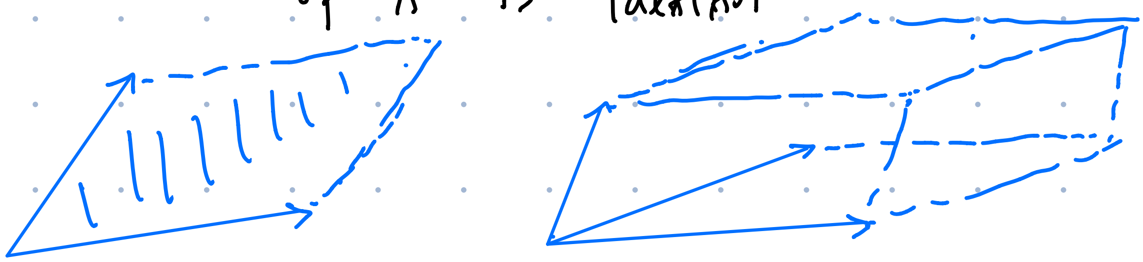
Last time:

- $\det(A) \neq 0 \iff A$ invertible.
- $\det(AB) = \det(A)\det(B)$.

Thm: Suppose A invertible.

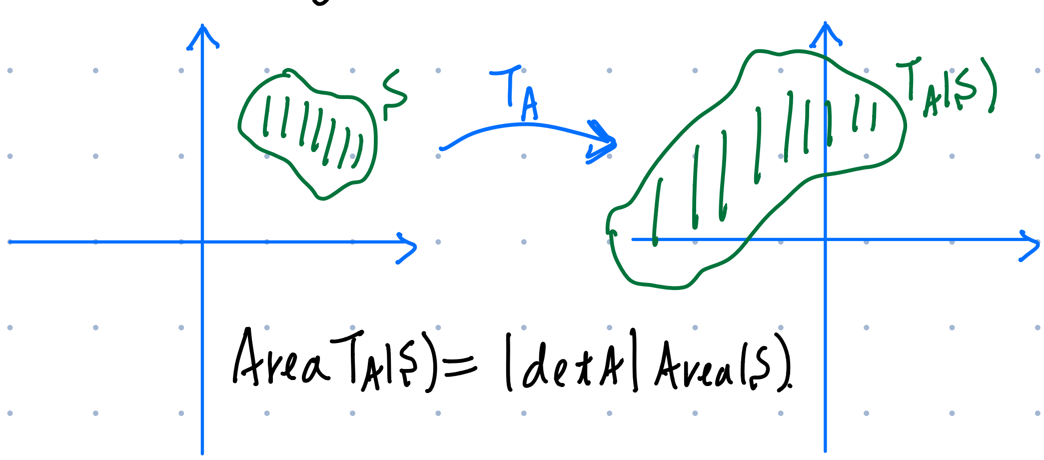
- A: 2×2, the area of the parallelogram spanned by the <u>rows</u> of A is $|\det(A)|$.
- A: 3×3 the volume of the parallelepiped spanned by the <u>rows</u> of A is $|\det(A)|$



Rmk: The same statement is true for <u>columns</u>, since $\det(A) = \det(A^T)$.

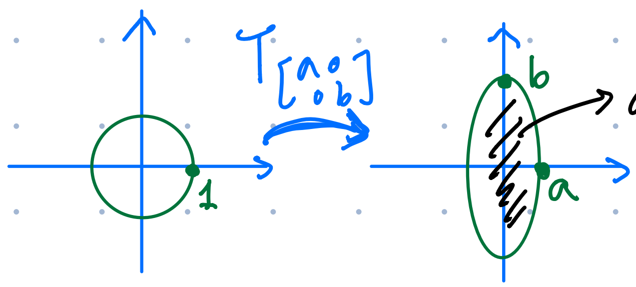
Thm: Suppose A: 2×2 invertible. $T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\vec{x} \longmapsto A\vec{x}$.

Then rate of change of area under T_A is $|\det(A)|$.



$$\text{Area } T_A(S) = |\det A| \text{ Area}(S)$$

e.g.:

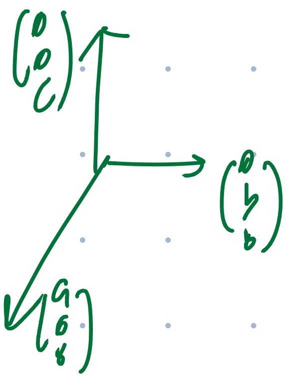


$$\text{area} = \left|\det\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right| \cdot \pi = ab\pi.$$

pf

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{3\times 3} \xrightarrow{\text{row operations}} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$\begin{cases} \text{vol} = |abc| \\ \det = abc \end{cases}$ $\square$



invertible

↓

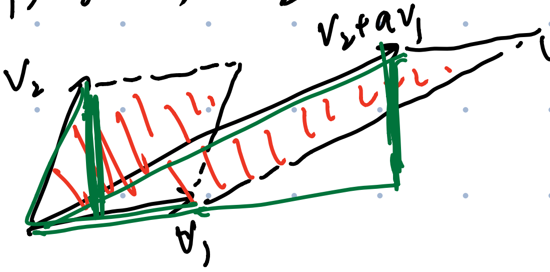
row operations to reduce to I_3

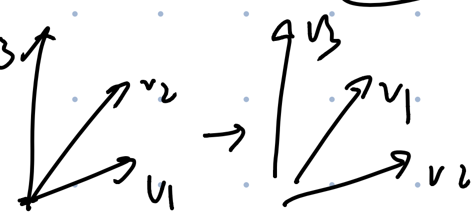
$\{v_1, v_2\}$

↓

$\{v_1, v_2 + av_1\}$

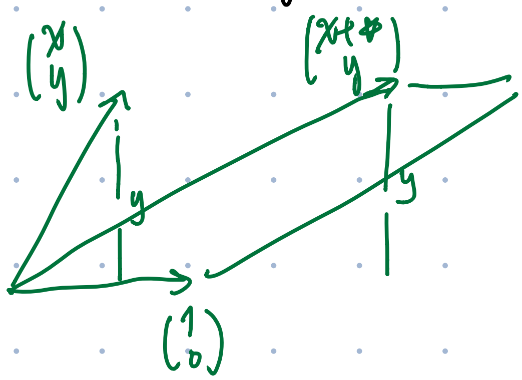
• $\begin{bmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{bmatrix}$, • $\begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{bmatrix}$ • $\cancel{\begin{bmatrix} c & & \\ & 1 & \\ & & 1 \end{bmatrix}}$





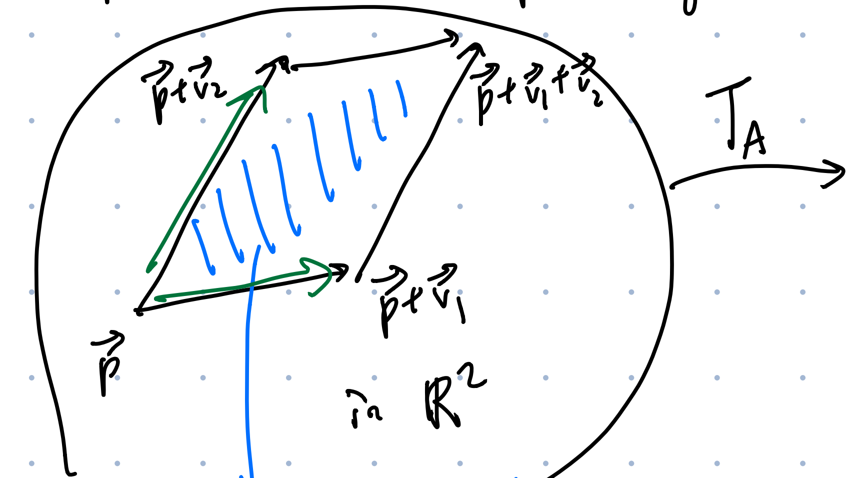
Claim: These two types of row operat's don't change the volume of the parallelepiped spanned by the rows

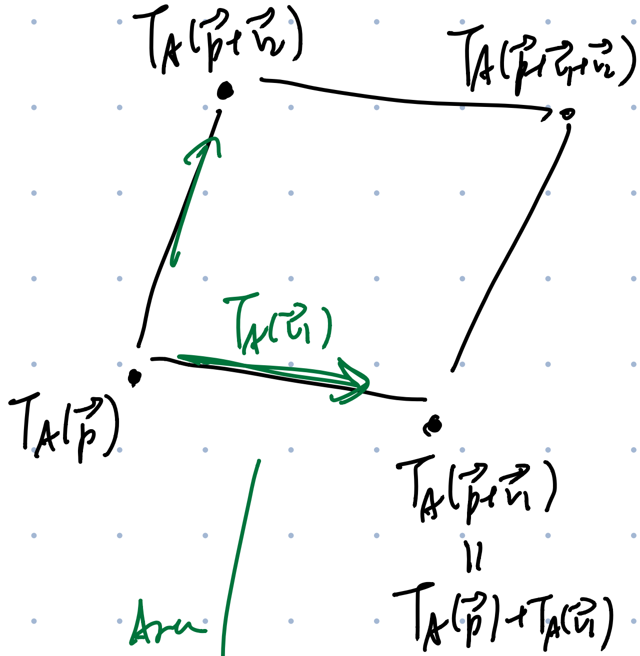
Claim: These two types of row operation also don't change $|\det(A)|$



Sketch of pf.

① prove it for all parallelograms.





$\{\vec{p}+s_1\vec{v}_1+s_2\vec{v}_2 \mid 0\le s_1,s_2\le 1\}$

Area $\downarrow$ $\qquad\qquad$ Area $\downarrow$

$$\left|\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\right| \qquad\qquad \left|\det\begin{bmatrix}T_A(\vec{v}_1) & T_A(\vec{v}_2)\end{bmatrix}\right|$$

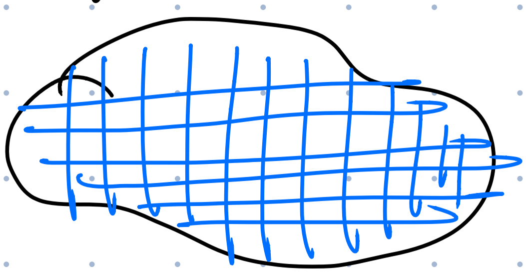
$$\updownarrow \qquad\qquad\qquad \|$$

$$\left|\det\begin{bmatrix}A\vec{v}_1 & A\vec{v}_2\end{bmatrix}\right|$$

$$\|$$

$$|\det(A)|\cdot\left|\det\left(\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\right)\right| = \left|\det\left(A\cdot\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\right)\right|$$

② for general region in $\mathbb{R}^2$



take a limit

Remark:

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$$\det\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2}\end{pmatrix}$$

Cramer's rule (useful for theoretic purpose)

A: $n\times n$ invertible. $\forall \vec{b}\in\mathbb{R}^n$, the sol of $A\vec{x}=\vec{b}$.

has entries:
$$x_i = \frac{\det A_i(\vec{b})}{\det A}.$$

where
$$A_i(\vec{b}) = \begin{array}{c} \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{i-1} & \vec{b} & \vec{a}_{i+1} & \cdots & \vec{a}_n \end{bmatrix} \\ \text{i-th} \end{array}$$

pf
$$\underline{I_i(\vec{x})} = \begin{array}{c} \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \vec{x} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \\ \text{i-th} \end{array} \qquad \underline{\det I_i(x) = x_i}$$

$$A\cdot I_i(\vec{x}) = A \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 \;\; \vec{x} & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{i-1} & \underset{\substack{\| \\ A\vec{x}}}{\vec{b}} & \vec{a}_{i+1} & \cdots & \vec{a}_n \end{bmatrix} = A_i(\vec{b})$$

$$\vec{a}_1 = A\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \det A_i(\vec{b}) = \det\left(A\, I_i(\vec{x})\right)$$
$$= \det(A)\,\det(I_i(\vec{x}))$$
$$= \det(A)\cdot x_i$$

Thm (Inverse formula)

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots \\ C_{12} & & & \\ \vdots & & & \end{pmatrix}$$

cofactors

$C_{ij} = (-1)^{i+j} \det A_{ij}$

pf $AA^{-1} = I = \begin{bmatrix} \vec{e}_1 \cdots \vec{e}_n \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Cramer's rule $\Rightarrow$ the (i,j)-th entry of A^{-1} is:

$$\frac{\det(A_i(\vec{e}_j))}{\det(A)}$$

$\|$ i-th comp. of $\vec{x}$

$$A \begin{bmatrix} & \vec{x} & \end{bmatrix} = I_n = \begin{bmatrix} \vec{e}_1 \cdots \vec{e}_j \cdots \vec{e}_n \end{bmatrix}$$

A^{-1}, j-th

$A\vec{x} = \vec{e}_j$

$$\det(A_i(\vec{e}_j)) = \det \begin{bmatrix} \vec{a}_1 \cdots \vec{a}_{i-1} & e_j & \vec{a}_{i+1} \cdots \vec{a}_n \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{11} & \cdots & a_{1,i-1} & 0 & a_{1,i+1} \cdots a_{1,n} \\ a_{21} & & & \vdots & \\ & & & 0 & \end{bmatrix}$$

$$j \quad \begin{pmatrix} \vdots & & \vdots & 0 & & \vdots \\ a_{n1} & & a_{n,i-1} & 0 & \cdots & a_{nn} \end{pmatrix}$$

$$= (-1)^{i+j} \det \underline{A_{ji}}$$

$$= C_{ji}$$

$\Rightarrow$ (i,j) entry of A^{-1} is $\dfrac{C_{ji}}{\det A}$

$$\Rightarrow \quad A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & \cdots & \\ C_{12} & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & \cdots & C_{nn} \end{pmatrix}. \quad \square$$

§ Vector spaces.

Def: A <u>vector space</u> is a set V of objects (called "<u>vectors</u>") with 2 operations: <u>addition</u> and <u>scalar multiplication</u> (by $\mathbb{R}$), s.t.

1) closed under operations:

   $\forall \vec{u}, \vec{v} \in V$, $c \in \mathbb{R}$, we have $\vec{u}+\vec{v} \in V$ and $c\vec{u} \in V$.

2) commutative & associativity of addition.

   $\vec{u}+\vec{v} = \vec{v}+\vec{u}$, $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$.

3) $\exists$ identity element of addition.

   $\exists\, \vec{0} \in V$ s.t. $\vec{v}+\vec{0} = \vec{v}$ $\forall \vec{v} \in V$.

4) $\exists$ additive inverse.

   $\forall \vec{v} \in V$ $\exists\, \vec{w} \in V$ s.t. $\vec{v}+\vec{w} = \vec{0}$

5) $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$, $(c+d)\vec{v} = c\vec{v}+d\vec{v}$, $c(d\vec{v}) = (cd)\vec{v}$, $1\vec{v} = \vec{v}$.

<u>Example of vector spaces?</u>

- $\mathbb{R}^n$. w/ standard vector addition & scalar multiplicat^n.
- $Poly_{\le n} = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0, \ldots, a_n \in \mathbb{R}\}$.
- S: set, $\{f: S \to \mathbb{R}\}$ $\quad f_1, f_2 : S \to \mathbb{R}$

  $$f_1 + f_2 : S \to \mathbb{R}$$
  $$x \mapsto f_1(x) + f_2(x)$$

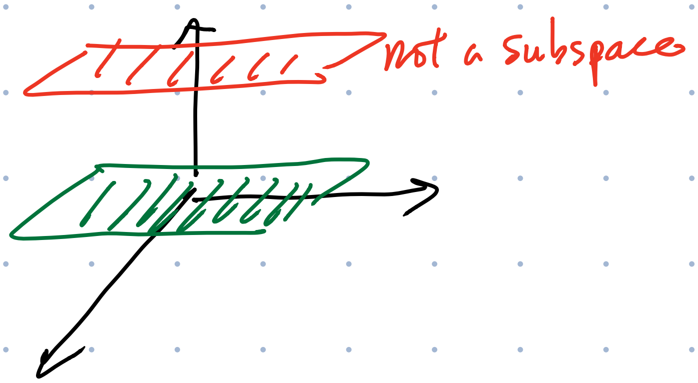
- $\mathcal{C}(\mathbb{R}) = \{\text{continuous fcn } \mathbb{R} \to \mathbb{R}\}$, $\mathcal{C}([0,1]) = \{\text{conti. fcn } [0,1] \to \mathbb{R}\}$
- $M_{m \times n}(\mathbb{R}) = \{\text{matrices of size } m \times n\}$
- $Poly = \{\text{poly.}\}$.

Def A *subspace* of a vector space V is a subset $H \subseteq V$ s.t.

1) $\vec{0} \in H$
2) $\forall \vec{u}, \vec{v} \in H, c \in \mathbb{R}, \quad \vec{u}+\vec{v} \in H, \ c\vec{u} \in H.$

Ex: A subspace of a v.s. is a v.s. itself.

e.g. $\mathbb{R}^3$ What are some possible subspaces?



e.g. $Poly_{\leq n} \subseteq Poly.$

e.g. $\{\vec{0}\} \subseteq V, \qquad V \subseteq V$

Def $\vec{v}_1, \dots, \vec{v}_k \in V$. v.s.

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\} \subseteq V$$

Ex $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is a subspace e.g. $A : m\times n$ $\quad A = [\vec{a}_1 \cdots \vec{a}_n]$

null space: $\quad \mathrm{Nul}(A) := \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \} \subseteq \mathbb{R}^n$

column space

$\mathrm{Col}(A) := \mathrm{Span}\{ \vec{a}_1, \dots, \vec{a}_n \} \subseteq \mathbb{R}^m$

Ex: Null space, Col. space are subspaces in $\mathbb{R}^n$, $\mathbb{R}^m$.

Note: $\mathrm{Nul}(A) = \{0\} \iff T_A$ inj.

$\mathrm{Col}(A) = \mathbb{R}^m \iff T_A$ surj.

e.g.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \qquad A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

$$\mathrm{Nul}(A) = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \}$$

$$= \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Def A fun $T: V \to W$ b/w vector spaces is called <u>linear transformation</u> if

- $T(\vec{v_1}) + T(\vec{v_2}) = T(\vec{v_1} + \vec{v_2})$
- $T(c\vec{v}) = cT(\vec{v})$.

Rmk In general, V, W can be "∞-dim'l",
so we can't represent T as a matrix

e.g. $T: \text{Poly} \longrightarrow \text{Poly}$.

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \longmapsto a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}.$$

Def $T: V \to W$

<u>Kernel</u>/<u>null space</u> of T:

$$\ker(T) := \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\} \subseteq V$$

<u>range</u>/<u>image</u> of T:

$$T(V) = \operatorname{Im}(T) := \{\vec{w} \in W \mid T(\vec{v}) = \vec{w} \text{ for some } \vec{v} \in V\} \subseteq W$$

Ex: These are subspaces in V or W.

(HW3 Q7: a more general statement)