#1: Let $V = \mathbb{R}^2$, and $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Define an inner product $\langle -, - \rangle_A$ on V by:

$$\langle \vec{x}, \vec{y} \rangle_A := \vec{x}^T A \vec{y}.$$

Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(1) Compute $\text{proj}_{\text{span}\{\vec{e}_1\}} \vec{v}$ and $\text{proj}_{\text{span}\{\vec{e}_2\}} \vec{v}$. (w.r.t. $\langle -, - \rangle_A$)

Is it true that $\vec{v} = \text{proj}_{\text{span}\{\vec{e}_1\}} \vec{v} + \text{proj}_{\text{span}\{\vec{e}_2\}} \vec{v}$?

(2) Is $\{\vec{e}_1, \vec{e}_2\}$ an orthonormal basis of V? (w.r.t. $\langle -, - \rangle_A$)

If not, find an orthonormal basis of V.

Sol^n: (1)

$$\text{proj}_{\text{span}\{\vec{e}_1\}} \vec{v} = \frac{\langle \vec{v}, \vec{e}_1 \rangle_A}{\|\vec{e}_1\|_A^2} \vec{e}_1 = \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$$

$$\text{proj}_{\text{span}\{\vec{e}_2\}} \vec{v} = \frac{\langle \vec{v}, \vec{e}_2 \rangle_A}{\|\vec{e}_2\|_A^2} \vec{e}_2 = \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

$$\vec{v} \neq \text{proj}_{\text{span}\, e_1} \vec{v} + \text{proj}_{\text{span}\, e_2} \vec{v}.$$

Note that this doesn't contradict with the thm we proved, since $\{\vec{e}_1, \vec{e}_2\}$ is NOT orthogonal w.r.t. $\langle -, - \rangle_A$.

(2) It's easy to check that $\{\vec{e_1}, \vec{e_2}\}$ isn't an orthonormal basis of V w.r.t. $\langle -,- \rangle_A$.

$$\left\langle \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} \right\rangle_A = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 2.$$

$$\left\langle \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}x\\y\end{bmatrix} \right\rangle_A = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 2x+y.$$

Hence $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}\right\}$ is orthogonal basis w.r.t. $\langle -,- \rangle_A$.

$$\left\langle \begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix} \right\rangle_A = \begin{bmatrix}1 & -2\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}1 & -2\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix} = 2$$

$\Rightarrow$ $\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-2\end{bmatrix}\right\}$ is an orthonormal basis. $\square$

#2: Let $V = \mathrm{Poly}_{\le 2}$. Consider the inner product

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\,dx.$$

Find an orthogonal basis of V.

Sol^n: Claim: $\{1, x, 3x^2-1\}$ is an orthogonal basis of V.

- $\langle 1, x\rangle = \int_{-1}^{1} x\,dx = 0.$
- $\langle 1, 3x^2-1\rangle = \int_{-1}^{1}(3x^2-1)\,dx = (x^3-x)\Big|_{-1}^{1} = 0.$
- $\langle x, 3x^2-1\rangle = \int_{-1}^{1} 3x^3-x\,dx = 0$

and it's clear that they form a basis of V. $\square$