(1) A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *contractive* if there exists $0 < K < 1$ such that

$$|f(x) - f(y)| \le K|x - y| \quad \text{holds for any} \quad x, y \in \mathbb{R}.$$

You'll show that any contractive map on $\mathbb{R}$ has a unique fixed point.

(a) Pick any $x_1 \in \mathbb{R}$. Construct a sequence (x_n) recursively via $x_{n+1} := f(x_n)$. Prove that such sequence (x_n) is a Cauchy sequence, therefore is convergent.

(b) Moreover, prove that the limit x^* of (x_n) is a *fixed point* of f, i.e. $f(x^*) = x^*$. (Hint: First, show that $\lim f(x_n) = x^*$ by the construction of (x_n). On the other hand, show that $\lim f(x_n) = f(x^*)$ by the fact that f is contractive.)

(c) Prove that f has a *unique* fixed point.

(a) Let $|x_2 - x_1| = M$, then $|x_3 - x_2| = |f(x_2) - f(x_1)| \le k|x_2 - x_1| = kM$,

Similarly, $|x_{n+1} - x_n| \le k^{n-1} \cdot M \quad \forall n.$

For any $\varepsilon > 0$, Choose $N > 0$ large enough s.t. $\dfrac{k^{N-1} \cdot M}{1-k} < \varepsilon.$

Then $\forall n > m > N$, we have:

$$\begin{aligned} |x_n - x_m| &\le |x_{m+1} - x_m| + |x_{m+2} - x_{m+1}| + \cdots \\ &\le k^{m-1} \cdot M + k^m \cdot M + \cdots \\ &= \frac{k^{m-1} \cdot M}{1-k} < \frac{k^{N-1} \cdot M}{1-k} < \varepsilon. \end{aligned} \quad (\text{Note: } 0<k<1)$$

Hence (x_n) is Cauchy. $\square$

(b) Let $\lim x_n = x^*$.

Then $x^* = \lim x_{n+1} = \lim f(x_n)$. by construction of (x_n).

Claim: $\lim f(x_n) = f(x^*)$. (then we have $f(x^*) = x^*$).

pf: Since $\lim x_n = x^*$, $\forall \varepsilon > 0$, $\exists N > 0$ s.t.

$$|x_n - x^*| < \varepsilon \quad \forall n > N$$

$$\Rightarrow |f(x_n) - f(x^*)| \le k|x_n - x^*| < k\varepsilon < \varepsilon. \quad \forall n > N.$$

$\square$

(c) If $f(x^*) = x^*$ and $f(y^*) = y^*$, then

$|x^* - y^*| = |f(x^*) - f(y^*)| \le k|x^* - y^*| < |x^* - y^*|$, unless $x^* = y^*$. $\square$

(2) Let $\{S_\alpha\}$ be a collection of (possibly infinitely many) subsets of a set S. Prove:

(a) The complement of union is the intersection of complements: $(\cup_\alpha S_\alpha)^c = \cap_\alpha (S_\alpha^c)$.

(b) The complement of intersection is the union of complements: $(\cap_\alpha S_\alpha)^c = \cup_\alpha (S_\alpha^c)$.

(a) $x \in \left(\bigcup_\alpha S_\alpha\right)^c \Leftrightarrow x \notin \bigcup_\alpha S_\alpha \Leftrightarrow x \notin S_\alpha \ \forall \alpha$

$\Leftrightarrow x \in S_\alpha^c \ \forall \alpha \Leftrightarrow x \in \bigcap_\alpha S_\alpha^c.$

(b) $x \in \left(\bigcap_\alpha S_\alpha\right)^c \Leftrightarrow x \notin \bigcap_\alpha S_\alpha \Leftrightarrow \exists \alpha$ s.t. $x \notin S_\alpha$

$\Leftrightarrow \exists \alpha$ s.t. $x \in S_\alpha^c \Leftrightarrow x \in \bigcup_\alpha S_\alpha^c.$ $\square$

(3) Prove that in a metric space:

(a) The union of (possibly infinitely many) open subsets is open.

(b) The intersection of *finitely many* open subsets is open.

(c) The intersection of (possibly infinitely many) closed subsets is closed.

(d) The union of *finitely many* closed subsets is closed.

(e) Find a counterexample of (a) if 'open' is replaced by 'closed'; find a counterexample of (c) if 'closed' is replaced by 'open'.

(a) $\{U_\alpha\}$ collection of open sets. Want: $\bigcup_\alpha U_\alpha$ open.

$\forall x \in \bigcup_\alpha U_\alpha$, $\exists \alpha$ s.t. $x \in U_\alpha$.

Since U_α open, $\exists r > 0$ s.t. $B_r(x) \subseteq U_\alpha$,

$\Rightarrow B_r(x) \subseteq U_\alpha \subseteq \bigcup_\alpha U_\alpha.$ $\square$

(b) $U_1, \dots, U_n$ open sets. Want: $\bigcap_{i=1}^n U_i$ open.

$\forall x \in \bigcap_{i=1}^n U_i$, $\exists r_1, \dots, r_n > 0$ s.t. $B_{r_i}(x) \subseteq U_i \ \forall 1 \le i \le n$.

Let $r = \min\{r_1, \dots, r_n\} > 0$. Then $B_r(x) \subseteq B_{r_i}(x) \subseteq U_i \ \forall i$.

$\Longrightarrow B_r(x) \subseteq \bigcap_{i=1}^n U_i$. $\square$

(c)(d): Follows from #2 and #3 (a)(b). $\square$

(e):

- $\{[-1+1/n, 1-1/n]\}_{n\in\mathbb{N}}$ collection of closed subsets of $\mathbb{R}$.

  $\bigcup_{n\in\mathbb{N}} [-1+1/n, 1-1/n] = (-1,1)$ NOT closed.

- $\{(-1/n, 1/n)\}_{n\in\mathbb{N}}$ collection of open subsets of $\mathbb{R}$.

  $\bigcap_{n\in\mathbb{N}} (-1/n, 1/n) = \{0\}$ NOT open.

(4) Let (S,d) be a metric space, and $K \subseteq S$ be a compact subset. Prove that K is bounded (i.e. there exist $x \in K$ and $R > 0$ such that $K \subseteq B_R(x)$).

$\{B_1(x) : x \in K\}$ is an open cover of K.

Since K cpt, $\exists\, x_1, \dots, x_n \in K$ s.t. $K \subseteq \bigcup_{i=1}^{n} B_1(x_i)$

Define $R := 1 + \max_{2\le i\le n} \{d(x_1, x_i)\}$.

Claim: $K \subseteq B_R(x_1)$. (therefore K is bdd.)

pf: Since $K \subseteq \bigcup_{i=1}^{n} B_1(x_i)$, $\forall x \in K$, $\exists\, 1 \le i \le n$

s.t. $x \in B_1(x_i)$.

- if $i = 1$, then $x \in B_1(x_1) \subseteq B_R(x_1)$
- if $2 \le i \le n$, then $d(x, x_1) \le d(x, x_i) + d(x_i, x_1)$
  $< 1 + d(x_i, x_1) \le R$.

$\Rightarrow x \in B_R(x_1)$. $\square$

(5) Let (S,d) be a metric space, $K \subseteq S$ be a compact subset, and $C \subseteq S$ be a closed subset. Prove that $C \cap K$ is a compact subset of S. (Hint: First show that $C \cap K$ is a closed subset of S. Now let $\{U_\alpha : \alpha \in I\}$ be an open cover of $C \cap K$, then $\{U_\alpha : \alpha \in I\} \cup \{(C \cap K)^c\}$ is an open cover of the compact set K.)

Claim: $C \cap K$ is closed subset of S.

pf: We proved in class that compact $\Rightarrow$ closed.
So K is a closed subset of S.
The claim then follows from #3(c). $\square$

Claim: $C \cap K$ is a cpt. subset of S.

pf: Let $\{U_\alpha\}$ be any open cover of $C \cap K$.
then $\{U_\alpha\} \cup \{(C \cap K)^c\}$ is an open cover of K.

$\Rightarrow \exists\ \alpha_1, \dots, \alpha_n$ s.t.

$$K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \cup (C \cap K)^c.$$

$$\Rightarrow C \cap K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \cup (C \cap K)^c.$$

$\Rightarrow C \cap K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ since
$C \cap K$ and $(C \cap K)^c$ are disjoint. $\square$

(6) Let $E \subseteq (S,d)$ be a subset of a metric space. Define the *Cantor–Bendixson derivative* of E:

$$E' := \{x \in S : x \text{ is a limit point of } E\}.$$

(a) Show that E' is a closed subset of S.

(b) Show that if $E' \neq \emptyset$ then E contains infinitely many elements.

(Recall that $x \in S$ is a limit point of E if for any $r > 0$, the intersection $B_r(x) \cap E$ contains at least a point other than x.)

(a) Claim: $(E')^c$ is open.

pf: Let $x \in (E')^c$. i.e. x is not a limit pt of E.

Then $\exists\, r > 0$ s.t. $B_r(x) \cap E$ is either ϕ or $\{x\}$.

We'll show that $B_{r/2}(x) \subseteq (E')^c$:

$\forall\, y \in B_{r/2}(x)$, one can use triangle ineq. to show that $B_{r/2}(y) \subseteq B_r(x)$.

$\Rightarrow B_{\min\{r/2,\, d(x,y)\}}(y) \cap E = \phi$.

$\Rightarrow y \in (E')^c$. $\square$

(b) Suppose x is a limit point of E.

$\Rightarrow B_1(x) \cap E$ contains at least a pt other than x, say $x_1 \in E$,

$\Rightarrow B_{d(x,x_1)}(x) \cap E$ contains at least a pt other than x, say $x_2 \in E$. then $x_2 \neq x$, $x_2 \neq x_1$ since $0 < d(x_2,x) < d(x_1,x)$

$\Rightarrow B_{\min\{d(x,x_1),\, d(x,x_2)\}}(x) \cap E$ contains a pt $x_3 \neq x$.

Similarly, x_3 is distinct from x_1, x_2.

one can construct ∞'ly pts in E inductively. $\square$