#1: For $a \in F$, $|a| = \begin{cases} a, & \text{if } a \geq 0. \\ -a, & \text{if } a \leq 0. \end{cases}$

Hence $a \leq |a|$ and $-a \leq |a|$ $\forall a \in F$.

Claim: $|a+b| \leq |a| + |b|$ $\forall a, b \in F$.

pf: $|a+b|$ is either $a+b$ or $-(a+b)$.

- $a+b \leq |a| + |b|$.
- $-(a+b) = (-a) + (-b) \leq |a| + |b|$. $\square$

Now we prove the desired statement by induction.
The statement is true for $n=2$.
Assume it's true for $n-1$. Then:

$$|a_1 + \cdots + a_n| \leq |a_1 + \cdots + a_{n-1}| + |a_n|$$

$$\underset{\text{inductive hypothesis.}}{\leq} |a_1| + \cdots + |a_{n-1}| + |a_n|. \ \square$$

#2: (a) $a \in F$ is a lower bound of S if $a \leq z$ $\forall z \in S$.

(b) $S \subseteq F$ is bounded below if there exists a lower bound of S

(c) $a \in F$ is the greatest lower bound of S if $a \geq a'$ for any lower bound a' of S.

(d) F satisfies the greatest lower bound property if for any $S \subseteq F$ bounded below (and nonempty), the greatest lower bound of S exists in F.

#3: We show that F satisfies the least upper bound property implies F satisfies the greatest lower bound property (the converse direction can be proved similarly).

Let $S \subseteq F$ be a nonempty set & bounded below.

So $\exists\, a \in F$ s.t. $a \leq z \quad \forall z \in S$.

$\Longrightarrow \quad -a \geq -z \quad \forall z \in S$.

$\Longrightarrow$ the set $T := \{-z \mid z \in S\} \subseteq F$ is bounded above

Since F satisfies the least upper bound property, the least upper bound of T exists in F, say it's $w \in F$.

Claim: $-w \in F$ is the greatest lower bound of S.

pf: • $-w$ is a lower bound of S:

we know $w \geq -z \quad \forall z \in S$, hence $z \leq -w \quad \forall z \in S$.

• $-w$ is the greatest lower bound of S:

Let $-w'$ be a lower bound of S.

Then $\quad -w' \leq z \quad \forall z \in S$.

$\Longrightarrow \quad w' \geq -z \quad \forall z \in S$.

$\Longrightarrow \quad w'$ is an upper bound of T.

$\Longrightarrow \quad w' \geq w$ since w is the least upper bound of T.

$\Longrightarrow \quad -w' \leq -w$. $\square$

#4: $\forall \varepsilon > 0$, $z - \varepsilon < z = \sup S$, hence $z-\varepsilon$ is not an upper bound of S. $\Rightarrow$ $\exists\, a \in S$ s.t. $z - \varepsilon < a$.

Since z is an upper bound of S. $\Rightarrow$ $z - \varepsilon < a \leq z$. $\square$

$a \in S$ may not be found so that $z - \varepsilon < a < z$.

For instance, let $S = \{0\} \subseteq \mathbb{R}$. Then $z = \sup S = 0$,

But $\forall \varepsilon > 0$, there is no $a \in S$ s.t. $-\varepsilon < a < 0$. $\square$

#5: Assume the contrary that $x > y$.

Take $\varepsilon = \dfrac{x-y}{2} > 0$.

Then $y + \varepsilon = \dfrac{x+y}{2} < x$. Contradiction. $\square$

#6: Let $S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$.

It's clear that 1 is an upper bound of S.

Assume the contrary that 1 is NOT the least upper bound of S.

Then $\exists\, \varepsilon > 0$ s.t. $1 - \varepsilon$ is an upper bound of S.

Since $\mathbb{N} \subseteq \mathbb{R}$ is not bounded above, $\exists\, N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$.

$\Rightarrow$ $1 - \frac{1}{N} > 1 - \varepsilon$. Contradiction. $\square$

#7: By assumption, $\exists N_1 > 0$ s.t. $a_n = b_n$ $\forall n > N_1$.

$\forall \varepsilon > 0$, $\exists N_2 > 0$ s.t. $n > N_2 \Rightarrow |a_n - a| < \varepsilon$.

Let $N = \max\{N_1, N_2\} > 0$,

then $n > N \Rightarrow |b_n - a| = |a_n - a| < \varepsilon$.

Hence $\lim_{n \to \infty} b_n = a$. $\square$

#8: Assume the contrary that $a > b$. Let $\varepsilon = \frac{a-b}{2} > 0$.

$\exists N_1 > 0$ s.t. $n > N_1 \Rightarrow |a_n - a| < \varepsilon$.

$\exists N_2 > 0$ s.t. $n > N_2 \Rightarrow |b_n - b| < \varepsilon$.

$\exists N_3 > 0$ s.t. $n > N_3 \Rightarrow a_n \leq b_n$

Let $N = \max\{N_1, N_2, N_3\} > 0$. Then

$n > N \Rightarrow \frac{a+b}{2} = a - \varepsilon < a_n \leq b_n < b + \varepsilon = \frac{a+b}{2}$.

Contradiction. $\square$

#9: By #4, $\forall n \in \mathbb{N}$, $\exists a_n \in S$ s.t. $z - \frac{1}{n} < a_n \leq z$.

Claim: $\lim_{n \to \infty} a_n = z$.

pf: $\forall \varepsilon > 0$, Choose $N > \frac{1}{\varepsilon}$. Then

$n > N \Rightarrow z - \varepsilon < z - \frac{1}{n} < a_n \leq z. \Rightarrow |a_n - z| < \varepsilon$. $\square$