HOMEWORK 6
MATH H54

Yu-Wei's Office Hours: Sunday 1-2:30pm and Thursday 12-1:30pm (PDT)
Michael's Office Hours: Monday 12-3pm (PDT)

Part I (no need to turn in)

This part of the homework provides some routine computational exercises. You don't have to turn in your solutions for this part, but being able to do the computations is vitally important for the learning process, so you definitely should do these practices before you start doing Part II of the homework.

The following exercises are from the corresponding sections of the UC Berkeley custom edition of Lay, Nagle, Saff, Snider, *Linear Algebra and Differential Equations*.

- **Exercise 6.2**: 15, 21
- **Exercise 6.7**: 21

Part II (Due October 20, 8am PDT)

Some ground rules:

- You have to submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW6**.
- The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own.
- You're allowed to use any result that is proved in the lecture. But if you'd like to use other results, you have to prove it first before using it.

Problems:

(1) Let V be an inner product space and $W = \mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$ be a subspace of V. Suppose that $\vec{x} \in V$ is orthogonal to each $\vec{v}_i$ for $1 \leq i \leq n$. Prove that $\vec{x}$ is orthogonal to every vector in W.

(2) Let V be an inner product space and $W \subseteq V$ be a subspace. (They could be infinite dimensional.)

   (a) Prove that the orthogonal complement
$$W^{\perp} := \{\vec{x} \in V \colon \langle \vec{x}, \vec{w} \rangle = 0 \text{ for any } \vec{w} \in W\}$$

is a subspace of V.

(b) Prove that $W \cap W^{\perp} = \{\vec{0}\}$. (Hint: Suppose $\vec{x} \in W \cap W^{\perp}$. Consider $\langle \vec{x}, \vec{x} \rangle$.)

(3) Let U be an orthogonal $n \times n$ matrix, i.e. the columns of U form an orthonormal basis of $\mathbb{R}^n$. Prove that the rows of U also form an orthonormal basis of $\mathbb{R}^n$.

(4) Suppose that U_1 and U_2 are both orthogonal matrices. Prove that the product U_1U_2 is also an orthogonal matrix.

(5) Let A be an invertible $n \times n$ matrix. For any $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, define

$$\langle \vec{v}_1, \vec{v}_2 \rangle_A := \vec{v}_1^T A^T A \vec{v}_2.$$

Prove that $\langle -, - \rangle_A$ is an inner product on $\mathbb{R}^n$.

(6) Let V be an inner product space (could be infinite dimensional). Prove that for any $\vec{v}_1, \vec{v}_2 \in V$,

$$||\vec{v}_1 + \vec{v}_2||^2 + ||\vec{v}_1 - \vec{v}_2||^2 = 2||\vec{v}_1||^2 + 2||\vec{v}_2||^2.$$

(7) Let V be an inner product space (could be infinite dimensional). Prove that for any $\vec{v}_1, \vec{v}_2 \in V$,

$$|\langle \vec{v}_1, \vec{v}_2 \rangle| \leq ||\vec{v}_1|| \cdot ||\vec{v}_2||.$$

When does the equality hold? (Hint: You can use the fact that $||\vec{x}|| \geq ||\mathrm{proj}_{\mathrm{Span}\{\vec{v}\}}\vec{x}||$ for any $\vec{x}, \vec{v}$, and the equality holds if and only if $\vec{x}$ and $\vec{v}$ are linearly dependent.)

(8) Let V be an inner product space (could be infinite dimensional). Prove that for any $\vec{v}_1, \vec{v}_2 \in V$,

$$||\vec{v}_1 + \vec{v}_2|| \leq ||\vec{v}_1|| + ||\vec{v}_2||.$$

(Hint: Take square on both sides, and apply the result in previous problem.)

(9) Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be any real monic polynomial. Find an $n \times n$ real matrix such that its characteristic polynomial is given by $(-1)^n p$. (This will be used to write an n-th order linear ordinary differential equation as a system of first order linear equations.) (Hint: Recall one of the problems in Quiz 3.)

(10) Let $V = \mathcal{C}[-\pi, \pi]$ be the vector space of continuous functions on $[-\pi, \pi]$. For $f, g \in V$, define

$$\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx.$$

(a) Prove that $(V, \langle -, - \rangle)$ is an inner product space.

(b) Prove that the set

$$\left\{ \frac{1}{\sqrt{2}}, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(kx), \sin(kx), \ldots \right\}$$

is an orthonormal set. (This will be useful for the discussions on Fourier series.)