Method of undetermined coefficients: $y'' + by' + cy = f.$ —— $(*)$

- If $f(t) = t^m e^{r_0 t}$,

  then there is a sol^n of $(*)$ of the form

$$y(t) = (A_m t^m + \cdots + A_0) \cdot t^s \cdot e^{r_0 t},$$

$\begin{cases} \bullet\ a_m t^m + a_{m-1} t^{m-1} + \cdots, \\ \bullet\ e^{r_0 t} \end{cases}$

$* e^{r_0 t}, \ * t e^{r_0 t}, \ * t^2 e^{r_0 t}$

where:

$s = 0$ if r_0 is not a root of $r^2 + br + c = 0$.

$s = 1$ if r_0 is a simple root of $r^2 + br + c = 0$.

$s = 2$ if r_0 is a double root of $r^2 + br + c = 0$.

- If $f(t) = t^m e^{\alpha t} \cos(\beta t)$ or $t^m e^{\alpha t} \sin(\beta t)$

  then there is a sol^n of $(*)$ of the form

$$y(t) = (A_m t^m + \cdots + A_0)\, t^s e^{\alpha t} \cos(\beta t) + (B_m t^m + \cdots + B_0)\, t^s e^{\alpha t} \sin(\beta t),$$

where:

$s = 0$ if $\alpha \pm i\beta$ is not a root of $r^2 + br + c = 0$,

$s = 1$ if $\alpha \pm i\beta$ are the roots of $r^2 + br + c = 0$.

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e.g. $y'' + 2y' + y = t^2 + t e^{-t}$, $\quad y(0) = 2$, $y'(0) = 1$.

If $\begin{cases} ① \ y_1'' + 2y_1' + y_1 = t^2 \\ ② \ y_2'' + 2y_2' + y_2 = t e^{-t}. \end{cases}$

then $(y_1 + y_2)'' + 2(y_1 + y_2)' + (y_1 + y_2) = t^2 + t e^{-t}$.

③ general sol^n of $y'' + 2y' + y = 0$. Say $\{y_3, y_4\}$ is a l.i. set of sol^{n}s of $y'' + 2y' + y = 0$.

$e^{-t}, \ t e^{-t}$

$\Rightarrow$ any sol^n of $y'' + 2y' + y = t^2 + t e^{-t}$ is of the form $(y_1 + y_2) + c_1 y_3 + c_2 y_4$.

① $y_1'' + 2y_1' + y_1 = t^2.$

Let $y_1(t) = A_2 t^2 + A_1 t + A_0$

$2y_1' = \overset{4}{\cancel{2}} A_2 t + 2A_1$

$y_1'' = 2A_2$

$$t^2 = \underbrace{A_2}_{=1} t^2 + \underbrace{(A_1 + 4A_2)}_{=0} t + \underbrace{(A_0 + 2A_1 + 2A_2)}_{=0}$$

$\Downarrow$ $A_1 = -4$ $\qquad$ $\Downarrow$ $A_0 = 6.$

$\Rightarrow$ $y_1 = t^2 - 4t + 6$ is a sol^n to

② $y_2'' + 2y_2' + y_2 = t e^{-t}.$

Since -1 is the double root of $r^2 + 2r + 1 = 0$
$\Leftrightarrow (r+1)^2$

Let $y_2(t) = (A_1 t + A_0) \cdot t^{\overset{2}{\|}s} \cdot e^{-t}$

$= (A_1 t + A_0)\, t^2 e^{-t}.$

$= (A_1 t^3 + A_0 t^2)\, e^{-t}$

$2y_2' = 2(3A_1 t^2 + 2A_0 t) e^{-t} - 2(A_1 t^3 + A_0 t^2) e^{-t}$

$= 2(-A_1 t^3 + (3A_1 - A_0) t^2 + 2A_0 t)\, e^{-t}.$

$y_2'' = (-3A_1 t^2 + (6A_1 - A_0) t + 2A_0) e^{-t}$
$+ (A_1 t^3 - (3A_1 - A_0) t^2 - 2A_0 t) e^{-t}$

$= (A_1 t^3 + (-6A_1 + A_0) t^2 + (6A_1 - 3A_0) t + 2A_0) e^{-t}$

$$t e^{-t} = e^{-t}\Big[\underbrace{(6A_1 + A_0)}_{=1} t + \underbrace{2A_0}_{=0}\Big]$$

$\Rightarrow$ $y_2 = \frac{1}{6} t^3 e^{-t}$ is a sol^n to the eq^n.

$\Rightarrow$ Any sol^{n} of $y'' + 2y' + y = t^2 + te^{-t}$ is of the form

$$y(t) = \underbrace{\left(t^2 - 4t + 6\right) + \frac{1}{6} t^3 e^{-t}}_{\text{is a particular sol}^{n} \text{ to}} + \underbrace{c_1 e^{-t} + c_2 t e^{-t}}_{\text{general sol}^{n} \text{ to } y''+2y'+y=0}.$$

Last step: Determine c_1, c_2 s.t. $y(0) = 2$, $y'(0) = 1$.

$$\begin{cases} 2 = y(0) = 6 + c_1. & \Rightarrow c_1 = -4 \\ 1 = y'(0) = -4 - c_1 + c_2 & \Rightarrow c_2 = 1. \end{cases}$$

$$y'(t) = (2t - 4) + \frac{1}{6}\left(3t^2 e^{-t} - t^3 e^{-t}\right)$$
$$- c_1 e^{-t} + c_2\left(e^{-t} - te^{-t}\right)$$

$\Rightarrow$ The sol^{n} to the initial value problem is:

$$\boxed{\left(t^2 - 4t + 6\right) + \frac{1}{6} t^3 e^{-t} - 4e^{-t} + te^{-t}.}$$

$\square$

Variation of parameters method can deal with solving non-homog. eq^{n} $\boxed{y'' + by' + cy = f}$ for general f.

- Say $\{y_1, y_2\}$ is a l.i. set of sol^{n}s of $y'' + by' + cy = 0$.

$\Rightarrow$ general sol^{n} to $y'' + by' + cy = 0$ is of the form

$$c_1 y_1(t) + c_2 y_2(t).$$

Idea: Try to find two functions $c_1(t)$ & $c_2(t)$

s.t. $y(t) := c_1(t)\, y_1(t) + c_2(t)\, y_2(t)$

satisfies $y'' + by' + cy = f$.

- $y'(t) = c_1' y_1 + c_1 y_1' + c_2' y_2 + c_2 y_2'$

Impose an extra condition on $c_1(t)$, $c_2(t)$:

$$\boxed{c_1' y_1 + c_2' y_2 = 0}$$

$c_1 y_1' + c_2 y_2'$

- $y''(t) = c_1' y_1' + c_1 y_1'' + c_2' y_2' + c_2 y_2''$.

$$\underline{y'' + by' + cy} = c_1' y_1' + c_1 y_1'' + c_2' y_2' + c_2 y_2''$$
$$\quad + b(c_1 y_1' + c_2 y_2')$$
$$\quad + c(c_1 y_1 + c_2 y_2)$$

$//$

f

$$= c_1' y_1' + c_2' y_2'.$$

Summary: If we can find $c_1(t)$, $c_2(t)$ s.t.

$$\begin{cases} c_1'(t)\, y_1(t) + c_2'(t)\, y_2(t) = 0 \\ c_1'(t)\, y_1'(t) + c_2'(t)\, y_2'(t) = f(t) \end{cases}$$

Then $y(t) = c_1(t)\, y_1(t) + c_2(t)\, y_2(t)$ satisfies

$$y'' + by' + cy = f.$$

$$\underbrace{\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}}_{\text{invertible } \forall t} \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \quad \forall t \in \mathbb{R}$$

invertible $\forall t$

by a proposition we proved before

(which uses the existence & uniqueness thm.)

$$\begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

$$= \frac{1}{y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t)} \begin{bmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{bmatrix} \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

$$= \frac{1}{y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t)} \begin{bmatrix} -y_2(t)\, f(t) \\ y_1(t)\, f(t) \end{bmatrix}$$

$$\begin{cases} c_1'(t) = \dfrac{-y_2 f}{y_1 y_2' - y_2 y_1'} \\[2ex] c_2'(t) = \dfrac{y_1 f}{y_1 y_2' - y_2 y_1'} \end{cases}$$

We can find $c_1(t)$, $c_2(t)$ that satisfies

just by doing integrations.

e.g. Find a sol^n $y(t): (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ of

$$y'' + y = \tan t = f$$

Use the variation of parameter method.

① $\{\cos t, \sin t\}$ is a l.i. set of sol^{ns} of $y''+y=0$.
$\quad\ \ \| \qquad \|$
$\quad\ y_1 \quad\ \ y_2$

② $$c_1'(t) = \frac{-y_2 f}{y_1 y_2' - y_2 y_1'} = \frac{-\sin^2 t / \cos t}{1}$$

$$c_2'(t) = \frac{y_1 f}{y_1 y_2' - y_2 y_1'} = \frac{\sin t}{1}$$

③ $$c_1(t) = \int \frac{-\sin^2 t}{\cos t}\, dt = \int \frac{-1+\cos^2 t}{\cos t}\, dt$$

$$= -\int \frac{1}{\cos t}\, dt + \int \cos t\, dt$$

$$= -\log|\sec t + \tan t| + \sin t + (\text{const.})$$

$$c_2(t) = \int \sin t\, dt = -\cos t + (\text{const.})$$

④ $y(t) = c_1(t)\, y_1(t) + c_2(t)\, y_2(t)$ is a sol^n to $y'' + y = \tan t$

$$= \left(-\log|\sec t + \tan t| + \sin t\right)\cos t + \left(-\cos t\right)\sin t$$

$$= -(\log|\sec t + \tan t|)\cdot \cos t + (\text{const.})\cdot \cos t + (\text{const.})\sin t.$$

§ Systems of first-order ordinary differential eq^{ns}

$$\begin{cases} x_1'(t) = 3x_1(t) + 2x_2(t) + e^t \\ x_2'(t) = x_1(t) - x_2(t) + t^3. \end{cases}$$

vector-valued fn.

$\vec{x}(t): \mathbb{R} \longrightarrow \mathbb{R}^2$

$t \mapsto \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\vec{x}'(t) = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} e^t \\ t^3 \end{bmatrix}$$

Rmk: e.g. models motion of three gravitationally interacting bodies.

$\vec{r}_i(t) = (x_i(t), y_i(t), z_i(t))$, m_i = mass $i = 1, 2, 3.$

$$\begin{cases} \vec{r}_1'' = -g m_2 \dfrac{\vec{r}_1 - \vec{r}_2}{\|\vec{r}_1 - \vec{r}_2\|^3} - g m_3 \dfrac{\vec{r}_1 - \vec{r}_3}{\|\vec{r}_1 - \vec{r}_3\|^3} \\ \vec{r}_2'' = -g m_1 \dfrac{\vec{r}_2 - \vec{r}_1}{\|\vec{r}_2 - \vec{r}_1\|^3} - g m_3 \quad \cdots \\ \vec{r}_3'' = \quad \cdots \end{cases}$$

Generally, there is no closed-form sol^n.

e.g. $y^{(n)}(t) + p_{n-1} y^{(n-1)}(t) + \cdots + p_1 y'(t) + p_0 y(t) = 0$

$$\begin{cases} x_1(t) = y(t) \\ x_2(t) = y'(t) \\ \vdots \\ x_n(t) = y^{(n-1)}(t) \end{cases}$$

$x_1' = y' = x_2$

$x_2' = x_3$

$\vdots$

$$\begin{aligned} x_n' &= y^{(n)}(t) \\ &= -p_{n-1} y^{(n-1)} - \cdots - p_0 y \\ &= -p_{n-1} x_n - \cdots - p_0 x_1. \end{aligned}$$

$$\begin{bmatrix} x_1' \\ \vdots \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & & -p_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

e.g.

$$\vec{x}'(t) = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \vec{x}(t).$$

$$\left(\begin{cases} x_1'(t) = \lambda_1 x_1(t) \\ x_2'(t) = \lambda_2 x_2(t) \\ \quad \vdots \\ x_n'(t) = \lambda_n x_n(t) \end{cases} \right)$$