This week: Applications of Cauchy's theorem.

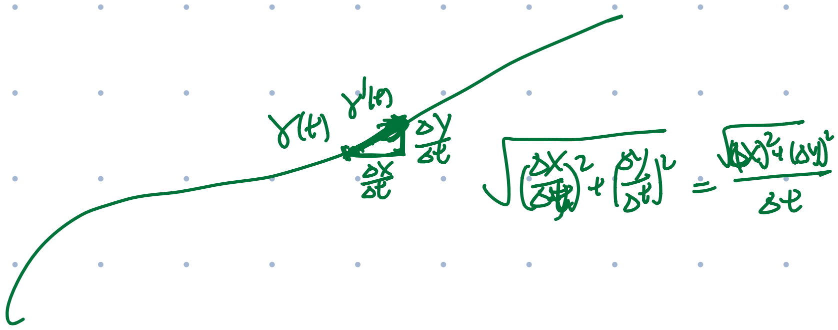
Next week: Proof of Cauchy's thm.

Rmk: • Textbook: γ curve, $z(t)$ - parametrization

• I'm using: γ curve, parametrization

Notion: (length). $\gamma : [a,b] \to \mathbb{C}$

Length of γ, $\text{length}(\gamma) := \int_a^b |\gamma'(t)|\, dt$



Rmk: $\text{length}(\gamma)$ is indep. of its parametrization:

$\gamma : [a,b] \to \mathbb{C}$

$\tilde{\gamma} : [c,d] \to \mathbb{C}$

$\gamma(\varphi(s)) = \tilde{\gamma}(s)$ $\varphi' > 0$

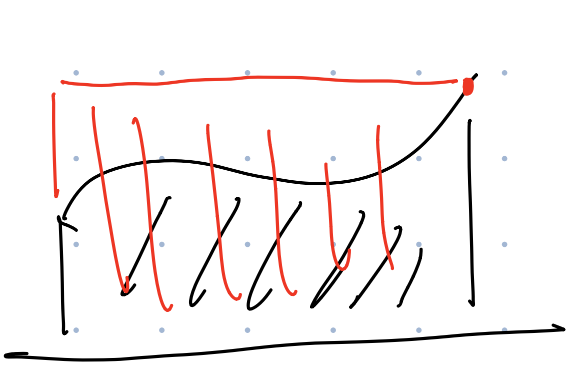
$$\begin{tikzcd}
{[a,b]} \arrow[r, "\gamma"] & \mathbb{C} \\
{[c,d]} \arrow[u, "\varphi"] \arrow[ur, "\tilde{\gamma}"'] &
\end{tikzcd}$$

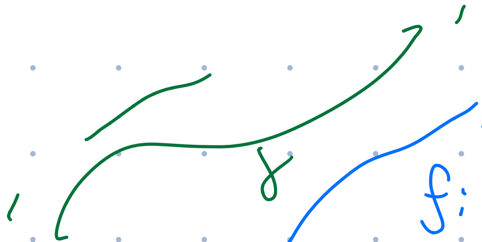
$$\int_a^b |\gamma'(t)|\, dt = \int_c^d |\gamma'(\varphi(s))|\, d\varphi(s)$$

$$= \int_c^d |\gamma'(\varphi(s))|\, \varphi'(s)\, ds$$

$$= \int_c^d |\gamma'(\varphi(s))\, \varphi'(s)|\, ds = \int_c^d |\tilde{\gamma}'(s)|\, ds$$

Fact: $\gamma:[a,b]\to\mathbb{C}$, $\boxed{\left|\int_\gamma f(z)\,dz\right| \le \max_{t\in[a,b]} |f(\gamma(t))|\cdot \text{length}(\gamma)}$





$f:[a,b]\to\mathbb{C}$

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$$

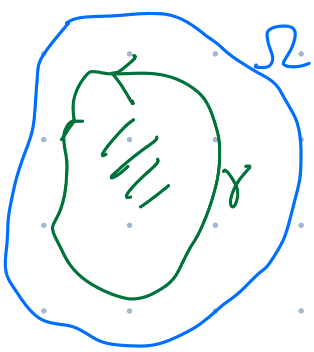
$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right|$$

$$\le \int_a^b \left|\underline{f(\gamma(t))}\gamma'(t)\right| dt$$

$$\le \left(\max_{t\in[a,b]} |f(\gamma(t))|\right)\cdot \underbrace{\int_a^b |\gamma'(t)|\,dt}_{\text{Length}(\gamma)}$$

$\square$

Thm (Cauchy): γ simple closed curve in $\mathbb{C}$



$\Omega\subseteq\mathbb{C}$ open, contains γ & its interior.

$f:\Omega\to\mathbb{C}$ holo.

$$\Longrightarrow \int_\gamma f(z)\,dz = 0$$

Augustin-Louis Cauchy

From Wikipedia, the free encyclopedia

"Cauchy" redirects here. For the lunar crater, see Cauchy (crater). For the statistical distribution, see Cauchy distribution. For the condition on sequences, see Cauchy sequence.

Baron **Augustin-Louis Cauchy** FRS FRSE (/koʊˈʃiː/;[1] French: [ogystɛ̃ lwi koʃi]; 21 August 1789 – 23 May 1857) was a French mathematician, engineer, and physicist who made pioneering contributions to several branches of mathematics, including mathematical analysis and continuum mechanics. He was one of the first to state and rigorously prove theorems of calculus, rejecting the heuristic principle of the generality of algebra of earlier authors. He almost singlehandedly founded complex analysis and the study of permutation groups in abstract algebra.

A profound mathematician, Cauchy had a great influence over his contemporaries and successors;[2] Hans Freudenthal stated: "More concepts and theorems have been named for Cauchy than for any other mathematician (in elasticity alone there are sixteen concepts and theorems named for Cauchy)."[3] Cauchy was a prolific writer; he wrote approximately eight hundred research articles and five complete textbooks on a variety of topics in the fields of mathematics and mathematical physics.

Augustin-Louis Cauchy	
 Cauchy around 1840. Lithography by Zéphirin Belliard after a painting by Jean Roller.	
Born	21 August 1789 Paris, France
Died	23 May 1857 (aged 67) Sceaux, France
Nationality	French

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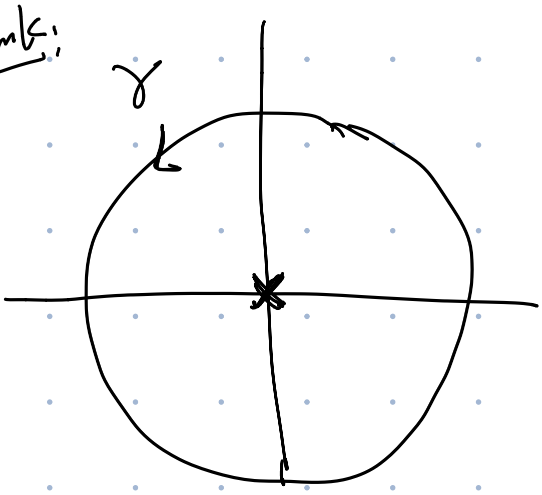
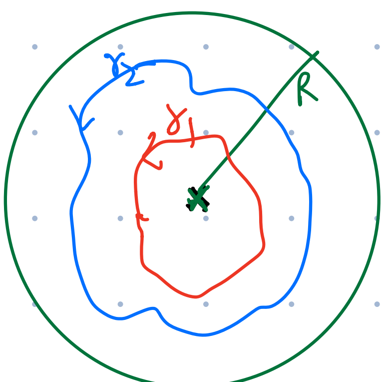
Rmk:



$f = \frac{1}{z}$ $\quad \Omega = \mathbb{C} \setminus \{0\}$

$$\int_\gamma f(z)\,dz = 2\pi i \neq 0$$

Rmk: ("keyhole argument")

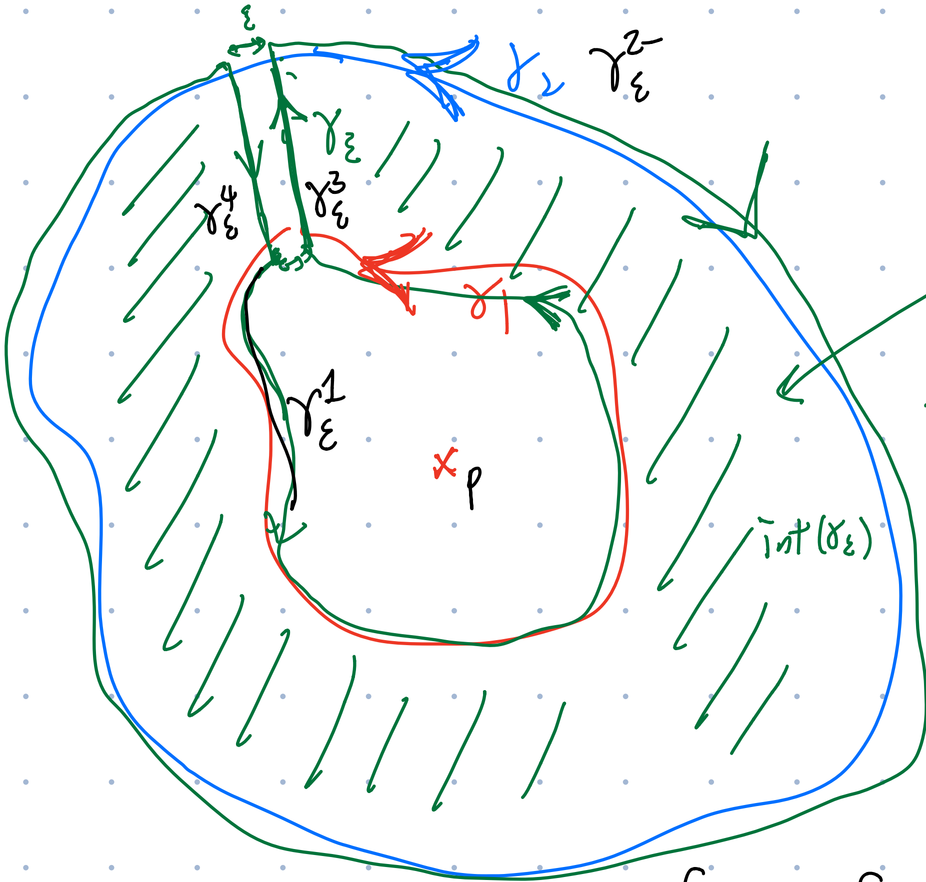


$\Omega = \mathbb{D}_0(R) \setminus \{0\}$, f: holo. on Ω.

Cauchy doesn't apply for

$$\underline{\int_{\gamma_1} f} \quad \text{and} \quad \underline{\int_{\gamma_2} f}.$$

But: we have: $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$



$f: \Omega \setminus \{p\} \to \mathbb{C}$ holo.

f is holo. on $\mathrm{int}(\gamma_\varepsilon)$

~~We can apply~~

$$\text{Cauchy} \implies \int_{\gamma_\varepsilon} f(z)\,dz = 0$$

$$\int_{\gamma_\varepsilon} f(z)\,dz = \int_{\gamma_\varepsilon^1} f + \int_{\gamma_\varepsilon^3} f + \int_{\gamma_\varepsilon^{2-}} f + \int_{\gamma_\varepsilon^4} f$$

Now, let $\varepsilon \to 0$, $\lim_{\varepsilon \to 0} \left(\int_{\gamma_\varepsilon^3} f + \int_{\gamma_\varepsilon^4} f \right) = 0$

$$\lim_{\varepsilon \to 0} \left(\int_{\gamma_\varepsilon^1} f \right) = \int_{\gamma_1} f$$

$$\lim_{\varepsilon \to 0} \left(\int_{\gamma_\varepsilon^{2-}} f \right) = -\int_{\gamma_2} f$$

$$\implies \int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz. \qquad \square$$

e.g. $\displaystyle\int_0^\infty \frac{1-\cos x}{x^2}\,dx = ??$

Why the integral conv.?

- as $x \to 0$, $\cos x = 1 - \frac{1}{2}x^2 + O(x^4)$

$$\frac{1-\cos x}{x^2} = \frac{1}{2} + O(x^2)$$

- as $x \to \infty$, $\left|\frac{1-\cos x}{x^2}\right| \le \frac{2}{x^2}$, $\displaystyle\int_1^\infty \frac{2}{x^2} < +\infty$

$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \lim_{\substack{R\to+\infty \\ \varepsilon\to 0}} \int_\varepsilon^R \frac{1-\cos x}{x^2}\,dx.$$

$e^{-ix} = \cos x - i\sin x$

$e^{ix} = \cos x + i\sin x$

$\cos x = \boxed{\dfrac{e^{ix}+e^{-ix}}{2}}$

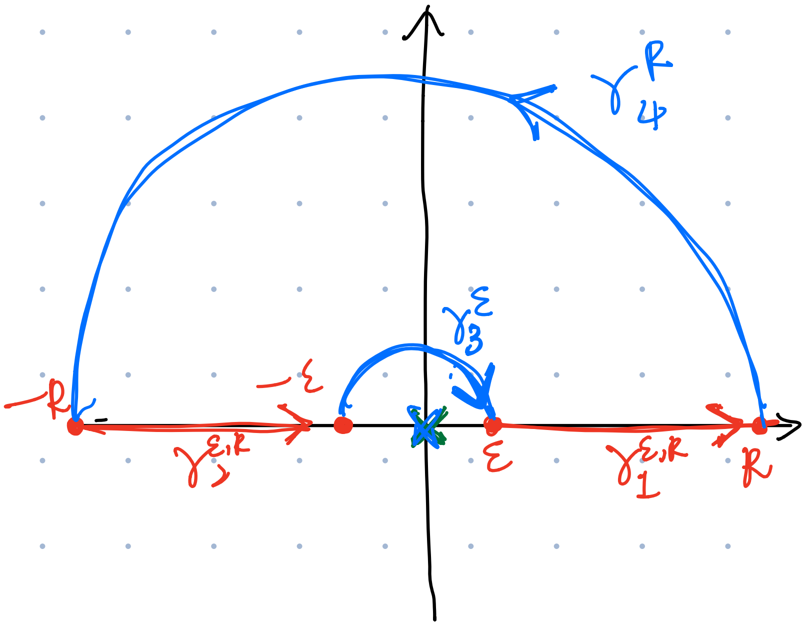
$\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

$$= \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_\varepsilon^R \frac{2-(e^{ix}+e^{-ix})}{2x^2}\,dx$$

$$= \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left(\int_\varepsilon^R \frac{1-e^{ix}}{2x^2}\,dx + \int_\varepsilon^R \frac{1-e^{-ix}}{2x^2}\,dx \right)$$

$$= \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left(\int_\varepsilon^R \frac{1-e^{ix}}{2x^2}\,dx + \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{2x^2}\,dx \right)$$

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left(\int_{\gamma_1} f + \int_{\gamma_2} f \right) = \lim \left(\boxed{\int_\varepsilon^R} + \int_{-R}^{-\varepsilon} \right) \left(\frac{1-e^{ix}}{2x^2}\,dx \right)$$



$$\boxed{f(z) = \frac{1-e^{iz}}{2z^2}}$$

holo. fcn on $\mathbb{C} \setminus \{0\}$

simple closed curve.

- f is holo. in the interior of $\gamma_2^{\varepsilon,R} \cup \gamma_3^\varepsilon \cup \gamma_1^{\varepsilon,R} \cup \gamma_4^R$.
- By Cauchy,

$$\int_{\gamma_2^{\varepsilon,R}} f + \int_{\gamma_3^\varepsilon} f + \int_{\gamma_1^{\varepsilon,R}} f + \int_{\gamma_4^R} f = 0$$

$\forall \varepsilon, R$.

- Therefore, we only need to compute:
  - $\boxed{\lim_{\varepsilon \to 0} \int_{\gamma_3^\varepsilon} f}$,
  - $\lim_{R \to \infty} \int_{\gamma_4^R} f = 0$

$$\int_{\gamma_4^R} f = \int_{\gamma_4^R} \frac{1-e^{iz}}{2z^2}\, dz$$

$\frac{d}{d\theta}\gamma(\theta) = \gamma'(\theta)$

$$= \int_0^\pi \frac{1-e^{i(Re^{i\theta})}}{2(Re^{i\theta})^2} (iRe^{i\theta})\, d\theta$$

$\gamma : [0,\pi] \to \mathbb{C}$

$\theta \mapsto Re^{i\theta}$

Claim: $\lim\limits_{R\to\infty} \int_{\gamma_4^R} f = 0$

Pf: $$\left|\int_{\gamma_4^R} f\right| = \left|\int_0^\pi \frac{1-e^{i(Re^{i\theta})}}{2(Re^{i\theta})^{\cancel{2}}} (i\cancel{R}\cancel{e^{i\theta}})\,d\theta\right|$$

$$\le \int_0^\pi \left|\frac{1-e^{iRe^{i\theta}}}{2(Re^{i\theta})} i\right| d\theta$$

$$= \int_0^\pi \frac{|1-e^{iRe^{i\theta}}|}{2R}\, d\theta \le \int_0^\pi \frac{2}{2R}\, d\theta = \frac{\pi}{R}$$

$$\left|e^{iRe^{i\theta}}\right| = \left|e^{iR(\cos\theta + i\sin\theta)}\right| = \left|e^{R(-\sin\theta + i\cos\theta)}\right|$$

$$= e^{-R\sin\theta}$$

$x, y \in \mathbb{R}$

$$\left(e^{x+iy} = e^x e^{iy} \Rightarrow |e^{x+iy}| = |e^x|\underbrace{|e^{iy}|}_{1} = e^x\right)$$

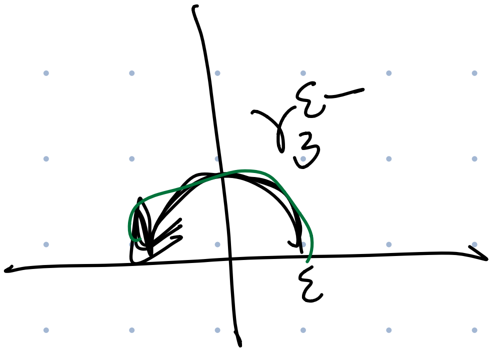
$$|1-e^{iRe^{i\theta}}| \le 1 + |e^{iRe^{i\theta}}|$$

$$= 1 + \underline{e^{-R\sin\theta}}$$

$R > 0$

$\sin\theta \ge 0$

$$\le 2$$



$$f = \frac{1-e^{iz}}{2z^2}$$

$$\gamma_3^{\varepsilon-}: [0,\pi] \to \mathbb{C}$$
$$\theta \mapsto \varepsilon e^{i\theta}$$

$$\int_{\gamma_3^{\varepsilon-}} f(z)\,dz = \int_0^{\pi} \frac{1-e^{i(\varepsilon e^{i\theta})}}{2(\varepsilon e^{i\theta})^{\cancel{2}}} \left(i\cancel{\varepsilon}\cancel{e^{i\theta}}\right) d\theta$$

$$= \frac{i}{2\varepsilon} \int_0^{\pi} \frac{1-e^{i\varepsilon e^{i\theta}}}{e^{i\theta}}\, d\theta$$

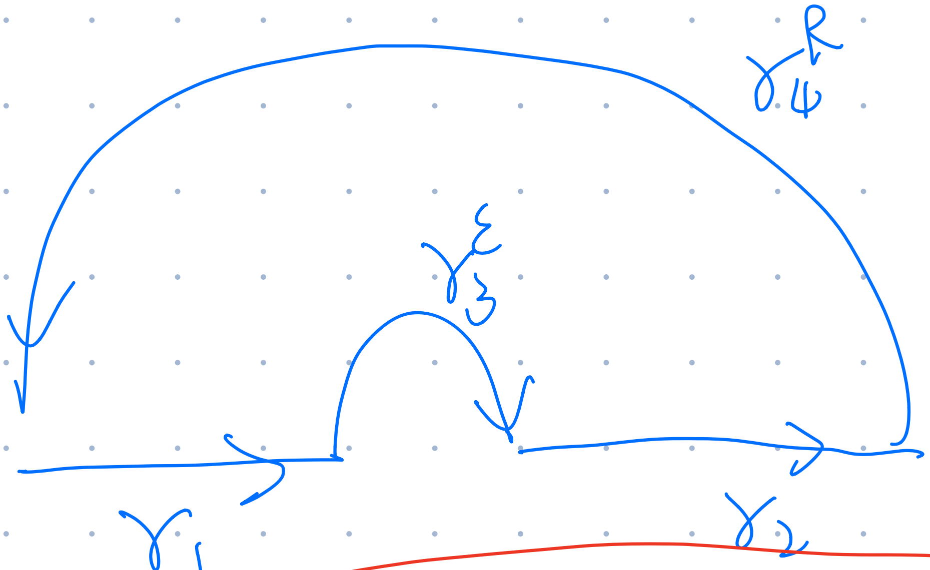
$$\left(e^{i\varepsilon e^{i\theta}} = 1 + i\varepsilon e^{i\theta} + O(\varepsilon^2) \right)$$

$$= \frac{i}{2\varepsilon} \int_0^{\pi} \frac{-i\varepsilon e^{i\theta} + O(\varepsilon^2)}{e^{i\theta}}\, d\theta$$

$$= \frac{i}{2\varepsilon}\left(-i\varepsilon\pi + O(\varepsilon^2) \right)$$

$$= \frac{\pi}{2} + O(\varepsilon)$$

So: $\lim_{\varepsilon \to 0} \int_{\gamma_3^{\varepsilon -}} f(z)\,dz = \frac{\pi}{2}$



$\int_0^\infty \frac{1-\cos x}{x^2}\,dx$

$\|$

Want: $\lim_{\substack{R\to\infty \\ \varepsilon \to 0}} \left(\int_{\gamma_1} f + \int_{\gamma_2} f \right) = \frac{\pi}{2}.$

$$\lim_{R\to\infty} \int_{\gamma_4^R} f = 0$$

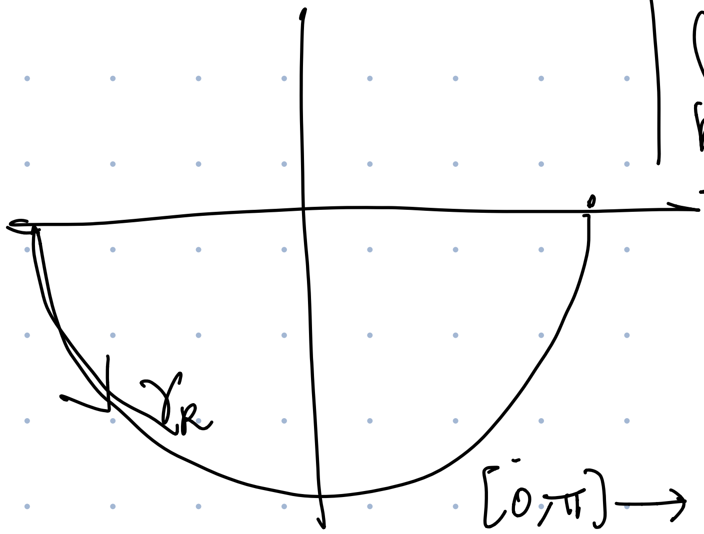
$$\lim_{\varepsilon\to 0} \int_{\gamma_3^\varepsilon} f = -\frac{\pi}{2}$$

$$\int_{\gamma_1} f + \int_{\gamma_2} f + \underbrace{\int_{\gamma_3^\varepsilon} f}_{\to -\frac{\pi}{2}} + \underbrace{\int_{\gamma_4^R} f}_{\to 0} = 0 \qquad \forall R, \varepsilon$$

Let $R\to\infty$, $\varepsilon \to 0$

Rmk: $f(z) = \dfrac{1-e^{iz}}{2z^2}$

$$\boxed{\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz}$$



$[0,\pi] \to \mathbb{C}$

$\theta \mapsto -Re^{i\theta}$

$$\int_0^{\pi} \frac{1-\boxed{e^{i(Re^{-i\theta})}}}{\cdots} \cdots \qquad \boxed{Re^{-i\theta}}$$

$$|e^{iRe^{-i\theta}}| = |e^{iR(\cos\theta - i\sin\theta)}| = |e^{R\sin\theta + iR\cos\theta}| = e^{R\sin\theta}$$

$$\left|\int_{\gamma_R} f\right| = \left|\int_0^{\pi} \frac{1-e^{i(-Re^{i\theta})}}{2(-Re^{i\theta})^{\cancel{2}}}(\cancel{-}i\cancel{R}\cancel{e^{i\theta}})\,d\theta\right|$$

$$\le \int_0^{\pi} \frac{|1-\boxed{e^{-iRe^{i\theta}}}|}{2R}\,d\theta \qquad (|i|=1)$$

$$|e^{-iRe^{i\theta}}| = |e^{-iR(\cos\theta + i\sin\theta)}| = |e^{R\sin\theta - iR\cos\theta}|$$

$$= e^{R\sin\theta}$$

Can't have upper bound of $e^{R\sin\theta}$ as $R \to +\infty$

So we can't conclude $\lim_{R\to\infty} \int_{\gamma_R} f = 0$ as before.
(in fact, this is not true)