What's linear algebra?

- In the most concrete form, it studies systems of <u>linear</u> eq'ns. like:

$$\begin{cases} x_1 + x_2 = 2 \\ -2x_1 + 3x_2 = 1. \end{cases}$$

- More abstractly, it studies "transformations" of "spaces" which carries "lines" to "lines".

  <u>e.g.</u>: Schrödinger eq'n:

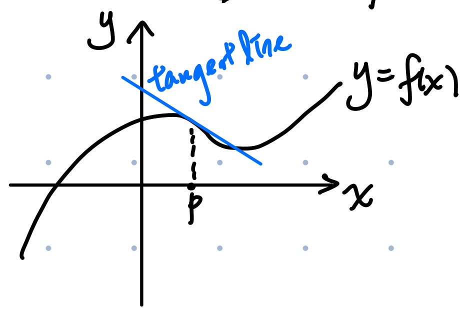
$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - V(x,t)\right) F(x,t) = 0.$$

  where: "space" $\mathcal{F}$:= space of functions $F(x,t)$.

  "transformation": $\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - V(x,t)\right) : \mathcal{F} \longrightarrow \mathcal{F}$

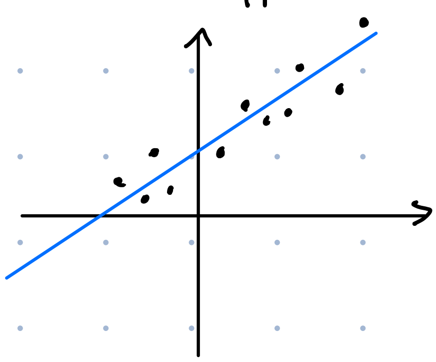
Some applications of linear algebra:

- To understand a map $f: X \longrightarrow Y$ at a point $p \in X$, we usually start with studying its "first order approximation", i.e. the "tangent map" $f_p: T_pX \longrightarrow T_{f(p)}Y$, which is a <u>linear</u> transformation.



  This is the most fundamental thing in multivariable calculus, differential geometry, ...

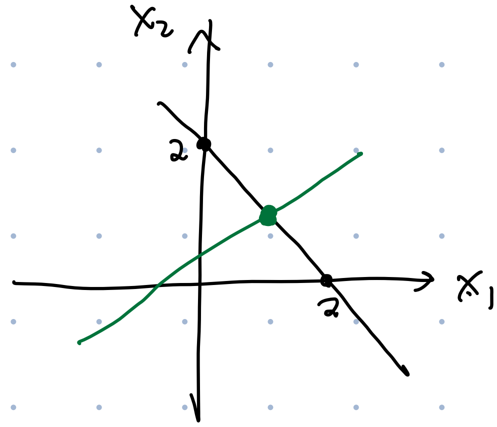
- Linear approximation



You'll leave this class equipped with a powerful conceptual framework on which the vast majority of math, science, engineering, ... depend.

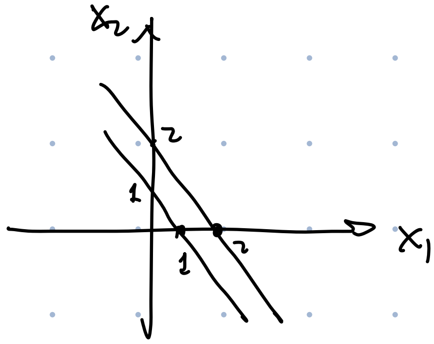
Warm-up

$$\begin{cases} x_1 + x_2 = 2 \\ -2x_1 + 3x_2 = 1. \end{cases}$$



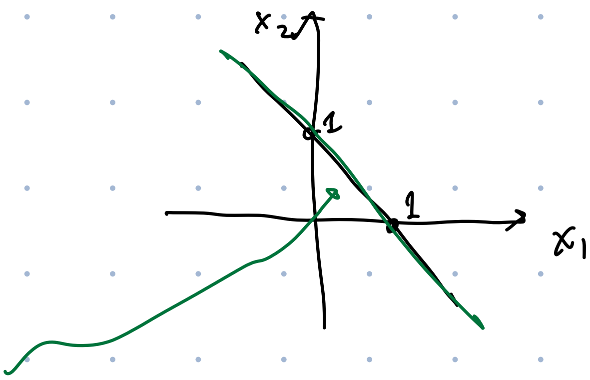
Any solutions?? $(1,1)$

e.g. $\begin{cases} x_1 + x_2 = 2 \\ x_1 + x_2 = 1 \end{cases}$



No solution.

e.g. $\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 2 \\ 3x_1 + 3x_2 = 3 \\ -x_1 - x_2 = -1 \end{cases}$

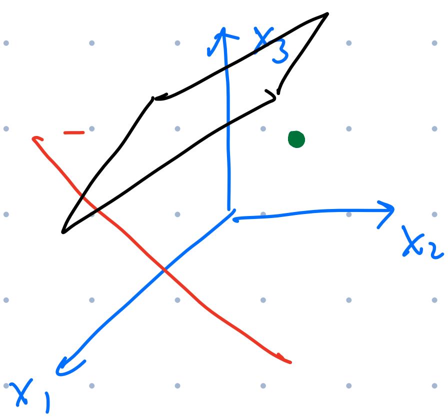


∞'ly many sol$^{\text{n}}$s

e.g. $0 \cdot x_1 + 0 \cdot x_2 = 0 \Leftarrow$ Any $(x_1, x_2) \in \mathbb{R}^2$ is a sol$^{\text{n}}$.

What about linear systems of 3 variables?
What can the sol$^{\text{n}}$ set look like? (geometrically)

$\forall$: for all
$\exists$: there exists
s.t.: such that.



$\mathbb{R}^3$, plane, line, pt, ϕ.

In HW, you'll prove that $\forall$ linear system, it has either no sol$^{\text{n}}$, a unique sol$^{\text{n}}$, or ∞'ly many sol$^{\text{n}}$s.

Def: System of linear eq^{n}s (linear system) in variables $x_1, \dots, x_n$:

$$(*)\ \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + \dots \quad \dots + a_{mn}x_n = b_m. \end{cases}$$

where each a_{ij}, b_i are all real numbers.

Def A vector in $\mathbb{R}^n$ is an ordered set of n real numbers, written as

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Def: We say a vector $\vec{v}$ is a sol^{n} of $(*)$ if

$$\begin{cases} a_{11}v_1 + \dots + a_{1n}v_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1}v_1 + \dots + a_{mn}v_n = b_m. \end{cases}$$

e.g.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & \text{— ①} \\ 3x_1 - 4x_2 - 5x_3 = 8 & \text{— ②} \\ 5x_1 - 2x_2 - 13x_3 = 18 & \text{— ③} \end{cases}$$

coefficient matrix of the linear system.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 3 & -4 & -5 & 8 \\ 5 & -2 & -13 & 18 \end{array}\right] \begin{array}{l} \text{①} \\ \text{②} \\ \text{③} \end{array}$$

augmented matrix of the linear system.

Replace ② by ② − 3 × ①:

the sol^n set is unchanged under this operation.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & -2 & -13 & 18 \end{array}\right]$$

Replace ③ by ③ − 5 × ①:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 8 & -18 & 18 \end{array}\right]$$

$2x_2 - 8x_3 = 8$

Divide ② by 2: $(R2 \to R2/2)$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 8 & -18 & 18 \end{array}\right]$$

Replace ③ by ③ − 8 × ② $(R3 \to R3 - 8R2)$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 14 & -14 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\downarrow R1 \to R1 + 2R2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \xleftarrow{R1 \to R1 + 7R3} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\downarrow R_2 \to R_2 + 4R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \to 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$\boxed{x_1 = 1}$
$x_2 = 0$
$x_3 = -1$

$\Rightarrow$ the system has a unique sol^{n} $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Row reduction algorithm.

(start with a linear system, we do a sequence of "row operations" s.t.

- the sol^{n} set is unchanged.
- the system is "simpler".)

Elementary row operations:

① Replace a row by the sum of the row itself and a multiple of another row.

② Multiply all entries of a row by a nonzero constant.

③ Exchange 2 rows

Step 1: find a leftmost nonzero entry

$$\begin{bmatrix} 0 & 0 & 0 & & \\ 0 & 0 & * & & \\ 0 & 0 & & & \\ \vdots & \vdots & * & & \\ 0 & 0 & * & & \end{bmatrix}$$

Step 2: Do ③ to make the entry we choose to be in the 1^{st} row.

$$\begin{bmatrix} 0 & 0 & * & \\ 0 & 0 & & \\ \vdots & \vdots & & \\ 0 & 0 & & \end{bmatrix}$$

Step 3: Do ② to make it $= 1$

$$\begin{bmatrix} 0 & 0 & 1 & \\ 0 & 0 & & \\ \vdots & \vdots & & \\ 0 & 0 & & \end{bmatrix}$$

Step 4: Do ①:

$$\begin{bmatrix} 0 & 0 & \boxed{1} & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & * & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 & \boxed{1} \\ & & & & & & 0 & 0 \\ & & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot positions

reduced echelon form