z) = f(1/z) - g_{prin}(z)$$

$= g(z) - g_{prin}(z)$ is holo. on $\{z: |z| \leq \delta\}$,

therefore bounded since $\{z: |z| \leq \delta\}$ is compact. $\square$

By Liouville thm, $\tilde{f}$ is a const. fcn.

Therefore, $f$ is a polynomial.
Note that any poly. of order $>1$ is not injective.
This concludes the proof. $\square$

**16.** Suppose $f$ and $g$ are holomorphic in a region containing the disc $|z| \leq 1$. Suppose that $f$ has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leq 1$. Let

$$f_\epsilon(z) = f(z) + \epsilon g(z).$$

Show that if $\epsilon$ is sufficiently small, then

(a) $f_\epsilon(z)$ has a unique zero in $|z| \leq 1$, and

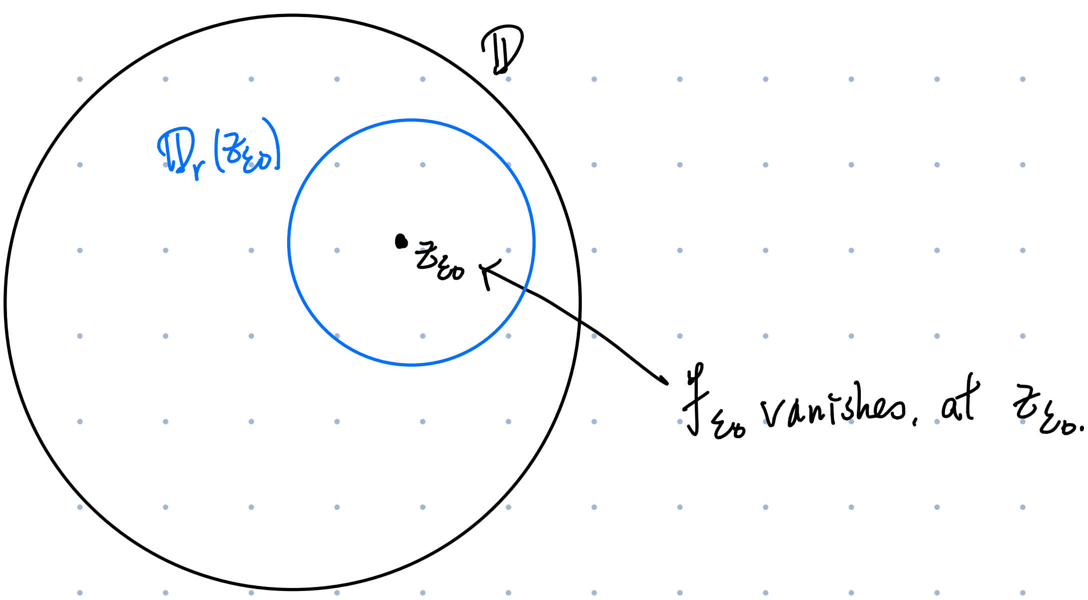
(b) if $z_\epsilon$ is this zero, the mapping $\epsilon \mapsto z_\epsilon$ is continuous.

(a) Since $f, g$ are conti. on $\partial\mathbb{D}$ and $f$ is nowhere vanishing on $\partial\mathbb{D}$, we can take $\varepsilon > 0$ small enough s.t.

$$\min_{\partial\mathbb{D}} |f| > \varepsilon \max_{\partial\mathbb{D}} |g(z)|.$$

Then Rouché's thm gives the result. $\square$

(b) For any such $\varepsilon_0 > 0$, and any $r > 0$ (s.t. $\mathbb{D}_r(z_{\varepsilon_0}) \subseteq \mathbb{D}$).



- $f_{\varepsilon_0}$ doesn't vanish on $\partial\mathbb{D}_r(z_{\varepsilon_0})$, by the same argument as (a), $\exists\,\delta>0$ s.t.

$$\min_{\partial\mathbb{D}_r(z_0)} |f_{\varepsilon_0}| > \delta \max_{\partial\mathbb{D}_r(z_0)} |g|.$$

- Hence $f_{\varepsilon_0} + \delta' g = f_{\varepsilon_0+\delta'}$ has a zero in $\mathbb{D}_r(z_{\varepsilon_0})$ $\forall |\delta'|<\delta$;

and the zero must be $z_{\varepsilon_0+\delta'}$ since $f_{\varepsilon_0+\delta'}$ has only one zero in $\mathbb{D}$.

- So we showed that $\forall r>0$, $\exists\,\delta>0$

s.t. $|\varepsilon-\varepsilon_0|<\delta \Longrightarrow |z_\varepsilon - z_{\varepsilon_0}| < r$,

i.e. the map $\varepsilon \longmapsto z_\varepsilon$ is continuous. $\square$

**17.** Let $f$ be non-constant and holomorphic in an open set containing the closed unit disc.

(a) Show that if $|f(z)|=1$ whenever $|z|=1$, then the image of $f$ contains the unit disc. [Hint: One must show that $f(z)=w_0$ has a root for every $w_0\in\mathbb{D}$. To do this, it suffices to show that $f(z)=0$ has a root (why?). Use the maximum modulus principle to conclude.]

(b) If $|f(z)|\geq 1$ whenever $|z|=1$ and there exists a point $z_0\in\mathbb{D}$ such that $|f(z_0)|<1$, then the image of $f$ contains the unit disc.

(a) Claim: $f(z)=0$ for some $z\in\mathbb{D}=\mathbb{D}_1(0)$.

pf: Suppose not. Then $g(z) := \dfrac{1}{f(z)}$ is holo. in some neighborhood of $\overline{\mathbb{D}}$.

- By max. modulus principle, $|g(z_0)| < \max_{\partial\mathbb{D}} |g| = 1$ $\forall z_0\in\mathbb{D}$,

since $f$ is nonconst.

- On the other hand, $|f(z_0)| < \max_{\partial D} |f| = 1$ $\forall z_0 \in \mathbb{D}$

again by max. principle. Contradiction. $\square$

- $\forall\, w_0 \in \mathbb{D}$, we have

$$|f(z)| = 1 > |w_0| \quad \forall z \in \partial\mathbb{D}.$$

By Rouché, # of zeros of $f$ in $\mathbb{D}$ = # of zeros of $f - w_0$ in $\mathbb{D}$

- By the claim, $f(z) = w$ has at least one sol$^{n}$ in $\mathbb{D}$. $\square$

(b) the same idea as (a) works. $\square$

(A) How many roots does $p(z) = z^4 - 6z + 3 = 0$ have in the annulus $1 < |z| < 2$?

$\underline{\underline{3}}$

- On $|z| = 1$,

$$|6z| = 6,$$
$$|z^4 + 3| \leq 4.$$

and $6z$ has one root in $\mathbb{D}_1(0)$.

- On $|z| = 2$,

$$|z^4| = 16.$$
$$|-6z + 3| \leq 12 + 3 = 15,$$

and $z^4$ has 4 roots in $\mathbb{D}_2(0)$. $\square$