HOMEWORK 8
MATH H54, FALL 2021

Due November 2, 11am

Some ground rules:

- Please submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW8**.
- The submission should be a **single PDF file**.
- Late homework will not be accepted/graded under any circumstances.
- Make sure the writing in your submission is clear enough. Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own, and acknowledge the students with whom you worked.
- **For True/False questions**: You have to prove the statement if your answer is "True"; otherwise, you have to provide an explicit counterexample and justification.
- You are allowed to use any result that is proved in the lecture. But if you would like to use other results, you have to prove it first before using it.

Problems:

(1) Let $p(\lambda)$ be the characteristic polynomial of an $n \times n$ orthogonal matrix. Prove that $\lambda^n p(\lambda^{-1}) = \pm p(\lambda)$.

(2) Let $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ be a real 2×2 symmetric invertible matrix. Prove the following statements.
   (a) A is positive definite if $\det A > 0$ and $a > 0$.
   (b) A is negative definite if $\det A > 0$ and $a < 0$.
   (c) A is indefinite if $\det A < 0$.
   (Hint: HW5 Problem 4.)

(3) (a) Prove that if B is an $m \times n$ matrix, then $B^T B$ is positive semidefinite.
   (b) Prove that if B is an $n \times n$ invertible matrix, then $B^T B$ is positive definite.
   (Hint: Consider the associate quadratic forms.)

(4) Prove that if A a positive definite matrix, then there exists a positive definite matrix B such that $A = B^T B$. (Hint: Write $A = PDP^T$, and write $D = CC^T$, where C is another diagonal matrix.)

(5) Let A and B be $n \times n$ positive definite matrices. Prove that $A + B$ also is positive definite. (Hint: Consider the associate quadratic forms.)

(6) Let A be an positive definite matrix. Prove that A^{-1} also is positive definite. (Hint: Consider eigenvalues.)

(7) Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and corresponding orthonormal eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. Prove that

$$\lambda_1 = \min_{\vec{x} \neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} \quad \text{and} \quad \lambda_n = \max_{\vec{x} \neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}.$$

(Hint: Write $A = PDP^T$.)

(8) Let A be a real symmetric matrix. Prove that

$$\operatorname{rank}(A) \cdot \operatorname{tr}(A^2) \geq (\operatorname{tr} A)^2.$$

(Hint: Consider eigenvalues and use the Cauchy–Schwartz inequality.)

(9) Let A be a real skew-symmetric matrix, i.e. $A = -A^T$. Prove that A^2 is a symmetric, negative semidefinite matrix. (Hint: Consider the associate quadratic form.)

(10) Let A be a real skew-symmetric matrix. Prove that $\mathbb{I} + A$ is invertible. (Hint: Use Problem 9.)