HOMEWORK 9
MATH 104, SECTION 2

Some ground rules:

- You have to submit your homework via **Gradescope** to the corresponding assignment. The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You're allowed to use any result that is proved in the lecture; but if you'd like to use other results, you have to prove them before using them.

Problem set (5 problems; due April 13 at 11am PT)

(1) For each of the following power series, find the radius of convergence and determine the exact interval of convergence.

$$(a)\ \sum \left(\frac{x}{n}\right)^n; \quad (b)\ \sum \left(\frac{(-1)^n}{n^2 \cdot 4^n}\right) x^n; \quad (c)\ \sum \left(\frac{(-1)^n}{n \cdot 4^n}\right) x^n; \quad (d)\ \sum x^{n!}.$$

(2) Suppose $\sum a_n x^n$ has finite radius of convergence $R > 0$ and $a_n \geq 0$ for all n. Prove that if the series converges at R, then it also converges at $-R$.

(3) Consider a power series $\sum a_n x^n$ with radius of convergence R. Prove that if $\limsup |a_n| > 0$, then $R \leq 1$.

(4) By mimicking what we discussed in class, prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

(Hint: First, we have $\sum (-1)^n x^{2n} = \frac{1}{1+x^2}$ for all $|x| < 1$.)

(5) Let (a_n) and (b_n) be two sequences of real numbers satisfying:

- The partial sums of (b_n) is bounded: there exists $L > 0$ such that $|b_1 + \cdots + b_k| < L$ for any k,
- $\lim a_n = 0$,
- $\sum |a_n - a_{n+1}|$ converges.

Prove that for any $k \in \mathbb{N}$, the series $\sum a_n^k b_n$ is convergent. (Hint: Same idea as the proof of Abel's theorem.)