**FINAL EXAM PRACTICE PROBLEMS**
**MATH H54, FALL 2021**

(1) Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a linearly independent set of vectors in a real vector space $V$. Prove that
$$\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \dots, \vec{v}_{n-1} + \vec{v}_n, \vec{v}_n + \vec{v}_1\}$$
is linearly independent if and only if $n$ is odd (not divisible by $2$).

(2) Let $A$ be a real $n \times n$ matrix. Prove that there exists a real $n \times n$ matrix $B$ such that $BA = 0$ (the zero matrix) and $\operatorname{rank}(A) + \operatorname{rank}(B) = n$. (Hint: First show that there exists an invertible matrix $P$ such that $PA$ is the reduced echelon form of $A$.) (Hint: Then find a square matrix $C$ such that $C(PA) = 0$ and $\operatorname{rank}(A) + \operatorname{rank}(C) = n$. Such $C$ should not be hard to construct, using the fact that $PA$ is of reduced echelon form.) (Hint: Finally, show that $B = CP$ has the desired properties.)

(3) Let $V$ be a finite dimensional real inner product space, and let $W \subseteq V$ be a subspace.
  (a) Define $T_W \colon V \to W$ to be the orthogonal projection onto $W$. Prove that for any $\vec{v}_1, \vec{v}_2 \in V$, one has $\langle \vec{v}_1, T_W(\vec{v}_2) \rangle = \langle T_W(\vec{v}_1), \vec{v}_2 \rangle$.
  (b) Conversely, suppose $T \colon V \to V$ is a linear transformation such that $T^2 = T$ and $\langle \vec{v}_1, T(\vec{v}_2) \rangle = \langle T(\vec{v}_1), \vec{v}_2 \rangle$ holds for any $\vec{v}_1, \vec{v}_2 \in V$. Prove that $T$ is the orthogonal projection onto its image $\operatorname{Im}(T)$. (Note: $T^2 = T \circ T$ denotes the composition of $T$ with itself.) (Hint: Plug in $\vec{v}_1 = T(\vec{v})$ for any $\vec{v} \in V$, and use the condition $T^2 = T$.)

(4) Let $W_1$ and $W_2$ be two subspaces of an $n$-dimensional real vector space $V$, satisfying $\dim(W_1) + \dim(W_2) = n$. Prove that there exists a linear transformation $T \colon V \to V$ such that
$$\operatorname{Ker}(T) = W_1 \text{ and } \operatorname{Im}(T) = W_2.$$
(Hint: Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis of $W_1$. To construct the transformation $T$, you might want to use the fact that $\{\vec{v}_1, \dots, \vec{v}_k\}$ can be extended to a basis $\{\vec{v}_1, \dots, \vec{v}_k, \dots, \vec{v}_n\}$ of $V$.)

(5) Let $A$ be an $n \times n$ matrix. Consider the linear transformation $T \colon \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ on the $n^2$-dimensional vector space $\mathrm{Mat}_{n\times n}(\mathbb{R})$ defined by $T(B) = AB$. Express $\det(T)$ in terms of $\det(A)$.

(6) Let $A$ be a square matrix with columns given by unit vectors. Prove that $|\det(A)| \leq 1$. When does the equality hold?

(7) Let $V$ be a finite-dimensional vector space, and let $T \colon V \to V$ be a diagonalizable linear transformation. Suppose $W \subseteq V$ is a subspace satisfying $T(W) \subseteq W$. Prove that the restriction $T|_W \colon W \to W$ also is diagonalizable.

(8) Consider a sequence of linear transformations between finite-dimensional vector spaces

$$\{0\} \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n \xrightarrow{T_n} \{0\}$$

Assume that $\mathrm{Im}(T_{i-1}) = \mathrm{Ker}(T_i)$ for all $1 \leq i \leq n$. What is the value of

$$\dim(V_1) - \dim(V_2) + \dim(V_3) - \cdots + (-1)^n \dim(V_n)?$$

(9) Let $A$ be a real $n \times n$ matrix. Prove that the following two statements are equivalent:

(a) $A^2 = A$;

(b) $\mathrm{rank}(A) + \mathrm{rank}(\mathbb{I}_n - A) = n$.

(10) Let $\{\vec{v}_1, \ldots, \vec{v}_k\}$ be an orthonormal set in a finite-dimensional inner product space $V$. Suppose that for any $\vec{v} \in V$ we have

$$||\vec{v}||^2 = \langle \vec{v}_1, \vec{v} \rangle^2 + \cdots + \langle \vec{v}_k, \vec{v} \rangle^2 .$$

Prove that $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis of $V$.

(11) Let $A$ be an $m \times n$ matrix and $B$ be an $n \times m$ matrix. Suppose that $\mathbb{I}_m - AB$ is invertible. Prove that $\mathbb{I}_n - BA$ also is invertible.

(12) Let $W_1$ and $W_2$ be subspaces of a vectors space $V$. Consider the union

$$W_1 \cup W_2 := \{x \in V : x \in W_1 \text{ or } x \in W_2\}.$$

Prove that if $W_1 \cup W_2$ is a subspace of $V$, then we must have $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(13) Let $T \colon V \to V$ be a linear transformation on a (possibly infinite-dimensional) vector space $V$. Suppose that every subspace of $V$ is invariant under $V$, i.e. $T(W) \subseteq W$ for any subspace $W \subseteq V$. Prove that $T$ is a scalar multiple of the identity transformation.

(5) Let $A$ be an $n \times n$ matrix. Consider the linear transformation $T\colon \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ on the $n^2$-dimensional vector space $\mathrm{Mat}_{n\times n}(\mathbb{R})$ defined by $T(B) = AB$. Express $\det(T)$ in terms of $\det(A)$.

$$T\colon M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R})$$
$$B \longmapsto AB$$

$$\boxed{A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}}$$

Choose a basis $\underbrace{\{e_{11}, e_{21}, \dots, e_{n1}, e_{12}, \dots, e_{n2}, \dots, e_{nn}\}}_{=\beta}$

$$\begin{bmatrix} 1 & 0 & \cdots \\ 0 & & \\ 0 & 0 & \\ 0 & & \\ \vdots & & \end{bmatrix} \quad \begin{bmatrix} 0 & & \\ 1 & & \\ 0 & 0 & \\ \vdots & & \\ 0 & & \end{bmatrix}$$

$$T(e_{11}) = A\begin{bmatrix} 1 & \\ 0 & \\ \vdots & 0 \\ 0 & \end{bmatrix} = \begin{bmatrix} | & & & \\ \vec{a}_1 & 0 & \cdots & 0 \\ | & & & \end{bmatrix} = \underline{a_{11}e_{11} + a_{21}e_{21} + \cdots + a_{n1}e_{n1}}$$

$$T(e_{21}) = A\begin{bmatrix} 0 & \\ 1 & \\ 0 & 0 \\ \vdots & \\ 0 & \end{bmatrix} = \begin{bmatrix} \vec{a}_2 & 0 & \cdots & 0 \end{bmatrix} = a_{12}e_{11} + a_{22}e_{21} + \cdots + a_{n2}e_{n1}$$

$$T_\beta = \begin{bmatrix} \boxed{\begin{matrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & A \\ \vdots & \vdots & \\ a_{n1} & a_{n2} & \end{matrix}} & & & \\ \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \vdots \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \boxed{A} & & \\ & & \ddots & \\ & & & \boxed{A} \end{bmatrix}$$

$$\det(T) = \det(A)^n.$$

$$\det \begin{bmatrix} \boxed{A}_{n\times n} & 0 \\ 0 & \boxed{B}_{m\times m} \end{bmatrix} = \det(A)\det(B)$$

$$\sum_{(\sigma_1,\sigma_2)\in S_n\times S_m} (-1)^{sgn(\sigma_1)+sgn(\sigma_2)} a_{1\sigma_1(1)}\cdots a_{n\sigma_1(n)}\, b_{1\sigma_2(1)}\cdots b_{m\sigma_2(m)}$$

$$= \left(\sum_{\sigma_1\in S_n} (-1)^{sgn(\sigma_1)} a_{1\sigma_1(1)}\cdots a_{n\sigma_1(n)}\right)\left(\sum_{\sigma_2\in S_m} (-1)^{sgn(\sigma_2)} b_{1\sigma_2(1)}\cdots b_{m\sigma_2(m)}\right)$$

(6) Let $A$ be a square matrix with columns given by unit vectors. Prove that $|\det(A)| \leq 1$. When does the equality hold?

- $A^TA = \begin{bmatrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{bmatrix} \Rightarrow \operatorname{Tr}(A^TA) = n = \sum \text{eigenvalues of } A^TA$

  $\operatorname{Tr}(A^TA) = \lambda_1 + \cdots + \lambda_n$

- $A^TA$ is symmetric positive semidefinite

  $\vec{x}^T A^T A \vec{x} = \langle A\vec{x}, A\vec{x}\rangle \geq 0 \quad \forall \vec{x}.$

  $\Rightarrow$ eigenvalues of $A^TA$ are non-negative. $\lambda_i \geq 0$

- $\det(A^TA) = \prod \text{eigenvalues of } A^TA = \lambda_1 \cdots \lambda_n.$

  $\det(A^TA) = \det(A)^2$

(AG ineq.): $\dfrac{\lambda_1 + \cdots + \lambda_n}{n} \geq \sqrt[n]{\lambda_1 \cdots \lambda_n}.$

$\dfrac{\lambda_1 + \cdots + \lambda_n}{n} = 1$

equality holds only if $\lambda_1 = \cdots = \lambda_n$.

$\Rightarrow \underbrace{\lambda_1 = \cdots = \lambda_n = 1.}$

$A^TA = PDP^T = I. \Rightarrow A$ is orthogonal

Another proof

$A = QR$, $Q$ orthogonal, $R = \begin{bmatrix} \|\vec{w}_1\| & & * \\ & \ddots & \\ 0 & & \|\vec{w}_n\| \end{bmatrix}$

$\{\vec{v}_1, \ldots, \vec{v}_n\} \xrightarrow{\text{G-S process}} \{\vec{w}_1, \ldots, \vec{w}_n\}$

columns of $A$

unit columns

$$\|\vec{w}_k\| = \|\mathrm{proj}_{\mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}^{\perp}} \vec{v}_k\| \le \|\vec{v}_k\| = 1.$$

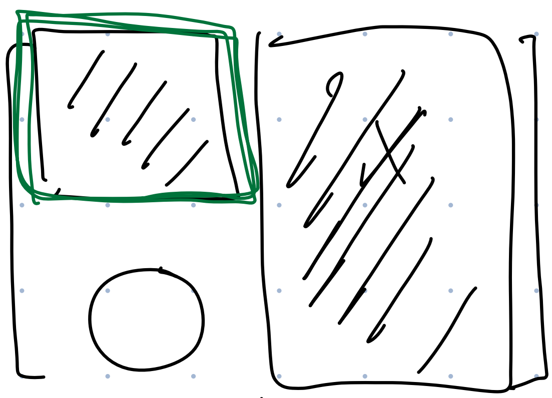
$$\Rightarrow 0 < \det R \le 1.$$

$$|\det Q| = 1. \quad \square$$

(7) Let $V$ be a finite-dimensional vector space, and let $T\colon V \to V$ be a diagonalizable linear transformation. Suppose $W \subseteq V$ is a subspace satisfying $T(W) \subseteq W$. Prove that the restriction $T|_W\colon W \to W$ also is diagonalizable.

$\{v_1, \ldots, v_k\}$ basis of $W$

$\{v_1, \ldots, v_k, \ldots, v_n\}$ basis of $V$



$= PDP^{-1}$

$V = \mathrm{Nul}(T-\lambda_1 I) \oplus \cdots \oplus \mathrm{Nul}(T-\lambda_k I)$ ~~$\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of $T$~~

- $\forall \vec{v} \in V$, $\exists!$ $\vec{v} = \vec{v}_1 + \cdots + \vec{v}_k$, $\vec{v}_i \in \mathrm{Nul}(T-\lambda_i I)$

Claim: $\forall \vec{w} \in W$, $\boxed{\vec{w} = \vec{w}_1 + \cdots + \vec{w}_k}$ $\boxed{\vec{w}_i \in W}$

$(\Rightarrow W = (\mathrm{Nul}(T-\lambda_1 I) \cap W) \oplus \cdots \oplus (\mathrm{Nul}(T-\lambda_k I) \cap W))$

Prove by contradiction.

$\vec{v} \in W$ is a generalized eigenvector of $\lambda$ of $T|_W$ but not an eigenvector of $T|_W$

$(T|_W - \lambda I_W)^{n>1} \vec{v} = \vec{0}$ but $T\vec{v} \ne \lambda \vec{v}$

$\vec{v}$ is also a generalized eigenvector of $T$ but not eigenvector.

pf: $T(\vec{w}) = \lambda_1 \vec{w}_1 + \cdots + \lambda_k \vec{w}_k \in W$

$\Rightarrow (\underbrace{\lambda_2 - \lambda_1}_{\neq 0}) \vec{w}_2 + \cdots + (\underbrace{\lambda_k - \lambda_1}_{\neq 0}) \vec{w}_k \in W$.

Continue this argument inductively $\Rightarrow$ each $\vec{w}_i \in W$.

(11) Let $A$ be an $m \times n$ matrix and $B$ be an $n \times m$ matrix. Suppose that $\mathbb{I}_m - AB$ is invertible. Prove that $\mathbb{I}_n - BA$ also is invertible.

$(\mathbb{I}_n - BA)\vec{x} = \vec{0} \Rightarrow \vec{x} = BA\vec{x}$.

$\Rightarrow A\vec{x} = ABA\vec{x}$.

$\Rightarrow (\underbrace{\mathbb{I} - AB}_{\text{invertible}}) A\vec{x} = \vec{0}$

$\Rightarrow A\vec{x} = \vec{0}$

$\Rightarrow \vec{x} = B\vec{0} = \vec{0}$. $\square$

(13) Let $T: V \to V$ be a linear transformation on a (possibly infinite-dimensional) vector space $V$. Suppose that every subspace of $V$ is invariant under $V$, i.e. $T(W) \subseteq W$ for any subspace $W \subseteq V$. Prove that $T$ is a scalar multiple of the identity transformation.

$\vec{v}_1, \vec{v}_2 \neq \vec{0}$, $T(\vec{v}_1) = c_1 \vec{v}_1$ Want: $c_1 = c_2$

$\vec{v}_1, \vec{v}_2$ l.i. $T(\vec{v}_2) = c_2 \vec{v}_2$

$\underline{\underline{\vec{v}_1 + \vec{v}_2}}$ $\quad T(\vec{v}_1 + \vec{v}_2) = c_3(\vec{v}_1 + \vec{v}_2)$

$$\|$$

$$c_1\vec{v}_1 + c_2\vec{v}_2$$

$$\Rightarrow \underbrace{(c_1 - c_3)}_{= 0}\vec{v}_1 = \underbrace{(c_3 - c_2)}_{= 0}\vec{v}_2 \qquad \square$$

(12) Let $W_1$ and $W_2$ be subspaces of a vectors space $V$. Consider the union

$$W_1 \cup W_2 := \{x \in V : x \in W_1 \text{ or } x \in W_2\}.$$

Prove that if $W_1 \cup W_2$ is a subspace of $V$, then we must have $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

$\vec{x}_1 \in W_1 \setminus W_2, \quad \vec{x}_2 \in W_2 \setminus W_1$

- $\boxed{\vec{x}_1 + \vec{x}_2} \in W_1 \cup W_2$ $\quad$ ($\vec{x}_1, \vec{x}_2 \in W_1 \cup W_2$ and $W_1 \cup W_2$ is a subspace)

- $\vec{x}_1 + \vec{x}_2 \notin W_1 \cup W_2$:

  $\vec{x}_1 + \vec{x}_2 \notin W_1$ since $\vec{x}_1 \in W_1$, $\vec{x}_2 \notin W_1$

  $\vec{x}_1 + \vec{x}_2 \notin W_2$ since $\vec{x}_2 \in W_2$, $\vec{x}_1 \notin W_2$. $\square$

(9) Let $A$ be a real $n \times n$ matrix. Prove that the following two statements are equivalent:

(a) $A^2 = A$;

(b) $\operatorname{rank}(A) + \operatorname{rank}(\mathbb{I}_n - A) = n$.

(a) $\Rightarrow$ (b) in HW

(b) $\Rightarrow$ (a):

$\dim \operatorname{Nul}(A) + \dim \operatorname{Nul}(A - \mathbb{I}_n) = n$

$\Rightarrow$ A is diagonalizable, and

$\underline{0, 1}$ are the only possible eigenvalues

$\Rightarrow A^2 = A.$

$A = PDP^{-1}$

$A^2 = PD^2P^{-1}$

$D = D^2$ since the diagonal entries are 0 or 1.