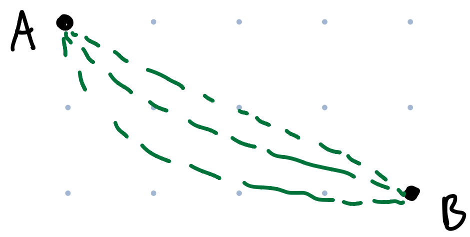
① Preview of differential eq^{ns}: Brachistochrone problem.



Find the path along which an object would slide (w/o friction) in the shortest possible time.

② Conway's well, lake, river, and weir.

Given $Q(x,y) = ax^2 + bxy + cy^2$, where $a, b, c \in \mathbb{Z}$.
For what $n \in \mathbb{Z}$ does there exist $(x,y) \in \mathbb{Z}$ st. $Q(x,y) = n$?

- t variable,
- $y = y(t)$, $\quad y'(t) = \frac{d}{dt} y(t)$, $\quad y''(t) = \frac{d}{dt}\left(\frac{d}{dt} y(t)\right), \ldots$

An ordinary diff'l eq'n (ODE) is of the form

$$F(t, y, y', \cdots y^{(n)}) = 0$$

e.g. $\quad y''(t) - t = 0$, $\quad \underline{y(t) = \frac{1}{6}t^3 + at + b}$, $a, b \in \mathbb{R}$

e.g. $\quad \boxed{y'(t) - y(t) = 0}$ $\quad y(t) = a e^t$, $a \in \mathbb{R}$

- $t_1, \cdots, t_n$ variables
- $\boxed{y = y(t_1, \ldots, t_n)}$

$$\frac{\partial}{\partial t_1} y(t_1^{(0)}, t_2^{(0)}) \qquad (t_1^{(0)}, t_2^{(0)})$$

$$\frac{\partial}{\partial t_i} y(t_1^{(0)}, \cdots, t_n^{(0)}) := \lim_{h \to 0} \frac{y(t_1^{(0)}, \ldots, t_i^{(0)} + h, \ldots, t_n^{(0)}) - y(t_1^{(0)}, \ldots, t_n^{(0)})}{h}$$

Partial differential eq'n:

$$F\left(t_1, \ldots, t_n, \frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_n}, \left(\frac{\partial}{\partial t_1}\right)^2, \cdots \right) = 0$$

$$\boxed{y: [0,a] \longrightarrow [0,b]}$$

$y(0) = 0, \quad y(a) = -b$

Derive formula for the traveling time from A to B along $(x, -y(x))$

Conservation of energy: $\underline{\frac{1}{2} m v(x)^2 - m g y(x) = 0}$

$$\Rightarrow \quad v(x) = \sqrt{2 g y(x)} \quad \forall x \in [0,a]$$

Let $s(x)$ = length of the path from $(0,0)$ to $(x, -y(x))$

Then the traveling time

$$T = \int dt = \int \frac{ds}{v} = \int \frac{\sqrt{(dx)^2 + (dy)^2}}{\sqrt{2 g y(x)}}$$

$$= \sqrt{(dx)^2 + \left(\frac{dy}{dx}\right)^2 (dx)^2}$$

dx  ds dy

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \frac{1}{\sqrt{2g}} \int_0^a \frac{\sqrt{1 + y'(x)^2}}{\sqrt{y(x)}}\, dx$$

Goal: Find y s.t. minimize

Space of fcns $\longrightarrow \mathbb{R}$

$$y \longmapsto \int_0^a \frac{\sqrt{1+y'(x)^2}}{\sqrt{y(x)}}\,dx$$

<u>Thm</u> (Euler-Lagrangian eq^{n})

If y^* is a minimizer of

$$F(y)=\int_0^a f(x, y(x), y'(x))\,dx$$

then y^* satisfies the E-L eq^{n}:

$$\frac{\partial}{\partial y} f(x,y,y') = \frac{d}{dx}\frac{\partial}{\partial y'} f(x,y,y')$$

<u>Shortest distance:</u>

$$L = \int ds = \int\sqrt{(dx)^2+(dy)^2}$$

$$= \int_0^a \sqrt{1+y'(x)^2}\,dx$$

To find the path w/ shortest distance,

$\iff$ find the minimizer of

$\longrightarrow \mathbb{R}$ $\quad f=\sqrt{1+(y')^2}$

$$y \longmapsto \int_0^a \sqrt{1+y'(x)^2}\,dx$$

By E-L, the minimizer should satisfy:

$$\frac{\partial}{\partial y}\left(\sqrt{1+(y')^2}\right) = \underbrace{\frac{d}{dx}\boxed{\frac{\partial}{\partial y'}\sqrt{1+(y')^2}}}_{=}$$

$$\frac{\partial}{\partial y}\left(\sqrt{1+(y')^2}\right) = 0$$

$$\frac{d}{dx}\,\frac{\cancel{2}\,y'(x)}{\cancel{2}\sqrt{1+(y'(x))^2}}$$

$$= \frac{y''(x)}{\left(1+(y')^2\right)^{3/2}}$$

$$\Rightarrow \quad y''(x) = 0$$

$\Rightarrow$ $y(x)$ is a linear fun

In our case, $f = \sqrt{\dfrac{1+(y')^2}{y}}$

$$\frac{\partial f}{\partial y} \overset{E\text{-}L}{=} \frac{d}{dx}\frac{\partial}{\partial y'} f$$

$$\frac{\partial f}{\partial y} = -\frac{\sqrt{1+(y')^2}}{2y^{3/2}}$$

$$\frac{d}{dx}\frac{\partial}{\partial y'} f = \frac{1}{\sqrt{y(1+(y')^2)}}\left(\frac{y''}{1+(y')^2} - \frac{(y')^2}{2y}\right)$$

$$\Rightarrow \boxed{\underline{2yy'' + (y')^2 + 1} = 0}$$

2nd order ODE in y

$$\Rightarrow \quad 0 = \underbrace{2yy'y'' + (y')^3}_{\frac{d}{dx}(y(y')^2)} + y'$$

$$= \frac{d}{dx}\left(\underline{y(y')^2 + y} \right)$$

$$\Rightarrow \quad y(y')^2 + y = C \quad \text{const.}$$

$$\Rightarrow \quad y' = \sqrt{\frac{C-y}{y}}$$

$$\Rightarrow \quad \sqrt{\frac{C-y}{y}} = \frac{dy}{dx}$$

$$\Rightarrow \quad \underline{dx} = \underline{\sqrt{\frac{y}{C-y}}\, dy}$$

$$\Rightarrow \quad \underline{x} = \int dx = \boxed{\int \sqrt{\frac{y}{C-y}}\, dy} + D$$

$$\underline{y = C\sin^2 t}$$

$$= \underline{2C\left(\frac{t}{2} - \frac{1}{4}\sin(2t)\right) + \overset{\text{const.}}{*}}$$

$$\Rightarrow \quad \begin{cases} \underline{x}(t) = Ct - \frac{C}{2}\sin(2t) + D \\ y(t) = \underline{C\sin^2 t} \end{cases}$$

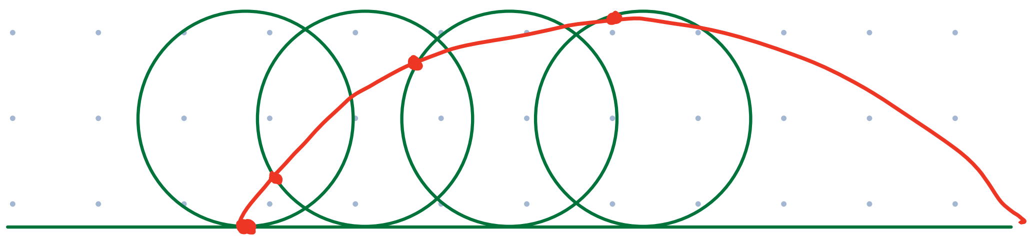
Plug in $t=0 \Rightarrow D=0$.

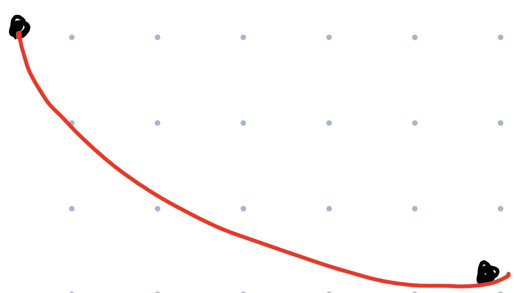
cycloid

the path can be parametrized by:

$$t \longmapsto \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C \begin{bmatrix} t - \frac{1}{2}\sin(2t) \\ \frac{1}{2} - \frac{1}{2}\cos(2t) \end{bmatrix}$$

that passes through the point (a, b)





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e.g. given $n \in \mathbb{Z}$, whether $\exists\ x, y \in \mathbb{Z}$

s.t. $n = x^2 + y^2 = (x+iy)(x-iy)$

Gauss $\boxed{\mathbb{Z}[i] = \{a+ib \mid a, b \in \mathbb{Z}\}}$ UFD

$$n = 2^{a}\underbrace{p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}}_{\text{mod } 4 \equiv 1}\ \underbrace{q_1^{b_1} \cdots q_n^{b_n}}_{\text{mod } 4 \equiv 3}$$

$a, b_1, \ldots, b_n$ even

Q: Given $Q\begin{pmatrix}x\\y\end{pmatrix} = ax^2+bxy+cy^2$, $a,b,c\in\mathbb{Z}$

$(x,y) = \vec{v} \in \mathbb{Z}^2$

- $Q(k\vec{v}) = k^2 Q(\vec{v})$

  $\Rightarrow$ we only need to know the values of Q for primitive integral vector $\vec{v}$.

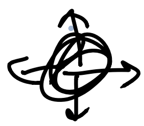
- $Q(\vec{v}) = Q(-\vec{v})$ $\vec{v} \leftrightarrow -\vec{v}$

Conway's topography of bases & superbases of $\mathbb{Z}^2$:

- say $\{\vec{f_1}, \vec{f_2}\} \subseteq \mathbb{Z}^2$ is an (integral) basis of $\mathbb{Z}^2$ if $\forall \vec{v} \in \mathbb{Z}^2$, $\exists k_1, k_2 \in \mathbb{Z}$ s.t. $\vec{v} = k_1\vec{f_1} + k_2\vec{f_2}$

  e.g. $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ ✓

  $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\right\}$ ✗ ← $\begin{bmatrix}0\\1\end{bmatrix}$ ✗

- $\{\pm\vec{f_1}, \pm\vec{f_2}\}$ 

  $\left\{\pm\begin{bmatrix}1\\0\end{bmatrix}, \pm\begin{bmatrix}0\\1\end{bmatrix}\right\}$

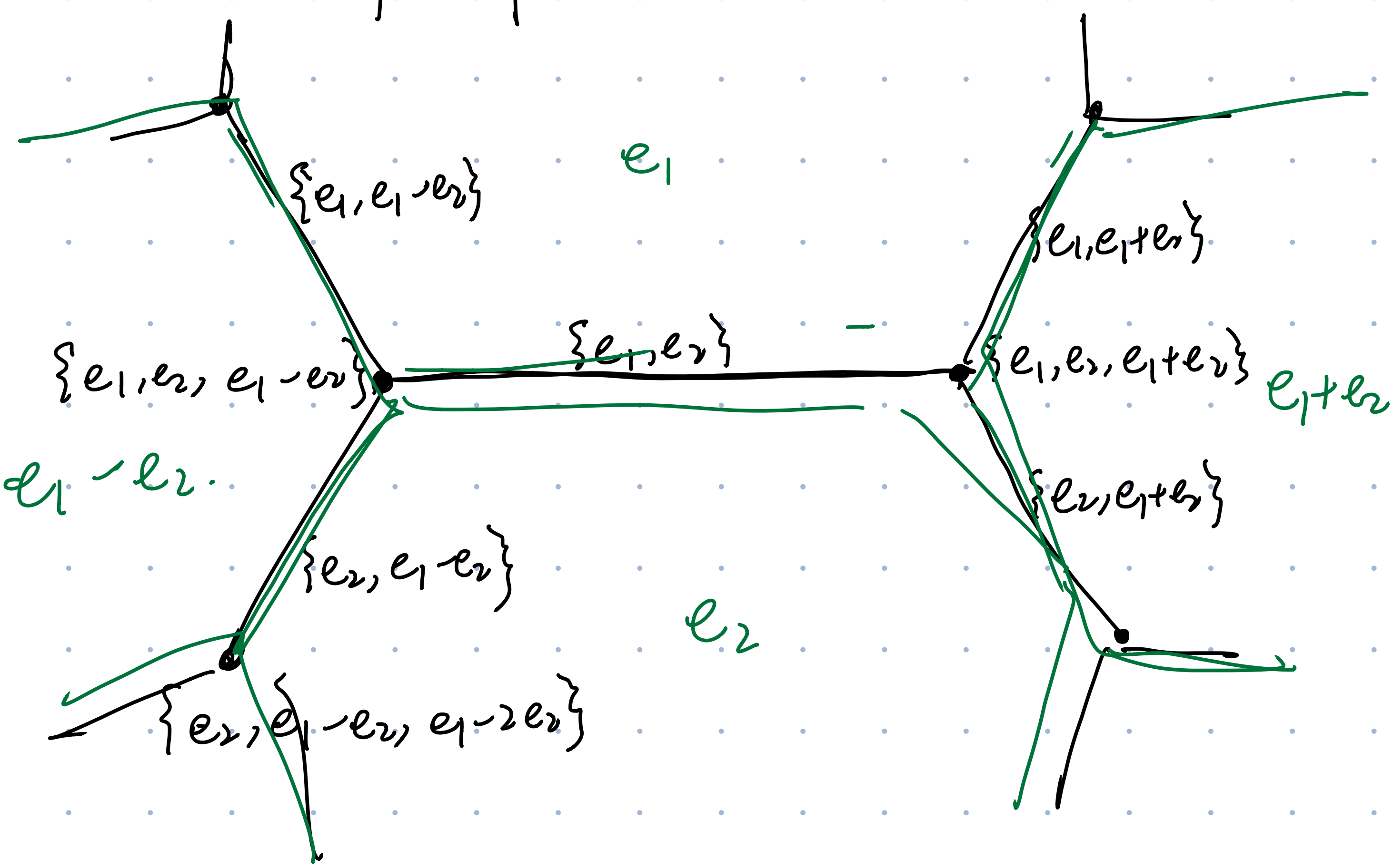
We say $\{\pm\vec{f_1}, \pm\vec{f_2}, \pm\vec{f_3}\}$ is a superbasis if $\{\vec{f_1}, \vec{f_2}\}$ is a basis and $\pm\vec{f_1} \pm \vec{f_2} \pm \vec{f_3} = \vec{0}$

e.g. $\left\{\pm\begin{bmatrix}1\\0\end{bmatrix}, \pm\begin{bmatrix}0\\1\end{bmatrix}, \pm\begin{bmatrix}1\\1\end{bmatrix}\right\}$ is a superbasis

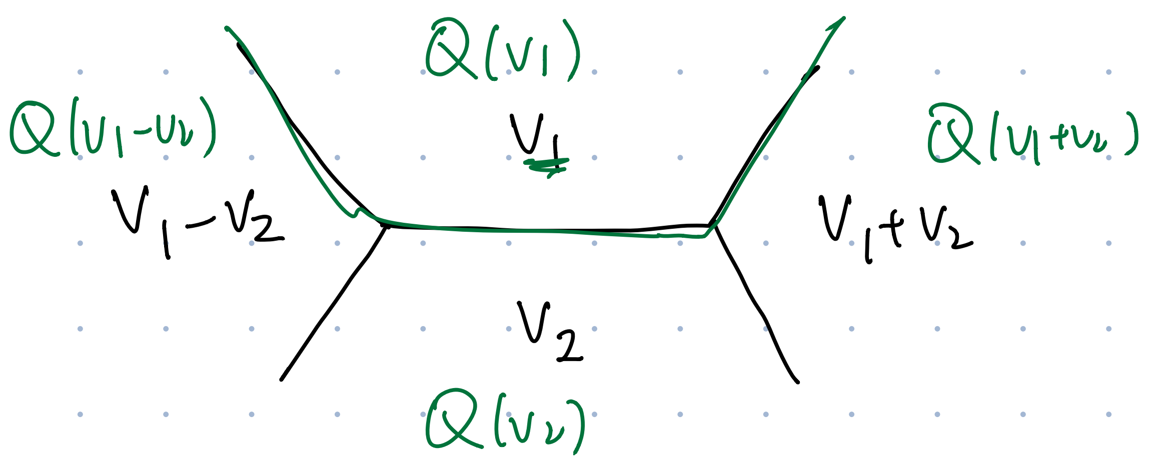
Facts: • any $\{\vec{f_1}, \vec{f_2}\}$ basis belongs to two superbasis.

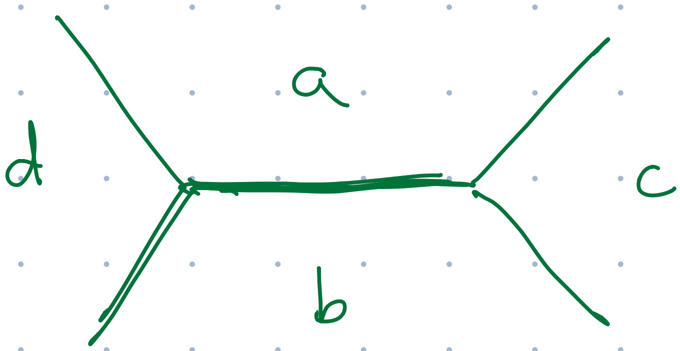
$$\{\vec{f_1}, \vec{f_2}, \vec{f_1}+\vec{f_2}\}, \quad \{f_1, f_2, f_1-f_2\}$$

• any superbasis contains 3 basis.



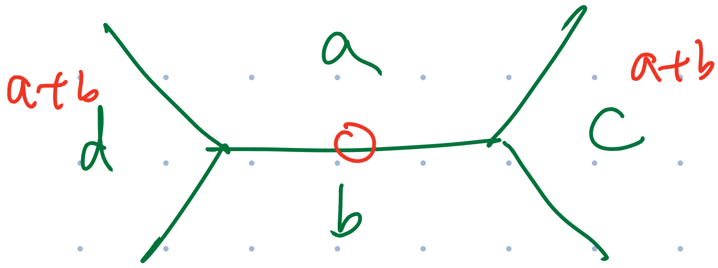
Lemma $Q(v_1+v_2)+Q(v_1-v_2)=2\big(Q(v_1)+Q(v_2)\big)$





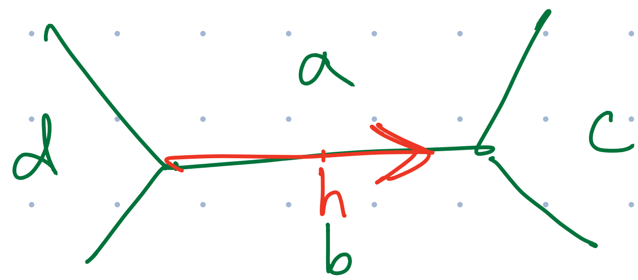
$$c+d = 2(a+b)$$

① If $c = d = a+b$



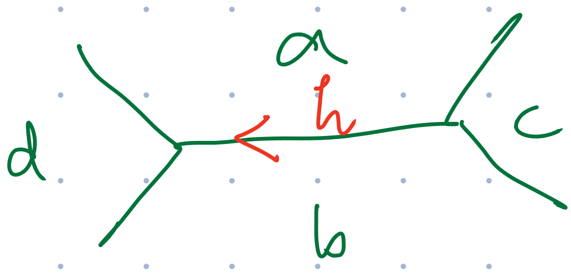
② If $c > d$

$$c = a+b+h$$
$$d = a+b-h$$

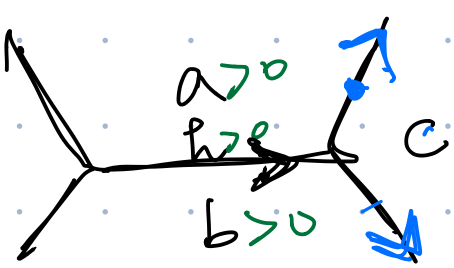


③ If $c < d$

$$c = a+b-h$$
$$d = a+b+h$$

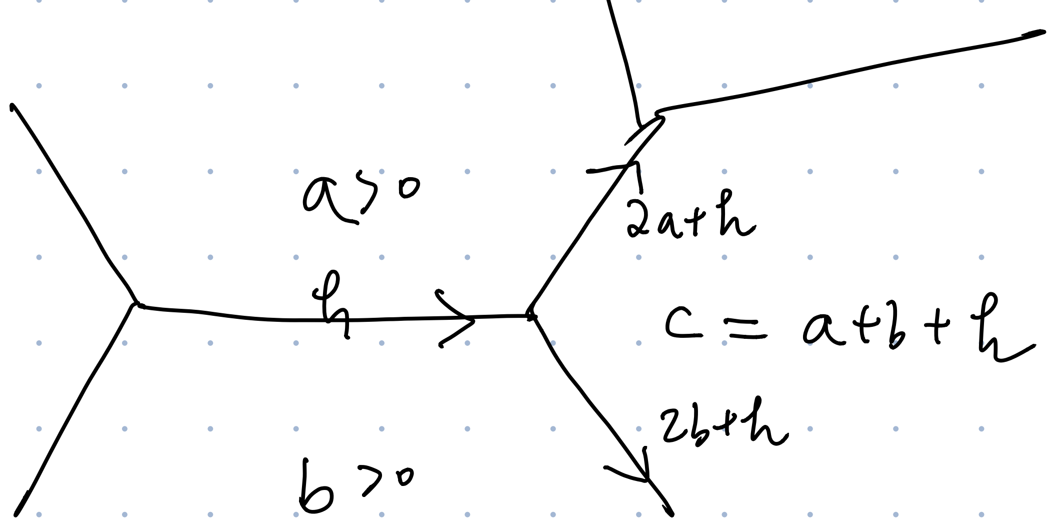


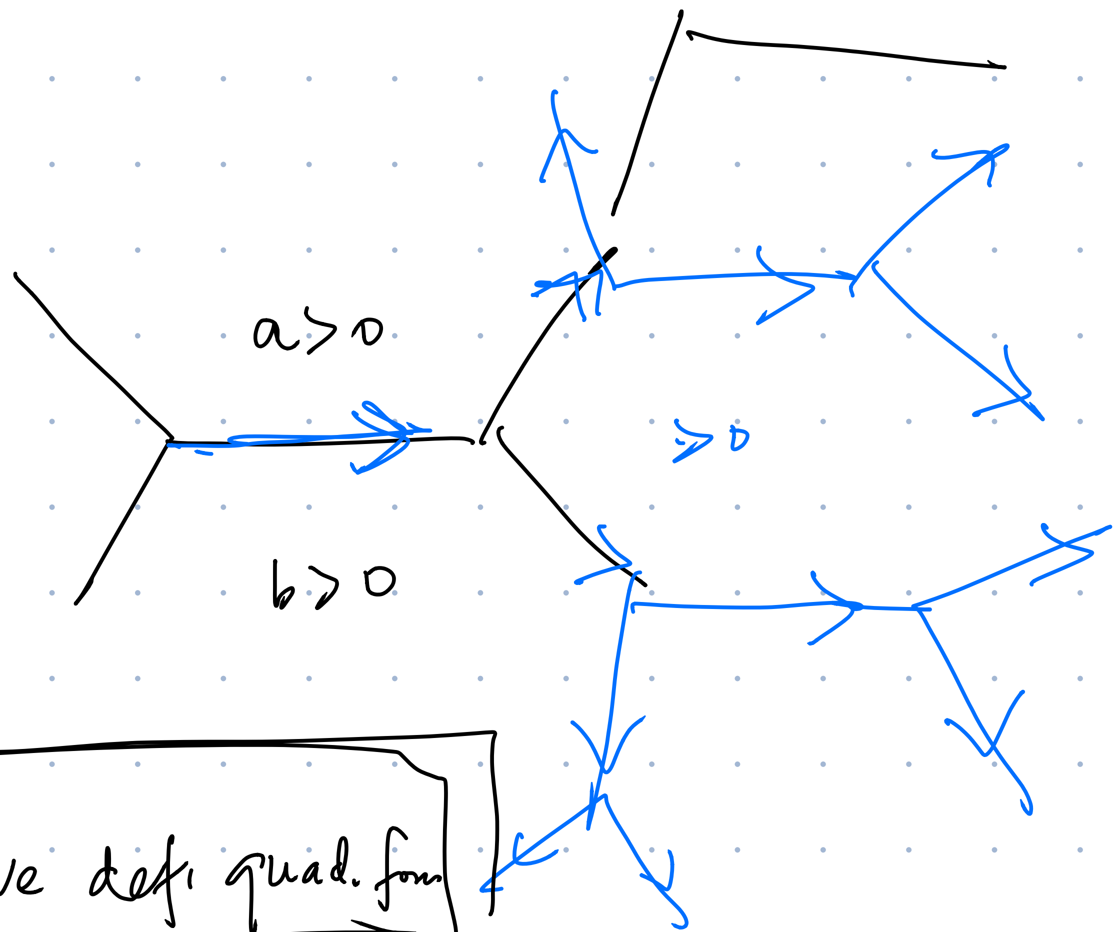
Lemma (Climbing Lemma)



$$b+x = 2(a+c)$$
$$= 2(2a+b+h)$$

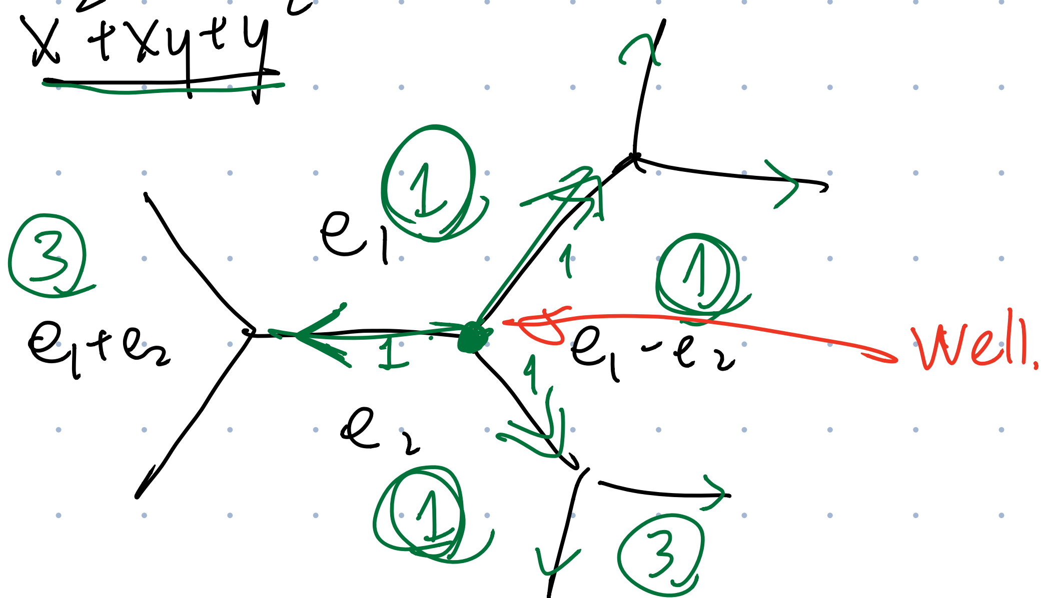
x: $\boxed{x = 4a+b+2h}$



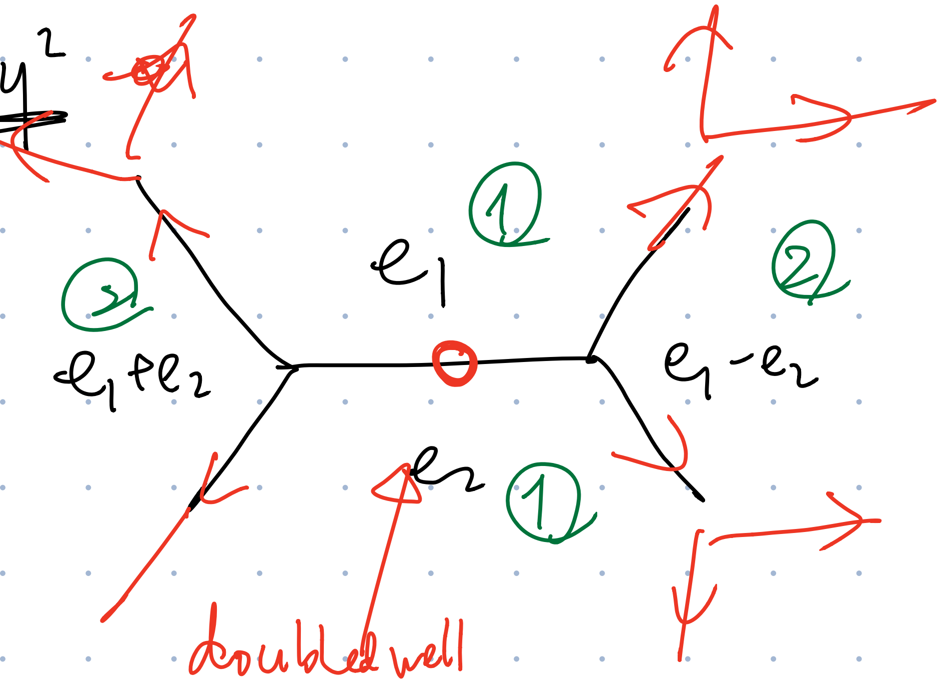


Positive defi quad. form

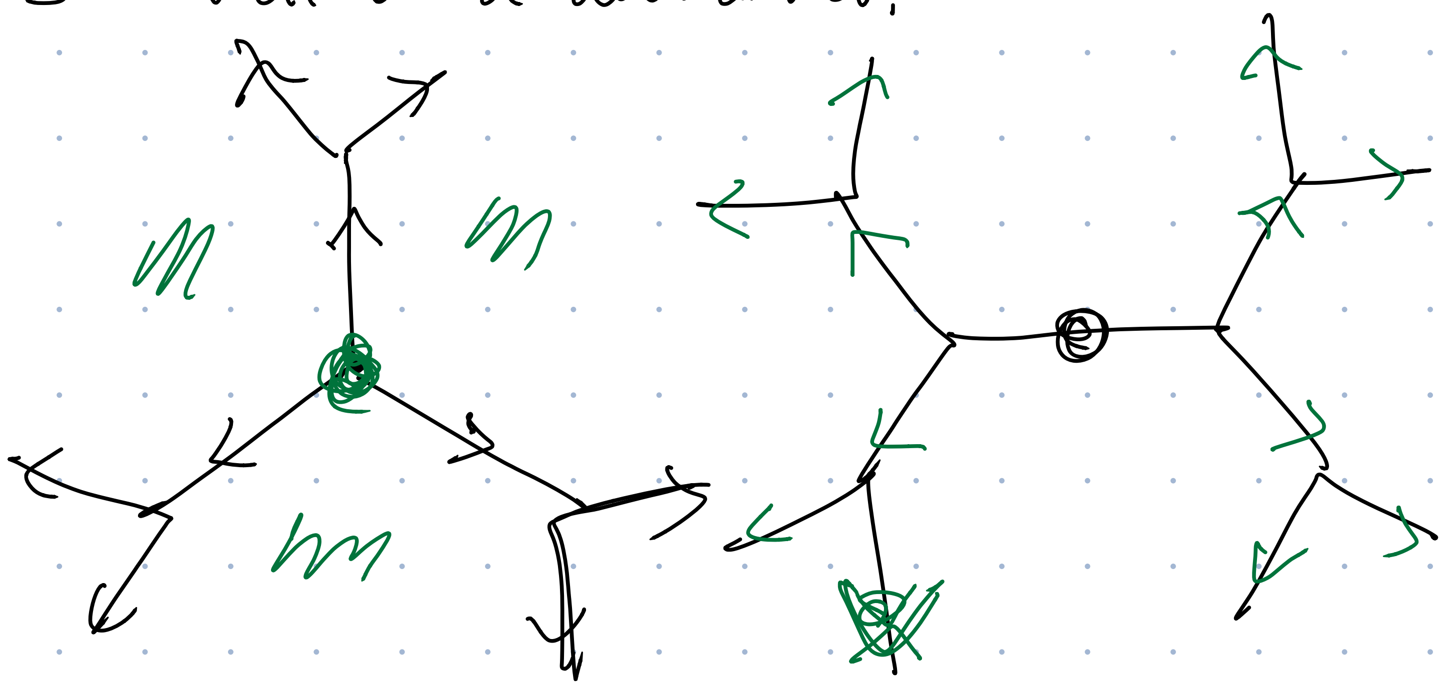
e.g. x^2+xy+y^2



e.g. x^2+y^2



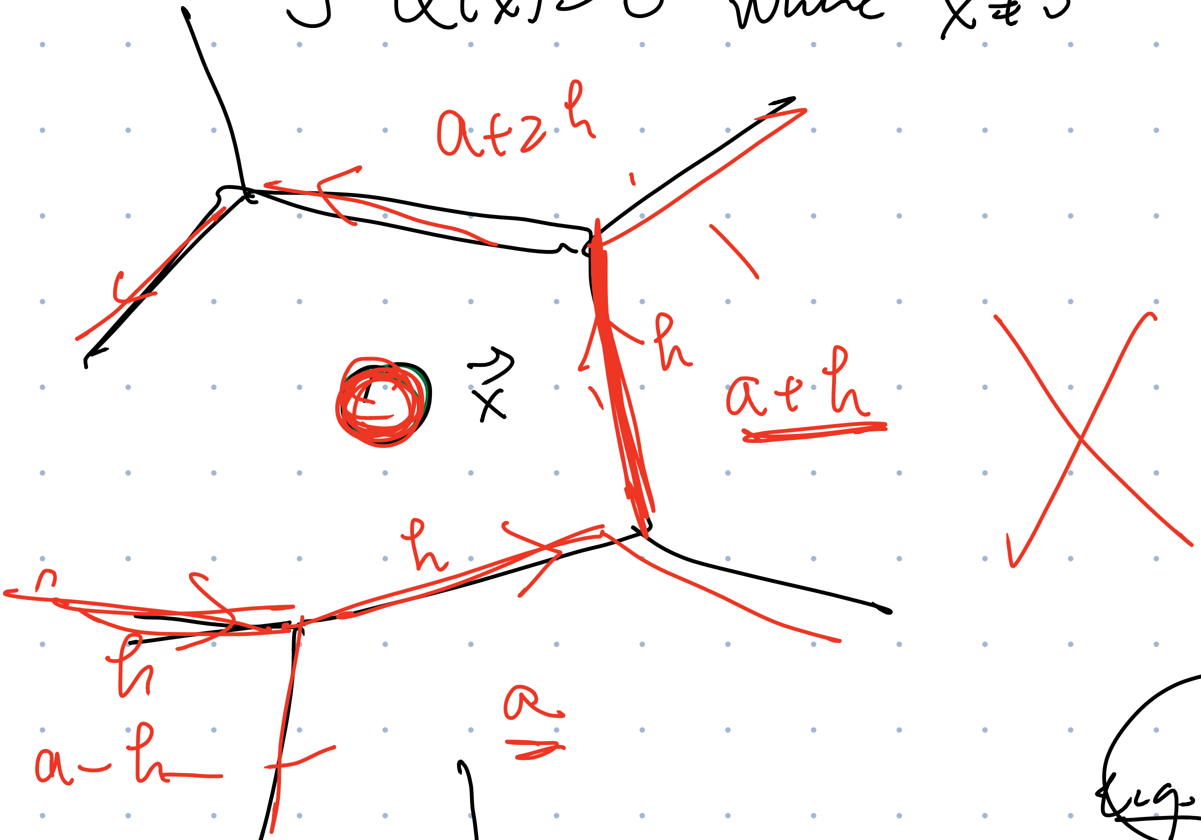
One can prove: $\forall$ positive defi. quad. form. $\exists$ a well or a doubled well.



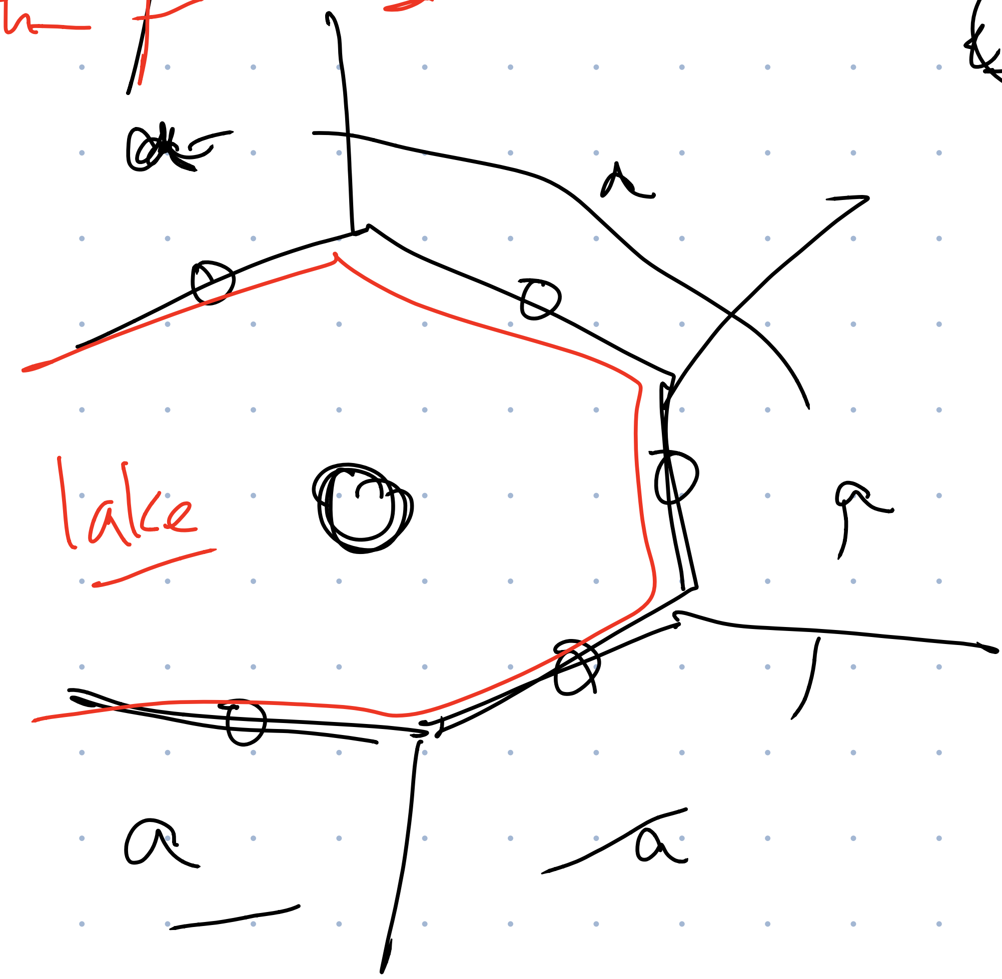
$$\boxed{Q(x,y) = n > 0}$$

positive semidefini:

$\exists\, Q(\vec{x}) = 0$ where $\vec{x} \neq \vec{0}$

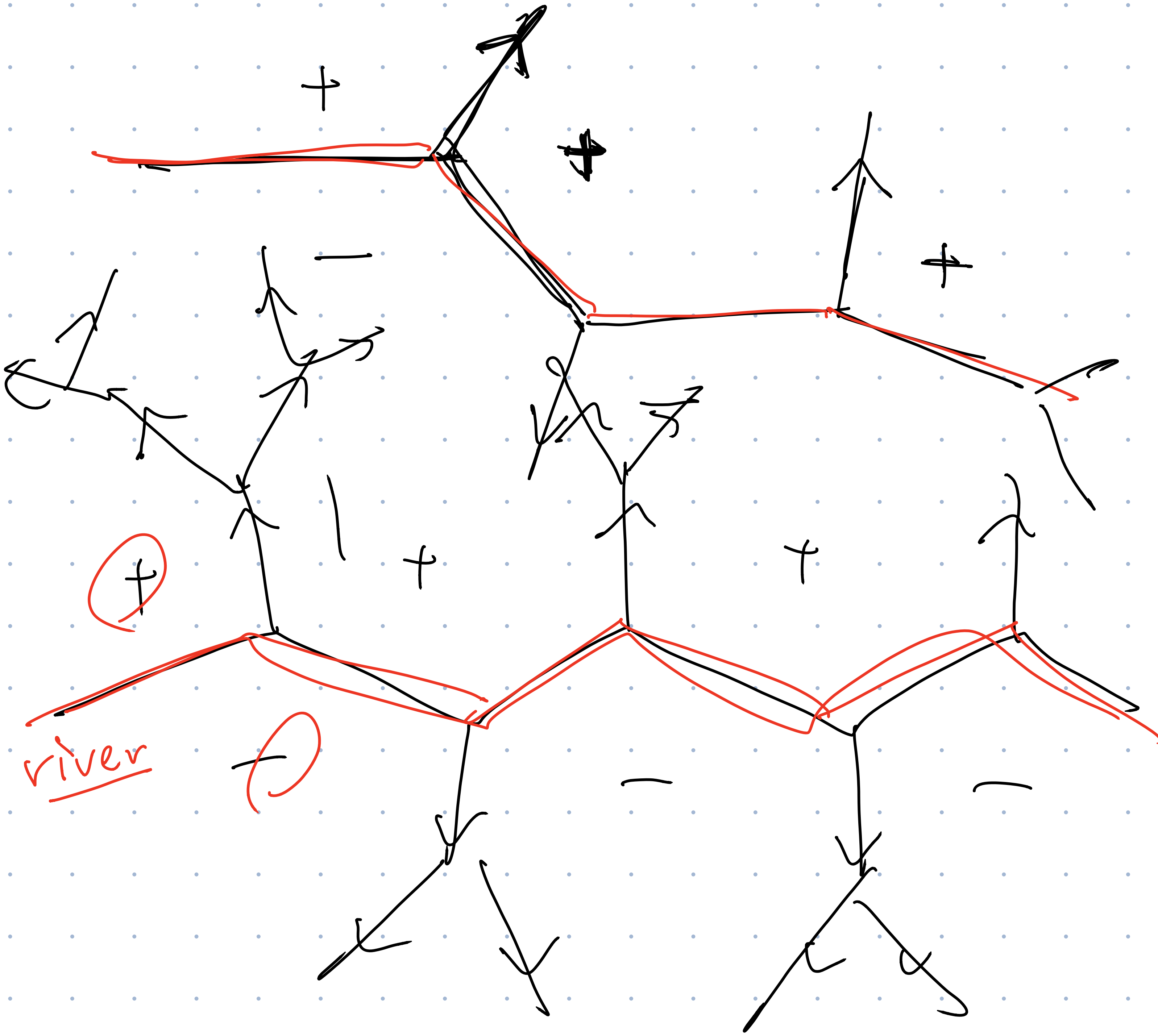


e.g. x^2

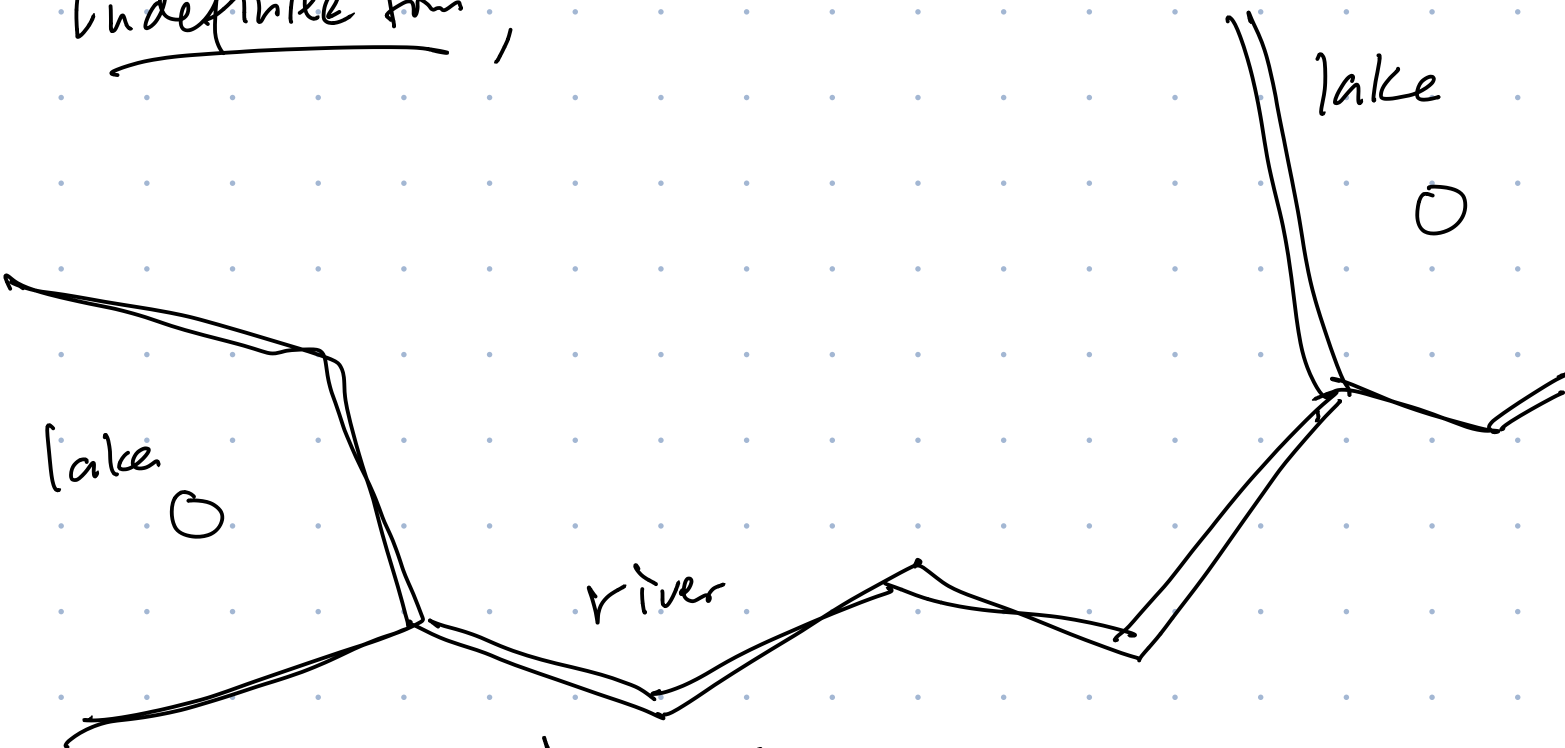


Indefinite form

Suppose $Q(\vec{x}) \neq 0$ for any $\vec{x} \in \mathbb{Z}^2$.



Indefinite form,



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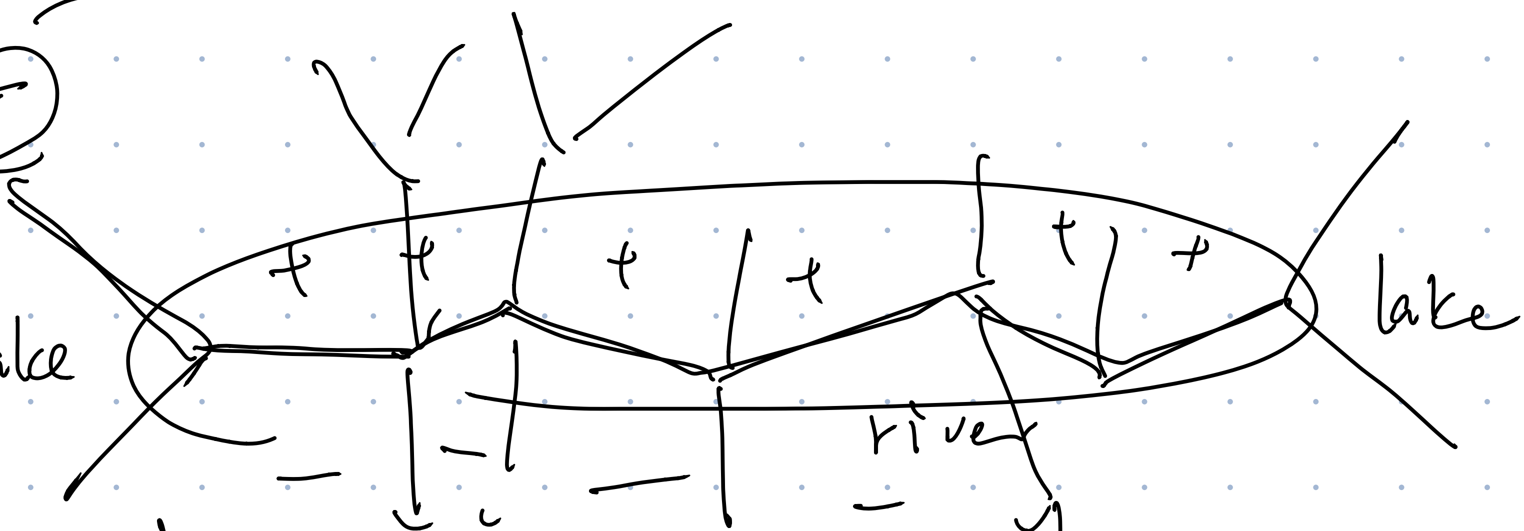

II