FINAL EXAM PRACTICE PROBLEMS
MATH 104, SECTION 2

(1) (a) Prove that there exists a unique real number $x \in \mathbb{R}$ satisfying $x = \cos x$.
   (b) Define a sequence of real numbers (a_n) as follows: Let a_1 be any real number satisfying $0 < a_1 < 1$, and define $a_2, a_3, \ldots$ recursively via $a_{n+1} := \cos(a_n)$. Prove that the sequence (a_n) is convergent, and the series $\sum a_n$ is divergent.

(2) Let $f\colon [a,b] \to \mathbb{R}$ be an integrable function. Prove that

$$\lim_{n\to\infty} \int_a^b f(x)\sin(nx)dx = 0.$$

Hint: First show that the statement is true for *step functions* (see Wikipedia for the definition of step functions). Then show that there exists a step function $S(x)$ such that $0 \le \int_a^b (f(x) - S(x))dx < \epsilon$.

(3) Let f, g, h be continuous functions on $[a,b]$ that are differentiable on (a,b). Consider

$$F(x) = \det \begin{pmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{pmatrix}.$$

   (a) Prove that F is also continuous on $[a,b]$ and differentiable on (a,b).
   (b) Prove that there exists $x_0 \in (a,b)$ such that $F'(x_0) = 0$.
   (c) Prove the following generalization of mean value theorem: If f and g are continuous functions on $[a,b]$ that are differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

(4) Let X be a compact metric space, and let $\mathcal{B}(X)$ be the set of real-valued bounded functions on X. For any $f, g \in \mathcal{B}(X)$, define

$$d_{\mathcal{B}}(f,g) := \sup_{x\in X} |f(x) - g(x)|.$$

We know that $(\mathcal{B}(X), d_{\mathcal{B}})$ is a metric space.
   (a) Prove that $\mathcal{B}(X)$ is a complete metric space, i.e. every Cauchy sequence in $\mathcal{B}(X)$ converges to some element in $\mathcal{B}(X)$.
   (b) Let $\mathcal{C}(X)$ be the set of real-valued continuous functions on X. Prove that $\mathcal{C}(X)$ is a closed subset of $\mathcal{B}(X)$.
   (c) Prove that a closed subset of a complete metric space is also complete, therefore concludes that $\mathcal{C}(X)$ is a complete metric space.

(5) Prove that the closed interval $[a,b]$ is not of measure zero in $\mathbb{R}$.
(Hint: Suppose there is a "bad" covering of $[a,b]$ by open intervals whose total length is less than $b-a$. First prove that you can assume the covering is finite. Take a bad covering $\{U_1,\dots,U_n\}$ consists of n open intervals. Then prove that there exists a bad covering consists of no more than $n-1$ open intervals. Show that this implies the existence of a bad covering consists of a single open interval, and get a contradiction.)

(6) Suppose that $f\colon [1,\infty)\to\mathbb{R}$ is uniformly continuous on $[1,\infty)$. Prove that there exists $M>0$ such that
$$\frac{|f(x)|}{x}\leq M \text{ holds for any } x\geq 1.$$

(7) (a) Find the domain $E\subset\mathbb{R}$ of pointwise convergence of the series
$$\sum_{n=1}^{\infty} e^{-nx}\cos(nx),$$
i.e. find all possible $x\in\mathbb{R}$ such that the above series converges.
(b) Prove or disprove: the series converges uniformly on E.

(8) Let $f\colon[0,1]\to\mathbb{R}$ be an increasing function.
(a) Prove that for any $a\in(0,1)$, the *left hand limit* $\lim_{x\to a^-} f(x)$ and the *right hand limit* $\lim_{x\to a^+} f(x)$ of f at a both exists. (See Ross, §20 for the definition.)
(b) Define $A := \{x\in[0,1]\colon f \text{ is not continuous at } x\}$. Prove that the set A is either finite or countable. (Hint: Define an injection from A to $\mathbb{Q}$ using (a).)

(9) Consider the function $f\colon\mathbb{R}^2\to\mathbb{R}$ defined by
$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$
One can regard $\mathbb{R}^2$ and $\mathbb{R}$ as metric spaces via the standard distance functions:
$$d((x_1,y_1),(x_2,y_2))=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$
Prove that:
(a) For any fixed $x\in\mathbb{R}$, the function $f_x\colon\mathbb{R}\to\mathbb{R}$ that sends y to $f(x,y)$ is continuous. Similarly, for any fixed $y\in\mathbb{R}$, the function $f_y\colon\mathbb{R}\to\mathbb{R}$ that sends x to $f(x,y)$ is also continuous.
(b) $f\colon\mathbb{R}^2\to\mathbb{R}$ is not a continuous function.

(10) Define a sequence of real numbers (a_n) by setting $a_1=1$ and
$$a_{n+1}=\sqrt{a_n^2+\frac{1}{2^n}} \text{ for } n\geq 1.$$
Prove that (a_n) is convergent.

(11) Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 2$, where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are real numbers. Suppose that all of the roots of $P(x)$ are real numbers.

Prove that all of the roots of its derivative $P'(x)$ are also real numbers.

(12) Let (f_n) be a sequence of real-valued function defined on a set X. Suppose that

- $f_n(x) \geq 0$ for any $x \in X$ and any $n \in \mathbb{N}$,
- $f_n(x) \geq f_{n+1}(x)$ for any $x \in X$ and any $n \in \mathbb{N}$,
- $\lim_{n\to\infty} \sup\{f_n(x) \colon x \in X\} = 0$.

Prove that the series of functions $\sum(-1)^n f_n(x)$ converges uniformly on X.

(13) Let $f \colon [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (-1)^k f\Big(\frac{k}{n}\Big) = 0.$$

(14) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that there exists $T > 0$ such that

$$f(x) = f(x+T) \text{ holds for any } x \in \mathbb{R}.$$

Prove that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = f(x_0 + \frac{T}{2})$.

(15) Let $a_1, a_2, \cdots, a_n$ be real numbers. Suppose that

$$|a_1 \sin x + a_2 \sin(2x) + \cdots + a_n \sin(nx)| \leq |\sin x| \text{ for any } x \in \mathbb{R}.$$

Prove that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$. (Hint: Let $f(x) = a_1 \sin x + a_2 \sin(2x) + \cdots + a_n \sin(nx)$ and consider $f'(0)$.)

(16) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function such that for any $r \in \mathbb{R}$, we have

$$\lim_{n\to\infty} f(\frac{r}{n}) = 0.$$

Prove or disprove: $\lim_{x\to 0} f(x) = 0$.

(17) Let f and g be continuous functions on $[a,b]$ that are differentiable on (a,b). Suppose that $f(a) = f(b) = 0$. Prove that there exists $x \in (a,b)$ such that $g'(x)f(x) + f'(x) = 0$.

(18) For a bounded function $f \colon [0,1] \to \mathbb{R}$, define

$$R_n \coloneqq \frac{1}{n} \sum_{k=1}^{n} f\Big(\frac{k}{n}\Big).$$

(a) Prove that if f is integrable, then $\lim_{n\to\infty} R_n = \int_0^1 f(x)dx$.

(b) Find an example of f that is not integrable, but $\lim_{n\to\infty} R_n$ exists.