HOMEWORK 10
MATH 104, SECTION 6

Office Hours (via Zoom): Tuesday and Wednesday 9:30-11am.

PROBLEM SET (10 PROBLEMS; DUE APRIL 9)

Submit your homework before the lecture on Thursday. *Late homework will not be accepted under any circumstances.* You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any.

Write clearly! Mastering mathematical writing is one of the goals of this course.

(1) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that the derivative $f'(x)$ exists for any $x \in \mathbb{R}$, but $f'\colon \mathbb{R} \to \mathbb{R}$ is not a continuous function.

(2) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Show that the Taylor series for f about $x = 0$ converges on $\mathbb{R}$, but it does not coincide with f on any open interval containing 0.

(3) Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

One can regard $\mathbb{R}^2$ and $\mathbb{R}$ as metric spaces via the standard distance functions:

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Prove that:

(a) For any fixed $x \in \mathbb{R}$, the function $f_x\colon \mathbb{R} \to \mathbb{R}$ that sends y to $f(x, y)$ is continuous. Similarly, for any fixed $y \in \mathbb{R}$, the function $f_y\colon \mathbb{R} \to \mathbb{R}$ that sends x to $f(x, y)$ is also continuous.

(b) $f\colon \mathbb{R}^2 \to \mathbb{R}$ is not a continuous function.

(4) We say a function $f\colon (a,b) \to \mathbb{R}$ is *strictly increasing* if $f(x) < f(y)$ for any $a < x < y < b$. Suppose f is differentiable on (a,b).

(a) Prove or disprove: If f is strictly increasing, then $f'(x) > 0$ for any $x \in (a,b)$.

(b) Prove or disprove: If $f'(x) > 0$ for any $x \in (a,b)$, then f is strictly increasing.

(Hint: Mean value theorem.)

(5) Consider the function $f(x) = \log(1+x)$ on $(-1,\infty)$.

(a) Compute the Taylor series for f about $x = 0$.

(b) Let $R_n(x)$ be the remainder of the Taylor series in part (a), i.e.

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}(x-0)^k.$$

Use Taylor's theorem to show that $\lim_{n\to\infty} R_n(1) = 0$, then obtain the formula

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

(6) Prove that the equation $e^x = 1 - x$ has a unique solution in $\mathbb{R}$.

(7) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying $|f(x) - f(y)| \le |x-y|^2$ for any $x, y \in \mathbb{R}$. Prove that f is a constant function.

(8) Let $f\colon (a,b) \to \mathbb{R}$ be an unbounded differentiable function. Prove that the derivative $f'\colon (a,b) \to \mathbb{R}$ is also unbounded.

(9) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function and is differentiable on $(0,1)$. Suppose that f satisfies:

- $f(0) = 0$.
- There exists $M > 0$ such that $|f'(x)| \le M|f(x)|$ for any $x \in (0,1)$.

Prove that $f(x) = 0$ for any $x \in [0,1]$.

(10) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, and fix a point $x_0 \in \mathbb{R}$. Consider the sequence $\{x_n\} \subset \mathbb{R}$ defined iteratively by $x_{n+1} = f(x_n)$. Suppose that $\lim_{n\to\infty} x_n = \ell \in \mathbb{R}$ converges, and suppose that $f'(\ell)$ exists. Prove that $|f'(\ell)| \le 1$.

Extra assumption: We assume the sequence (x_n) has the following property: For any $N > 0$, there exists $n > N$ such that $x_n \neq \ell$ (Otherwise there are counterexamples to the statement).