HOMEWORK 6
MATH H54, FALL 2021

Due October 19, 11am

Some ground rules:

- Please submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW6**.
- The submission should be a **single PDF file**.
- Late homework will not be accepted/graded under any circumstances.
- Make sure the writing in your submission is clear enough. Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own, and acknowledge the students with whom you worked.
- **For True/False questions**: You have to prove the statement if your answer is "True"; otherwise, you have to provide an explicit counterexample and justification.
- You are allowed to use any result that is proved in the lecture. But if you would like to use other results, you have to prove it first before using it.

Problems:

(1) Let V be an inner product space and $W = \mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$ be a subspace of V. Suppose that $\vec{x} \in V$ is orthogonal to each $\vec{v}_i$ for $1 \leq i \leq n$. Prove that $\vec{x}$ is orthogonal to every vector in W.

(2) Let V be an inner product space and $W \subseteq V$ be a subspace. (They could be infinite dimensional.)

(a) Prove that the orthogonal complement

$$W^{\perp} := \{\vec{x} \in V : \langle \vec{x}, \vec{w} \rangle = 0 \text{ for any } \vec{w} \in W\}$$

is a subspace of V.

(b) Prove that $W \cap W^{\perp} = \{\vec{0}\}$. (Hint: Suppose $\vec{x} \in W \cap W^{\perp}$. Consider $\langle \vec{x}, \vec{x} \rangle$.)

(3) Let U be an orthogonal $n \times n$ matrix, i.e. the columns of U form an orthonormal basis of $\mathbb{R}^n$. Prove that the rows of U also form an orthonormal basis of $\mathbb{R}^n$.

(4) Suppose that U_1 and U_2 are both orthogonal matrices. Prove that the product $U_1 U_2$ is also an orthogonal matrix.

(5) Let A be an invertible $n \times n$ matrix. For any $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, define

$$\langle \vec{v}_1, \vec{v}_2 \rangle_A := \vec{v}_1^T A^T A \vec{v}_2.$$

Prove that $\langle -, - \rangle_A$ is an inner product on $\mathbb{R}^n$.

(6) Let V be an inner product space (could be infinite dimensional). Prove that for any $\vec{v}_1, \vec{v}_2 \in V$,
$$||\vec{v}_1 + \vec{v}_2||^2 + ||\vec{v}_1 - \vec{v}_2||^2 = 2||\vec{v}_1||^2 + 2||\vec{v}_2||^2.$$

(7) Let V be an inner product space (could be infinite dimensional). Prove that for any $\vec{v}_1, \vec{v}_2 \in V$,
$$|\langle \vec{v}_1, \vec{v}_2 \rangle| \leq ||\vec{v}_1|| \cdot ||\vec{v}_2||.$$
When does the equality hold? (Hint: You can use the fact that $||\vec{x}|| \geq ||\mathrm{proj}_{\mathrm{Span}\{\vec{v}\}}\vec{x}||$ for any $\vec{x}, \vec{v}$, and the equality holds if and only if $\vec{x}$ and $\vec{v}$ are linearly dependent.)

(8) Let V be an inner product space (could be infinite dimensional). Prove that for any $\vec{v}_1, \vec{v}_2 \in V$,
$$||\vec{v}_1 + \vec{v}_2|| \leq ||\vec{v}_1|| + ||\vec{v}_2||.$$
(Hint: Take square on both sides, and apply the result in previous problem.)

(9) Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be any real monic polynomial. Find an $n \times n$ real matrix such that its characteristic polynomial is given by $(-1)^n p$. (This will be used later to write an n-th order linear ordinary differential equation as a system of first order linear equations.) (Hint: Consider matrices of the form
$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \star & \star & \cdots & \cdots & \star \end{bmatrix}.)$$

(10) Let $V = \mathcal{C}[-\pi, \pi]$ be the vector space of continuous functions on $[-\pi, \pi]$. For $f, g \in V$, define
$$\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx.$$

(a) Prove that $(V, \langle -, - \rangle)$ is an inner product space.

(b) Prove that the set
$$\left\{ \frac{1}{\sqrt{2}}, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(kx), \sin(kx), \ldots \right\}$$
is an orthonormal set. (This will be useful later for the discussions on Fourier series.)