(1) Let E be a nonempty, closed, and bounded subset of $\mathbb{R}$. Prove that $\sup E$ and $\inf E$ both belong to E.

- $\sup E$, $\inf E$ exists and are real numbers since E is bounded.

Claim: $z := \sup E \in E$. (the proof for $\inf E$ is similar).

pf:
- Assume that $z \notin E$.
- $\forall \varepsilon > 0$, $\exists x \in E$ s.t. $z - \varepsilon < x \leq z$. (HW1 #4),

  and since $x \in E$, $z \notin E$, we have $x \neq z$. $\Longrightarrow z - \varepsilon < x < z$.

  $\Longrightarrow x \in B_\varepsilon(z) \setminus \{z\}$.
- Hence, z is a limit point of E.
- $z \in \overline{E} = E$. since E is closed. Contradiction. $\square$

(2) Consider the following two functions on $\mathbb{R}$:

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

For each of the functions, prove or disprove that it is continuous at the point $x = 0$.

Claim: f is continuous at $x=0$.

pf: $\forall \varepsilon > 0$, take $\delta = \varepsilon$, then

$$|f(x) - f(0)| = |x \sin(\tfrac{1}{x})| \leq |x| < \varepsilon \quad \forall\, 0 < |x| < \delta = \varepsilon. \quad \square$$

Claim: g is not continuous at $x=0$.

pf: consider $x_n = \dfrac{1}{2n\pi + \pi/2}$, $g(x_n) = 1$.

$\lim x_n = 0$, but $\lim g(x_n) = 1 \neq 0 = g(0)$. $\square$

(3) Let $\epsilon > 0$ be a positive number. In each case, find a $\delta > 0$ (which should depend on ϵ) such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon \text{ holds.}$$

(a) $f(x) = \frac{1}{x}$; $x_0 = 1$.
(b) $f(x) = \sqrt{|x|}$; $x_0 = 0$.
(c) $f(x) = \sqrt{x}$; $x_0 = 1$.

(a) Let $\delta = \min\{ \frac{1}{2}, \frac{\varepsilon}{3} \} > 0$.

Then $\forall\ |x - x_0| = |x-1| < \delta$, we have:

- $|x-1| < \delta \leq \frac{1}{2} \implies |x| > \frac{1}{2}$
- $|f(x) - f(x_0)| = \left| \frac{1}{x} - 1 \right| = \dfrac{|x-1|}{|x|} < \dfrac{\delta}{1/2} < \varepsilon.$ $\square$

(b) Let $\delta = \varepsilon^2 > 0$.

Then $\forall\ |x - x_0| = |x| < \delta$, we have:

$$|f(x) - f(x_0)| = \sqrt{|x|} < \sqrt{\delta} = \varepsilon. \ \square$$

(c) Let $\delta = \min\{ \frac{1}{2}, \varepsilon \} > 0$

Then $\forall\ |x - x_0| = |x-1| < \delta$, we have:

- $|x-1| < \delta \leq \frac{1}{2} \implies |x| > \frac{1}{2}$.
- $|f(x) - f(1)| = |\sqrt{x} - 1| = \dfrac{|x-1|}{\sqrt{x} + 1} \leq \dfrac{\delta}{1/\sqrt{2} + 1} < \varepsilon.$ $\square$

(4) Suppose f, g are real-valued continuous functions on the closed interval $[a, b]$, and $f(a) < g(a)$ and $f(b) > g(b)$. Prove that $f(c) = g(c)$ for some $c \in (a, b)$.

- By #5, $h(x) := f(x) - g(x)$ is continuous on $[a,b]$.
- $h(a) < 0$, $h(b) > 0$. By Intermediate value thm, $\exists\, c \in (a,b)$ s.t. $h(c) = 0$. $\implies f(c) = g(c)$. $\square$

(5) Prove the following generalization of Ross, Theorem 17.4: Let (X,d) be any metric space, and let $f,g : X \to \mathbb{R}$ be two real-valued functions that are continuous at $x_0 \in X$. Prove that the functions $f+g$ and fg are both continuous at x_0. Moreover, if $g(x_0) \neq 0$, then f/g is also continuous at x_0. (The proofs are very similar, so you can pick one of $f+g, fg, f/g$ and prove it.)

Let's prove $f+g$ is continuous at x_0:

$\forall (x_n)$ in X converging to x_0,

- since f is conti. at x_0, we have: $\lim_{n\to\infty} f(x_n) = f(x_0)$.
- since g ——— " ———: $\lim_{n\to\infty} g(x_n) = g(x_0)$.
- By limit thm.,

$$\lim_{n\to\infty}\left(f(x_n)+g(x_n)\right) = f(x_0)+g(x_0). \quad \square$$

(6) Prove the following generalization of Ross, Theorem 17.5: Let $(X,d_X),(Y,d_Y),(Z,d_Z)$ be three metric spaces and let $f : X \to Y$ and $g : Y \to Z$ be two maps among them. Define the composite function $g \circ f : X \to Z$ via $(g\circ f)(x) := g(f(x))$. Prove that if f is continuous at $x_0 \in X$ and g is continuous at $f(x_0) \in Y$, then the composition $g \circ f$ is continuous at x_0.

$\forall (x_n)$ seq. in X converging to x_0.

- since f is conti. at x_0, we have: $\lim f(x_n) = f(x_0)$.
- since g is conti. at x_0, we have: $\lim g(f(x_n)) = g(f(x_0))$.

$\square$.