FIRST MIDTERM PRACTICE PROBLEMS
MATH 104, SECTION 2

(1) Let $a_n = \sqrt{n^2+1} - n$. Prove that (a_n) is convergent based on the definition. (You're not allowed to use any theorem for this problem.)

(2) Let (a_n) be a sequence of real numbers where $a_1 = 1$ and
$$a_{n+1} = \frac{n}{n+3} a_n^2 \quad \text{for} \quad n \geq 1.$$
Prove that (a_n) is convergent and find the limit.

(3) Let $a_n = (n!)^{1/n}$. Is (a_n) convergent or divergent? (Recall that $n! = 1 \cdot 2 \cdot \dots \cdot n$.)

(4) Let (a_n) be a sequence of real numbers. Suppose that (a_n^3) is convergent. Prove that (a_n) is convergent.

(5) Let (a_n) be a sequence of real numbers satisfying
$$0 \leq a_{n+m} \leq a_n + a_m \quad \text{for any} \quad n, m \in \mathbb{N}.$$
Define $b_n := \frac{a_n}{n}$ for each n. Prove that the sequence (b_n) is convergent. (Hint: First prove that (b_n) is bounded. Let $z := \limsup b_n$. There exists a subsequence (b_{k_n}) such that $\lim b_{k_n} = z$. For any $m \in \mathbb{N}$, you can write $k_n = \ell_n m + r_n$ where $0 \leq r_n < m$. Then try to show that $z \leq b_m$ by taking $n \to \infty$ for certain inequality obtained from the assumption.)

(6) Consider the metric space $\mathbb{R}$ with the usual distance function $d(x,y) = |x - y|$. Prove or disprove the following statements.
  (a) $\mathbb{Q} \subseteq \mathbb{R}$ is an open subset.
  (b) $\mathbb{Q} \subseteq \mathbb{R}$ is a closed subset.