Solitons in the infinite Relativistic Toda lattice

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M - seminar

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Plan :

- Relativistic Toda lattice
- Coxeter-Toda lattices, Poisson Lie groups GL_∞
- Discrete (factorization) dynamics
- Solitons in an infinite relativistic Toda lattice

Relativistic Toda system (Toda latte) (Ruijsenaars, 1987, 1990; Ruijsenaars, Schneider, 1986)

Hamiltonian dynamical system on $\mathbb{R}_c^{\infty,\infty}$: soliton

$p, q : \mathbb{Z} \to \mathbb{R}$, $\{p_n, q_m\} = \delta_{n,m}$, p_n, q_n - variables, c constant $c \in \mathbb{R}$

$p_n \to 0$, $q_n \to c$ as $n \to \pm\infty$, (rapidly)

$f(x) \to c$, $x \to \pm\infty$

The Hamiltonian

$$(1)\ \checkmark \quad \boxed{H_\varepsilon = \sum_{n\in\mathbb{Z}} \left(e^{\varepsilon p_n} + \varepsilon^2 e^{\varepsilon(p_n+p_{n+1}) + q_{n-1} - q_{n+1}} - 1 - \varepsilon^2\right)}$$

convergent series on $\mathbb{R}_c^{\infty,\infty}$.

Faddeev Takht ......

As $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0} \frac{H_\varepsilon}{\varepsilon^2} = \sum_{n\in\mathbb{Z}} \left(\frac{p_n^2}{2} + e^{q_{n-1}-q_{n+1}}\right) \quad \text{(Toda limit)}$$

Toda Hamiltonian

Equation of motion: $\{p_n(t), q_n(t)\}$

$$\dot{p}_n(t) = -\frac{\partial H_\varepsilon}{\partial q_n}(p(t), q(t)) = \{H_\varepsilon, p_n\}$$

$$\dot{q}_n(t) = \frac{\partial H_\varepsilon}{\partial p_n}(p(t), q(t)) = \{H_\varepsilon, q_n\}$$

ε - parametrizes a family of integrable systems (one for each ε).

The goal is to construct solutions

$\{q_n(t), p_n(t)\}$ ⟶ the soliton solution

discret time version: $\alpha : \mathbb{R}_c^{\infty,\infty} \circlearrowleft$ Poisson mapping (later) $x(t) = \alpha^t(\alpha)$

Poisson commuting Hamiltonians:

$(M_{2n}, \omega) \leftarrow$ Lagr.

$\downarrow$

B_n $\quad I_1, \dots, I_n$

$C(M_{2n}) \supset I$

$I \simeq C(B)$

Two linear operators on $\mathbb{R}^\infty = \mathbb{R}^{\mathbb{Z}}$

$$\begin{cases} (L^+\varphi)_n \overset{def}{=} \varphi_n e^{\frac{\varepsilon p_n}{2}} + \varepsilon e^{q_n - q_{n+1}} \varphi_{n+1}, \\ (L^-\varphi)_n \overset{def}{=} \varphi_n e^{-\frac{\varepsilon p_n}{2}} - \varepsilon e^{q_{n-1} - q_n} \varphi_{n-1}, \end{cases}$$

$$\begin{cases} (L_0^+\varphi)_n = \varphi_n + \varepsilon \varphi_{n+1} \\ (L_0^-\varphi)_n = \varphi_n - \varepsilon \varphi_{n-1} \end{cases}$$

$$H_\varepsilon = \mathrm{Tr}_{\mathbb{R}^\infty}\left(L^+(L^-)^{-1} - (L_0^+(L_0^-)^{-1}) \right)$$

$$H_k = \mathrm{Tr}^{reg}_{\mathbb{R}^\infty}\left((L^+(L^-)^{-1})^k \right), \text{ equivalently, } H_{\omega_k} = \mathrm{Tr}^{reg}_{\Lambda^k \mathbb{R}^\infty}\left(L^+(L^-)^{-1} \right),$$

$$\boxed{\begin{array}{l} \underline{\text{Thm}}.\ \{H_{\omega_k}\} \text{ is a Poisson commuting family} \\ \text{of functions on } \mathbb{R}^{\infty,\infty}_c. \text{ Each } H_{\omega_k} \text{ is a} \\ \text{convergent series.} \end{array}}$$

For Toda yes
For R.Toda yes

Double Bruhat cells

G - simple, complex, algebraic, $B \subset G$ Borel

B^- - opposite Borel, $H \subset B \subset G$ Cartan

$N(H) \subset G$, $H \subset N(H)$ normal, $N(H)/H \simeq W$, Weyl group

$$G = \bigsqcup_{u \in W} BuB = \bigsqcup_{v \in W} B^- v B^-$$ Bruhat decompositions

$$G^{u,v} = BuB \cap B^- v B^-, \quad G = \bigsqcup_{u,v} G^{u,v}$$ double Bruhat cells

Cluster coordinates on $G^{u,v}$:

- $u = s_{i_1} \cdots s_{i_\ell}$, $v = s_{j_1} \cdots s_{j_m}$ reduced decompositions
- $\{k_1, \dots, k_{\ell+m}\}$ = a shuffle of $\{i_1 \dots i_\ell\}\{j_1 \dots j_m\}$

$$G^{u,v}_{\{k\}} = \{ H \exp(t_1 X_{k_1}) \cdots \exp(t_{m+\ell} X_{k_{\ell+m}}) \}, \quad t_i \neq 0$$

coord. charts

$i = 1 \dots r$
H_i, e_α, f_β
e_i, f_i

$x_k = e_{i_\alpha}$ if $k = i_\alpha$; $x_k = f_{j_\beta}$ if $k = j_\beta$

Thm. (Fomin - Zelevinsky) ① $G^{u,v}_{\{k\}} \subset G^{u,v}$ is a coordinate atlas for $G^{u,v}$. Transition functions are mutations of cluster variables (sum of two monomials)

② Let $G_{\geq 0}$ be totally nonnegative part of the split real form. $((SL_n)_{\geq 0} = \{\text{all minors} \geq 0\})$.

$\underline{G^{u,v}_{>0} = G^{u,v} \cap G_{\geq 0}} = G^{u,v}_{\{k\}} \cap G_{\geq 0}$ for any $\{k\}$

i.e. $\underline{G^{u,v}_{>0} \simeq \mathbb{R}^{r+l+m}}$ $l = l(u)$, $m = l(v)$,

Example: $SL_2 = \underline{SL_2^{s,s}} \sqcup \underline{SL_2^{s,1}} \sqcup \underline{SL_2^{1,s}} \sqcup \underline{SL_2^{1,1}}$

$\underline{SL_2^{s,s}} = \{\exp(te)H\exp(sf)\} \cup \{\exp(sf)H\exp(te)\} = \{\begin{pmatrix} * & \neq 0 \\ \neq 0 & * \end{pmatrix}\}$

$= \{\begin{pmatrix} \neq 0 & \neq 0 \\ \neq 0 & * \end{pmatrix}\} \cup \{\begin{pmatrix} * & \neq 0 \\ \neq 0 & \neq 0 \end{pmatrix}\}$) , ...

Standard Poisson-Lie structure

$G = SL_n$ (for Toda this is enough)

$\mathbb{C}^n \otimes \mathbb{C}^n$, $r = \frac{1}{2}\sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i<j} e_{ij} \otimes e_{ji}$, (Cherednik, Drinfeld, ...)

$\{g_{ij}, g_{kl}\} = [r, g \otimes g]_{ij,kl}$, **claim**: Poisson bracket and Poisson Lie (compatible with $G \times G \to G$)

or

$n \times n$ $\quad \{g \otimes 1, 1 \otimes g\} = [r, g \otimes g]$,

Poisson commuting functions: $\chi_V(g) = tr_V(\pi_V(g))$

$\{\chi_V(g), \chi_W(g)\} = Tr_{V \otimes W}(\{g \otimes 1, 1 \otimes g\}) =$

$= Tr_{V \otimes W}([r, g \otimes g]) = 0$

cyclic prop. of the trace.

Construction of integrable systems: G - Poisson manifold

- Fix $S \subset G$ a symplectic leaf
- $H_v = \chi_v|_S$, $\{H_v, H_w\} = 0$, this Poisson commuting ring I_S has rank $\leq$ rank(G)

When $\dim(S) = 2\,\mathrm{rank}(I_S)$ we have a Liouville integrable system.

Thm (Hodges-Levasseur; Vaksman-Soibelman, compact version, ...) Double Bruhat cells are homogeneous Poisson varieties fibered by symplectic leaves of dimension

$S^{u,v} \subset G^{u,v}$

$$\boxed{\dim(S^{u,v}) = \ell(u) + \ell(v) + \dim(\ker(uv^{-1}))}$$

Example: The only example relevant to rel. Toda

$c = s_1 \cdots s_{n-1}$, $u, v = c$ Coxeter elements

$$\exp(t_{n-1}e_{n-1})\exp(t_{n-2}e_{n-2})\cdots\exp(t_1 e_1) = \begin{pmatrix} 1 & \neq 0 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \neq 0 \\ 0 & & & 1 \end{pmatrix} \text{ if } t_i \neq 0$$

$$\exp(s_1 f_1)\cdots\exp(s_{n-2}f_{n-2})\exp(s_{n-1}f_{n-1}) = \begin{pmatrix} 1 & & & 0 \\ \neq 0 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & \neq 0 & 1 \end{pmatrix} \text{ if } s_i \neq 0$$

$$G^{c,c}_{id} = \left\{ \begin{pmatrix} * & \neq 0 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \neq 0 \\ 0 & & & * \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ \neq 0 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & \neq 0 & 1 \end{pmatrix}^{-1} \right\},$$ similarly for other shuffles.

coord. chart corresp. to trivial shuffle

<u>Corollary</u>

$$G^{c,c}_{>0} = \{ L^+ (L^-)^{-1} \} = \{ (L^-)^{-1} L^+ \} = \cdots \simeq \mathbb{R}^{3(n-1)}$$

Because cluster coordinates in this case are global

$$L^+ = \begin{pmatrix} \diagdown\!\diagdown & 0 \\ 0 & \end{pmatrix} \text{ (entries } > 0\text{)}, \quad L^- = \begin{pmatrix} \diagdown\!\diagdown & 0 \\ 0 & \end{pmatrix} \text{ (entries } < 0\text{)}$$

$$L^+ = \begin{pmatrix} a_1 & b_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & b_{n-1} \\ 0 & & & a_n \end{pmatrix}, \quad L^- = \begin{pmatrix} a_1^{-1} & & & 0 \\ -c_1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & -c_{n-1} & a_n^{-1} \end{pmatrix}, \qquad b_i, c_i\ a_i > 0$$

Poisson Casimirs b_i/c_i, all symplectic leaves are isomorphic. Cluster Darbou coordinates:

$$\frac{a_i}{a_{i-1}} = e^{\frac{\varepsilon p_i}{2}}, \quad b_i = \varepsilon e^{q_i - q_{i+1}}, \quad c_i = \varepsilon e^{q_{i-1} - q_i} \quad \boxed{\frac{b_i}{c_i} = 1}$$

Mutations induce discrete time dynamics on $G^{c,c}_{>0}$

In particular, the factorization dynamics:

$$\boxed{L^+_{t+1} (L^-_{t+1})^{-1} = (L^-_t)^{-1} L^+_t} \leftarrow L^+ (L^-)^{-1} = (\tilde{L}^-)^{-1} \tilde{L}^+,$$

$x(t) = \alpha^t(x)$ $\quad \alpha : G^{c,c}_{id, >0} \circlearrowleft$ symplectomorphis

SL_n "$n \to \infty$"

GL_∞ with the standard Poisson Lie structure

- $GL_\infty = \{ \dots \exp(t e_{ij}) \dots \}^{gl_\infty}$, possibly infinite product

  $GL_\infty^{small} = \text{ind limit of } GL_N$, finite products

- Bruhat decomposition, $W = S_\infty$, $S_\infty^{small} = \text{ind limit } S_N$

- Double Bruhat cells, positivity

← defined

$GL^{u,v}_{\infty,id} \subset GL^{u,v}_\infty$

- Standard Poisson Lie structure

$$r = \frac{1}{2}\sum_{i \in \mathbb{Z}} e_{ii} \otimes e_{ii} + \sum_{i<j} e_{ij} \otimes e_{ji},$$

$$\{g \otimes 1, 1 \otimes g\} = \varepsilon [r, g \otimes g]$$

if g are as above the sum is finite

- $GL_\infty^{u,v} \subset GL_\infty$ fibered over an ∞-dim torus

  fibers = symplectic leaves (.....)

- $u = v = c$ = Coxeter element $c = \cdots s_{i+1} s_i \cdots$

  (translation on $\mathbb{Z}$)

Coxeter-Toda symplectic leaves

$$GL_\infty^{c,c} = \underset{\substack{\text{shuffles of} \\ \mathbb{Z}\times\mathbb{Z}}}{\bigsqcup} GL_\infty^{c,c}(sh),$$

Coxeter double Bruhat cell

$$L^+ = \begin{pmatrix} \stackrel{\geq 0}{\diagdown\!\diagdown} & {\scriptstyle\geq 0}\; 0 \\ 0 & \end{pmatrix}, \quad L^- = \begin{pmatrix} {\scriptstyle \geq 0} & 0 \\ 0 & \stackrel{\diagdown\!\diagdown}{\scriptstyle \leq 0} \end{pmatrix}$$

positive part $\simeq GL_\infty^{c,c}(sh) = \{ \underline{L^+(L^-)^{-1}} \}$ all minor positive

$$L^+ = \begin{array}{cl} \left[\begin{matrix} \ddots & & \\ & \boxed{\begin{matrix} \frac{A_n}{A_{n-1}}, & B_n \\ 0 & \frac{A_{n+1}}{A_n} \end{matrix}} & \\ & & \ddots \end{matrix}\right] & \begin{matrix} n \\ n+1 \end{matrix} \end{array}, \quad L^- = \begin{array}{cl} \left[\begin{matrix} \ddots & & \\ & \boxed{\begin{matrix} \frac{D_n}{D_{n-1}}, & 0 \\ C_n, & \frac{D_{n+1}}{D_n} \end{matrix}} & \\ & & \ddots \end{matrix}\right] & \begin{matrix} n \\ n+1 \end{matrix} \end{array},$$

Poisson Casimirs B_n/C_n

$$\underline{L^+(L^-)^{-1}} = \begin{pmatrix} \ddots & & & 0 \\ & \boxed{\begin{matrix} 1 & W_n \\ 0 & 1 \end{matrix}} & W_{n+1} & \\ 0 & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \ddots & & & 0 \\ & \boxed{\begin{matrix} \frac{A_n D_{n-1}}{A_{n-1} D_n} & 0 \\ 0 & \frac{A_{n+1} D_n}{A_n D_{n+1}} \end{matrix}} & & \\ 0 & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & \\ \vdots & \boxed{\begin{matrix} 1 & 0 \\ V_n & 1 \end{matrix}} & & 0 \\ 0 & & V_{n+1} & 1 \\ & & \ddots & \ddots \end{pmatrix}$$

$$W_n = B_n \frac{A_n}{A_{n+1}}, \quad V_n = -C_n \frac{D_{n-1}}{D_n} \qquad \frac{B_n}{C_n} \text{ - Casimirs}$$

Darboux coordinates on symplectic leaf with $\underline{\frac{B_n}{C_n} = 1}$

$$\frac{D_{n-1}}{D_n} \underset{\text{our choice}}{=} \frac{A_n}{A_{n-1}} = e^{\frac{\varepsilon p_n}{2}}, \qquad B_n = C_n = \varepsilon e^{\frac{q_n - q_{n+1}}{2}},$$

$$\{p_n, q_m\} = \delta_{n,m},$$

Define variables

$$x_n^+ = -B_n C_n \frac{D_{n-1} A_n}{D_n A_{n+1}}, \quad x_n^- = -B_n C_n \frac{A_{n-1} D_n}{A_n D_{n+1}}$$

When x_n^+, x_n^- real and <0, they are coord. on the positive part of the sympl. leaf.

In p, q coordinates:

$$① \begin{cases} x_n^+ = -\varepsilon^2 e^{q_n - q_{n+1} + \varepsilon(p_n - p_{n+1})} \\ x_n^- = -\varepsilon^2 e^{q_n - q_{n+1} - \varepsilon(p_n - p_{n+1})} \end{cases}$$

Poisson brackets

1 $\{x^+_n, x^+_m\} = \{x^-_n, x^-_m\} = 0$

$\{x^+_n, x^-_m\} = \varepsilon\, C_{n,m}\, x^+_n x^-_m \,,$ $\qquad C_{nm} = \begin{cases} 2 \,, & n = m \\ -1 \,, & n = m \pm 1 \\ 0 \,, & \text{otherwise} \end{cases}$

$S_c = \{ p : \mathbb{Z} \to \mathbb{R},\ q : \mathbb{Z} \to \mathbb{R} \,;\ \text{s.t. } q_n \to c,\ p_n \to 0,\ n \to \pm\infty \}$

rapidly

(asymptotically the lattice is static, $q_n = c$ & $p_n = 0$)

Hamiltonians in the relativistic Toda lattice

Denote $(L^+(L^-)^{-1})_0$ the asymptotical value of $L^+(L^-)^{-1}$

$$L^+(L^-)^{-1} = \begin{pmatrix} \ddots & \ddots & & \\ & 1 & \varepsilon & & 0 \\ & & 1 & \varepsilon & \\ & & & 1 & \varepsilon \\ & 0 & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \ddots & & & & \\ \ddots & 1 & & & 0 \\ & -\varepsilon & 1 & & \\ & & -\varepsilon & 1 & \\ & 0 & & -\varepsilon & \ddots \end{pmatrix}^{-1}$$

Poisson commuting Hamiltonians:

$$\begin{cases} H_V = \mathrm{Tr}_V\left(L^+(L^-)^{-1} - (L^+(L^-)^{-1})_0 \right), \\ H_{RT} = H_{\mathbb{C}^2} = \sum_n \left(e^{\varepsilon p_n} \left(1 + \varepsilon^2 e^{\frac{\varepsilon}{2}(p_{n-1} - p_n) + q_{n-1} - q_n}\right) - \right. \\ \qquad \left. - (1 + \varepsilon^2) \right) \end{cases}$$

$(H_{RT} = \underline{S_{+1}}$ in Ruijsenaars notations). S_{-1}

Factorization dynamics

On the positive part of each symplectic leaf of $GL_\infty^{c,c} \subset GL_\infty$ we have

$$L^+(L^-)^{-1} = \alpha(L^-)^{-1}\alpha(L^+), \quad \alpha: S_c \to S_c$$

symplectomorphism

Thm (Hoffman, Kellendonk, Kutz, R., 1999) for G simple

$$\alpha(x_i^+) = x_i^-, \quad \alpha(x_i^-) = \frac{(x_i^-)^2}{x_i^+} \prod_j (1 - x_j^-)^{-c_{ij}}$$

Discret time discrete space infinite dimensional dynamical system; ∞ many integrals. Is it integrable?

Spectral problem

(L. Faddeev, L. Takhtajan
Hamiltonian methods
...)

Let L^+, L^- be as above

$$L^+(L^-)^{-1}x = z^2 x$$

. charact. spectrum
values of z...

$\{p_n, q_n\} \rightleftarrows$ Spec.

action
angle variab

$\varphi_n = z^n x_n$,

$$\boxed{(*) \quad (b_n z - b_n^{-1} z^{-1})\varphi_n = c_{n-1}\varphi_n + c_n \varphi_n}$$

$b_n = e^{\frac{\varepsilon}{2} p_n}$, $c_n = \varepsilon e^{\frac{q_n - q_{n+1}}{2}}$, $z \in \mathbb{R}$,

$n \in \mathbb{Z}$, $q_n \to c$, $p_n \to 0$, (fast) as $n \to \pm\infty$

Jost solutions:

$\phi_n(\omega) \to \omega^n$, $n \to +\infty$; $\psi_n(\omega) \to \omega^{-n}$, $n \to -\infty$

$|w| = 1$, $\boxed{\underset{z \in \mathbb{R}}{z - z^{-1}} = \underset{|w|=1}{\varepsilon (w + w^{-1})}}$ ($\varepsilon \to 0$, Toda limit)

Thm Jost solutions exist, unique and holomorphic in the unit disc $|w| < 1$.

periodic rel. Toda
f. d. int. system
80-90

When $|w| = 1$ we have linear dependence

$$\psi_n(w) = \alpha(w)\, \phi_n(w^{-1}) + \beta(w)\, \phi_n(w),$$

$$\phi_n(w) = \alpha(w^{-1})\, \psi_n(w^{-1}) - \beta(w^{-1})\, \psi_n(w),$$

For $|w| \le 1$

$$\phi_n(w) = \sum_{m=n}^{\infty} K(n,m)\, w^m$$

Thm $K(n,m)$ can be computed recursively from (*),

$$K(n,n) = e^{c - q_n - \varepsilon \sum_{m<n} p_m}$$

. . . .

Define the Wronskian of two Jost solutions as

$$W(n) = \phi_n(\bar{\omega}^{-1})\,\phi_{n+1}(\omega) - \phi_{n+1}(\bar{\omega}^{-1})\,\phi_n(\omega)$$

Thm. (i) $W(n) = \dfrac{\varepsilon(\omega - \bar{\omega}^{-1})}{c_n}$,

(ii) $\alpha(\omega) = \dfrac{c_n}{\varepsilon(\omega - \bar{\omega}^{-1})}\left(\psi_n(\omega)\,\phi_{n+1}(\omega) - \psi_{n+1}(\omega)\,\phi_n(\omega)\right)$

Spectrum of L (of (*))

when $p_n \to 0$, $q_n \to c$ sufficiently fast

<u>Thm</u>. Spectrum = continuous $\sqcup$ discrete

discrete = finitely many eigenvalues ~ zeroes of $\alpha(\omega)$

<u>Spectral data:</u>

finitely many

$$\boxed{\left\{ R(\omega) = \frac{\beta(\omega)}{\alpha(\omega)},\ |\omega| = 1 \right\} \sqcup \left\{ \begin{array}{l} \omega_j,\ |\omega_j| < 1,\ \alpha(\omega_j) = 0 \\ c_j - \text{normalization coeff.} \end{array} \right\}}$$

<u>Inverse spectral problem</u>

<u>Theorem</u> (a version of Gelfand-Levitan-Marchenko theorem)

Coefficients $K(n,m)$ of Jost solutions are the unique solution to

$$\begin{cases} \delta_{n,m}\, \varkappa(n,m) + F(n+m) + \sum\limits_{m'>n} \varkappa(n,m')\, F(m+m') = 0 \\[2ex] \dfrac{1}{K(n,n)^2} = 1 + F(2n) + \sum\limits_{m>n} \varkappa(n,m)\, F(n+m) \end{cases}$$

where $F(n) = \underset{\text{cont.}}{F_c(n)} + \underset{\text{discrete}}{F_B(n)}$

$$F_c(n) = \frac{1}{2\pi i} \int_{|\omega|=1} R(\omega)\, \omega^{n-1}\, d\omega\,, \quad F_B(n) = \sum_j c_j^2\, \varepsilon(1-\omega_j^{-2})\, \omega_j^n\,,$$

Taking into account that $K(n,n) = e^{c-q_n}, \ldots$

we can reconstruct p_n, q_n from the spectral data.

One soliton solution

(a) Spectral data : $R(\omega)=0$, $\omega_1 = e^{-\gamma} \in \mathbb{R}$,

$\boxed{F_c = 0 ,}$ $\boxed{F_\ell(m) = c_1^2 \varepsilon (1-\omega_1^{-2})\, \omega_1^m}$

Solution for $x(n,m)$, $K(n,n)$:

$$x(n,m) = -\ \varepsilon\,(1-\omega_1^{-2})\,\omega_1^m \ \frac{c_1^2\,\omega_1^n}{1-e^{2\delta}\omega_1^{2n}} \ , \quad e^{\delta} = \varepsilon^{1/2} c_1$$

$$K(n,n) = \sqrt{\frac{1-e^{2\delta}\,\omega_1^{2(n-1)}}{1-e^{2\delta}\,\omega_1^{2n}}}$$

From here

$$\boxed{e^{q_{n-1}-q_n} = 1 - \frac{\operatorname{sh}^2(\gamma)}{\operatorname{sh}^2(\gamma(n-1)-\delta)} \ ,}$$

(b) One soliton solution for the factorization dynamics

$$\boxed{e^{q_{n-1}(t) - q_n(t)} = 1 - \frac{sh^2(\gamma)}{sh^2(\gamma(n-1) - \beta t - \delta)}} ,$$

$$e^{\beta} = 1 + \varepsilon \omega_1^{-1} z_1 , \quad e^{\delta} = \varepsilon^{1/2} c_1 ,$$

(c) similarly N-soliton solutions.

Integrability

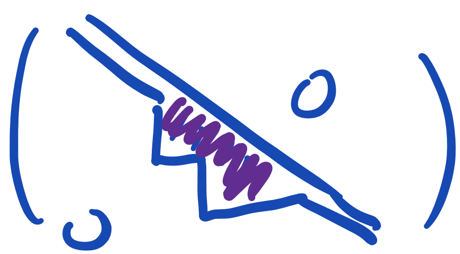
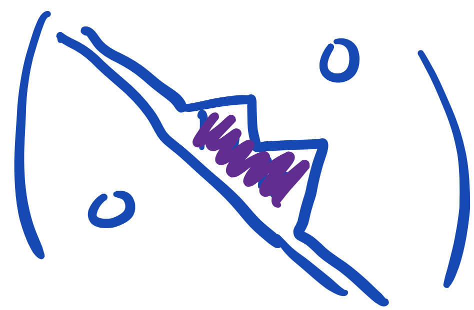
$$\{p_n, q_n\} \longleftrightarrow \text{Spectral data}$$

Spectral data = action-angle variables
(as in Toda)

∞ - dimensional analytic integrable system
(as KdV, Toda, ...)

Open problems

- other symplectic leaves $\quad S^{u,v} \quad$ finite $\quad u = u_0 c$, $v = u_0 c$

$L^- =$  , $L^+ =$ 

∞ - dim superintegrable systems with $\uparrow$
solitons