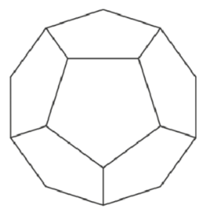
PROJECTS OF ALGEBRAIC COMBINATORICS II

This is a preliminary version of the research projects, subject to change later.

Project 1. Let D be a dodecahedron which centered at the origin of $\vec{0} \in \mathbb{R}^3$.

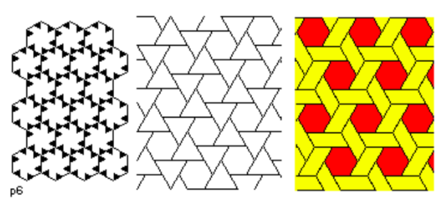


Project 1: Part (A). Find the symmetry group $\mathrm{Aut}(D)$ (which is a subgroup of $\mathrm{O}(3, \mathbb{R})$) and the rotational symmetry group $\mathrm{Aut}^+(D) \coloneqq \mathrm{Aut}(D) \cap \mathrm{SO}(3, \mathbb{R})$.

Project 1: Part (B). Complete the proof of the classification theorem: Any finite subgroup of $\mathrm{SO}(3, \mathbb{R})$ is isomorphic to one of the following groups:

- cyclic group $\mathbb{Z}/n\mathbb{Z}$,
- dihedral group D_{2n},
- the rotational symmetry group of a tetrahedron, a cube, or a dodecahedron.

Project 2. The goal of this project is to give a full classification of wallpaper groups $G \subseteq \mathrm{Isom}(\mathbb{R}^2)$ up to isomorphisms.

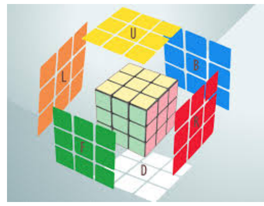


Project 2: Part (A). Classify G (up to isomorphisms) when L_G is of the following types: (i) centered rectangular; (ii) square; (iii) hexagonal.

Project 2: Part (B). Complete the classification: Prove that, up to isomorphisms,

- there is a unique G with point group $\overline{G}$ isomorphic to either $\{e\}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, or D_{12};
- there are two non-isomorphic G with point group $\overline{G}$ isomorphic to either D_6 or D_8;
- there are four non-isomorphic G with point group $\overline{G}$ isomorphic to either $\mathbb{Z}/2\mathbb{Z}$ or D_4.

Project 3. Let U, D, F, B, L, R denote the following operations on the Rubik's cube:



- U: Rotate the upward face counterclockwisely by 90 degrees.
- D: Rotate the downward face counterclockwisely by 90 degrees.
- F: Rotate the front face counterclockwisely by 90 degrees.
- B: Rotate the back face counterclockwisely by 90 degrees.
- L: Rotate the left face counterclockwisely by 90 degrees.
- R: Rotate the right face counterclockwisely by 90 degrees.

Note that these six operations act on the set of 54 facets of the cube. Therefore, each of them would give a non-trivial element of the symmetric group S_{54}. The *Rubik's cube group* is defined to be the subgroup of S_{54} generated by these six operations.

Project 3: Part (A). Find the order of the Rubik's cube group.

Project 3: Part (B).* Find the group structure of the Rubik's cube group. *(Hint: Express the group as certain direct products and semi-direct products of some well-known groups.)*

Project 4. Let us start with some definitions.

- A subset $P \subseteq \mathbb{R}^n$ is called *convex* if any segment between points in the set, is contained in the set.
- A subset $P \subseteq \mathbb{R}^n$ is an *n-dimensional polytope* if it is a closed and bounded subset of $\mathbb{R}^n$, and is bounded by $(n-1)$-dimensional hyperplanes and has non-empty interior.
- An n-dimensional polytope is called *regular* if, for each $j = 0, 1, \ldots, n-1$ its symmetry group acts transitively on the j-faces of it.

Project 4: Part (A). Classify all convex, regular 4-dimensional polytope.

Project 4: Part (B).* Find the symmetry groups of the convex regular 4-dimensional polytopes. *(Hint: Among all regular polytopes, the symmetry groups of half of them are easier to describe. For these cases, express them as certain direct products and semi-direct products of some well-known groups. For the other half... try your best!)*