#1: Let A be an orthogonal matrix. then $A^{-1} = A^T$.

$\Rightarrow \det(A) = \pm 1$.

$\Rightarrow$
$$\begin{aligned}
\lambda^n p_A\left(\frac{1}{\lambda}\right) &= \lambda^n \cdot \det\left(A - \frac{1}{\lambda} I\right) \\
&= \det(\lambda A - I) \\
&= \pm \det(\lambda A^T A - A^T) \\
&= \pm \det(A^T - \lambda I) \\
&= \pm \det(A - \lambda I) \\
&= \pm p_A(\lambda). \quad \square
\end{aligned}$$

#2: Let λ_1, λ_2 be eigenvalues of A.

then $\det(A) = \lambda_1 \lambda_2 = ad - b^2 \neq 0$.

$\operatorname{tr}(A) = \lambda_1 + \lambda_2 = a + d$.

- If $\det(A) > 0$, and $a > 0$,

  $\Rightarrow$ • $ad > b^2 \geq 0 \Rightarrow d > 0$.

  • $\lambda_1 \lambda_2 > 0$,

  $\lambda_1 + \lambda_2 = a + d > 0$.

  $\Rightarrow \lambda_1, \lambda_2 > 0$, hence A is positive definite.

- If $\det(A) > 0$ and $a < 0$,

  $\Rightarrow ad > b^2 \geq 0 \Rightarrow d < 0$.

$\Rightarrow \lambda_1 \lambda_2 > 0$ and $\lambda_1 + \lambda_2 < 0$.

$\Rightarrow \lambda_1, \lambda_2 < 0$, hence A is negative definite

- if $\det(A) < 0$, then $\lambda_1 \lambda_2 < 0$,

  so A is indefinite. $\square$

#3:

(a) $\vec{x}^T B^T B \vec{x} = \langle B\vec{x}, B\vec{x} \rangle \geq 0 \quad \forall \vec{x}$.

(b) If $\vec{x} \neq \vec{0}$, then $B\vec{x} \neq \vec{0}$ since B is invertible,

hence $\vec{x}^T B \vec{x} = \langle B\vec{x}, B\vec{x} \rangle > 0$. $\square$

#4: Let $A = PDP^T$ be an orthogonal diagonalization.

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$ where $\lambda_1, \dots, \lambda_n > 0$.

Let $C = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$, and $B = PCP^T$. $\square$

#5: $\forall \vec{x} \neq \vec{0}$, then $\vec{x}^T A \vec{x} > 0$, $\vec{x}^T B \vec{x} > 0$,

hence $\vec{x}^T (A+B) \vec{x} > 0$. $\square$

#6: $A = PDP^T$, where $P^T = P^{-1}$, $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $\lambda_1, \dots, \lambda_n > 0$.

$$\Rightarrow A^{-1} = P \begin{bmatrix} 1/\lambda_1 & & \\ & \ddots & \\ & & 1/\lambda_n \end{bmatrix} P^T.$$ positive definite.

#7: $A = PDP^T$ orthogonal diagonalization.

Then

$$\min_{\vec{x} \neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \min_{\vec{x} \neq \vec{0}} \frac{\vec{x}^T P D P^T \vec{x}}{\|\vec{x}\|^2}.$$

$$= \min_{\vec{x} \neq \vec{0}} \frac{(P^T \vec{x})^T D (P^T \vec{x})}{\|P^T \vec{x}\|^2} = \min_{\vec{y} \neq \vec{0}} \frac{\vec{y}^T D \vec{y}}{\|\vec{y}\|^2}.$$

since P is orthogonal,

since P^T is invertible

$$= \min_{(y_1, \dots, y_n) \neq (0, \dots, 0)} \frac{\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2}{y_1^2 + \cdots + y_n^2} = \lambda_1,$$

Similarly, $$\max_{\vec{x} \neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \lambda_n. \quad \square$$

#8:

$$A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^T, \quad P^T = P^{-1}.$$

Let $\lambda_1, \dots, \lambda_k$ be the nonzero eigenvalues,
Then

$$\text{rank}(A) = k.$$

$$\text{tr}(A) = \lambda_1 + \dots + \lambda_k.$$

$$\text{tr}(A^2) = \lambda_1^2 + \dots + \lambda_k^2.$$

Cauchy-Schwartz $\Rightarrow$ $k(\lambda_1^2 + \dots + \lambda_k^2) \geq (\lambda_1 + \dots + \lambda_k)^2$. $\square$

#9: • $(A^2)^T = A^T A^T = (-A)(-A) = A^2$.

• $\vec{x}^T A^2 \vec{x} = -\vec{x}^T (A^T A) \vec{x} \leq 0$. $\square$

#10: By #9, $I - A^2$ is positive definite

$\Rightarrow$ $\det(I - A^2) \neq 0$.

$$\det(I - A^2) = \det(I - A)\det(I + A). \quad \square$$