Math H54 First Midterm October 3, 2019

Name: ____

- You have 80 minutes to complete the exam.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	20	
2	20	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	100	

1. (4 points each) Determine if each statement is TRUE or FALSE, and give a short justification.

   (a) Let A and B be two $n \times n$ square matrices. If AB is an invertible matrix, then A and B must be invertible as well.

   (b) If the determinant of a 4×4 matrix A is 0, then $\dim \operatorname{Nul}(A) = 0$.

   (c) Let A and B be two $n \times n$ square matrices. We have $(AB)^T = A^T B^T$.

   (d) The set $\{p(t) : p(0) = 0 \text{ or } p(1) = 0\}$ of polynomials p satisfying $p(0) = 0$ or $p(1) = 0$ with the usual addition and scalar multiplication forms a vector space.

   (e) A set of n linearly independent vectors in $\mathbb{R}^n$ must form a basis of $\mathbb{R}^n$.

2. (10 points each) Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & -1 \\ 1 & -1 & 2 & 4 & -1 \\ 2 & -2 & 1 & 5 & -2 \end{pmatrix}.$$

(a) Find a basis of the column space $\mathrm{Col}(A)$.

(b) Find a basis of the null space $\mathrm{Nul}(A)$.

3. (10 points) Recall that for any $m \times n$ matrix A, one can associate a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ that sends $\vec{x}$ to $A\vec{x}$. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{pmatrix}.$$

Find a matrix C such that $T_B \circ T_C = T_A$ (i.e. $T_B(T_C(\vec{x})) = T_A(\vec{x})$ for any $\vec{x} \in \mathbb{R}^3$).

4. (10 points) Let

$$A = \begin{pmatrix} a & 1 & 0 \\ 1 & b & 1 \\ 0 & 1 & c \end{pmatrix},$$

where a, b, c are real numbers. Suppose that the transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ sends the unit ball $\mathbb{B} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ to certain three dimensional region $T_A(\mathbb{B})$ with volume $\mathrm{Vol}(T_A(\mathbb{B})) = \frac{32}{3}\pi$. What can you say about the numbers a, b, c? Be as explicit as possible.
(Recall that $\mathrm{Vol}(\mathbb{B}) = \frac{4}{3}\pi$.)

5. (10 points) Let $\mathcal{B} = \{-1+t, 1-2t\}$ and $\mathcal{C} = \{13-5t, 5-2t\}$ be two bases of the vector space P_1 of polynomials of degree ≤ 1. Suppose the coordinate vector of an element $\vec{x} \in P_1$ with respect to $\mathcal{B}$ is $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Find its coordinate vector $[\vec{x}]_{\mathcal{C}}$ with respect to $\mathcal{C}$.

6. (5 points each)

(a) Let A be an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be its associated linear transformation. Suppose that T_A is <u>injective</u>. What can you say about the relation between m and n? What can you say about the dimensions of the column space $\mathrm{Col}(A)$ and the null space $\mathrm{Nul}(A)$? Provide justifications for your answers.

(b) Same problems as in part (a), but replace <u>injective</u> by <u>surjective</u>.

7. (10 points) Let V be a vector space and $\{v_1, \ldots, v_n\}$ be a linearly independent set in V. Suppose that $w \in V$ and $w \notin \text{Span}\{v_1, \ldots, v_n\}$. Prove that $\{v_1, \ldots, v_n, w\}$ is a linearly independent set.

8. (10 points) Let A be an $m \times n$ matrix and B be an $n \times n$ square matrix. Show that $\dim \operatorname{Nul}(AB) \geq \dim \operatorname{Nul}(A)$. What can you say about the relation between $\dim \operatorname{Nul}(AB)$ and $\dim \operatorname{Nul}(A)$ if B is invertible?
(Hint: You may use the rank theorem.)