

What's linear algebra?

- In the most concrete form, it studies systems of linear eqns. like:

$$\begin{cases} x_1 + x_2 = 2 \\ -2x_1 + 3x_2 = 1. \end{cases}$$

- More abstractly, it studies "transformations" of "spaces" which carries "lines" to "lines".

e.g. Schrödinger eq'n:

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - V(x,t) \right) F(x,t) = 0.$$

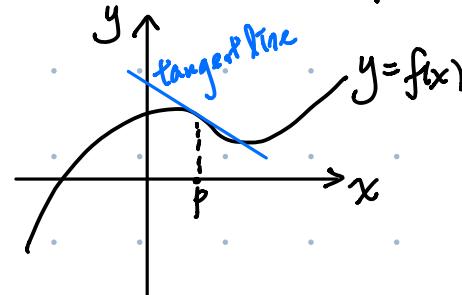
where: "space" of := space of functions $F(x,t)$.

"transformation": $(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - V(x,t)) : \mathcal{F} \rightarrow \mathcal{F}$

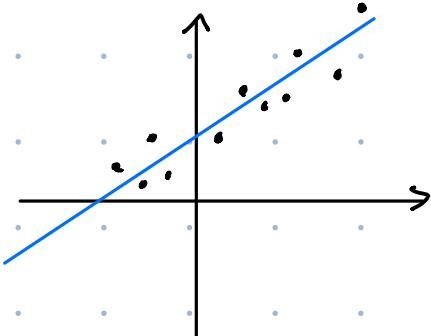
Some applications of linear algebra:

- To understand a map $f: X \rightarrow Y$ at a point $p \in X$, we usually start with studying its "first order approximation", i.e. the "tangent map" $f_p: T_p X \rightarrow T_{f(p)} Y$, which is a linear transformation.

This is the most fundamental thing
in multivariable calculus, differential
geometry,



- Linear approximation



You'll leave this class equipped with a powerful conceptual framework on which the vast majority of math, science, engineering, ... depend.

Warm-up:

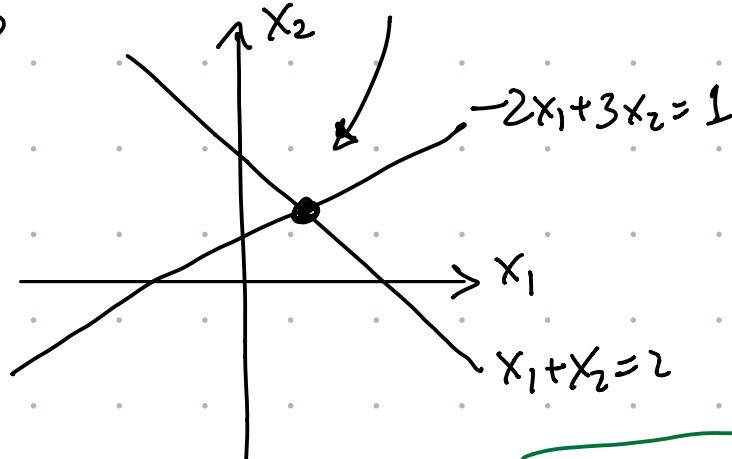
e.g. $\begin{cases} x_1 + x_2 = 2 \\ -2x_1 + 3x_2 = 1. \end{cases}$

Any solutions?

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

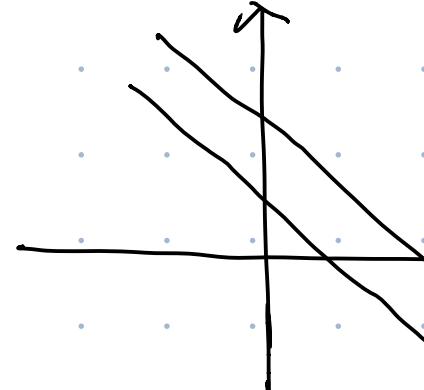
solves the linear system

Geometrically?



e.g. $\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}$

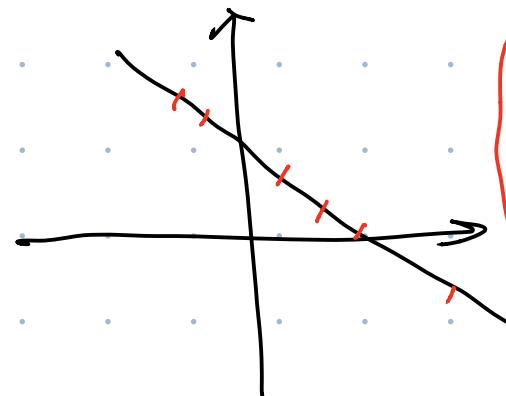
NO sol'n



For a system of linear eqns in 2 variables, the sol'n set is either empty, a pt, a line, whole \mathbb{R}^2

e.g. $\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 2 \\ 3x_1 + 3x_2 = 3 \\ -x_1 - x_2 = -1 \end{cases}$

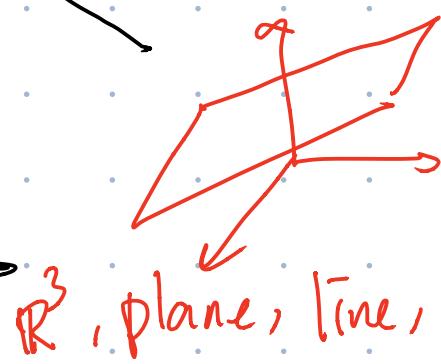
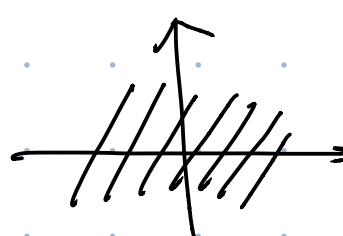
∞ many sol'n's



Q: What about system of linear eqns w/ 3 variables?

e.g. "trivial" linear eqn:

$$\begin{cases} 0x_1 + 0x_2 = 0 \end{cases}$$



\mathbb{R}^3 , plane, line, pt, φ

In HW you'll show that \forall linear system, it can have either
 for any

no solⁿs, a unique solⁿ, or ∞ many solⁿs.

Def System of linear eqns (Linear system) in variables x_1, \dots, x_n :

$$(*) \quad \left\{ \begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m. \end{array} \right.$$

Where each a_{ij}, b_i are all real numbers (\mathbb{R})

Def A vector in \mathbb{R}^n is an ordered list of n real numbers,
written as

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Def We say a vector \vec{v} is a solⁿ of (*) if

$$\left\{ \begin{array}{l} a_{11}v_1 + \dots + a_{1n}v_n = b_1 \\ \vdots \\ a_{m1}v_1 + \dots + a_{mn}v_n = b_m. \end{array} \right.$$

e.g.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & \text{--- } ① \\ 3x_1 - 4x_2 - 5x_3 = 8 & \text{--- } ② \\ 5x_1 - 2x_2 - 13x_3 = 18 & \text{--- } ③ \end{cases}$$

It's more convenient to write down the system without the variables:

1	-2	1	0
3	-4	-5	8
5	-2	-13	18

augmented matrix.

coefficient matrix.

Replace ② with ② - 3 × ①:

$$\begin{aligned} ②: \quad & 3x_1 - 4x_2 - 5x_3 = 8 \\ - 3 \times ①: \quad & \cancel{3x_1 - 6x_2 + 3x_3 = 0} \\ & 2x_2 - 8x_3 = 8 \end{aligned}$$

sol'n set of ② is the same as the sol'n set of

1	-2	1	0
0	2	-8	8
5	-2	-13	18

Replace ③ with ③ - 5 × ①:

$$\begin{aligned} ③: \quad & 5x_1 - 2x_2 - 13x_3 = 18 \\ 5 \times ①: \quad & \cancel{5x_1 - 10x_2 + 5x_3 = 0} \\ & 8x_2 - 18x_3 = 18 \end{aligned}$$

1	-2	1	0
0	2	-8	8
0	8	-18	18

(2')

Replace ②' by ②'/2:

$$2x_2 - 8x_3 = 8$$



$$x_2 - 4x_3 = 4$$

1	-2	1	0
0	1	-4	4
0	8	-18	18

(1'')
(2'')
(3'')

Replace ③^{ll} by ①^{ll} + 2 × ②^{ll}:

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 8 & -18 & 18 \end{array} \right]$$

Replace ③^{ll} by ③^{ll} - 8 × ②^{ll}

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 14 & -14 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

$$\begin{aligned} & -18 - 4 \times (-8) \\ & 18 - 8 \times 4 \quad 8 + 7(-1) \\ & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right] \quad 4 + (-1) \times 4 \end{aligned}$$

$$\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$$

Row reduction algorithm:

(Start w/ a linear system, we do some "row operations" to get a "simpler" linear system, while the solⁿ set is unchanged)

Elementary row operations

- ① Replace a row by the sum of itself and a multiple of another row.
- ② multiply all entries of a row by a nonzero constant.
- ③ Interchange 2 rows

- ① find a left-most nonzero entry
- ② Do ③ to make the nonzero entry to be in the 1st row.

- ③ Do ② to make it = 1.
- ④ Do ① to make the remaining column = 0

The image shows a hand-drawn diagram on dot-grid paper. At the top, there is a sequence of symbols: two circles, followed by a dot, then a circle, an asterisk, another asterisk, and a circle. Below this, there is another row of symbols: three circles, followed by a dot, then two circles, an asterisk, and a circle. To the right of the second row, the number '1' is written vertically. Below these rows, there are several vertical lines and circles connected by arrows, forming a flowchart-like structure. On the far right, there is a large rectangular box with a wavy bottom edge.

pivot positions.

Reduced echelon form.

Case 1: The last column of β is pivot

An augmented matrix with 6 rows and 6 columns. The first two columns are zero vectors. The third column has entries 1, 0, 0, 0, 0, 0. The fourth column has entries 0, 1, 0, 0, 0, 0. The fifth column has entries 0, 0, 1, 0, 0, 0. The sixth column is the right-hand side, with entries 1, 1, 1, 1, 1, 1. The third row is circled in red. The fourth row is circled in blue. The fifth row is circled in blue. The right-hand side is circled in red.

$$\begin{array}{cccc|c} 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$0x_1 + 0x_2 + \dots + 0x_n = 1$
NO sol'n!!

Case 2: The last column is not pivot.

An augmented matrix with 6 rows and 6 columns. The first two columns are zero vectors. The third column has entries 1, 0, 0, 0, 0, 0. The fourth column has entries 0, 1, 0, 0, 0, 0. The fifth column has entries 0, 0, 1, 0, 0, 0. The sixth column is the right-hand side, with entries 1, 1, 1, 1, 1, 1. The first row is circled in red. The second row is circled in blue. The third row is circled in blue. The right-hand side is circled in red.

$$\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \end{array}$$

Q: In which case does the linear system have sol'n??

Q: In case 2, how to find sol'n?

An augmented matrix with 6 rows and 6 columns. The first two columns are zero vectors. The third column has entries 1, 0, 0, 0, 0, 0. The fourth column has entries 0, 1, 0, 0, 0, 0. The fifth column has entries 0, 0, 1, 0, 0, 0. The sixth column is the right-hand side, with entries 8, 9, 10, 0, 0, 0. The first row is circled in green. The second row is circled in green. The third row is circled in green. The right-hand side is circled in green.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

x_1 x_2 x_3 x_4 x_5 x_6

$$\left. \begin{array}{l} x_2 + 3x_4 + 5x_6 = 8 \\ x_3 + 4x_4 + 6x_6 = 9 \\ x_5 + 7x_6 = 10 \end{array} \right\} \quad \begin{array}{l} x_2 = 8 - 3x_4 - 5x_6 \\ x_3 = 9 - 4x_4 - 6x_6 \\ x_5 = 10 - 7x_6 \end{array}$$