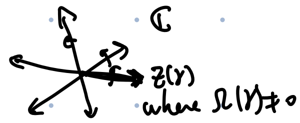
Today: – Riemann-Hilbert problem from BPS data

– Introduction to mirror symmetry.

Recap:



$\begin{pmatrix}\text{CY3 cat.}\\ \text{with a}\\ \text{stability}\\ \text{condition}\end{pmatrix}$ $\rightsquigarrow$ BPS data $\rightsquigarrow$ wall-crossing automorphism $S(\ell)$ on twisted torus T for each active ray ℓ. $\rightsquigarrow$ RH problem.

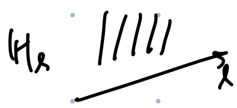
$\mathcal{D}$, $\sigma=(Z,P)$
$Z:\Gamma\to\mathbb{C}$

$(\Gamma,\langle -,-\rangle)\rightsquigarrow T$
Z
$\Omega(\gamma)$: count σ-stable objects of class γ

$\begin{pmatrix}M=\text{Stab},\\ M_{cpx},\ M_{Kah}\text{ of}\\ \text{a CY3-fold}\end{pmatrix}$ $\rightsquigarrow$ family of BPS data over M, satisfies wall-crossing formula. $\rightsquigarrow$ family of solutions of the RH problems. $\rightsquigarrow$ "τ-function" on M.

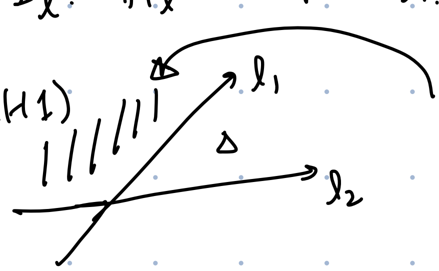
Bridgeland's observation: For certain family of BPS data, the asymptotic behavior of $\log\tau$ reproduces (part of) the $g=0$ Gopakumar-Vafa invariants on a CY3-fold X.

RH problem: Fix $\xi\in T$.



For each non-active ray ℓ, we want to find a holomorphic function $\Phi_\ell:\mathbb{H}_\ell\to T$ s.t.

(RH1)



$$\Phi_{\ell_1}=S(\Delta)\circ\Phi_{\ell_2}$$

$S(\Delta)=\overrightarrow{\prod_{\ell\in\Delta}}S(\ell)$

(RH2) $\exp\left(2\pi i Z(\gamma)/t\right)\cdot x_\gamma\left(\Phi_\ell(t)\right)\longrightarrow\xi(\gamma)$ as $t\to 0$ in $\mathbb{H}_\ell$ $\quad\forall\gamma\in\Gamma$.

(RH3) $|t|^{-k}<\left|x_\gamma\left(\Phi_\ell(t)\right)\right|<|t|^{k}$ as $t\to\infty$ in $\mathbb{H}_\ell$.

Notation: $\Phi_{\ell,\gamma}(t) := x_\gamma(\Phi_\ell(t)) \quad : \mathbb{H}_\ell \to \mathbb{C}$

$$\Psi_{\ell,\gamma}(t) := \exp(2\pi i Z(\gamma)/t)\, x_\gamma(\Phi_\ell(t))\, \xi(\gamma)^{-1}.$$

Rmk: • To solve $\{\Phi_\ell\}_\ell \Leftrightarrow$ To solve $\{\Phi_{\ell,\gamma}\}_{\ell,\gamma}$ or $\{\Psi_{\ell,\gamma}\}_{\ell,\gamma}$.

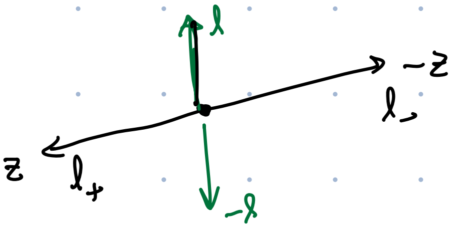
• β is a null class ($\langle\alpha,\beta\rangle = 0\ \forall \alpha$ active), then $\Psi_{\ell,\beta} \equiv 1$ for any solution of RH problem.

---

e.g. (doubled A_1-quiver) Fix $z \in \mathbb{C}^*$.

- $\Gamma = \mathbb{Z}\gamma \oplus \mathbb{Z}\gamma^\vee$, $\quad \langle \gamma^\vee, \gamma \rangle = 1$.
- $Z(a\gamma, b\gamma^\vee) := az$
- $\Omega(a\gamma, b\gamma^\vee) := \begin{cases} 1 & \text{if } (a,b) = (\pm 1, 0) \\ 0 & \text{otherwise.} \end{cases}$

$\leadsto$
- active class: $(\pm 1, 0) = \pm\gamma$.
- active rays:



- γ is a null class, $\gamma^\vee$ is not a null class.

We want to solve for $\Phi_{\ell,\gamma^\vee}$ for $\ell \neq \ell_\pm$

(RH1) $\Rightarrow$ $\Psi_{\ell,\gamma^\vee}$ has analytic continuation to $\mathbb{C}\setminus \ell_-$

$\Phi_{-\ell,\gamma^\vee}$ has analytic continuation to $\mathbb{C}\setminus \ell_+$.

Wall-crossing condition: $\Phi_{-\ell} = S(\ell_+)\circ \Phi_{\ell}.$

$$x_{\gamma^\vee}\circ \Phi_{-\ell} = x_{\gamma^\vee}\circ S(\ell_+)\circ \Phi_{\ell}.$$

$$x_{\gamma^\vee}\circ S(\ell_+) = x_{\gamma^\vee}(1-x_\gamma)^{\langle\gamma^\vee,\gamma\rangle} = x_{\gamma^\vee}(1-x_\gamma)$$

$$x_{\gamma^\vee}(\Phi_{-\ell}(t)) = x_{\gamma^\vee}(\Phi_{\ell}(t)) - x_{\gamma^\vee}(\Phi_{\ell}(t))\, x_\gamma(\Phi_\ell(t))$$

$$\Phi_{-\ell,\gamma^\vee}(t) = \Phi_{\ell,\gamma^\vee}(t) - \Phi_{\ell,\gamma^\vee}(t)\,\Phi_{\ell,\gamma}(t)$$

$$= \Phi_{\ell,\gamma^\vee}(t)\left(1-\underline{\Phi_{\ell,\gamma}(t)}\right)$$

Since γ is null, $\Psi_{\ell,\gamma}(t)\equiv 1.$

$$\Rightarrow \Phi_{\ell,\gamma}(t) = \exp\left(-2\pi i\, \underbrace{Z(\gamma)}_{= z\in\mathbb{C}^*}/t\right)\underbrace{\xi(\gamma)}_{=\xi\in\mathbb{C}^*}$$

$\boxed{\Psi_{\ell_\pm,\gamma^\vee}}$

RH problem: Find $x_\pm(t): \mathbb{C}^*\setminus \ell_\mp \to \mathbb{C}^*$ holomorphic functions s.t.

1) $$x_-(t) = \begin{cases} x_+(t)\cdot\left(1-\xi\, e^{-2\pi i z/t}\right) & \text{for } t\in \mathbb{H}_{\ell_+} \\ x_+(t)\left(1-\xi^{-1} e^{2\pi i z/t}\right) & \text{for } t\in \mathbb{H}_{\ell_-}\end{cases}$$

2) $x_\pm(t) \longrightarrow 1$ as $t\to 0$ in $\mathbb{C}^*\setminus \ell_\mp$

3) $|t|^{-k} < |x_\pm(t)| < |t|^{k}$ as $t\to\infty$ in $\mathbb{C}^*\setminus \ell_\mp$.

Rmk: Solution is unique (if exists):

Thm: When $\xi = 1$. $\exists!$ solution $X_+(t) = \Lambda(\frac{Z}{t})$, $X_-(t) = \Lambda(\frac{-Z}{t})^{-1}$,

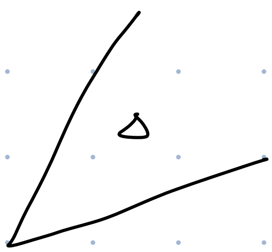
where $\Lambda(w) = \dfrac{e^w \, \Gamma(w)}{\sqrt{2\pi} \; w^{w-\frac{1}{2}}}$ holomorphic on $\mathbb{C}^* \setminus \mathbb{R}_{<0}$.

Thm $\xi = 1$, (Γ, Z, Ω): finite (finitely many active classes), uncoupled ($\langle \gamma_1, \gamma_2 \rangle = 0$ $\forall \gamma_1, \gamma_2$ active), integral ($\Omega(\gamma) \in \mathbb{Z}$), then

$$\Psi_{\ell,\beta}(t) = \prod_{\substack{Z(\gamma) \in \mathbb{H}_\ell \\ \gamma \text{ active}}} \Lambda\left(\frac{Z(\gamma)}{t}\right)^{\Omega(\gamma)\langle \beta, \gamma \rangle}$$

Def A <u>variation of BPS data</u> over a complex manifold M if

- Γ fixed.
- $Z_p(\gamma) \in \mathbb{C}$ varies holomorphically in $p \in M$ $\forall \gamma \in \Gamma$.
- satisfies wall-crossing formula:



$S_p(\Delta) = \overrightarrow{\prod_{\ell \subseteq \Delta}} S_p(\ell)$ is constant in $p \in M$, as long as $\partial \Delta$ is not active.

- the map $\begin{array}{ccc} M & \longrightarrow & \mathrm{Hom}(\Gamma, \mathbb{C}) \\ p & \longmapsto & Z_p \end{array}$ is a local isomorphism.

<u>Z-function (on M)</u>: Choose a basis $\Gamma = \langle \gamma_1, \cdots, \gamma_n \rangle$, then this gives a local coordinate $\{ z_i = Z(\gamma_i) \}$ of M.

For a family of solutions of RH problems $\Psi_\ell(p,t)$: $\overset{p}{\underset{}{M}} \times \overset{t}{\mathbb{H}_\ell} \to T$.

say $\tau_\ell : M \times \mathbb{H}_\ell \longrightarrow \mathbb{C}^*$ is a τ-function if:

$$\frac{\partial \log \Psi_{\ell,\gamma_i}}{\partial t} = \sum_j \langle \gamma_i, \gamma_j \rangle \frac{\partial \log \tau_\ell}{\partial z_j}.$$

Thm (Bridgeland) $(\Gamma, Z_p, \Omega_p)_{p\in M}$ variation of finite, uncoupled, integral BPS data. $\rightsquigarrow$ unique solution $\forall p \in M$ of RH problem. $\rightsquigarrow \tau_\ell$

$$\tau_\ell(p,t) = \prod_{Z_p(\gamma)\in\mathbb{H}_\ell} \Upsilon\left(\frac{Z(\gamma)}{t}\right)^{\Omega(\gamma)},$$

where $\Upsilon(w) := \dfrac{e^{-\zeta'(-1)}\, e^{\frac{3}{4}w^2}\, G(w+1)}{(2\pi)^{w/2}\, w^{w^2/2}}$ holo. on $\mathbb{C}\setminus\mathbb{R}_{\le 0}$

(main property: $\frac{d}{dw}\log \Upsilon(w) = w\frac{d}{dw}\log\Lambda(w)$)

(G - Barnes G-function, double Gamma function
$G(w+1) = \Gamma(w)G(w)$, $G(1)=1$, $G(n) = (n-2)!\,(n-3)!\cdots 1!$)

Rmk: $$\log \tau_\ell(p,t) \xrightarrow[\substack{t\to 0\\ \text{in } \mathbb{H}_\ell}]{} \frac{1}{24}\sum_{Z_p(\gamma)\in\mathbb{H}_\ell} \Omega_p(\gamma)\log\left(\frac{t}{Z_p(\gamma)}\right)$$
$$+ \boxed{\sum_{g\ge 2}\sum_{Z_p(\gamma)\in\mathbb{H}_\ell} \frac{\Omega_p(\gamma) B_{2g}}{4g(2g-2)}\left(\frac{t}{Z_p(\gamma)}\right)^{2g-2}}.$$

$g=0$ GV invariants of CY3.

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}$$

$$G(z) = e^{\frac{1}{12} - \zeta'(-1,z)} \cdot \Gamma(z)^{z-1} \cdot C.$$

GW potential function of CY 3-fold X:

$$F(x,\lambda) = \sum_{g\geq 0}\sum_{\beta\in H_2(X,\mathbb{Z})} GW(g,\beta)\, x^{\beta}\, \lambda^{2g-2} = \underset{\substack{\uparrow \\ \beta=0}}{F_0(x,\lambda)} + \underset{\substack{\uparrow \\ \beta\neq 0}}{\tilde{F}(x,\lambda)}$$

$$F_0(x,\lambda) = a_0(x)\lambda^{-2} + a_1(x) + \sum_{g\geq 2} \boxed{\chi(X)\underbrace{\frac{(-1)^{g-1} B_{2g} B_{2g-2}}{4g(2g-2)(2g-2)!}}_{} \lambda^{2g-2}}$$

Hodge integral on $\overline{M}_g$

(Faber-Pandharipande)

$$\tilde{F}(x,\lambda) = \sum_{g\geq 0}\sum_{\beta\neq 0} GV(g,\beta) \sum_{k\geq 1}\frac{1}{k}\left(2\sin\left(\frac{k\lambda}{2}\right)\right)^{2g-2} x^{k\beta}.$$

consider only $g=0$ GV invar. $\longrightarrow$ $\displaystyle\sum_{\beta\neq 0} GV(0,\beta)\sum_{k\geq 1}\frac{1}{k}\underbrace{\left(2\sin\left(\frac{k\lambda}{2}\right)\right)^{-2}}_{} x^{k\beta}$

$$\downarrow$$

$$\frac{1}{x^2} - \frac{1}{12} + \sum_{g\geq 2}\frac{(-1)^{g-1}B_{2g}}{2g(2g-2)!}x^{2g-2}$$

$$= b_0(x)\lambda^{-2} + b_1(x) + \sum_{g\geq 2}\boxed{\left(\sum_{\beta\neq 0} GV(0,\beta)\frac{(-1)^{g-1}B_{2g}}{2g(2g-2)!} Li_{3-2g}(x^{\beta})\right)\lambda^{2g-2}}$$

$$\left(Li_k(x) = \sum_{n\geq 1}\frac{x^n}{n^k}\right)$$

$\mathcal{A} = Coh_{\leq 1}^{CY3}(X)$

Consider the following family of BPS data:

base of family of BPS data $\to$ $M = M^{\mathbb{C}}_{Kah}(X) \ni \omega_{\mathbb{C}} := B + i\omega$, $B,\omega\in H^{1,1}(X,\mathbb{R})$, ω-ample

$\to \Gamma = H_2(X,\mathbb{Z}) \oplus \underset{\substack{\| \\ H_0(X,\mathbb{Z})}}{\mathbb{Z}}. \ni (\beta, n)$, $\langle -,- \rangle \equiv 0.$

- $Z_{\omega_{\mathbb{C}}}(\beta, n) := \beta \cdot \omega_{\mathbb{C}} - n$.
- For $\gamma \in \Gamma$, $\Omega(\gamma) \in \mathbb{Q}$ defined by Joyce–Song using moduli stack of semistable objects in $\mathcal{A} = \mathrm{Coh}_{\leq 1}(X)$.
  - These BPS numbers are independent of $\omega_{\mathbb{C}}$.

Fact: $S(\Delta)$ determines the BPS numbers $\Omega(\gamma)$ for γ s.t. $Z(\gamma) \in \Delta$

In our case, $\langle -, - \rangle \equiv 0$. $\Rightarrow$ wall-crossing automorphism are all trivial

$\Rightarrow$ $\Omega(\gamma)$ independent of $\omega_{\mathbb{C}}$

$$\Omega(0, n) = -\chi(X) \quad \forall n \in \mathbb{Z} \setminus \{0\}$$

(Conj): $\Omega(\beta, n) = GV(0, \beta) \quad \beta \neq 0.$

Conclusion: If we plug these (Z, Ω) into the asymptotic expansion of $\log \tau$, under the change of variables:

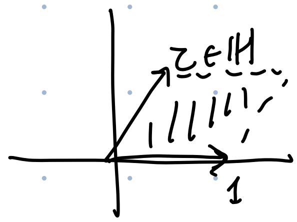
$$\lambda = 2\pi t, \quad x^{\beta} = \exp(2\pi i \beta \cdot \omega_{\mathbb{C}}),$$

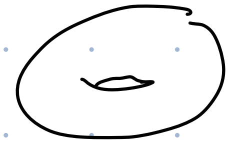
it reproduces the $g=0$ contribution to the GV generating fn.

Mirror symmetry: (CY $\leftrightarrow$ CY, Fano $\leftrightarrow$ LG)

upshot: duality between complex structures & symplectic structures.

e.g. elliptic curve $E = \mathbb{C}/\Lambda$



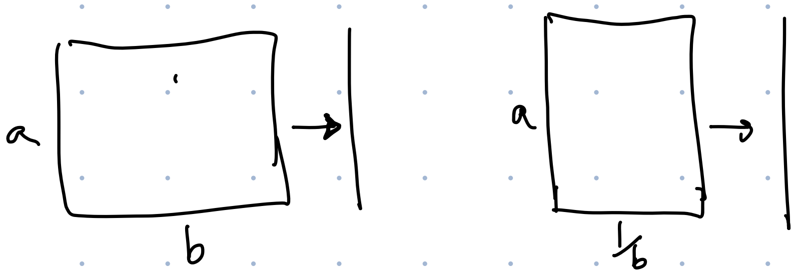


Mirror symmetry: $\exists\, E^\vee$ s.t. complex (E) "$=$" symplectic $(E^\vee)$
symplectic (E) "$=$" complex $(E^\vee)$

- Complex geometry of E is governed by $\tau \in \mathbb{H}$
- symplectic manifold (X^{2n}, ω)
  $\uparrow$ non-degenerate closed 2-form.

symplectic geometry of E $\rightsquigarrow$ volume form on E.

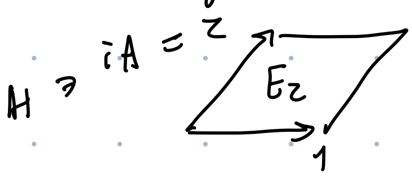
e.g.

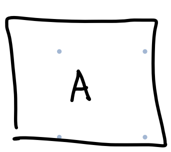


Complex:	a/b	ab
Symplectic:	ab	a/b

Strominger-Yau-Zaslow conjecture: "Mirror symmetry is T-duality"
(Later)

Rmk: For general $\tau \in \mathbb{H}$, one needs to consider "complexified volume form".





Area $= A \in \mathbb{R}_{>0} + i\mathbb{R}$
$\omega = A\, dx \wedge dy$

Kontsevich's Homological mirror symmetry conjecture: $D^b Coh(E_\tau) \cong D^\pi Fuk(E, \omega)$

We'll examine the morphisms of certain objects, and recover basic identity of theta functions.

$D^b Coh(E_\tau)$: We'll choose 3 objects: $\underset{\mathcal{O}}{\overset{\|}{\mathcal{L}_0}}, \mathcal{L}_1, \mathcal{L}_2$

- consider $E_\tau = \mathbb{C}/\langle 1, \tau\rangle \xrightarrow[\simeq]{\exp(2\pi i\,\cdot)} \mathbb{C}^*/u \sim qu =: E_q$, $q = e^{2\pi i \tau}$.

$$z \longmapsto \exp(2\pi i z)$$

- for any holomorphic function $\varphi: \mathbb{C}^* \to \mathbb{C}^*$, we define a line bundle $\mathcal{L}_q(\varphi) \to E_q$:

$$\mathcal{L}_q(\varphi) = \mathbb{C}^* \times \mathbb{C} / (u, v) \sim (qu, \varphi(u) v).$$

- Choose a particular line bundle on E_q of degree 1 ($\exists$ global section that vanishes at exactly 1 point).

$$\varphi_0(u) := q^{-\frac{1}{2}} u^{-1}. \quad \rightsquigarrow \quad \mathcal{L}_q(\varphi_0) \to E_q.$$

Then $\theta(\tau, z) := \sum_{n\in\mathbb{Z}} e^{\pi i (n^2\tau + 2nz)} = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} u^n$

is a section of $\mathcal{L}_q(\varphi_0)$: (θ vanishes at $\frac{1+\tau}{2}$ in the fundamental domain of $\langle 1, \tau\rangle$)

$$\sum_n q^{\frac{1}{2}n^2 + n} u^n = \sum_n q^{\frac{1}{2}(n+1)^2 - \frac{1}{2}} u^n = \sum_n q^{\frac{1}{2}n^2 - \frac{1}{2}} u^{n-1}$$

$$\left(u, \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} u^n\right) \sim \left(qu, \varphi_0(u) \sum_n q^{\frac{1}{2}n^2} u^n\right)$$

$$= \left(qu, \underbrace{q^{-\frac{1}{2}} u^{-1} \sum_n q^{\frac{1}{2}n^2} u^n}\right)$$

$$\mathcal{L}_0 = \mathcal{O}, \quad \mathcal{L}_1 = \mathcal{L}_q(\varphi_0), \quad \mathcal{L}_2 = \mathcal{L}_q(\varphi_0)^{\otimes 2}.$$

Theta functions:

$$\theta[a, z_0](\tau, z) := \sum_{n\in\mathbb{Z}} \exp\left\{ \pi i \left[(n+a)^2 \tau + 2(n+a)(z+z_0) \right] \right\}.$$

Fact: $\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) = \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) \cong H^0(E_\tau, \mathcal{L}_1) = \langle \theta[0,0](\tau, z) \rangle_{\mathbb{C}}$.

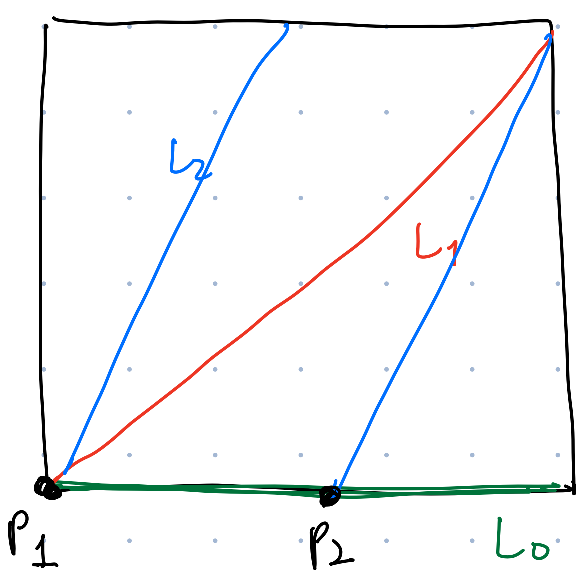
$\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_2) = H^0(E_\tau, \mathcal{L}_2) = \langle \theta[0,0](2\tau, 2z), \theta[\tfrac{1}{2}, 0](2\tau, 2z) \rangle_{\mathbb{C}}$

$D^\pi \mathrm{Fuk}(E, \omega)$: $\qquad (X^{2n}, \omega)$

Objects: Roughly, Lagrangian submanifolds $L^n \subseteq (X^{2n}, \omega)$ with $\omega|_L \equiv 0$.

(+ extra data).

In our case, any real curve in E.



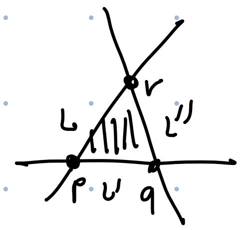
Morphism $\mathrm{Hom}(L, L') = \bigoplus_{p \in L \cap L'} \mathbb{C}\{p\}$

$\mathrm{Hom}(L_0, L_1) = \mathrm{Hom}(L_1, L_2) = \mathbb{C}\cdot p_1$

$\mathrm{Hom}(L_0, L_2) = \mathbb{C}p_1 \oplus \mathbb{C}p_2.$

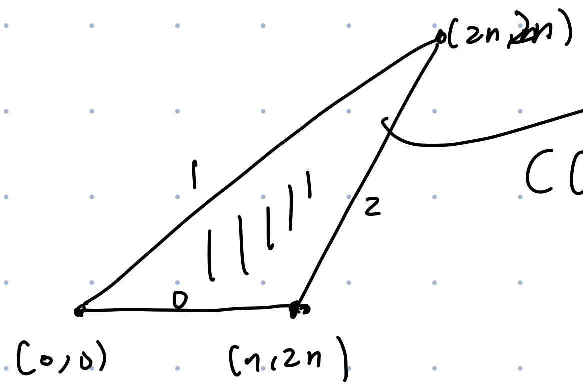
Composition of morphisms:

$$m_2: \mathrm{Hom}(L, L') \otimes \mathrm{Hom}(L', L'') \longrightarrow \mathrm{Hom}(L, L'')$$
$$p \otimes q \longmapsto \sum_{r \in L \cap L''} C(p,q,r)\, r$$

where $C(p,q,r) := \sum \exp\left(2\pi i \,\mathrm{Area}_\omega\left(\right.\right.$  $\left.\left.\right)\right)$

e.g. $\mathrm{Hom}(L_0, L_1) \otimes \mathrm{Hom}(L_1, L_2) \longrightarrow \mathrm{Hom}(L_0, L_2)$
$$P_1 \otimes P_1 \longmapsto \underline{C(P_1,P_1,P_1)}\, P_1 + \underline{C(P_1,P_1,P_2)}\, P_2$$

$C(P_1,P_1,P_1)$ ← all Δ with vertices are integer points, and edges have slopes 0, 1, 2.

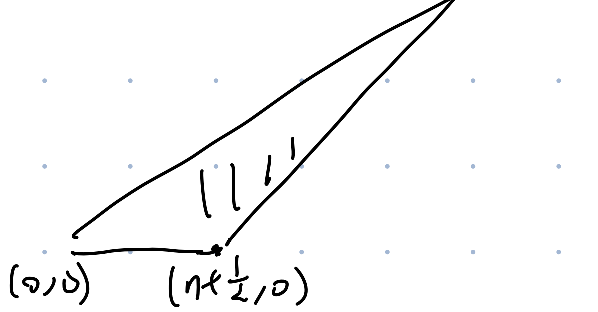


$$C(P_1,P_1,P_1) = \sum_{n \in \mathbb{Z}} \exp\left(2\pi i \,\mathrm{Area}_\omega\left(\right.\right.$$  $$\left.\left.\right)\right)$$

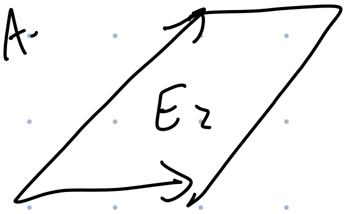
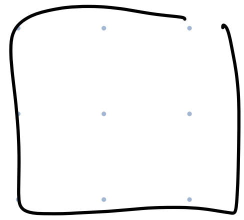
$$= \sum_{n \in \mathbb{Z}} \exp(2\pi i A n^2)$$

$$= \sum_{n \in \mathbb{Z}} \exp(2\pi Z n^2)$$

$$= \theta[0,0](2Z, 0).$$



$$C(P_1, P_1, P_2) = \theta\left[\tfrac{1}{2}, 0\right](2Z, 0).$$

Complex	Symplectic
$\tau = iA$ 	 $\omega = A\, dx \wedge dy$
$\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$	L_0, L_1, L_2.

$$\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) \cong \langle \theta[0,0](\tau, z) \rangle_{\mathbb{C}} \cong \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2)$$

$$\mathrm{Hom}(L_0, L_1) \cong \mathbb{C} \cdot p_1 \cong \mathrm{Hom}(L_1, L_2)$$

$$\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_2) \cong \langle \theta[0,0](2\tau, 2z), \theta[\tfrac{1}{2},0](2\tau, 2z) \rangle_{\mathbb{C}}$$

$$\mathrm{Hom}(L_0, L_2) = \mathbb{C} p_1 \oplus \mathbb{C} p_2$$

$$m_2(p_1 \otimes p_1) = \theta[0,0](2\tau, 0)\, p_1 + \theta[\tfrac{1}{2}, 0](2\tau, 0)\, p_2$$

$$\xRightarrow{\text{HMS}} \theta[0,0](\tau, z)^2 = \theta[0,0](2\tau, 0)\, \theta[0,0](2\tau, 2z) + \theta[\tfrac{1}{2},0](2\tau, 0)\, \theta[\tfrac{1}{2},0](2\tau, 2z)$$

(addition formula of theta functions)

Rmk. Theta functions form a particular basis of $H^0(\mathcal{L})$, which in this case are mirror to the intersections of the Lagrangians.

Gross, Hacking, Keel, Kontsevich, ...'s idea of canonical basis

X = CY, $\mathcal{L}$ - ample l.b. $\check{X}$: mirror of X.

$$\mathrm{Hom}(\mathcal{O}_X, \mathcal{L}) \overset{\text{Mirror}}{\longleftrightarrow} \mathrm{Hom}(L_{\mathcal{O}_X}, L_{\mathcal{L}}) = \bigoplus_{p \in L_{\mathcal{O}_X} \cap L_{\mathcal{L}}} \mathbb{C} \cdot p$$

$\mathrm{Hom}(\mathcal{O}_X, \mathcal{L}) = H^0(\mathcal{L})$

$\mathrm{Hom}(L_{\mathcal{O}_X}, L_{\mathcal{L}})$ has a canonical basis given by the intersection points

There should be a canonical basis of $H^0(\mathcal{L})$.