Def: A <u>metric space</u> (S,d) is a set S with a <u>distance fcn</u> $d: S\times S \longrightarrow \mathbb{R}_{\geq 0}$ s.t.

1) $d(x,x)=0 \quad \forall x\in S.$
2) $d(x,y)>0 \quad \forall x,y\in S$ and $x\neq y.$
3) $d(x,y)=d(y,x) \quad \forall x,y\in S$
4) $d(x,z)\leq d(x,y)+d(y,z) \quad \forall x,y,z\in S$

Def: (S,d) metric space, $x\in S$, $r>0$

$$B_r(x) := \{y\in S : d(x,y)<r\} \subseteq S$$

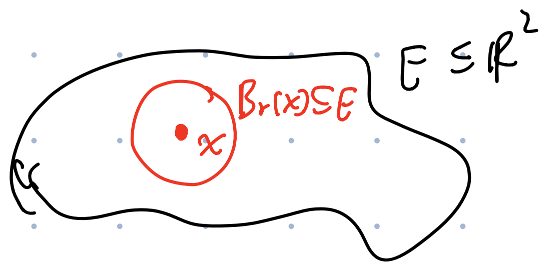
<u>e.g.</u> $(\mathbb{R}, d_{std})$. $B_1(3) = (2,4)$

<u>Notation</u>:
- $(2,4) := \{x\in\mathbb{R} \mid 2<x<4\}$ "<u>open</u>" interval
- $[2,4] := \{x\in\mathbb{R} \mid 2\leq x\leq 4\}$ "closed" interval

$(2,4]$

<u>Def</u> (S,d) metric space. $E\subseteq S$ subset
We say $E\subseteq S$ is <u>open</u> if $\forall x\in E$, $\exists r>0$

s.t. $B_r(x)\subseteq E$.



e.g. Is $[0,1] \subseteq \mathbb{R}$ open?

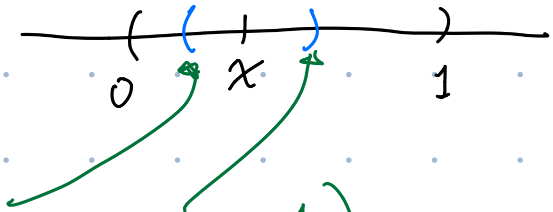
No: $0 \in [0,1]$

But for any $r>0$, $B_r(0) = (-r, r) \not\subseteq [0,1]$

e.g. Is $(0,1) \subseteq \mathbb{R}^2$ open?

Yes: $\forall x \in (0,1)$, we need to show: $\exists r>0$ s.t. $B_r(x) \subseteq (0,1)$

$r := \min\left\{\frac{x}{2}, \frac{1-x}{2}\right\} > 0$

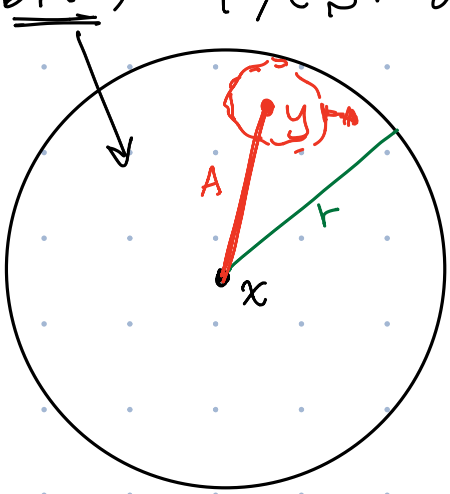


$\Rightarrow \underbrace{\boxed{B_r(x)}}_{=(x-r,\, x+r)} \subseteq (0,1)$

$(\because 0 < x-r,\ x+r < 1)$

e.g. (S,d) metric space, $x \in S$, $r>0$

$B_r(x) = \{y \in S : d(x,y) < r\}$ is an open subset in S.



$\forall y \in B_r(x)$, we need to find $s>0$ s.t. $B_s(y) \subseteq B_r(x)$

Let's say $d(x,y) = A < r$

Define $s := r - A > 0$

Claim: $B_s(y) \subseteq B_r(x)$:

pf $\forall z \in B_s(y)$, $d(y,z) < s$.

$d(z,x) \le d(z,y) + d(y,x) < s + A = r \Rightarrow z \in B_r(x)$. □

Def: $E \subseteq (S,d)$ is closed if its complement $E^C = \{x \in S : x \notin E\}$ is open.

e.g. Is $[0,1] \subseteq \mathbb{R}$ closed?

Yes. $\mathbb{R} \setminus [0,1] = (-\infty, 0) \cup (1, \infty)$ is open.

e.g. $(0,1) \subseteq \mathbb{R}$ not closed, b/c

$(0,1)^C = (-\infty, 0] \cup [1, \infty)$ is not open.

e.g. $(0,1] \subseteq \mathbb{R}$ is not open, not closed.

e.g. (S,d) any metric space, S is open, ϕ = empty set is open

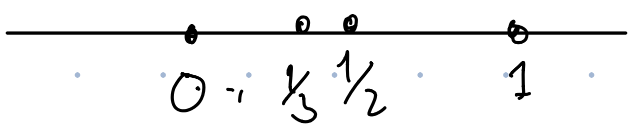
$\Rightarrow S$ & ϕ are both open & closed.

e.g. Is $\{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$ closed?

e.g. $\{1\} \subseteq \mathbb{R}$ is closed.
$(-\infty, 1) \cup (1, \infty)$ open

No

$0 \notin E = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$



$0 \in E^C$,

But $\forall r > 0$, $B_r(0) \cap E \neq \phi$.

$\Longrightarrow B_r(0) \not\subseteq E^C \quad \forall r > 0$

$\Longrightarrow E^C$ is not open

e.g. $\{\frac{1}{n} : n\in\mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$ <u>closed</u>

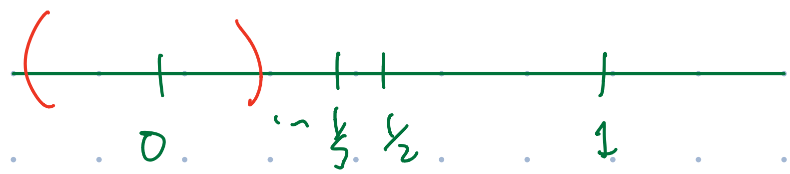
(Exercise)

<u>Rmk</u>: If $E \subseteq S$ is closed, then it should contain all of its "limit points".

<u>Def</u> $(S,d) \supseteq E$.

say $x\in S$ is a <u>limit point of E</u> if

$\forall r>0$, $B_r(x)\cap E$ contains a point y s.t. $y\neq x$.



<u>Def</u>: The <u>closure</u> of $E\subseteq S$ is:

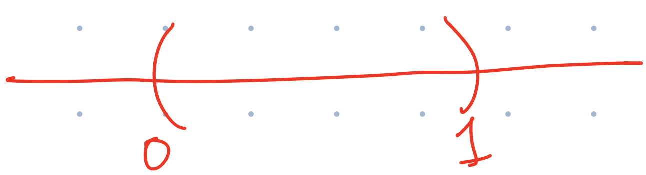
$$\overline{E} := E \cup \{\text{limit points of } E\}.$$

<u>Rmk</u>: ① a limit point of E may not be an element in E.

e.g. $\boxed{(0,1)\subseteq\mathbb{R}}$

E

What are the limit pts of E?? $[0,1]$



② a pt in E may not be a limit pt of E.

e.g. $1\in\{\frac{1}{n} : n\in\mathbb{N}\}$ is not a limit pt

In fact, the only limit pt of the subset $\{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is the pt. $\{0\}$,

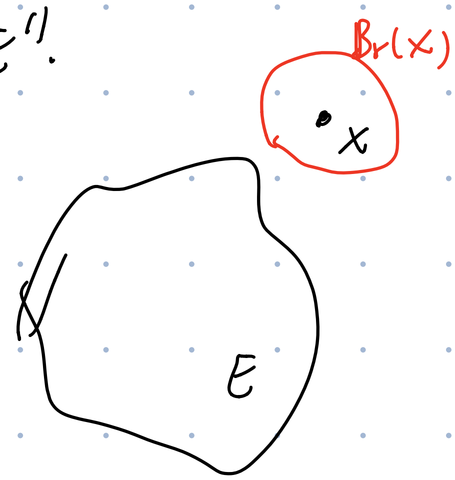
Thm: A subset $E \subseteq (S,d)$ is closed $\Leftrightarrow$ $E = \overline{E}$

pf ($\Rightarrow$) Assuming E is closed

Want to show: "$\{$limit pts of $E\} \subseteq E$"

$\Updownarrow$

"if $x \notin E$, then x is not a limit pt of E".



Let $x \notin E$, $\Leftrightarrow$ $x \in E^c$,

E^c is open since E is closed,

$\exists\, r > 0$ s.t. $B_r(x) \subseteq E^c$

$\Rightarrow$ $B_r(x) \cap E = \phi$

$\Rightarrow$ x is not a limit pt of E. $\square$

($\Leftarrow$) Assuming $E = \overline{E}$, (i.e. E contains all the limit pts of E)

Want to show: "E is closed" $\Leftrightarrow$ "E^c is open"

Let $x \in E^c$, $x \notin E = \overline{E}$

$\Rightarrow$ x is not a limit pt of E.

$\Rightarrow$ $\exists\, r > 0$ s.t. $B_r(x) \cap E$ is either empty or $\cancel{\{x\}}$

$\Rightarrow$ $\exists\, r > 0$ s.t. $B_r(x) \cap E = \phi$ $\Rightarrow$ $B_r(x) \subseteq E^c$ $\square$

§ Compact subsets

Def $E \subseteq (S, d)$.

Let $\{U_\alpha : \alpha \in I\}$ be a collection of open subsets of S.

- say $\{U_\alpha : \alpha \in I\}$ is an open cover of E if:

$$E \subseteq \left(\bigcup_{\alpha \in I} U_\alpha \right)$$

(i.e. $\forall x \in E$, $\exists \alpha \in I$ s.t. $x \in U_\alpha$)

- say the open cover has a finite subcover if $\exists$ finitely many $\alpha_1, \ldots, \alpha_n$ s.t.

$$E \subseteq (U_{\alpha_1} \cup \cdots \cup U_{\alpha_n})$$

Def $K \subseteq (S, d)$ is a compact subset if every open cover of K has a finite subcover.

Non-examples:

- $\mathbb{N} = \{1, 2, 3, \ldots\} \subseteq \mathbb{R}$ is not compact.

Consider $\{U_n = (n - \frac{1}{3}, n + \frac{1}{3})\}_{n \in \mathbb{N}}$. open cover of $\mathbb{N}$

$$\| \\ \{(1-\tfrac{1}{3}, 1+\tfrac{1}{3}), (2-\tfrac{1}{3}, 2+\tfrac{1}{3}), \ldots\ldots\}$$

But any finite collection of these open intervals does not cover $\mathbb{N}$.

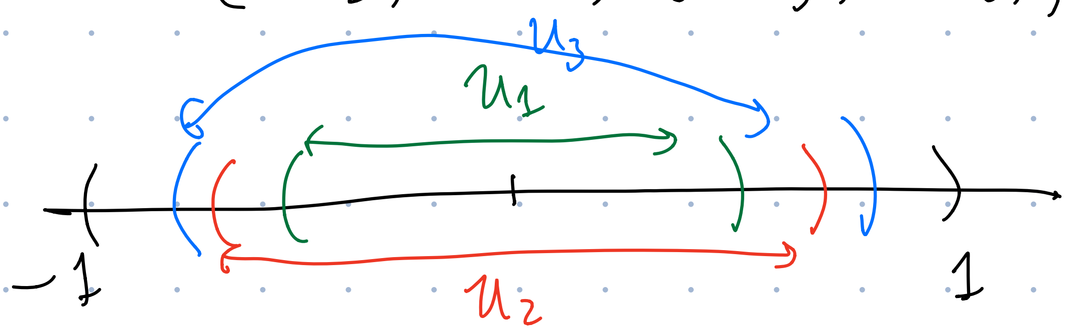
(i.e. $\{U_n\}_{n\in\mathbb{N}}$ has no finite subcover)

$\Longrightarrow$ $\mathbb{N}$ is not compact.

- Is $E = (-1,1) \subseteq \mathbb{R}$ compact??

No: Consider $\left\{ U_n = \left(-1+\frac{1}{n+1},\ 1-\frac{1}{n+1}\right) \right\}_{n\in\mathbb{N}}$.

$\left(-1+\frac{1}{2},\ 1-\frac{1}{2}\right),\ \left(-1+\frac{1}{3},\ 1-\frac{1}{3}\right),\ \ldots$



Ex: Show that $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of E.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \ (-1,1)$

Ex: There is no finite subcover of $\{U_n\}$

$\Longrightarrow$ $(-1,1)$ is not compact.