FINAL EXAM SOLUTION
MATH 104, SECTION 6, SPRING 2020

1

Define a sequence of real numbers (a_n) by setting $a_1 = 1$ and

$$a_{n+1} = \sqrt{a_n^2 + \frac{1}{2^n}} \quad \text{for} \quad n \geq 1.$$

Prove that (a_n) is convergent.
(Hint: Try to estimate $|a_{n+1} - a_n|$, then use the Cauchy criterion.)

Solution. It is easy to show that $a_n \geq 1$ for any n and that (a_n) is strictly increasing. We have

$$0 < a_{n+1} - a_n = \frac{a_{n+1}^2 - a_n^2}{a_{n+1} + a_n} = \frac{1/2^n}{a_{n+1} + a_n} \leq \frac{1}{2^{n+1}}.$$

Hence for any $n, m > 0$, we have

$$0 < a_{n+m} - a_n \leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+m}} < \frac{1}{2^n}.$$

For any $\epsilon > 0$, choose N large so that $2^N > \frac{1}{\epsilon}$. Then for any $m > n > N$, we have

$$0 < a_m - a_n < \frac{1}{2^n} < \frac{1}{2^N} < \epsilon.$$

Hence (a_n) is convergent by Cauchy criterion.

2

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 2$, where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are real numbers. Suppose that all of the roots of $P(x)$ are real numbers.

Prove that all of the roots of its derivative $P'(x)$ are also real numbers. (Warning: Remember that $P(x)$ could have roots with multiplicities > 1.)

Solution. Let $r_1 < \cdots < r_k$ be the roots of P, with multiplicities $m_1, \ldots, m_k$, respectively. Then $\sum_{i=1}^k m_i = n$.

We first claim that if r_i is a root of P with multiplicity $m_i > 1$, then r_i is a root of P' with multiplicity $m_i - 1$. One can write $P(x) = (x - r_i)^{m_i} Q(x)$, where $Q(x)$ is a real polynomial with $Q(r_i) \neq 0$. Then

$$P'(x) = m_i (x - r_i)^{m_i - 1} Q(x) + (x - r_i)^{m_i} Q'(x).$$

It is then clear that r_i is a root of P' with multiplicity $m_i - 1$.

Secondly, we claim that for each $1 \leq i \leq k-1$, there exists some $s_i \in (r_i, r_{i+1})$ such that $P'(s_i) = 0$. This simply follows from the fact that $P(r_i) = P(r_{i+1}) = 0$ and the Mean Value Theorem.

Finally, we claim that the roots of P' found in the previous two claims are all the roots of P', therefore all the roots of P' are real numbers. Let $I \subset \{1, \ldots, k\}$ be the index set such that $m_i > 1$ for $i \in I$. Then the number of roots of P' found in the first two claims (counted with multiplicities) is:

$$\begin{aligned} k - 1 + \sum_{i \in I}(m_i - 1) &= (k - |I|) + (\sum_{i \in I} m_i) - 1 \\ &= (\sum_{i \notin I} m_i) + (\sum_{i \in I} m_i) - 1 \\ &= n - 1. \end{aligned}$$

Since P' is a polynomial of degree $n-1$, this proves the claim.

3

Let (f_n) be a sequence of real-valued function defined on a set X. Suppose that

- $f_n(x) \geq 0$ for any $x \in X$ and any $n \in \mathbb{N}$,
- $f_n(x) \geq f_{n+1}(x)$ for any $x \in X$ and any $n \in \mathbb{N}$,
- $\lim_{n\to\infty} \sup\{f_n(x) \colon x \in X\} = 0$.

Prove that the series of functions $\sum(-1)^n f_n(x)$ converges uniformly on X.

Solution. Observe that $\sum(-1)^n f_n(x)$ is an alternating series for each $x \in X$, hence converges pointwisely to $f(x) \in \mathbb{R}$. Recall from the proof of alternating series test that

$$|f(x) - \sum_{n=1}^{k}(-1)^n f_n(x)| \leq f_k(x)$$

for any $x \in X$ and any $k > 0$.

For any $\epsilon > 0$, there exists $N > 0$ such that

$$\sup\{f_k(x) \colon x \in X\} < \epsilon \text{ for any } k > N.$$

Hence for any $k > N$, we have

$$|f(x) - \sum_{n=1}^{k}(-1)^n f_n(x)| \leq f_k(x) \leq \sup\{f_k(x) \colon x \in X\} < \epsilon.$$

This proves that $\sum(-1)^n f_n(x) \to f(x)$ converges uniformly on X.

4

(1) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \left(\int_0^1 |f(x)|^n dx \right)^{1/n} = \sup \Big\{ |f(x)| \colon x \in [0,1] \Big\}.$$

(Hint: For any $\epsilon > 0$, show that there exists some subinterval of $[0,1]$ such that the value of $|f|$ on this subinterval is at least $\sup\{|f(x)|\} - \epsilon$. Then use this to estimate the left hand side.)

(2) Let $g\colon [0,1] \to \mathbb{R}$ be a positive and continuous function. Give a similar expression of

$$\lim_{n\to-\infty} \left(\int_0^1 |g(x)|^n dx \right)^{1/n}$$

and justify your answer.
(Hint: This is not hard to find using the previous part.)

Solution. (1) Since f is a continuous function on a compact set, the supremum of $|f(x)|$ is attained, say by $x_0 \in [0,1]$, i.e. $|f(x_0)| = \sup \Big\{ |f(x)| \colon x \in [0,1] \Big\}$. If $|f(x_0)| = 0$, then f is the constant zero function, and the result is obvious. Suppose that $|f(x_0)| > 0$. For any $0 < \epsilon < |f(x_0)|$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x_0)| - \epsilon < |f(x)| \leq |f(x_0)|.$$

Therefore

$$|f(x_0)|^n \geq \int_0^1 |f(x)|^n dx \geq \delta(|f(x_0)| - \epsilon)^n.$$

Thus

$$|f(x_0)| \geq \left(\int_0^1 |f(x)|^n dx \right)^{1/n} \geq \delta^{1/n}(|f(x_0)| - \epsilon).$$

Hence

$$|f(x_0)| \geq \limsup_{n\to\infty} \left(\int_0^1 |f(x)|^n dx \right)^{1/n} \geq \liminf_{n\to\infty} \left(\int_0^1 |f(x)|^n dx \right)^{1/n} \geq |f(x_0)| - \epsilon.$$

These inequalities hold for any $\epsilon > 0$, so

$$\limsup_{n\to\infty} \left(\int_0^1 |f(x)|^n dx \right)^{1/n} = \liminf_{n\to\infty} \left(\int_0^1 |f(x)|^n dx \right)^{1/n} = |f(x_0)|.$$

Hence

$$\lim_{n\to\infty} \left(\int_0^1 |f(x)|^n dx \right)^{1/n} = |f(x_0)| = \sup \Big\{ |f(x)| \colon x \in [0,1] \Big\}.$$

(2) We claim that

$$\lim_{n\to-\infty}\Big(\int_0^1 |g(x)|^n dx\Big)^{1/n} = \inf\Big\{|g(x)| \colon x\in[0,1]\Big\}.$$

Since g is a positive function, one can consider its inverse

$$f(x) := \frac{1}{g(x)},$$

which is also a positive continuous function. By Part (1), we have

$$\lim_{n\to\infty}\Big(\int_0^1 \frac{1}{|g(x)|^n} dx\Big)^{1/n} = \sup\Big\{\frac{1}{|g(x)|} \colon x\in[0,1]\Big\}.$$

Note that g is a positive continuous function on a compact set, hence the infimum of $|g|$ is attained, therefore is positive. Therefore

$$\sup\Big\{\frac{1}{|g(x)|} \colon x\in[0,1]\Big\} = \frac{1}{\inf\Big\{|g(x)| \colon x\in[0,1]\Big\}}.$$

Hence

$$\begin{aligned}
\lim_{n\to-\infty}\Big(\int_0^1 |g(x)|^n dx\Big)^{1/n} &= \lim_{n\to\infty}\Big(\int_0^1 |g(x)|^{-n} dx\Big)^{-1/n} \\
&= \Big(\lim_{n\to\infty}\Big(\int_0^1 \frac{1}{|g(x)|^n} dx\Big)^{1/n}\Big)^{-1} \\
&= \Big(\sup\Big\{\frac{1}{|g(x)|} \colon x\in[0,1]\Big\}\Big)^{-1} \\
&= \inf\Big\{|g(x)| \colon x\in[0,1]\Big\}.
\end{aligned}$$

5

Let (X,d) be a metric space and $E\subset X$ be a nonempty subset. Define a function $f\colon X\to[0,\infty)$ by:

$$f(x) := \inf\{d(x,y)\colon y\in E\}.$$

Prove that f is uniformly continuous on X.

Solution. Let $\delta>0$ and suppose that $x_1,x_2\in X$ has distance $d(x_1,x_2)<\delta$. We claim that $|f(x_1)-f(x_2)|<2\delta$. (This proves that f is uniformly continuous.) Since $f(x_1)=\inf\{d(x_1,y)\colon y\in E\}\geq 0$, there exists $y\in E$ such that $d(x_1,y)<f(x_1)+\delta$. Hence

$$\delta > d(x_1,x_2) \geq d(x_2,y)-d(y,x_1) > f(x_2)-f(x_1)-\delta.$$

Therefore $f(x_2)-f(x_1)<2\delta$. One can use the same argument to show that $f(x_1)-f(x_2)<2\delta$. This proves the claim.

6

Let $S_1 = (\mathbb{R}, d_{\text{std}})$ be the standard metric space of real numbers, i.e. $d_{\text{std}}(x, y) = |x - y|$. Let $S_2 = (\mathbb{R}, d_0)$ be the metric space whose elements are still the real numbers, but equips with a different distance function:

$$d_0(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

(1) Describe all the open subsets of S_2, and justify your answer.
(2) Describe all the compact subsets of S_2, and justify your answer.
(3) Describe all the continuous functions $f : S_2 \to S_1$, and justify your answer.
(4) Describe all the continuous functions $f : S_1 \to S_2$, and justify your answer. (Hint: What are the connected subsets of S_2?)

(Warning: Do NOT simply copy and paste the definition of open, compact, or continuous. Give more explicit descriptions.)

Solution. (1) Any subset of S_2 is an open subset of S_2. Firstly, observe that any point $x \in \mathbb{R}$ is an open subset of S_2, since the open ball of radius $\frac{1}{2}$ centered at x consists of a single element: $B_{\frac{1}{2}}(x) = \{x\}$. Secondly, one can consider any subset $E \subset S_2$ as union of open subsets:

$$E = \cup_{x \in E} \{x\},$$

hence E is open.

(2) A subset $E \subset S_2$ is compact if and only if E consists of finitely many elements. Firstly, it is clear that any finite subset $E \subset S_2$ is compact, since any open cover of E has a finite sub-cover. Conversely, suppose $E \subset S_2$ consists of infinitely many elements. Then

$$\{\{x\}\}_{x \in E}$$

gives an open cover of E, since we proved in Part (1) that any point is an open subset of S_2. However, this open cover of E does not admit any finite sub-cover. Hence any infinite set $E \subset S_2$ is not compact.

(3) Any function $f : S_2 \to S_1$ is continuous. Recall that $f : S_2 \to S_1$ is continuous if and only if the preimage $f^{-1}(U) \subset S_2$ is open for any open subset $U \subset S_1$. This condition is always satisfied because any subset in S_2 is open by Part (1).

(4) A function $f : S_1 \to S_2$ is continuous if and only if f is a constant function. Firstly, it is clear that any constant function is continuous. Conversely,

suppose that $f\colon S_1 \to S_2$ is continuous. Then the image of a connected subset of S_1 under f is a connected subset of S_2. In particular, the whole range $f(\mathbb{R}) \subset S_2$ is connected in S_2. By Part (1), any nonempty subset $E \subset S_2$ is disconnected unless E consists of a single element. Hence $f(\mathbb{R})$ is a single element, i.e. f is a constant function.

7

Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (-1)^k f\Big(\frac{k}{n}\Big) = 0.$$

Solution. f is uniformly continuous on $[0,1]$ since $[0,1]$ is compact. Hence for any $\epsilon > 0$, there exists $N > 0$ such that

$$|x-y| < \frac{1}{N},\ x, y \in [0,1] \implies |f(x) - f(y)| < \epsilon.$$

Consider any $n > N$. We have

$$\begin{aligned}\Big|\frac{1}{n}\sum_{k=1}^{n}(-1)^k f\Big(\frac{k}{n}\Big)\Big| &\le \frac{1}{n}\Big(\Big| - f(\frac{1}{n}) + f(\frac{2}{n})\Big| + \Big| - f(\frac{3}{n}) + f(\frac{4}{n})\Big| + \cdots\Big)\\ &< \frac{1}{n}\Big(\epsilon \cdot \frac{n}{2} + |f(1)|\Big) = \frac{\epsilon}{2} + \frac{|f(1)|}{n}.\end{aligned}$$

Let $n \to \infty$, we obtain

$$\frac{-\epsilon}{2} \le \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(-1)^k f\Big(\frac{k}{n}\Big) \le \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(-1)^k f\Big(\frac{k}{n}\Big) \le \frac{\epsilon}{2}.$$

Since these inequalities hold for any $\epsilon > 0$, we have

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(-1)^k f\Big(\frac{k}{n}\Big) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(-1)^k f\Big(\frac{k}{n}\Big) = 0.$$

Hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (-1)^k f\Big(\frac{k}{n}\Big) = 0.$$

8

Suppose that $f\colon [1,\infty) \to \mathbb{R}$ is uniformly continuous on $[1,\infty)$. Prove that there exists $M > 0$ such that

$$\frac{|f(x)|}{x} \le M \text{ holds for any } x \ge 1.$$

Solution. Since f is uniformly continuous, there exists $\delta > 0$ such that

$$|x-y| < \delta,\ x, y \geq 1 \implies |f(x) - f(y)| < 1.$$

For any $x \geq 1$, there exists $x_0 \in [1, 1+\delta]$ such that $x - x_0 = \frac{N\delta}{2}$ for some $N \in \mathbb{N}$. Then

$$|f(x) - f(x_0)| \leq \sum_{i=1}^{N} \Big| f\Big(x - \frac{(i-1)\delta}{2}\Big) - f\Big(x - \frac{i\delta}{2}\Big)\Big| < N < \frac{2x}{\delta}.$$

Since f is continuous on the compact set $[1, 1+\delta]$, there exists $N' > 0$ such that

$$|f(x_0)| < N' \text{ for any } x_0 \in [1, 1+\delta].$$

Hence for any $x \geq 1$, we have

$$\frac{|f(x)|}{x} < \frac{\frac{2x}{\delta} + N'}{x} = \frac{2}{\delta} + \frac{N'}{x} \leq \frac{2}{\delta} + N'.$$

Let M be the constant $M \coloneqq \frac{2}{\delta} + N'$.

9

Prove that the closed interval $[a, b]$ is not of measure zero in $\mathbb{R}$.
(Hint: Suppose there is a "bad" covering of $[a, b]$ by open intervals whose total length is less than $b - a$. First prove that you can assume the covering is finite. Take a bad covering $\{U_1, \ldots, U_n\}$ consists of n open intervals. Then prove that there exists a bad covering consists of no more than $n-1$ open intervals. Show that this implies the existence of a bad covering consists of a single open interval, and get a contradiction.)

Solution. We say a covering of $[a, b]$ by (at most countably many) open intervals is *bad* if its total length is less than $b - a$. To show that $[a, b]$ is not of measure zero, it suffices to show that *there is no bad coverings*.

Assume the contrary that $\{U_i\}$ is a bad covering of $[a, b]$. Since $[a, b]$ is compact, there is a finite sub-covering $\{U_1, \ldots, U_n\}$ of $\{U_i\}$ that covers $[a, b]$. Note that the total length of $\{U_1, \ldots, U_n\}$ can not be larger than the total length of $\{U_i\}$, hence $\{U_1, \ldots, U_n\}$ is a bad covering.

We claim that if $n \geq 2$, then there exists a bad covering of $[a, b]$ which consists of less than n open intervals. Without loss of generality, suppose that $a \in U_1$. Write $U_1 = (a - \epsilon, c)$ where $\epsilon > 0$ and $c > a$. Suppose that $c > b$, then U_1 covers the whole interval $[a, b]$ and this proves the claim. Otherwise, suppose $c \leq b$. Then there is another open interval in $\{U_2, \ldots, U_n\}$ that contains c, say $c \in U_2$. Write $U_2 = (c - \delta, d)$ where $\delta > 0$ and $d > c$. Now we define $V \coloneqq (a - \epsilon, d) = U_1 \cup U_2$. Then $\{V, U_3, \ldots, U_n\}$ is a covering of $[a,b]$ by $n-1$ open intervals, and the total length is less than $b-a$ since

$$\text{length}(V)+\sum_{i=3}^{n}\text{length}(U_i)=\sum_{i=1}^{n}\text{length}(U_i)-\delta<\sum_{i=1}^{n}\text{length}(U_i)<b-a,$$

so it is a bad covering. This proves the claim.

By induction, this proves that there exists a bad covering of $[a,b]$ consists of a single open interval $U=(c,d)$. Then we have $c<a<b<d$, hence

$$\text{length}(U)=d-c>b-a.$$

This contradicts with $\{U\}$ is a bad covering since its total length is greater than $b-a$.

10

Let $a>0$ be any positive real number. Prove that there exists a unique continuous function $f\colon[-a,a]\to\mathbb{R}$ such that for any $x\in[-a,a]$ the following equality holds:

$$f(x)=1+\frac{1}{\pi}\int_{-a}^{a}\frac{1}{1+(x-y)^2}f(y)dy.$$

Moreover, prove that this function f is positive, i.e. $f(x)>0$ for any $x\in[-a,a]$.
(Hint: You can use the contraction mapping theorem on complete metric spaces we mentioned in class, i.e. any contraction map on a complete metric space has a unique fixed point.)

Solution. Consider the metric space

$$\mathcal{C}([-a,a]):=\{f\colon[-a,a]\to\mathbb{R}\text{ continuous function}\}$$

equipped with the distance function given by the sup norm (cf. HW12 #1). By HW12 #1(d), this gives a complete metric space. Consider the following function

$$F\colon\mathcal{C}([-a,a])\to\mathcal{C}([-a,a]),\quad f\mapsto F(f),$$

where $F(f)$ is defined by

$$(F(f))(x)=1+\frac{1}{\pi}\int_{-a}^{a}\frac{1}{1+(x-y)^2}f(y)dy.$$

Then

$$\begin{aligned}d(F(f),F(g))&=\sup_{x\in X}\frac{1}{\pi}\Big|\int_{-a}^{a}\frac{1}{1+(x-y)^2}(f(y)-g(y))dy\Big|\\&\le d(f,g)\frac{1}{\pi}\int_{-a}^{a}\frac{1}{1+y^2}dy\end{aligned}$$

Observe that
$$K_a := \frac{1}{\pi}\int_{-a}^{a}\frac{1}{1+y^2}dy < 1$$
since
$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+y^2}dy = 1.$$
Hence F is a contraction map on the complete metric space $\mathcal{C}([-a,a])$, therefore has a unique fixed point. This proves the first part of the question.

Now we prove that the function f which gives the fixed point is a positive function. Since f is continuous on the compact set $[-a,a]$, the infimum of its value is achieved, say by $x_0 \in [-a,a]$. Then
$$f(x_0) = 1 + \frac{1}{\pi}\int_{-a}^{a}\frac{1}{1+(x_0-y)^2}f(y)dy \geq 1 + f(x_0)\frac{1}{\pi}\int_{-a}^{a}\frac{1}{1+(x_0-y)^2}dy.$$
Since
$$\frac{1}{\pi}\int_{-a}^{a}\frac{1}{1+(x_0-y)^2}dy < 1,$$
we obtain $f(x_0) > 0$. Therefore $f(x) \geq f(x_0) > 0$ for any $x \in [-a,a]$.