4/16/2020.

Riemann-Lebesgue thm

$f: [a,b] \longrightarrow \mathbb{R}$ bdd.

$A = \{x \in [a,b] : f$ is disconti. at $x\}$.

f is integrable $\Leftrightarrow$ A has measure zero.

Recall oscillation of f at x_0

X - metric space

$f: X \longrightarrow \mathbb{R}$

$$\operatorname{osc}(f; U) := \sup_{x \in U} f(x) - \inf_{x \in U} f(x)$$

$U \subset X$ open

$$\operatorname{osc}(f; x_0) := \lim_{\delta \to 0} \operatorname{osc}(f; B_\delta(x_0))$$

Ex f conti. at $x_0 \Leftrightarrow \operatorname{osc}(f; x_0) = 0$.

Thm (A) $f: [a,b] \to \mathbb{R}$ bdd integrable.

Then $A = \{x \in [a,b] \mid \operatorname{osc}(f; x) > 0\}$ has measure zero.

$$A_k := \{x \in [a,b] \mid \operatorname{osc}(f; x) \geq \tfrac{1}{k}\}$$

$$A = \bigcup_{k=1}^{\infty} A_k$$

Ex If A_k has measure zero $\forall k=1,2,\cdots$

then $\bigcup_{k=1}^{\infty} A_k$ has measure zero.

It suffices to show $\forall \ell$,

A_ℓ has measure zero.

i.e., $\forall \varepsilon>0$ find open intervals that covers A_ℓ, with total length $<\varepsilon$

$$\sum_{k,\ [t_{k-1},t_k]\cap A_\ell \neq \phi} (t_k-t_{k-1})\cdot \frac{1}{\ell}$$

Idea: f: int.

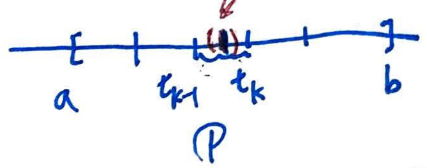
$$\sum (t_k-t_{k-1})\left(\sup_{[t_{k-1},t_k]} f - \inf_{[t_{k-1},t_k]} f\right)$$

$\forall \varepsilon>0$, $\exists P$

s.t. $U(f,P) - L(f,P) < \varepsilon/\ell$

$$\sup_{x\in[t_{k-1},t_k]} f(x) - \inf_{x\in[t_{k-1},t_k]} f(x) \geq \frac{1}{\ell}$$

$x\in A_\ell$



Small modification:

When $t_k \in A_\ell$.

Thm(B) $f: [a,b] \to \mathbb{R}$ bdd fcn

$A = \{x \in [a,b] : \operatorname{osc}(f,x) > 0\}$.

If A has measure zero,

then f is integrable.

Idea $\forall \varepsilon > 0$, Find P

s.t. $U(f,P) - L(f,P) < \varepsilon$

$$\| $$

$$\boxed{\sum (t_k - t_{k-1}) \left(\sup_{[t_{k-1},t_k]} f - \inf_{[t_{k-1},t_k]} f \right)}$$

for Each k,

If $\operatorname{osc}(f,x)$ small $< \delta$

If $\{\operatorname{osc}(f,x) \geq \delta\} = \boxed{A_\delta}$

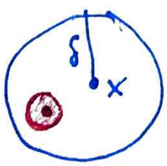
Lemma $A_s := \{x \in [a,b] : \operatorname{osc}(f,x) \geq s\}$ is compact.

pf Suffices to show A_s is closed. i.e. A_s^c is open

$$A_s^c = \{x \in [a,b] : \operatorname{osc}(f,x) < s\}$$

$x \in A_s^c$, $\operatorname{osc}(f,x) = \lim_{\delta \to 0} \operatorname{osc}(f, B_\delta(x)) = t < s$, where $B_\delta(x) = (x-\delta, x+\delta)$

$\exists \delta > 0$ s.t. $\operatorname{osc}(f, B_\delta(x)) < \frac{t+s}{2} < s$



Claim: $B_\delta(x) \subset A_s^c$ □

pf Thm (B) (A measure zero $\Rightarrow$ f int.)

$\forall \varepsilon > 0$.

$$A_{\frac{\varepsilon}{2(b-a)}} = \left\{ x \in [a,b] : osc(f;x) \geq \frac{\varepsilon}{2(b-a)} \right\}$$

$\cap$

A

- cpt.
- measure zero.,

$\exists \{I_i\}$ open interval $i = 1, 2, \dots, N$

s.t. $\cdot\ A_{\frac{\varepsilon}{2(b-a)}} \subset \bigcup_{i=1}^{N} I_i$.

$\cdot\ \sum \text{length}(I_i) < \frac{\varepsilon}{2(M-m)}$

$\sup_{x \in [a,b]} f(x)$ $\quad \inf_{x \in [a,b]} f(x)$

Idea

$\boxed{\bigcup_{i=1}^{N} I_i}$ total length $< \frac{\varepsilon}{2(M-m)}$

$(t_k - t_{k-1})$ (sup−inf)

M−m

$\boxed{[a,b] \setminus \bigcup_{i=1}^{N} I_i}$ $(t_k - t_{k-1})$ (sup−inf)

$\frac{\varepsilon}{2(b-a)}$

$x \in [a,b] \setminus \bigcup_{i=1}^{N} I_i$ $\quad osc(f;x) < \frac{\varepsilon}{2(b-a)}$

$\Rightarrow \exists\, \delta_x > 0$ s.t.

$$osc(f; B_{\delta_x}(x)) < \frac{\varepsilon}{2(b-a)}$$

$$B_{\delta_x}(x) = (x - \delta_x, x + \delta_x)$$

$\{(x-\delta_x, x+\delta_x)\}_{x \in \boxed{[a,b] \setminus \bigcup_{i=1}^{N} I_i}}$

open cover of $\underbrace{[a,b] \setminus \bigcup_{i=1}^{N} I_i}_{\text{cpt}}$

$\exists$ finite subcover

$\{(x_1-\delta_1, x_1+\delta_1), \cdots, (x_k-\delta_k, x_k+\delta_k)$

~~δ_0~~ ~~$(x_1-\delta_1, x_1+\delta_1)$~~

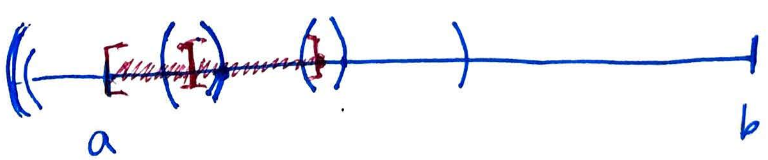
$\boxed{\text{osc}(f; \text{each of them}) < \frac{\varepsilon}{2(b-a)}}$

$\left\{ \begin{array}{c} (x_1-\delta_1, x_1+\delta_1), \cdots, (x_k-\delta_k, x_k+\delta_k), \\ I_1, \cdots, I_N \end{array} \right\}$ osc $< \frac{\varepsilon}{2(b-a)}$ length $< \frac{\varepsilon}{2(M-m)}$

covers $[a,b]$

Define $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$

s.t. each $[t_{i-1}, t_i]$ is completely in one of these open intervals



Divide $\{[t_{i-1}, t_i]\}$ into 2 categories.

$$I = \{1 \le i \le n \mid [t_{i-1}, t_i] \text{ is in } I_j \text{ for some } j\}$$
$1, \ldots, N$

$$J = \{1 \le i \le n \mid [t_{i-1}, t_i] \text{ is in } (x_j - \delta_j, x_j + \delta_j)\}$$
for some $j = 1, \ldots, k$

$U(f, P) - L(f, P)$ $< \varepsilon$

$$= \sum (t_{\cancel{k}} - t_{i-1})\left(\sup_{x \in [t_{i-1}, t_i]} f(x) - \inf_{x \in [t_{i-1}, t_i]} f(x)\right)$$

$$= \sum_I (\quad)(\quad)$$

$$+ \sum_J (\quad)(\quad)$$

For $i \in I$,

$$\sum_{i \in I} (t_i - t_{i-1})(\underbrace{\sup f(x) - \inf f(x)}_{\substack{\wedge \\ M-m}})$$

$$\leq (M-m) \cdot \frac{\varepsilon}{2(M-m)} = \frac{\varepsilon}{2}$$

For $i \in J$, $[t_{i-1}, t_i] \subset (x_j - \delta_j, x_j + \delta_j)$

$$\sum_{i \in J} (t_i - t_{i-1}) (\sup_{t_{i-1}, t_i} f - \inf f) \quad osc < \frac{\varepsilon}{2(b-a)}$$

$$< \frac{\varepsilon}{2(b-a)} \sum_{i \in J} (t_i - t_{i-1}) \le \frac{\varepsilon}{2} \quad \square$$

$$\sum_{i \in J} (t_i - t_{i-1}) \le b-a$$

Next time

$$\begin{cases} x_1' = f_1(x_1, \dots, x_m) \\ \vdots \\ x_m' = f_m(x_1, \dots, x_m) \end{cases}$$

$x_1(t), \dots, x_m(t)$ on some small nbd of 0.

$f_i : \mathbb{R}^m \to \mathbb{R}$

$$\begin{cases} x_1'(t) = f_1(x_1(t), \dots, x_m(t)) \\ \vdots \\ x_m'(t) = f_m(x_1(t), \dots, x_m(t)) \end{cases}$$

"Lipschitz conti."

$(x_1(0), \dots, x_m(0)) = p \in \mathbb{R}^m$