**Def:** An <u>inner product space</u> is a pair $(V, \langle -,- \rangle)$ where:

- $V$ is a real vector space
- $\langle -,- \rangle : V \times V \longrightarrow \mathbb{R}$ (called the <u>inner product</u>) s.t.

1) $\langle \vec{v}_1, \vec{v}_2 \rangle = \langle \vec{v}_2, \vec{v}_1 \rangle \quad \forall\, \vec{v}_1, \vec{v}_2 \in V.$
2) $\langle \vec{v}_1 + \vec{v}_2, \vec{v}_3 \rangle = \langle \vec{v}_1, \vec{v}_3 \rangle + \langle \vec{v}_2, \vec{v}_3 \rangle \quad \forall\, \vec{v}_1, \vec{v}_2, \vec{v}_3 \in V$
3) $\langle c\vec{v}_1, \vec{v}_2 \rangle = c \langle \vec{v}_1, \vec{v}_2 \rangle \quad \forall\, \vec{v}_1, \vec{v}_2 \in V,\ c \in \mathbb{R}$
4) $\langle \vec{v}, \vec{v} \rangle \geq 0. \quad \forall\, \vec{v} \in V$
5) $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$.

**Rmk:** Inner product $\rightsquigarrow$ length, angle, orthogonality, w.r.t. an inner product.

**Rmk:** Fix $V$, the choice of inner product is not unique.

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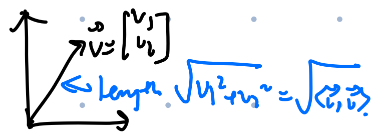
**e.g.** $V = \mathbb{R}^n$.

$\langle \vec{v}, \vec{v} \rangle = v_1^2 + \cdots + v_n^2 \geq 0$




$\langle \vec{v}, \vec{w} \rangle := v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$

$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$

"standard Inner product of $\mathbb{R}^n$"

**Rmk:** There are other possible inner products on $\mathbb{R}^n$ (HW)

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**Def:** $(V, \langle -,- \rangle)$ inner product space, $\vec{v} \in V$

Define the <u>length</u> of $\vec{v}$ w.r.t. (with respect to) the inner product $\langle -,- \rangle$ to be:

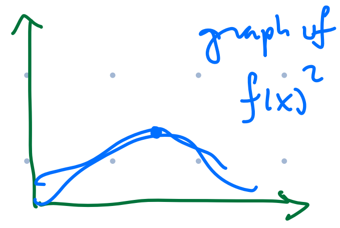
$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} \geq 0.$$

e.g. $V = \mathcal{C}[a,b] = \{ \text{continuous functions } f : [a,b] \to \mathbb{R} \}$ ← v.s.

For $f, g \in V$, we define

$$\langle f, g \rangle := \int_a^b f(x) g(x) \, dx.$$

$$\langle f, f \rangle = \int_a^b f(x)^2 \, dx \geq 0.$$

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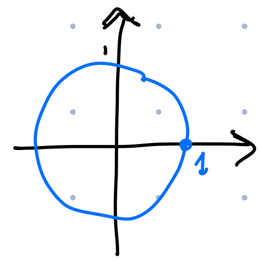
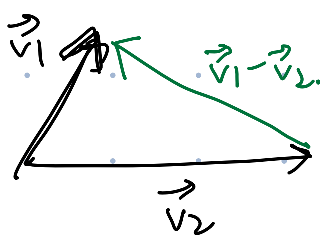
Rmk: $(??, \|\vec{v}\| |c|)$

- $\|c\vec{v}\| = \sqrt{\langle c\vec{v}, c\vec{v} \rangle} = \sqrt{c^2 \langle \vec{v}, \vec{v} \rangle}$
  $= |c| \cdot \sqrt{\langle \vec{v}, \vec{v} \rangle} = |c| \cdot \|\vec{v}\|.$




Def

- Say a vector $\vec{v} \in V$ is an unit vector if $\|\vec{v}\| = 1$.
- distance between $\vec{v}_1$ and $\vec{v}_2$: $\|\vec{v}_1 - \vec{v}_2\|$

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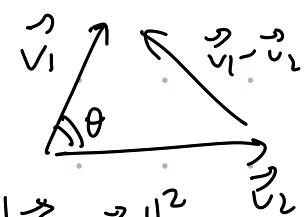
e.g. In $\mathbb{R}^2$, w/ std. inner product




$$\|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 - 2\|\vec{v}_1\| \|\vec{v}_2\| \cos\theta = \|\vec{v}_1 - \vec{v}_2\|^2$$

$$\Rightarrow -2\|\vec{v}_1\| \|\vec{v}_2\| \cos\theta = \|\vec{v}_1 - \vec{v}_2\|^2 - \|\vec{v}_1\|^2 - \|\vec{v}_2\|^2$$

$$= \langle \vec{v}_1 - \vec{v}_2, \vec{v}_1 - \vec{v}_2 \rangle - \langle \vec{v}_1, \vec{v}_1 \rangle - \langle \vec{v}_2, \vec{v}_2 \rangle$$

$$= \cancel{\langle \vec{v}_1, \vec{v}_1 \rangle} - \langle \vec{v}_1, \vec{v}_2 \rangle - \langle \vec{v}_2, \vec{v}_1 \rangle + \cancel{\langle \vec{v}_2, \vec{v}_2 \rangle}$$
$$- \cancel{\langle \vec{v}_1, \vec{v}_1 \rangle} - \cancel{\langle \vec{v}_2, \vec{v}_2 \rangle}$$

$$= -2 \langle \vec{v}_1, \vec{v}_2 \rangle$$

$$\Rightarrow \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cos\theta = \langle \vec{v}_1, \vec{v}_2 \rangle$$

$$\Rightarrow \boxed{\cos\theta = \frac{\langle \vec{v}_1, \vec{v}_2 \rangle}{\|\vec{v}_1\| \cdot \|\vec{v}_2\|}} \Leftarrow$$

define the angle between $\vec{v}_1, \vec{v}_2$ in any inner product space by this formula.

In particular, $\theta = \frac{\pi}{2} \Rightarrow \cos\theta = 0 \Rightarrow \langle \vec{v}_1, \vec{v}_2 \rangle = 0$

"$\vec{v}_1, \vec{v}_2$ are orthogonal"

**Def** $(V, \langle -,- \rangle)$ an inner product space.

Say $\vec{v}_1, \vec{v}_2$ are <u>orthogonal</u> if $\langle \vec{v}_1, \vec{v}_2 \rangle = 0$.

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<u>Pythagorean thm</u> If $\vec{v}_1, \vec{v}_2$ are orthogonal, then $\|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 = \|\vec{v}_1 + \vec{v}_2\|^2$.

<u>pf</u> $\|\vec{v}_1 + \vec{v}_2\|^2 = \langle \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 \rangle$

$$= \langle \vec{v}_1, \vec{v}_1 \rangle + \underbrace{\cancel{\langle \vec{v}_1, \vec{v}_2 \rangle}}_{0} + \underbrace{\cancel{\langle \vec{v}_2, \vec{v}_1 \rangle}}_{0} + \langle \vec{v}_2, \vec{v}_2 \rangle$$

b/c $\vec{v}_1, \vec{v}_2$ orthogonal

$$= \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2.$$

Def $(V, \langle -,- \rangle)$ inner product space

$W \subseteq V$ subspace

The <u>orthogonal complement</u> $W^\perp \subseteq V$ <u>of $W$</u>:

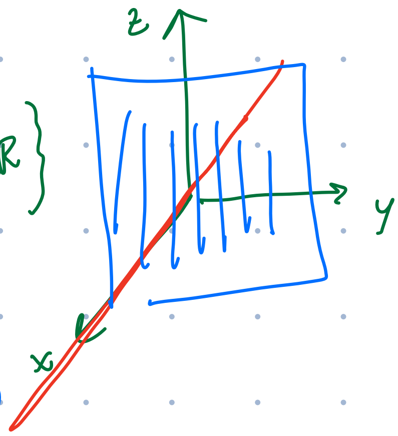
$$W^\perp := \{\vec{x} \in V \mid \langle \vec{x}, \vec{w} \rangle = 0 \quad \forall\, \vec{w} \in W\}$$

e.g. $V = \mathbb{R}^3$, std. inner product.

$$W = \mathrm{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\} = \left\{\begin{bmatrix}a\\0\\0\end{bmatrix} : a \in \mathbb{R}\right\}$$

$W^\perp = ??$

$$= \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \underbrace{\left\langle \begin{bmatrix}x\\y\\z\end{bmatrix}, \begin{bmatrix}a\\0\\0\end{bmatrix} \right\rangle}_{x \cdot a} = 0 \ \forall a \right\}$$




$$= \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x = 0\right\} = \mathrm{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

Rmk: (HW): • $W^\perp$ is also a subspace of $V$.

• $W \cap W^\perp = \{\vec{0}\}$.

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e.g. $A$: $m \times n$ matrix.

$$(\mathrm{Col}(A))^\perp = \mathrm{Nul}(A^T)$$

$$(\mathrm{Row}(A))^\perp = \mathrm{Nul}(A)$$

$$\underbrace{\begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_m & - \end{bmatrix}}_{A} \begin{bmatrix} \\ w \\ \\ \end{bmatrix} = \begin{bmatrix} \langle v_1, w \rangle \\ \langle v_2, w \rangle \\ \vdots \\ \langle v_m, w \rangle \end{bmatrix}$$

$$\vec{w} \in (\mathrm{Row}(A))^\perp \iff \langle \vec{w}, \vec{v}_i \rangle = 0 \ \forall i$$

$$\iff A\vec{w} = 0$$

$$\iff \vec{w} \in \mathrm{Nul}(A)$$

**Def** Say $\{\vec{v}_1, \dots, \vec{v}_n\}$ (nonzero vectors) is an orthogonal set if $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ $\forall\, i \neq j$

**Thm** Any orthogonal set is linearly independent.

pf. $\{\vec{v}_1, \dots, \vec{v}_n\}$ orthogonal set.

$$a_1\vec{v}_1 + \dots + a_n\vec{v}_n = \vec{0}$$

$$\langle \vec{v}_1, a_1\vec{v}_1 + \dots + a_n\vec{v}_n \rangle = \langle \vec{v}_1, \vec{0} \rangle = 0$$

$$\|$$

$$a_1 \langle \vec{v}_1, \vec{v}_1 \rangle + a_2 \cancelto{0}{\langle \vec{v}_1, \vec{v}_2 \rangle} + \dots + a_n \cancelto{0}{\langle \vec{v}_1, \vec{v}_n \rangle}.$$

$$\|$$

$$a_1 \|\vec{v}_1\|^2 \quad \Rightarrow \quad a_1 = 0$$

□

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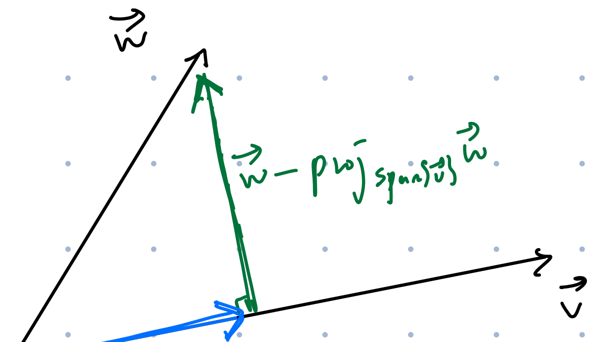
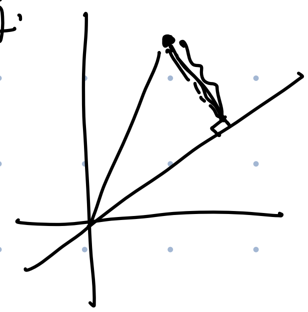



e.g.



$$\mathrm{proj}_{\mathrm{span}\{\vec{v}\}}\vec{w} = c\vec{v} = \frac{\langle \vec{w}, \vec{v} \rangle}{\|\vec{v}\|^2} \cdot \vec{v}.$$

$$\langle \vec{w} - \underbrace{\mathrm{proj}_{\mathrm{span}\{\vec{v}\}}}_{c\vec{v}}\vec{w}, \quad \vec{v} \rangle = 0$$

$$\Rightarrow \quad \langle \vec{w}, \vec{v} \rangle - \langle c\vec{v}, \vec{v} \rangle = 0$$

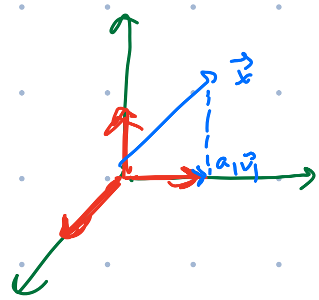
$$\Rightarrow \quad \langle \vec{w}, \vec{v} \rangle - c \langle \vec{v}, \vec{v} \rangle = 0 \quad \Rightarrow \quad c = \frac{\langle \vec{w}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}$$

**Thm** $(V, \langle -,- \rangle)$ inner product space, $\{\vec v_1, \dots, \vec v_n\}$ orthogonal basis
(it's a basis & orthogonal set)

Then $\forall\, \vec x \in V$, we have

$$\vec x = a_1 \vec v_1 + \dots + a_n \vec v_n,$$

where $$a_i = \frac{\langle \vec v_i, \vec x\rangle}{\|\vec v_i\|^2}.$$




Rmk. This is not true if $\{\vec v_1, \dots, \vec v_n\}$ is not orthogonal.

Pf.
- $\vec y = \vec x - (a_1\vec v_1 + \dots + a_n \vec v_n)$ (want to show: $\vec y = \vec 0$)
- $\langle \vec y, \vec v_1\rangle = \langle \vec x - (a_1 \vec v_1 + \dots + a_n \vec v_n), \vec v_1\rangle$

$$= \langle \vec x, \vec v_1\rangle - a_1\langle \vec v_1, \vec v_1\rangle - \dots - \cancelto{0}{a_2\langle \vec v_2, \vec v_1\rangle} - \dots - \cancelto{0}{\dots}$$

$$= \langle \vec x, \vec v_1\rangle - \frac{\langle \vec v_1, \vec x\rangle}{\|\vec v_1\|^2}\|\vec v_1\|^2 = 0$$

Similarly, $\langle \vec y, \vec v_i\rangle = 0 \quad \forall i$

- $\vec y = b_1 \vec v_1 + \dots + b_n \vec v_n$ for some $b_1, \dots, b_n$.

$\langle \vec y, \vec y\rangle = b_1\langle \vec y, \vec v_1\rangle + \dots + b_n\langle \vec y, \vec v_n\rangle = 0$

$\Rightarrow \vec y = \vec 0$. $\square$

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**Def** Say $\{\vec v_1, \dots, \vec v_n\}$ is an <u>orthonormal set</u> if it's an orthogonal set & each $\vec v_i$ is an unit vector.

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**Def:** $A$: $n \times n$ ~~square~~ matrix. $A$ is called <u>orthogonal</u> if $A^T A = I_n$.

$\Leftrightarrow$ the columns of $A$ is an orthonormal set.