Today: Symmetric matrices, quadratic forms.
Thursday: Preview of differential equations, ...
Next week: Q & A for $2^{nd}$ midterm, & $2^{nd}$ midterm.

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**Symmetric matrices** $A = A^T$, $A$: real.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \ddots & \\ \vdots & & \ddots \end{bmatrix} \quad a_{ij} = a_{ji} \ \forall i, j$$

$f: \mathbb{R}^n \to \mathbb{R}$, $x_0$

$\frac{\partial f}{\partial x_1}\Big|_{x_0}, \ldots, \frac{\partial f}{\partial x_n}\Big|_{x_0}$

$\Downarrow$

$$\text{Hess}(f) \quad \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \boxed{\frac{\partial^2 f}{\partial x_1 \partial x_2}} & \cdots \\ \boxed{\frac{\partial^2 f}{\partial x_2 \partial x_1}} & \ddots & \\ \vdots & \cdots & \end{pmatrix}$$

$f: \mathbb{R} \to \mathbb{R}$

$f'$

$f''$

**Prop:** $A^T = A \Rightarrow$ any 2 eigenvectors with distinct eigenvalues are orthogonal.

**Pf** Suppose $\lambda_1 \neq \lambda_2$ are eigenvalues of $A$.

$$A\vec{v}_1 = \lambda_1 \vec{v}_1, \quad A\vec{v}_2 = \lambda_2 \vec{v}_2, \quad \vec{v}_1, \vec{v}_2 \neq \vec{0}.$$

Want to show: $\langle \vec{v}_1, \vec{v}_2 \rangle = 0$.

$$\langle A\vec{v}_1, \vec{v}_2 \rangle = (A\vec{v}_1)^T \vec{v}_2 = \vec{v}_1^T A^T \vec{v}_2$$

$\lambda_1 \langle \vec{v}_1, \vec{v}_2 \rangle \ \Vert$ ; $\Vert$ A symmetric

$$\lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle = \langle \vec{v}_1, A\vec{v}_2 \rangle = \vec{v}_1^T A \vec{v}_2$$

$\square$

**Def** Say a matrix $A$ is <u>orthogonally diagonalizable</u> if $\exists$ orthonormal eigenbasis of $A$.

($\Leftrightarrow$ $\exists$ orthogonal matrix $P$ and a diagonal matrix $D$ s.t. $A = PDP^T$) $P^T = P^{-1}$

$\boxed{\text{Thm: } A \text{ is orthogonally diagonalizable} \iff A \text{ is symmetric}}$

($\Rightarrow$) clear since $D = D^T$

($\Leftarrow$)

<u>Want to show</u>: If $A$ is symmetric, then $\exists$ an orthonormal eigenbasis of $A$.

We'll prove this in two steps.

1) If $A$ symmetric and diagonalizable, then $A$ is orthogonally diagonalizable.

2) If $A$ symmetric, then $A$ is diagonalizable.

<u>Proof of 1)</u>: Suppose $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of $A$.

For each eigenspace $\mathrm{Nul}(A - \lambda_i I)$, we pick an orthonormal basis $\{v_i^{(1)}, \dots, v_i^{(m_i)}\}$. $m_i$ = mult($\lambda_i$)

(Since $A$ is diagonalizable, $\dim \mathrm{Nul}(A - \lambda_i I) = \mathrm{mult}(\lambda_i)$)

Consider $\{v_1^{(1)}, \dots, v_1^{(m_1)}, v_2^{(1)}, \dots, v_2^{(m_2)}, \dots, v_k^{(1)}, \dots, v_k^{(m_k)}\}$.

• (this set consists of exactly $n$ vectors.)

Claim: This is an orthonormal eigenbasis of $A$.
(follows from the previous Proposition). $\square$

Proof of 2): (Any symmetric matrix is diagonalizable).

Induction on the size of $A$.

$1\times 1 \to$ clear

Assume any $(n-1)\times(n-1)$ symmetric matrix is diagonalizable,

$A: n\times n$, $A^T = A$.

- Pick any eigenvector $\vec{v} \neq \vec{0}$ of $A$, $\|\vec{v}\| = 1$, $A\vec{v} = \lambda\vec{v}$.
- Choose $\{\vec{v}_2, \dots, \vec{v}_n\}$ s.t. $\{\vec{v}_1, \dots, \vec{v}_n\}$ orthonormal set. ($\vec{v}_1 = \vec{v}$)
- $Q = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ orthogonal, $Q^T Q = I$.

$Q^T A Q$

- $AQ = A\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{bmatrix}$

$$Q^T A Q = \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \lambda\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \lambda & \boxed{\overset{0}{* \; * \; * \; * \; \cdots}} \\ 0 & \\ 0 & \\ \vdots & A' = (A')^T \\ 0 & \end{bmatrix}$$

- Since $A = A^T$, so $Q^T A Q$ is also symmetric

- By inductive hypothesis, $A'$ is diagonalizable.

  By 1st step, $A'$ is orthogonally diagonalizable,

  so $\exists$ $\underset{(n-1)\times(n-1)}{P}$ orthogonal, $D$ : diagonal

  st. $A' = PDP^T$

$$\Rightarrow \quad Q^T A Q = \left[\begin{array}{c|c} \lambda & 0\,0\,0\,\cdots \\ \hline 0 & \\ 0 & \\ \vdots & PDP^T \\ \vdots & \end{array}\right]$$

$$= \underbrace{\left[\begin{array}{c|c} 1 & 0\,0\,\cdots \\ \hline 0 & \\ \vdots & P \\ \vdots & \end{array}\right]}_{\substack{\| \\ R \\ \text{orthogonal}}} \underbrace{\left[\begin{array}{c|c} \lambda & 0\cdots \\ \hline 0 & \\ \vdots & D \\ \vdots & \end{array}\right]}_{\substack{\uparrow \\ \text{diagonal}}} \underbrace{\left[\begin{array}{c|c} 1 & 0\cdots \\ \hline 0 & \\ \vdots & P^T \\ \vdots & \end{array}\right]}_{R^T}$$

$$\Rightarrow \quad A = (\underbrace{QR}_{\substack{\| \\ S \\ \text{orthogonal}}}) \underbrace{\left[\begin{array}{c|c} \lambda & 0\cdots \\ \hline 0 & \\ \vdots & D \\ \vdots & \end{array}\right]}_{\substack{\| \\ D' \text{ diagonal}}} \underbrace{(QR)^T}_{\substack{\| \\ S^T}}$$

$$= S D' S^T \qquad \square$$

Rmk: $A$ symmetric $\Rightarrow$ all eigenvalues are real

$$A\vec{v} = \lambda\vec{v}, \quad \underset{\in \mathbb{C}^n}{\vec{v}} \neq \vec{0}, \quad \lambda \in \mathbb{C}$$

(want to show: $\underline{\underline{\lambda \in \mathbb{R}}}$)

Consider $\overline{\vec{v}}^T A \vec{v}$

$$\overline{\overline{\vec{v}}^T A \vec{v}} = \vec{v}^T \overline{A}\, \overline{\vec{v}} = \vec{v}^T A \overline{\vec{v}}$$

$\overline{A} = A$ real

$\vec{v}^T A \overline{\vec{v}} = \overline{\vec{v}}^T A \vec{v}$ ($A$ symmetric)

$$\overline{\vec{v}}^T A \vec{v}^T = \vec{v}^T A^T \overline{\vec{v}}$$

$$\Rightarrow \overline{\vec{v}}^T A \vec{v} \in \mathbb{R}$$

$$\Rightarrow \lambda \boxed{\overline{\vec{v}}^T \vec{v}} \in \mathbb{R}$$

$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad v_i \in \mathbb{C}$

$$\overline{\vec{v}}^T \vec{v} = \begin{bmatrix} \overline{v}_1 & \overline{v}_2 & \cdots & \overline{v}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1 \overline{v}_1 + \cdots + v_n \overline{v}_n$$

$$= |v_1|^2 + \cdots + |v_n|^2 \in \mathbb{R}$$

$$\Rightarrow \lambda \in \mathbb{R}$$

## Quadratic forms

homogeneous polynomials of deg 2 in $x_1, \dots, x_n$ variables

$$Q(x_1, \cdots, x_n) = \sum_{i=1}^{n} b_i x_i^2 + \sum_{1 \le i < j \le n} b_{ij} x_i x_j$$

$$\left\{ \begin{array}{c} \text{Symmetric matrices} \\ n \times n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Quad forms} \\ \text{in } x_1, \dots, x_n \end{array} \right\}$$

$$A \longmapsto Q_A(\vec{x}) := \vec{x}^T A \vec{x}$$

$$= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & & \\ \vdots & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \end{bmatrix}$$

$$\begin{aligned} = \; & a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\ + \; & a_{21}x_1x_2 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\ & \vdots \\ + \; & a_{n1}x_1x_n + a_{n2}x_2x_n + \cdots + a_{nn}x_n^2 \end{aligned}$$

$$= \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{1 \le i < j \le n} (\overbrace{a_{ij} + a_{ji}}^{= 2a_{ij} \text{ since } A^T = A}) x_i x_j$$

e.g. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

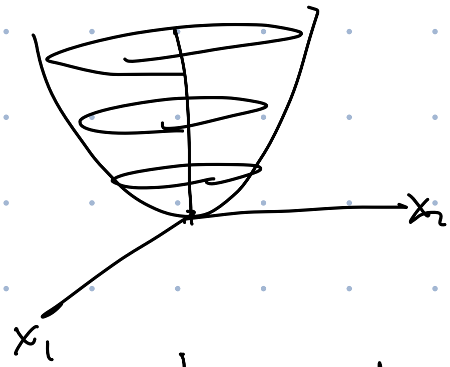
$$Q_A(\vec{x}) = \vec{x}^T A \vec{x} = x_1^2 + x_2^2 + 2x_1x_2$$

Conversely,

$$\begin{bmatrix} b_1 & \frac{b_{12}}{2} & \frac{b_{13}}{2} & \cdots & \\ \frac{b_{12}}{2} & b_2 & & & \\ \frac{b_{13}}{2} & & b_3 & & \\ \vdots & & & \ddots & \\ & & & & b_n \end{bmatrix} \longleftarrow\!\!| \quad \sum_{i=1}^{n} b_i x_i^2 + \sum_{1 \le i < j \le n} b_{ij} x_i x_j$$

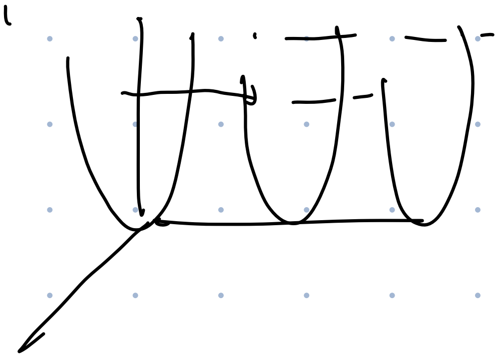
e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q_A(\vec{x}) = x_1^2 + x_2^2$

positive definite




e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $Q_A(\vec{x}) = x_1^2$

positive semidefinite



e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $Q_A(\vec{x}) = x_1^2 - x_2^2$

Indefinite

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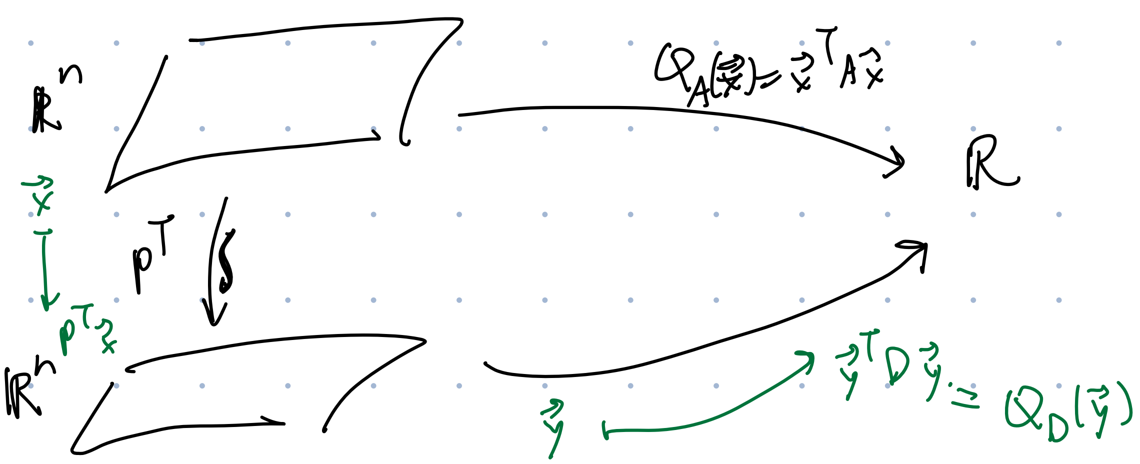
$$Q_A(\vec{x}) = \vec{x}^T A \vec{x} = \vec{x}^T P D P^T \vec{x}$$



e.g. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. $Q_A(\vec{x}) = x_1^2 + x_2^2 + 2x_1x_2$

$$= PDP^T, \text{ where } P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\vec{y} = P^T\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1 - x_2}{\sqrt{2}} \\ \frac{x_1 + x_2}{\sqrt{2}} \end{bmatrix}.$$

$$Q_A(\vec{x}) = \vec{y}^T D \vec{y} = [y_1 \ y_2] \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 2y_2^2 \ \left( = (x_1 + x_2)^2 \right)$$

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**Def**: $A$ symmetric $Q_A(\vec{x}) = \vec{x}^T A \vec{x}$.

- Say $A$ is <u>positive definite</u> if $Q_A(\vec{x}) > 0 \ \forall \vec{x} \neq \vec{0}$
- ——— <u>positive semidefinite</u> if $Q_A(\vec{x}) \geq 0 \ \forall \vec{x}$
- <u>negative (semi) definite</u>
- Say $A$ is <u>indefinite</u> if it's not of any of the above cases