1. Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$.

(b) $1/z = \overline{z}$.

(c) $\mathrm{Re}(z) = 3$.

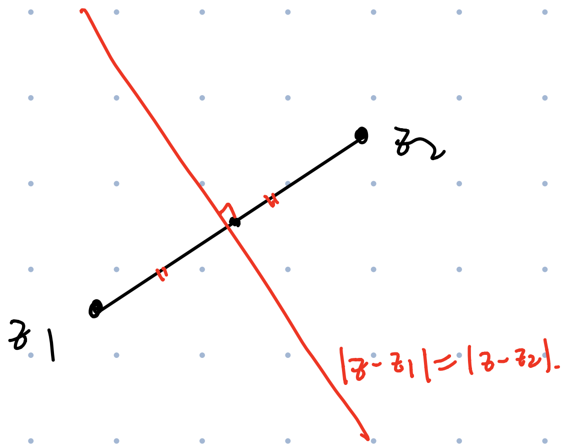
(d) $\mathrm{Re}(z) > c$, (resp., $\geq c$) where $c \in \mathbb{R}$.

(e) $\mathrm{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$.

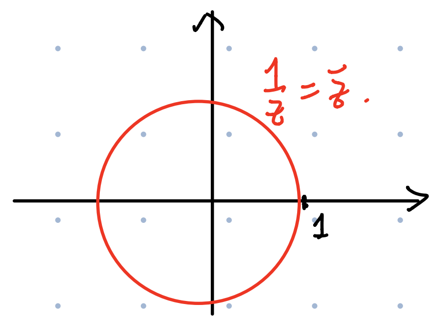
(f) $|z| = \mathrm{Re}(z) + 1$.

(g) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$.

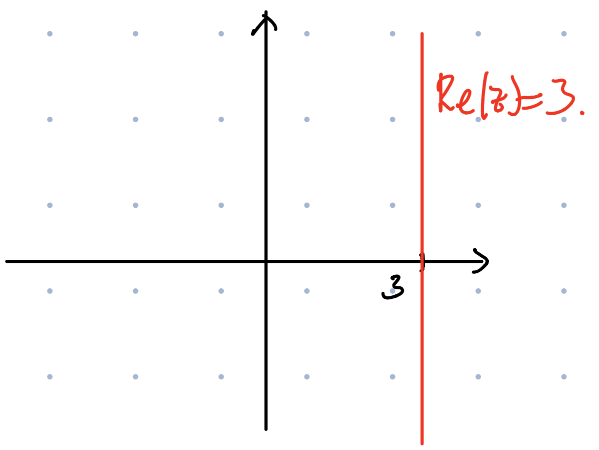
(a)



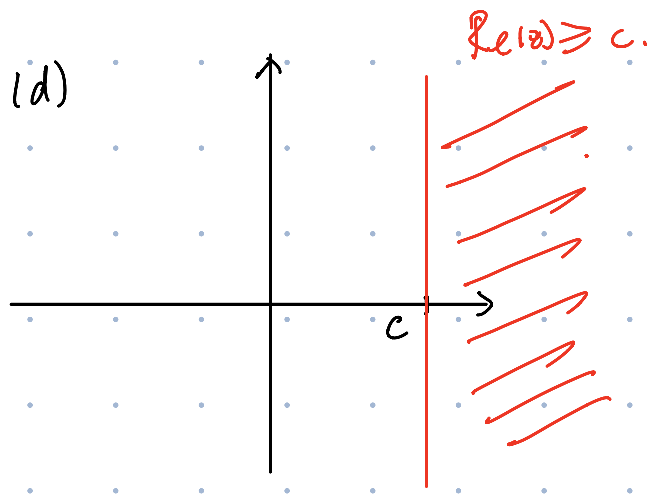
(b) $1 = z\overline{z} = |z|^2$.



(c)



(d)



(e) $\mathrm{Re}(az+b) > 0 \iff \mathrm{Re}(az) > \mathrm{Re}(-b)$.

Write $a = \mathrm{Re}(a) + i\,\mathrm{Im}(a)$, $z = x + iy$.

then $\mathrm{Re}(az) = \mathrm{Re}(a)x - \mathrm{Im}(a)y$.

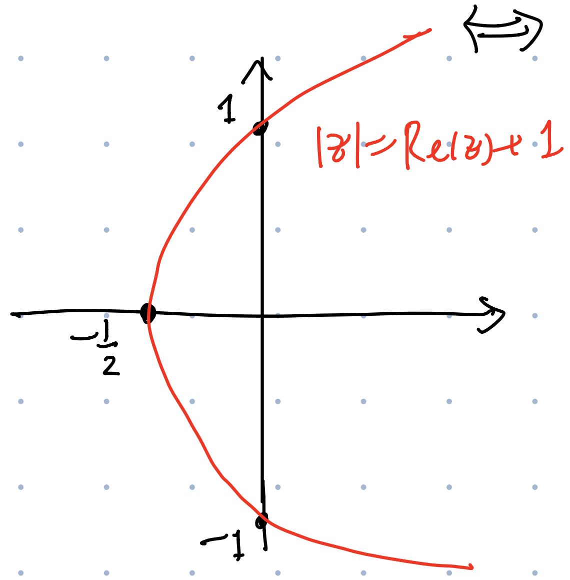
So $\mathrm{Re}(az+b) > 0$ is a half-plane determined by

$\mathrm{Re}(a)x - \mathrm{Im}(a)y > \mathrm{Re}(-b)$

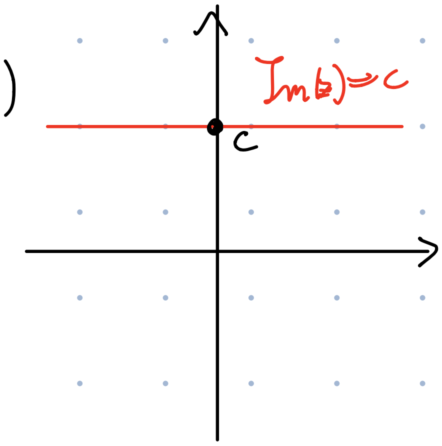
(f) $|z| = \mathrm{Re}(z) + 1 \iff \mathrm{Re}(z) \geq -1$ and $|z|^2 = (\mathrm{Re}(z)+1)^2$

$$\underbrace{|z|^2}_{=\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} = \underbrace{(\mathrm{Re}(z)+1)^2}_{=\mathrm{Re}(z)^2 + 2\mathrm{Re}(z) + 1}$$

$\iff \mathrm{Im}(z)^2 = 2\mathrm{Re}(z) + 1.$



(g)



2. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product in $\mathbb{R}^2$. In other words, if $Z = (x_1, y_1)$ and $W = (x_2, y_2)$, then

$$\langle Z, W \rangle = x_1 x_2 + y_1 y_2.$$

Similarly, we may define a Hermitian inner product $(\cdot, \cdot)$ in $\mathbb{C}$ by

$$(z, w) = z\overline{w}.$$

The term Hermitian is used to describe the fact that $(\cdot, \cdot)$ is not symmetric, but rather satisfies the relation

$$(z, w) = \overline{(w, z)} \quad \text{for all } z, w \in \mathbb{C}.$$

Show that

$$\langle z, w \rangle = \frac{1}{2}[(z, w) + (w, z)] = \mathrm{Re}(z, w),$$

where we use the usual identification $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$.

$\langle z, w \rangle = x_1 x_2 + y_1 y_2$

$(z, w) = z\overline{w} = (x_1 + iy_1)(x_2 - iy_2) = (x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)$

$(w, z) = w\overline{z} = (x_2 + iy_2)(x_1 - iy_1) = (x_1 x_2 + y_1 y_2) + i(x_1 y_2 - x_2 y_1)$

Then it's clear that

$\langle z, w \rangle = \frac{1}{2}\big((z, w) + (w, z)\big) = \mathrm{Re}(z, w).$ $\square$

4. Show that it is impossible to define a total ordering on $\mathbb{C}$. In other words, one cannot find a relation $\succ$ between complex numbers so that:

(i) For any two complex numbers z, w, one and only one of the following is true: $z \succ w$, $w \succ z$ or $z = w$.

(ii) For all $z_1, z_2, z_3 \in \mathbb{C}$ the relation $z_1 \succ z_2$ implies $z_1 + z_3 \succ z_2 + z_3$.

(iii) Moreover, for all $z_1, z_2, z_3 \in \mathbb{C}$ with $z_3 \succ 0$, then $z_1 \succ z_2$ implies $z_1 z_3 \succ z_2 z_3$.

[Hint: First check if $i \succ 0$ is possible.]

Assume that there exists a total ordering $\succ$ on $\mathbb{C}$.
By (i), we must have either $i \succ 0$ or $0 \succ i$.

- Suppose $i \succ 0$.
  By (iii), we have $-1 = i^2 \succ 0^2 = 0$,
  and $1 = (-1)^2 \succ 0^2 = 0$.
  By (ii), we have $0 = (-1)+1 \succ 0+1 = 1$.
  contradiction.
- Suppose $0 \succ i$. By (ii), we have $-i \succ 0$.
  Then we can get a contradiction by the same argument. $\square$

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7. The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

(a) Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

[Hint: Why can one assume that z is real? It then suffices to prove that

$$(r-w)(r-\overline{w}) \leq (1-rw)(1-r\overline{w})$$

with equality for appropriate r and $|w|$.]

(b) Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

(i) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.

(ii) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective. [Hint: Calculate $F \circ F$.]

<u>7.(a):</u> Let $|z| < 1$, $|w| < 1$,
and let $\theta = \arg(z)$.
Define $z' := e^{-i\theta} z$, $w' := e^{-i\theta} w$.
Then $z' \in \mathbb{R}$,

$$|w' - z'| = |e^{-i\theta}(w-z)| = |w-z|$$

$$|1 - \overline{w'} z'| = |1 - e^{i\theta}\overline{w} e^{-i\theta} z| = |1 - \overline{w} z|$$

Hence, it suffices to consider the case where z is replaced by z' and w is replaced by w', i.e. we may assume $z \in \mathbb{R}$.

It suffices to prove $|w-z|^2 < |1-\bar{w}z|^2$, if $|z|<1, |w|<1$

$$|w-z|^2 = (w-z)(\bar{w}-z) = |w|^2 + z^2 - z(w+\bar{w})$$

$$|1-\bar{w}z|^2 = (1-\bar{w}z)(1-wz) = 1 - z(w+\bar{w}) + |w|^2 z^2$$

$$\Leftrightarrow \quad (1-|w|^2)(1-z^2) > 0.$$

Also, the equality holds if $|z|=1$ or $|w|=1$. $\square$

(b). (i) For any $w \in \mathbb{D}$, $|w|<1$.
Hence $\forall z \in \mathbb{D}$, $|F(z)| = \left|\frac{w-z}{1-\bar{w}z}\right| < 1$. by part (a).

$1-\bar{w}z \neq 0$ since $|\bar{w}z| = |w||z| < 1$.

Hence F is holo. on $\mathbb{D}$ by Prop. 2.2 (iii). $\square$

(ii) is easy to check.

(iii) follows from part. (a).

(iv) We claim that the inverse of F is F, i.e.
$F(F(z)) = z \quad \forall z \in \mathbb{D}$:

$$F(F(z)) = \frac{w - F(z)}{1-\bar{w}F(z)} = \frac{w - \frac{w-z}{1-\bar{w}z}}{1-\bar{w}\cdot\frac{w-z}{1-\bar{w}z}}$$

$$= \frac{w(1-\bar{w}z)-(w-z)}{1-\bar{w}z - \bar{w}(w-z)} = \frac{z(1-|w|^2)}{1-|w|^2} = z,$$
$\square$

<u>Note</u>: These maps will be useful when we discuss conformal mappings.

10. Show that

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \triangle,$$

where $\triangle$ is the **Laplacian**

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\cdot\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\cdot\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$= \frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x\partial y} - i\frac{\partial^2}{\partial y\partial x} + \frac{\partial^2}{\partial y^2} = \Delta.$$

The other equality can be checked similarly. $\square$

11. Use Exercise 10 to prove that if f is holomorphic in the open set Ω, then the real and imaginary parts of f are **harmonic**; that is, their Laplacian is zero.

f hol. $\Rightarrow \frac{\partial}{\partial \bar{z}} f = 0 \Rightarrow \Delta f = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} f = 0.$

$\Downarrow$

$$\frac{\partial}{\partial z}\bar{f} = \overline{\frac{\partial f}{\partial \bar{z}}} = 0. \Rightarrow \Delta \bar{f} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\bar{f} = 0.$$

Hence $\Delta(\mathrm{Re} f) = \Delta\left(\frac{f+\bar{f}}{2}\right) = 0,$

$\Delta(\mathrm{Im} f) = \Delta\left(\frac{f-\bar{f}}{2i}\right) = 0.$ $\square$

Alternatively, write $f = u + iv$.

C-R relation $\Rightarrow u_x = v_y,\ u_y = -v_x$

$\Rightarrow u_{xx} = v_{yx} = v_{xy} = -u_{yy}. \Rightarrow \Delta u = 0.$

$v_{xx} = -u_{yx} = -u_{xy} = -v_{yy} \Rightarrow \Delta v = 0.$ $\square$