Nielsen realization for Bridgeland stability conditions on K3 surfaces

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A general problem

Let $\mathcal{D}$ be a triangulated category (e.g. $D^b\mathrm{Coh}(X)$). Study

the group of autoequivalences $\mathrm{Aut}(\mathcal{D})$.

(Why? It contains $\mathrm{Aut}(X)$, and "hidden symmetries" like spherical twists)

- Complexity: categorical entropy, categorical polynomial entropy
- Group structures?
- Spaces that it acts on, ideally: hyperbolic space, CAT(0) space, etc.
- Classifications (e.g. finite order, "reducible", "pseudo-Anosov", etc.): via entropy, or via its action on certain spaces, etc.

Next, we explain "Nielsen realization" in the title –

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Nielsen–Thurston classification

- Σ: Riemann surface
- $\mathrm{MCG}(\Sigma) = \mathrm{Diff}(\Sigma)/\text{isotopy}$: mapping class group
- each mapping class is either:
  - finite order
  - reducible
  - pseudo-Anosov

For instance –

- elements of $\mathrm{MCG}(T^2) = \mathrm{SL}(2, \mathbb{Z})$ are either:
  - elliptic (finite order)
  - parabolic (Dehn twist)
  - hyperbolic (two transverse foliations: stretch one by $\lambda > 1$, compress the other by $1/\lambda$)

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Realization problem

- Nielsen asked (1923): Let $G \subseteq \mathrm{MCG}(\Sigma)$ be a finite subgroup. Does there always exist a lifting $G \subseteq \mathrm{Diff}(\Sigma)$?
  (Recall that $\mathrm{MCG}(\Sigma) = \mathrm{Diff}(\Sigma)/\text{isotopy}$).
- Kerckhoff (1983): Yes! Moreover, there exists a metric g such that $G \subseteq \mathrm{Isom}(\Sigma, g)$. Or equivalently, G fixes a point in $\mathrm{Teich}(\Sigma)$.
  (There is a natural action of $\mathrm{MCG}(\Sigma)$ on $\mathrm{Teich}(\Sigma)$, e.g. $\mathrm{MCG}(T^2) = \mathrm{SL}(2, \mathbb{Z})$ acts on $\mathrm{Teich}(T^2) = \mathbb{H}$.)
  (Rephrase: any finite subgroup of $\mathrm{MCG}(\Sigma)$ can be realized as symmetries with respect to a metric on Σ.)
- Farb–Looijenga (2021) also proved similar statements for K3 surfaces (under certain conditions), where g is replaced by complex structure or Ricci-flat metric on the K3 surface.

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Let $\mathcal{D}$ be a triangulated category, and $G \subseteq \mathrm{Aut}(\mathcal{D})$ be a finite subgroup. Does there exist $\sigma \in \mathrm{Stab}(\mathcal{D})$ such that $\Phi \cdot \sigma = \sigma$ for all $\Phi \in G$?

(Rephrase: any finite subgroup of $\mathrm{Aut}(\mathcal{D})$ can be realized as symmetries with respect to a stability condition on $\mathcal{D}$.)

- There is a whole dictionary of analogy between Teichmüller theory and stability conditions on triangulated categories (Haiden, Katzarkov, Kontsevich, Bridgeland, Smith, etc.). This problem is the categorical version of the Nielsen realization problem.
- When $\mathcal{D} = D^b\mathrm{Coh}(X)$, stability conditions on $\mathcal{D}$ are roughly Kähler structures on X; so this problem is similar to (but not quite the same) the mirror problem of Farb–Looijenga.

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A dictionary of analogy

(after Gaiotto, Moore, Neitzke; Bridgeland, Smith; Dimitrov, Haiden, Katzarkov, Kontsevich, etc.)

Riemann surface Σ	Triangulated category $\mathcal{D}$
curve C	object E
$C_1 \cap C_2$	$\mathrm{Hom}(E_1, E_2)$
metric g	Bridgeland stability condition σ
geodesics	semistable objects
length $\ell_g(C)$	mass $m_\sigma(E)$
$\mathrm{MCG}(\Sigma)$	$\mathrm{Aut}(\mathcal{D})$
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Many analogous statements in Teichmüller theory can be proved in the categorical setting for $\mathcal{D} = D^b\mathrm{Coh}$(elliptic curve). An interesting general question is whether some of these can be generalized to dim ≥ 2.

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A categorical realization problem II

Let $G \subseteq \mathrm{Aut}(\mathcal{D})/[1]$ be a finite subgroup.
Does there exist $\sigma \in \mathrm{Stab}(\mathcal{D})/\mathbb{C}$ such that $\Phi \cdot \sigma = \sigma$ for all $\Phi \in G$?

- This statement is stronger than the previous one.
- There are many examples of $\mathcal{D}$ where there are not many interesting finite order elements in $\mathrm{Aut}(\mathcal{D})$, but there are many interesting finite order elements in $\mathrm{Aut}(\mathcal{D})/[1]$.
- The shift functor [1] (or rather [2]) has no classical counterpart in Teichmüller theory, so it is natural to consider this stronger question.

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Main theorems (F.–Lai, 2023)

- The answer is yes, for $\mathcal{D} = D^b\mathrm{Coh}(X)$ where X is a curve, a (twisted) abelian surface, a generic twisted K3 surface, or a K3 surface of Picard number $\rho = 1$.
- For K3 surfaces of $\rho = 1$, we obtain:
  - classification and counting formula of the conjugacy classes of finite subgroups of $\mathrm{Aut}(\mathcal{D})$ and $\mathrm{Aut}(\mathcal{D})/[1]$;
  - one-to-one correspondence between $\{$maximal finite subgroups of $\mathrm{Aut}(\mathcal{D})/[1]\}$ and $\{$elliptic points of $\mathrm{Stab}^\dagger_{\mathsf{red}}(\mathcal{D})/\mathbb{C}\}$ (analogue: one-to-one correspondence between $\{$maximal finite subgroups of $\mathrm{PSL}(2,\mathbb{Z})\}$ and $\{$elliptic points of $\mathbb{H}\}$)

Here, $\mathrm{Stab}^\dagger_{\mathsf{red}}(\mathcal{D}) = \{(Z,P) \mid Z^2 = 0 \text{ in } N(\mathcal{D}) \otimes \mathbb{C}\} \subseteq \mathrm{Stab}^\dagger(\mathcal{D})$.

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(Still in the case of K3 surfaces of $\rho = 1$)

$\Phi \in \mathrm{Aut}(\mathcal{D})$ can be classified into (modulo quotienting certain subgroup):

- finite order up to shifts
- reducible, which further classified into:
  - "(-2)-reducible": spherical twists T_S
  - "0-reducible": which fixes a class $w \in N(\mathcal{D})$ with $w^2 = 0$ (e.g. $\otimes\mathcal{O}(1)$)
- hyperbolic: $\rho([\Phi]_{N(\mathcal{D})}) > 1$

Modulo certain conjectures regarding the (polynomial) entropy of the reducible autoequivalences, we expect the following trichotomy:

- finite order if and only if $h_{\mathrm{cat}} = h_{\mathrm{poly}} = 0$
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Strategy and Difficulties

For K3 or abelian surfaces, Bridgeland (2008) showed that there is an $\mathrm{Aut}(\mathcal{D})$-equivariant covering map

$$\mathrm{Stab}^{\dagger}_{\mathsf{red}}(\mathcal{D})/\mathbb{C} \xrightarrow{\pi} Q_0^+(\mathcal{D})$$

where $Q_0^+(\mathcal{D}) = \{v \in \mathbb{P}(N(\mathcal{D}) \otimes \mathbb{C}) \mid v^2 = 0, v\overline{v} > 0\} \setminus \bigcup_{\delta^2=-2} \delta^{\perp}$.

- For abelian surfaces, there is no spherical objects in $\mathcal{D}$, so:
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For K3 or abelian surfaces, Bridgeland (2008) showed that there is an $\mathrm{Aut}(\mathcal{D})$-equivariant covering map

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where $Q_0^{+}(\mathcal{D}) = \{v \in \mathbb{P}(N(\mathcal{D}) \otimes \mathbb{C}) \mid v^2 = 0, v\overline{v} > 0\} \setminus \bigcup_{\delta^2=-2} \delta^{\perp}$.

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Suppose X is a K3 surface of $\rho = 1$ and degree $2n$.

- We have $Q_0^+(\mathcal{D}) \cong \mathbb{H} \setminus$ "(-2)-points".
- By Dolgachaev (1996) and Kawatani (2014), the action of $\mathrm{Aut}(\mathcal{D})$ on $Q_0^+(\mathcal{D})$ factors through $\mathrm{Im}(\mathrm{Aut}(\mathcal{D}) \xrightarrow{f} \mathrm{PSL}(2, \mathbb{R})) = \Gamma_0^+(n)$ the Fricke modular group, where $\Gamma_0^+(n) = \left\langle \Gamma_0(n), \begin{bmatrix} & -1/\sqrt{n} \\ \sqrt{n} & \end{bmatrix} =: \omega_n \right\rangle$.

We showed that the following statements are equivalent:

- $f(\Phi)$ fixes a (-2)-point in $\mathbb{H}$
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- $\Phi = T_S \Psi$ for some spherical object S and some $\Psi \in \mathrm{Deck}(\pi)$

Moreover, we showed that autoequivalences of the form $T_S \Psi$ must be of infinite order in $\mathrm{Aut}(\mathcal{D})/[1]$. This resolves the first issue.

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Lifting of fixed points

- Let $\Phi \in \mathrm{Aut}(\mathcal{D})/[1]$ be of finite order, then the previous discussion shows that it fixes a point in $Q_0^+(\mathcal{D})$.
- Kawatani (2019): $\pi_1(Q_0^+(\mathcal{D})) \cong \star_{\mathsf{free}} T_S^2$.
- Bayer–Bridgeland (2017): $\mathrm{Stab}^\dagger_{\mathsf{red}}(\mathcal{D})/\mathbb{C}$ is contractible.
- Combining these two results, we have $\mathsf{Deck}(\pi) \cong \star_{\mathsf{free}} T_S^2$.
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A few further problems

- Do 0-reducible autoequivalences have zero entropy? $(h(\otimes\mathcal{O}(1)) = 0)$
- Generalize realization results to:
  - general special cubic fourfolds $\mathrm{Ku}(X)$
  - K3 surfaces of Picard number $\rho \geq 2$
  - $\cdots$?

Thank you for your attention!

Reference: F.–Lai, *Nielsen realization problem for Bridgeland stability conditions on generic K3 surfaces*, arXiv:2302.12663