SECOND MIDTERM PRACTICE PROBLEMS
MATH H54, FALL 2021

(1) Consider the symmetric matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

(2) Let $M_{2\times 2}(\mathbb{R})$ be the set of all real 2×2 matrices. It is naturally a vector space with the standard matrix addition and scalar multiplication. Consider the function $\langle -, - \rangle : M_{2\times 2}(\mathbb{R}) \times M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ given by

$$\langle A, B\rangle = \mathrm{tr}(AB^T).$$

  (a) Show that $(M_{2\times 2}(\mathbb{R}), \langle -, - \rangle)$ is an inner product space.
  (b) Construct an orthonormal basis (with respect to the inner product $\langle -, - \rangle$) of the subspace of $M_{2\times 2}(\mathbb{R})$ spanned by $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
  (c) Consider another function $\langle -, - \rangle_2 : M_{2\times 2}(\mathbb{R}) \times M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by $\langle A, B\rangle_2 = \mathrm{tr}(AB)$. Does $\langle -, - \rangle_2$ give an inner product on the vector space $M_{2\times 2}(\mathbb{R})$?

(3) Find all possible 5×5 real symmetric matrices A satisfying $A^3 - 2A = 4\mathbb{I}_5$.

(4) Let W_1 and W_2 be two subspaces of a finite dimensional inner product space V.
  (a) Prove that $W_1^\perp \cap W_2^\perp = (W_1 + W_2)^\perp$.
  (b) Prove that $\dim(W_1) - \dim(W_1 \cap W_2) = \dim(W_2^\perp) - \dim(W_1^\perp \cap W_2^\perp)$.

(5) Let A and B be two square matrices.
  (a) Suppose that $\lambda \neq 0$ is an eigenvalue of AB. Prove that λ is also an eigenvalue of BA.
  (b) Does the same statement hold for $\lambda = 0$?

(6) Let $(V, \langle -, - \rangle)$ be an inner product space, and let $T\colon V \to V$ be a linear transformation. Suppose that $||T(\vec{x})|| = ||\vec{x}||$ for any $\vec{x} \in V$. Prove that

$$\langle T(\vec{x}), T(\vec{y})\rangle = \langle \vec{x}, \vec{y}\rangle \quad \text{for any } \vec{x}, \vec{y} \in V.$$

(7) Let $A_1, \ldots, A_k$ be $n\times n$ real symmetric matrices. Suppose that $A_1^2 + \cdots + A_k^2 = 0$ (the zero matrix). Prove that $A_1 = \cdots = A_k = 0$ (the zero matrix).

(8) Let A be an $n\times n$ diagonalizable matrix with $n-1$ distinct eigenvalues. Prove that for any $\vec{v} \in \mathbb{R}^n$, the set $\{\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}\}$ is linearly dependent.