4/9/2020

- Fundamental thms of calculus
- countable sets, measure zero sets in $\mathbb{R}$

FTC (I): f conti. on $[a,b]$, diff^{ble} on (a,b)

f': (bdd) integrable on $[a,b]$

$$\rightarrow \int_a^b f'(x)\,dx = f(b) - f(a)$$

pf ~~$\forall \varepsilon > 0$~~, $\exists$ P ~~s.t. $U(f',P) - L(f',P) < \varepsilon$~~

$P \rightarrow \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$

$$f(b) - f(a) = (f(t_n) - f(t_{n-1})) + (f(t_{n-1}) - f(t_{n-2})) + \cdots + (f(t_1) - f(t_0))$$

$$= \sum_{k=1}^{n} (f(t_k) - f(t_{k-1}))$$

$$f(t_k) - f(t_{k-1}) = \boxed{f'(x_k)\cdot(t_k - t_{k-1})}$$

for some $x_k \in (t_{k-1}, t_k)$

$$\boxed{\inf_{x \in [t_{k-1}, t_k]} f'(x)(t_k - t_{k-1})} \le f'(x_k)\cdot(t_k - t_{k-1}) \le \boxed{\sup_{x \in [t_{k-1}, t_k]} f'(x)\cdot(t_k - t_{k-1})}$$

Sum up $k = 1, \ldots, n$:

$$L(f', P) \le f(b) - f(a) \le U(f', P) \quad \forall P$$

Since f' is int. $\Rightarrow \int_a^b f'(x)\,dx = f(b) - f(a)$. $\square$

Coro (Integral by parts)

u, v conti. $[a,b]$ diff^{ble} on (a,b)

u', v' int. $[a,b]$

$$\int_a^b u(x)v'(x) + \int_a^b u'(x)v(x) = u(b)v(b) - u(a)v(a)$$

pf FTC(I): $f(x) = u(x)v(x)$

FTC (II):

1) f: (bdd) integrable $[a,b]$.

Ex: then f is int. on $[a,x]$ $\forall x \in [a,b]$

$$F(x) := \int_a^x f(t)\,dt \quad \forall x \in [a,b]$$

Then F is conti. on $[a,b]$

2) Moreover, if f conti. at $x_0 \in [a,b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$

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pf 1) $x < y$ in $[a,b]$

Ex f is int., then $|f|$ is int. and

$$|F(x) - F(y)| = \left|\int_x^y f(t)\,dt\right| \leq \int_x^y |f(t)|\,dt \leq M(y-x)$$

Since f is bdd, i.e. $\exists M > 0$ s.t. $|f(x)| < M$ $\forall x \in [a,b]$

$\forall \varepsilon > 0$, if $|x-y| < \frac{\varepsilon}{M}$, then $|F(x) - F(y)| < \varepsilon$.

$\Rightarrow$ F is unif. conti. on $[a,b]$. $\square$

pf 2)

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right|$$

$$= \left|\frac{\int_{x_0}^{x} f(t)\,dt}{x-x_0} - \underbrace{f(x_0)}_{\parallel}\right|$$

$$\frac{1}{x-x_0}\int_{x_0}^{x} f(x_0)\,dt$$

$$= \left|\frac{1}{x-x_0}\int_{x_0}^{x} (f(t)-f(x_0))\,dt\right| < \varepsilon$$

$\square$

Since f conti. at x_0,

$\forall \varepsilon > 0,\ \exists \delta > 0$

s.t. $|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$.

If $0 < |x-x_0| < \delta$,

$$\left|\int_{x_0}^{x} (f(t)-f(x_0))\,dt\right|$$

$$\le \left|\int_{x_0}^{x} \underbrace{|f(t)-f(x_0)|}_{\overset{\wedge}{\varepsilon}}\,dt\right|$$

$$< \varepsilon \cdot |x-x_0|$$

Def

$|A| = |B|$

- Two sets A, B have the same cardinality

If $\exists\, f : A \longrightarrow B$ that is both

injective and surjective $\rightarrow$ (bijective)

($a_1 \neq a_2$ in A then $f(a_1) \neq f(a_2)$)

($\forall b \in B$. $\exists a \in A$ s.t. $f(a) = b$)

- "$|A| \leq |B|$": $\exists\, g : A \to B$ injective
- "$|A| < |B|$": $\exists\, h : A \to B$ injective, but $\nexists\, k : A \to B$ bijective.

Def. An infinite set A is countable if $|A| = |\mathbb{N}| = |\{1, 2, 3, \dots\}|$

- An infinite set A is uncountable if $|A| > |\mathbb{N}|$

Rmk:

$\hookrightarrow$

$\twoheadrightarrow$

- $|A| \leq |B|$ and $|B| \leq |A| \Rightarrow |A| = |B|$

  $f : A \underset{inj}{\hookrightarrow} B$ $\quad g : B \underset{inj}{\hookrightarrow} A$ $\quad \exists\, h : A \to B$ bij.

  Schröder-Berstein thm (cf wiki)

- Axiom of Choice $\Leftrightarrow$ $\forall A, B$ sets either $|A| \leq |B|$ or $|B| \leq |A|$

e.g. $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$ is countable

$$
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathbb{N} \\
0 & \longmapsto & 1 \\
1 & \longmapsto & 2 \\
-1 & \longmapsto & 3 \\
2 & \longmapsto & 4 \\
-2 & \longmapsto & 5 \\
 & \vdots & \\
n>0 & \longmapsto & 2n \\
n<0 & \longmapsto & -2n+1
\end{array}
$$

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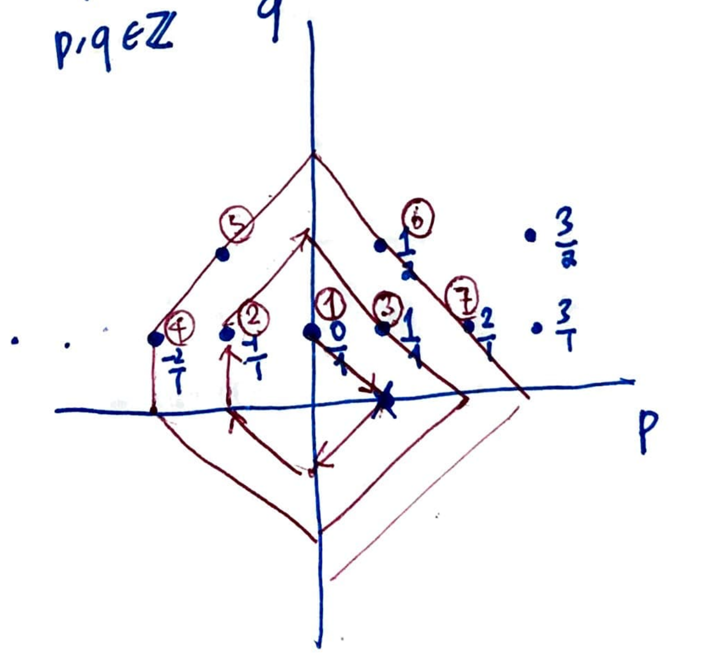
e.g. $|\mathbb{Q}| = |\mathbb{N}|$

$\frac{p}{q}$ $(p,q)=1$, $q>0$, $p,q \in \mathbb{Z}$

assign elt in $\mathbb{Q}$ an elt in $\mathbb{Z}^2 \subset \mathbb{R}^2$



∞ sets

$\exists\ S_1, S_2, S_3, \dots$

s.t. $|S_1| < |S_2| < |S_3| < \dots$

<u>Def</u> For any set A, define the <u>power set</u> $P(A)$ to be the set of all subsets of A, (including ϕ and A itself)

<u>e.g.</u> $A = \{1,2,3\}$

$$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

<u>Cantor</u>: $|A| < |P(A)|$

pf • $f: A \longrightarrow P(A)$ injective

$$a \longmapsto \{a\}$$

• <u>Claim</u>: $\boxed{\text{Any } f: A \to P(A) \text{ is NOT surjective.}}$

<u>Want</u>: $\exists\ B \in P(A)$ s.t. $B \neq f(a)\ \forall a \in A$

a subset of A

Define $B := \{a \in A \mid a \notin \underbrace{f(a)}_{\text{a subset of } A}\}$

For any $a \in A$, either $a \in B$
or $a \notin B$

① $a \in B \Rightarrow a \notin f(a)$

$\Rightarrow B \neq f(a)$

② $a \notin B \Rightarrow a \in f(a)$

$\Rightarrow B \neq f(a)$

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e.g. $P(\mathbb{N})$ is uncountable.
