Today: Abel's thm, Weierstrass approx. thm, differentiation.

- Say $f(x) = \sum a_n x^n$, has radius of conv. $R > 0$
- last time, we proved: $\forall\ 0 < R' < R$, $\underline{\sum a_n x^n}$ conv. unif. on $[-R', R']$

$\Rightarrow$ f is conti. on $(-R, R)$

Abel's thm: Suppose $\sum a_n x^n$ has r.o.c. $R > 0$, and suppose $\sum a_n x^n$ converges at $x = R$, (or $-R$) then $f(x) = \sum a_n x^n$ is continuous at $x = R$ (or $-R$)

$-R \quad 0 \quad R$

pf: We may assume $R = 1$. (otherwise, rescale $x \mapsto Rx$)

Want: f is conti. on $[0, 1]$

$$\Uparrow$$

$$\underline{\sum a_n x^n \text{ conv. unif. on } [0,1]}$$

$$\Updownarrow$$

Want to prove this

$$\boxed{\forall \varepsilon > 0,\ \exists N > 0 \text{ s.t. } \left| \sum_{k=m}^{n} a_k x^k \right| < \varepsilon \quad \forall n \geq m > N,\ \forall x \in [0,1]}$$

Another simplificat^{n}: we may assume $f(1) = \sum a_n = 0$
(otherwise $f \mapsto f - a_0$)

$$\boxed{\sum_{k=m}^{n} a_k x^k}$$

Let $s_k = \sum_{l=0}^{k} a_l$

$= a_0 + \cdots + a_k$

then $a_k = s_k - s_{k-1}$

$$= \sum_{k=m}^{n} (s_k - s_{k-1}) x^k$$

$$= \sum_{k=m}^{n} \left(s_k x^k - s_{k-1} x^k\right)$$

$$S_m x^m - S_{m-1} x^m$$
$$S_{m+1} x^{m+1} - S_m x^{m+1} \quad = S_m (x^m - x^{m+1})$$
$$S_{m+2} x^{m+2} - S_{m+1} x^{m+2} \quad = S_{m+1} (x^{m+1} - x^{m+2})$$
$$\vdots$$
$$S_n x^n - S_{n-1} x^n$$

$$= \underset{①}{-S_{m-1} x^m} + \underset{②}{S_n x^n}$$

$$+ \underset{③}{\sum_{k=m}^{n-1} s_k (x^k - x^{k+1})}$$

$\underline{\sum a_n = 0}$ $\quad \forall \varepsilon > 0, \ \exists N > 0$ s.t. $|S_n| < \varepsilon/3 \quad \forall n > N$.

$$|①| = |S_{m-1} x^m| < \varepsilon/3 \quad \forall m > N, \ x \in [0,1]$$

$$|②| = |S_n x^n| < \varepsilon/3 \quad \forall n > N, \ x \in [0,1]$$

$$|③| = \left|\sum_{k=m}^{n-1} s_k (x^k - x^{k+1})\right| = |1-x| \left|\sum_{k=m}^{n-1} s_k x^k\right|$$

$$\leq |1-x| \sum_{k=m}^{n-1} |s_k| |x|^k \qquad \forall n, m > N,$$

$$< \cancel{|1-x|} \cdot \varepsilon \cdot \underbrace{\sum_{k=m}^{n-1} x^k}_{\frac{1-x^{n-m}}{\cancel{1-x}}} \qquad \forall x \in [0,1]$$

$$< \varepsilon/3$$

$$\Rightarrow \left|\sum_{k=m}^{n} a_k x^k\right| < \varepsilon \quad \forall n, m > N, \ x \in [0,1]. \ \square$$

Weierstrass Approximation Thm.

Every continuous fcn on $[a,b] \subseteq \mathbb{R}$ can be uniformly approximated by polynomials.

i.e. $\forall f : [a,b] \to \mathbb{R}$ conti.

$\exists (P_n)$: seq. of poly.

s.t. $P_n \to f$ unif. on $[a,b]$.

Def: X metric space, $E \subseteq X$ subset.

We say E is <u>dense in X</u> if $\overline{E} = X$

$$\overline{E} = E \cup \{\text{limit pts of } E \text{ in } X\}.$$

e.g. $X = (\mathbb{R}, d_{std})$

$E = \mathbb{Q} \subseteq X$ dense in X.

Define

- $\mathcal{C}([a,b])$ = the set of real-valued conti fcns. on $[a,b]$ $\left(f : [a,b] \xrightarrow[\text{conti.}]{} \mathbb{R}\right)$
- $d(f,g) := \sup_{x \in [a,b]} |f(x) - g(x)|$

This is a metric space.

- $\underset{\substack{\uparrow \\ \text{conti. fcn on } [a,b]}}{(f_n)} \longrightarrow \underset{\uparrow}{f}$ unif. $\iff f_n \to f$ in $(\mathcal{C}([a,b]), d)$

Weierstrass Approx. thm.: is equivalent to:

$$\{\text{polynomials}\} \subseteq \mathcal{C}([a,b])$$

is dense.

X = cpt metric space.

$\mathcal{C}(X)$ = real-valued conti. fcns on X,

$$d(f,g) := \sup_{x \in X} |f(x) - g(x)|$$

↑ metric space.

unital

Def: $A \subseteq \mathcal{C}(X)$ is called an unital subalgebra of $\mathcal{C}(X)$ if:

- A is a $\mathbb{R}$-vector subspace of $\mathcal{C}(X)$ (i.e. it's closed under addition and scalar multiplication)
- A is closed under multiplication
- A contains $\mathbb{1}$ (the const. fcn which sends every elts in X ~~to~~ 1)
  $\begin{pmatrix} \mathbb{1}: X \to \mathbb{R} \\ x \mapsto 1. \end{pmatrix}$

e.g. $\{\text{poly.}\} \subseteq \mathcal{C}([a,b])$ is an unital subalg. of $\mathcal{C}[a,b]$

e.g. $\{\text{const. fcns}\} \subseteq \mathcal{C}(X)$ is an unital subalg.

Def: $A \subseteq \mathcal{C}(X)$ separate points if $\forall\, x \neq y$ in X $\exists\, f \in A$ s.t. $f(x) \neq f(y)$.

Stone-Weierstrass thm: X = cpt metric space.

$A \subseteq \mathcal{C}(X)$ unital subalg. that separate points.

Then A is dense in $\mathcal{C}(X)$.

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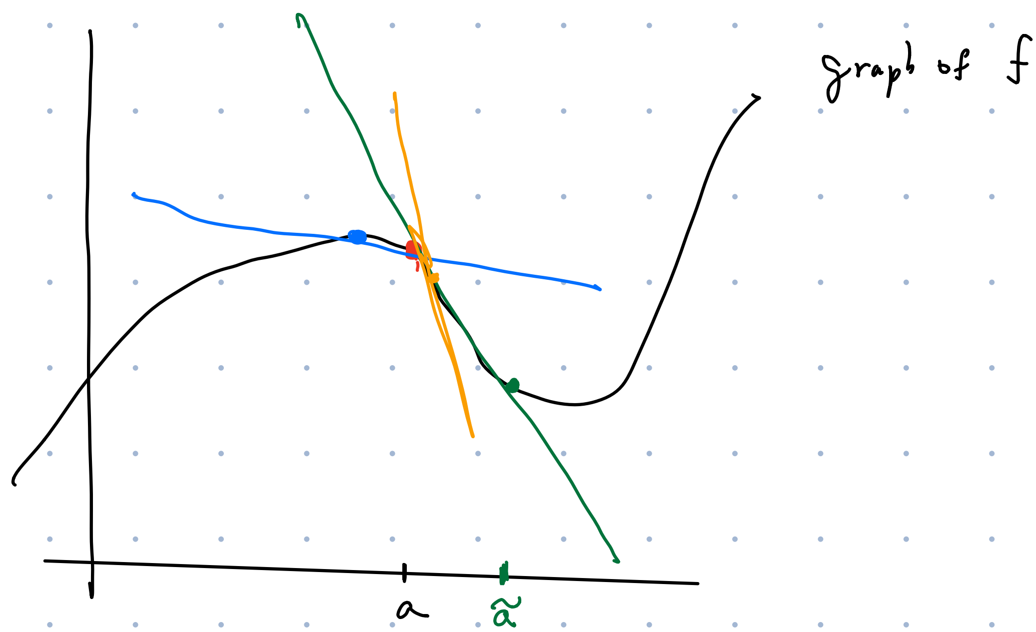
§ Differentiation:



intuitively, "$f'(a)$" = "slope of the tangent line to the graph of f at the point $(a, f(a))$"

= "the limit of slope of the secant line: $\displaystyle\lim_{\tilde a \to a} \frac{f(\tilde a) - f(a)}{\tilde a - a}$ connects $(a, f(a))$ and $(\tilde a, f(\tilde a))$ as $\tilde a \to a$"

fcn in $\tilde a$

Def: (limit of fcns). Say "$\lim_{x \to a} F(x) = L$" if

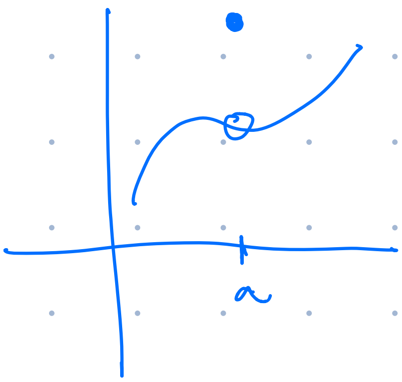
$\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $0 < |x-a| < \delta \Rightarrow |F(x) - L| < \varepsilon$.

Rmk: F doesn't have to be defined at a,

even if F is defined at a, $\lim_{x \to a} F(a)$ may not exist,

and may not be $= F(a)$



Rmk: If F is conti. at a,

then $\lim_{x \to a} F(x) = F(a)$.

Def: $f : I \to \mathbb{R}$, I: open interval containing $a \in \mathbb{R}$.

We say f is <u>differentiable at a</u> if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists and is finite}$$

In this case, we define $f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

e.g. $f(x) = x^n : \mathbb{R} \to \mathbb{R}$ $\boxed{f'(a) = ??}$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{\cancel{(x-a)}(x^{n-1} + x^{n-2} a + x^{n-3} a^2 + \cdots + a^{n-1})}{\cancel{x-a}}$$

$$= n a^{n-1}. \quad \square$$

Thm: If f is differentiable at a, then f is conti. at a.

$$\Downarrow$$

pf:

- $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. $= f'(a)$

- $\lim_{x\to a} (x-a) = 0$

- $$f'(a)\cdot 0 = \left(\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a} (x-a)\right)$$
  $$= \lim_{x\to a}\left[\left(\frac{f(x)-f(a)}{x-a}\right)(x-a)\right]$$
  $$= \lim_{x\to a} (f(x)-f(a))$$
  $$f'(a)\cdot 0 = 0$$

- $\lim_{x\to a} f(x) = f(a)$, i.e. f is conti. at a. $\square$