#1;

(a) $c_1 e^{-\frac{2}{3}t} + c_2 e^{\frac{1}{2}t}$. where $c_1, c_2 \in \mathbb{R}$.

(b) $c_1 e^{-\frac{5}{3}t} + c_2 t e^{-\frac{5}{2}t}$

(c) $c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t)$.

(d) (i) Find a sol^{n}: Let $y(t) = \alpha\cos(t) + \beta\sin(t)$.

Then $y'' + 4y = 3(\alpha\cos(t) + \beta\sin(t))$

$\Rightarrow$ $y(t) = \frac{1}{3}(\sin t - \cos t)$ is a sol^{n}.

(ii) general sol^{n} of $y'' + 4y = 0$: $c_1\cos(2t) + c_2\sin(2t)$

$\Rightarrow$ general sol^{n}: $c_1\cos(2t) + c_2\sin(2t) + \frac{1}{3}(\sin t - \cos t)$.

(e)
- general sol^{n} of $y'' - 2y' + y = 0$: $c_1 e^t + c_2 t e^t$.
- let $y_1(t) = e^t$, $y_2(t) = te^t$, $f(t) = t^{-1}e^t$.

use the method of variation of parameters:

$$v_1(t) = \int \frac{-t^{-1}e^t \cdot te^t}{e^t(e^t + te^t) - e^t \cdot te^t}\,dt = -t + \text{const}.$$

$$v_2(t) = \int \frac{t^{-1}e^t \cdot e^t}{e^t(e^t + te^t) - e^t \cdot te^t}\,dt = \log|t| + \text{const}.$$

$\Rightarrow$ general sol^{n}: $c_1 e^t + c_2 te^t + t\log|t|e^t$

(f)
- general sol^{n} of $y'' + 16y = 0$: $c_1\cos(4t) + c_2\sin(4t)$.
- let $y_1(t) = \cos(4t)$, $y_2(t) = \sin(4t)$, $f(t) = \sec(4t)$

$$v_1(t) = \int \frac{-\sec(4t)\sin(4t)}{4}\,dt = -\frac{1}{4}\int \tan(4t)\,dt$$

$$= \frac{1}{16} \log|\cos(4t)| + \text{const}.$$

$$v_2(t) = \int \frac{\sec(4t)\cos(4t)}{4} dt = \frac{1}{4}t + \text{const}.$$

$\Rightarrow$ $\frac{1}{16}\left(\log|\cos(4t)|\right)\cdot\cos(4t) + \frac{1}{4}t\sin(4t)$ is a sol^{n}

$\Rightarrow$ general sol^{n}: $c_1\cos(4t) + c_2\sin(4t)$
$+ \frac{1}{16}\left(\log|\cos(4t)|\right)\cdot\cos(4t) + \frac{1}{4}t\sin(4t)$

#2:

(a) general sol^{n}: $c_1 e^{5t} + c_2 e^{-t} = y(t)$

$$y'(t) = 5c_1 e^{5t} - c_2 e^{-t}$$

$$\begin{cases} 3 = y(-1) = c_1 e^{-5} + c_2 e \\ 9 = y'(-1) = 5c_1 e^{-5} - c_2 e. \end{cases}$$

$\Rightarrow$ $c_1 = 2e^5$, $c_2 = -e^{-1}$.

$\Rightarrow$ $y(t) = 2e^{5t+5} - e^{-t-1}$.

(b) general sol^{n}: $y(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

$$y'(t) = c_1(-e^{-t}\cos t - e^{-t}\sin t) + c_2(-e^{-t}\sin t + e^{-t}\cos t)$$

$$\begin{cases} 2 = y(0) = c_1 \\ 1 = y'(0) = c_1(-1) + c_2(1) \end{cases} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = 3 \end{cases}$$

$\Rightarrow \ y(t) = 2e^{-t}\cos t + 3e^{-t}\sin t.$

(c) • general sol^{n} of $y'' + y' - 12y = 0$: $c_1 e^{-4t} + c_2 e^{3t}$

• a sol^{n} of $y'' + y' - 12y = e^{t}$: $\frac{-1}{10} e^{t}$.

• a sol^{n} of $y'' + y' - 12y = e^{2t}$: $\frac{-1}{6} e^{2t}$

• a sol^{n} of $y'' + y' - 12y = -1$: $\frac{1}{12}$

$\Rightarrow$ general sol^{n}:

$$y = c_1 e^{-4t} + c_2 e^{3t} - \frac{1}{10} e^{t} - \frac{1}{6} e^{2t} + \frac{1}{12}$$

$$y' = -4c_1 e^{-4t} + 3c_2 e^{3t} - \frac{1}{10} e^{t} - \frac{1}{3} e^{2t}$$

$$\begin{cases} 1 = y(0) = c_1 + c_2 - \frac{1}{10} - \frac{1}{6} + \frac{1}{12} = c_1 + c_2 - \frac{11}{60} \\ 3 = y'(0) = -4c_1 + 3c_2 - \frac{1}{10} - \frac{1}{3} = -4c_1 + 3c_2 - \frac{13}{30} \end{cases}$$

$$\Rightarrow \begin{cases} c_1 + c_2 = \frac{71}{60} \\ -4c_1 + 3c_2 = \frac{103}{30}. \end{cases} \Rightarrow \begin{cases} c_1 = \frac{1}{60} \\ c_2 = \frac{7}{6} \end{cases}$$

$$\Rightarrow y(t) = \frac{1}{60} e^{-4t} + \frac{7}{6} e^{3t} - \frac{1}{10} e^{t} - \frac{1}{6} e^{2t} + \frac{1}{12}.$$

#3: (a) $y(t) = c_1 \cos t + c_2 \sin(t)$.

(b) $\begin{cases} 2 = y(0) = c_1 \\ 0 = y(\frac{\pi}{2}) = c_2. \end{cases} \Rightarrow$ $y(t) = 2\cos t$ is the unique sol^{n}.

(c) $\begin{cases} 2 = y(0) = c_1 \\ 0 = y(\pi) = -c_1 \end{cases}$ has no sol^{n}.

(d) $\begin{cases} 2 = y(0) = c_1 \\ -2 = (\pi) = -c_1 \end{cases}$

hence $y(t) = 2\cos t + c_2 \sin t$ is a sol^{n} for any $c_2 \in \mathbb{R}$.

#4: (a) general sol^{n}: $c_1 e^{t} + c_2 e^{-t} = y(t)$

$$y'(t) = c_1 e^{t} - c_2 e^{-t}.$$

- cosh: $\begin{cases} 1 = y(0) = c_1 + c_2 \\ 0 = y'(0) = c_1 - c_2 \end{cases} \Rightarrow c_1 = c_2 = \frac{1}{2}$

$$\cosh t = \frac{e^{t} + e^{-t}}{2}$$

- sinht $\begin{cases} 0 = y(0) = c_1 + c_2 \\ 1 = y'(0) = c_1 - c_2 \end{cases} \Rightarrow c_1 = \frac{1}{2}, \quad c_2 = \frac{-1}{2}.$

$$\sinh t = \frac{e^{t} - e^{-t}}{2}.$$

It's easy to check that $(\cosh t)' = \sinh t$ and $(\sinh t)' = \cosh t$.

(b) $c_1 e^{t} + c_2 e^{-t} = (c_1+c_2)\cosh t + (c_1-c_2)\sinh t$

#5: $\underline{b=5}$: general sol^{n}: $y(t) = c_1 e^{-4t} + c_2 e^{-t}$.

$y'(t) = -4c_1 e^{-4t} - c_2 e^{-t}$.

$$\begin{cases} 1 = y(0) = c_1 + c_2 \\ 0 = y'(0) = -4c_1 - c_2 \end{cases} \Rightarrow \begin{cases} c_1 = -\frac{1}{3} \\ c_2 = \frac{4}{3} \end{cases}$$

$$y_5(t) := -\frac{1}{3} e^{-4t} + \frac{4}{3} e^{-t}$$

$\underline{b=4}$: general sol^{n}: $y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$.

$y'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$

$$\begin{cases} 1 = y(0) = c_1 \\ 0 = y'(0) = -2c_1 + c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 2. \end{cases}$$

$$y_4(t) = e^{-2t} + 2t e^{-2t}.$$

$\underline{b=2}$: general sol^{n}: $y(t) = c_1 e^{-t} \cos(\sqrt{3}t) + c_2 e^{-t} \sin(\sqrt{3}t)$

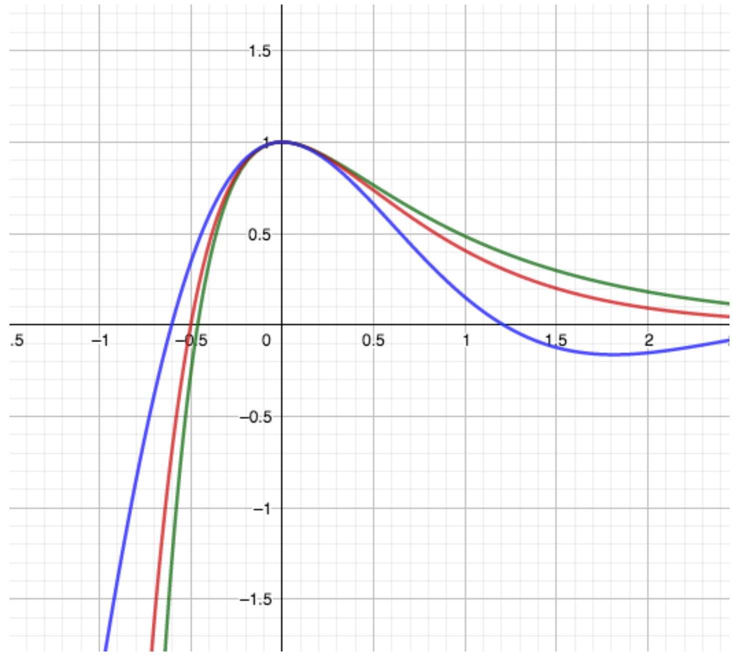
$y'(t) = -c_1 e^{-t} \cos(\sqrt{3}t) = \sqrt{3} c_1 e^{-t} \sin(\sqrt{3}t)$

$- c_2 e^{-t} \sin(\sqrt{3}t) + \sqrt{3} c_2 e^{-t} \cos(\sqrt{3}t)$

$$\begin{cases} 1 = y(0) = c_1 \\ 0 = y'(0) = -c_1 + \sqrt{3} c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = \frac{1}{\sqrt{3}} \end{cases}$$

$$y_2(t) = e^{-t} \cos(\sqrt{3}t) + \frac{1}{\sqrt{3}} e^{-t} \sin(\sqrt{3}t)$$

Sketch:



#6: (a) This is straightforward.

(b) general sol^n: $y(t) = c_1 \cos(t) + c_2 \sin(t)$.

$y'(t) = -c_1 \sin t + c_2 \cos t$.

$$\begin{cases} \sin x = y(0) = c_1 \\ \cos x = y'(0) = c_2 . \end{cases}$$

$$\Rightarrow \quad y(t) = \sin x \cos t + \cos x \sin t$$

(c) By the uniqueness thm, we have

$$\sin(x+t) = \sin x \cos t + \cos x \sin t .$$

#7: (a) $\{e^t \cos t, e^t \sin t\}$.

(b) The roots of the auxiliary eq'n are $1 \pm i$.

Hence $\underline{b = -2}$ and $\underline{c = 2}$.

$f(t)$ is a fcn s.t. $y'' - 2y' + 2y = f(t)$
has a sol^{n} given by t^2+1.

$$\Rightarrow f(t) = (t^2+1)'' - 2(t^2+1)' + 2(t^2+1)$$
$$= 2 - 4t + 2t^2 + 2$$
$$= 2t^2 - 4t + 4.$$

#8: Let $y_1(t) = \cos(t)$, $y_2(t) = \sin(t)$

$$v_1(t) = \int_0^t -f(s)\sin(s)\,ds, \quad v_2(t) = \int_0^t f(s)\cos(s)\,ds.$$

a sol^{n} is given by:

$$v_1(t)\,y_1(t) + v_2(t)\,y_2(t)$$
$$= \int_0^t -f(s)\sin(s)\cos(t) + f(s)\cos(s)\sin(t) \quad ds$$
$$= \int_0^t f(s)\sin(t-s)\,ds.$$

#9: $y(t)$ is either of the form

$$c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad r_1 \neq r_2 \text{ in } \mathbb{R}$$
or $$c_1 e^{rt} + c_2 t e^{rt} \qquad r \in \mathbb{R}.$$

(1) $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ for some $r_1 \neq r_2$ in $\mathbb{R}$,
$c_1, c_2 \in \mathbb{R}$, (not all zero).

The statement is clear if $c_1 = 0$ or $c_2 = 0$.
Now suppose $c_1 c_2 \neq 0$. Then

$$y(t_0) = 0 \iff \frac{-c_1}{c_2} = e^{(r_2 - r_1) t_0} \quad \text{—— } (*)$$

Since $r_2 \neq r_1$ and $\exp(T)$ is injective,
$(*)$ has at most 1 sol$^{\underline{n}}$.

(2) $y(t) = c_1 e^{rt} + c_2 t e^{rt}$ for some $r, c_1, c_2 \in \mathbb{R}$,
c_1, c_2 not all 0.

Again, the statement is clear if $c_1 = 0$ or $c_2 = 0$.
Now suppose $c_1 c_2 \neq 0$, then

$$y(t_0) = 0 \iff \frac{-c_1}{c_2} = t. \quad \square$$