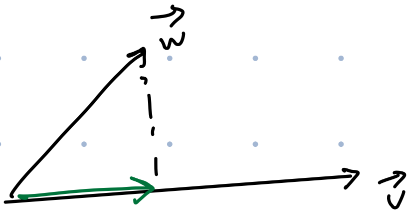
Recall:

- 

$$\operatorname{proj}_{\operatorname{span}\{\vec{v}\}} \vec{w} = \frac{\langle \vec{w}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}.$$

- $(V, \langle -,- \rangle)$ inner product space, $\{\vec{v}_1, \dots, \vec{v}_n\}$ orthogonal basis (it's a basis & orthogonal set)

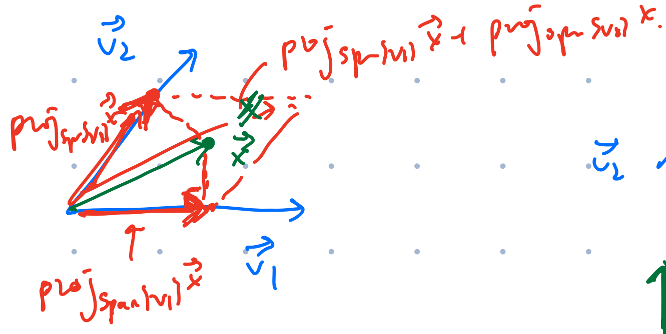
Then $\forall \vec{x} \in V$, we have $\vec{x} = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n$,

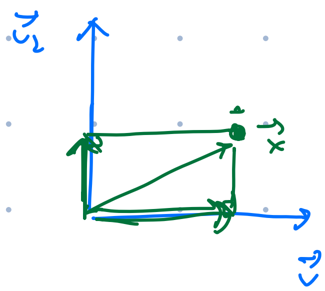
where $a_i = \dfrac{\langle \vec{v}_i, \vec{x} \rangle}{\|\vec{v}_i\|^2}$.

i.e., $\vec{x} = \operatorname{proj}_{\operatorname{span}\{\vec{v}_1\}} \vec{x} + \dots + \operatorname{proj}_{\operatorname{span}\{\vec{v}_n\}} \vec{x}$.

Rmk: the statement doesn't hold if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is NOT orthogonal.

e.g. $V = \mathbb{R}^2$





Def Say $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an <u>orthonormal set</u> if it's an orthogonal set & each $\vec{v}_i$ is an unit vector.

Rmk: If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is orthogonal, then $\left\{ \dfrac{\vec{v}_1}{\|\vec{v}_1\|}, \dots, \dfrac{\vec{v}_n}{\|\vec{v}_n\|} \right\}$ is orthonormal.

$$\left\langle \frac{\vec{v}}{\|\vec{v}\|}, \frac{\vec{v}}{\|\vec{v}\|} \right\rangle = \frac{1}{\|\vec{v}\|^2} \langle \vec{v}, \vec{v} \rangle = 1$$

Rmk: $\mathcal{U}$: $m\times n$ w/ orthonormal columns, $\mathcal{U} = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix}$

$\iff \mathcal{U}^T\mathcal{U} = I_n.$

$$\mathcal{U}^T\mathcal{U} = \begin{bmatrix} - \vec{u}_1 - \\ \vdots \\ - \vec{u}_n - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \langle \vec{u}_1, \vec{u}_1 \rangle & \langle \vec{u}_1, \vec{u}_2 \rangle & \cdots & \langle \vec{u}_1, \vec{u}_n \rangle \\ \langle \vec{u}_2, \vec{u}_1 \rangle & & & \vdots \\ \vdots & & & \vdots \\ \langle \vec{u}_n, \vec{u}_1 \rangle & & \cdots & \langle \vec{u}_n, \vec{u}_n \rangle \end{bmatrix}$$

$\langle x, y \rangle_{\mathbb{R}} \overset{??}{=} \langle T_{\mathcal{U}}x, T_{\mathcal{U}}y \rangle$

$$\boxed{T_{\mathcal{U}}: \underset{\langle,\rangle}{\mathbb{R}^n} \longrightarrow \underset{\langle,\rangle}{\mathbb{R}^m}}$$

Thm: $\boxed{\mathcal{U}: m\times n.}$ $\left(\mathcal{U}^T\mathcal{U} = I_n.\right)$ Then:

$\forall \vec{x}, \vec{y} \in \mathbb{R}^n$, we have

$$\langle \mathcal{U}\vec{x}, \mathcal{U}\vec{y} \rangle_{\mathbb{R}^m} = \langle \vec{x}, \vec{y} \rangle_{\mathbb{R}^n}$$

(i.e. $T_{\mathcal{U}}$ respects the inner product structures on $\mathbb{R}^n, \mathbb{R}^m$)
(standard)

pf: $\langle \vec{x}, \vec{y} \rangle_{\mathbb{R}^n} = \vec{x}^T \vec{y}.$

$$\langle \mathcal{U}\vec{x}, \mathcal{U}\vec{y} \rangle = (\mathcal{U}\vec{x})^T(\mathcal{U}\vec{y}) = \left(\vec{x}^T\mathcal{U}^T\right)(\mathcal{U}\vec{y})$$

$$= \vec{x}^T \underbrace{\mathcal{U}^T\mathcal{U}}_{= I_n} \vec{y} = \vec{x}^T\vec{y} = \langle \vec{x}, \vec{y} \rangle. \quad \square$$

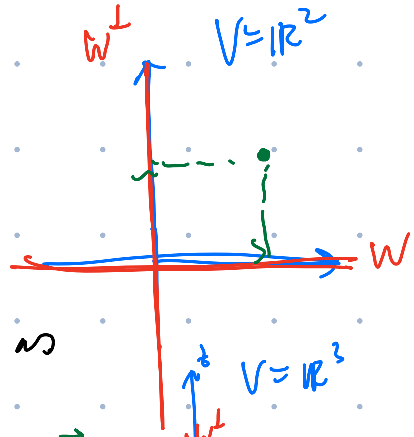
Def: A: $n\times n$ ~~square~~ matrix. A is called __orthogonal__ if $A^TA = I_n$.

$\iff$ the columns of A is an orthonormal set.

$\implies T_A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ respects the inner product on $\mathbb{R}^n$.

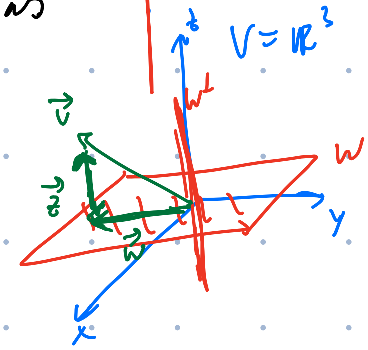
$$\langle A\vec{x}, A\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^n.$$

Orthogonal decomposition theorem.



Suppose V: finite dim'l inner product space

$W \subseteq V$ subspace.



Every vector $\vec{v} \in V$ can be uniquely written as

$$\vec{v} = \boxed{\vec{w}} + \vec{z}$$

where $\vec{w} \in W$ and $\vec{z} \in W^\perp$.

$(\vec{w} := \mathrm{proj}_W \vec{v})$

More concretely, if $\{\vec{u}_1, \dots, \vec{u}_n\}$ is an orthogonal basis of W, then

$$\vec{w} = \frac{\langle \vec{v}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 + \cdots + \frac{\langle \vec{v}, \vec{u}_n \rangle}{\|\vec{u}_n\|^2} \vec{u}_n$$

and $\vec{z} = \vec{v} - \vec{w}$.

Pf: Choose $\vec{w}$ and $\vec{z}$ as above

need to check: 1) $\vec{w} \in W$. ✓

2) $\vec{z} \in W^\perp$. $\Longleftrightarrow$ $\langle \vec{z}, \vec{u}_i \rangle = 0 \quad \forall\, 1 \le i \le n$

orthogonal to $\vec{u}_i$ except $\vec{u}_i$ itself.

$$\langle \vec{z}, \vec{u}_i \rangle = \langle \vec{v} - \vec{w}, \vec{u}_i \rangle$$

$$= \left\langle \vec{v} - \left(\frac{\langle \vec{v}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 + \cdots + \frac{\langle \vec{v}, \vec{u}_n \rangle}{\|\vec{u}_n\|^2} \vec{u}_n \right), \vec{u}_i \right\rangle$$

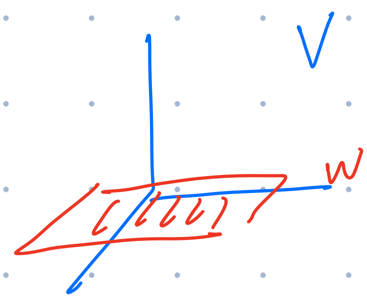
$$= \langle \vec{v}, \vec{u}_i \rangle - \underbrace{\left\langle \frac{\langle \vec{v}, \vec{u}_i \rangle}{\|\vec{u}_i\|^2} \vec{u}_i, \vec{u}_i \right\rangle}_{\frac{\langle \vec{v}, \vec{u}_i \rangle}{\cancel{\|\vec{u}_i\|^2}} \cancel{\langle \vec{u}_i, \vec{u}_i \rangle}} = 0.$$

Uniqueness: $\vec{v} = \vec{w}_1 + \vec{z}_1 = \vec{w}_2 + \vec{z}_2$, $\quad \vec{w}_1, \vec{w}_2 \in W$, $\vec{z}_1, \vec{z}_2 \in W^\perp$.

$$\Rightarrow \underbrace{\vec{w}_1 - \vec{w}_2}_{\in W} = \underbrace{\vec{z}_2 - \vec{z}_1}_{\in W^\perp}$$

Since $W \cap W^\perp = \{\vec{0}\}$ $\Rightarrow$ $\vec{w}_1 - \vec{w}_2 = \vec{z}_2 - \vec{z}_1 = \vec{0}$. $\square$

Rmk: Suppose $\vec{v} \in W$, $\mathrm{proj}_W \vec{v} = \vec{v}$.



Thm: V: finite dim'l inner product space, $W \subseteq V$.

$$\dim W + \dim W^\perp = \dim V$$

pf: W $\quad \{\vec{a}_1, \dots, \vec{a}_n\}$ basis of W

$W^\perp$ $\quad \{\vec{b}_1, \dots, \vec{b}_m\}$ basis of $W^\perp$

Claim: $\{\vec{a}_1, \dots, \vec{a}_n, \vec{b}_1, \dots, \vec{b}_m\}$ is a basis of V.

① they span V:

orthogonal decomp. Thm.: $\forall \vec{v} \in V$, $\exists!$ $\vec{w} \in W$, $\vec{z} \in W^\perp$

st. $\vec{v} = \underbrace{\vec{w}}_{\in \mathrm{Span}\{\vec{a}_1, \dots, \vec{a}_n\}} + \underbrace{\vec{z}}_{\in \mathrm{Span}\{\vec{b}_1, \dots, \vec{b}_m\}} \in \mathrm{Span}\{\vec{a}_1, \dots, \vec{a}_n, \vec{b}_1, \dots, \vec{b}_m\}$

② l.i.:

Suppose $x_1\vec a_1+\cdots+x_n\vec a_n+y_1\vec b_1+\cdots+y_m\vec b_m=\vec 0$

$\Rightarrow \underbrace{x_1\vec a_1+\cdots+x_n\vec a_n}_{W} = \underbrace{-(y_1\vec b_1+\cdots+y_m\vec b_m)}_{W^\perp} = \vec 0$

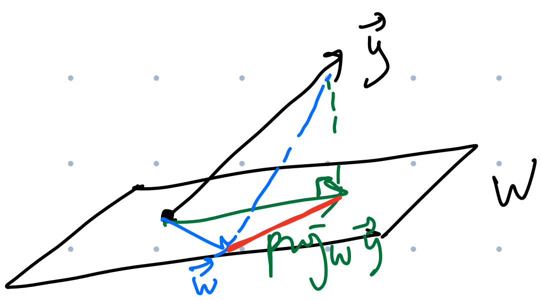
$\Rightarrow$ all x_i's, y_i's $=0$. □

<u>Best approximation thm</u>, $W \subseteq V$ $\quad \vec y \in V$.

↳ " $\text{proj}_W \vec y$ is the closest point in W to $\vec y$ "

i.e. $\|\vec y - \text{proj}_W \vec y\| < \|\vec y - \vec w\|$

for any $\vec w \in W \setminus \{\text{proj}_W \vec y\}$.



pf: Consider $\text{proj}_W \vec y - \vec w \neq \vec 0$

$$\langle \underbrace{\text{proj}_W \vec y - \vec w}_{W}, \underbrace{\vec y - \text{proj}_W \vec y}_{W^\perp} \rangle = 0$$

By Pythagorean thm, we have:

$$\|\vec y - \text{proj}_W \vec y\|^2 + \|\underbrace{\text{proj}_W \vec y - \vec w}_{>0}\|^2 = \|(\vec y - \text{proj}_W \vec y) + (\text{proj}_W \vec y - \vec w)\|^2$$

$$= \|\vec y - \vec w\|^2.$$ □

Gram-Schmidt process

V: finite dim'l inner product space

$\{\vec{v}_1, \dots, \vec{v}_n\}$ basis of V.

Goal: find an orthogonal basis of V.
(orthonormal)

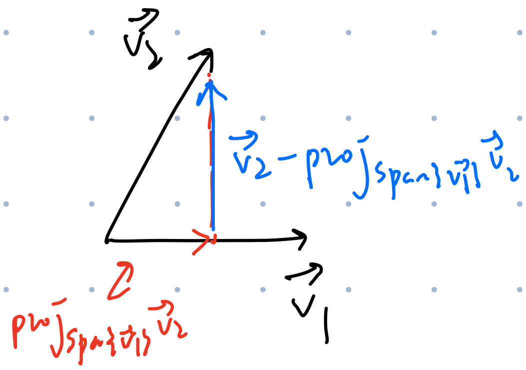
1) $\vec{v}_1 =: \vec{w}_1$

2) $\{\vec{v}_1, \vec{v}_2\}$

Idea, replace $\vec{v}_2$ by "$\vec{w}_2$"

s.t.

- $\langle \vec{v}_1, \vec{w}_2 \rangle = 0$
- $\mathrm{Span}\{\vec{v}_1, \vec{v}_2\} = \mathrm{Span}\{\vec{v}_1, \vec{w}_2\}$.



Define $\vec{w}_2 := \vec{v}_2 - \mathrm{proj}_{\mathrm{span}\{\vec{v}_1\}} \vec{v}_2$

- $\langle \vec{v}_1, \vec{w}_2 \rangle = \left\langle \vec{v}_1, \vec{v}_2 - \dfrac{\langle \vec{v}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \right\rangle$

$$= \langle \vec{v}_1, \vec{v}_2 \rangle - \frac{\langle \vec{v}_2, \vec{v}_1 \rangle}{\cancel{\|\vec{v}_1\|^2}} \cancel{\langle \vec{v}_1, \vec{v}_1 \rangle}$$

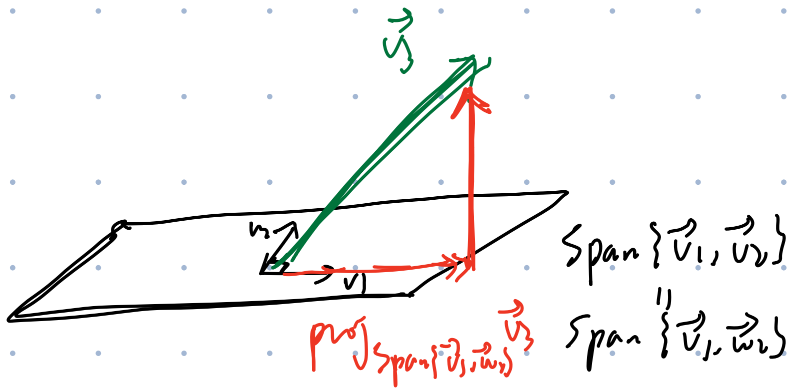
$$= 0.$$

- $\mathrm{Span}\{\vec{v}_1, \vec{w}_2\} = \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$

$\quad \|$

$\vec{v}_2 - * \vec{v}_1$

3) $\{\vec{v}_1, \vec{w}_2, \vec{v}_3\}$.



Define $\vec{w}_3 := \vec{v}_3 - \operatorname{proj}_{\operatorname{Span}\{\vec{v}_1, \vec{w}_2\}} \vec{v}_3$

- $\{\vec{v}_1, \vec{w}_2, \vec{w}_3\}$ is orthogonal.
- $\operatorname{Span}\{\vec{v}_1, \vec{w}_2, \vec{v}_3\} = \operatorname{Span}\{\vec{v}_1, \vec{w}_2, \vec{w}_3\}$

$$\vec{w}_3 = \vec{v}_3 - * \vec{v}_1 - * \vec{v}_2$$

Continue this process inductively,

$\rightsquigarrow$ orthogonal basis $\{\vec{v}_1, \vec{w}_2, \vec{w}_3, \dots, \vec{w}_n\}$ of V.

($\rightsquigarrow$ orthonormal basis $\left\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \dots, \frac{\vec{w}_n}{\|\vec{w}_n\|}\right\}$ of V)

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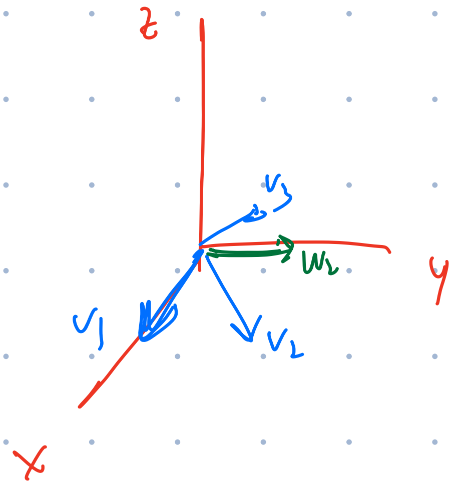
e.g. $\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ in $\mathbb{R}^3$

Replace $\vec{v}_2$ by $\vec{w}_2 = \vec{v}_2 - \operatorname{proj}_{\operatorname{Span}\{\vec{v}_1\}} \vec{v}_2$



$$= \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{\overbrace{\left\langle \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\rangle}^{=1}}{\underbrace{\left\| \begin{bmatrix}1\\0\\0\end{bmatrix} \right\|^2}_{=1}} \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

Replace $\vec{v}_3$ by $\vec{w}_3 = \vec{v}_3 - \overbrace{\text{proj}_{\text{span}\{\underbrace{\vec{v}_1, \vec{w}_2}_{\text{orthogonal}}\}} \vec{v}_3}^{\frac{\langle \vec{v}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\langle \vec{v}_3, \vec{w}_2 \rangle}{\|\vec{w}_2\|^2} \vec{w}_2}$

$$= \begin{bmatrix}1\\1\\1\end{bmatrix} - \left(\frac{\overbrace{\left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\rangle}^{=1}}{\underbrace{\left\| \begin{bmatrix}1\\0\\0\end{bmatrix} \right\|^2}_{=1}} \begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{\overbrace{\left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\rangle}^{=1}}{\underbrace{\left\| \begin{bmatrix}0\\1\\0\end{bmatrix} \right\|^2}_{=1}} \begin{bmatrix}0\\1\\0\end{bmatrix} \right)$$

$$= \begin{bmatrix}0\\0\\1\end{bmatrix}.$$