$$\begin{cases} u_t(x,t) = \beta u_{xx}(x,t), & \forall x \in (0,L),\ t>0. \quad \text{(Heat eq}^{\text{n}}\text{)} \\ u(0,t) = u(L,t) = 0 & \forall t>0. \\ u(x,0) = f(x) & \forall x \in (0,L) \end{cases}$$

$\boxed{\text{Recap}}$

- If $f(x) = \sum_n c_n \sin\left(\frac{n\pi x}{L}\right)$.

  then $u(x,t) = \sum_n c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$ is the sol$^{\text{n}}$.

- In general, $f(x)$ can't be written as a finite sum of these sine fcns. But we can find an (infinite) sequence $\{c_n\} \subseteq \mathbb{R}$ s.t.

$$\lim_{N\to+\infty} \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \qquad \forall x \in (0,L)$$

(provided f is continuous)., using Fourier series.

Def: Let f be a piecewise conti. fcn. on $[-L,L]$, the Fourier series of f: is:

$$\tilde{f}(x) := \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}\right),$$

where
$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}\,dx, \quad k \geq 0,$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx. \quad k \geq 1.$$

Recall: $\left\{ \frac{1}{\sqrt{2}}, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \sin\frac{2\pi x}{L}, \cdots \right\}.$

is an orthonormal basis, w.r.t. $\langle f, g\rangle := \frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx$.

The components in $\tilde{f}$: $\left\{ \frac{a_0}{2},\ a_1 \cos\frac{\pi x}{L},\ b_1 \sin\frac{\pi x}{L},\ a_2 \cos\frac{2\pi x}{L}, \dots \right\}$

are the orthogonal projection of f onto these basis vectors.

Final — Standard computational problems

① different eq[ns] 3 problems × 10 pts

max. for part ① is 20 pts

② linear alg. (80 pts)

9 problem

(you only need to answer correctly 6 / 7 problems to get 80 pts)

$\swarrow$ 14 $\quad \downarrow$ 12

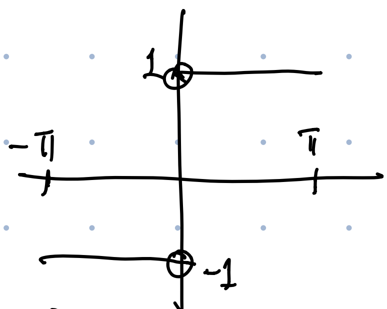
24 hrs window

or

12/16 3-6pm PST

Choose some problem on PDEs from textbook (Yu-Wei)

e.g. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

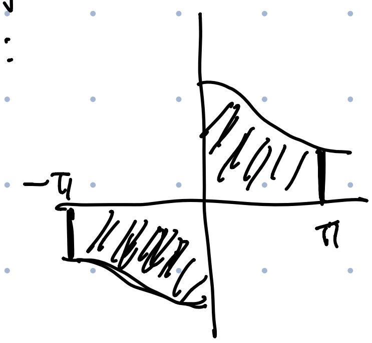


odd fcn: $f(x) = -f(-x)$

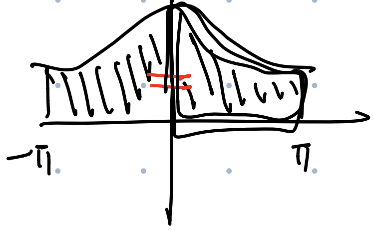
even fcn: $f(x) = f(-x)$

Compute the Fourier series of f on $[-\pi, \pi]$:

$$\Rightarrow a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overset{\text{odd}}{f(x)}\ \overset{\text{even}}{\cos(kx)}}_{\text{odd fcn.}}\,dx = 0$$



$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overset{\text{odd}}{f(x)}\ \overset{\text{odd}}{\sin(kx)}}_{\text{even}}\,dx.$$



$$= \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin(kx)\, dx.$$

$$\cancel{\left.\frac{-\cos(kx)}{k}\right|_0^{\pi}}$$

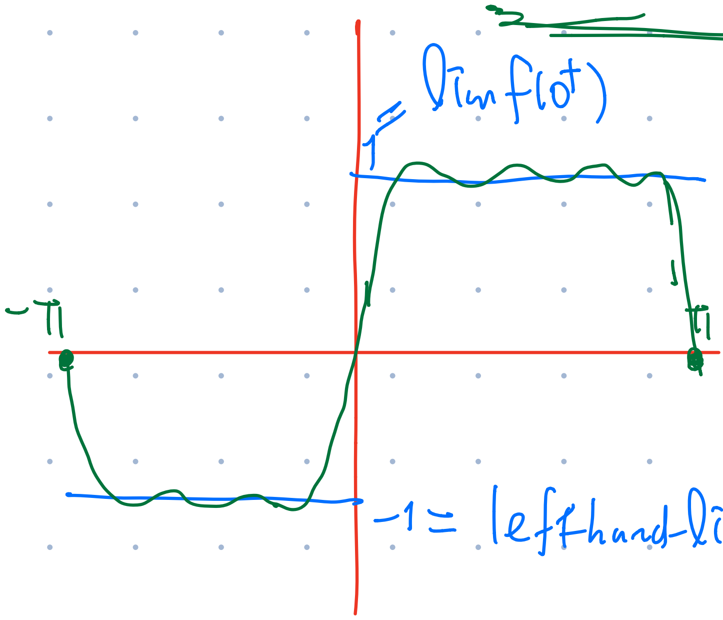
$$= \frac{2}{\pi}\left(\frac{\overbrace{-\cos(k\pi)}^{\begin{cases}-1, & k \text{ even} \\ 1, & k \text{ odd.}\end{cases}}}{k} - \frac{\overbrace{-\cos(k\cdot 0)}^{-1}}{k} \right)$$

$$= \begin{cases} 0 & k \text{ even} \\ \frac{4}{k\pi} & k \text{ odd.} \end{cases}$$

$\Rightarrow$ The Fourier series of f on $[-\pi, \pi]$ is:

$$\hat{f}(x) = \sum_{\substack{k \geq 1 \\ k \text{ odd}}} \frac{4}{k\pi} \sin(kx)$$

$$= \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots \right)$$



$-1 =$ lefthand limit of f at $0 = \lim f(0^-)$

Thm If f, f' piecewise conti, on $[-L, L]$ then

- If $x \in (-L, L)$, then $\lim_{N\to\infty}\left(\frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)\right) = \frac{1}{2}\left(f(x^-) + f(x^+)\right)$
- if $x = \pm L$, $\quad = \frac{1}{2}\left(f(-L^+) + f(L^-)\right)$

Ex: $f(x) = x^2$. Compute $\tilde{f}$ on $[-\pi, \pi]$

We'll get: $\quad \tilde{f}(x) = \boxed{\frac{\pi^2}{3} + \sum_{n\geq 1} \frac{(-1)^n}{n^2} \cos(nx)} = f(x) = x^2$

One can use the conv. thm. to prove Euler's formula: $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$

Obs: ① If f is an odd fcn on $[-L, L]$,

then $\tilde{f}$ only consists of $\sin\frac{k\pi x}{L}$

Recall: In heat eq[12], we're given a fcn $f(x)$ on $(0, L)$,

we'd like to write $f = \sum c_k \sin\frac{k\pi x}{L}$

[1] "odd extension" Define

$$f_o(x) := \begin{cases} f(x) & \text{if } x \in (0, L) \\ -f(-x) & \text{if } x \in (-L, 0) \end{cases}$$

Then f_o is automatically an odd fcn on $(-L, 0) \cup (0, L)$

Then we compute the Fourier series of f_o on $(-L,L)$

$$a_k = \frac{1}{L}\int_{-L}^{L} \overset{\text{odd}}{f_o(x)}\, \overset{\text{even}}{\cos\frac{k\pi x}{L}}\, dx = 0$$

$$b_k = \frac{1}{L}\int_{-L}^{L} \overset{\text{odd}}{f_o(x)}\, \overset{\text{odd}}{\sin\frac{k\pi x}{L}}\, dx = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\, dx$$

$$\Rightarrow \tilde{f}(x) = \sum_{k\geq 1} b_k \sin\frac{k\pi x}{L}$$

Summary: If f is conti. on $(0,L)$,

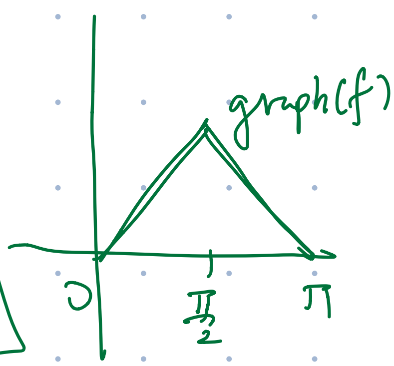
then $$\sum_{k\geq 1} b_k \sin\frac{k\pi x}{L} = f(x) \quad \forall\, x\in(0,L)$$

where $$b_k := \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\, dx.$$

$$\Rightarrow u(x,t) = \sum_n b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$ solves the heat eq'n.

---

e.g.
$$\begin{cases} u_t = \overset{\beta}{2}\, u_{xx}, & 0<x<\pi,\quad t>0 \\ u(0,t) = u(\pi,t) = 0, & t>0 \\ u(x,0) = \boxed{f(x) = \begin{cases} x, & 0<x<\frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2}<x<\pi \end{cases}} \end{cases}$$



To solve this, we only need to compute

$$u(x,t) = \sum_k \frac{4}{\pi k^2}\sin\left(\frac{k\pi}{2}\right)\sin(kx)\, e^{-2k^2 t}$$

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx. = \boxed{\frac{4}{\pi k^2}\sin\left(\frac{k\pi}{2}\right)}$$

$$= \frac{2}{\pi}\left(\int_0^{\pi/2} x\sin(kx)\,dx + \int_{\pi/2}^{\pi}(\pi-x)\sin(kx)\,dx\right)$$

$$\frac{-\cos(kx)}{k}$$

$$x\,\frac{-\cos(kx)}{k}\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{-\cos(kx)}{k}\,dx$$

$$(\pi-x)\frac{-\cos(kx)}{k}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$-\frac{\pi}{2}\,\frac{\cos\left(\frac{k\pi}{2}\right)}{k} + \frac{1}{k}\int_0^{\frac{\pi}{2}}\cos(kx)\,dx$$

$$-\int_{\frac{\pi}{2}}^{\pi} \frac{\cos(kx)}{k}\,dx$$

$$\frac{\sin(kx)}{k}\Big|_0^{\frac{\pi}{2}}$$

$$\frac{\sin\left(\frac{k\pi}{2}\right)}{k}$$

$$\cancel{\frac{\pi}{2}\,\frac{\cos\left(k\frac{\pi}{2}\right)}{k}}$$

$$\|$$

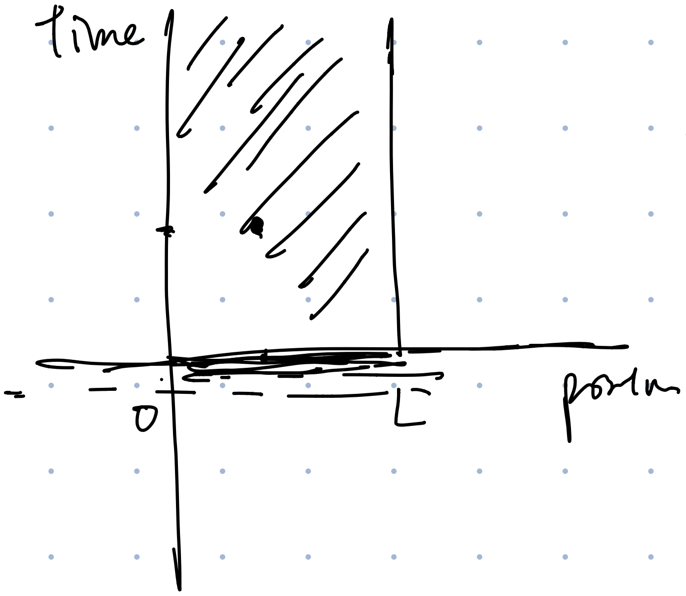
$$\cancel{-\frac{\pi\cos\left(\frac{k\pi}{2}\right)}{2k}} + \frac{\sin\left(\frac{k\pi}{2}\right)}{k^2}$$

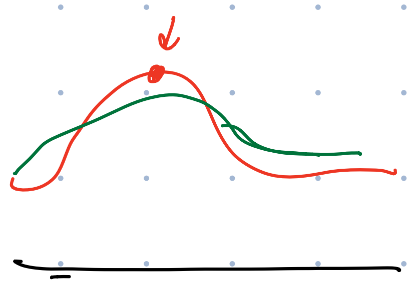
$$-\frac{1}{k}\,\frac{\sin(kx)}{k}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$\frac{1}{k^2}\sin\left(\frac{k\pi}{2}\right) =$$

Max. principle for heat eq^{n}: If $u(x,t)$ is a sol^{n}, then

$$\max_{\substack{t\ge 0\\ 0\le x\le L}} u(x,t) = \max_{0\le x\le L} u(x,0)$$





We can use max. principle to prove the uniqueness:

$$\begin{cases} u_t = \beta u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Suppose u_1, u_2 are sol^{n}s

$\boxed{w = u_1 - u_2}$

Apply max. principle

$\Downarrow$

$$\boxed{\begin{cases} w_t = \beta w_{xx} \\ w(0,t) = w(L,t) = 0 \\ \boxed{w(x,0) = 0} \end{cases}}$$

$\underline{W(x,t) \le 0 \quad \forall x, \forall t}$

$$\Downarrow$$

$$\boxed{u_1(x,t) \le u_2(x,t) \quad \forall x \quad \forall t}$$

Switch the role of u_1 & u_2 $\Rightarrow$ $u_1 \equiv u_2$.

---

e.g.

Variation of Boundary condⁿ.

$$\begin{cases} u_t = \beta u_{xx} & t>0, 0<x<L \\ u_x(0,t) = u_x(L,t) = 0, & t>0 \\ u(x,0) = f(x), & 0<x<L \end{cases}$$

(heat doesn't go in/out at the ends)

Again, try separation of variables, suppose

$$u(x,t) = X(x)T(t)$$

Again, $u_t = \beta u_{xx}$ will imply:

$$\begin{cases} \boxed{X''(x) + \lambda X(x) = 0} \\ T'(t) + \beta\lambda T(t) = 0 \end{cases}$$

The boundary condⁿ $\Rightarrow$ $\boxed{X'(0) = X'(L) = 0.}$

(Last time: $X(0) = X(L) = 0$)

• If $\lambda < 0$, then $r^2 + \lambda = 0$ has 2 distinct real roots r_1, r_2

$$X(x) = \ast\, e^{r_1 x} + \ast\, e^{r_2 x}$$

$$\begin{cases} 0 = \underline{X'(0)} = \ast\, r_1 + \ast\, r_2 \\ 0 = X'(L) = \end{cases}$$

$\boxed{\text{To be continued ...}}$