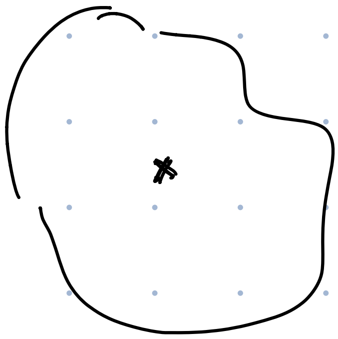
Def: $f : \Omega \to \mathbb{C}$ holo.

Say f has an <u>isolated singularity at z_0</u> if

- $z_0 \notin \Omega$.
- $\exists\, r > 0$ s.t. $\mathbb{D}_r^\times(z_0) = \{ z \in \mathbb{C} \mid 0 < |z - z_0| < r \} \subseteq \Omega$.



e.g. $\frac{1}{z}$ has an isolated sing at 0

eg $\frac{1}{(z-1)(z+1)}$ has two iso. sing. at ± 1.

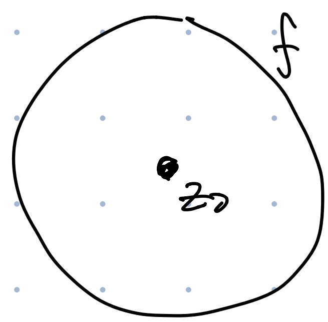
Classification of isolated singularities

- removable singularity
- pole
- essential singularity

Def z_0 is an isolated sing. of f, say it's a <u>removable sing.</u> if $\exists\, L \in \mathbb{C},\ r > 0$ st.

$$F(z) = \begin{cases} f(z), & \text{if } z \in \mathbb{D}_r^\times(z_0) \\ L, & \text{if } z = z_0 \end{cases}$$

is holo. in $\mathbb{D}_r(z_0)$.



Q: $\frac{\sin z}{z}$ has an isolated sing at $z = 0$.

Is it removable?? Yes.:

$$F(z) = \begin{cases} \sin z / z & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$ holo. on $\mathbb{C}$

Rmk: If z_0 is a removable sing of f, then f is bounded near z_0

We'll show the converse is true!!

Def: f has a pole at z_0 if $\exists\, r > 0$ s.t.

1) f is holo. & non-vanishing on $\mathbb{D}_r^\times(z_0)$

2) $$F(z) = \begin{cases} \frac{1}{f(z)}, & z \in \mathbb{D}_r^\times(z_0) \\ 0, & z = z_0 \end{cases}$$

is holo. on $\mathbb{D}_r(z_0)$.

e.g.: $f(z) = \frac{1}{z^n}$, $n \geq 1$, has a pole at $z = 0$

Rmk: If z_0 is a pole of f, then

$|f(z)| \longrightarrow +\infty$ as $z \longrightarrow z_0$.

We'll show the converse is true!

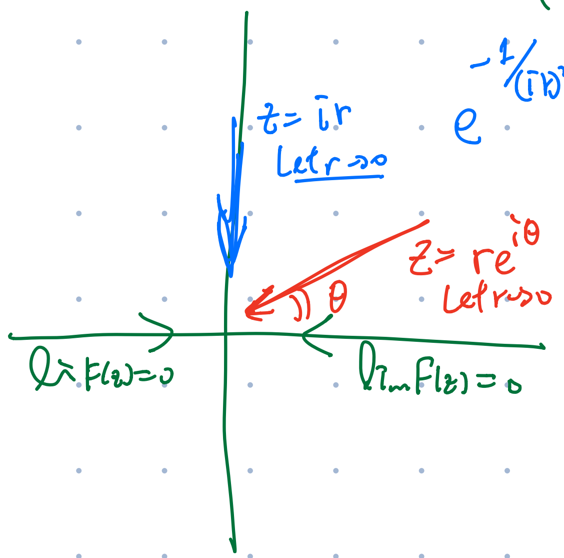
e.g. $\boxed{e^{\frac{1}{z^2}}}$ has iso. sing at 0.

removable? pole? neither?

- not removable, b/c it's not bdd near 0
- not pole:

$$F(z) = \begin{cases} e^{-1/z^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

<u>holo. at 0??</u>



$$e^{-1/(ir)^2} = \boxed{e^{1/r^2}} \to +\infty \text{ as } r \to 0$$

$$e^{\frac{-1}{r^2 e^{i2\theta}}} = e^{-\frac{1}{r^2} e^{-i2\theta}}$$

$$= e^{-\frac{1}{r^2}(\cos 2\theta - i \sin 2\theta)}$$

$$= e^{-\frac{1}{r^2}\cos 2\theta} \cancel{\left(\cos\left(\frac{1}{r^2}\sin 2\theta\right) + i \sin\left(\frac{1}{r^2}\sin 2\theta\right)\right)}$$

Def: An isolated sing which is not removable, not a pole, is called an <u>essential singularity</u>

We'll show that if z_0 is an essential sing of f, then $\forall \varepsilon > 0$,

$$f\left(\mathbb{D}_\varepsilon^\times(z_0)\right) \subseteq \mathbb{C}$$

dense.

$\Big($ f has ~~zero~~ at z_0, $\exists r > 0$, nonvanishing g on $\mathbb{D}_r(z_0)$, $n \geq 1$

w/ $f(z) = (z - z_0)^n g(z) \quad \forall z \in \mathbb{D}_r(z_0)$ $\Big)$

Thm f has a pole at z_0, then $\exists\, r>0$, and nonvanishing holo. fcn $h(z)$ on $\mathbb{D}_r(z_0)$, $n \geq 1$

s.t. $$f(z) = \frac{h(z)}{(z-z_0)^n} \quad \text{on} \quad \mathbb{D}_r^\times(z_0).$$

n — order of the pole

pf Since f has a pole at z_0, $\exists\, r'>0$

s.t. 1) f is holo, nonvanishing on $\mathbb{D}_{r'}^\times(z_0)$

2) $$F(z) = \begin{cases} 1/f(z), & z \in \mathbb{D}_{r'}^\times(z_0) \\ 0, & z = z_0 \end{cases}$$ is holo.

- z_0 is a zero of the holo. fcn. $F(z)$.
- $\exists\, 0<r<r'$, g: nonvanishing holo. in $\mathbb{D}_r(z_0)$, $n \geq 1$

s.t. $$F(z) = (z-z_0)^n g(z) \quad \text{on} \quad \mathbb{D}_r(z_0).$$

$$\Rightarrow f(z) = \frac{1}{(z-z_0)^n} \cdot \frac{1}{g(z)} \quad \text{on} \quad \mathbb{D}_r^\times(z_0)$$

$\frac{1}{g(z)} = h(z)$ — nonvanishing holo. fcn on $\mathbb{D}_r(z_0)$. $\square$

- $h(z)$ nonvanishing holo. on $\mathbb{D}_r(z_0)$.

$$h(z) = a_{-n} + a_{-n+1}(z-z_0) + a_{-n+2}(z-z_0)^2 + \cdots$$

power series exp. of h in $\mathbb{D}_r(z_0)$

where $$a_{k-n} = \frac{1}{k!} h^{(k)}(z_0)$$

- So $$f(z) = \frac{h(z)}{(z-z_0)^n}$$

"principal part of f at z_0"

$$= \underbrace{\boxed{\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}}}$$

$$+ \underbrace{a_0 + a_1(z-z_0) + \cdots\cdots}$$

holo. in $\mathbb{D}_r(z_0)$

on $\mathbb{D}_r^{\times}(z_0)$

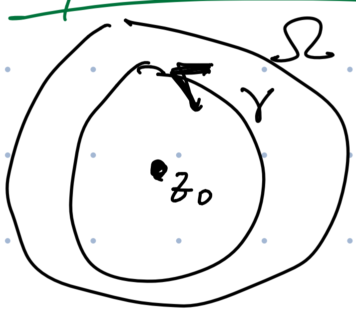
Def: The residue of f at a pole z_0 (of order n) is

$$\operatorname*{res}_{z=z_0} f := a_{-1}$$

$$= \boxed{\frac{1}{(n-1)!} h^{(n-1)}(z_0)}$$

$$= \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} \Big(\underbrace{(z-z_0)^n f(z)}_{= h(z)}\Big)\Big|_{z_0}$$

Why is residue important?



$$\int_\gamma f(z)\,dz = \int_\gamma \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + (\text{holo.})\,dz$$

$$= 2\pi i \cdot \operatorname*{Res}_{z=z_0} f$$

$$\int_\gamma (\text{holo})\, dz = 0, \qquad \int_\gamma \frac{dz}{(z-z_0)^n} = 0 \quad \forall n \geq 2$$

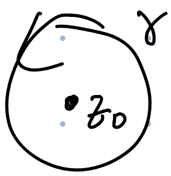
$$\int_\gamma \frac{dz}{z-z_0} = 2\pi i$$

Special case: $n=1$;

$$f(z) = \frac{h(z)}{z-z_0}$$

If $h(z) = a_0 + a_1(z-z_0) + a_2 \ldots$ $\rightarrow$ $\boxed{\frac{a_0}{z-z_0}} + a_1 + a_2(z-z_0) + \ldots$

$$\operatorname*{Res}_{z=z_0} f = h(z_0)$$



Cauchy integral formula

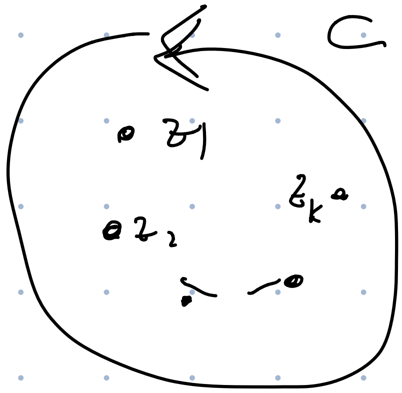
$$\int_\gamma f(z)\, dz = \int_\gamma \frac{h(z)}{z-z_0}\, dz = 2\pi i\, h(z_0)$$

$$= 2\pi i \operatorname*{Res}_{z=z_0} f$$

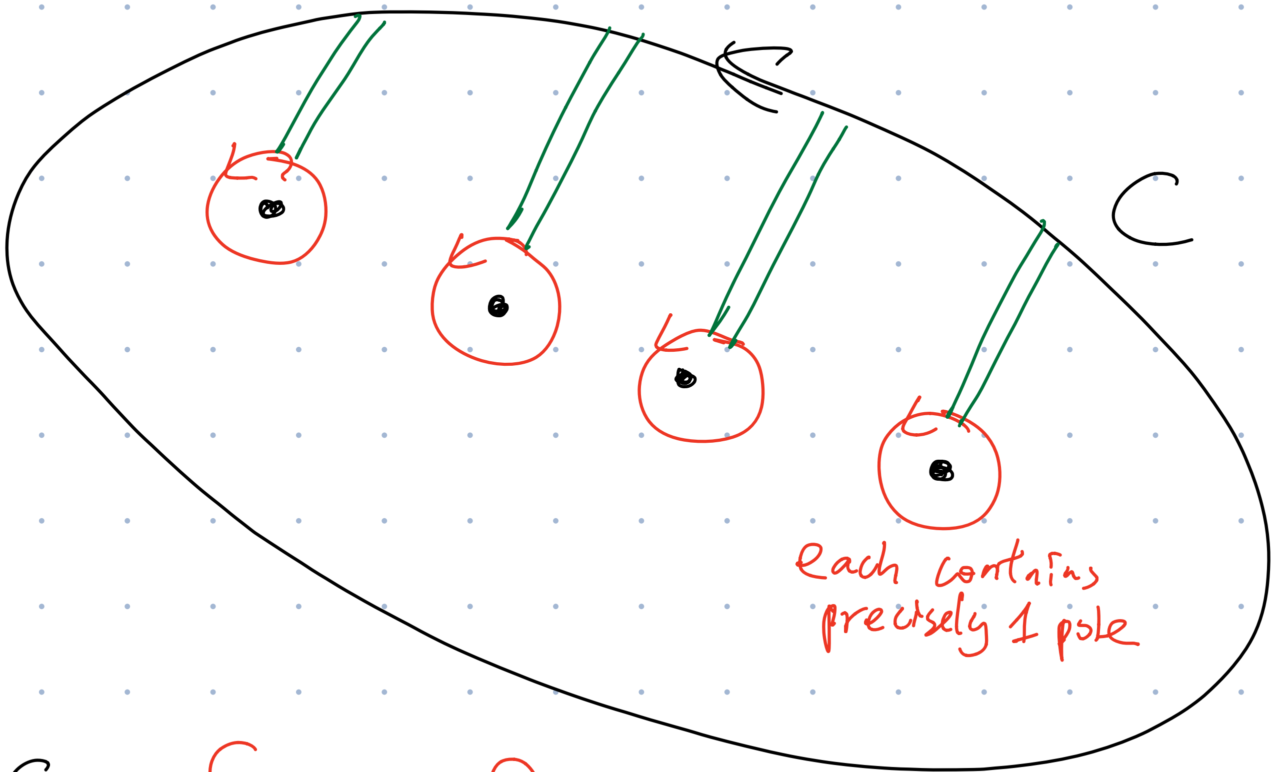
Residue Thm: $f: \Omega \to \mathbb{C}$ holo. except at poles $z_1, \ldots, z_k, \ldots$

$C \subseteq \Omega$ simple closed curve oriented positively

- Suppose $z_1, \ldots, z_k$ are the poles in the interior of C.
- C doesn't pass through any poles.



$$\int_C f(z)\,dz = 2\pi i \sum_{i=1}^{k} \operatorname*{Res}_{z=z_i} f$$



$$\int_C = \int_{\bigcirc} + \int_{\bigcirc} + \int_{\bigcirc} + \int_{\bigcirc}$$

$$\int_{\bigcirc} = 2\pi i \operatorname*{Res}_{z=z_i} f$$