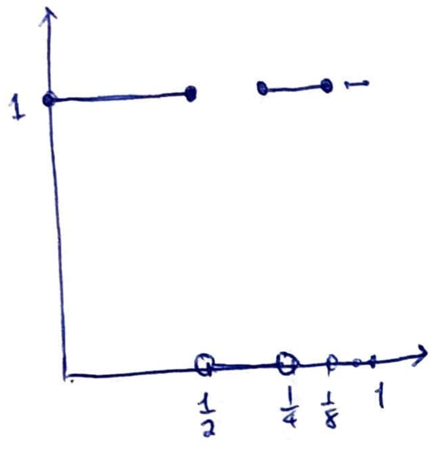
HW11 sol'n

\#1 The function looks like:



$\forall \varepsilon > 0$, take N large s.t. $\frac{2N+1}{2^{2N}} < \varepsilon$.

Consider the partition $P = \{0 = t_0 < t_1 < \cdots < t_{2^{2N}} = 1\}$, where $t_i = \frac{i}{2^{2N}}$.

Then

$$\sup_{x \in [t_{k-1}, t_k]} f(x) - \inf_{x \in [t_{k-1}, t_k]} f(x) = \begin{cases} 1 & \text{if } \begin{cases} \frac{k-1}{2^{2N}} = 1 - \frac{1}{2^{2l+1}} \text{ for some } l = 0, \dots, N-1, \text{ OR} \\ \frac{k}{2^{2N}} = 1 - \frac{1}{2^{2l}} \text{ for some } l = 1, \dots, N \text{ OR} \\ k = 2^{2N} \end{cases} \\ 0 & \end{cases}$$

$$\Rightarrow \quad U(f,P) - L(f,P) = \frac{2N+1}{2^{2N}} < \varepsilon.$$

Hence f is integrable.

$$\int_0^1 f(x)\,dx = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots = \frac{2}{3}. \qquad \square$$

2:

(a) $\forall \varepsilon > 0$, $\exists P = \{0 = t_0 < \cdots < t_\ell = 1\}$

st. $U(f,P) - L(f,P) < \varepsilon$

$\forall n \in \mathbb{N}$, and $\forall 1 \le k \le n$, $\exists 1 \le i \le \ell$ st. $\frac{k}{n} \in [t_{i-1}, t_i]$.

$$\Rightarrow \inf_{x \in [t_{i-1}, t_i]} f(x) \le f\left(\tfrac{k}{n}\right) \le \sup_{x \in [t_{i-1}, t_i]} f(x).$$

Hence

$$R_n = \frac{1}{n} \sum_{k=1}^{n} f\left(\tfrac{k}{n}\right) \le \frac{1}{n} \sum_{i=1}^{\ell} \#\left\{1 \le k \le n \,\middle|\, \tfrac{k}{n} \in [t_{i-1}, t_i]\right\} \cdot \sup_{x \in [t_{i-1}, t_i]} f(x)$$

One can check that $\forall i$,

$$n(t_i - t_{i-1}) - 1 < \#\left\{1 \le k \le n \,\middle|\, \tfrac{k}{n} \in [t_{i-1}, t_i]\right\} \le n(t_i - t_{i-1}) + 1$$

Hence.

$$R_n \le \frac{1}{n} \sum_{i=1}^{\ell} \left(n(t_i - t_{i-1}) + 1\right) \sup_{x \in [t_{i-1}, t_i]} f(x)$$

$$= \sum_{i=1}^{\ell} (t_i - t_{i-1}) \sup_{x \in [t_{i-1}, t_i]} f(x) + \frac{1}{n} \sum_{i=1}^{\ell} \sup_{x \in [t_{i-1}, t_i]} f(x)$$

$$\le U(f,P) + \frac{\ell}{n} \left(\sup_{x \in [0,1]} f(x)\right) =: M.$$

$$= U(f,P) + \frac{\ell M}{n}.$$

Similarly, $R_n \ge L(f,P) - \frac{\ell M}{n}$. So we have:

$$L(f,P) - \frac{\ell M}{n} \le R_n \le U(f,P) + \frac{\ell M}{n}.$$

$$< L(f,P) + \frac{\ell M}{n} + \varepsilon$$

$\int_0^1 f$

$\|$

Let $n \to \infty$,

$$\Rightarrow L(f,P) \le \lim_{n \to \infty} R_n \le L(f,P) + \varepsilon. \quad \forall \varepsilon > 0. \text{ Hence } \lim_{n\to\infty} R_n = L(f,P)$$

$\square$

#3 See Ross Thm 33.5.

#4 $\displaystyle\int_0^1 \frac{f(x)}{f(x)+f(1-x)}\,dx = \int_0^1 \frac{f(1-x)}{f(x)+f(1-x)}\,dx.$

and $\displaystyle\int_0^1 \frac{f(x)}{f(x)+f(1-x)} + \frac{f(1-x)}{f(x)+f(1-x)}\,dx = \int_0^1 1\,dx = 1.$

$\displaystyle\Rightarrow \int_0^1 \frac{f(x)}{f(x)+f(1-x)}\,dx = \boxed{\frac{1}{2}}.$

#5 Since f is integrable, $\forall \varepsilon>0$, $\exists P$ "$\{a=t_0<\cdots<t_n=b\}$" s.t. $U(f,P)-L(f,P)<\varepsilon/2$

$\displaystyle L(f,P) = \sum_{i=1}^{n} (t_i - t_{i-1}) \inf_{x\in[t_{i-1},t_i]} f(x).$

Define $\displaystyle \varsigma(x) := \begin{cases} \inf\limits_{x\in[t_{i-1},t_i]} f(x) & \text{if } x\in[t_{i-1},t_i). \\ f(b) & \text{if } x=b=t_n. \end{cases}$

Then:

- $\varsigma(x)$ is a step function
- $f(x) \geq \varsigma(x)$ $\forall x\in[a,b]$.
- $\int_a^b \varsigma(x)\,dx = L(f,P)$.
- $0 \leq \int_a^b (f(x)-\varsigma(x))\,dx = \int_a^b f(x)dx - L(f,P) < \varepsilon/2$

Suppose we prove $\lim_{n\to\infty} \int_a^b \varsigma(x)\sin(nx)dx = 0$ for any step fcn ς,

then $\left|\int_a^b f(x)\sin(nx)dx\right| \leq \left|\int_a^b (f(x)-\varsigma(x))\sin(nx)dx\right| + \left|\int_a^b \varsigma(x)\sin(nx)dx\right|$

$\displaystyle \leq \underbrace{\int_a^b (f(x)-\varsigma(x))dx}_{<\,\varepsilon/2} + \underbrace{\left|\int_a^b \varsigma(x)\sin(nx)dx\right|}_{<\,\varepsilon/2 \text{ for } n>N}$ (we can find such $N>0$)

This concludes the proof.

So it suffices to show:

<u>Claim:</u> $\lim_{n\to\infty} \int_a^b s(x)\sin(nx)\,dx = 0 \quad \forall$ step fcn s.

By the definition of step fcns, it suffices to prove the statement for <u>constant fcns</u>.

$$\lim_{n\to\infty} \int_{a'}^{b'} c\sin(nx)\,dx = \lim_{n\to\infty} c\,\frac{-1}{n}\Big(\underbrace{\cos(nb') - \cos(na')}_{\text{bdd by } 2.}\Big)$$

$\longrightarrow 0$ as $n \to \infty$. □

#6.

(a) Consider $f \equiv 0$ and $g \equiv 1$ on $[0,1]$,

Then
- $d_\infty(f,g) = 1$.
- $Tf = 0$, $(Tg)(x) = x$.
- $d_\infty(Tf, Tg) = 1$.

Hence T is not a contraction. □

(b). Observe that $f \equiv 0$ satisfies $Tf = f$.

Suppose $Tf = f$ for $f \in \mathcal{C}[0,1]$,

By FTC $\Rightarrow$ $f(x) = f'(x) \ \forall x \in [0,1]$.

Define $g(x) = f(x) \cdot e^{-x}$ on $[0,1]$.

Then $f(x) = g(x) \cdot e^{x}$.

$f'(x) = e^{x}(g(x) + g'(x)) \quad \Rightarrow \quad g'(x) \equiv 0$

$\Rightarrow \quad g(x)$ const.

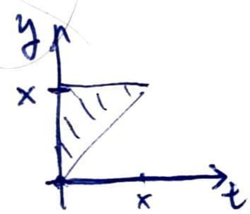
So f is of the form $f = C\cdot e^x$.

$$C = f(0) = (Tf)(0) = \int_0^0 ce^t\,dt = 0.$$

$$\Rightarrow f \equiv 0. \quad \square$$

(c)
$$d_\infty(T^2 f, T^2 g) = \sup_{x\in[0,1]} \left|(T^2 f)(x) - (T^2 g)(x)\right|$$

$$= \sup_{x\in[0,1]} \left| \int_0^x (Tf)(y)\,dy - \int_0^x (Tg)(y)\,dy \right|$$



$$= \sup_{x\in[0,1]} \left| \int_0^x \left(\int_0^y f(t)\,dt\right) dy - \int_0^x \left(\int_0^y g(t)\,dt\right) dy \right|$$

$$= \sup_{x\in[0,1]} \left| \int_0^x \left(\int_0^y (f(t)-g(t))\,dt\right) dy \right|$$

$$\le \sup_{x\in[0,1]} \int_0^x \int_0^y d_\infty(f,g)\,dt\,dy$$

$$= \sup_{x\in[0,1]} \frac{x^2}{2} d_\infty(f,g) = \frac{1}{2} d_\infty(f,g).$$

Hence T^2 is a contraction (with $K = \frac{1}{2}$). $\square$

#7: By Weierstrass approx. thm., $\exists$ polynomials $P_n(x)$

s.t. $P_n \longrightarrow f$ unif. on $[a,b]$. Let $M = \sup_{x\in[a,b]} f(x) < +\infty$

$\forall \varepsilon > 0$, $\exists N > 0$ s.t. $|P_n(x) - f(x)| < \varepsilon/M$ $\forall x \in [a,b]$ $\forall n > N$.

$\Rightarrow |(f\cdot P_n)(x) - (f^2)(x)| < \varepsilon$ $\forall x \in [a,b]$ $\forall n > N$.

$\Rightarrow f\cdot P_n \longrightarrow f^2$ unif. on $[a,b]$.

$\Rightarrow \lim_{n\to\infty} \int_a^b P_n(x) f(x)\,dx = \int_a^b f(x)^2\,dx$

By the assumption, we have $\int_a^b P(x) f(x) = 0 \quad \forall$ poly. P.

Hence $\int_a^b f(x)^2 dx = \lim_{n\to\infty} \int_a^b P_n(x) f(x) = 0$.

Since f is conti., one can show that $f \equiv 0$. (why?) $\square$

#8.

$$0 \le \int_a^b \left(\int_a^b (f(x)g(y) - f(y)g(x))^2 \, dx \right) dy$$

$$= \int_a^b \left(\int_a^b \left(f(x)^2 g(y)^2 + f(y)^2 g(x)^2 - 2 f(x) f(y) g(x) g(y)\right) dx \right) dy$$

$$= \left(\int_a^b g(y)^2 dy\right)\left(\int_a^b f(x)^2 dx\right) + \left(\int_a^b f(y)^2 dy\right)\left(\int_a^b g(x)^2 dx\right)$$

$$- 2 \left(\int_a^b f(x) g(x) dx\right)\left(\int_a^b f(y) g(y) dy\right)$$

$$= 2\left(\int_a^b f^2\right)\left(\int_a^b g^2\right) - 2\left(\int_a^b fg\right)^2 . \quad \square$$

#9: For any $f \in \mathcal{A}$ and any $b \in \mathbb{R}$,

$$(3b+2)^2 = \left(\int_0^1 f(x)(x+b)\,dx\right)^2 \le \left(\int_0^1 f^2\right)\left(\int_0^1 (x+b)^2 dx\right) = \left(\int_0^1 f^2\right)\cdot\left(b^2+b+\tfrac{1}{3}\right)$$

$$\Rightarrow \int_0^1 f^2(x)\,dx \ge \frac{3(3b+2)^2}{3b^2+3b+1} \qquad \forall f \in \mathcal{A},\ b \in \mathbb{R}$$

$$\Rightarrow \int_0^1 f^2(x)\,dx \ge \sup_{b\in\mathbb{R}} \frac{3(3b+2)^2}{3b^2+3b+1} = 12 \qquad \forall f \in \mathcal{A}.$$

not hard to show.

When $f(x) = 6x$, Check that: $f \in \mathcal{A}$ and $\int_0^1 f^2 = 12$

$$\Rightarrow \min_{f\in\mathcal{A}} \int_0^1 f^2 = 12. \quad \square$$