## Announcement:

- my OH today: 12-~~1:30~~pm $\longrightarrow$ 5-6:30 pm PDT.

  (this is an one-time change)

## Recap:

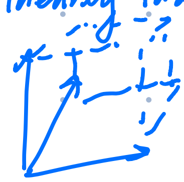
$$\mathbb{R}^n \underset{T_{A^{-1}}}{\overset{T_A}{\rightleftarrows}} \mathbb{R}^n$$

$T_{A^{-1}} \circ T_A = id = T_A \circ T_{A^{-1}}$

- Define matrix product $AB$: $T_{AB} = T_A \circ T_B$.
- Invertible matrix $A_{n\times n}$: $\exists A^{-1}$ s.t. $A^{-1}A = AA^{-1} = I_n$. (generalization of $a \in \mathbb{R}\setminus 0$, $a^{-1}a = 1 = aa^{-1}$)
- We proved last time that: If $A: n\times n$ is invertible, then $T_A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is bijective (inj. & surj.)

## Today:

- Prove the converse statement.
- Define <u>determinant</u> of a square matrix.

some applications of determinant:
- check invertibility, linearly indep.
- compute volume of 
- change of variable formula in multivariable calculus.
- ......

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## Recap: <u>elementary matrices</u>:

the effect of left multiplication by the matrices:

$$\begin{array}{cc} & \begin{array}{ccccc} & j & & & \end{array} \\ i & \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & a & & \ddots & \\ & & & & 1 \end{bmatrix} \end{array}$$

replace the $i$th row by ($i$th row) + $a$($j$th row)

$$\begin{array}{cc} & \begin{array}{ccc} & i & j \end{array} \\ & \begin{bmatrix} 1 & & \\ & \ddots & \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ i \\ \\ j \\ \\ \end{array}\begin{bmatrix} \ddots & & & & \\ & 0 & & 1 & \\ & & 1 \ddots 1 & & \\ & 1 & & 0 & \\ & & & & 1 \ddots 1 \end{bmatrix}$$

swap ($i$-th row) & ($j$-th row)

$$\begin{array}{c} \\ i \\ \\ \end{array}\begin{bmatrix} 1 \ddots 1 & & \\ & a\neq 0 & \\ & & 1 \ddots 1 \end{bmatrix}$$

multiply the ($i$-th row) by $a \neq 0$

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Q: What happens if we multiply the elementary matrices on the right?

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & 2a_{11}+a_{12} \\ a_{21} & 2a_{21}+a_{22} \end{pmatrix}$$

"column operation"

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**Fact:** ~~These~~ elementary matrices are invertible.

$$\boxed{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \underbrace{2 \quad 0}_{} & & 1 \end{bmatrix}\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix}$$

replace the 3rd row of $M$ by 3rd row of $M$ + 2 (1st row of $M$)

Suppose $A$ is the inverse of $\begin{bmatrix} 1 & & \\ & 1 & \\ 2 & & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = I_3$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ 2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -2 & & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 = I, \quad \text{i.e.} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1/2 & & \\ & 1 & \\ & & 1 \end{bmatrix} = I_n$$

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Thm $A$: $n \times n$ invertible $\Leftrightarrow$ $A$ is row equivalent to $I_n$

($\Leftrightarrow$ $A$ has pivots in each column/row)

($\Leftrightarrow$ $T_A$ is bijective)

$$A \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

pf ($\Rightarrow$) Last time

($\Leftarrow$) $\exists$ $E_1, \dots, E_k$ elementary matrices st.

$$\boxed{E_1 \cdots E_k A = I_n}$$

$$\Rightarrow \underbrace{E_1^{-1} E_1}_{I_n} E_2 \cdots E_k A = E_1^{-1} \cdot I_n = E_1^{-1}$$

$$\parallel$$

$$E_2 \cdots E_k A$$

$$\boxed{\begin{aligned} E_1 E_2 \cdots E_k A &= I_n \\ E_1 \cdots E_k I_n &= A^{-1} \end{aligned}}$$

$$\Rightarrow E_3 \cdots E_k A = E_2^{-1} E_1^{-1}$$

$$\Rightarrow A = E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1}$$

$\Rightarrow$ $A$ is invertible,

and $\boxed{A^{-1} = E_1 E_2 \cdots E_k.}$ $\square$

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Thm (Invertible Criterion) $A$: $n \times n$ matrix. The following are equivalent:

① $A$ is invertible ($\exists B$ st. $BA = AB = I_n$.)

② $T_A$ is surjective.

②' $A$ has pivot in each row.

②'' $A\vec{x} = \vec{b}$ has sol$^{n}$ $\forall \vec{b} \in \mathbb{R}^n$

③ $T_A$ is injective

③' $A$ has pivot in each column

③'' $A\vec{x}=\vec{0}$ has only the trivial sol$^{n}$

③''' The columns of $A$ are l.i.

④ $\exists$ $n\times n$ $B$ s.t. $BA=I_n$

⑤ $\exists$ $n\times n$ $C$ s.t. $AC=I_n$

pf ① $\Leftrightarrow$ ② $\Leftrightarrow$ ③ proved.

① $\Rightarrow$ ④, ⑤ clear.

$\boxed{④ \Rightarrow ③''}$ :

We'd like to prove: "$A\vec{x}=\vec{0} \Rightarrow \vec{x}=\vec{0}$"

The condition we have: $BA=I_n$

$$\vec{x}=I_n\vec{x}=BA\vec{x}=B\vec{0}=\vec{0}$$

$\boxed{⑤ \Rightarrow ②''}$ :

We'd like to prove: "$\forall \vec{b}$, $A\vec{x}=\vec{b}$ has a sol$^{n}$"

The cond$^{n}$ we have: $AC=I_n$

$A\vec{c}_1=\vec{e}_1=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$, ..., $A\vec{c}_n=\vec{e}_n$

$$\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \vec{b} = b_1\vec{e}_1 + \cdots + b_n\vec{e}_n$$

$$A\left(\boxed{\underbrace{b_1\vec{c}_1 + \cdots + b_n\vec{c}_n}_{C\vec{b}}}\right) = b_1\vec{e}_1 + \cdots + b_n\vec{e}_n = \vec{b}.$$

$\cdot\ \vec{b}$

$$AC\vec{b} = I_n\vec{b} = \vec{b}$$

$$\begin{array}{c} \| \\ A(\boxed{C\vec{b}}) \end{array}$$

$\vec{x} = C\vec{b}$ is a sol'n to $A\vec{x} = \vec{b}$.

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How to compute $A^{-1}$??

$$\boxed{E_1E_2\cdots E_k}\, A = I_n$$

$$E_1\cdots E_k\, I_n = A^{-1}$$

$$\begin{bmatrix} A & | & I_n \end{bmatrix}_{n \times 2n}$$

$\downarrow$ elementary row operations $E_1 \cdots E_k$

$$\begin{bmatrix} I_n & | & A^{-1} \end{bmatrix}$$

e.g. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right] \quad \boxed{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

e.g. $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right]$

Assume $a \neq 0$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible $\Leftrightarrow ad-bc \neq 0$

$\Downarrow$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\downarrow$

$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$

$\downarrow$

$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$

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In general, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

- $ad-bc=0$, $\left\{\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right\}$ l.d.
- $\boxed{ad-bc} \neq 0$, $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

**Def** A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is <u>invertible</u> if

$\exists$ a function $S: \mathbb{R}^n \to \mathbb{R}^n$

st. $T(S(\vec{x})) = \vec{x} = S(T(\vec{x}))$ $\forall \vec{x} \in \mathbb{R}^n$

i.e. $T \circ S = \mathrm{id}_{\mathbb{R}^n} = S \circ T$.

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**Thm.** Suppose $T = T_A$ for $A$: $n \times n$

Then
- $T$ is invertible $\iff$ $A$ is invertible.
- In this case, $S(\vec{x}) = A^{-1}\vec{x}$ is the unique fcn st. $S(T(\vec{x})) = \vec{x} = T(S(\vec{x}))$ $\forall \vec{x} \in \mathbb{R}^n$

pf: ($\Rightarrow$) $T(\vec{x}) = T_A(\vec{x}) = A\vec{x}$.

$\vec{x} = T(S(\vec{x})) = A(S(\vec{x}))$ $\forall \vec{x}$

$\Rightarrow$ $T_A$ is surjective $\Rightarrow$ $A$ invertible.
(proved before)

($\Leftarrow$) Define $S(\vec{x}) = A^{-1}\vec{x}$.

Check $S(T(\vec{x})) = A^{-1}(A\vec{x}) = \vec{x}$.

$T(S(\vec{x})) = A(A^{-1}\vec{x}) = \vec{x}$.

<u>Uniqueness of $S$:</u> Suppose $S'$

$$\left.\begin{aligned} T(S(\vec{x})) &= \vec{x} = S(T(\vec{x})) \\ T(S'(\vec{x})) &= \vec{x} = S'(T(\vec{x})) \end{aligned}\right\} \forall \cancel{\vec{x}}$$

① each $\vec{v} \in \mathbb{R}^n$, $\exists$ $\vec{w} \in \mathbb{R}^n$ st. $T(\vec{w}) = \vec{v}$

② $\vec{w} = S(T(\vec{w})) = S(\vec{v}) \Rightarrow S(\vec{v}) = S'(\vec{v})\ \forall \vec{v}$
by ①
$\vec{w} = S'(T(\vec{w})) = S'(\vec{v}) \Rightarrow S = S'$

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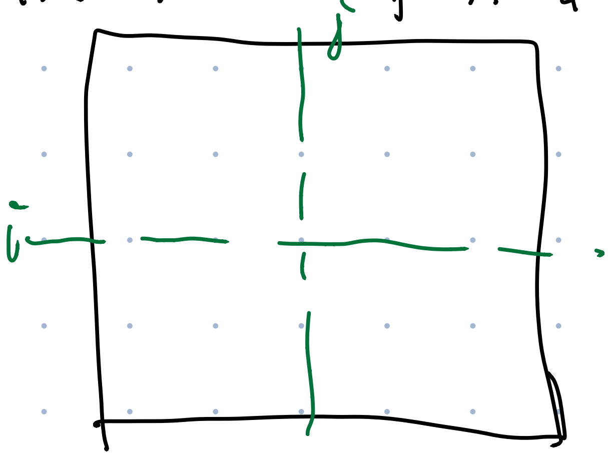
## Determinant (of a square matrix)

- $1 \times 1$: $[a_{11}]$ $\quad \det([a_{11}]) := a_{11}$
- $2 \times 2$: $\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) := a_{11}a_{22} - a_{12}a_{21}$

**Def**: $A: n \times n$, define $\boxed{A_{ij}}$ to be the $(n-1)\times(n-1)$ matrix w/ the $i$-th row of $A$ & $j$-th column of $A$ removed.




e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\quad A_{21} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$

**Def** Determinant $\det(A)$ is defined inductively as follows:

Suppose we've defined $\det(\cdot)$ for any $(n-1)\times(n-1)$ matrix

Now suppose $A: n \times n$

Define the <u>(i,j)-cofactor</u> of $A$ as:

$$C_{ij} := (-1)^{i+j} \det A_{ij}$$

Pick any row $a_{i1}, \dots, a_{in}$ or any column $a_{1j}, \dots, a_{nj}$ of $A$

<u>Define</u>

$$\det(A) := a_{i1} C_{i1} + \dots + a_{in} C_{in}$$

or

$$\det(A) = a_{1j} C_{1j} + \dots + a_{nj} C_{nj}$$

(cofactor expression of the $i$th row or $j$th column)

<u>Thm</u> det is well-defined, i.e. indep. of which column/row we choose to do the cofactor exp.

<u>e.g.:</u> $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\boxed{a_{11}a_{22} - a_{12}a_{21}}$$

$$\det(A) = a_{21} \boxed{C_{21}} + a_{22} C_{22}$$

$$C_{21} = (-1)^{2+1} \det A_{21} = -a_{12}$$

$$C_{22} = (-1)^{2+2} \det A_{22} = a_{11}$$

eg:

$$A = \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$C_{11} = (-1)^{1+1} \det A_{11} = a_{22}a_{33} - a_{23}a_{32}$$

$$C_{12} = \boxed{(-1)^{1+2}} \det A_{12} = a_{23}a_{31} - a_{21}a_{33}$$

$$C_{13} = (-1)^{1+3} \det A_{13} = a_{21}a_{32} - a_{22}a_{31}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$
$$+ a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}$$
$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Rmk: "diagonal" ↘ "diagonal" ↙

But this is NOT true if size > 3!!

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In general,

$$\begin{bmatrix} \bullet & & & \\ & & \bullet & \\ & \bullet & & \\ & & & \bullet \end{bmatrix}$$

$$\det(A) = \sum_{\sigma} (-1)^{\mathrm{sign}(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ bij.