$$\begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}$$

Plug in: $\underline{x_n = 0}$: $\Rightarrow$ $\det = x_1 x_2 \cdots x_{n-1} \det\left(\quad \right)_{n-1 \times n-1}$

$$x_1 x_2 \cdots x_{n-1} \prod_{n-1 \geq i > j \geq 1} (x_i - x_j)$$

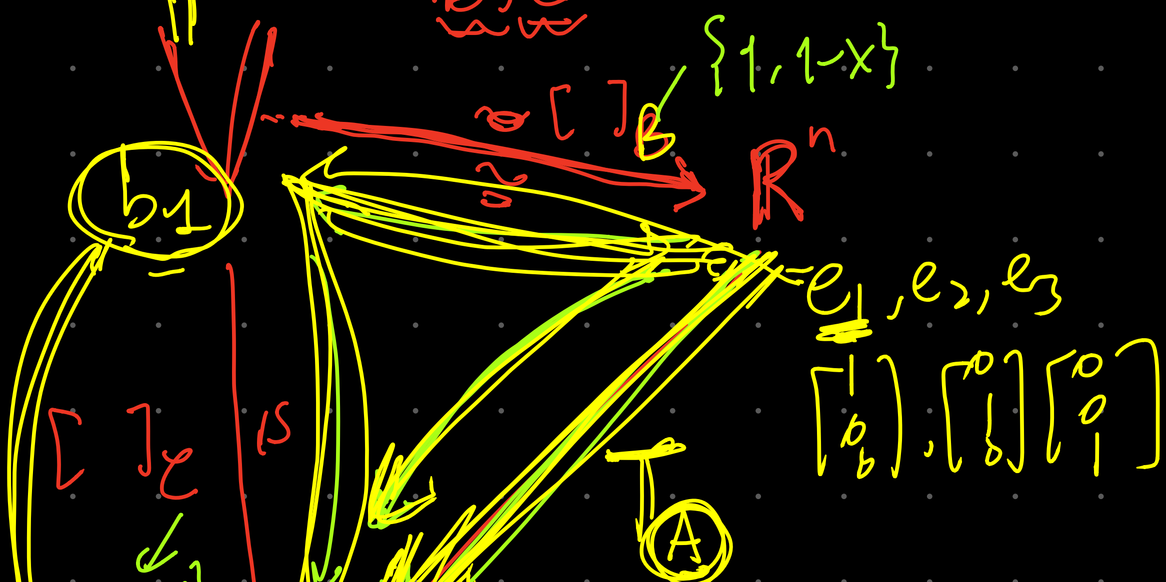
$$\boxed{(x_n - x_1)\cdots(x_n - x_{n-1})} \cdot C$$

$\mathbb{R}^2$

$\cong \{1, x\}$

$\text{Poly}_{\leq 1}$

$\mathcal{B}, \mathcal{C}$ $\quad \underline{\dim V = n}$



$\{1, 1+x\}$

$\mathbb{R}^n$

$\underline{\mathbb{R}^3}$

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$b_1 \quad b_2 \quad b_3$

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$c_1 \quad c_2 \quad c_3$

$$\underline{[b_1]_{\mathcal{C}}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$b_1 = x\begin{bmatrix} c_1 \end{bmatrix} + y\begin{bmatrix} c_2 \end{bmatrix} + z\begin{bmatrix} c_3 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \underline{\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}} \begin{bmatrix} b_1 \end{bmatrix}_{\mathcal{C}}$$

$$\overline{T}_A(e_1) = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^{-1} b_1$$

$$A = C^{-1} B$$

$$T : V \longrightarrow W.$$

$$\dim V < +\infty$$

$$\underset{n}{\dim V} = \dim \ker T + \underset{m}{\dim \operatorname{Im} T}$$

---

(1) $\boxed{V \xrightarrow[k_{ij}]{\sim} \mathbb{R}^n}$

$\mathbb{R}^m \quad m \le n$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\text{surjective}", "\dim\ker = n-m"'] \arrow[d, phantom, "\cong"] & \mathbb{R}^m \arrow[d, phantom, "\cong"] \\
\boxed{V} \arrow[r, "\text{surjective}"] \arrow[dr, "T"'] & \boxed{T(V)} \arrow[d, phantom, "\subseteq"] \\
 & W
\end{tikzcd}$$

$\to$ finite dim'l.

b/c $\dim T(V) \le \dim V = n$

by ~~rank thm~~ rank-nullity thm for matrices

$l = n - m$

dim = n

$$\begin{tikzcd}
\ker(T') \arrow[r] \arrow[d, "\cong"] & \mathbb{R}^n \arrow[rr, "T'"] \arrow[d, "\cong"] & & \mathbb{R}^m \\
\ker(T) \arrow[r] & V \arrow[rr, "T"] & & T(V)
\end{tikzcd}$$

$$\begin{array}{ccc} v & \longmapsto & 0 \\ \updownarrow & & \updownarrow \;\; \| \\ v' & \xrightarrow{T} & 0 \end{array}$$

dim ??

$\alpha_i \longmapsto w_i$

~~Claim~~: $T: \underline{V} \longrightarrow W$

$\{v_1, \dots, v_n\}$ basis of $\ker V$

$\{w_1, \dots, w_m\}$ basis of $T(V)$

Suppose $\alpha_i \in V$, $T(\alpha_i) = w_i$

Claim: $\{v_1, \dots, v_n, \alpha_1, \dots, \alpha_m\}$ is a basis of V.

(1) $\{v_1, \dots, v_n, \alpha_1, \dots, \alpha_m\}$ l.i. ??

$$\boxed{c_1 v_1 + \dots + c_n v_n + d_1 \alpha_1 + \dots + d_m \alpha_m = 0}$$

$= 0_V$

$$\Rightarrow T(0) = T(c_1 v_1 + \cdots + c_n v_n + d_1 \alpha_1 + \cdots + d_m \alpha_m)$$

$$0 = \qquad = d_1 w_1 + \cdots + d_m w_m \quad \text{in } W$$

② $\operatorname{Span}\{v_1, \ldots, v_n, \alpha_1, \ldots, \alpha_m\} = V$??

$\forall v \in V \qquad \boxed{T(v)} \in T(V) = \operatorname{Span}\{w_1, \ldots, w_m\}$

$$T(v) = c_1 w_1 + \cdots + c_m w_m$$

$$= c_1 T(\alpha_1) + \cdots + c_m T(\alpha_m)$$

$$= T(c_1 \alpha_1 + \cdots + c_m \alpha_m)$$

$$T(\boxed{v - c_1 \alpha_1 - \cdots - c_m \alpha_m}) = 0$$

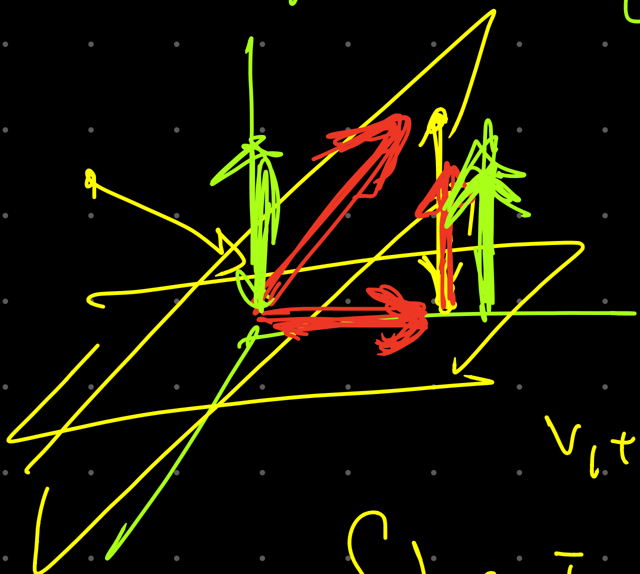
$$\ker(T) = \operatorname{Span}\{v_1, \ldots, v_n\}$$

$$v - c_1 \alpha_1 - \cdots - c_m \alpha_m = d_1 v_1 + \cdots + d_n v_n$$

$v = c_1\alpha_1 + \dots + c_m\alpha_m + d_1v_1 + \dots + d_nv_n$

$\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^3$ $\quad$ $T^2 = T$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \longmapsto \begin{bmatrix}x\\y\\0\end{bmatrix}$$



$v_1 + v_2 = v$

$$\begin{cases} \ker T + \operatorname{Im} T = V \\ \ker T \cap \operatorname{Im} T = \{0\} \end{cases}$$

$$\Rightarrow \boxed{V = \operatorname{Ker}(T) \oplus \operatorname{Im}(T)}$$

$W_1, W_2 \subseteq V$

Claim: $V = W_1 + W_2$

$W_1 \cap W_2 = \{0\}$

$\Rightarrow \forall v \in V, \exists!\, v_1 \in W_1,\ v_2 \in W_2$

s.t. $v = v_1 + v_2$

$$V = v_1 + v_2$$
$$= v_1' + v_2'$$
$$v_1, v_1' \in W_1$$
$$v_2, v_2' \in W_2$$
$$v_1 - v_1' = v_2' - v_2$$