Outline: (I) recap: slope stability of vector bundles.

(II) stability conditions on abelian categories

(III) stability conditions on triangulated categories

(IV): motivation / digression on mirror symmetry

(V): examples

(VI): connections with Teichmüller theory, rotation theory, Stokes phenomenon, ⋯.

(I) recap: slope stability of vector bundles.

X - smooth projective variety $/\mathbb{C}$

ω - ample class

E - vector bundle $/X$ $\rightsquigarrow$ $\mathrm{rank}(E)$

$$\deg_\omega(E) := c_1(E).\omega^{\dim X - 1}.$$

$$\rightsquigarrow \text{slope } \mu_\omega(E) := \frac{\deg_\omega(E)}{\mathrm{rk}(E)}$$

Rmk: • can also define μ_ω for coherent sheaves on X.

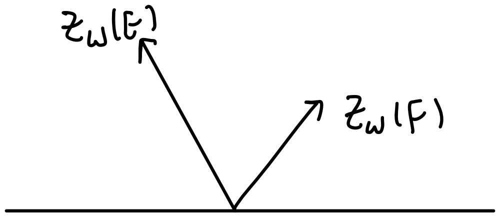
• if E is torsion sheaf, then $\mu_\omega(E) := +\infty$

Def: $E \in \mathrm{Coh}(X)$ is called (semi) stable if

$$\mu(F) \underset{(\leq)}{<} \mu(E) \quad \text{for any proper subsheaf } 0 \neq F \subseteq E.$$

Rmk: $E \in \mathrm{Coh}(X)$ define $Z_\omega(E) := -\deg(E) + i\,\mathrm{rk}(E) \in \mathbb{C}$

$\uparrow$ (central charge of E)

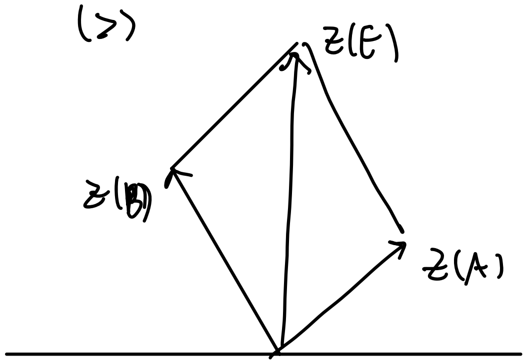


$\Leftrightarrow \quad \mu_\omega(F) < \mu_\omega(E)$

Fact. (see-saw) $\quad 0 \to A \to E \to B \to 0$ short exact seq. in $\mathrm{Coh}(X)$

If $\mu(A) < \mu(E)$, then $\mu(E) < \mu(B)$

$(>) \qquad\qquad\qquad (>)$



Fact. If $A, B \neq 0$ both semistable, $\mu(A) > \mu(B)$

then $\mathrm{Hom}(A, B) = 0$.

pf Suppose $\exists\ A \xrightarrow{f \neq 0} B$, $\quad \underset{\neq}{\overset{0}{}} Q := \mathrm{Im}(f) \subseteq B$

- B is semistable $\Rightarrow \mu(Q) \leq \mu(B)$

-
$$\begin{tikzcd} & A \arrow[r, two heads] \arrow[rr, bend left, "f"] & Q \arrow[r, hook] & B \\ K \arrow[ur] \end{tikzcd}$$

$0 \to K \to A \to Q \to 0$ short exact seq.

A s.s. $\Rightarrow \mu(K) \leq \mu(A) \xRightarrow[\text{see-saw}]{} \mu(A) \leq \mu(Q)$ $\square$

Thm (Harder-Narasimhan property) $0 \neq E \in Coh(X)$,

$\exists\ 0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$.

s.t. $A_i := E_i / E_{i-1}$ semistable, $\mu(A_1) > \mu(A_2) > \cdots > \mu(A_n)$

$i = 1, \ldots, n$

Rmk:

- Harder-Narasimhan filtration is unique.
- $Coh^{\omega}_{\mu}(X) := \{ E \in Coh(X) \mid E$ is μ_ω-semistable, $\mu_\omega(E) = \mu$ or $E \cong 0 \}$

  is an abelian subcategory of $Coh(X)$,

  where the simple objects are the stable coherent sheaves.

Upshot: ω-ample class $\rightsquigarrow$ μ_ω slope $\rightsquigarrow$ $\{Coh^{\omega}_{\mu}(X)\}_{\mu}$ "nice refinement" of $Coh(X)$.

"Kähler/symplectic input" into the "complex-geometric cat." $Coh(X)$

(II) Stability conditions on abelian categories $\mathcal{A}$

Def $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ group homomorphism

$K_0(\mathcal{A})$: abelian gp. generated by $[E]$, $E \in \mathcal{A}$, modulo relations $[A] + [B] = [C]$ if $\exists\ 0 \to A \to C \to B \to 0$ short exact seq.

We say is a stability function if $Z(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$ $\forall E \neq 0$

Def: Denote $D := -\operatorname{Re} Z$, $R := \operatorname{Im} Z$, $M := \frac{D}{R}$

Say $0 \neq E \in \mathcal{A}$ is <u>(semi)stable</u> (w.r.t. Z) if

$$M(F) \underset{(\leq)}{<} M(E) \quad \forall\, 0 \neq F \subsetneq E$$

Def Say a stability fcn <u>satisfies the Harder-Narasimhan property</u> if $\forall\, 0 \neq E \in \mathcal{A}$, $\exists\, 0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$

s.t. $A_i := E_i / E_{i-1}$ is Z-s.s. and $M(A_1) > \cdots > M(A_n)$.

e.g.: $X =$ smooth projective curve, $\mathcal{A} = \operatorname{Coh}(X)$

$Z(E) := -\deg(E) + i\, \operatorname{rk}(E)$ is a stab. fcn w/ HN property

Rmk: For $\dim X \geq 2$, Toda proved:

$$\nexists \quad \begin{tikzcd} K_0(\operatorname{Coh}(X)) \arrow[rr, "Z"', "\text{stab. fcn.}"] \arrow[dr, "ch"'] & \circlearrowleft & \mathbb{C} \\ & H^*(X, \mathbb{Q}) \arrow[ur] & \end{tikzcd}$$

e.g. $\mathcal{A} = \mathrm{Rep}(\bullet \longrightarrow \bullet)$ Objects: $\{ V_1 \xrightarrow{T} V_2 \}$

Choose $z_1, z_2 \in \mathbb{H} \cup \mathbb{R}_{<0}$.

Define $Z_{z_1,z_2}(V_1 \xrightarrow{T} V_2) := z_1 \dim V_1 + z_2 \dim V_2 \in \mathbb{H} \cup \mathbb{R}_{<0}$

<u>What are stable objects w.r.t. Z_{z_1,z_2}?</u>

- stable objs must be indecomposable; the only indecomposable objects are: $\boxed{\mathbb{C} \longrightarrow 0}$, $\boxed{0 \to \mathbb{C}}$, $\underbrace{\mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C}}_{E}$.

  stable w.r.t. any stab. cond'n: $S_1 = (\mathbb{C} \to 0)$, $S_2 = (0 \to \mathbb{C})$

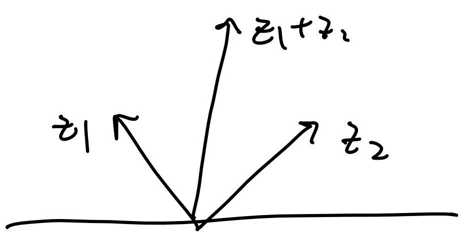
Q: Is E stable w.r.t. Z_{z_1,z_2}?

$0 \to S_2 \to E \to S_1 \to 0$

$\Updownarrow$

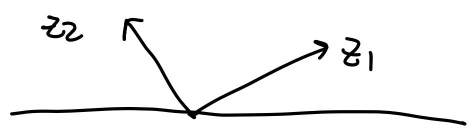
$$\underset{\substack{\| \\ \mathrm{Arg}(z_2)}}{\mathrm{Arg}\, Z_{z_1,z_2}(S_2)} < \underset{\substack{\| \\ \mathrm{Arg}(z_1+z_2)}}{\mathrm{Arg}\, Z_{z_1,z_2}(E)} \iff \mathrm{Arg}(z_2) < \mathrm{Arg}(z_1)$$

<u>Case 1</u>:



3 stable objects

<u>Case 2</u>;



2 stable objects

$\mathrm{Arg}(z_1) = \mathrm{Arg}(z_2)$

(II) Bridgeland stab. cond$^{\underline{n}}$ on Δ cat.

Rmk: There could be many hearts (of bounded t-structures) on a Δ-cat. For instance, there are many well-known examples of derived equivalent varieties ($D^b(X) \cong D^b(Y)$.)

$$\begin{array}{ccc} D^b(X) & \cong & D^b(Y) \\ \cup\!\!| & & \cup\!\!| \\ \mathrm{Coh}(X) & & \mathrm{Coh}(Y) \end{array}$$

"all possible hearts of a Δ cat" is a discrete notion, but (the space of) Bridgeland stab. cond$^{\underline{n}}$ gives a way to move from one heart to another

Bridgeland stab. cond$^{\underline{n}}$ on a Δ-cat. D:

- Fix Γ: finitely generated free abel gp. $\cong \mathbb{Z}^n$, a norm $\|\cdot\|$ on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, fix a gp homom. $cl: K_0(D) \to \Gamma$ ($K_0(D) = K_0(\mathcal{A})$)

- A Bridgeland stability condition is a pair $(Z, \mathcal{A})$, where
  - $Z: \Gamma \longrightarrow \mathbb{C}$ group homomorphism, $\mathcal{A} \subseteq D$ is a heart
  - $Z \circ cl$ is a stability function on $\mathcal{A}$ with HN property.
  - $\sup \left\{ \dfrac{\|cl(E)\|}{|Z(cl(E))|} \;\middle|\; E \in \mathcal{A} \text{ semistable} \right\} < +\infty$

Rmk: An equivalent definition of stab. cond^{n} on D:

$\sigma = (Z, P)$, where:

- $Z: \Gamma \to \mathbb{C}$ gp homom.
- $P = \{P(\phi)\}_{\phi \in \mathbb{R}}$, $P(\phi) \subseteq D$ full additive subcat.

  $\nwarrow$ semistable objects of phase ϕ.

satisfies:

- $Z(E) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$ if $E \in P(\phi)$
- $P(\phi+1) = P(\phi)[1]$.
- $\mathrm{Hom}(A_1, A_2) = 0$ if $A_i \in P(\phi_i)$ and $\phi_1 > \phi_2$
- $\forall\, 0 \neq E \in D$, $\exists$ $0 = E_0 \to E_1 \to \cdots \to E_n = E$ with $A_1, \dots, A_n$ the cones, $\phi^+ = \phi_1$, $\phi^- = \phi_n$

$m_\sigma(E) := \sum_i |Z_\sigma(A_i)|$

where $A_i \in P(\phi_i)$ and $\phi_1 > \cdots > \phi_n$

- $\sup \left\{ \frac{\|cl(E)\|}{|Z(cl(E))|} \;\middle|\; 0 \neq E \in \bigcup_\phi P(\phi) \right\} < +$

There is a generalized metric on $\underline{\underline{Stab(D)}}$:

$$d(\sigma_1, \sigma_2) := \sup_{E \neq 0} \left\{ |\phi_1^+(E) - \phi_2^+(E)|,\ |\phi_1^-(E) - \phi_2^-(E)|,\ \left|\log \frac{m_{\sigma_1}(E)}{m_{\sigma_2}(E)}\right| \right\}$$

$\in [0, +\infty]$

Thm (Bridgeland) $\mathrm{Stab}_\Gamma(D) \xrightarrow{Z} \mathrm{Hom}(\Gamma, \mathbb{C})$

is a local homeomorphism.

In particular, $\mathrm{Stab}_\Gamma(D)$ is a complex manifold

of $\dim = \mathrm{rank}(\Gamma)$.

(IV): motivation / digression on mirror symmetry

Kontsevich's homological mirror symmetry conjecture:

X - CY $\qquad\qquad \check{X}$: mirror CY

$$\begin{array}{ccc} D^b\mathrm{Coh}(X) & & D^b\mathrm{Coh}(\check{X}) \\ & \times & \\ D^\pi\mathrm{Fuk}(X) & & D^\pi\mathrm{Fuk}(\check{X}) \end{array}$$

Obj: Lagr. submflds of X.

X CY, I, ω (cpx, sympl.) $\xrightarrow{\sigma\text{-model}}$ $N=(2,2)$ SCFT

(symplectic) A-model $\rightsquigarrow D^\pi\mathrm{Fuk}(X,\omega)$

B-model (complex) $\rightsquigarrow D^b\mathrm{Coh}(X, I)$

<u>Idea</u> Stability conditions on $\qquad$ recover the other half of the info.

<u>Conj</u> (Bridgeland, Joyce) $D^\pi\mathrm{Fuk}(X,\omega)$.

Choose a holo. top form $\Omega_X \rightsquigarrow$ a stab. cond$^{\underline{n}}$ on $D^\pi\mathrm{Fuk}(X,\omega)$,

where $Z(L) = \int_L \Omega_X$, $P(\phi) = \{\text{slag of phase } \phi\}$

§ Some group actions on $\mathrm{Stab}(D)$

1) $F \in \mathrm{Aut}(D)$

$$F \cdot (Z, P) := \left(Z \circ [F]^{-1}, P'(\phi) \right)$$

where $P'(\phi) := F(P(\phi))$.

2) $g = (T, f) \in \widetilde{GL}^{+}(2, \mathbb{R}) \curvearrowright \mathrm{Stab}(D)$.

(where $T \in GL^{+}(2, \mathbb{R})$, $f : \mathbb{R} \longrightarrow \mathbb{R}$ increasing,
$f(\phi + 1) = f(\phi) + 1$.
induced maps on $\mathbb{R}^2 \setminus \{0\} / \mathbb{R}_{>0} \cong S^1 \cong \mathbb{R} / 2\mathbb{Z}$.)

$$(Z, P) \cdot g := \left(T^{-1} \circ Z, P'' \right), \text{ where } P''(\phi) := P(f(\phi))$$

$D = D^b \mathrm{Coh}(C)$
$\uparrow$
curve.

$$\begin{tikzcd} K_0(D) \arrow[rr, "Z"] \arrow[dr] & \circlearrowleft & \mathbb{C} \\ & \Gamma = N(D) \cong H^0(C) \oplus H^2(C) \arrow[ur] & \end{tikzcd}$$

$D^b \mathrm{Rep}(\cdot \rightrightarrows \cdot)$
$\wr\!\wr$

1) $\mathrm{Stab}(D^b(\mathbb{P}^1)) \cong \mathbb{C}^2$ (Okada)

2) $\mathrm{Stab}(D^b(C_{g \geq 1})) \cong \widetilde{GL}^{+}(2, \mathbb{R}) \cong \mathbb{C} \times \mathbb{H}$.

(Bridgeland: $g = 1$, Macri: $g \geq 1$)

§ One application:

- Bridgeland:

$$\begin{array}{c} \mathrm{Stab}^{\dagger}(D^b\mathrm{Coh}(K3)) \\ \downarrow Z \\ \mathcal{P}_0^{+}(X) \end{array}$$

covering map.

$\leftarrow$ period domain

- Conj
  - $\mathrm{Aut}(D)$ preserves $\mathrm{Stab}^{\dagger}(D)$.
  - $\mathrm{Stab}^{\dagger}(D)$ is simply connected.

isometries on Mukai lattice

$$\Rightarrow \quad 1 \to \pi_1 \mathcal{P}_0^{+}(X) \to \mathrm{Aut}(D^b(X)) \to \mathrm{Aut}^{+} H^{*}(X,\mathbb{Z}) \to 1$$

- Bayer-Bridgeland: (K3 surface, $\rho = 1$)

  $\mathrm{Stab}^{\dagger}(D)$ is contractible.

  $\rightsquigarrow$ nice description of $\mathrm{Aut}(D)$.

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§ Parallel between stability conditions & Teichmüller theory

(Gaiotto-Moore-Neitzke, Bridgeland-Smith, Haiden-Katzarkov-Kontsevich)

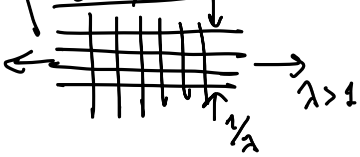
Riemann surface S	Δ cat. D
curve C	object E.
$C_1 \cap C_2$	$\mathrm{Hom}(E_1, E_2)$
$MCG(S) \curvearrowright \mathrm{Teich}(S)$	$\mathrm{Aut}(D) \curvearrowright \mathrm{Stab}(D)$

pseudo-Anosov (generic)

reducible

finite order

locally on S,



metric g	stab. cond^{ns} σ
geodesic/straight line	semistable objects
length $l_g(C)$	mass $m_\sigma(E)$

Thurston: S, f, $\forall g$ $\exists\ \lambda_1 > \cdots > \lambda_n$

s.t. $\forall C \subseteq S$, $\exists\ \lambda_i$ s.t. $\limsup\limits_{n\to\infty} l_g(f^n C)^{1/n} = \lambda_i$

f is pseudo-Anosov $\iff$ $n = 1$ and $\lambda_1 > 1$.