Math H54 (Math 49) Final Exam December 17, 2019

Name: Solution

- You have 170 minutes to complete the exam (3:10pm – 6:00pm).
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using. Exception: if you are asked to prove a theorem, you are not allowed to use it!
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total		

1. (10 points) Find a function $y(t)$ satisfying

$$y''(t) - 2y'(t) + 2y(t) = 2t - 2; \quad y(0) = 4, \quad y'(0) = 2.$$

c.f. Problem 8 of H54 final.

2. (10 points) Find a function $y(t)$ satisfying

$$y''(t) + 2y'(t) + y(t) = t^2 + te^{-t}; \quad y(0) = 2, \quad y'(0) = 1.$$

① Find a particular sol'n of $y'' + 2y' + y = t^2$.

$\rightsquigarrow \quad y_1 = t^2 - 4t + 6$

② - - - - - - - - - $y'' + 2y' + y = te^{-t}$.

$\rightsquigarrow \quad y_2 = \frac{1}{6} t^3 e^{-t}$

③ Homogeneous sol'n: $y'' + 2y' + y = 0$.

$\rightsquigarrow \quad y_h = c_1 e^{-t} + c_2 te^{-t}$.

$\Rightarrow$ general sol'n: $c_1 e^{-t} + c_2 te^{-t} + t^2 - 4t + 6 + \frac{1}{6} t^3 e^{-t}$

$2 = y(0) = c_1 + 6 \quad \Rightarrow \quad c_1 = -4.$

$1 = y'(0) = -c_1 + c_2 - 4 \quad \Rightarrow \quad c_2 = 1$

$\Rightarrow \underline{\underline{y(t)}} = -4e^{-t} + te^{-t} + t^2 - 4t + 6 + \frac{1}{6} t^3 e^{-t}.$

$\square$

3. In this problem, you will find three functions $y_1(t), y_2(t), y_3(t)$ satisfying

$$y_1'(t) = 4y_2(t), \quad y_2'(t) = 4y_1(t) + 3y_3(t), \quad y_3'(t) = 3y_2(t)$$

and the initial conditions

$$y_1(0) = 0, \quad y_2(0) = -10, \quad y_3(0) = 0.$$

(a) (2 points) Write the differential equations as a matrix equation $\vec{y}\,'(t) = A\vec{y}(t)$ for some matrix A, and write the initial conditions as a vector equation $\vec{y}(0) = \vec{b}$ for some vector $\vec{b}$.

cf. Problem 9 of H54 final.

(b) (3 points) Find a diagonalization of A.

(c) (3 points) Find a fundamental matrix of $\vec{y}\,'(t) = A\vec{y}(t)$.

(d) (2 points) Finish the problem, i.e. find the functions $y_1(t), y_2(t), y_3(t)$.

4. (10 points) Recall that the uniqueness theorem for system of linear differential equations states that for any real $n \times n$ matrix A, any $t_0 \in \mathbb{R}$ and any $\vec{x}_0 \in \mathbb{R}^n$, there exists a unique vector valued function $\vec{x}(t) : \mathbb{R} \to \mathbb{R}^n$ satisfying

$$\vec{x}\,'(t) = A\vec{x}(t) \quad \text{and} \quad \vec{x}(t_0) = \vec{x}_0.$$

Use the uniqueness theorem to prove that if $\{\vec{x}_1(t), \ldots, \vec{x}_n(t)\}$ is a linearly independent set of solutions of the homogeneous equation $\vec{x}\,'(t) = A\vec{x}(t)$, then the Wronskian

$$W[\vec{x}_1, \ldots, \vec{x}_n](t) = \det \begin{bmatrix} | & & | \\ \vec{x}_1(t) & \cdots & \vec{x}_n(t) \\ | & & | \end{bmatrix}$$

is not zero for any $t \in \mathbb{R}$.

cf. Problem 1(e) in H54 final

5. (10 points; 2 parts)

(a) Find the Fourier sine series of the function $f(x) = x(x - \pi)$ on $0 < x < \pi$.
Hint: Integration by parts, with patience.

c.f. Problem 10 in H54 final

(b) Find a function $u(x,t)$ satisfying

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$
$$u(0,t) = u(\pi,t) = 0, \quad t > 0,$$
$$u(x,0) = x(x-\pi), \quad 0 < x < \pi.$$

6. (10 points) Find a function $u(x,t)$ satisfying

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R} \quad t > 0,$$
$$u(x,0) = \sin x$$
$$\frac{\partial u}{\partial t}(x,0) = \sin x$$

Hint: Recall d'Alembert's solution for wave equation looks like $A(x+\alpha t) + B(x-\alpha t)$.

$u(x,t) = A(x+3t) + B(x-3t)$ satisfies $\partial_t^2 u = 9\partial_x^2 u$.

$$\begin{cases} A(x) + B(x) = \sin x. \\ 3A'(x) - 3B'(x) = \sin x \end{cases} \Rightarrow A(x) - B(x) = \int_{x_0}^{x} \frac{1}{3}\sin(s)\,ds + C$$

$$\Rightarrow \quad A(x) = \frac{1}{2}\left(\sin x + \int_{x_0}^{x} \frac{1}{3}\sin(s)\,ds + C\right)$$
$$B(x) = \frac{1}{2}\left(\sin x - \int_{x_0}^{x} \frac{1}{3}\sin(s)\,ds + C\right)$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left(\sin(x+3t) + \sin(x-3t)\right) + \frac{1}{6}\int_{x-3t}^{x+3t} \sin(s)\,ds$$

$$= \frac{1}{2}\left(\sin(x+3t) + \sin(x-3t)\right) + \frac{1}{6}\left(-\cos(x+3t) + \cos(x-3t)\right)$$

$$= \sin x \cos(3t) + \frac{1}{3}\sin x \sin(3t)$$