Summary so far: What do we know about holo. fcns?

1) def^n of holo fcn: "complex derivative" exists.

   (Is $f'(z)$ conti.? holo.?)

2) the power series fcn $\sum a_n (z-z_0)^n$ is holo. within its radius of convergence

   (Do all holo. fcn, admit such expression, at least locally?)

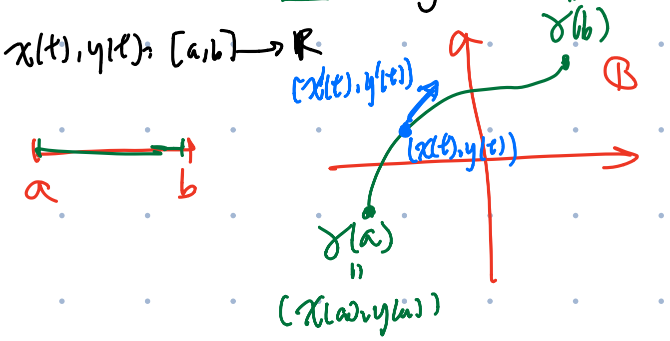
We'll give affirmative answers to both questions. (& even better!)

Main tool: Integration along curve.

Def: A parametrized smooth curve in $\mathbb{C}$ is a map

$$\gamma : [a,b] \longrightarrow \mathbb{C}, \quad \gamma(t) = x(t) + i\,y(t) \quad (x(t), y(t))$$

$x(t), y(t) : [a,b] \to \mathbb{R}$



s.t.

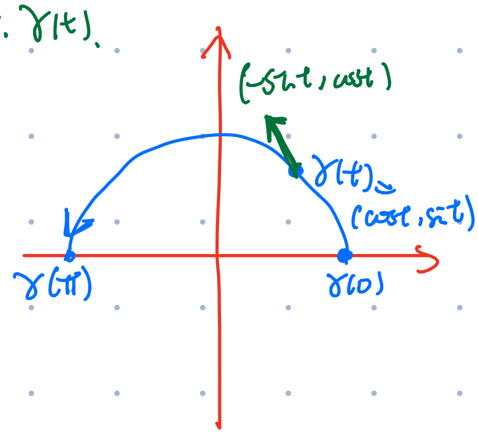
- $x(t), y(t)$ are differentiable & $x'(t), y'(t)$ are continuous.
- $\gamma'(t) = (x'(t), y'(t)) \neq (0,0) \quad \forall t.$

  tangent direction of γ at the pt. $\gamma(t)$.

e.g. $\gamma : [0, \pi] \longrightarrow \mathbb{C}$

$$t \longmapsto e^{it} = \cos t + i \sin t$$

$$\gamma'(t) = -\sin t + i \cos t$$



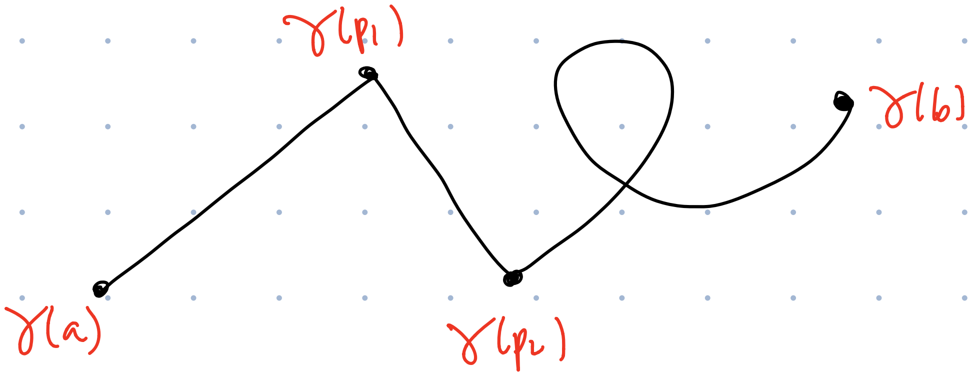
Def: A <u>parametrized piecewise smooth curve</u> is a conti. map

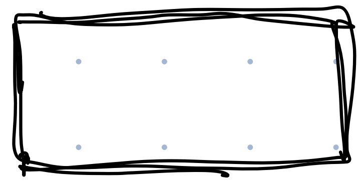
$\gamma : [a,b] \longrightarrow \mathbb{C}$

s.t. $\exists\ a < p_1 < \cdots < p_n < b$.

s.t. $\gamma\big|_{[a,p_1]}, \gamma\big|_{[p_1,p_2]}, \ldots, \gamma\big|_{[p_n,b]}$ are

parametrized smooth curves.





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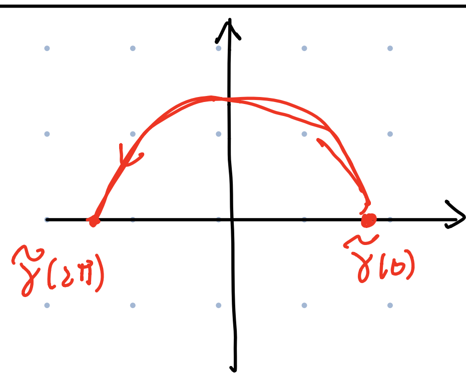
e.g. $\tilde{\gamma} : [0, 2\pi] \longrightarrow \mathbb{C}$

$s \longmapsto e^{i s/2}$



Def: Say two parametrized smooth curves $\gamma : [a,b] \to \mathbb{C}$, $\tilde{\gamma} : [c,d] \to \mathbb{C}$

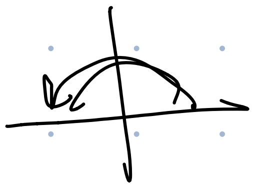
are <u>equivalent</u>, if $\exists\ \varphi : [c,d] \longrightarrow [a,b]$ bijection

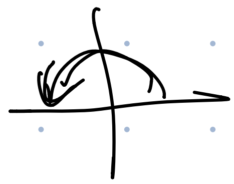
s

s.t. $\varphi'(s) > 0$ and $\gamma(\varphi(s)) = \tilde{\gamma}(s)\ \forall s \in [c,d]$

$$\begin{tikzcd} {[a,b]} \arrow[r, "\gamma"] & \mathbb{C} \\ {[c,d]} \arrow[u, "\varphi"] \arrow[ur, "\tilde{\gamma}"'] & \end{tikzcd}$$

e.g. $\gamma : [0,\pi] \to \mathbb{C}$, $t \mapsto e^{it}$, $\tilde{\gamma} : [0,2\pi] \to \mathbb{C}$, $s \mapsto e^{is/2}$





$$\begin{tikzcd} {[0,\pi]} \arrow[r, "\gamma"] & \mathbb{C} \\ {[0,2\pi]} \arrow[u, "\varphi"] \arrow[ur, "\tilde{\gamma}"'] & \end{tikzcd}$$

$t \longmapsto e^{it}$

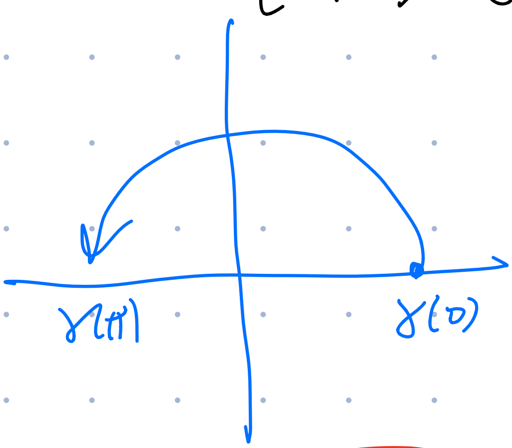
$s \longmapsto e^{is/2}$

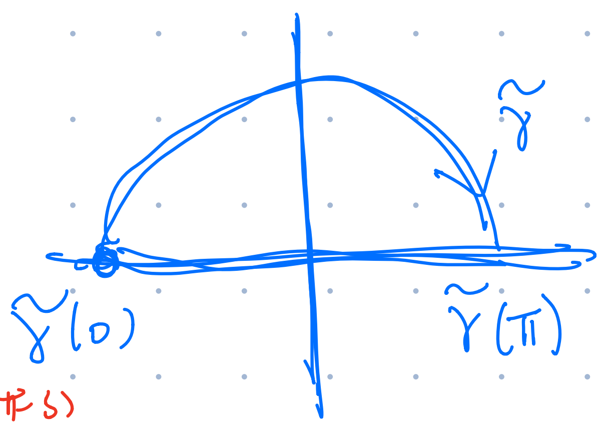
$s/2 \longleftarrow s$

$\varphi' = \frac{1}{2} > 0$

γ & $\tilde{\gamma}$ are equivalent.

e.g. $\gamma : [0,\pi] \to \mathbb{C}$, $t \mapsto e^{it}$; $\tilde{\gamma} : [0,\pi] \to \mathbb{C}$, $t \mapsto e^{i(\pi - t)}$





$\longrightarrow e^{i(\pi - s)}$

$$\begin{tikzcd} {[0,\pi]} \arrow[r, "\gamma"] & \mathbb{C} \\ & {[0,\pi]} \arrow[ul, "\varphi"] \arrow[u, "\tilde{\gamma}"'] \end{tikzcd}$$

$t \longmapsto e^{it}$

$\pi - s \leftarrow\!\shortmid\; s$

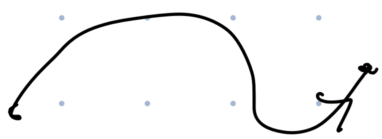
$s \longmapsto e^{i(\pi - s)}$

$\varphi'(s) = -1 < 0$

γ & $\tilde{\gamma}$ are NOT equivalent

Def A <u>(piecewise) smooth curve</u> is an equivalent class of parametrized (piecewise) smooth curve.

Def A curve is <u>simple</u> is it doesn't have any self-intersection.



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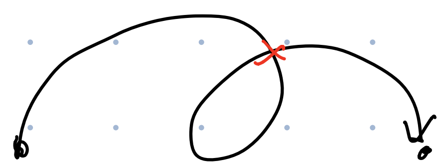
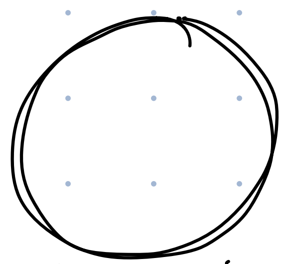


not simple.

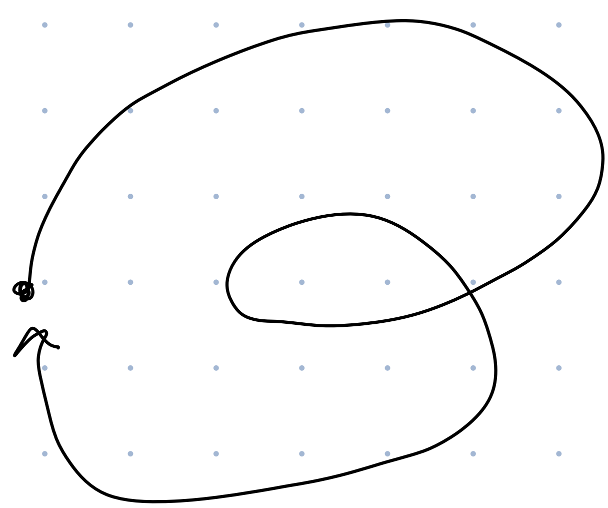
Def: A curve is <u>closed</u> if its start pt = its end pt.
i.e. say $\gamma : [a,b] \longrightarrow \mathbb{C}$ is any parametrization of the curve,
then $\gamma(a) = \gamma(b)$



simple & closed

$\gamma(a)$

$\gamma(b)$

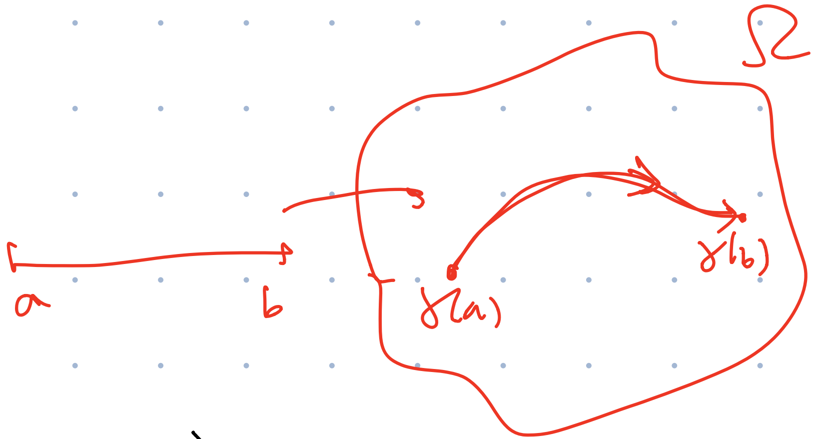


Setting:

$\Omega \subseteq \mathbb{C}$ open connected subset in $\mathbb{C}$

$f: \Omega \to \mathbb{C}$ conti. fcn.

$\gamma: [a,b] \longrightarrow \Omega$ (piecewise) smooth curve.



Def (integration of f along γ):

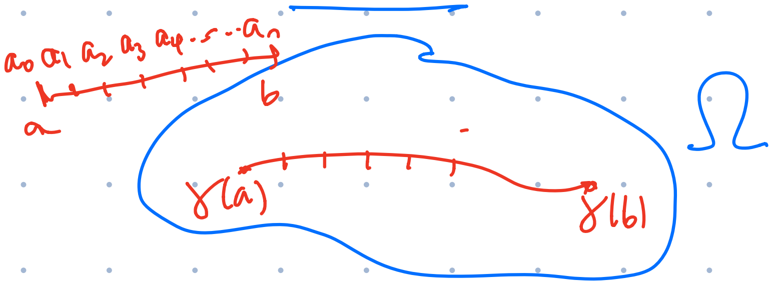
$$\underbrace{\int_\gamma f(z)\,dz}_{\text{notation}} := \int_a^b \underbrace{f(\gamma(t))\cdot\gamma'(t)}_{\text{cpx-valued fcn in }[a,b]}\,dt$$

(Rmk: If $g:[a,b] \longrightarrow \mathbb{C}$, $g = \underset{\uparrow}{u} + i\underset{\uparrow}{v}$ (Re(g), Im(g))

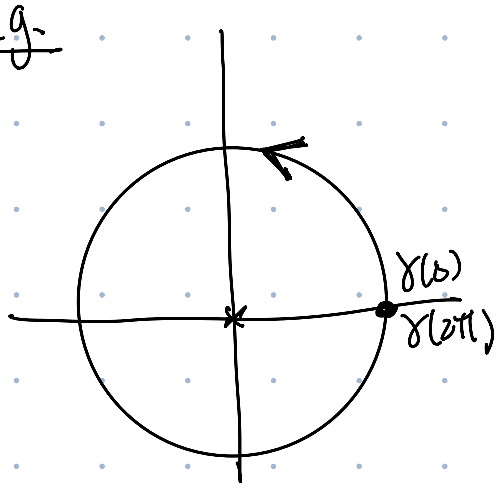
$$\int_a^b g(t)\,dt := \left(\int_a^b u(t)\,dt\right) + i\left(\int_a^b v(t)\,dt\right)$$)

Rmk

$$\underline{\int_\gamma f(z)\,dz} = \lim_{n\to\infty}\left(\sum_{i=0}^{n} f\left(\gamma\left(\frac{a_{i-1}+a_i}{2}\right)\right)(a_i - a_{i-1})\right)$$



e.g.



$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$$
$$t \longmapsto e^{it}$$

$$\underline{\Omega = \mathbb{C} \setminus \{0\}}$$

$f(z) = \dfrac{1}{z}$ is conti.. on $\mathbb{C}\setminus\{0\}$
~~(holo.)~~

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} f(\underbrace{\gamma(t)}_{= e^{it}}) \cdot \gamma'(t)\,dt$$

$$= \int_0^{2\pi} \frac{1}{\cancel{e^{it}}} \cdot (i\,\cancel{e^{it}})\,dt$$

$$= 2\pi i.$$

(<u>HW:</u> $\displaystyle\oint_{\underline{|z|=R}} \frac{1}{z}\,dz = 2\pi i \quad \forall R > 0$)

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$$
$$t \longmapsto R \cdot e^{it}$$

<u>Q</u>: If γ & $\tilde{\gamma}$ are equivalent, i.e.

$$\begin{tikzcd}
{[a,b]} \arrow[r, "\gamma"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C} \\
{[c,d]} \arrow[u, "\varphi"] \arrow[ur, "\tilde{\gamma}"'] &
\end{tikzcd}$$

$\varphi' > 0$

$$\tilde{\gamma} = \gamma \circ \varphi$$

then $\displaystyle\int_a^b f(\gamma(t))\gamma'(t)\,dt \overset{??}{\underset{\text{yes}}{=}} \boxed{\int_c^d f(\tilde\gamma(s))\tilde\gamma'(s)\,ds.}$

(i.e. does the integration depend on the parametrization of the curve??) no

Integration of f along $\tilde\gamma$

$$\int_c^d f(\tilde\gamma(s))\tilde\gamma'(s)\,ds$$

$$= \int_c^d f(\gamma(\varphi(s)))\,\frac{d}{ds}\gamma(\varphi(s))\,ds$$

$\varphi(c)=a$
$\varphi(d)=b$

$\boxed{t=\varphi(s)}$

$$= \int_c^d f(\gamma(\varphi(s)))\,\gamma'(\varphi(s))\,\varphi'(s)\,ds$$

$$= \int_c^d f(\gamma(\varphi(s)))\,\gamma'(\varphi(s))\,d\varphi(s)$$

$$\overset{\text{change of variable}}{\underset{t=\varphi(s)}{=}} \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

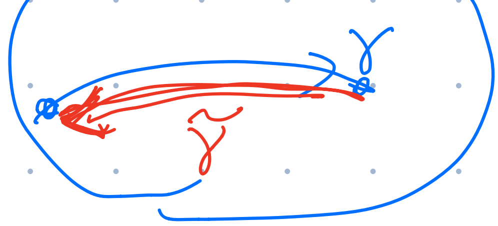
$\shortparallel$

Integration of f along γ.

$\Rightarrow$ the integration along curves doesn't depend on the parametrization of the curve.

Rmk (HW) $\tilde\gamma$ is the "same curve" as γ but in opposite direction,

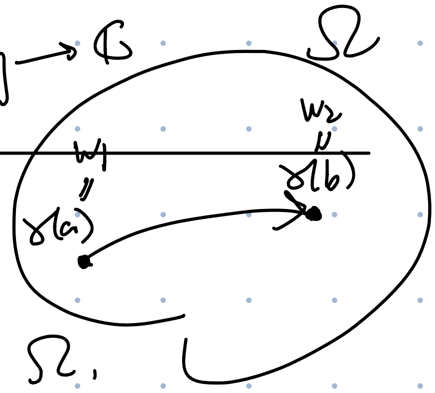
not equiv.
$$\begin{cases} \gamma : [a,b] \longrightarrow \mathbb{C} \\ \tilde{\gamma} : [-b,-a] \longrightarrow \mathbb{C} \\ \quad t \longmapsto \gamma(-t) \end{cases}$$



$\to$ is in the opposite direction of γ

$$\int_{\gamma} f(z)\,dz = -\int_{\tilde{\gamma}} f(z)\,dz$$

$\gamma : [a,b] \to \mathbb{C}$



Thm If f is conti. in Ω,
γ is a piecewise smooth curve in Ω,
starting at w_1, ending at w_2

If f is primitive in Ω (i.e. $\exists\, F : \Omega \to \mathbb{C}$ holo. s.t. $f = F'$)

then $$\int_{\gamma} f(z)\,dz = F(w_2) - F(w_1)$$

pf: $\gamma : [a,b] \longrightarrow \mathbb{C}$, $\gamma(a) = w_1$, $\gamma(b) = w_2$

$$\int_{\gamma} f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$

$$= \int_a^b \underline{F'(\gamma(t)) \cdot \gamma'(t)}\,dt$$

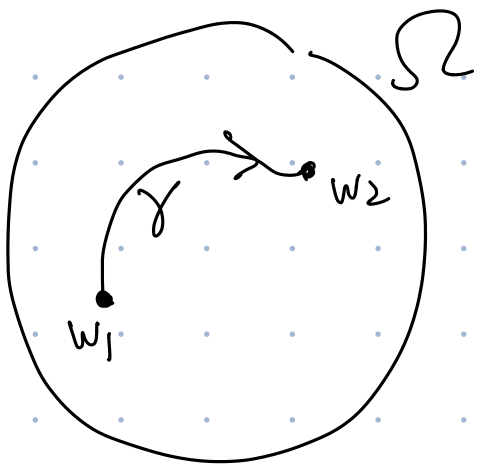
$$= \int_a^b \frac{d}{dt}\big(F(\gamma(t))\big)\,dt \quad \text{by Chain Rule.}$$

$$= F(\gamma(b)) - F(\gamma(a)) \text{ by FTC}$$
$$= F(w_2) - F(w_1). \quad \square$$

Coro: If f is primitive in Ω, γ is a <u>closed</u> curve in Ω, then $\int_\gamma f(z)\,dz = 0$. $\square$

Coro: If f is holo in Ω, and $f' = 0$ then f is constant in Ω.

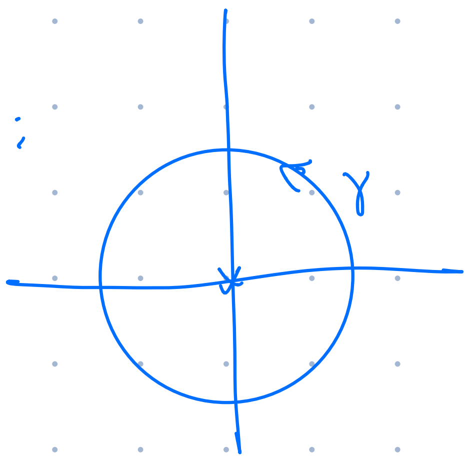
pf:



$$\underbrace{\int_\gamma \overbrace{f'(z)}^{0}\,dz}_{\substack{\| \\ \text{Thm}}}$$
$$f(w_2) - f(w_1)$$

$$\Rightarrow f(w_1) = f(w_2). \quad \square$$

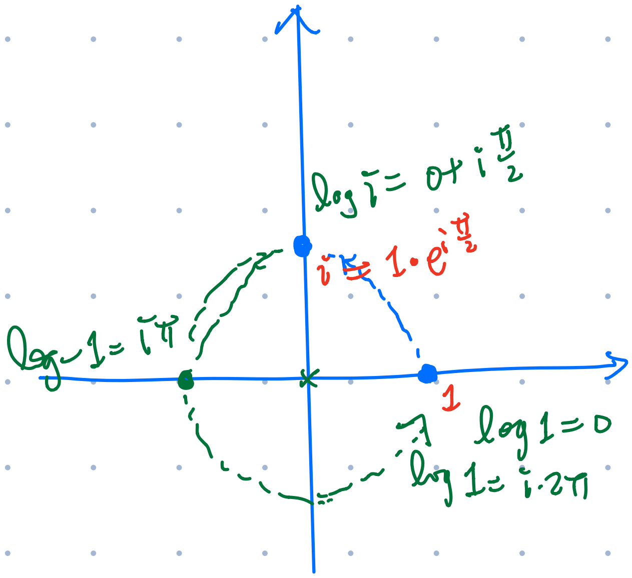
Q:



$$\int_\gamma \frac{1}{z}\,dz = 2\pi i$$

This doesn't contradict w/ Coro. b/c f is NOT primitive in $\mathbb{C} \setminus \{0\}$

"$\log z$" is not well-defined in $\mathbb{C} \setminus \{0\}$



on $\mathbb{C}\setminus\{0\}$

$\boxed{\log z = w}$

$w = x + iy$

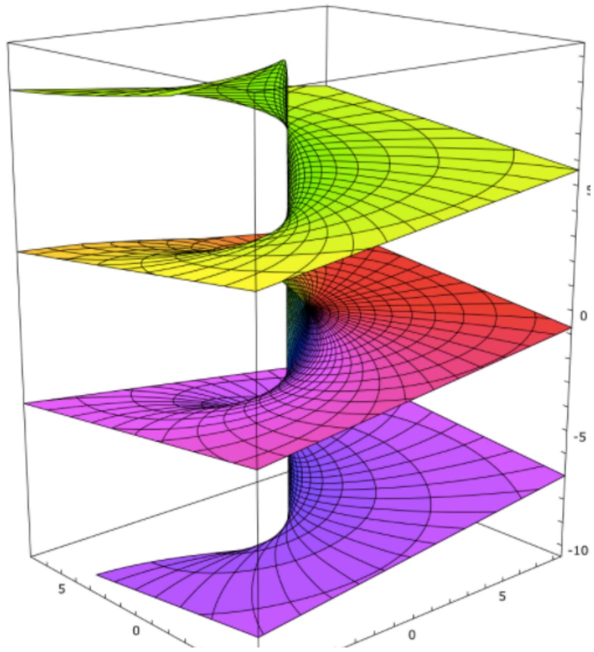
$z = e^w$

$re^{i\theta} = e^x e^{iy}$

$z \neq 0$

$0 < r = e^x$ $\quad x = \log r$

$\theta = y \mod 2\pi\mathbb{Z}$



$\log z$ is well-defined on "universal cover" of $\mathbb{C}\setminus\{0\}$