Def D: triangulated category.

An object $G \in Ob(D)$ is a split generator if $\forall\, 0 \neq E \in D$,

$$\exists\quad \begin{tikzcd}[column sep=small] 0 \arrow[r] & \cdots\cdots \arrow[r] & E \oplus E' \\ G[n_1] \arrow[u] & & G[n_k] \arrow[u] \end{tikzcd}$$

exact Δ's in D

for some integers $n_1, \dots, n_k$.

e.g. X - smooth projective variety of dimension d, $\mathcal{L}$ - ample line bundle.

$\bigoplus_{i=0}^{d} \mathcal{L}^{\otimes i}$ is a split generator of $D^b Coh(X)$ (Orlov)

Def: A function $\mu : Ob(D) \longrightarrow \mathbb{R}_{\geq 0}$ is a mass function if

- if $A \to B \to C \to A[1]$ is an exact Δ, then $\mu(A) + \mu(C) \geq \mu(B)$.
- $\mu(E) \leq \mu(E \oplus F)$
- $\mu(0) = 0$

Idea $F : D \to D$ endofunctor of D.

We'd like to define the entropy of F by

$$\text{"}\; h(F) := \lim_{n\to\infty} \frac{1}{n} \log \mu(F^n G) \text{"}.$$

Some possible mass functions: Complexity function,

$\sum_{k \in \mathbb{Z}} \dim Hom(E, -[k])$,

Bridgeland stability condition.

Def (Dimitrov–Haiden–Katzarkov–Kontsevich) $E_1, E_2 \in \mathrm{Ob}(D)$,

we define <u>complexity of E_2 relative to E_1</u>: $\delta_t(E_1, E_2) : \mathbb{R} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$

$$\delta_t(E_1, E_2) := \begin{cases} 0 & \text{if } E_2 = 0 \\ \inf \left\{ \sum_i e^{n_i t} \;\middle|\; \begin{tikzcd}[column sep=small] 0 \arrow[rr] & & \cdots \arrow[rr] \arrow[dl] & & E_2 \oplus E' \arrow[dl] \\ & E_1[n_1] \arrow[ul] & & E_1[n_k] \arrow[ul] & \end{tikzcd} \right\} & \text{if } E_2 \in \langle E_1 \rangle \\ +\infty & \text{if } E_2 \notin \langle E_1 \rangle . \end{cases}$$

<u>Basic properties</u>:

- $\delta_t(E_1, E_3) \leq \delta_t(E_1, E_2)\, \delta_t(E_2, E_3)$. If $E_2 \neq 0$.
- $\delta_t(E_1, E_2 \oplus E_3) \leq \delta_t(E_1, E_2) + \delta_t(E_1, E_3)$
- $\delta_t(F(E_1), F(E_2)) \leq \delta_t(E_1, E_2)$ $\forall\, F: D \to D'$ exact functor.
- $\delta_t(E, B) \leq \delta_t(E, A) + \delta_t(E, C)$ $\forall\, A \to B \to C \to A[1]$ exact Δ
- $\delta_t(E_1, E_2) \leq \delta_t(E_1, E_2 \oplus E_3)$.

(In particular, $\delta_t(G, -)$ is a mass function for a split generator G.)

<u>Ex</u>: $D = D^b(k)$ bounded derived category of finite dimensional vector spaces/k

- k is a split generator.
- $\delta_t(k, \underset{\substack{\cap \\ D}}{E}) = \sum_{n \in \mathbb{Z}} \dim(H^n E) \cdot e^{-nt}$

Assumption:
- $\mathcal{D}$ is saturated. (satisfied by all the examples we'll discuss)
- F is not virtually zero. i.e. $F^n \neq 0 \quad \forall n \geq 1$.

  (in particular $\Rightarrow$ $F^n G \neq 0 \quad \forall n \geq 1$, G - split generator).

Def (DHKK) $F : \mathcal{D} \to \mathcal{D}$ is an endofunctor of a triangulated cat. $\mathcal{D}$, with a split generator G. We define the entropy function of F to be:

$$\begin{aligned} h_t(F) : \mathbb{R} &\longrightarrow \mathbb{R} \cup \{-\infty\} \\ t &\longmapsto h_t(F), \end{aligned}$$

where $$h_t(F) := \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, F^n G).$$

① limit exists:

$$\begin{aligned} \delta_t(G, F^{n+m} G) &\leq \delta_t(G, F^n G)\, \delta_t(F^n G, F^{n+m} G) \\ &\leq \delta_t(G, F^n G)\, \delta_t(G, F^m G) \end{aligned}$$

$$\Rightarrow \quad \log \delta_t(G, F^{n+m} G) \leq \log \delta_t(G, F^n G) + \log \delta_t(G, F^m G)$$

i.e. $\{\log \delta_t(G, F^n G)\}_{n \geq 1}$ is a subadditive sequence

Apply Fekete's lemma: ($\{a_n\}$ subadditive $\Rightarrow$ $\lim \frac{a_n}{n}$ exists, and $\lim \frac{a_n}{n} = \liminf \frac{a_n}{n}$)

$$\Rightarrow \quad \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, F^n G) \text{ exists}, \in \mathbb{R} \cup \{-\infty\}.$$

② <u>entropy is independent of the choice of G:</u>

$$\delta_t(G', F^nG') \le \delta_t(G',G)\,\delta_t(G,F^nG)\,\delta_t(F^nG, F^nG')$$
$$\le \delta_t(G',G)\,\delta_t(G,F^nG)\,\delta_t(G,G')$$

$$\Rightarrow \underset{\downarrow\, n\to+\infty}{\frac{\log \delta_t(G',F^nG')}{n}} \le \underbrace{\frac{\log\delta_t(G',G)+\log\delta_t(G,G')}{n}}_{\to\, 0} + \underset{\downarrow\, n\to+\infty}{\frac{\log \delta_t(G,F^nG)}{n}}$$

$\square$

<u>Rmk:</u>

- Later, we'll show that $h_t(F)\in\mathbb{R}$ (i.e. $h_t(F)\neq-\infty$)
- $h_{cat}(F) := h_0(F) \geq 0$ by definition of the entropy fn.
- $h_t(F) = \lim_{n\to\infty}\frac{1}{n}\log\delta_t(G,F^nG')$ where G, G' are split generators

$$\Big(\delta_t(G,F^nG') \le \delta_t(G,F^nG)\,\delta_t(F^nG,F^nG') \le \delta_t(G,F^nG)\,\delta_t(G,G')$$
$$\delta_t(G,F^nG) \le \delta_t(G,F^nG')\,\delta_t(F^nG',F^nG) \le \delta_t(G,F^nG')\,\delta_t(G',G)\Big)$$

<u>Thm</u> (DHKK)

$\sum_k \dim\mathrm{Hom}(G,-[k])e^{-kt}$

$$h_t(F) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{k\in\mathbb{Z}}\dim\mathrm{Hom}(G,F^nG'[k])\,e^{-kt}.$$

<u>Lemma</u> $\exists\, C_1(t), C_2(t): \mathbb{R}\to\mathbb{R}_{>0}$ (depend on G) s.t.

$$C_1(t)\,\delta_t(G,E) \le \delta_t(k, \mathrm{Hom}^\bullet(G,E)) \le C_2(t)\,\delta_t(G,E) \quad \forall E.$$

Then, the theorem follows from taking $E = F^n G'$, since

$$\delta_t(\Bbbk, \mathrm{Hom}^\bullet(G,E)) = \sum_k \dim \mathrm{Hom}(G, E[k])\, e^{-kt}$$

<u>Sketch of proof of lemma:</u>

- Consider the functor $\mathrm{Hom}^\bullet(G,-): \mathcal{D} \to D^b(\Bbbk)$.

$$\delta_t(\Bbbk, \mathrm{Hom}^\bullet(G,E)) \leq \delta_t(\Bbbk, \mathrm{Hom}^\bullet(G,G))\, \delta_t(\mathrm{Hom}^\bullet(G,G), \mathrm{Hom}^\bullet(G,E))$$

$$\leq \underbrace{\delta_t(\Bbbk, \mathrm{Hom}^\bullet(G,G))}_{\overset{\text{!!}}{C_2(t)}}\, \delta_t(G, E)$$

- $$\delta_t(G,E) \leq \underbrace{\delta_t(G, G\otimes \mathrm{Hom}^\bullet(G,E))}_{\overset{\text{\rotatebox{90}{$=$}}}{\delta_t(\Bbbk, \mathrm{Hom}^\bullet(G,E))}}\, \underbrace{\delta_t(G\otimes \mathrm{Hom}^\bullet(G,E), E)}_{\overset{\text{\rotatebox{90}{$\leq$}}}{\delta_t(G\otimes \mathrm{Hom}^\bullet(G,-), \mathrm{id}_{\mathcal{D}})}}$$

by considering the functor $D^b(\Bbbk) \xrightarrow{G\otimes -} \mathcal{D}$

by considering the functor:
$$\begin{aligned} \mathrm{Fun}(\mathcal{D}, \mathcal{D}) &\xrightarrow{(E)} \mathcal{D} \\ \Phi &\longmapsto \Phi(E) \end{aligned}$$

is finite, by the assumption that $\mathcal{D}$ is saturated.

<u>Basic properties:</u>

- $h_t(F^n) = n\, h_t(F) \quad \forall n \geq 1$
- $h_t([1]) = t$
- If $F_1 F_2 = F_2 F_1$, then $h_t(F_1F_2) \leq h_t(F_1) + h_t(F_2)$
- $h_t(F[n]) = h_t(F) + nt$

<u>Lemma</u> (commutativity). $h_t(F_1F_2) = h_t(F_2F_1)$

$$\delta_t(G, (F_1F_2)^n G) \leq \delta_t(G, F_1G) \cdot \delta_t(F_1G, F_1(F_2F_1)^{n-1}G) \cdot \delta_t(F_1(F_2F_1)^{n-1}G, (F_1F_2)^n G)$$

$$\leq \delta_t(G, F_1G)\, \delta_t(G, (F_2F_1)^{n-1}G)\, \delta_t(G, F_2G)$$

$\square$

<u>Cor:</u> If F_1 is an autoequivalence, then $h_t(F_1F_2F_1^{-1}) = h_t(F_2)$

i.e. entropy is a class function on $\mathrm{Aut}(D)$.

<u>Lemma</u> (Inverse) If D admits a Serre functor S, $F \in \mathrm{Aut}(D)$, then $h_t(F^{-1}) = h_{-t}(F)$

pf $\quad \varepsilon_t(E_1, E_2) := \sum_k \dim \mathrm{Hom}(E_1, E_2[k])\, e^{-kt}$

$$h_t(F) = \lim_{n\to\infty} \frac{1}{n} \log \varepsilon_t(G, F^n G')$$

$$\varepsilon_t(G, F^{-n}G) = \varepsilon_t(F^n G, G) = \varepsilon_{-t}(G, SF^n G)$$

$$= \varepsilon_{-t}(G, F^n \underbrace{SG}_{\text{also a split generator}})$$

$\square$

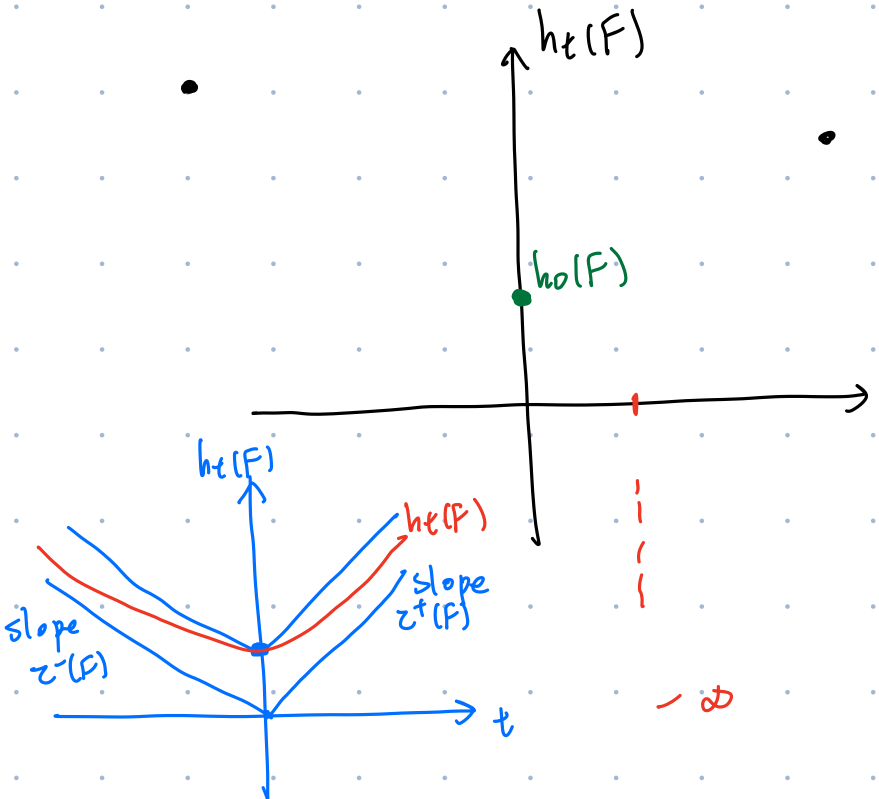
Thm (convexity and finiteness of $h_t(F)$).

$h_t(F)$ is a real-valued convex function in t.

pf.

- $h_t(F) = \lim_{n\to\infty} \frac{1}{n} \log \underbrace{\sum_k \dim \mathrm{Hom}(G, F^n G'[k])\, e^{-kt}}_{\log\left(\sum a_i e^{k_i t}\right) \text{ is a convex functn.},\ a_i \geq 0}$

  is convex.

- 

- If $h_t(F) = -\infty$ for some $t>0$, then $h_t(F) = +\infty$ for some $t<0$ by the convexity of $h_t(F)$. But this contradicts with $h_t(F) \in \mathbb{R} \cup \{-\infty\}$. $\square$

Thm (linear growth rate of $h_t(F)$ as $t \to \pm\infty$: Shifting numbers)

- $\lim_{t\to\pm\infty} \frac{h_t(F)}{t}$ exist $=: \tau^{\pm}(F)$

- $\lim_{n\to\infty} \frac{\boxed{\max/\min \{k \in \mathbb{Z} \mid \mathrm{Hom}(G, F^n G[-k]) \neq 0\}}}{n}$ exist, and $= \tau^{\pm}(F)$

  $=: \varepsilon^{\pm}(G, F^n G)$

- $t \cdot \tau^+(F) \leq h_t(F) \leq h_0(F) + t \cdot \tau^+(F)$ if $t \geq 0$.
  $t \cdot \tau^-(F) \leq h_t(F) \leq h_0(F) + t \cdot \tau^-(F)$ if $t \leq 0$.

Pf For $t>0$:

$$\sum_k \dim \mathrm{Hom}(G, F^n G[k]) e^{-kt}$$
$$\|$$
$$e^{\varepsilon^+(G,F^nG)t} \leq \underline{\varepsilon_t(G,F^nG)} \leq \left(\sum_k \dim \mathrm{Hom}(G, F^n G[k])\right) \cdot e^{\varepsilon^+(G,F^nG)t}$$

$$\xRightarrow{\frac{1}{n}\log(-)} t \cdot \limsup_{n\to\infty} \frac{\varepsilon^+(G,F^nG)}{n} \leq \limsup_{n\to\infty} \frac{1}{n} \log \varepsilon_t(G,F^nG)$$
$$\|$$
$$\lim_{n\to\infty} \frac{1}{n} \log \varepsilon_t(G,F^nG) = h_t(F)$$
$$\|$$
$$\liminf_{n\to\infty} \frac{1}{n} \log \varepsilon_t(G,F^nG)$$
$$\geq$$
$$\liminf_{n\to\infty} \frac{1}{n}\left(\log\left(\sum_k \dim \mathrm{Hom}(G, F^n G[k])\right) + \varepsilon^+(G,F^nG)t\right)$$
$$\|$$
$$h_0(F) + t \liminf_{n\to\infty} \frac{\varepsilon^+(G,F^nG)}{n}$$

$$t \cdot \varepsilon^+(F) \leq h_t(F) \leq h_0(F) + t \cdot \varepsilon^+(F)$$
$$\Uparrow$$

$$\Rightarrow t \cdot \limsup_{n\to\infty} \frac{\varepsilon^+(G,F^nG)}{n} \leq h_t(F) \leq h_0(F) + t \cdot \liminf_{n\to\infty} \frac{\varepsilon^+(G,F^nG)}{n}$$

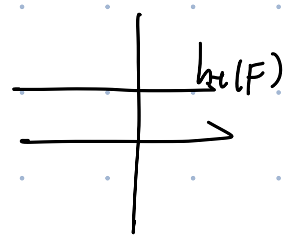
$$\xRightarrow[t\to+\infty]{-/t} \limsup_{n\to\infty} \frac{\varepsilon^+(G,F^nG)}{n} \leq \liminf \frac{h_t(F)}{t} \leq \limsup \frac{h_t(F)}{t} \leq \liminf_{n\to\infty} \frac{\varepsilon^+(G,F^nG)}{n}$$

$\Rightarrow$ $\displaystyle\lim_{n\to\infty} \frac{\varepsilon^+(G,F^nG)}{n}$ and $\displaystyle\lim_{t\to+\infty} \frac{h_t(F)}{t}$ both exist, and coincide.

$$\overset{!!}{\varepsilon^+(F)}$$

$\square$

Lemma Suppose $\exists$ split generator G and $M > 0$ s.t.



$$\mathrm{Hom}(G, F^n G[k]) = 0 \quad \forall\, |k| > M,\ n \geq 0.$$

Then $h_t(F)$ is a constant function. (i.e. $\tau^{\pm}(F) = 0$)

pf
$$\varepsilon_0(G, F^n G) \cdot e^{-M|t|} \leq \varepsilon_t(G, F^n G) \leq \varepsilon_0(G, F^n G) \cdot e^{M|t|}$$

Take $\frac{1}{n}\log(-)$, $n \to +\infty$, we'll get:

$$h_0(F) \leq h_t(F) \leq h_0(F). \quad \square$$

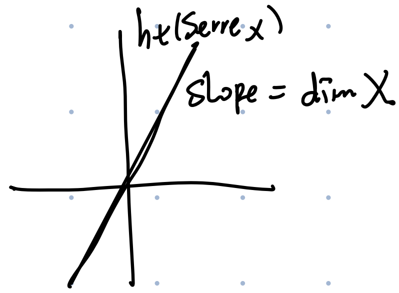
e.g. X. smooth projective variety of dim. d. L: line bundle, $h_t(-\otimes L) = 0$ $\forall t$

pf Choose $G = \bigoplus_{i=0}^{d} \mathcal{O}(i)$. By lemma, $h_t(-\otimes L)$ is a constant fcn.

$$\underbrace{\sum_k \dim \mathrm{Hom}(G, G \otimes L^{\otimes n}[k])}$$
$$\|$$
$$\dim H^k(X, G^\vee \otimes G \otimes L^{\otimes n})$$
$$\downarrow$$

at most polyonimal growth of degree d.

$\Rightarrow$ $h_t(F) = 0$.



e.g. X - smooth projective variety of dim. d.

Serre functor $S = - \otimes \omega_X[d]$ of $D^b \mathrm{Coh}(X)$

$\Rightarrow$ $h_t(S) = dt$.

<u>Thm</u> (Kikuta–Takahashi) X - smooth projective variety $/\mathbb{C}$. $f: X \to X$

$f^* = \mathbb{L}f^* : D^b\mathrm{Coh}(X) \circlearrowleft$

$$h_{cat}(f^*) = h_{top}(f)$$
$$= \log \rho([f^*])$$

numerical Grothendieck grp of D.

where: $[f^*] : N(D) \longrightarrow N(D)$ induced by f^*

ρ is the spectral radius of the linear map $[f^*]$.

<u>Sketch of proof:</u>

- Choose $G = \bigoplus_{k=1}^{d+1} \mathcal{O}(k)$, $G^* = \bigoplus_{k=1}^{d+1} \mathcal{O}(-k)$, ($d = \dim X$)
- $\varepsilon_0(G, (f^*)^n G^*) \overset{\text{Kodaira vanishing}}{=\!=} (-1)^d \underbrace{\chi(G, (f^*)^n G^*)}_{\sum_k (-1)^k \dim \mathrm{Hom}(G, (f^*)^n G^*[k])}$

$\mathrm{Hom}(G, (f^*)^n G^*[*])$
$= H^*(X, G^* \otimes (f^*) G^*)$

$$\left(\Rightarrow h_{cat}(f^*) = \lim_{n\to\infty} \frac{1}{n} \log \varepsilon_0(G, (f^*)^n G^*) \right.$$
$$= \lim_{n\to\infty} \frac{1}{n} \log |\chi(G, (f^*)^n G^*)|$$
$$\leq \log \rho([f^*]).$$

In general, we have a Yomdin-type inequality: $h_{cat}(F) \geq \log \rho(N(F))$ (we'll prove later).

$$\left. \Rightarrow h_{cat}(f^*) = \log \rho([f^*]). \right)$$

- $\chi(G, (f^*)^n G^*) \overset{HRR}{=} \int_X ch(G^*)\, ch((f^*)^n G^*)\, td(X)$

- $ch(G^*) = ch\left(\bigoplus_{i=1}^{d+1} \mathcal{L}^{-i}\right) = ch(\mathcal{L}^{-1}) + \cdots + ch(\mathcal{L}^{-(d+1)})$

$(\mathcal{O}(1) = \mathcal{L})$

$$= ch(\mathcal{L}^{-1}) + ch(\mathcal{L}^{-1})^2 + \cdots + ch(\mathcal{L}^{-1})^{d+1}$$

$$= e^{c_1(\mathcal{L}^{-1})} + e^{2c_1(\mathcal{L}^{-1})} + \cdots + e^{(d+1)c_1(\mathcal{L}^{-1})}$$

$$= (d+1) - (1+\cdots+(d+1))c_1(\mathcal{L}) + \left(\frac{1^2+\cdots+(d+1)^2}{2}\right) c_1(\mathcal{L})^2 + \cdots$$

- $(-1)^d \chi(G, (f^*)^n G^*) = \sum_{r=0}^{d} \sum_{q=0}^{d-r} C_{r,q} \int_X (f^*)^n c_1(\mathcal{L})^q \cup c_1(\mathcal{L})^{d-r-q} \cup td_r(X)$

where $C_{r,q} \in \mathbb{Q}$, and $C_{0,q} > 0$

Recall Dynamical degree: (q-th dynamical deg.)

$$d_q(f) := \limsup_{n\to\infty} \left(\int_X (f^*)^n c_1(\mathcal{L})^q \cup c_1(\mathcal{L})^{d-q} \right)^{\frac{1}{n}}$$

- Decompose $H^{q,q}(X, \mathbb{C})$ into f^*-invariant subspaces:

$$H^{q,q}(X,\mathbb{C}) = H^{q,q}(X)_{\lambda_1, m_1} \oplus \cdots \oplus H^{q,q}(X)_{\lambda_\ell, m_\ell} \quad \text{(Jordan blocks)}$$

$$(\underset{r_q(f)}{|\lambda_1|}, \underset{l_q(f)}{m_1}) \geq (|\lambda_2|, m_2) \geq \cdots \geq (|\lambda_\ell|, m_\ell)$$

Dinh-Sibony:
$$\limsup_{n\to\infty} \frac{\int_X (f^*)^n c_1(\mathcal{L})^q \cup c_1(\mathcal{L})^{d-q}}{n^{l_q(f)-1}\, r_q(f)^n} < +\infty, \quad \limsup_{n\to\infty} \frac{n^{l_q(f)-1}\, r_q(f)^n}{\int_X (f^*)^n c_1(\mathcal{L})^q \cup c_1(\mathcal{L})^{d-q}} < +\infty$$

In particular, $d_q(f) = \lim_{n\to\infty} \int_X (f^*)^n c_1(\mathcal{L})^q \cup c_1(\mathcal{L})^{d-q} = r_q(f)$.

Gromov-Yomdin: $h_{top}(f) = \log r(f) = \log \max_q r_q(f)$

$r(f)$: spectral radius of f on $H^*(X, \mathbb{C})$

$\log \max_q r_q(f) = \log \max_q d_q(f)$

Choose p to be: $(r_p(f), \ell_p(f)) = \max_q (r_q(f), \ell_q(f))$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathcal{E}_0(G, F^n G^*) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{r=0}^{d} \sum_{q=0}^{d-r} C_{r,q} \int_X (f^*)^n c_1(\mathcal{L})^q \cup c_1(\mathcal{L})^{d-q-r} td_r(X)$$

$$= \log d_p(f) + \underbrace{\lim_{n\to\infty} \frac{1}{n} \log \sum_r \sum_q C_{r,q} \frac{\int_X (f^*)^n c_1(\mathcal{L})^q c_1(\mathcal{L})^{d-q-r} td_r(X)}{\int_X (f^*)^n c_1(\mathcal{L})^p \cup c_1(\mathcal{L})^{d-p}}}_{=0}$$

$$h_{cat}(F) = \log d_p(f) = \log \max_q d_q(f) = h_{top}(f)$$

$\square$