SECOND MIDTERM SOLUTION
MATH H54

(1) (15 points) Consider the symmetric matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. (You have to write down every steps of your calculations, not just the final answer.)

Solution.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}^T.$$

(2) Let $M_2(\mathbb{R})$ be the set of all real 2×2 matrices. It is a vector space with the standard matrix addition and scalar multiplication. Consider the function $\langle -, - \rangle : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to \mathbb{R}$ given by

$$\langle A, B \rangle := \operatorname{tr}(AB^T),$$

where $A, B \in M_2(\mathbb{R})$ and tr denotes the trace function. It is not hard to check that $\langle -, - \rangle$ gives an inner product on $M_2(\mathbb{R})$.

(a) (15 points) Construct an orthonormal basis (with respect to $\langle -, - \rangle$) for the subspace of $M_2(\mathbb{R})$ spanned by $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(b) (5 points) Consider another function $\langle -, - \rangle_2 : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to \mathbb{R}$ given by

$$\langle A, B \rangle_2 := \operatorname{tr}(AB).$$

Does $\langle -, - \rangle_2$ give an inner product on $M_2(\mathbb{R})$ as well? Prove your answer.

Solution. (a)

$$\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix} \right\}.$$

(b) No.

$$\left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle_2 = -2 < 0.$$

(3) True or False. For each of the following statements, either prove the statement, or give an explicit counterexample.

(a) (5 points) Let A be a square matrix. "If A^2 is diagonalizable, then so is A."

(b) (5 points) Let A be a square matrix. "If A is diagonalizable, then so is A^2."
(c) (15 points) "There does not exist an orthogonal matrix such that 2 is one of its eigenvalues."

Solution. (a) False. Counterexample: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.
(b) True. If $A = PDP^{-1}$, then $A^2 = PD^2P^{-1}$.
(c) True. Let A be an orthogonal matrix. Then $||A\vec{v}|| = ||\vec{v}||$ for any $\vec{v}$. Hence any eigenvalue $\lambda \in \mathbb{C}$ of A satisfies $|\lambda| = 1$.

(4) (20 points) Let $(V, \langle -, - \rangle)$ be an inner product space, and let $T \colon V \to V$ be a linear transformation. Suppose that $||T(\vec{x})|| = ||\vec{x}||$ for any $\vec{x} \in V$. Prove that

$$\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle \quad \text{for any } \vec{x}, \vec{y} \in V.$$

(Hint: Consider $||T(\vec{x}+\vec{y})||^2 = ||\vec{x}+\vec{y}||^2$.)
Solution. $||\vec{x}+\vec{y}||^2 = \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle = ||\vec{x}||^2 + ||\vec{y}||^2 + 2\langle x, y \rangle$. Similarly, we have $||T(\vec{x}+\vec{y})||^2 = ||T(\vec{x})||^2 + ||T(\vec{y})||^2 + 2\langle T(x), T(y) \rangle$. The assumption that $||T(\vec{x})|| = ||\vec{x}||$ for any $\vec{x} \in V$ therefore implies that $\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$ for any $\vec{x}, \vec{y} \in V$.

(5) (20 points) Let $A_1, \ldots, A_k$ be $n \times n$ real symmetric matrices. Suppose that $A_1^2 + \cdots + A_k^2 = 0$ (the zero matrix). Prove that $A_1 = \cdots = A_k = 0$ (the zero matrix). (Hint: Consider $\vec{x}^T(A_1^2 + \cdots + A_k^2)\vec{x}$.)
Solution. We have $0 = \vec{x}^T(A_1^2 + \cdots + A_k^2)\vec{x} = \vec{x}^T A_1^T A_1 \vec{x} + \cdots \vec{x}^T A_k^T A_k \vec{x} = ||A_1\vec{x}||^2 + \cdots + ||A_k\vec{x}||^2$ for any $\vec{x}$. Hence $A_i\vec{x} = 0$ for any $1 \le i \le k$ and $\vec{x}$. Thus $A_i = 0$ for any i.

(6) (20 points) Let A be an $n \times n$ diagonalizable matrix with $n-1$ distinct eigenvalues. Prove that for any $\vec{v} \in \mathbb{R}^n$, the set $\{\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}\}$ is linearly dependent.
Solution. Write $A = PDP^{-1}$. Note that the statements "the set $\{\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}\}$ is linearly dependent for any $\vec{v}$" and "the set $\{P\vec{v}, PD\vec{v}, \ldots, PD^{n-1}\vec{v}\}$ is linearly dependent for any $\vec{v}$" are equivalent (why?). Also, since P is invertible, the above statement is equivalent to "the set $\{\vec{v}, D\vec{v}, \ldots, D^{n-1}\vec{v}\}$ is linearly dependent for any $\vec{v}$" (why?).

Let $\vec{v} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T$ and let $\lambda_1, \ldots, \lambda_n$ be the diagonal entries of D. Consider the $n \times n$ matrix with columns $\{\vec{v}, D\vec{v}, \ldots, D^{n-1}\vec{v}\}$:

$$\begin{bmatrix} v_1 & \lambda_1 v_1 & \cdots & \lambda_1^{n-1} v_1 \\ v_2 & \lambda_2 v_2 & \cdots & \lambda_2^{n-1} v_2 \\ \vdots & \vdots & & \vdots \\ v_n & \lambda_n v_n & \cdots & \lambda_n^{n-1} v_n \end{bmatrix}.$$

Since A has repeat eigenvalues, so this matrix has linearly dependent rows, hence not invertible. Therefore the set $\{\vec{v}, D\vec{v}, \ldots, D^{n-1}\vec{v}\}$ is linearly dependent for any $\vec{v}$.