3/12/2020

Last time X - any set $f_n : X \longrightarrow \mathbb{R}$, $f : X \longrightarrow \mathbb{R}$

$f_n \to f$ pointwise if $\forall x_0 \in X$, $f_n(x_0) \to f(x_0)$

$f_n \to f$ uniformly if $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $|f_n(x_0) - f(x_0)| < \varepsilon$ $\forall n > N$, $x_0 \in X$

"An "a priori" way to know whether the seq. f_n conv. (unif.) to some fcn. f, without knowing what f is"

c.f. Seq. of real numbers (a_n)

conv. $\Longleftrightarrow$ Cauchy & we don't have to know the limit beforehand.

Def X, $f_n : X \longrightarrow \mathbb{R}$ We say (f_n) is uniformly Cauchy if $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $|f_n(x) - f_m(x)| < \varepsilon$ $\forall n, m > N$. $\forall x \in X$

HW unif. conv. $\Rightarrow$ unif. Cauchy.

Thm (f_n) unif. Cauchy $\Leftrightarrow$ (f_n) conv. unif. to some $f : X \longrightarrow \mathbb{R}$

pf ($\Rightarrow$) To determine f, we need to determine $f(x)$ $\forall x \in X$

(If $f_n \to f$, then $f_n(x) \to f(x)$ $\forall x \in X$)

Does $(f_n(x))$ conv.? (for any x)

Yes. b/c $(f_n(x))$ is a Cauchy seq.

(by (f_n) is unif. Cauchy)

We define $f(x) := \lim\limits_{n\to\infty} f_n(x)$.

2 Show "$f_n \to f$ uniformly"

$\forall \varepsilon > 0$, $\exists N > 0$ s.t. $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$ $\forall x \in X$ $\forall n, m > N$

$\Rightarrow$ $f_n(x) - \frac{\varepsilon}{2} < f_m(x) < f_n(x) + \frac{\varepsilon}{2}$ $\forall x \in X$ $\forall n, m > N$

As $m \to \infty$,

$$f_n(x) - \frac{\varepsilon}{2} \leq f(x) \leq f_n(x) + \frac{\varepsilon}{2} \quad \forall x \in X \quad \forall n > N$$

e.g. $0 < \frac{1}{n}$

$0 = \lim \frac{1}{n}$

$\Rightarrow$ $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$ $\forall x \in X$, $\forall n > N$

$\Leftrightarrow$ $f_n \to f$ unif. $\square$

~~Recall~~ Recall $(a_n) \subset \mathbb{R}$ $\quad \sum a_n$

Series of fcns Given a seq. of fcns $f_n : X \to \mathbb{R}$ (X: a set)

Partial sum $\quad S_n : X \longrightarrow \mathbb{R}$

$$x \longmapsto \sum_{k=1}^{n} f_k(x)$$

If $(S_n(x))$ conv. for every $x \in X$, then we define the series of fcn.

$$\sum_{n=1}^{\infty} f_n \quad \text{by} \quad \left(\sum_{n=1}^{\infty} f_n\right)(x) := \lim_{n\to\infty} S_n(x)$$

$X \to \mathbb{R}$

cf) Cauchy criterion for series $\sum a_n$:
$\Leftrightarrow \forall \varepsilon > 0, \exists N > 0$ s.t. $\left|\sum_{k=m}^{n} a_k\right| < \varepsilon \quad \forall n \geq m > N$

We say the series $\sum_{n=1}^{\infty} f_n$ conv. unif.

if the seq. of fcns $(S_n(x))$ conv. unif.

$\Leftrightarrow$ (S_n) is unif. Cauchy, i.e. $\forall \varepsilon > 0, \exists N > 0$

s.t. $\left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| < \varepsilon \quad \forall x \in X, \forall n, m > N.$

$\Leftrightarrow$ $\forall \varepsilon > 0, \exists N > 0$ s.t. $\left|\sum_{k=m}^{n} f_k(x)\right| < \varepsilon \quad \forall x \in X \quad \forall n \geq m > N.$

Coro If $\sum f_n$ conv. unif., then

$$\lim_{n\to\infty} \sup\{|f_n(x)| : x\in X\} = 0$$

$\Rightarrow \lim_{n\to\infty} |f_n(x)| = 0 \quad \forall x\in X$

pf In Cauchy criterion take $n=m$:

$\Rightarrow \forall \varepsilon>0, \exists N>0$ s.t. $|f_n(x)| < \varepsilon/2 \quad \forall x\in X \ \forall n>N$

$$\Downarrow$$

$$\sup\{|f_n(x)| : x\in X\} \le \varepsilon/2 < \varepsilon$$

e.g. $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ but $\sup = 1$

$\Rightarrow \lim_{n\to\infty} \sup\{|f_n(x)| : x\in X\} = 0.$ $\square$

Rmk This is sometimes useful to show $\sum f_n$ doesn't conv. unif.

Weierstrass M-test: Suppose $(M_n)\subset \mathbb{R}$, $M_n \ge 0$, $\sum M_n < +\infty$ and (f_n), $f_n : X \longrightarrow \mathbb{R}$, $|f_n(x)| \le M_n \ \forall x\in X$

Then $\sum f_n$ conv. unif.

pf $\forall \varepsilon>0, \exists N>0$ s.t. $0 \le \sum_{k=m}^{n} M_k < \varepsilon \quad \forall n\ge m>N$

$\Rightarrow \forall n\ge m>N, \forall x\in X,$

$$\left|\sum_{k=m}^{n} f_k(x)\right| \le \sum_{k=m}^{n} |f_k(x)| \le \sum_{k=m}^{n} M_k < \varepsilon \quad \square$$

e.g. Consider $f_0 = 1$, $f_1 = x$, $f_2 = x^2$, ...

$$\text{"}\sum_{n=0}^{\infty} f_n(x)\text{"} = \text{"}\underbrace{1 + x + x^2 + x^3 + \cdots}_{\text{conv. if } x \in (-1,1)}\text{"}$$

Can define $\displaystyle\sum_{n=0}^{\infty} f_n(x)$ on $x \in (-1,1)$

$$= \frac{1}{1-x}$$

<u>Q</u>: Does $\displaystyle\sum_{n=0}^{\infty} f_n(x)$ conv. unif. on $x \in (-1,1)$?

<u>No</u> $\sup\{|f_n(x)| : x \in (-1,1)\} = \sup\{|x^n| : x \in (-1,1)\} = 1$

So $\lim_{n\to\infty} \sup\{|f_n(x)| : x \in (-1,1)\} \not\to 0$

$\Rightarrow$ $\sum f_n$ doesn't conv. unif.

<u>Q</u>: Does $\sum f_n(x)$ conv. unif. on $x \in [-R,R]$ for $R < 1$?

<u>Yes</u> $|f_n(x)| = |x^n| \le R^n \quad \forall x \in [-R,R]$

b/c $R < 1$, so $\sum R^n < +\infty$

By Weierstrass M-test $\Rightarrow$ $\sum f_n(x)$ conv. unif. $\square$

(Pretend we don't know what $\sum f_n$ is)

$\Rightarrow$ $\sum f_n$ is conti. on $[-R, R]$ for any $R < 1$

$\Rightarrow$ $\sum f_n$ is conti. on $(-1, 1)$

($\forall x \in (-1,1)$, $\exists R < 1$ st. $x \in [-R, R]$)

Q: X cpt ~~s~~ metric space $f_n : X \longrightarrow \mathbb{R}$ (conti.?)

$(f_n) \to f$ pointwise $\overset{?}{\Rightarrow}$ unif.

Power series $f_0 = a_0$, $f_1 = a_1 x$, $f_2 = a_2 x^2$, ...

(partial sums $S_n = a_0 + a_1 x + \cdots + a_n x^n$)

Power series " $\sum_{n=0}^{\infty} a_n x^n$ " $\boxed{\text{When does it conv.?}}$

Thm $\beta := \limsup |a_n|^{1/n}$ $R := 1/\beta$ $\left(\begin{array}{l} \beta = 0 \to R = +\infty \\ \beta = +\infty \to R = 0 \end{array} \right)$

Then $\sum a_n x^n$ conv. for $|x| < R$ $\leftarrow$ radius of conv. of the power series $\sum a_n x^n$

$\sum a_n x^n$ div. for $|x| > R$

pf Root test $\Rightarrow$

If $\underbrace{\limsup |a_n x^n|^{1/n} = |x| \cdot \limsup |a_n|^{1/n} = |x| \cdot \beta}$ < 1, then $\sum a_n x^n$ conv.

$$\Updownarrow$$

$$|x| < R$$

Same for the second statement. $\square$