(A) Let $\Lambda \subseteq \mathbb{C}$ be a lattice. Suppose z_1, z_2 are two complex numbers such that $\wp(z_1) \neq \wp(z_2)$ and $z_1, z_2, z_1 \pm z_2 \notin \Lambda$. In this problem, you'll prove the addition theorem for the $\wp$-function

$$\wp(z_1) + \wp(z_2) + \wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2.$$

(1) Let $f(z) = \wp'(z) - (a\wp(z) + b)$. There exists a unique pair of complex numbers a, b such that $f(z_1) = f(z_2) = 0$. Show that

$$a = \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}.$$

(2) By analyzing the poles of f in the fundamental domain, show that $f(z_1 + z_2) = 0$.

(3) Consider the following polynomial of degree 3:

$$F(X) = 4X^3 - g_2 X - g_3 - (aX + b)^2.$$

Show that $\wp(z_1), \wp(z_2), \wp(z_1 + z_2)$ are the roots of F, then prove the addition theorem for the $\wp$-function.

(B) Prove that

$$\sum_{1 \leq n^2 + m^2 \leq R^2} \frac{1}{n^2 + m^2} = 2\pi \log R + O(1) \quad \text{as } R \to \infty.$$

(This is part of Exercise 3 in the textbook.)

(C) Let $\tau \in \mathbb{H}$ be an element in the upper half-plane. Denote

$$\wp(z, \tau) = \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left(\frac{1}{(z + m + n\tau)^2} - \frac{1}{(m + n\tau)^2}\right).$$

Prove that for any integers $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ $\left(\text{i.e. } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})\right)$,

$$\wp\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 \wp(z, \tau).$$