HOMEWORK 8
MATH 104, SECTION 2

Some ground rules:

- You have to submit your homework via **Gradescope** to the corresponding assignment. The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You're allowed to use any result that is proved in the lecture; but if you'd like to use other results, you have to prove them before using them.

Problem set (5 problems; due March 30 at 11am PT)

(1) Let X be a set, and (f_n) be a sequence of functions $f_n \colon X \to \mathbb{R}$.
  (a) Suppose that (f_n) converges to $f \colon X \to \mathbb{R}$ uniformly and each (f_n) is bounded. Prove that f is also bounded.
  (b) Find an example of (f_n) converges to $f \colon X \to \mathbb{R}$ pointwisely and each (f_n) is bounded, but f is unbounded.

(2) Let X be a set, and (f_n) be a sequence of functions $f_n \colon X \to \mathbb{R}$. Prove that if (f_n) converges to some function $f \colon X \to \mathbb{R}$ uniformly, then (f_n) is uniformly Cauchy.

(3) Let X be a set. Consider the set $\mathcal{B}(X)$ consisting of real-valued *bounded* functions $f \colon X \to \mathbb{R}$. For $f_1, f_2 \in \mathcal{B}(X)$, define
$$d(f_1, f_2) \coloneqq \sup_{x \in X} |f_1(x) - f_2(x)| .$$
Prove that $(\mathcal{B}(X), d)$ is a metric space.

(4) Consider the sequence of functions (f_n) defined by $f_n(x) = \frac{nx}{1+nx}$ for $x \geq 0$.
  (a) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \geq 0$.
  (b) Let $a > 0$. Prove or disprove: (f_n) converges uniformly to f on $[a, \infty)$.
  (c) Prove or disprove: (f_n) converges uniformly to f on $[0, \infty)$.

(5) Let X be a compact metric space, and (f_n) be a sequence of continuous functions $f_n \colon X \to \mathbb{R}$. Suppose that
  - (f_n) converges pointwisely to a continuous function $f \colon X \to \mathbb{R}$.
  - $f_{n+1}(x) \leq f_n(x)$ for any $x \in X$ and $n \in \mathbb{N}$.

Prove that (f_n) converges uniformly to f on X.
(Hint: Define $g_n := f_n - f$. Consider the set $E_n := \{x \in X : g_n(x) < \epsilon\}$. Show that $E_1 \subset E_2 \subset E_3 \subset \cdots$ and that $X = \cup E_n$.)