#1: Find $\vec{x}(t): \mathbb{R} \longrightarrow \mathbb{R}^3$ s.t.

$$\vec{x}'(t) = \begin{bmatrix} 4 & 0 & 0 \\ 3 & -2 & 0 \\ -3 & 6 & 4 \end{bmatrix} \vec{x}(t), \qquad \vec{x}(0) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

Sol^n: eigenvalues of the matrix: 4 (multi. 2), and -2.

(4): $\mathrm{Nul}\begin{bmatrix} 0 & 0 & 0 \\ 3 & -6 & 0 \\ -3 & 6 & 0 \end{bmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

(−2): $\mathrm{Nul}\begin{bmatrix} 6 & 0 & 0 \\ 3 & 0 & 0 \\ -3 & 6 & 6 \end{bmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$$\Rightarrow \quad X(t) := \begin{bmatrix} 2e^{4t} & 0 & 0 \\ e^{4t} & 0 & -e^{-2t} \\ 0 & e^{4t} & e^{-2t} \end{bmatrix} \text{ is a fund. matrix.}$$

Need to find $\vec{c} \in \mathbb{R}^3$ s.t. $X(0)\,\vec{c} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$.

$$X(0) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \Rightarrow \vec{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \text{ sol}^n \quad \vec{x}(t) = \begin{bmatrix} 2e^{4t} \\ e^{4t} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^{4t} \end{bmatrix} + \begin{bmatrix} 0 \\ -e^{-2t} \\ e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{4t} \\ e^{4t} - e^{-2t} \\ e^{4t} + e^{-2t} \end{bmatrix}. \qquad \square$$

#2: Find $\vec{x}(t): \mathbb{R} \longrightarrow \mathbb{R}^3$ s.t.

$$\vec{x}\,'(t) = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \vec{x}(t), \qquad \vec{x}(0) = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

Sol:

$$\det \begin{pmatrix} 3-\lambda & 0 & 0 \\ -1 & 1-\lambda & 1 \\ 2 & 0 & 1-\lambda \end{pmatrix} = (3-\lambda)(1-\lambda)^2$$

③: $$\mathrm{Nul} \begin{pmatrix} 0 & 0 & 0 \\ -1 & -2 & 1 \\ 2 & 0 & -2 \end{pmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

①: $$\mathrm{Nul} \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\longrightarrow$ NOT diagonalizable.

$$\mathrm{Nul}(A - \mathbb{I})^2 = \mathrm{Nul} \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

$$= \mathrm{Nul} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$e^{tA} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e^{t} e^{t(A-\mathbb{I})} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= e^{t} \left(\mathbb{I} + t(A-\mathbb{I}) \right) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= e^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t e^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \quad X(t) = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{t} & te^{t} \\ e^{3t} & 0 & e^{t} \end{bmatrix}$$ is a fund. matrix.

Find $\vec{c} \in \mathbb{R}^3$ s.t. $\underbrace{X(0)}_{=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \vec{c} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

$$\Rightarrow \vec{c} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

$\Rightarrow$ the sol^{n} $\vec{x}(t) = \begin{bmatrix} 2e^{3t} \\ 3e^{t}(t+1) \\ 2e^{3t} + 3e^{t} \end{bmatrix}$. $\square$