

# GEOMETRY AND SYMMETRY, FALL 2022

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If you'd like to read more or find more exercise problems, here are some recommended reference books:

- M. A. Armstrong, *Groups and Symmetry*
- M. Artin, *Algebra*
- E. G. Rees, *Notes on Geometry*

## 1. OVERVIEW OF THE COURSE

### Lecture 1

We will explore the *symmetries* of various *geometric spaces* in this course. The spaces that we will consider include: the Euclidean spaces  $\mathbb{R}^2$ ,  $\mathbb{R}^3$ , the spheres  $S^1$ ,  $S^2$ , the hyperbolic space  $\mathbb{H}^2$ , and some of their interesting subsets.

Let us consider the following shapes in  $\mathbb{R}^2$ . Intuitively, we know that a square



is “more symmetric” than a rectangle, and a rectangle is “more symmetric” than an arbitrary 4-gon. In order to make sense of these statements, we have to define what are *symmetries* of these shapes.

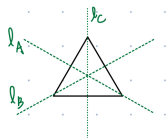
Each of the geometric spaces that we will consider ( $\mathbb{R}^2$ ,  $\mathbb{R}^3$ ,  $S^1$ ,  $S^2$ ,  $\mathbb{H}^2$ , etc.) has a natural metric (i.e. distance  $d(x, y)$  between any two points  $x, y$ ). The symmetries that we are interested in are the *isometries* (i.e. distance-preserving functions) of these spaces. For instance, an isometry of  $\mathbb{R}^2$  is a function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $d(f(x), f(y)) = d(x, y)$  for any  $x, y \in \mathbb{R}^2$ .

**Definition 1.1.** Let  $S \subseteq \mathbb{R}^2$  be a subset of  $\mathbb{R}^2$ . An isometry  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is called a *symmetry of  $S$*  if we have  $f(S) = S$ , i.e.

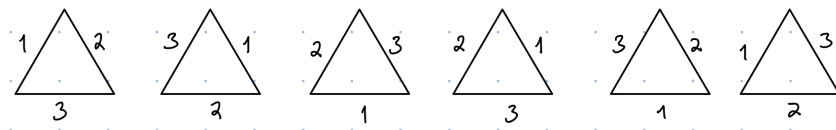
- for any  $p \in S$ , we have  $f(p) \in S$ ; and
- for any  $q \in S$ , there exists  $p \in S$  such that  $f(p) = q$ .

*Example.* Let us look at an easy example: an equilateral triangle. It has two kinds of symmetries:

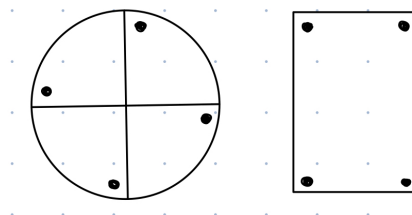
- Rotational symmetries: one can rotate the triangle by  $\frac{2\pi}{3}$ ,  $\frac{4\pi}{3}$ , or  $2\pi$  without changing its appearance.
- Reflection symmetries: there are three “mirror lines” through which we can reflect the shape without changing its appearance.



The easiest way to study the symmetries of a shape is by *counting*. In this example, it's easy to check that there are 6 symmetries. If we put labels on the edges of the triangle, then the effect of these symmetries look like:



However, counting alone is usually not good enough.



*Example.* Both of the following shapes have 4 symmetries. The shape on the left has 4 rotational symmetries (by  $\frac{\pi}{2}$ ,  $\pi$ ,  $\frac{3\pi}{2}$ ,  $2\pi$ ), but no reflection symmetries. In contrast, the shape on the right has 2 rotational symmetries and 2 reflection symmetries. How can we distinguish them?

As we'll see later in this course, *group theory* provides rigorous tools to describe the symmetries of shapes. For any shape (or any geometric object), the set of its symmetries has a natural *group structure*. In the example above, although the sets of symmetries of both shapes have 4 elements, but their underlying group structures are different, and that's how we can tell them apart (e.g. consider the *orders* of elements in these two groups).

Another important tool that we will encounter is basic *linear algebra*, in particular *matrices* or *matrix groups*. The reason is that certain matrix groups ( $O(2, \mathbb{R})$ ,  $O(3, \mathbb{R})$ ,  $SL(2, \mathbb{R})$ ,  $SL(2, \mathbb{C})$ , etc.) act naturally as isometries on the spaces that we are interested in like  $\mathbb{R}^2$ ,  $\mathbb{R}^3$ ,  $S^1$ ,  $S^2$ ,  $\mathbb{H}^2$ . For instance, you'll show in the homework that any isometry of the Euclidean space  $\mathbb{R}^n$  is a composition of a translation and a linear transformation.

Now we mention some examples that we'll be studying in this course.

*Example.* Consider a regular  $n$ -gon  $P_n$  in  $\mathbb{R}^2$ . It is not hard to show that  $P_n$  has  $2n$  symmetries. We'll:

- discuss the group structure of the symmetry group of  $P_n$  (the resulting group is called the *dihedral group*  $D_n$ ;
- prove that any finite subgroup of  $O(2, \mathbb{R})$  is either a cyclic group or the symmetry group of a regular  $n$ -gon.

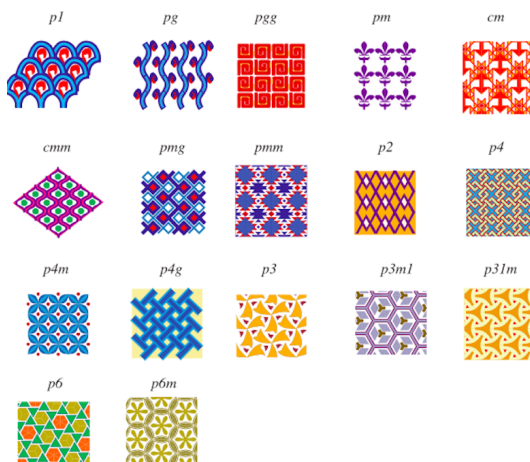
*Example.* An important class of examples of subsets in  $\mathbb{R}^3$ , that possess many symmetries, are the Platonic solids (regular polyhedrons). We will:

- prove that there are only five of them;
- study their symmetry groups;



- more importantly, prove that any finite subgroup of  $\text{SO}(3, \mathbb{R})$  is either a cyclic group, a dihedral group, or the (rotational) symmetry group of one of the Platonic solids.

*Example.* A *wallpaper* is a mathematical object that covers the whole  $\mathbb{R}^2$  by repeating a pattern indefinitely. The symmetry group of a wallpaper is called a *wallpaper group* (we will provide more precise definition later on). We will show that there are exactly 17 different wallpaper groups.



In later parts of this course, we'll discuss the isometries of some important non-Euclidean spaces, including the Riemann sphere  $S^2$  (whose isometries are certain *Möbius transformations*), and the hyperbolic plane  $\mathbb{H}$  (whose isometries form the *modular group*).

## 2. A CRASH COURSE ON BASIC GROUP THEORY

**2.1. Binary operators.** Before discussing the actual definition of a *group*, let us first consider a more general notion of *binary operators*.

**Definition 2.1.** Let  $S$  be a set. A *binary operator* on  $S$  is a function

$$\circ: S \times S \rightarrow S.$$

*Example.* Addition on the set of positive integers (denoted by  $\mathbb{N}$ ), or the set of integers (denoted by  $\mathbb{Z}$ ), or the set of rational numbers (denoted by  $\mathbb{Q}$ ) or the set of real numbers (denoted by  $\mathbb{R}$ ), is a binary operator. Same for multiplication.

*Non-example.* Subtraction on the set of positive integers is *not* a binary operator. Division on the set of integers is *not* a binary operator.

**Definition 2.2.** Let  $(S, \circ)$  be a set with a binary operator. We say an element  $e \in S$  is an *identity element* if  $e \circ a = a \circ e = a$  for any  $a \in S$ .

*Example.* The element  $0 \in \mathbb{Z}$  is an identity element of  $(\mathbb{Z}, +)$ . The element  $1 \in \mathbb{Z}$  is an identity element of  $(\mathbb{Z}, \times)$ .

*Non-example.*  $(\mathbb{N}, +)$  has no identity element.

*Exercise.* Prove that any set with a binary operator  $(S, \circ)$  has at most one identity element.

**Definition 2.3.** Let  $(S, \circ, e)$  be a set with a binary operator and an identity element. We say an element  $a' \in S$  is an *inverse* of  $a \in S$  if  $a \circ a' = a' \circ a = e$ .

*Example.* For  $(\mathbb{Z}, +)$ , the inverse of  $a \in \mathbb{Z}$  is given by  $-a$ . For  $(\mathbb{R}, \times)$ , the inverse of  $a \in \mathbb{R}$  is given by  $1/a$ , provided that  $a \neq 0$ .

*Non-example.* For  $(\mathbb{Z}, \times)$ , any element  $a \in \mathbb{Z}$  has no inverse unless  $a = \pm 1$ .

**Definition 2.4.** Let  $(S, \circ)$  be a set with a binary operator. We say  $(S, \circ)$  is *associative* if  $(a \circ b) \circ c = a \circ (b \circ c)$  holds for any  $a, b, c \in S$ .

*Exercise.* Let  $(S, \circ, e)$  be a set with an associative binary operator and an identity element. Prove that any element in  $S$  has at most one inverse.

Most of the examples that we'll be discussing are associative. Here is a non-example (which we will not encounter in this course):

*Non-example.* The cross product  $\times$  on  $\mathbb{R}^3$  is *not* associative. Rather, it satisfies the *Jacobi identity*

$$\vec{v}_1 \times (\vec{v}_2 \times \vec{v}_3) + \vec{v}_2 \times (\vec{v}_3 \times \vec{v}_1) + \vec{v}_3 \times (\vec{v}_1 \times \vec{v}_2) = 0$$

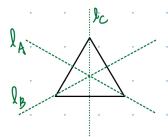
**Definition 2.5.** Let  $(S, \circ)$  be a set with a binary operator. We say  $(S, \circ)$  is *commutative* if  $a \circ b = b \circ a$  for any  $a, b \in S$ .

*Warning.* Many of the examples that we'll consider are *not* commutative.

*Non-example.* Consider the set of all six geometric transformations that give the symmetries of an equilateral triangle:

$$S = \left\{ \text{rotate } 0, \text{rotate } \frac{2\pi}{3}, \text{rotate } \frac{4\pi}{3}, \text{reflect along } \ell_A, \text{reflect along } \ell_B, \text{reflect along } \ell_C \right\}.$$

(note: rotations are typically assumed to be counterclockwise)

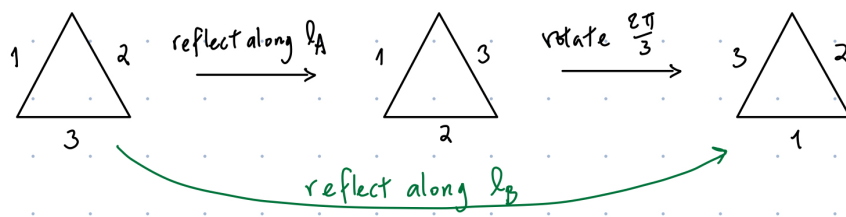


There is a binary operation on  $S$  given by *composing* these geometric transformations:

$$\circ: S \times S \rightarrow S,$$

where  $a \circ b \in S$  is the transformation given by “do  $b$ , and then do  $a$ ”. For instance, we have

$$\left( \text{rotate } \frac{2\pi}{3} \right) \circ \left( \text{reflect along } \ell_A \right) = \text{reflect along } \ell_B.$$

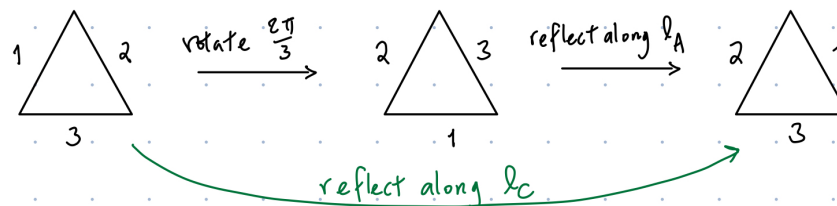


On the other hand, by reversing the order one gets

$$\left( \text{reflect along } \ell_A \right) \circ \left( \text{rotate } \frac{2\pi}{3} \right) = \text{reflect along } \ell_C.$$

This shows that  $(S, \circ)$  is *not* commutative.

*Non-example.* Another important class of groups that we will discuss is the *matrix groups*. They are *not* commutative in most cases.



## 2.2. Groups.

**Definition 2.6.** Let  $(G, \circ)$  be a set with a binary operator. It is called a *group* if it satisfies the following conditions:

- (1) It is associative.
- (2) It has the identity element (which will usually be denoted by  $e$ ,  $e_G$ ,  $1$ , or  $1_G$ ).
- (3) Any element  $a \in G$  has an inverse (which will be denoted by  $a^{-1} \in G$ ).

*Remark 2.7.* Here are some notions that we will be using frequently:

- If a group  $(G, \circ)$  is commutative, then it is called an *abelian group*.
- We'll use  $|G|$  to denote the number of elements in the set  $G$ , and will call it the *order* of  $G$ . Note that the order of a group could be infinite in general.
- We quite often would omit “ $\circ$ ”, and simply denote  $a \circ b$  by  $ab$ , denote  $a \circ a$  by  $a^2$ , denote  $a \circ a \circ a$  by  $a^3$ , and so on.

*Example.* Consider the set of integers modulo  $n$

$$\mathbb{Z}/n\mathbb{Z} := \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}.$$

Addition and multiplication are well-defined on  $\mathbb{Z}/n\mathbb{Z}$ . It's not hard to show that  $(\mathbb{Z}/n\mathbb{Z}, +)$  is an abelian group of order  $n$ , with the identity given by  $\bar{0}$ .

*Example.* Consider the subset of  $\mathbb{Z}/n\mathbb{Z}$  consisting of elements that are coprime with  $n$ :

$$(\mathbb{Z}/n\mathbb{Z})^* := \{\bar{m} \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}.$$

It's not hard to show that  $((\mathbb{Z}/n\mathbb{Z})^*, \times)$  is an abelian group, with the identity given by  $\bar{1}$ .

*Example.* The set of all integers  $\mathbb{Z}$  under addition is an example of an abelian group with infinite order.

*Example.* The set  $\{0\}$  under addition is an example of a group with only one element (a trivial group).

*Example.* Let  $G_1$  and  $G_2$  be two groups. Consider the set

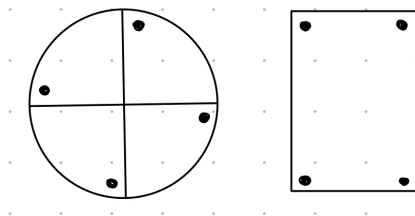
$$G_1 \times G_2 := \{(g_1, g_2) : g_1 \in G_1 \text{ and } g_2 \in G_2\}.$$

Define a binary operator on  $G_1 \times G_2$  as follows:

$$(g_1, g_2) \circ (g'_1, g'_2) := (g_1 \circ g'_1, g_2 \circ g'_2).$$

It's not hard to show that  $(G_1 \times G_2, \circ)$  is also a group. It's called the *direct product* of  $G_1$  and  $G_2$ .

*Example.* Let's come back to the following examples again. As discussed ear-



lier, the symmetries of a shape form a group, where the binary operation is given by composition. The symmetry group of the first shape is

$$G_1 := \left\{ \text{rotate } 0, \text{ rotate } \frac{\pi}{2}, \text{ rotate } \pi, \text{ rotate } \frac{3\pi}{2} \right\}.$$

One thing we might notice about this group is that all elements of the group can be obtained by taking one element of the set, and combining it different number of times. Let's denote rotate  $\frac{\pi}{2}$  by  $a$ . Then  $G_1$  can be rewritten as

$$G_1 = \{e, a, a^2, a^3\}.$$

Notice that  $a^4 = e$  since rotate  $2\pi$  is the same as rotate 0, i.e. the identity map. The same is true for  $\mathbb{Z}/4\mathbb{Z}$  (under addition) if one lets  $a = \bar{1}$  and note that  $a^4 = \bar{4} = \bar{0} = e$  in  $\mathbb{Z}/4\mathbb{Z}$ . In fact, we'll see that the symmetry group of the first shape and  $\mathbb{Z}/4\mathbb{Z}$  are *isomorphic*, which means that they are essentially the same group.

On the other hand, the symmetry group of the second shape is

$$G_2 := \left\{ \text{rotate } 0, \text{ rotate } \pi, \text{ reflect along } \ell_1, \text{ reflect along } \ell_2 \right\}.$$



It's not hard to see that there is no element  $a \in G_2$  such that  $G_2 = \{e, a, a^2, a^3\}$ . Therefore,  $G_2$  and  $G_1$  are not isomorphic. In fact, one can show that  $G_2$  is isomorphic to the direct product  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

**2.3. Homomorphisms.** For any mathematical structure (like groups), it is crucially important to understand how two structures of the same type (like two groups) are related in a meaningful way. Functions that bridge such two structures are called *homomorphisms*. (In the Ancient Greek language, “homo-” means “same”, and “morphe” means “form” or “shape”.) In general, a homomorphism is a function between two mathematical structures of the same type, that preserves the operations of the structures.

**Definition 2.8.** Let  $G$  and  $H$  be two groups. A function  $f: G \rightarrow H$  is called a *homomorphism* if for any  $g_1, g_2 \in G$  we have

$$f(g_1 g_2) = f(g_1) f(g_2)$$

Furthermore, a homomorphism that is both injective and surjective is called an *isomorphism*. In this case, we'll use the notation “ $G \cong H$ ”.

In other words, a homomorphism is a function that is compatible with the binary operations on the two groups.

*Exercise.* Let  $f: G \rightarrow H$  be a group homomorphism. Prove that

- It preserves the identity:  $f(e_G) = e_H$ .
- It preserves the inverses:  $f(g^{-1}) = f(g)^{-1}$  for any  $g \in G$ .

*Example.* We considered the symmetry group

$$G_1 := \left\{ \text{rotate } 0, \text{ rotate } \frac{\pi}{2}, \text{ rotate } \pi, \text{ rotate } \frac{3\pi}{2} \right\} = \{e, a, a^2, a^3\}$$

where  $a^4 = e$ . One can define a function

$$G_1 \rightarrow \mathbb{Z}/4\mathbb{Z}$$

by sending  $e \mapsto \bar{0}$ ,  $a \mapsto \bar{1}$ ,  $a^2 \mapsto \bar{2}$ , and  $a^3 \mapsto \bar{3}$ . It's an easy exercise to show that this function is an isomorphism.

*Remark 2.9.* A convenient way to present a group is by choosing elements that *generate* the group (which means that any element of the group can be

written as a product of some of these generators and their inverses), and a set of *relations* among these generators. For instance,  $\mathbb{Z}/4\mathbb{Z}$  can be presented by

$$\mathbb{Z}/4\mathbb{Z} = \langle a : a^4 = e \rangle,$$

which means that one can find an element  $a \in \mathbb{Z}/4\mathbb{Z}$  such that any element in  $\mathbb{Z}/4\mathbb{Z}$  can be written as a power of  $a$ , and it satisfies  $a^4 = e$  (it's not hard to see that  $a$  can be chosen to be  $\bar{1}$  or  $\bar{3}$  in this case).

**Definition 2.10.** A group  $G$  that can be generated by a single element  $g$  is called a *cyclic* group (i.e. any element of  $G$  is of the form  $g^k$  for some  $k \in \mathbb{Z}$ ).

**Definition 2.11.** Let  $g$  be an element in a group  $G$ . If there exists a positive integer  $n$  such that  $g^n = e$ , then the smallest possible  $n$  satisfying  $g^n = e$  is called the *order* of  $g$ . If such  $n$  does not exist, then we say  $g$  is of infinite order.

*Exercise.* Let  $G$  be a cyclic group, and say it can be generated by an element  $g \in G$ .

- If  $g$  is of finite order, say  $\text{order}(g) = n$ . Prove that  $G \cong \mathbb{Z}/n\mathbb{Z}$ .
- If  $g$  is of infinite order, then prove that  $G \cong \mathbb{Z}$ .

Therefore, any cyclic group is isomorphic to either  $\mathbb{Z}$  or  $\mathbb{Z}/n\mathbb{Z}$  for some positive integer  $n$ .

*Exercise.* Prove that  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  is not a cyclic group.

*Example.* Let  $D_n$  be the symmetry group of a regular  $n$ -gon. It is not hard to show that  $D_n$  is generated by rotation by  $2\pi/n$  (which we'll denote by  $r$ ), and a reflection (which we'll denote by  $s$ ). The group  $D_n$  is of order  $2n$ , with elements given by

$$D_n = \{1, r, r^2, \dots, r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\}.$$

The generators  $r$  and  $s$  satisfy the relations  $r^n = s^2 = 1$  and  $s^{-1}rs = r^{-1}$ .

$$\begin{aligned} D_n &= \langle r, s \mid r^n = s^2 = 1, s^{-1}rs = r^{-1} \rangle \\ &= \langle r, s \mid r^n = s^2 = (rs)^2 = 1 \rangle. \end{aligned}$$

*Remark 2.12.* Since  $D_n$  is not commutative, it is not isomorphic to the direct product  $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . On the other hand, it is isomorphic to the *semi-direct product* of its order 2 *subgroup*  $\langle s \rangle$  and its order  $n$  *normal subgroup*  $\langle r \rangle$ :  $D_n \cong \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z}$ . We'll introduce these notations later on.

## 2.4. Subgroups.

**Definition 2.13.** Let  $G$  be a group. We say a subset  $H \subseteq G$  is a *subgroup* if:

- (1) it is closed under the binary operation of  $G$ : for any  $a, b \in H$ , we have  $ab \in H$ ;
- (2) it contains the identity element of  $G$ :  $e_G \in H$ ;
- (3) it is closed under taking inverse: for any  $a \in H$ , we have  $a^{-1} \in H$ .

*Exercise.* A subgroup  $H \subseteq G$  is itself a group, with the binary operator and the identity element inherit from  $G$ .

*Example.* For any group  $G$ , the subset  $\{e_G\} \subseteq G$  is always a subgroup, called the *trivial* subgroup of  $G$ . Also, the group  $G$  itself is a subgroup of  $G$ .

*Example.* For any positive integer  $n$ , the subset  $n\mathbb{Z} \subseteq (\mathbb{Z}, +)$  is a subgroup.

*Exercise.* Let  $G$  be a group and  $g_1, \dots, g_n$  be elements of  $G$ . Prove that the following two statements are equivalent:

- any  $g \in G$  can be written as  $g = g_{i_1}^{a_1} g_{i_2}^{a_2} \cdots g_{i_k}^{a_k}$  for some  $i_1, \dots, i_k \in \{1, \dots, n\}$  and  $a_1, \dots, a_k \in \mathbb{Z}$ ;
- the smallest subgroup of  $G$  that contains  $g_1, \dots, g_n$  is the group  $G$  itself.

In this case, we say  $\{g_1, \dots, g_n\}$  *generates* the group  $G$ .

If  $H$  is a subgroup of  $G$ , then one can break  $G$  up into pieces, each of which looks like  $H$ . These pieces are called *cosets* of  $H$ , and they arise by “multiplying”  $H$  by elements of  $G$ .

**Definition 2.14.** Let  $G$  be a group and  $H \subseteq G$  be a subgroup. A *left coset* of  $H$  in  $G$  is a subset of the form

$$gH = \{gh \mid h \in H\} \text{ for some } g \in G.$$

The element  $g$  is called a *representative* of the coset  $gH$ . The collection of all left cosets is denoted by  $G/H$ . Its order  $|G/H|$  is called the *index* of  $H$  in  $G$ , and will sometimes be denoted by  $[G : H]$ .

Similarly, a *right coset* is a subset of the form

$$Hg = \{hg \mid h \in H\} \text{ for some } g \in G.$$

The collection of all right cosets is denoted by  $H \backslash G$ .

*Example.* Consider the subgroup  $n\mathbb{Z} \subseteq (\mathbb{Z}, +)$ . Since the group  $(\mathbb{Z}, +)$  is abelian, its left cosets and right cosets are identical. It is clear that the subgroup has exactly  $n$  cosets  $\bar{0}, \bar{1}, \dots, \overline{n-1}$ , where  $\bar{i} = i + n\mathbb{Z}$  consists of integers  $\equiv i$  modulo  $n$ . Hence  $n\mathbb{Z} \subseteq \mathbb{Z}$  is a subgroup of index  $n$ .

## Lecture 2

*Exercise.* The representative of a coset is *not* unique. In fact, show that a coset  $gH$  can be represented by any element of the form  $gh$  where  $h \in H$ .

*Exercise.* Consider the subgroup  $\mathbb{Z} \subseteq (\mathbb{R}, +)$ . The set of cosets  $\mathbb{R}/\mathbb{Z}$  can be identified with  $S^1$ , the unit circle in  $\mathbb{R}^2$ : Points of the circle are of the form  $e^{2\pi i\theta}$  where  $\theta \in \mathbb{R}$ . Show that the map  $t \mapsto e^{2\pi it}$  gives a bijection between  $\mathbb{R}/\mathbb{Z}$  and  $S^1$ .

**Proposition 2.15.** *Let  $H \subseteq G$  be a subgroup. Prove that for any two cosets  $aH$  and  $bH$ , we have:*

- either  $aH$  and  $bH$  are disjoint:  $aH \cap bH = \emptyset$ ,
- or  $aH$  and  $bH$  are exactly the same:  $aH = bH$ .

*Proof.* Suppose  $aH$  and  $bH$  are not disjoint. Then there exists  $h_1, h_2 \in H$  such that  $ah_1 = bh_2$ . For any  $h \in H$ , we have

$$ah = a(h_1h_1^{-1})h = b(h_2h_1^{-1}h) \in bH.$$

Hence  $aH \subseteq bH$ . Similarly, one can show that  $bH \subseteq aH$ . Therefore  $aH = bH$ .  $\square$

**Theorem 2.16** (Lagrange). *Let  $G$  be a finite group, and  $H \subseteq G$  be a subgroup. Then  $|G|$  is divisible by  $|H|$ . Moreover, we have  $|G| = |H|[G : H]$ .*

*Proof.* Since  $g \in gH$ , any element of  $G$  belongs to a left coset of  $H$ . Then the previous proposition shows that  $G$  is the disjoint union of the left cosets of  $H$ . Since each coset has exactly  $|H|$  elements, we can conclude that  $|G| = |H|[G : H]$ .  $\square$

*Exercise.* Let  $G$  be a finite group and  $g$  be an element of  $G$ . Prove that the order of  $g$  divides the order of  $G$ .

*Remark 2.17.* In the example  $n\mathbb{Z} \subseteq (\mathbb{Z}, +)$ , one can notice that the set of all cosets  $\{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$  also has a natural group structure inherits from the group structure on  $(\mathbb{Z}, +)$ : one defines  $\bar{i} + \bar{j}$  to be  $\overline{i+j}$ .

However, the set of all left cosets does *not* always admit a group structure! Let  $H \subseteq G$  be a subgroup and  $a, b \in G$  be two elements in  $G$ . It is tempting to define a group structure on  $G/H$  simply by declaring “ $aH \circ bH = (ab)H$ ”. In order for this definition to make sense, we need to show that, if  $a'$  is a representative of  $aH$  and  $b'$  is a representative of  $bH$ , then  $a'b'H = abH$ . This is equivalent to, for any  $a, b \in G$  and  $h_1, h_2 \in H$ , one needs  $ah_1bh_2H = abH$ , or equivalently,  $b^{-1}h_1b \in H$ . This is equivalent to the condition that for any  $g \in G$  one needs  $gH = Hg$ , i.e. the left and right cosets of  $H$  in  $G$  coincide, which is *not* true in general.

**Definition 2.18.** A subgroup  $H \subseteq G$  is called *normal* if  $gH = Hg$  for any  $g \in G$ .

By the previous remark, if  $H \subseteq G$  is a normal subgroup, then the set of (left) cosets  $G/H$  admits a group structure inherit from  $G$ : let  $aH$  and  $bH$  be two cosets, then  $aH \circ bH := (ab)H$  gives a well-defined group structure on  $G/H$ . The resulting group  $G/H$  is called the *quotient group*.

**Theorem 2.19** (First isomorphism theorem). *Let  $f: G \rightarrow H$  be a group homomorphism. Define*

$$\text{Ker}(f) := \{g \in G \mid f(g) = 1_H\} \subseteq G$$

and

$$\text{Im}(f) := \{h \in H \mid h = f(g) \text{ for some } g \in G\} \subseteq H.$$

Then

- (1)  $\text{Ker}(f)$  is a normal subgroup of  $G$ .
- (2)  $\text{Im}(f)$  is a subgroup of  $H$ .
- (3) There is an isomorphism between  $G/\text{Ker}(f)$  and  $\text{Im}(f)$ .

*Proof.* It is not hard to show that  $\text{Ker}(f) \subseteq G$  and  $\text{Im}(f) \subseteq H$  are subgroups (homework). To show that  $\text{Ker}(f) \subseteq G$  is normal, one needs to show that for any  $g \in \text{Ker}(f)$  and  $g' \in G$ , we have  $g'gg'^{-1} \in \text{Ker}(f)$ . This is true because

$$f(g'gg'^{-1}) = f(g')f(g)f(g'^{-1}) = f(g')f(g)^{-1} = 1_H.$$

Now we define a map  $\bar{f}$  from  $G/\text{Ker}(f)$  to  $\text{Im}(f)$ : For any coset  $g\text{Ker}(f)$ , we define  $\bar{f}(g\text{Ker}(f)) := f(g)$ . This is a well-defined function on the set of cosets  $G/\text{Ker}(f)$ , because any representative of  $g\text{Ker}(f)$  is of the form  $gg'$  for some  $g' \in \text{Ker}(f)$ , and we have  $f(gg') = f(g)f(g') = f(g)$ . It is not hard to

check that  $\bar{f}: G/\text{Ker}(f) \rightarrow \text{Im}(f)$  is a surjective group homomorphism. It is also injective: if  $\bar{f}(g_1\text{Ker}(f)) = \bar{f}(g_2\text{Ker}(f))$ , then we have  $f(g_1) = f(g_2)$ , or equivalently  $g_2^{-1}g_1 \in \text{Ker}(f)$ . Hence the cosets  $g_1\text{Ker}(f) = g_2\text{Ker}(f)$  coincide.  $\square$

**2.5. Symmetry groups.** For any set  $X$ , a *permutation* of  $X$  is a bijective function  $f: X \rightarrow X$ . The *symmetric group*  $S_X$  *defined over*  $X$  is the set of all permutations of  $X$ , equipped with the group structure given by compositions. In particular, when  $X$  is a finite set of  $n$  elements  $\{1, 2, \dots, n\}$ , its symmetric group would be denoted by  $S_n$ . It is not hard to see that  $|S_n| = n!$ .

*Remark 2.20.* One of the reasons that symmetric groups are important is that, any group is isomorphic to a subgroup of a symmetric group (Cayley's theorem). More specifically, one can show that any group  $G$  is isomorphic to a subgroup of the symmetric group  $S_G$  whose elements are the permutations of the underlying set of  $G$ . Explicitly, for each  $g \in G$ , we define a permutation of  $G$  (called left multiplication)  $\ell_g: G \rightarrow G$  by  $\ell_g(x) := gx$ . It is an easy exercise to check that the map  $G \rightarrow S_G$  given by  $g \mapsto \ell_g$  is an injective group homomorphism. Hence  $G$  is isomorphic to the image of  $G \rightarrow S_G$ , which is a subgroup of  $S_G$ . In particular, if  $G$  is a finite group of order  $n$ , then this argument shows that  $G$  is isomorphic to a subgroup of  $S_n$ .

*Remark 2.21.* Symmetric groups will also arise naturally when we discuss the symmetry groups of Platonic solids. Let  $G$  be the symmetry group of a tetrahedron  $T$ . It is not hard to see that any symmetry of  $T$  sends a vertex of  $T$  to a vertex (not necessarily the same one); in other words, it gives rise to a permutation of the four vertices of  $T$ . This gives a group homomorphism  $\rho: \text{Aut}(T) \rightarrow S_4$ . Note that  $\rho$  is injective (why?), hence the symmetry group  $\text{Aut}(T)$  is isomorphic to a subgroup of the symmetric group  $S_4$ . (In fact, one can use the *orbit-stabilizer theorem* to show that  $\text{Aut}(T) \cong S_4$ .)

Any element of  $S_n$  can be represented by Cauchy's "two-line notation". Let  $\sigma \in S_n$  be a permutation of the set  $\{1, 2, \dots, n\}$ . Then we'll write

$$\sigma = \begin{bmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n). \end{bmatrix}$$

As usual, the composition  $\sigma_1\sigma_2 \in S_n$  is given by  $k \mapsto \sigma_1(\sigma_2(k))$ , i.e. first apply  $\sigma_2$  then apply  $\sigma_1$ . For instance, verify that

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Permutations are also often written in *cycle notation* (“decomposition into disjoint cycles”). To write down  $\sigma \in S_n$  in cycle notation, one proceeds as follows:

- Write an open bracket then select an arbitrary element  $x \in \{1, \dots, n\}$ , and write down:  $(x$
- Then trace the orbit of  $x$ : write down its value under successive applications of  $\sigma$ :  $(x \ \sigma(x) \ \sigma^2(x) \dots$
- Repeat until the value return to  $x$ , and write down a closing parenthesis rather than  $x$ :  $(x \ \sigma(x) \ \sigma^2(x) \dots)$
- Continue with any element  $y$  that is not yet written down, and proceed in the same way:  $(x \ \sigma(x) \ \sigma^2(x) \dots)(y \ \sigma(y) \dots)$
- Repeat until all elements of  $\{1, \dots, n\}$  are written in one of the cycles.

For instance,

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 6 & 2 & 3 & 5 \end{bmatrix} = (1)(24)(365) = (24)(365).$$

Here  $\sigma(1) = 1$  forms an 1-cycle, which is often omitted.

A 2-cycle is called a *transposition*. An important fact is that any element  $\sigma \in S_n$  can be written as a product of transpositions. To see this, it suffices to show that any cycle can be written as a product of transpositions, as any  $\sigma$  is a product of cycles. This can be easily verified:

$$(i_1 i_2 \dots i_k) = (i_1 i_k)(i_1 i_{k-1}) \dots (i_1 i_2).$$

It is not hard to see that there is no unique way to represent a permutation by a product of transpositions. For instance,  $(123) = (13)(12) = (12)(23) = (12)(23)(13)(13)$ . However, the *parity* (i.e. even or odd) of the numbers of transpositions of such representations is unique. (For instance,  $(123)$  can not be written as the product of odd number of transpositions.) This permits the *parity of a permutation* to be a well-defined notion.

The key idea of the proof is to define a group homomorphism

$$\text{sgn}: S_n \rightarrow \{+1, -1\} \text{ (under multiplication)}$$

so that all transpositions map to  $-1$ . Indeed, if we can find such a homomorphism, then for any representation  $\sigma = \tau_1 \cdots \tau_k$  where  $\tau_i$ 's are transpositions, we have

$$\text{sgn}(\sigma) = \text{sgn}(\tau_1) \cdots \text{sgn}(\tau_k) = (-1)^k.$$

This shows that the parity of  $k$  is independent of the choice of the decomposition.

Now, to define such group homomorphism  $\text{sgn}$ , we consider the Vandermonde polynomial

$$P(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

For  $\sigma \in S_n$ , define

$$\text{sgn}(\sigma) := \frac{P(x_{\sigma(1)}, \dots, x_{\sigma(n)})}{P(x_1, \dots, x_n)}.$$

Observe that the polynomials  $P(x_1, \dots, x_n)$  and  $P(x_{\sigma(1)}, \dots, x_{\sigma(n)})$  have the same factors except for the signs, therefore  $\text{sgn}(\sigma) = \pm 1$ . It defines a group homomorphism  $\text{sgn}: S_n \rightarrow \{\pm 1\}$  since

$$\begin{aligned} \text{sgn}(\sigma_1 \sigma_2) &= \frac{P(x_{\sigma_1(\sigma_2(1))}, \dots, x_{\sigma_1(\sigma_2(n))})}{P(x_1, \dots, x_n)} \\ &= \frac{P(x_{\sigma_1(\sigma_2(1))}, \dots, x_{\sigma_1(\sigma_2(n))})}{P(x_{\sigma_2(1)}, \dots, x_{\sigma_2(n)})} \cdot \frac{P(x_{\sigma_2(1)}, \dots, x_{\sigma_2(n)})}{P(x_1, \dots, x_n)} \\ &= \text{sgn}(\sigma_1) \text{sgn}(\sigma_2). \end{aligned}$$

Also, it is easy to check that  $\text{sgn}$  sends any transposition to  $-1$ . This finishes the proof.

**Definition 2.22.** The subset of  $S_n$  consisting of all *even* permutations will be denoted by  $A_n$ . It is a *normal subgroup* of  $S_n$  since it is the kernel of the group homomorphism  $\text{sgn}$ . The group  $A_n$  is called the *alternating group* (of  $n$  elements).

*Exercise.* Show that  $A_n \subseteq S_n$  is a normal subgroup of index 2; it has two cosets, one of them consists of all even permutations, the other consists of all odd permutations.



*Remark 2.23.* The *center*  $Z(G)$  of a group  $G$  is defined to be

$$Z(G) = \{g \in G \mid gh = hg \text{ for any } h \in G\} \subseteq G.$$

It is not hard to show that  $Z(G)$  is a subgroup of  $G$ . The center measures the *commutativity* of the group: for instance, if  $G$  is abelian then  $Z(G) = G$ . In the homework, you'll show that the symmetric group  $S_n$  has trivial center  $Z(S_n) = \{e\}$  if  $n \geq 3$ .

## Lecture 3

**2.6. Group actions.** We will be interested in groups  $G$  that act as symmetries of a set  $X$  (for instance, the symmetry group of a tetrahedron acting on the set of its vertices). Let us introduce the formal definition of group actions.

**Definition 2.24.** We say that a group  $G$  *acts on a set*  $X$  if there is a map

$$G \times X \rightarrow X; \quad (g, x) \mapsto g \cdot x$$

satisfying:

- $e_G \cdot x = x$  for any  $x \in X$ ,
- $g \cdot (h \cdot x) = (gh) \cdot x$  for any  $g, h \in G$  and  $x \in X$ .

The dot “ $\cdot$ ” is sometimes omitted when the context is clear.

*Exercise.* Show that to give a group action of  $G$  on  $X$  is equivalent to give a group homomorphism  $\rho: G \rightarrow S_X$ . (Hint: Relate them by  $g \cdot x = \rho(g)(x)$ .)

*Example.* The symmetric group  $S_n$  acts on the set  $\{1, \dots, n\}$ .

*Example.*  $\text{Isom}(\mathbb{R}^n)$  acts on  $\mathbb{R}^n$ .

*Example.*  $O(n, \mathbb{R})$  acts on the unit sphere  $S^{n-1} \subseteq \mathbb{R}^n$ , where

$$S^{n-1} = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| = 1\}.$$

*Example.* The dihedral group  $D_n$  acts on the set of vertices of a regular  $n$ -gon, which gives a group homomorphism  $D_n \rightarrow S_n$ . Similarly, the symmetry group of a Platonic solid  $P$  acts on the set of its vertices.

*Example.* Let  $G$  be a group. The left multiplication action of  $G$  on itself is defined to be

$$G \times G \rightarrow G; \quad (g, h) \mapsto g \cdot h := gh.$$

Equivalently, it's a group homomorphism

$$G \rightarrow S_G; \quad g \mapsto L_g,$$

where  $L_g(h) := gh$ .

*Exercise.* Check that the right multiplication  $g \cdot h := hg$  is *not* an action of  $G$  on itself. Instead,  $g \cdot h := hg^{-1}$  is an action of  $G$  on itself.

*Example.* Let  $G$  be a group. An important action of  $G$  on itself is the *conjugacy action*:

$$G \times G \rightarrow G; \quad (g, h) \mapsto g \cdot h := ghg^{-1}.$$

Equivalently, the conjugacy action is given by

$$G \rightarrow S_G; \quad g \mapsto \text{Ad}_g,$$

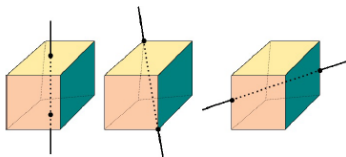
where  $\text{Ad}_g(h) := ghg^{-1}$ . Note that the permutation  $\text{Ad}_g: G \rightarrow G$  is in fact a group isomorphism.

Two elements  $h$  and  $h'$  of  $G$  are said to be *conjugate* if there exists  $g \in G$  such that  $h' = ghg^{-1}$ . In this case, we say  $h$  and  $h'$  belong to the same conjugacy class. The study of conjugacy classes of non-abelian groups is fundamental for the study of their structure.

*Exercise.* Elements in the same conjugacy class behave similarly in many ways. For instance, prove that  $\text{order}(h) = \text{order}(ghg^{-1})$ .

*Exercise.* Let  $G$  be a group and  $g \in G$ . Show that  $g$  is in the center  $Z(G)$  of  $G$  if and only if the conjugacy class of  $g$  consists of a single element  $\{g\}$ .

*Example.* Let  $C$  be a cube in  $\mathbb{R}^3$  centered at the origin. Denote  $\text{Aut}^+(C)$  the *rotational symmetric group* of  $C$ . Each element of  $\text{Aut}^+(C)$  is a rotation that fixes a line through the origin, and sends the cube  $C$  to itself. For instance:



- identity map;
- rotate  $\pi/2, \pi, 3\pi/2$  along the first (left-most) line: there are 3 such lines, so this gives in total 9 elements of  $\text{Aut}^+(C)$ ;
- rotate  $2\pi/3, 4\pi/3$  along the second line: there are 4 such lines, so this gives in total 8 elements of  $\text{Aut}^+(C)$ ;
- rotate  $\pi$  along the third line: there are 6 such lines, so this gives in total 6 elements of  $\text{Aut}^+(C)$ .

Hence  $|\text{Aut}^+(C)|$  is at least 24.

On the other hand, observe that  $\text{Aut}^+(C)$  gives an action on the set of the four main diagonals of  $C$ , therefore induces a group homomorphism

$$\rho: \text{Aut}^+(C) \rightarrow S_4.$$

One can show that  $\rho$  is injective (this is not a trivial observation: one needs to show that the antipodal map  $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3)$  is *not* a rotation). Now, combining with the fact that  $|\text{Aut}^+(C)| \geq 24$ , we can conclude that  $\rho$  is an isomorphism  $\text{Aut}^+(C) \cong S_4$ .

**Definition 2.25.** Let  $X$  be a set admitting a group action by  $G$ . For any  $x \in X$ , define its *orbit* to be

$$\text{orb}(x) := \{g \cdot x \mid g \in G\} \subseteq X.$$

It sometimes is also denoted by  $Gx$ .

The subset of  $G$  fixing  $x$

$$\text{Stab}(x) := \{g \in G \mid g \cdot x = x\}$$

is called the *stabilizer* of  $x$ , which is a subgroup of  $G$  (why?).

*Exercise.* Determine the orbits and stabilizers of the examples of group actions we mentioned above.

*Exercise.* Let  $X$  be a set admitting a group action by  $G$ . Let  $\text{orb}(x)$  and  $\text{orb}(y)$  be two orbits of the action. Prove that either  $\text{orb}(x) = \text{orb}(y)$  or  $\text{orb}(x) \cap \text{orb}(y) = \emptyset$ .

In other words, a group  $G$  acting on a set  $X$  decomposes  $X$  into disjoint union of the orbits of the action. The set of all orbits is denoted by  $X/G$ .

**Theorem 2.26.** Let  $X$  be a set admitting a group action by  $G$ . Let  $g \in G$  and  $x \in X$ .

- (1)  $\text{Stab}(gx) = g\text{Stab}(x)g^{-1}$ . In other words, the stabilizers of points on the same orbit are conjugate to each other.
- (2) (*Orbit-stabilizer theorem*) There is a bijective map between the orbit  $\text{orb}(x)$  and the set of left cosets  $G/\text{Stab}(x)$ . In particular, if  $|G|$  is finite then  $|G| = |\text{Stab}(x)||\text{orb}(x)|$ .

*Proof.* The first statement follows from

$$h \in \text{Stab}(gx) \Leftrightarrow hgx = gx \Leftrightarrow g^{-1}hgx = x \Leftrightarrow g^{-1}hg \in \text{Stab}(x).$$

To prove the second statement, consider the map

$$f: G \rightarrow \text{orb}(x); \quad g \mapsto gx.$$

The map is clearly surjective. For any two elements  $g_1, g_2 \in G$ ,

$$f(g_1) = f(g_2) \Leftrightarrow g_1x = g_2x \Leftrightarrow g_2^{-1}g_1x = x \Leftrightarrow g_2^{-1}g_1 \in \text{Stab}(x) \Leftrightarrow g_1 \in g_2\text{Stab}(x).$$

Hence  $f(g_1) = f(g_2)$  if and only if  $g_1$  and  $g_2$  lie in the same coset for the stabilizer subgroup  $\text{Stab}(x) \subseteq G$ . This proves the second statement.  $\square$

*Example.* Given a cube, we would like to put  $\{1, 2, 3, 4, 5, 6\}$  on its faces to make it a dice. How many different dice can we build (up to rotational symmetry)?

In terms of group actions, let  $X$  be the set of all possible ways of labeling  $\{1, 2, 3, 4, 5, 6\}$  on a dice, and  $G$  be the rotational symmetric group of the cube  $\text{Aut}^+(C)$  which acts naturally on  $X$ . The question is equivalent to counting the number of orbits of this group action. Observe that  $\text{Stab}(x) = \{e\}$  for any  $x \in X$ . By the orbit-stabilizer theorem, we have  $|\text{orb}(x)| \cdot 1 = |\text{Aut}^+(C)| = 24$  for each  $x \in X$ , i.e. each orbit has exactly 24 elements. The set  $X$  has  $6!$  elements, so the number of orbits is  $6!/24 = 30$ .

**Theorem 2.27** (Burnside's lemma). *Let  $X$  be a finite set admitting a group action by a finite group  $G$ . For any  $g \in G$ , denote  $X^g = \{x \in X \mid gx = x\}$  the collection of points fixed by  $g$ . Then the number of disjoint orbits satisfies*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

*Proof.* Consider the set of pairs

$$Z = \{(g, x) \in G \times X \mid gx = x\}.$$

On the one hand, for each  $g_0 \in G$  there exists  $|X^{g_0}|$  many elements in  $X$  such that  $(g_0, x) \in Z$ . Hence  $|Z| = \sum_{g \in G} |X^g|$ . On the other hand, for each  $x_0 \in X$ , there are  $|\text{Stab}(x_0)|$  many elements in  $G$  such that  $(g, x_0) \in Z$ . Hence

$$|Z| = \sum_{x \in X} |\text{Stab}(x)| = \sum_{x \in X} \frac{|G|}{|\text{orb}(x)|}.$$

Denote  $O_1, \dots, O_k$  the orbits of  $X$  under the  $G$ -action, where  $k = |X/G|$ . Then

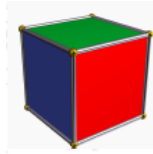
$$\sum_{x \in X} \frac{|G|}{|\text{orb}(x)|} = |G| \sum_{i=1}^k \sum_{x \in O_i} \frac{1}{|\text{orb}(x)|} = |G| \sum_{i=1}^k 1 = |G| |X/G|.$$

Therefore, we have

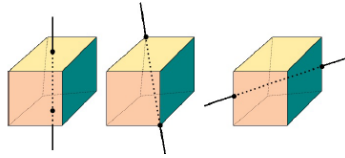
$$|G| |X/G| = |Z| = \sum_{g \in G} |X^g|.$$

□

*Example.* How many different ways are there to color the faces of a cube with three colors (up to rotational symmetry)? Let  $X$  be the set of all possible



colorings of the cube, and let  $G = \text{Aut}^+(C)$ . The problem is equivalent to calculating the number of orbits  $|X/G|$ . By Burnside's lemma, it suffices to compute the size of the fixed point sets for each element of  $G$ .



- identity map: fixes all colorings, there are  $3^6$  of them;
- rotate  $\pi/2, 3\pi/2$  along lines of the first type (6 such rotations): each fixes  $3^3$  colorings;
- rotate  $\pi$  along lines of the first type (3 such rotations): each fixes  $3^4$  colorings;
- rotate  $2\pi/3, 4\pi/3$  along lines of the second type (8 such rotations): each fixes  $3^2$  colorings;
- rotate  $\pi$  along lines of the third type (6 such rotations): each fixes  $3^3$  colorings.

By Burnside's lemma, we have

$$|X/G| = \frac{1}{24} (1 \cdot 3^6 + 6 \cdot 3^3 + 3 \cdot 3^4 + 8 \cdot 3^2 + 6 \cdot 3^3) = 57.$$

3. PLATONIC SOLIDS AND FINITE SUBGROUPS OF  $\text{SO}(3, \mathbb{R})$ 

## 3.1. Classification of the Platonic solids.

**Definition 3.1.** A *Platonic solid* is a convex polyhedron satisfying the following conditions:

- (1) all its faces are convex regular polygons, and are congruent (identical in shape and size);
- (2) none of its faces intersect except at their edges;
- (3) the same number of faces meet at each of its vertices.

Each Platonic solid is completely determined by two numbers  $p$  and  $q$ , where

- $p$  is the number of edges (or equivalently, vertices) of each face;
- $q$  is the number of faces (or equivalently, edges) that meet at each vertex.

**Fact 3.2.** *There are only five Platonic solids.*



<i>Polyhedron</i>	<i>Vertices <math>V</math></i>	<i>Edges <math>E</math></i>	<i>Faces <math>F</math></i>	$(p, q)$
<i>Tetrahedron</i>	4	6	4	$(3, 3)$
<i>Cube</i>	8	12	6	$(4, 3)$
<i>Octahedron</i>	6	12	8	$(3, 4)$
<i>Dodecahedron</i>	20	30	12	$(5, 3)$
<i>Icosahedron</i>	12	30	20	$(3, 5)$

*Proof.* We would like to show that there is no other possible  $(p, q)$  that can be used to form a Platonic solid.

**A topological proof:** It is not hard to see that  $pF = 2E$  and  $qV = 2E$ . By the Euler's formula  $V - E + F = 2$ , one obtains

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E} > \frac{1}{2}.$$

Also, note that  $p$  and  $q$  must both be at least 3. One can then check that there are only 5 possibilities for  $(p, q)$ .

**A geometric proof:** Consider any vertex of a Platonic solid. The angle between two edges that meet at a vertex is  $\pi - \frac{2\pi}{p}$ . Now we use the fact that at each vertex of a convex polyhedron, the total (among the adjacent faces) of the angles between their respective adjacent sides is strictly less than  $2\pi$ . Therefore, we have

$$q \left( \pi - \frac{2\pi}{p} \right) < 2\pi,$$

which is equivalent to  $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$  that we obtained in the previous proof.  $\square$

**3.2. Symmetry groups of the Platonic solids.** In order to study the symmetry group  $\text{Aut}(P)$  of a Platonic solid  $P$ , one can move the solid so that its center is located at the origin  $\vec{0} = (0, 0, 0) \in \mathbb{R}^3$ . Then, any isometry of  $\mathbb{R}^3$  that fixes  $P$  must also fix the origin.

**Definition 3.3.** The *orthogonal group*  $O(3, \mathbb{R})$  is defined as:

$$O(3, \mathbb{R}) := \left\{ f \in \text{Isom}(\mathbb{R}^3) \mid f(\vec{0}) = \vec{0} \right\},$$

which consists of isometries of  $\mathbb{R}^3$  that fix the origin  $\vec{0} = (0, 0, 0)$  of  $\mathbb{R}^3$ .

*Exercise.* Prove that  $O(3, \mathbb{R})$  is a subgroup of  $\text{Isom}(\mathbb{R}^3)$ .

**Definition 3.4.** A *rotation* of  $\mathbb{R}^3$  about the origin is a map  $f \in O(3, \mathbb{R})$  such that

- $f$  fixes a line  $\ell$  through the origin (called the *axis of rotation*), and
- $f$  rotates the two-dimensional plane through the origin orthogonal to  $\ell$ .

*Exercise (Hard).* Prove that the composition of two rotations is still a rotation. Therefore, the set of all rotations about the origin forms a subgroup of  $O(3, \mathbb{R})$ .

**Notation.** The group of all rotations about the origin of  $\mathbb{R}^3$  will be denoted by  $SO(3, \mathbb{R})$ . We will call it the *rotation group* for now. (Its official name is the *special orthogonal group* in dimension 3. Both  $O(3, \mathbb{R})$  and  $SO(3, \mathbb{R})$  are important examples of *matrix groups*. We will discuss their further properties later.)

Given a Platonic solid  $P$  centered at the origin  $\vec{0} \in \mathbb{R}^3$ , we would like to study:

- the symmetry group  $\text{Aut}(P)$ , which is a subgroup of the orthogonal group  $O(3, \mathbb{R})$ ;
- the intersection  $\text{Aut}(P) \cap SO(3, \mathbb{R})$ , consisting of rotations that fix the solid  $P$ , will be called the *rotational symmetry group* of  $P$ , and will be denoted by  $\text{Aut}^+(P)$ .

We will prove that any finite subgroup of  $SO(3, \mathbb{R})$  is either cyclic, dihedral, or  $\text{Aut}^+(P)$  for some Platonic solid  $P$ . Therefore, the Platonic solids not only classify the regular polyhedrons in  $\mathbb{R}^3$ , but also provide a classification of finite subgroups of the rotation groups in dimension 3.

**The tetrahedron:** Let us first study the symmetry group of the tetrahedron  $T$  (centered at the origin). Any symmetry of  $T$  permutes its vertices, hence we have a group homomorphism

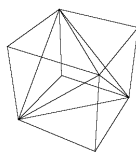
$$\rho: \text{Aut}(T) \rightarrow S_4.$$

Note that  $\rho$  is injective: if a symmetry fixes all four vertices, then it is the identity. On the other hand, consider the action of  $\text{Aut}(T)$  on the set of vertices  $\{1, 2, 3, 4\}$ . For any vertex, it is clear that its orbit consists of four elements, and its stabilizer consists of six elements. Hence  $|\text{Aut}(T)| = 24$  by the orbit-stabilizer theorem, and  $\rho$  is therefore an isomorphism.

#### Lecture 4

*Exercise.* Prove that  $|\text{Aut}^+(T)| = 12$  and  $\text{Aut}^+(T) \cong A_4$ . Can you identify the corresponding rotations?

**The cube:** There are two tetrahedra embedded in a cube  $C$ , with vertices at the vertices of the cube, denoted by  $T^+$  and  $T^-$  (see the figure below). Here



are some useful observations:

- Any symmetry of  $C$  either maps each of the two tetrahedra onto itself, or interchanges the tetrahedra.



- The symmetry  $J: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  which sends  $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3)$  would interchange  $T^+$  and  $T^-$ . It is an exercise to show that  $J$  commutes with any other symmetry of the cube (i.e.  $J \circ f = f \circ J$  for any symmetry  $f$ ); or equivalently,  $J$  is in the center of  $\text{Aut}(C)$ .

One can then define a map (which can be checked is a group homomorphism)

$$\rho: \text{Aut}(C) \rightarrow \text{Aut}(T^+) \times \mathbb{Z}/2\mathbb{Z}$$

where:

- $\rho(f) = (f, \bar{0})$  if  $f(T^+) = T^+$ ;
- $\rho(f) = (J \circ f, \bar{1})$  if  $f(T^+) = T^-$ .

It is again easy to show that  $\rho$  is injective, and moreover is isomorphic by the orbit-stabilizer theorem. Hence  $\text{Aut}(C) \cong S_4 \times (\mathbb{Z}/2\mathbb{Z})$ .

*Exercise.* Verify that the map  $\rho$  is indeed a group homomorphism.

*Exercise.* Prove that  $\text{Aut}^+(C) \cong S_4$ . Can you identify the corresponding rotations?

**The dodecahedron:** By the orbit-stabilizer theorem, it is not hard to show that the rotational symmetry group of a dodecahedron has order  $|\text{Aut}^+(D)| = 60$ . Let us try to identify these 60 rotations that fix the dodecahedron  $D$ :

- the identity map;
- for each pair of opposite faces (there are 6 such pairs), there are 4 rotations (by multiples of  $2\pi/5$ ) about their centers;
- for each pair of opposite edges (there are 15 such pairs), there is 1 rotation (by  $\pi$ ) about their centers;
- for each pair of opposite vertices (there are 10 such pairs), there are 2 rotations (by multiples of  $2\pi/3$ ) about the line connecting them.

These add up to  $1 + 6 \times 4 + 15 \times 1 + 10 \times 2 = 60$ , which therefore are all the elements of  $\text{Aut}^+(D)$ .

Consider the alternating group  $A_5$ . It has 24 elements that are 5-cycles, 20 elements that are 3-cycles, and 15 elements that are of the form  $(\star\star)(\star\star)$ . This suggests that we might have  $\text{Aut}^+(D) \cong A_5$ . Let us try to prove it.

Similar to the two tetrahedra embedded in a cube, there are five cubes embedded in a dodecahedron  $D$  in such a way that any symmetry of the



dodecahedron permutes these five embedded cubes, therefore gives a group homomorphism

$$\rho: \text{Aut}(D) \rightarrow S_5.$$

However, the homomorphism  $\rho$  is *not* surjective: in fact, one can check that any symmetry of  $D$  gives an *even* permutations of the set of the five cubes. So we actually get a homomorphism

$$\rho: \text{Aut}(D) \rightarrow A_5.$$

Moreover, one can check that the kernel of  $\rho$ , namely symmetries of  $D$  that fix each of the five cubes, consists of the identity and the map  $J$  that sends  $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3)$ . Hence  $\rho$  induces an injective homomorphism  $\text{Aut}(D) / \{1, J\} \hookrightarrow A_5$ . By the orbit-stabilizer theorem, one can show that  $|\text{Aut}(D)| = 120$ . Therefore by counting the orders of these groups, we have

$$\text{Aut}(D) / \{1, J\} \cong A_5.$$

On the other hand, consider the composition  $\tau$  of group homomorphisms

$$\tau: \text{Aut}^+(D) \hookrightarrow \text{Aut}(D) \rightarrow \text{Aut}(D) / \{1, J\}.$$

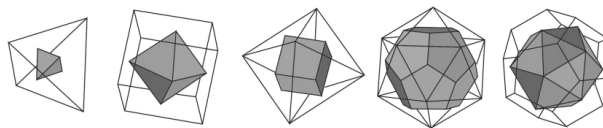
Since  $J$  is not a rotation, the composition  $\tau$  is injective. Again by counting the orders of these groups, we obtain

$$\text{Aut}^+(D) \cong \text{Aut}(D) / \{1, J\} \cong A_5.$$

Now, let us study the full symmetry group  $\text{Aut}(D)$ . Since  $|\text{Aut}(D)| = 120$ , we know that  $\text{Aut}(D)$  contains  $\text{Aut}^+(D) \cong A_5$  as an index two subgroup. It is tempting to believe that  $\text{Aut}(D)$  is isomorphic to  $S_5$ . However, this is *not* true! One way to see this is by observing that  $J \neq 1$  is in the center  $\text{Aut}(D)$ , but we also know that the center of  $S_5$  is trivial, hence  $\text{Aut}(D) \not\cong S_5$ . In fact, it is not hard to show the following.

*Exercise.* Prove that  $\text{Aut}(D) \cong A_5 \times (\mathbb{Z}/2\mathbb{Z})$ .

*Remark 3.5.* Every polyhedron has a *dual polyhedron* with faces and vertices interchanged. One can construct the dual polyhedron by taking the vertices of the dual to be the centers of the faces of the original figure. Connecting the centers of adjacent faces in the original forms the edges of the dual and thereby interchanges the number of faces and vertices while maintaining the number of edges. The dual of every Platonic solid is another Platonic solid, so we can arrange the five solids into dual pairs (where the tetrahedron is self-dual).



*Exercise.* The symmetry group of any polyhedron coincides with the symmetry group of its dual. (This is not hard to show by examining the construction of the dual polyhedron.) Therefore, there are only three symmetry groups associated with the Platonic solids rather than five.

We summarize the symmetry groups of the Platonic solids as follows.

Polyhedron	Aut	Aut <sup>+</sup>
Tetrahedron	$S_4$	$A_4$
Cube/Octahedron	$S_4 \times (\mathbb{Z}/2\mathbb{Z})$	$S_4$
Dodecahedron/Icosahedron	$A_5 \times (\mathbb{Z}/2\mathbb{Z})$	$A_5$

**3.3. Finite subgroups of the rotation group  $\text{SO}(3, \mathbb{R})$ .** Before proving the classification result regarding the rotation group in three dimension, let us begin with a two-dimensional result.

**Theorem 3.6.** *Let  $G$  be a finite subgroup of  $\text{O}(2, \mathbb{R})$ . Then  $G$  is isomorphic to either a cyclic group or a dihedral group.*

*Proof.* Any element of  $\text{O}(2, \mathbb{R})$  acts naturally on the unit circle  $S^1 \subseteq \mathbb{R}^2$ . Let  $g \in G$  be a non-identity element. It is not hard to show that  $g$  is either a rotation (when  $g$  does not fix any point of  $S^1$ ), or a reflection (when  $g$  fixes at least a point of  $S^1$ ).

First, suppose that all elements of  $G$  are rotations. Write  $r_\theta \in \text{O}(2, \mathbb{R})$  for the counterclockwise rotation by  $\theta$ , where  $0 \leq \theta < 2\pi$ . Choose  $r_\phi \in G$  with the smallest positive  $\phi$  (it is possible since  $G$  is finite). We claim that  $G$  is the

cyclic group generated by  $r_\phi$ . Let  $r_\theta \in G$ , and write  $\theta = m\phi + \psi$  where  $m \in \mathbb{N}$  and  $0 \leq \psi < \phi$ . Then  $r_\psi = (r_\phi)^{-m}r_\theta \in G$ . Therefore  $\psi = 0$  by the minimality of  $\phi$ . Hence  $r_\theta = (r_\phi)^m$ .

Second, suppose  $G$  contains a reflection  $s$ . Let  $H \subseteq G$  be the subgroup consisting of rotations (including the identity). By the first case, we have  $H = \{1, r, \dots, r^{n-1}\}$  for some positive integer  $n$ . Consider any other reflection  $s' \in G$ . One can show that the composition of any two reflections is a rotation, hence  $ss' \in H$ . So  $s' = sr^k$  for some  $0 \leq k \leq n-1$ . This shows that

$$G = \{1, r, \dots, r^{n-1}, s, sr, \dots, sr^{n-1}\}.$$

It is easy to show that a rotation  $r$  and a reflection  $s$  satisfy the relation  $sr = r^{-1}s$ . Hence we get  $G \cong D_n$ .  $\square$

*Exercise.* Prove (geometrically) that  $O(2, \mathbb{R})$  is a subgroup of  $SO(3, \mathbb{R})$ .

**Theorem 3.7.** *Let  $G$  be a finite subgroup of  $SO(3, \mathbb{R})$ . Then  $G$  is isomorphic to precisely one of the following groups:*

- cyclic group  $\mathbb{Z}/n\mathbb{Z}$ ,
- dihedral group  $D_n$ ,
- $A_4$ : the rotational symmetry group of a tetrahedron,
- $S_4$ : the rotational symmetry group of a cube (or a octahedron),
- $A_5$ : the rotational symmetry group of a dodecahedron (or a icosahedron).

*Proof.* Observe that  $G \subseteq SO(3, \mathbb{R})$  acts on the unit sphere  $S^2 \subseteq \mathbb{R}^3$ . Each rotation (other than  $e \in SO(3, \mathbb{R})$ ) gives two poles on the unit sphere which are the intersection of the axis of rotation with the unit sphere. Let  $X \subseteq S^2$  denote the set of all poles of all the elements in  $G \setminus \{e\}$ . We claim that  $G$  acts on the set  $X$ . To see this, let  $g \in G$  and  $x \in X$ . Say  $x$  is a pole for  $h \in G \setminus \{e\}$  (i.e.  $h(x) = x$ ). Then  $(ghg^{-1})(gx) = ghx = gx$ . Hence  $gx$  is a pole for  $ghg^{-1} \neq e$ , so  $gx \in X$ .

Now the idea of the proof is to apply Burnside's lemma to the action of  $G$  on  $X$ , and show that  $X$  has to be a particularly nice configuration of points on the sphere.

Let  $N$  be the number of orbits of the  $G$ -action on  $X$ . Choose a representative from each orbit, say  $x_1, \dots, x_N \in X$ . Observe that the identity  $e$  fixes every

pole, and each  $g \neq e$  fixes exactly two poles. By Burnside's lemma, we have

$$\begin{aligned} N &= \frac{1}{|G|} (|X| + (|G| - 1) \cdot 2) \\ &= \frac{1}{|G|} \left( 2(|G| - 1) + \sum_{i=1}^N |\text{orb}(x_i)| \right) \end{aligned}$$

By orbit-stabilizer theorem, we have

$$\begin{aligned} 2 \left( 1 - \frac{1}{|G|} \right) &= N - \sum_{i=1}^N \frac{|\text{orb}(x_i)|}{|G|} \\ &= N - \frac{1}{|\text{Stab}(x_i)|} \\ &= \sum_{i=1}^N \left( 1 - \frac{1}{|\text{Stab}(x_i)|} \right) \end{aligned}$$

Since  $|\text{Stab}(x_i)| \geq 2$  for each  $i$ , it is then easy to deduce that  $N \leq 3$ . Clearly there is no solution with  $N = 1$ , so  $N$  is either 2 or 3.

Suppose  $N = 2$ . Then

$$2 - \frac{2}{|G|} = 2 - \frac{1}{|\text{Stab}(x_1)|} - \frac{1}{|\text{Stab}(x_2)|} \leq 2 - \frac{2}{|G|}.$$

Hence  $\text{Stab}(x_1) = \text{Stab}(x_2) = G$  and  $|\text{orb}(x_1)| = |\text{orb}(x_2)| = 1$ . In other words,  $X$  consists of two unit vectors that are fixed by all elements of  $G$ . Suppose that one of the vectors is  $u$ , then the other must be  $-u$ . Therefore, any element of  $G$  has its axis of rotation given by  $u$  (or equivalently  $-u$ ), and rotates the two-dimensional plane orthogonal to  $u$ . By what we discussed earlier about finite subgroups of  $O(2, \mathbb{R})$ ,  $G$  is a cyclic group.

Suppose  $N = 3$ . Let  $|\text{Stab}(x_1)| \geq |\text{Stab}(x_2)| \geq |\text{Stab}(x_3)| \geq 2$ . Then

$$\frac{1}{|\text{Stab}(x_1)|} + \frac{1}{|\text{Stab}(x_2)|} + \frac{1}{|\text{Stab}(x_3)|} = 1 + \frac{2}{|G|}.$$

This implies that  $3/|\text{Stab}(x_3)| > 1$ , hence  $|\text{Stab}(x_3)| = 2$ . Therefore

$$\frac{1}{|\text{Stab}(x_1)|} + \frac{1}{|\text{Stab}(x_2)|} = \frac{1}{2} + \frac{2}{|G|}.$$

This implies that  $2/|\text{Stab}(x_2)| > 1/2$ , hence  $|\text{Stab}(x_2)|$  is either 2 or 3. There are four possible cases:

- (a)  $|\text{Stab}(x_2)| = 2$ .
- (b)  $|\text{Stab}(x_2)| = 3$  and  $|\text{Stab}(x_1)| = 3$ . ( $|G| = 12$ )
- (c)  $|\text{Stab}(x_2)| = 3$  and  $|\text{Stab}(x_1)| = 4$ . ( $|G| = 24$ )
- (d)  $|\text{Stab}(x_2)| = 3$  and  $|\text{Stab}(x_1)| = 5$ . ( $|G| = 60$ )

**Case (a):** Suppose  $|\text{Stab}(x_2)| = 2$ . If  $|\text{Stab}(x_1)| = 2$ , then we get  $|G| = 4$ . It is an easy exercise to show that any group of four elements is isomorphic to either  $\mathbb{Z}/4\mathbb{Z}$  or  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong D_2 = \langle r, s \mid r^2 = s^2 = (rs)^2 = 1 \rangle$ .

If  $|\text{Stab}(x_1)| = M \geq 3$ , then  $|G| = 2M$  and the orbit of  $x_1$  consists of two elements. The stabilizer  $\text{Stab}(x_1)$  is a finite group of  $M$  rotations about the line  $\ell_{x_1}$  through  $x_1$ , so it is a cyclic group of  $M$  elements generated by a rotation  $R \in G$ . Denote the plane orthogonal to  $\ell_{x_1}$  and passes through the origin by  $P_{x_1}$ .

First, one observes that  $\text{orb}(x_1) = \{x_1, -x_1\}$ , since they are the only two elements of  $X$  that have  $M$  elements in their stabilizers. Now, we consider the action of  $\text{Stab}(x_1) = \langle R \rangle$  on  $x_2$ . The points  $\{x_2, Rx_2, \dots, R^{M-1}x_2\}$  would form a regular  $M$ -gon in a plane orthogonal to  $\ell_{x_1}$ . We claim they actually lie in  $P_{x_1}$ .

## Lecture 5

Upcoming topics:

- finish the proof of the classification result
- a crash course (or review) of linear algebra (linear transformations, determinants, eigenvectors/eigenvalues, orthogonal matrices, etc.)
- identify rotations in  $\mathbb{R}^3$  with the special orthogonal group

□