Mirror Symmetry and Rigid Structures of Generalized K3 Surfaces

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Overview

Interplay of

- generalized Calabi-Yau geometry (4-dim):
  unification of CY geometry and symplectic geometry
- mirror symmetry:
  duality between complex geometry and symplectic geometry

Generalized CY geometry brings a new insight into ”rigid structure” of K3 surfaces. In particular, it solves the problem of MS for singular K3 surfaces.

Generalized CY structures (4-dim)

M: C^∞-manifold underlying a K3 surface,
$A^{2*}_{\mathbb{C}}(M) = \oplus_{i=0}^{2} A^{2i}_{\mathbb{C}}(M)$: even diff forms with $\mathbb{C}$-coeff with Mukai pairing

$$\langle \varphi, \psi \rangle = \varphi_2 \wedge \psi_2 - \varphi_0 \wedge \psi_4 - \varphi_4 \wedge \psi_0 \in A^4_{\mathbb{C}}(M)$$

where φ_i denotes the degree i part of φ.

Definiton 2.1 (generalized CY structure (4-dim), Hitchin)

A <u>generalized CY structure</u> on M is a closed form $\varphi \in A^{2*}_{\mathbb{C}}(M)$ such that

$$\langle \varphi, \varphi \rangle = 0, \quad \langle \varphi, \overline{\varphi} \rangle > 0$$

($E_\varphi = \{(v, \xi) \in T_M \oplus T^*_M \mid \iota(v)\varphi + \xi \wedge \varphi = 0\}$ generalized complex structure)

Generalized CY structures (4-dim)

- symplectic form ω, $\varphi = e^{\sqrt{-1}\omega} = 1 + \sqrt{-1}\omega - \frac{1}{2}\omega^2$.

$$\langle e^{\sqrt{-1}\omega}, e^{\sqrt{-1}\omega} \rangle = \langle 1 + \sqrt{-1}\omega - \frac{1}{2}\omega^2, 1 + \sqrt{-1}\omega - \frac{1}{2}\omega^2 \rangle = 0,$$
$$\langle e^{\sqrt{-1}\omega}, e^{-\sqrt{-1}\omega} \rangle = 2\omega^2 > 0.$$

- hol 2-form w.r.t complex structure σ, $\varphi = \sigma$.

$$\langle \sigma, \sigma \rangle = 0,$$
$$\langle \sigma, \overline{\sigma} \rangle = \sigma \wedge \overline{\sigma} > 0.$$

B-field transform

$B \in A^2_{\mathbb{C}}(M)$ acts on $A^{2*}_{\mathbb{C}}(M)$ by the exterior product of e^B:

$$e^B\varphi = (1 + B + \frac{1}{2}B \wedge B) \wedge \varphi.$$

This action is orthogonal w.r.t. the Mukai pairing

$$\langle e^B\varphi, e^B\psi\rangle = \langle\varphi, \psi\rangle.$$

A real closed 2-form is called a <u>B-field</u>.

Theorem 2.2

For a B-field B and a gCY structure φ, the B-field transform $e^B\varphi$ is a gCY structure.

Classification of gCY structures

Theorem 2.3 (Hitchin)

Let φ be a gCY structure.

- (type A) $\varphi_0 \neq 0$: $\exists$ a symplectic form ω, a B-field B,

$$\varphi = e^B(\varphi_0 e^{\sqrt{-1}\omega}) = \varphi_0 e^{B+\sqrt{-1}\omega}$$

- (type B) $\varphi_0 = 0$: $\exists$ a hol 2-form σ (w.r.t. a complex str), a B-field B,

$$\varphi = e^B\sigma = \sigma + \sigma \wedge B \ (= \sigma + \sigma \wedge B^{0,2})$$

Definiton 2.4

gCY structures φ, φ' are isomorphic if $\exists$ an exact B-field B and $f \in \mathrm{Diff}_*(M)$ such that $\varphi = e^B f^*\varphi'$.

$\mathrm{Diff}_*(M) = \mathrm{Ker}(\mathrm{Diff}(M) \to O(H^2(M, \mathbb{Z})))$.

Unification of A- and B-structures

A fascinating aspect of gCY structures is the occurrence of the complex structure σ and symplectic structure $e^{\sqrt{-1}\omega}$ in the same moduli.

Example 2.5 (Hitchin)

For a hol 2-form σ, the real and imaginary parts $\mathrm{Re}(\sigma), \mathrm{Im}(\sigma)$ are symplectic forms. A family of gCY structures of type A

$$\varphi_t = te^{\frac{1}{t}(\mathrm{Re}(\sigma)+\sqrt{-1}\mathrm{Im}(\sigma))} = t(1+\frac{1}{t}\sigma+\frac{1}{2t^2}\sigma^2) = t+\sigma$$

converges, as $t \to 0$, to the gCY structure σ of type B.

The B-fields interpolate between gCY structures of type A and B.

Kähler structure

For a gCY structure φ, define a distribution P_φ of real 2-planes by :

$$P_\varphi = \mathbb{R}\mathrm{Re}\varphi \oplus \mathbb{R}\mathrm{Im}\varphi \subset A^*(M).$$

gCY structures φ and φ' are called orthogonal if P_φ and $P_{\varphi'}$ are pointwise orthogonal $P_\varphi \perp P_{\varphi'}$. This is a stronger condition than $\langle \varphi, \varphi' \rangle = 0$.

Definiton 2.6 (Kähler)

A gCY structure φ is called Kähler if $\exists$ another gCY structure φ' orthogonal to φ. Such φ' is called a Kähler structure for φ.

A Kähler structure for $\varphi = \sigma$ is of the form $\varphi' = \varphi_0' e^{B+\sqrt{-1}\omega}$. The orthogonality reads

$$\sigma \wedge B = \sigma \wedge \omega = 0.$$

Therefore B is a closed real $(1,1)$-form and $\pm\omega$ is a Kähler form w.r.t. σ.

HyperKähler structure

Recall that a Kähler form ω on a K3 surface is a hyperKähler form if for some $C \in \mathbb{R}$

$$2\omega^2 = C\sigma \wedge \overline{\sigma}.$$

Definiton 2.7 (hyperKähler)

A gCY structure φ is hyperKähler if $\exists$ a Kähler structure φ' such that

$$\langle \varphi, \overline{\varphi} \rangle = \langle \varphi', \overline{\varphi'} \rangle.$$

Such φ' is called a hyperKähler structure for φ.

- $\langle e^{\sqrt{-1}\omega}, e^{-\sqrt{-1}\omega} \rangle = 2\omega^2$, $\langle \sigma, \overline{\sigma} \rangle = \sigma \wedge \overline{\sigma}$.
- If φ' a (hyper)Kähler structure for φ, then $e^B\varphi'$ is a (hyper)Kähler structure for $e^B\varphi$.

Classification of hyperKähler structures

(details are not important)

- $\underline{\varphi = \sigma}$: a hyperKähler structure is $\varphi' = \lambda e^{B+\sqrt{-1}\omega}$, where B is a closed $(1,1)$-form and $\pm\omega$ is a hyperKähler form such that

$$2|\lambda|^2\omega^2 = \sigma \wedge \overline{\sigma}.$$

- $\underline{\varphi = \lambda e^{\sqrt{-1}\omega}}$: a hyperKähler structure is either
  - $\varphi' = \sigma$, where $\pm\omega$ is a hyperKähler form,
  - $\varphi' = \lambda' e^{B' + \sqrt{-1}\omega'}$ such that
    - $\omega \wedge \omega' = \omega \wedge B' = \omega' \wedge B = 0$, $B'^2 = \omega^2 + \omega'^2$,
    - $|\lambda|^2\omega^2 = |\lambda'|^2\omega'^2$.

Any hyperKähler structure is a B-field transform of one of the above cases. There are 3 cases:

(type A, type B), (type B, type A), (type A, type A)

Generalized K3 surfaces

Definiton 2.8

A generalized K3 surface is a pair (φ, φ') of gCY structures such that φ is a hyperKähler structure for φ'.

- A K3 surface $S = M_\sigma$ with a hyperKähler form ω is considered as a gK3 surface $(e^{\sqrt{-1}\omega}, \sigma)$.
- gK3 surfaces (φ, φ') and (ψ, ψ') are called isomorphic if $\exists$ $f \in \mathrm{Diff}_*(M)$ and exact $B \in A^2(M)$ such that

$$(\varphi, \varphi') = e^B f^*(\psi, \psi') = (e^B f^* \psi, e^B f^* \psi').$$

Isom classes are classified by cohomology classes

gK3 surfaces and SCFT moduli space

Theorem 2.9 (Huybrechts)

$\mathfrak{M}_{\mathrm{HK}} = \left(\mathrm{Met}^{\mathrm{HK}}(M)/\mathrm{Diff}_*(M)\right) \times H^2(M,\mathbb{R})$:
moduli space of the B-field shifts of the hyperKähler metrics

$$\begin{tikzcd}
\mathfrak{M}_{\mathrm{K3}} \times H^2(M,\mathbb{R}) \arrow[r, hook, "\iota"] \arrow[dr, "S^2"'] & \mathfrak{M}_{\mathrm{gK3}} \arrow[r, "\mathrm{per}_{\mathrm{gK3}}", "(\varphi,\varphi')\mapsto(P_{[\varphi]},P_{[\varphi']})"'] \arrow[d] & \mathrm{Gr}^{po}_{2,2}(H^*(M,\mathbb{R})) = \mathfrak{M}_{(2,2)} \arrow[d, "S^2\times S^2"] \\
& \mathfrak{M}_{\mathrm{HK}} \arrow[r, "\mathrm{per}_{\mathrm{HK}}"] & \mathrm{Gr}^{po}_{4}(H^*(M,\mathbb{R})) = \mathfrak{M}_{(4,4)}
\end{tikzcd}$$

Mirror symmetry for K3 surfaces is an involution of the SCFT moduli spaces (Aspinwall-Morrison).

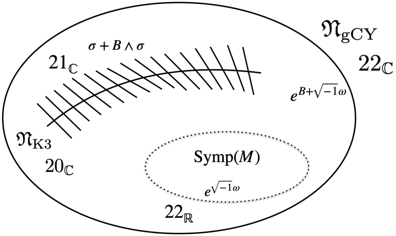
Period domains and period maps

$\mathfrak{M}_{\mathrm{gCY}} = \{\mathbb{C}\varphi\}/\cong$: moduli space of gCY structures of hyperKähler type

Theorem 2.10 (Huybrechts)

$$\begin{array}{ccl}
\mathfrak{M}_{\mathrm{gCY}} & \xrightarrow[\mathbb{C}\varphi \to [\phi]]{\mathfrak{per}_{\mathrm{gCY}}} & \widetilde{\mathfrak{D}} = \{[\varphi] \in \mathbb{P}(H^*(M,\mathbb{C})) \mid \langle \varphi, \varphi \rangle = 0, \langle \varphi, \overline{\varphi} \rangle > 0\} \\
\cup & & \qquad\qquad\qquad \cup \\
\mathfrak{M}_{\mathrm{K3}} & \xrightarrow[\mathbb{C}\sigma \to [\sigma]]{\mathfrak{per}_{\mathrm{K3}}} & \mathfrak{D} = \{[\sigma] \in \mathbb{P}(H^2(M,\mathbb{C})) \mid \langle \sigma, \sigma \rangle = 0, \langle \sigma, \overline{\sigma} \rangle > 0\}
\end{array}$$

$\mathfrak{per}_{\mathrm{gCY}}$: étale surjective



K3 surfaces and lattices

Mirror symmetry for a (classical) K3 surface S is very subtle because the complex and Kähler structures are somewhat mixed in $H^2(S,\mathbb{C})$.

A conventional formulation of mirror symmetry for K3 surfaces is given by Dolgachev in terms of sublattices of $H^*(S,\mathbb{Z}) \cong U^{\oplus 4} \oplus E_8^{\oplus 2}$:

- Néron-Severi lattice:

$$NS(S) = \{\delta \in H^2(S,\mathbb{Z}) \mid \langle \delta, [\sigma] \rangle = 0\}$$

- algebraic lattice:

$$NS'(S) = H^0(S,\mathbb{Z}) \oplus NS(S) \oplus H^4(S,\mathbb{Z}) \cong NS(S) \oplus U.$$

- transcendental lattice:

$$T(S) = NS'(S)^{\perp} \subset H^*(S,\mathbb{Z})$$

Mirror symmetry for K3 surfaces

Definiton 3.1 (Dolgachev)

Given $M \subset \Lambda_{K3} = U^{\oplus 3} \oplus E_8^{\oplus 2}$ of sign $(1, \mu)$, assume that $\exists\, N$ such that

$$M^{\perp} = N \oplus U.$$

Then the family $\mathcal{S}$ of M-pol K3 surfaces and the family $\mathcal{S}^{\vee}$ of N-pol K3 surfaces are <u>mirror symmetric</u>.

For generic M-pol K3 surface S and N-pol K3 surface $S^{\vee}$,

$$NS'(S) \cong M \oplus U \cong T(S^{\vee}), \quad T(S) \cong N \oplus U \cong NS'(S^{\vee}),$$

duality of algebraic and transcendental cycles.

Mirror symmetry for K3 surfaces

$$
\begin{array}{cccccc}
 & & & NS'(S) & & \\
 & & & 1 & & \\
 & & 0 & & 0 & \\
T(S) & 1 & & 20 & & 1 \\
 & & 0 & & 0 & \\
 & & & 1 & &
\end{array}
$$

$$
\underset{NS'(S)}{\overset{T(S^\vee)}{\underline{\overline{M \oplus U}}}} \oplus \underset{T(S)}{\overset{NS'(S^\vee)}{\underline{\overline{U \oplus N}}}}
$$

Drawbacks

The conventioanl formulation has drawbacks:

- $NS'(S)$ and $T(S)$ are not symmetric.
- The assumption $M^{\perp} = N \oplus U$ does not hold in general:
  - singular K3 surface, where $T(S)$ is of sign $(2, 0)$.

	singular K3 surface	??
Kähler	20-dim	0-dim
complex	0-dim	20-dim

  - $M^{\perp} = N \oplus U(k)$

The problems are caused by $H^0(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}) \cong U$.

Algebraic and transcendental lattices

We define sublattices of $H^*(M,\mathbb{Z})$ reflecting a gCY structure.

Definiton 3.2

The algebraic and transcendental lattices of a gK3 surface $X = (\varphi, \varphi')$ are defined respectively by

$$\widetilde{NS}(X) = \{\delta \in H^*(M,\mathbb{Z}) \mid \langle \delta, [\varphi'] \rangle = 0\},$$
$$\widetilde{T}(X) = \{\delta \in H^*(M,\mathbb{Z}) \mid \langle \delta, [\varphi] \rangle = 0\}.$$

- $\widetilde{NS}(X)$ and $\widetilde{T}(X)$ are defined on an equal footing.

$$2 \leq \mathrm{rank}(\widetilde{NS}(X)),\ \mathrm{rank}(\widetilde{T}(X)) \leq 22.$$

- In general, pt and $[M]$ are no longer "algebraic".

Complex and Kähler rigidity

Definiton 4.1

A gK3 surface $X = (\varphi, \varphi')$ is called

- complex rigid if φ' is of type B and $\mathrm{rank}(\widetilde{NS}(X)) = 22$.
- Kähler rigid if φ is of type A and $\mathrm{rank}(\widetilde{T}(X)) = 22$.

Theorem 4.2

A complex rigid gK3 surface is of the form $e^{B'}(\lambda e^{B+\sqrt{-1}\omega}, \sigma)$:

- M_σ: singular K3 surface
- $B \in H^{1,1}(M_\sigma, \mathbb{R})$,
- $B' \in H^2(M, \mathbb{Q})$,
- $\pm\omega$ is a Kähler form w.r.t. σ.

Glipmse of Kähler rigidity

S: K3 surface, $NS(S) = \mathbb{Z}H, H^2 = 2n > 0$.

$$v_1 = (1, 0, -n), \quad v_2 = (0, H, 0) \in NS'(S)$$

Then

$$\begin{aligned} e^{\sqrt{-1}H} &= (1, \sqrt{-1}H, -n) \\ &= v_1 + \sqrt{-1}v_2 \in (\mathbb{Z}v_1 + \mathbb{Z}v_2)_{\mathbb{C}} \subsetneq NS'(S)_{\mathbb{C}}. \end{aligned}$$

On the other hand, for $\epsilon^2 \notin \mathbb{Q}$

$$\begin{aligned} e^{\sqrt{-1}\epsilon H} &= (1, \sqrt{-1}\epsilon H, -\epsilon^2 n) \\ &= (1, 0, -\epsilon^2 n) + \sqrt{-1}\epsilon(0, H, 0) \\ &= (1, 0, 0) - \epsilon^2(0, 0, n) + \sqrt{-1}\epsilon(0, H, 0) \in NS'(S)_{\mathbb{C}} \end{aligned}$$

$\rightsquigarrow$ Kähler rigidity

Mukai lattice polarization

Definiton 4.3 (Mukai lattice polarization)

For $\kappa, \lambda \geq 2$ such that $\kappa + \lambda = 24$, and even lattices K and L of signature $(2, \kappa - 2)$ and $(2, \lambda - 2)$, a pair (X, j) of

- a gK3 surface $X = (\varphi, \varphi')$,
- a primitive embedding $j : K \oplus L \hookrightarrow H^*(M, \mathbb{Z})$ such that
  - $K \subset \widetilde{NS}(X)$ and $K_{\mathbb{C}}$ contains gCY structure of type A,
  - $L \subset \widetilde{T}(X)$ and $L_{\mathbb{C}}$ contains gCY structure of type B.

is called a $\underline{(K, L)\text{-polarized}}$ gK3 surface.

"polarization $\subset$ lattice polarization $\subset$ Mukai lattice polarization"

Mirror symmetry for gK3 surfaces

Definiton 4.4

The family $\mathcal{X}$ of (K, L)-pol gK3 surfaces and the family $\mathcal{Y}$ of (L, K)-pol gK3 surfaces are mirror symmetric.

For generic (K, L)-pol gK3 surface X and (L, K)-pol gK3 surface Y,

$$\widetilde{NS}(X) \cong K \cong \widetilde{T}(Y), \quad \widetilde{T}(X) \cong L \cong \widetilde{NS}(Y),$$

duality between algebraic and transcendental cycles w.r.t. gCY structures.

MS for complex and Kähler rigid gK3 surfaces

For $n > 0$, consider $K = \langle -2n \rangle^{\oplus 2} \oplus U \oplus E_8^{\oplus 2}, \quad L = \langle 2n \rangle^{\oplus 2}$.

- The family $\mathcal{X}$ of (K, L)-pol gK3 surfaces is given by

$$\mathcal{X} = \{X = (e^{B+\sqrt{-1}\omega}, \sigma)\}$$

  where $T(M_\sigma) = L$, and $B, \omega \in NS(M_\sigma)_{\mathbb{R}}$. They are singular K3 surfaces with complexified Kähler parameters $B + \sqrt{-1}\omega \in NS(M_\sigma)_{\mathbb{C}}$.
- The family $\mathcal{Y}$ of (L, K)-pol gK3 surfaces has a 19-dim subfamily of K3 surfaces of the form

$$\{Y = (e^{\sqrt{-1}H}, \sigma^\vee)\}$$

  where $NS(M_{\sigma^\vee}) = \mathbb{Z}H$ such that $H^2 = 2n$.

MS for complex and Kähler rgid gK3 surfaces

In summary, for $K = \langle -2n \rangle^{\oplus 2} \oplus U \oplus E_8^{\oplus 2}$, $L = \langle 2n \rangle^{\oplus 2}$,

- (K, L)-pol gK3 surfaces = singular K3 surfaces
- (L, K)-pol gK3 surfaces $\supset$ pol K3 surfaces (S, H) with $H^2 = 2n$

	(K, L)-pol gK3	(L, K)-pol gK3
A-deform	20-dim	0-dim
B-deform	0-dim	20-dim

The new formulation is compatible with Aspinwall-Morrison's description of the moduli space $\mathfrak{M}_{(2,2)} = \mathrm{Gr}^{po}_{2,2}(H^*(M, \mathbb{R}))$.

謝謝! Thank you!

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