## Last time:

linear system $\longleftrightarrow$ associated augmented matrix.

$$\left[\begin{array}{ccccc|c} * & * & * & * & * & * \\ * & & & & & \\ \vdots & & & & & \\ * & * & \cdots & & * & * \end{array}\right]$$

they won't change the solution set. $\rightarrow$ elementary row operations

$\Downarrow$

reduced echelon form.

## How to read the sol$^{n}$'s of a reduced echelon form?

**Case 1:** The last column has a pivot.

$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow 0\cdot x_1 + 0\cdot x_2 + \cdots + 0\cdot x_6 = 1.$$

No sol$^{n}$ !!

**Case 2:** The last column has no pivots.

$$\begin{array}{cccccc} \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \end{array}\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 2 & 0 & 4 & 7 \\ 0 & 0 & 1 & 3 & 0 & 5 & 8 \\ 0 & 0 & 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ & \uparrow & \uparrow & & \uparrow & \end{array}$$

$$\begin{cases} x_2 + 2x_4 + 4x_6 = 7 \\ x_3 + 3x_4 + 5x_6 = 8 \\ x_5 + 6x_6 = 9 \end{cases} \Longleftrightarrow \begin{cases} x_2 = 7 - 2x_4 - 4x_6 \\ x_3 = 8 - 3x_4 - 5x_6 \\ x_5 = 9 - 6x_6 \end{cases}$$

Observe:
- for the "non-pivot variables" $(x_1, x_4, x_6)$, we can assign them to be any real number.
- for any choice of $x_1, x_4, x_6 \in \mathbb{R}$, there is a unique choice

of $x_2, x_3, x_5$ that makes $\begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix}$ a sol$^n$. of the system.

In other words, the sol$^n$ set of the system is given by:

$$\left\{ \vec{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix} \in \mathbb{R}^6 \;\middle|\; \begin{array}{l} x_1, x_4, x_6 \text{ any real numbers, and.} \\ x_2 = 7 - 2x_4 - 4x_6 \\ x_3 = 8 - 3x_4 - 5x_6 \\ x_5 = 9 \qquad\;\; - 6x_6 \end{array} \right\}$$

**Summary:**

- a linear system has a sol$^n$ $\Leftrightarrow$ the last column of its augmented matrix has no pivots.
- when it has a sol$^n$, the sol$^n$ is <u>unique</u> $\Leftrightarrow$ each column in the coefficient matrix has a pivot.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccccc|c} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ \vdots & 0 & 0 & 1 & * & * \\ \vdots & \vdots & \vdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

has a unique sol$^n$ $\qquad$ sol$^n$ is <u>not</u> unique

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Q: How do we characterise the vectors $\vec{b}$ s.t. $[\,A \mid \vec{b}\,]$ has a sol$^n$?

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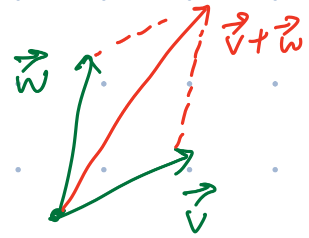
## § Linear combinations of vectors.

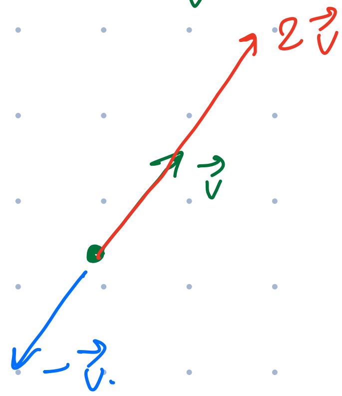
Def: $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$

**Sum:** $\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix} \in \mathbb{R}^n$




**Scalar multiplication:** $c \in \mathbb{R}$,

$$c \cdot \vec{v} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix} \in \mathbb{R}^n.$$




**Def:** $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$.

Then, for any $c_1, \dots, c_k \in \mathbb{R}$, the vector

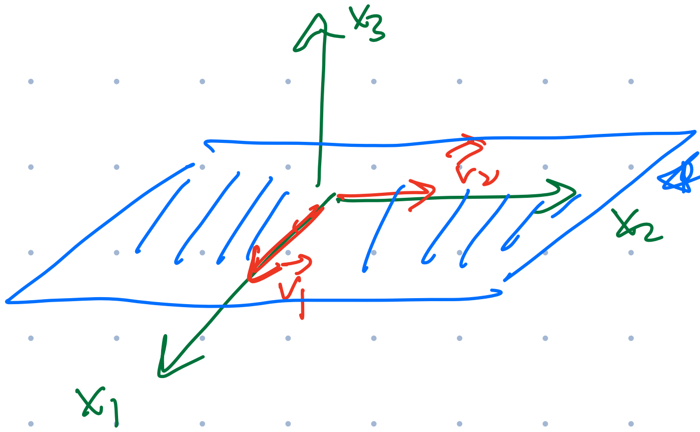
$$c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k$$

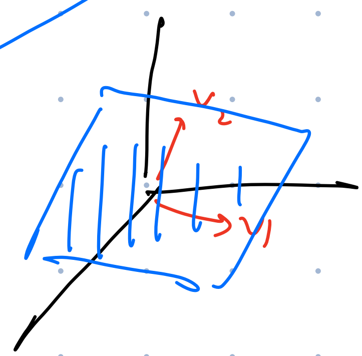
is a linear combination of $\vec{v}_1, \dots, \vec{v}_k$ (of weights $c_1, \dots, c_k$)

e.g.: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$

Q: What are all the linear combinations of $\vec{v}_1, \vec{v}_2$?

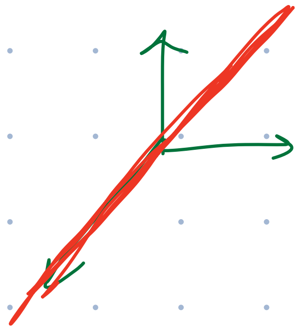
$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$



Q: What are all the linear combinations of $\vec{v}_1$?

$$c_1\vec{v}_1 = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}, \quad c_1 \in \mathbb{R}$$



**Def:** $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} := \{\text{linear combinations of } \vec{v}_1, \dots, \vec{v}_k\}$

$$= \{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}.$$

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e.g. $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$

Q: Is $\vec{b} \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$?

$$\Updownarrow$$

$\exists$?? $c_1, c_2 \in \mathbb{R}$ s.t. $\boxed{c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{b}.}$

$$c_1\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ -2c_1 \\ -5c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ 5c_2 \\ 6c_2 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ -2c_1 + 5c_2 \\ -5c_1 + 6c_2 \end{bmatrix}$$

$$\Updownarrow$$

Does $\begin{cases} c_1 + 2c_2 = 7 \\ -2c_1 + 5c_2 = 4 \\ -5c_1 + 6c_2 = -3 \end{cases}$ have a sol$^{n}$??

$$\Updownarrow$$

augmented matrix $\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right]$

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{b}$

Observation: $\vec{b} \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$

$\iff \left[\begin{array}{ccc|c} | & & | & | \\ \vec{v}_1 & \cdots & \vec{v}_k & \vec{b} \\ | & & | & | \end{array}\right]$ has a sol$^{n}$.

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Def: (matrix-vector product).

$A$: $m \times n$ matrix. $= \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$. $\quad \vec{a}_i \in \mathbb{R}^m$

$\vec{x} \in \mathbb{R}^n \qquad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

Define: $\quad A\vec{x} := x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n \in \mathbb{R}^m.$

e.g. $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 8 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 9 \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} 7 \cdot 1 + 8 \cdot 3 + 9 \cdot 5 \\ 7 \cdot 2 + 8 \cdot 4 + 9 \cdot 6 \end{bmatrix} \in \mathbb{R}^2$$

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ doesn't make sense!

**Rmk:** Given a $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$ $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$.

Then

$$\left[ \vec{a}_1 \; \cdots \; \vec{a}_n \;\middle|\; \vec{b} \right] \text{ has a sol}^{n}$$

$$\iff \quad \vec{b} \in \operatorname{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}.$$

$$\iff \quad \exists\, \vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{b}.$$

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Given an $m \times n$ matrix $A$, we can define a function

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\vec{x} \longmapsto A\vec{x}.$$

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**Rmk:** The map $T_A$ encodes the info of the solvability of linear eq'ns $A\vec{x} = \vec{b}$.

$$[A \mid \vec{b}] \text{ has sol}^{n} \iff \vec{b} \text{ is in the } \underline{\text{image}} \text{ of } T_A.$$

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We'll be studying criterions of <u>injectivity</u> and <u>surjectivity</u> of $T_A$.

For a function $f : X \to Y$.

- $f$ is <u>injective</u> if $\forall x_1, x_2 \in X$, $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$

$$\left(\begin{array}{l} x_1 \bullet \xrightarrow{f} \bullet \\ x_2 \bullet \nearrow_{f} \end{array} \quad \text{doesn't happen}\right)$$

• $f$ is surjective if $\forall y \in Y$, $\exists x \in X$ s.t. $f(x) = y$.

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**Prop**: $A$: $m \times n$, $\quad T_A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\vec{x} \longmapsto A\vec{x}.$$

- $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$, we have $T_A(\vec{u}+\vec{v}) = T_A(\vec{u}) + T_A(\vec{v})$.
- $\forall c \in \mathbb{R}$, we have $T_A(c\vec{v}) = c\,T_A(\vec{v})$

(i.e. $T_A$ is a $\boxed{\text{linear transformation}}$)

**pf**: $T_A(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$

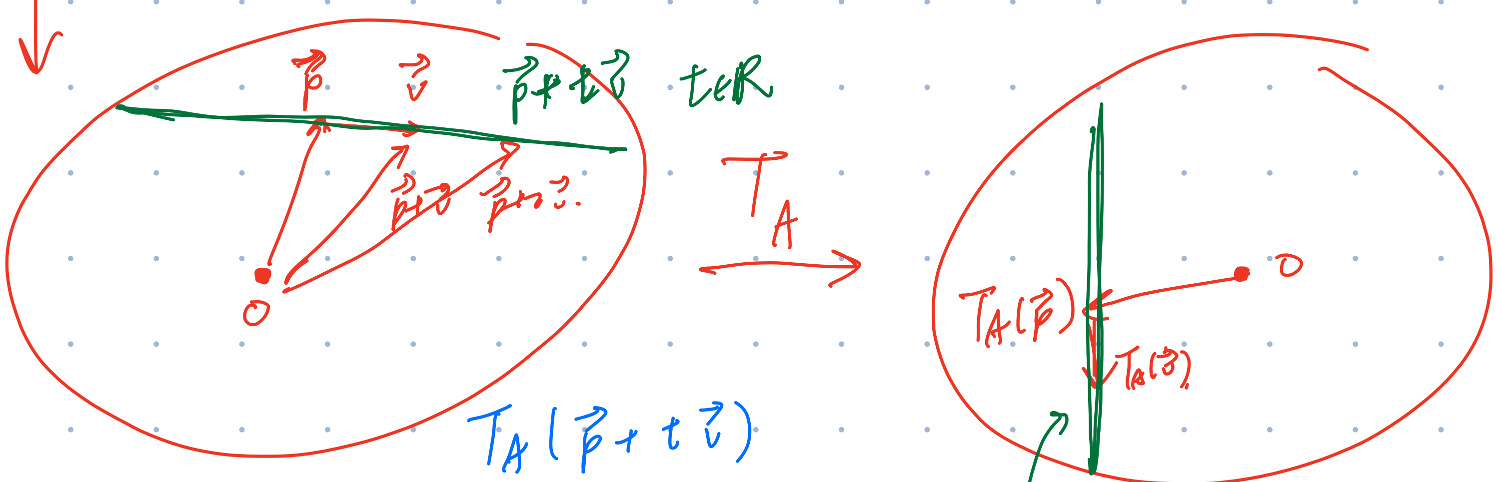
$$A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \boxed{(u_1+v_1)\vec{a}_1 + \cdots + (u_n+v_n)\vec{a}_n}$$

$$T_A(\vec{u}) = A\vec{u} = u_1\vec{a}_1 + \cdots + u_n\vec{a}_n$$

$$T_A(\vec{v}) = A\vec{v} = v_1\vec{a}_1 + \cdots + v_n\vec{a}_n.$$

$\square$




$$T_A(\vec{p}+t\vec{v}) = T_A(\vec{p}) + T_A(t\vec{v}) = T_A(\vec{p}) + t\,T_A(\vec{v})$$

Q: When is $\boxed{T_A : \mathbb{R}^n \to \mathbb{R}^m \text{ surjective}}$ ??

$\iff \forall \vec{b} \in \mathbb{R}^m, \ \exists \vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$.

$\iff \forall \vec{b} \in \mathbb{R}^m$, the system $[A | \vec{b}]$ has a sol$^{n}$.

e.g. $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & 8 \\ 3 & 5 & 8 & 13 \end{bmatrix}$ Is $T_A$ surjective?

$\Updownarrow$

whether $\forall \vec{b} \in \mathbb{R}^3$, $[A | \vec{b}]$ has a sol$^{n}$?

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & b_1 \\ 2 & 3 & 5 & 8 & b_2 \\ 3 & 5 & 8 & 13 & b_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - 2b_1 \\ 0 & 2 & 2 & 4 & b_3 - 3b_1 \end{array}\right]$$

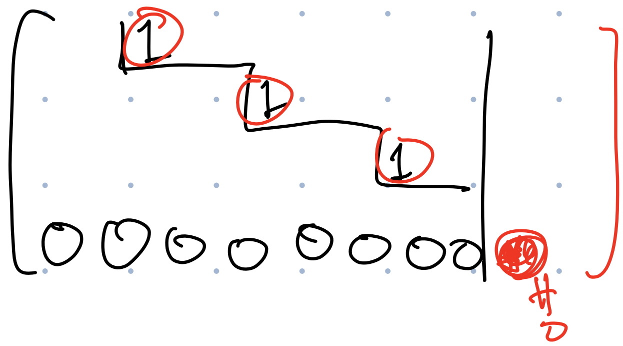
$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & \boxed{b_3 - 2b_1 - 2(b_2 - 2b_1)} \end{array}\right]$$

$0x_1 + 0x_2 + 0x_3 + 0x_4$

$\exists\, b_1, b_2, b_3$ s.t. $b_3 - 2b_1 - 2(b_2 - 2b_1) \neq 0$

So, $T_A$ is not surjective

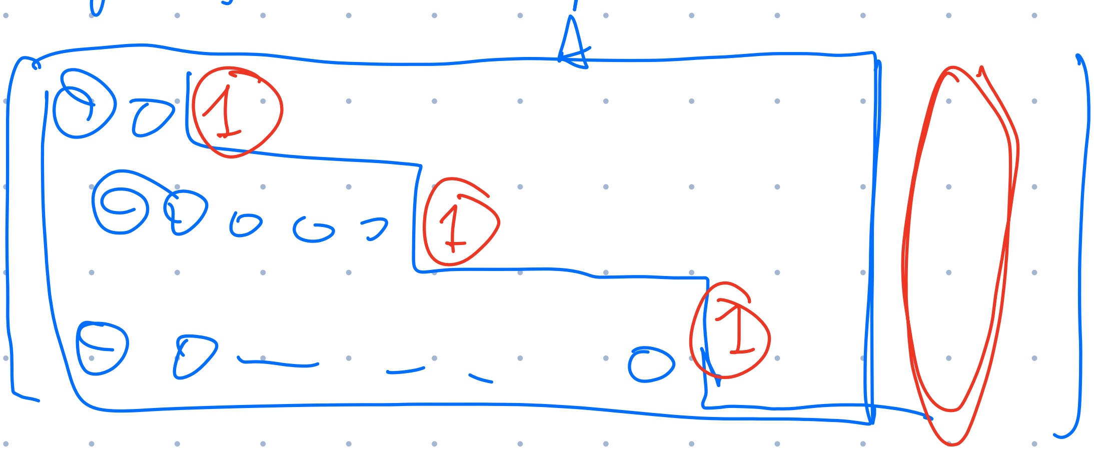
More generally,



**Observation:** If $A$ doesn't have pivot in each row, then $\exists\, \vec{b} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$ has no sol$^n$.

($\Leftrightarrow$ $T_A$ is not surjective).

Conversely, if $A$ has pivots in each row,



then $[A \mid \vec{b}]$ doesn't have pivot in the last column. $\forall\, \vec{b} \in \mathbb{R}^m$

$\Rightarrow$ $[A \mid \vec{b}]$ has a sol$^n$ $\forall\, \vec{b} \in \mathbb{R}^m$.

$\Leftrightarrow$ $T_A$ is surjective.

**Conclusion:** $T_A$ is surjective $\Leftrightarrow$ $A$ has pivot in each row.

**Thm:** $A$: $m \times n$. Then the following are equivalent:

- $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is surjective.
  $\vec{x} \longmapsto A\vec{x}$
- $A\vec{x} = \vec{b}$ has a sol$^n$ $\forall \vec{b} \in \mathbb{R}^m$
- $A$ has pivot in each row.
- $\forall \vec{b} \in \mathbb{R}^m$, $\vec{b}$ is a linear combination of the columns $\vec{a}_1, \dots, \vec{a}_n$ of $A$
- $\mathrm{Span}\{\vec{a}_1, \dots, \vec{a}_n\} = \mathbb{R}^m$