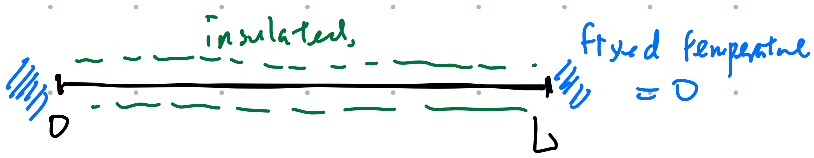
This week: 1-dimensional $\underbrace{\text{heat equation}}$; Fourier series.

a "partial differential equation":

$$\underbrace{u_t(x,t)}_{\frac{\partial}{\partial t} u(x,t)} = \beta \underbrace{u_{xx}(x,t)}_{\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} u(x,t)\right)}$$

Heat equation: Models heat flow in a 1-dimensional object.



Let $u(x,t) :=$ temperature of the rod at $x \in [0, L]$, time $t \geq 0$.

$$\begin{cases} \text{initial condition (time } t=0\text{):} & u(x,0) = f(x) \quad (\text{continuous, } f(0)=f(L)=0) \\ \text{boundary condition:} & u(0,t) = u(L,t) = 0 \quad \forall t \geq 0. \end{cases}$$

Fourier's law $\Rightarrow$ $u_t(x,t) = \beta u_{xx}(x,t)$ for some $\beta > 0$.

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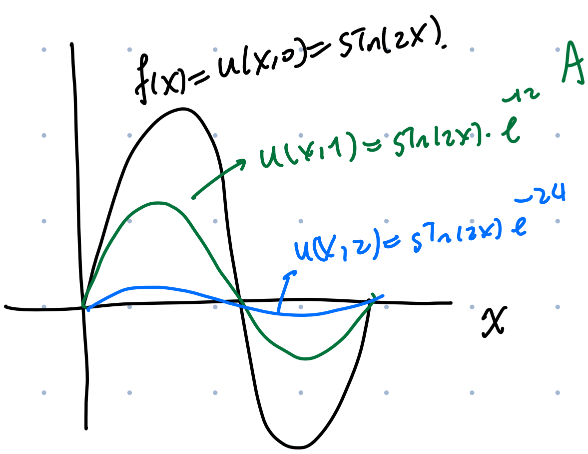
e.g. $L = \pi$, $f(x) = \sin(2x)$, $\beta = 3$.



A solution:

$$u(x,t) = \sin(2x) \cdot e^{-3 \cdot 2^2 t}$$

$u(x,t)$ converges (uniformly) to the zero fn as $t \to +\infty$.

Uniqueness of sol^{n} of heat eq^{n}: (energy method)

Suppose u_1, u_2 are both sol^{ns} to:

$$\begin{cases} u_t = \beta u_{xx}. & \underline{\beta > 0} \\ u(0,t) = u(L,t) = 0 & \forall t \geq 0. \\ u(x,0) = f(x) & \forall x \in [0,L] \end{cases}$$

Consider $w := u_1 - u_2$.

$$\Rightarrow \begin{cases} w_t = \beta w_{xx} \\ w(0,t) = w(L,t) = 0 & \forall t \geq 0. \\ w(x,0) = 0 & \forall x \in [0,L] \end{cases}$$

(Goal: $w = 0$.)

$$E(t) := \frac{1}{2} \int_0^L w(x,t)^2 \, dx. \quad \geq 0$$

- $$\frac{dE}{dt} = \frac{1}{2} \int_0^L \frac{\partial}{\partial t} \left(w(x,t)^2 \right) dx.$$

$$= \frac{1}{\cancel{2}} \int_0^L \cancel{2}\, w \cdot \underbrace{w_t}_{\beta w_{xx}} \, dx. = \beta \int_0^L w \cdot w_{xx} \, dx.$$

$$= \beta \left(\underbrace{w \cdot w_x \Big|_0^L}_{w(L,t) w_x(L,t) - w(0,t) w_x(0,t)} - \int_0^L (w_x)^2 dx \right) \leq 0$$

$w(\cancel{L,t})^{0} w_x(L,t) - w(\cancel{0,t})_{0} w_x(0,t)$

- $E(0) = \frac{1}{2}\int_0^L \underbrace{w(x,0)}_{\substack{\| \\ 0}}{}^2\,dx = 0$

$\Longrightarrow E(t) \equiv 0 \quad \forall t.$

$\Longrightarrow w \equiv 0.$ $\square$

$$\begin{cases} u_t = \beta u_{xx}, & \underline{\beta > 0} \\ u(0,t) = u(L,t) = 0 & \forall t \geq 0. \\ u(x,0) = f(x) & \forall x \in [0,L] \end{cases}$$

Method of separation of variables.

Assume
$$u(x,t) = X(x) \cdot T(t)$$
Try to find $X(x)$, $T(t)$ so that $\begin{cases} u_t = \beta u_{xx} \\ u(0,t) = u(L,t) = 0. \end{cases}$

$$X(x) \cdot T'(t) = \beta X''(x) \cdot T(t)$$

$$\Rightarrow \frac{T'(t)}{\beta T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \quad \lambda = \text{constant}.$$

$$\Rightarrow \begin{cases} X''(x) = -\lambda X(x) \\ T'(t) = -\beta\lambda T(t). \end{cases}$$

$$\begin{matrix} u(0,t) = & u(L,t) = 0 \\ \| & \| \\ X(0)T(t) & X(L)T(t) \end{matrix}$$

Assuming $u(x,t)$ is not the zero function, then $X(0) = X(L) = 0$

Want to find $X(x)$ and λ s.t.

$$\begin{cases} X''(x) = -\lambda X(x). \\ X(0) = X(L) = 0. \end{cases}$$

(HW: $y'' + by' + cy = 0$. If the auxiliary eq^{n} $r^2 + br + c = 0$ has two real roots. Then a nonzero sol^{n} can take the value 0 at most once) $\Rightarrow$ $\lambda > 0$, and

general sol^{n} to $X'' + \lambda X = 0$ is:

$$X(x) = \underbrace{\cancel{c_1 \cos(\sqrt{\lambda}\, x)}}_{0} + c_2 \sin(\sqrt{\lambda}\, x)$$

$$\begin{cases} 0 = X(0) = c_1 \\ 0 = X(L) = \underbrace{c_2}_{\neq 0} \underbrace{\boxed{\sin(\sqrt{\lambda} \cdot L)}}_{= 0} \end{cases}$$

So we should chose $\lambda = \left(\frac{n\pi}{L}\right)^2$ for some $n = 1, 2, 3, \ldots$

and $X(x) = c \sin\left(\frac{n\pi}{L} \cdot x\right)$

We still need to deal with:

- $T'(t) = -\beta \lambda T(t)$ $\rightarrow$ $T(t) = \text{const.}\, e^{-\beta \left(\frac{n\pi}{L}\right)^2 t}$
- $u(x, 0) = f(x)$

$$\begin{aligned} &\parallel \\ &X(x)\, T(0) \\ &\parallel \\ \text{const. } &\sin\left(\frac{n\pi}{L} \cdot x\right) \end{aligned}$$

cond. 1

If $f(x) = \overset{\text{const.}}{C_n} \sin\left(\frac{n\pi}{L} x\right)$,
then

$$u(x,t) = C_n \sin\left(\frac{n\pi}{L} \cdot x\right) e^{-\beta \left(\frac{n\pi}{L}\right)^2 t}$$

is the sol^{n}

If $f(x) = \sum_{n \geq 1} c_n \sin\left(\frac{n\pi}{L} x\right)$,

then $u(x,t) = \sum_{n \geq 1} c_n \sin\left(\frac{n\pi}{L} x\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$.

is the sol^{n} of the heat eq^{n}.

Question: Can we write (or express a limit) any continuous function f as $\sum_{n \geq 1} c_n \sin\left(\frac{n\pi}{L} x\right)$

(Yes; using Fourier series)

Fourier series $\approx$ orthogonal projection of functions to a particular orthonormal basis in $\mathcal{C}[-L,L]$.

$\mathcal{C}[-L,L]$, inner product: $\langle f, g \rangle = \frac{1}{L} \int_{-L}^{L} f(x) g(x)\, dx$.

$$\left\{ \frac{1}{\sqrt{2}}, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \cdots \right\}$$

forms an orthonormal set.

projection of f onto $\langle \frac{1}{\sqrt{2}} \rangle$:

$$\left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \frac{1}{\sqrt{2}}\, dx \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx.$$

$$\left\langle f, \cos\frac{n\pi x}{L} \right\rangle \cos\frac{n\pi x}{L} = \frac{1}{L}\left(\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\, dx \right) \cdot \cos\frac{n\pi x}{L}.$$

<u>Def</u>: f piecewise conti. fun. on $[-L, L]$, the <u>Fourier series</u> of f is:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L} \right)$$

where

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}\,dx, \qquad k \geq 0.$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx \qquad k \geq 1.$$