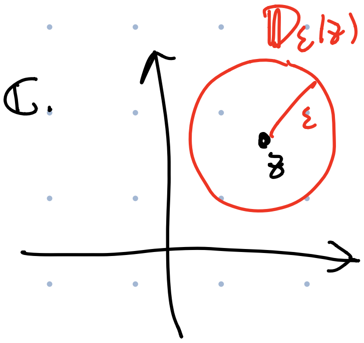
Today: Holomorphic functions, Review of topology

We'll work in $\mathbb{C}$.

- <u>open disk</u> of radius $\varepsilon > 0$ centered at $z \in \mathbb{C}$.

$$\mathbb{D}_\varepsilon(z) := \{ w \in \mathbb{C} \mid |z - w| < \varepsilon \}.$$



- $\Omega \subseteq \mathbb{C}$ is <u>open</u> if

$$\forall z \in \Omega, \ \exists \varepsilon > 0 \ \text{s.t.} \ \mathbb{D}_\varepsilon(z) \subseteq \Omega.$$

such that.

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§ Holomorphic functions

<u>Def</u>: $\Omega \subseteq \mathbb{C}$ open subset. $f : \Omega \longrightarrow \mathbb{C}$

Say f is <u>holomorphic</u> at a point $z_0 \in \Omega$

If

$$\boxed{\lim_{\substack{h \to 0 \\ h \in \mathbb{C} \setminus \{0\}}} \frac{f(z_0 + h) - f(z_0)}{h} \ \text{exists.}}$$

More precisely, there exists $\boxed{L} \in \mathbb{C}$ s.t.

derivative of f at $z_0 = f'(z_0)$

$\forall \varepsilon > 0, \ \exists \delta > 0$

$$\text{s.t.} \quad \underset{h \in \mathbb{C}}{0 < |h| < \delta} \implies \left| \frac{f(z_0 + h) - f(z_0)}{h} - L \right| < \varepsilon.$$

- f is <u>holo. in Ω</u> if it's holo. at all pt. $z_0 \in \Omega$

e.g. $f:\mathbb{C}\longrightarrow\mathbb{C}$, $f(z)=z^n$, $n\in\mathbb{N}$,

$$\frac{f(z+h)-f(z)}{h}=\frac{(z+h)^n-z^n}{h}$$

$$=\frac{\left(\cancel{z^n}+\binom{n}{1}z^{n-1}\cancel{h}+\cdots+\binom{n}{q}z\,h^{n-1}+h^{n}\right)-\cancel{z^n}}{\cancel{h}}$$

$$=\binom{n}{1}z^{n-1}+\binom{n}{2}z^{n-2}h^{2-1}+\cdots+h^{n-1}$$

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=n\,z^{n-1}.$$

So $f=z^n$ is holomorphic, and $f'(z)=n\,z^{n-1}$.

e.g. $f:\mathbb{C}\longrightarrow\mathbb{C}$

$z\longmapsto\bar{z}$

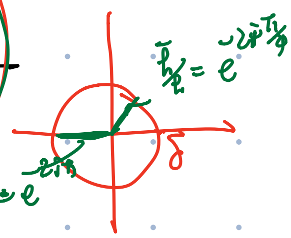
$(x+iy\longmapsto x-iy)$

$\forall\varepsilon>0,\ \exists\delta>0$ s.t. $0<|h|<\delta\Rightarrow\left|\frac{f(z+h)-f(z)}{h}-L\right|<\varepsilon$

$\qquad\qquad\qquad\qquad\qquad\left|\frac{\bar h}{h}-L\right|$

L

$$\frac{f(z+h)-f(z)}{h}=\frac{\overline{z+h}-\bar z}{h}=\frac{\bar h}{h}$$

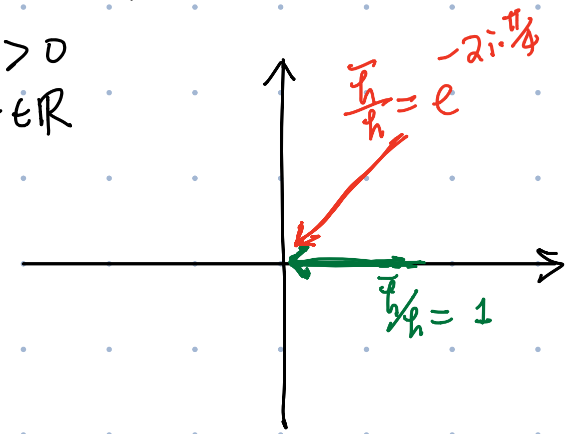


Write $h=\varepsilon e^{i\theta}$, $\bar h=\varepsilon e^{-i\theta}$,

$\varepsilon>0$

$\theta\in\mathbb{R}$

$$\frac{\bar h}{h}=\frac{\varepsilon e^{-i\theta}}{\varepsilon e^{i\theta}}=e^{-2i\theta}$$



If we let h approaches 0 along different direction,
the limit of $\frac{f(z+h)-f(z)}{h}$ is different

$\Longrightarrow$ f is NOT holomorphic.

Ex:

- $\frac{1}{z}$ is holo. on $\mathbb{C}\setminus\{0\}$
- $\bar{z}^n$ is not holo., $|z|$ is not holo.
- If f, g holo. $\Rightarrow$ $f+g$, fg holo., $g\circ f$ holo.
  If $g(z_0)\neq 0$ then f/g is holo. at z_0

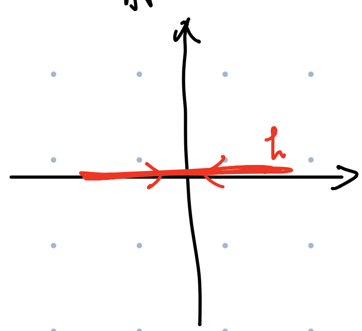
§ Necessary condition for holomorphicity.

Suppose f is holo. at z_0,

$$\lim_{\substack{h\to 0\\ h\in\mathbb{C}\setminus\{0\}}} \frac{f(z_0+h)-f(z_0)}{h} \quad \text{exists.}$$

- $f:\mathbb{C}\to\mathbb{C}$ view as $\mathbb{R}^2\longrightarrow\mathbb{R}^2$

$$\frac{\partial f}{\partial x} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}\setminus\{0\}}} \frac{f(x+h,y)-f(x,y)}{h}$$



$x+i(y+h) = z+ih$

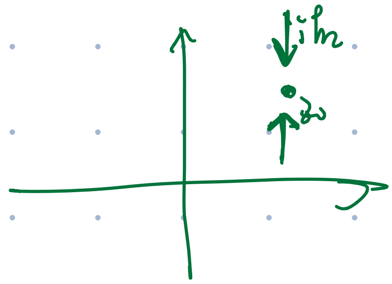
$$\frac{\partial f}{\partial y} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}\setminus\{0\}}} \frac{f(x,y+h)-f(x,y)}{h}.$$

- Suppose f is holo. at z_0, $\lim_{\substack{h\to 0\\ h\in\mathbb{C}\setminus\{0\}}} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)$ exists.

Let's write $h = h_1 + i h_2$, $h_1, h_2 \in \mathbb{R}$

$$\frac{\partial f}{\partial x}(z_0) = \lim_{\substack{h_1 \to 0 \\ h_1 \in \mathbb{R}\setminus\{0\}}} \frac{f(z_0+h_1)-f(z_0)}{h_1} = f'(z_0)$$

(let h approaching 0 in the real axis)



$$f'(z_0) = \lim_{\substack{h_2 \to 0 \\ h_2 \in \mathbb{R}\setminus\{0\}}} \frac{f(z_0+ih_2)-f(z_0)}{ih_2}$$

$$= \frac{1}{i} \underline{\lim_{\substack{h_2 \to 0 \\ h_2 \in \mathbb{R}\setminus\{0\}}} \frac{f(z_0+ih_2)-f(z_0)}{h_2}} = \frac{1}{i}\frac{\partial f}{\partial y}(z_0)$$

$\Longrightarrow$ a necessary condition for f to be holo. at z_0 is:

$$\frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0) = f'(z_0)$$

$f : \mathbb{C} \to \mathbb{C}$

$z \mapsto \mathrm{Re}(f(z))$ $\quad$ $z \mapsto \mathrm{Im}(f(z))$

Denote $u : \mathbb{C} \to \mathbb{R}$, $v : \mathbb{C} \to \mathbb{R}$ to be the real & imaginary part of f.

$\boxed{f = u + iv}$ (We can still define $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$)

$$\frac{\partial f}{\partial x}(z_0) = \underbrace{\frac{1}{i}\frac{\partial f}{\partial y}(z_0)}$$

$$\frac{\partial f}{\partial x}(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)$$

$$\frac{1}{i}\frac{\partial f}{\partial y}(z_0) = \frac{1}{i}\left(\frac{\partial u}{\partial y}(z_0) + i\frac{\partial v}{\partial y}(z_0)\right) = \frac{\partial v}{\partial y}(z_0) - i\frac{\partial u}{\partial y}(z_0)$$

$\left(\frac{1}{i} = -i\right)$

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}(z_0) = \dfrac{\partial v}{\partial y}(z_0) \\ \dfrac{\partial v}{\partial x}(z_0) = -\dfrac{\partial u}{\partial y}(z_0) \end{cases}$$

Cauchy-Riemann equations

e.g. $f(z) = z$

$u(z) = \mathrm{Re}(z)$, $v(z) = \mathrm{Im}(z)$

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto (x,y)$

$\begin{cases} u(x,y) = x \\ v(x,y) = y \end{cases}$

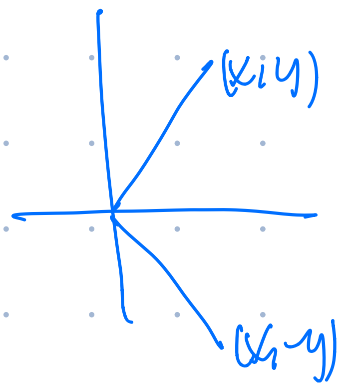
$$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y} = 1$$

$$\frac{\partial v}{\partial x} = 0 = \frac{\partial u}{\partial y} = 0$$

e.g. $f(z) = \bar{z}$

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto (x,-y)$



$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$

$\Rightarrow$ (u,v) doesn't satisfy the C-R eq^{ns}.

$\Rightarrow$ f is NOT holo.

Thm 2.4: $\begin{cases} u, v \text{ satisfy C-R eq}^{ns} \\ u, v \text{ are continuously differentiable} \end{cases}$ $\Longrightarrow$ $f = u + iv$ is holo.

(the partial derivatives are cont.)

NOT true w/o e.g. Ex. 12

Notation: $\frac{\partial f}{\partial x}(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}\setminus\{0\}}} \frac{f(z_0+h) - f(z_0)}{h}$

$$\frac{\partial f}{\partial y}(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}\setminus\{0\}}} \frac{f(z_0+ih) - f(z_0)}{h}$$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

$$\boxed{\frac{\partial f}{\partial z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

f is holo. at z_0 $\Longrightarrow \begin{cases} \frac{\partial f}{\partial \bar{z}}(z_0) = 0 \\ \frac{\partial f}{\partial z}(z_0) = f'(z_0) \end{cases}$

$\left(\text{b/c } f \text{ holo} \Rightarrow \underline{\frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0)} = f'(z_0) \right)$

$\Downarrow$

$$\frac{\partial}{\partial \bar{z}} f\Big|_{z_0} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) f = 0$$

$$\frac{\partial}{\partial z} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) f = f'(z_0)$$

$\mathbb{D}_\varepsilon(z)$ open disc

$\overline{\mathbb{D}_\varepsilon(z)}$ closed disc $= \{ w \in \mathbb{C} \mid |w-z| \le \varepsilon \}$

- $\Omega \subseteq \mathbb{C}$ closed if $\mathbb{C} \setminus \Omega$ is open

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- $\Omega \subseteq \mathbb{C}$ bounded if $\exists R > 0$ s.t. $|z| < R \ \forall z \in \Omega$
- $\Omega \subseteq \mathbb{C}$ compact if it's closed & bdd

Equivalently:

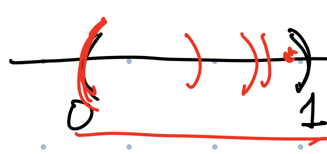
① Ω is sequentially compact: if $\forall \{z_n\} \subseteq \Omega$, $\exists$ subseq. $\{z_{n_k}\}$ s.t. $\lim z_{n_k}$ exists in Ω

② (finite open subcover:) Ω is cpt if $\forall$ open covering $\{U_\alpha\}_{\alpha \in I}$ of Ω

$\Omega \subset \bigcup U_\alpha$

$\exists$ a finitely many such U_α, $U_1, \ldots, U_n$

s.t. $\Omega \subseteq \bigcup_{i=1}^{n} U_i$



$\{U_n = (0, 1 - \frac{1}{n})\}_{n \in \mathbb{N}}$

Note: For any metric space, for $\mathbb{R}^n$ (Heine-Borel thm.)

① $\Longleftrightarrow$ ② $\Longrightarrow$ closed & bounded

Continuity for maps btw metric spaces

$$f : (X, d_X) \longrightarrow (Y, d_Y)$$

is continuous if one of the following equiv. cond's hold

①: $\forall \underset{\text{open}}{U \subseteq Y}$, $\boxed{f^{-1}(U)} \subseteq X$ is open

②: $\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$.

③: $\forall \{x_n\} \subseteq X$ s.t. $\lim x_n = x_0 \Rightarrow \lim f(x_n) = f(x_0)$

Prop: $f : X \longrightarrow Y$ conti. $\underset{\text{cpt}}{K \subseteq X}$

then $\underset{\text{cpt}}{f(K) \subseteq Y}$