Today:
- Fundamental thm of calculus.
- (un)countable sets.

Thm Any monotone bounded fn. $f: [a,b] \to \mathbb{R}$ is integrable.

pf: Say $f: [a,b] \to \mathbb{R}$ bdd increasing.

Want: $\forall \varepsilon > 0$, $\exists$ $P = \{t_0 < \cdots < t_n\}$ partition of $[a,b]$

s.t. $U(f,P) - L(f,P) < \varepsilon$.

$$\underbrace{U(f,P) - L(f,P)}_{} = \sum_{i=1}^{n} (t_i - t_{i-1}) \cdot \Big(\underbrace{\boxed{\sup_{x \in [t_{i-1}, t_i]} f(x)}}_{= f(t_i)} - \underbrace{\boxed{\inf_{x \in [t_{i-1}, t_i]} f(x)}}_{= f(t_{i-1})} \Big)$$

$$< \tilde{\varepsilon} \underbrace{\sum_{i=1}^{n} \big(f(t_i) - f(t_{i-1}) \big)}_{} = \tilde{\varepsilon} \cdot (f(b) - f(a)) = \varepsilon$$

$$\sum_{i=1}^{n} \big(f(t_i) - f(t_{i-1}) \big) = \cancel{f(t_1)} - f(t_0) + \cancel{f(t_2)} - \cancel{f(t_1)} + \cancel{f(t_3)} - \cancel{f(t_2)} + \cdots + f(t_n) - \cancel{f(t_{n-1})}$$

with $t_0 = a$, $t_n = b$.

Suppose we choose P s.t. $0 < t_i - t_{i-1} < \tilde{\varepsilon}$ $\forall i$, where $\tilde{\varepsilon} = \dfrac{\varepsilon}{f(b) - f(a)}$.

$\square$

Rmk:
- Any monotone bdd fcn has at most countably many disconti. pts.
- countable sets are of measure zero

FTC (I):
- f: conti. on $[a,b]$, differentiable on (a,b).
- f': bdd. integrable.

$$\Longrightarrow \int_a^b f'(x)\,dx = f(b) - f(a)$$

pf: P partition of $[a,b]$, given by $a = t_0 < t_1 < \cdots < t_n = b$.

$$f(b) - f(a) = \big(f(t_n) - f(t_{n-1})\big) + \big(f(t_{n-1}) - f(t_{n-2})\big) + \cdots + \big(f(t_1) - f(t_0)\big)$$

$\exists\, s_n \in (t_{n-1}, t_n)$ $\quad \|$ MVT $\qquad \exists\, s_{n-1} \in (t_{n-2}, t_{n-1})$

$$f'(s_n)\cdot(t_n - t_{n-1}) \qquad f'(s_{n-1})(t_{n-1} - t_{n-2}) \qquad \cdots$$

$$\Big(\inf_{x \in [t_{n-1}, t_n]} f'(x)\Big)\cdot(t_n - t_{n-1}) \le f'(s_n)\cdot(t_n - t_{n-1}) \le \Big(\sup_{x \in [t_{n-1}, t_n]} f'(x)\Big)\cdot(t_n - t_{n-1})$$

$$\Longrightarrow \quad L(f', P) \le f(b) - f(a) \le U(f', P). \quad \forall P$$

$$\Longrightarrow \quad L(f') \le f(b) - f(a) \le U(f').$$

Since f' is assumed to be integrable, so $L(f') = U(f')$

$$\Longrightarrow \quad f(b) - f(a) = L(f') = U(f') = \int_a^b f'(x)\,dx. \quad \square$$

FTC (II):

1) f: (bdd) integrable on $[a,b]$.

$$F(x) := \int_a^x f(t)\,dt \qquad \forall x \in [a,b]$$

Then F is conti. on $[a,b]$.

2) Moreover, if f conti. at $x_0 \in [a,b]$,

then F is differentiable at x_0,

and $F'(x_0) = f(x_0)$

Say $|f(x)| < M \quad \forall x \in [a,b]$

pf 1): $x_1 < x_2$ in $[a,b]$

$$|F(x_1) - F(x_2)| = \left| \int_a^{x_1} f(t)\,dt - \int_a^{x_2} f(t)\,dt \right|$$

$$= \left| \int_{x_1}^{x_2} f(t)\,dt \right| \underset{(HW)}{\leq} \int_{x_1}^{x_2} |f(t)|\,dt$$

$$< M(x_2 - x_1)$$

$\Rightarrow$ F is Lipschitz conti. on $[a,b]$

($\Rightarrow$ unif. conti $\Rightarrow$ conti.).

---

pf 2) f conti. at x_0

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right|$$

$$= \left| \frac{\int_a^{x} f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} - f(x_0) \right|$$

$$= \left| \frac{\int_{x_0}^{x} f(t)\,dt}{x - x_0} - f(x_0) \right|$$

$$= \left| \frac{\int_{x_0}^{x} f(t)\,dt}{x-x_0} - \frac{\int_{x_0}^{x} f(x_0)\,dt}{x-x_0} \right|$$

$$= \frac{1}{|x-x_0|} \cdot \left| \int_{x_0}^{x} \big(f(t)-f(x_0)\big)\,dt \right|$$

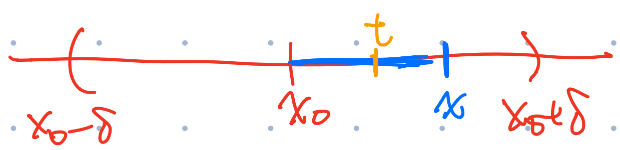
(Since f is conti. at x_0, $\forall \varepsilon>0,\ \exists \delta>0$ s.t. $|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$)

If $0<|x-x_0|<\delta$, then

$$\left| \frac{F(x)-F(x_0)}{x-x_0} - f(x_0) \right| = \frac{1}{|x-x_0|} \cdot \left| \int_{x_0}^{x} \big(f(t)-f(x_0)\big)\,dt \right|$$

$$\leq \frac{1}{|x-x_0|} \left| \int_{x_0}^{x} \overbrace{|f(t)-f(x_0)|}^{<\varepsilon}\,dt \right|$$

(Since x is in the δ-nbd of x_0,



)

$$< \frac{1}{|x-x_0|} \cdot \varepsilon \cdot |x-x_0| = \varepsilon.$$

$\Longrightarrow \lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0}$ exists and equals to $f(x_0)$. $\square$

Rmk: Some easy corollaries of FTC (I)(II):

- integration by parts (34.2)
- change of variables (34.4)

Def: Say two sets A, B <u>have the same cardinality</u> (denoted this by $|A| = |B|$) if there exists a function $f: A \to B$ that is bijective.

bijective = injective & surjective

injective: $\forall x_1 \neq x_2$ in A, we have $f(x_1) \neq f(x_2)$ in B

surjective: $\forall y \in B$, $\exists x \in A$ s.t. $f(x) = y$

Def: Write "$|A| \leq |B|$" if $\exists\, g: A \to B$ injective.

Def: Write "$|A| < |B|$" if $\exists\, i: A \to B$ injective but $\nexists\, j: B \to A$ injective.

Rmk:

- If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$ (very non-trivial!! Schröder-Bernstein thm.; wiki)
- <u>axiom of choice</u> $\Longleftrightarrow$ "$\forall A, B$ sets, either $|A| \leq |B|$ or $|B| \leq |A|$".

Def:
- An infinite set A is called <u>countable</u> if $|A| = |\mathbb{N}| = |\{1,2,3,4,\dots\}|$
- An infinite set A is <u>uncountable</u> if $|A| > |\mathbb{N}|$

---

e.g. $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ countable.

$$\mathbb{Z} \longrightarrow \mathbb{N} \quad \text{bijection.}$$

$$\begin{aligned}
0 &\longmapsto 1 \\
-1 &\longmapsto 2 \\
1 &\longmapsto 3 \\
-2 &\longmapsto 4 \\
2 &\longmapsto 5 \\
&\ \vdots \\
-n &\longmapsto 2n \\
n &\longmapsto 2n+1
\end{aligned}$$

---

e.g. $\mathbb{Q}$ countable

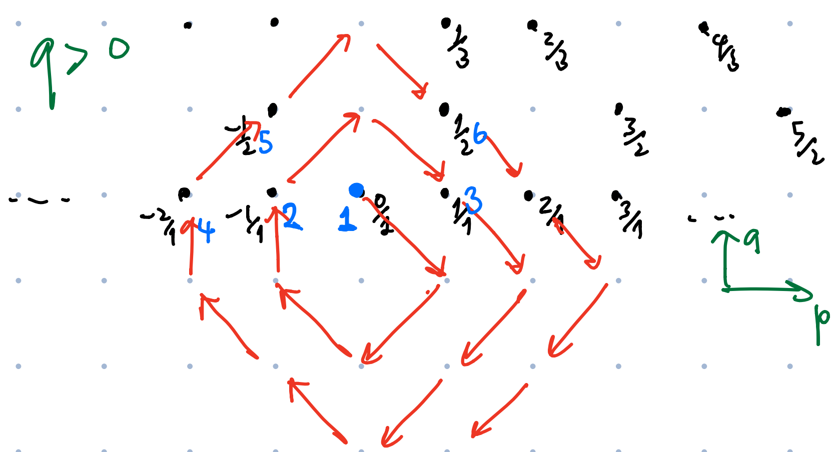
$\mathbb{Q}$

$\downarrow$

$\frac{p}{q}$ $\quad \gcd(p,q)=1, \quad q>0$

$\downarrow$

$\mathbb{N} = \{1, 2, 3, \dots\}$



Rmk: $\exists$ infinite sets $S_1, S_2, S_3, \ldots$

s.t. $|S_1| < |S_2| < |S_3| < |S_4| < \cdots$

Def: For any set A, define the power set $P(A)$ to be the set of all subsets of A. (including ϕ and A)

e.g. $A = \{1,2,3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Cantor: $|A| < |P(A)|$

pf: • $|A| \leq |P(A)|$, i.e. $\exists$ injection $f: A \to P(A)$

$a \mapsto \{a\}$.

• Claim: Any map $\boxed{f: A \to P(A)}$ is not surjective

Want: Find $B \in P(A)$ s.t. $B \neq f(a)$ $\forall a \in A$.

Define $B := \{a \in A \mid a \notin f(a)\} \subseteq A$

$f(a)$ is an elt in $P(A)$, i.e. is a subset of A.

We'll prove that $B \neq f(a)$ $\forall a \in A$.

① $a \in B$

$\Rightarrow a \notin f(a) \Rightarrow B \neq f(a)$

② $a \notin B$

$\Rightarrow a \in f(a) \Rightarrow B \neq f(a)$. $\square$

E.g. $|P(\mathbb{N})| > |\mathbb{N}| \Rightarrow P(\mathbb{N})$ is uncountable.