(1) (a) (10 points) Prove that the series

$$\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^3$$

is convergent for any $x \in \mathbb{R}$.

(b) (20 points) Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^3$$

is a continuous function on $\mathbb{R}$.

1(a):

- If $x=0$, then the series clearly converges to 0.
- If $x \neq 0$, $$\lim_{n\to\infty}\left|\frac{\left(\frac{x^{n+1}}{(n+1)!}\right)^3}{\left(\frac{x^n}{n!}\right)^3}\right| = \lim_{n\to\infty}\frac{|x|^3}{(n+1)^3} = 0,$$

hence the series conv. by ratio test.

1(b):

Claim: The series $\sum\left(\frac{x^n}{n!}\right)^3$ conv. uniformly on $[-R,R]$. $\forall R>0$.

pf:
- For $x\in[-R,R]$, $\left|\left(\frac{x^n}{n!}\right)^3\right| \leq \left(\frac{R^n}{n!}\right)^3$.
- $\sum\left(\frac{R^n}{n!}\right)^3$ converges by the ratio test as in part (a).
- By Weierstrass M-test, the claim follows. □

- Since $\sum_{n=0}^{N}\left(\frac{x^n}{n!}\right)^3$ is continuous for any $N>0$, by the claim, we have $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^3$ is continuous on $[-R,R]$ $\forall R>0$. [unif. limit of conti. fcn. is conti.]

- $\forall x_0\in\mathbb{R}$, $\exists R>0$ s.t. $|x_0|<R$. Hence $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^3$ is continuous at x_0. for any $x_0\in\mathbb{R}$. Therefore conti. on $\mathbb{R}$. □

Note: The series $\sum \left(\frac{x^n}{n!}\right)^3$ does not converge uniformly on $\mathbb{R}$.

since
$$\sup\left\{\left|\frac{x^n}{n!}\right|^3 : x \in \mathbb{R}\right\} = +\infty \quad \text{for any } n.$$

(Recall that if the series conv. unif. on $\mathbb{R}$, then we should have:
$$\lim_{n\to\infty}\left(\sup\left\{\left|\frac{x^n}{n!}\right|^3 : x \in \mathbb{R}\right\}\right) = 0.)$$

(2) (20 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$ satisfying $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

- f is continuous on the compact set $[0, 2\pi]$, therefore f is uniformly continuous on $[0, 2\pi]$, i.e.

  $\forall \varepsilon > 0$, $\exists \tilde{\delta} > 0$ s.t.

  if $\begin{cases} x, y \in [0, 2\pi], \\ |x-y| < \tilde{\delta} \end{cases}$ then $|f(x) - f(y)| < \varepsilon/2$.

- Since $f(x+2\pi) = f(x)$ $\forall x \in \mathbb{R}$, we also have: $\forall n \in \mathbb{Z}$,

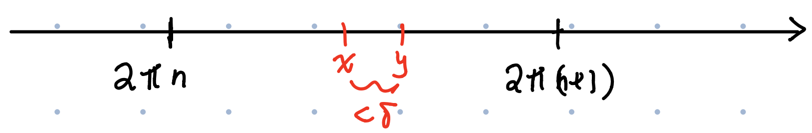
  If $\begin{cases} x, y \in [2\pi n, 2\pi(n+1)] \\ |x-y| < \tilde{\delta} \end{cases}$ then $|f(x) - f(y)| < \varepsilon/2$.

- Define $\delta = \min\{\tilde{\delta}, \pi\} > 0$.

  Claim: if $\begin{cases} x, y \in \mathbb{R} \\ |x-y| < \delta \end{cases}$, then $|f(x) - f(y)| < \varepsilon$.

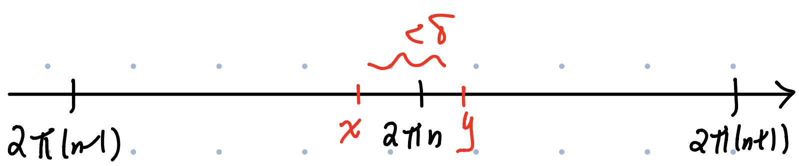
  pf: Since $|x-y| < \delta \le \pi$, one of the following two situations happen:

Case 1: $x, y \in [2\pi n, 2\pi(n+1)]$ for some $n \in \mathbb{Z}$.



then $|f(x) - f(y)| < \varepsilon/2$.

Case 2:



Then $|f(x) - f(y)| \le |f(x) - f(2\pi n)| + |f(y) - f(2\pi n)|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$

(3) Let $S_1 = (\mathbb{R}, d_{\text{std}})$ be the standard metric space of real numbers, i.e. $d_{\text{std}}(x, y) = |x - y|$. Let $S_2 = (\mathbb{R}, d_0)$ be the metric space whose elements are still the real numbers, but equips with a different distance function:

$$d_0(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

(a) (10 points) Characterize all the open subsets of S_2, and justify your answer.
(b) (10 points) Characterize all the compact subsets of S_2, and justify your answer.
(c) (15 points) Characterize all the continuous functions $f\colon S_2 \to S_1$, and justify your answer.
(d) (15 points) Characterize all the continuous functions $f\colon S_1 \to S_2$, and justify your answer.

(Warning: Do NOT simply copy and paste the definition of open, compact, or continuous. Give more explicit descriptions.)

(a) Claim: Any subset of S_2 is open.

pf: $\forall x_0 \in S_2$, $B_{1/2}(x_0) = \{x_0\}$.

Hence the one-point set $\{x_0\}$ is open in S_2 $\forall x_0 \in S_2$

Since any union of open subsets is open, the claim follows. $\square$

(b) Claim: $E \subseteq S_2$ is compact $\Leftrightarrow$ E is a finite set.

pf: It's clear by the definition of compactness that any finite set is compact.

Now if $E \subseteq S_2$ is an infinite set,

$$E = \{x_\alpha \mid \alpha \in I\} \subseteq S_2.$$

Then $E = \bigcup_{\alpha \in I} \{x_\alpha\}$ is an open cover of E (since any subset of S_2 is open), which doesn't admit any finite subcover. So E is not compact. $\square$

(c) <u>Claim:</u> Any function $f: \S_2 \longrightarrow \S_1$ is continuous.

pf: "$f: \S_2 \to \S_1$ continuous" $\iff$ "$\forall\, \mathcal{U} \underset{\text{open}}{\subseteq} \S_1,\ \underset{\text{is open}}{f^{-1}(\mathcal{U}) \subseteq \S_2}$".

Since any subset of $\S_2$ is open, the claim follows. $\square$

(d) <u>Claim:</u> $f: \S_1 \longrightarrow \S_2$ continuous $\iff$ f is a constant function.

pf: First, it's easy to show using (a) that if $E \subseteq \S_2$ is a nonempty <u>connected</u> subset, then E consists of a single point. (if $E = \{x_0\} \cup \{x_\alpha : \alpha \in I\}$, then the open sets $\{x_0\}$ and $\{x_\alpha : \alpha \in I\}$ separate E).

Suppose $f: \S_1 \longrightarrow \S_2$ is conti., since $\S_1 = (\mathbb{R}, d_{std})$ is connected, we have $f(\S_1) \subseteq \S_2$ is connected., therefore must be a single point. $\square$