MATHEMATICS FROM EXAMPLES, SPRING 2023

INSTRUCTOR: YU-WEI FAN

Course Description. Examples in mathematics are like phenomena in physics. They play a vital role in the historical development of mathematics and are the driving force behind profound mathematical concepts and methods. Important theorems in modern mathematics often come from the understanding and research of some basic examples. The goal of this course is to provide the motivation and intuition behind abstract mathematical concepts by introducing some interesting examples.

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1. Overview of the course

Example 1.1. Let $x \in (0,1)\backslash\mathbb{Q}$ be an irrational number. It can be written uniquely as a continued fraction

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}}}$$

where $a_1, a_2, \ldots$ are positive integers.

How often does a positive integer k appear in such an expression?

It turns out that for any given k, the frequency of k appearing in the continued fraction expression of x is the same for *almost every* $x \in (0,1)\backslash\mathbb{Q}$, and is given by the following formula

$$\lim_{n\to\infty} \frac{\#\{i \mid a_i = k, 1 \le i \le n\}}{n} = \frac{1}{\log 2} \log\left(\frac{(k+1)^2}{k(k+2)}\right).$$

In order to prove this, we will introduce some basic ideas of *measure theory* and *ergodic theory.*

Example 1.2. Consider the following *necklace-splitting problem.* Two thieves have stolen a precious necklace (opened, with two ends), on which there are d kinds of stones (diamonds, sapphires, rubies, etc.), an even number of each kind. The thieves do not know the values of stones of various kinds, so they want to divide the stones of each kind evenly. They would like to achieve this by as few cuts as possible. The question is, what is the minimum amount of cuts needed to divide the stones of each kind evenly?

It is not hard to show that at least d cuts is necessary: place the stones of the first kind first, then the stones of the second kind, and so on. The *necklace theorem* shows that d cuts is always sufficient. Surprisingly, all known proofs of this theorem are *topological.*

Example 1.3. Let $C \subseteq \mathbb{R}^2$ be a simple closed curve. One considers the following *Rectangular Peg Problems.*

- Does there always exist four points on C such that they form the vertices of a rectangle?
- A much harder question: Fix a rectangle R. Does there always exist four points on C such that they form the vertices of a rectangle which is similar to R?

The first question was answered positively by Vaughan in 1981, which uses some basic *topology*. The second question was also answered positively quite recently by Greene and Lobb [9]; their proof involves more advanced tools from *symplectic geometry*, which is beyond the scope of this course.

Example 1.4. Which positive integers n can be written as the sum of two squares $n = x^2 + y^2$?

To answer this question, it is natural to introduce the *ring* of *Gaussian integers* $\mathbb{Z}[i]$, since one has the factorization $x^2 + y^2 = (x + iy)(x - iy)$. The question then reduced to studying the properties of the ring $\mathbb{Z}[i]$.

Example 1.5. One can consider a more refined question: How many ways can a positive integer n be written as the sum of two (or more) squares?

The problem is closely related to the *theta function*, which is a function defined for a complex variable $\tau \in \mathbb{H}$ on the upper half plane:

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 \tau} = \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \text{where} \quad q = \exp(2\pi i \tau).$$

Let us define $r_2(n)$ to be the number of ways that n can be written as the sum of two squares

$$r_2(n) = \#\{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = n\}.$$

It is not hard to see that

$$\theta(\tau)^2 = \sum_{n=0}^{\infty} r_2(n) q^n.$$

The problem then reduces to understand $\theta(\tau)^2$. It turns out that $\theta(\tau)^2$ is a *modular form of weight* 1 *for the congruence subgroup* $\Gamma_1(4) \subseteq \mathrm{SL}(2, \mathbb{Z})$, and we can use the theory of modular forms to obtain an explicit formula of $r_2(n)$. In fact, the same method also applies to the sum of $2k$ square numbers for any positive integer k, where we can use modular forms to obtain explicit formula of $r_{2k}(n)$.

Example 1.6. Is the rope in the following figure knotted? Motivated by this sort of questions, we will introduce various *knot invariants* and their *categorifications*, and discuss what kinds of information are encoded by them. The construction of the categorification involves ideas including *cobordism categories* and *topological quantum field theory*, which are of independent interests.

Example 1.7. In 1696, Johann Bernoulli posed the problem of the brachistochrone (from ancient Greek, which means "shortest time") as a challenge to the mathematicians of his day: Given two points A and B in a plane, where B is lower and not directly below A, what is the curve traced out by a point acted on only by gravity, which starts from A and reaches B in the *shortest time*?

This problem is widely regarded as the founding problem of the *calculus of variations*, which study ways of finding the curve, or surface, minimizing a given integral. We will discuss the approach developed by Euler (in 1736) and Lagrange (in 1755) to deal with general problems of this kind.

Example 1.8. Let $P = (p_1, \dots, p_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial function, i.e. each coordinate $p_1, \dots, p_n$ is a polynomial in $\mathbb{C}^n$. It was proved independently by Grothendieck (1966) and Ax (1968) that if P is injective then it is bijective. In fact, this theorem can be generalized to any *algebraic variety* over an *algebraically closed field*.

The method of proof is really noteworthy: it showcases the idea that *finitely many* algebraic relations in fields of characteristic 0 can be translated into algebraic relations over finite fields with large characteristics. Thus, one can use the arithmetic of finite fields to prove a statement about $\mathbb{C}$ even though there is no homomorphism from any finite field to $\mathbb{C}$. This is a great example of applications of techniques from *model theory* in *mathematical logic*.

Example 1.9. Let a and m be integers that are relatively prime. Are there infinitely many primes in the sequence

$$a, a+m, a+2m, \dots?$$

This was conjectured to be true by Legendre, and later proved by Dirichlet in 1837 with his *L-series*. This theorem is believed to represent the beginning of rigorous *analytic number theory*. In fact, Dirichlet shows a stronger result that the "density" of the subset

$$\{\text{prime } p \mid p \equiv a \pmod{m}\} \subseteq \{\text{prime } p\}$$

is $1/\varphi(m)$. In other words, the prime numbers are equally distributed among different classes modulo m which are relatively prime to m.

Example 1.10. In 1657, Fermat wrote letters to his friend de Bessy, his Dutch correspondent van Schooten, and English mathematicians Wallis and Brouncker. In the letters, Fermat invited them to solve some curious mathematical problems. The central questions are concerned with certain quadratic equations of the form

$$x^2 - Ny^2 = 1, \qquad x, y \in \mathbb{Z}_{>0}.$$

To Wallis and Brouncker, he challenged them with the cases $N = 151$ and $N = 313$; but to his countryman de Bessy, he merely demanded answers for the cases $N = 61$ and $N = 109$, "so as not to give him too much trouble".

More generally, this problem can be interpreted as understanding the values of integral binary quadratic forms, such as $3x^2 + 6xy - 5y^2$. We will give a quick tour to the concept of *Conway's topograph*, with his *wells, rivers, lakes*, and *weirs*, and see how these help us with answering the problem.

Example 1.11. The dilogarithm function is defined by the power series

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad \text{for } |z| < 1.$$

The definition (and the name) come from the analogy with the Taylor series of the ordinary logarithm around 1

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad \text{for } |z| < 1,$$

which leads similarly to the definition of the *polylogarithm*

$$\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \qquad \text{for } |z| < 1, \qquad m = 1, 2, \ldots.$$

The dilogarithm function is one of the simplest non-elementary functions one can imagine. It is also one of the strangest. Almost all of its appearances in mathematics, and almost all the formulas relating to it, have something of the fantastical in them. We will discuss its relations with *hyperbolic* 3*-manifolds*, *quantum dilogarithm identity*, and *wall-crossing formula of stability conditions.*

Example 1.12. Let us consider the power series

$$\sum_{k=0}^{\infty}(-1)^k k! x^{k+1}.$$

Clearly, it is divergent for any $x \neq 0$, which makes it seems uninteresting. However, this power series and certain series of this sort, actually appears "in nature". For instance, the series could represent the solution of an ordinary differential equation, or gives the value of a physical quantity of interest, such as the energy. Many mathematicians and physicists have recently become interested in these series due to their appearance in numerous topics at the forefront of research, including: gauge theory of singular connections, quantization of symplectic and Poisson manifolds, Floer homology and Fukaya categories, knot invariants, wall-crossing and stability conditions in algebraic geometry, perturbative expansions in quantum field theory, etc.

We will discuss an approach toward making sense of this divergent issue, via the method of *Borel summation.* Along the way, we will see interesting phenomenons like *resurgence*, *Stokes phenomenon*, and relate it back to the *wall-crossing formula.*