Plan:
- following 2 weeks: conformal maps, Riemann mapping thm.
- last 2 weeks: elliptic fcns, modular forms.

Conformal maps

Def $U, V \subseteq \mathbb{C}$ open subsets.

Say $f: U \to V$ is <u>conformal</u> or <u>biholomorphic</u>

if:
- f is bijective. (injective + surjective)
- both f and f^{-1} are holomorphic.

We say U & V are <u>conformally equivalent</u> or <u>biholomorphic</u>
if $\exists$ $f: U \to V$ biholo.

Rmk: If U, V are biholo. (i.e. $\exists$ $f: U \to V$ biholo.)
then
- $\{\text{holo. fcns on } U\} \xleftrightarrow{1-1} \{\text{holo. fcns on } V\}$

$$U \xrightarrow{g} \mathbb{C} \longmapsto V \xrightarrow{f^{-1}} U \xrightarrow{g} \mathbb{C}$$

$$U \xrightarrow{f} V \xrightarrow{h} \mathbb{C} \longleftarrow\!\shortmid\; V \xrightarrow{h} \mathbb{C}$$

- $\{\text{holo. fcns } U \to U\} \xleftrightarrow{1-1} \{\text{holo. fcns } V \to V\}$

$$U \xrightarrow{g} U \longmapsto$$

$$\begin{tikzcd} V \arrow[r, dashed] \arrow[d, "f^{-1}"'] & V \\ U \arrow[r, "g"] & U \arrow[u, "f"'] \end{tikzcd}$$

open unit disk

Q: What are all the open subsets of $\mathbb{C}$ that are biholo to $\mathbb{D}$?

In HW, you'll show that if $\mathcal{U} \underset{\text{biholo}}{\cong} \mathbb{D}$, then $\mathcal{U}$ must be simply connected and connected.

(you can prove it by only knowing $\exists f: \mathcal{U} \to \mathbb{D}$

where • f is bijective

• f, f^{-1} are continuous)

(some topological constrains on $\mathcal{U}$ if $\mathcal{U}$ is biholo. to $\mathbb{D}$.)

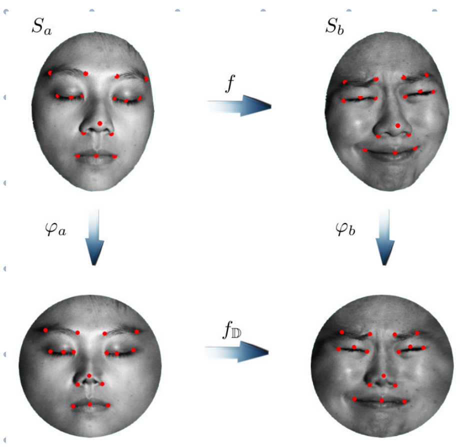
Q: Are $\mathbb{C}$ and $\mathbb{D}$ biholo.??

(i.e. $\exists f: \mathbb{C} \to \mathbb{D}$ biholo.??)

No, by Liouville.

Riemann mapping thm. $\Omega \subsetneq \mathbb{C}$ nonempty open subset, connected, simply connected.

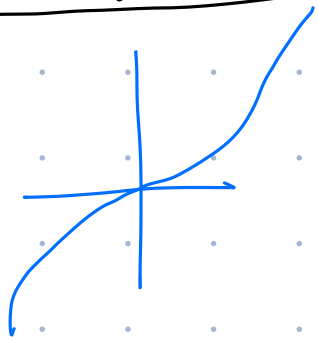
Then Ω is biholo. to $\mathbb{D}$.



Thm $U, V \subseteq \mathbb{C}$ open, $f: U \to V$ bijective, holo,
then $f^{-1}: V \to U$ is also holo.
($\Rightarrow$ f is biholo.)

Rmk: Not true / $\mathbb{R}$.

$f(x) = x^3$.



bijective
differentiable

But $f^{-1}(x) = x^{1/3}$ is not differentiable (at $x=0$)

Prop: $f: U \to V$ injective, holo, $\Rightarrow$ $f'(z) \neq 0$ $\forall z \in U$.

Prop $\Rightarrow$ Thm ($f: U \to V$ bij. holo, Want: $f^{-1}: V \to U$ holo.)

- $g = f^{-1}: V \to U$

$$w_0 \longleftrightarrow z_0,$$
$$w \longleftrightarrow z$$

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\frac{w - w_0}{g(w) - g(w_0)}} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}$$

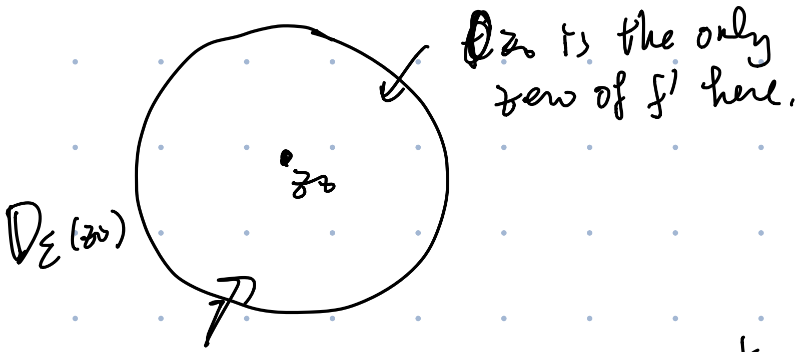
$$\to \frac{1}{f'(z_0)} \quad \text{as} \quad w \to w_0$$

$\Rightarrow$ $\overset{f^{-1}}{\overset{\parallel}{g}}$ is holo., and $g'(w_0) = \dfrac{1}{f'(z_0)}$. $\square$

Pf of Prop: ($f: U \to V$ inj. holo, then $f' \neq 0$)

Suppose $f'(z_0) = 0$ for some $z_0 \in U$.

- z_0 is an isolated zero of f'. (b/c f is injective)



$$f(z) = f(z_0) + a(z - z_0)^k + (z - z_0)^{k+1} g(z).$$

where $a \neq 0$, $k \geq 2$, g holo.

- We can shrink the nbhd further so that

$$|a(z-z_0)^k| > |(z-z_0)^{k+1} g(z)| \quad \forall z \in \partial D_\varepsilon(z_0)$$

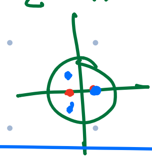
Since

$$\lim_{z \to z_0} \left| \frac{(z-z_0)^{k+1} g(z)}{a(z-z_0)^k} \right| = \lim_{z \to z_0} \frac{|g(z)| \cdot |z - z_0|}{|a|} = 0$$

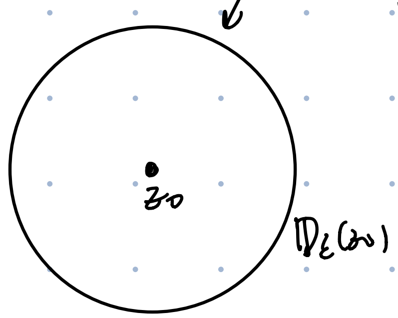
Idea: when we take small enough nbhd of z_0, $\boxed{f(z_0) + a(z-z_0)^k}$ dominates $(z-z_0)^{k+1} g(z)$

Consider $\boxed{f(z_0) + a(z-z_0)^{k \geq 2}}$

e.g. $z^{k \geq 2}$ is not injective in any nbd of 0.



$$|a(z-z_0)^k| > |(z-z_0)^{k+1} g(z)| \quad \forall z \in \partial D_\varepsilon(z_0)$$



Choose $w \neq 0$ small enough, s.t.

$$|a(z-z_0)^k| > |(z-z_0)^{k+1} g(z) - w|$$

holds on $z \in \partial D_\varepsilon(z_0)$.

- $f(z) - \underset{-w}{f(z_0)} = a(z-z_0)^k + (z-z_0)^{k+1} g(z) - w$

By Rouché,

$\#$ of zeros of $a(z-z_0)^k$ in $D_\varepsilon(z_0)$ $= k \geq 2$

$= \#$ of zeros of $f(z) - \underset{-w}{f(z_0)}$ in $D_\varepsilon(z_0)$

$\Rightarrow \#$ of zeros of $\boxed{f(z) - \underset{-w}{f(z_0)}}$ in $D_\varepsilon(z_0)$ $= k \geq 2$

$\downarrow$

has zero of order k at z_0

$\rightarrow$ this doesn't give any contradiction.

Now, we have:

$\#$ of zeros of $f(z) - f(z_0) - w$ in $D_\varepsilon(z_0)$ ≥ 2.

counts multiplicity

(for some $w \neq 0$)

<u>Claim</u>: zeros of $f(z) - f(z_0) - w$ in $D_\varepsilon(z_0)$ are all simple (order 1)

pf: If $\tilde{z}$ is a zero of $f(z) - f(z_0) - w$ w/ order ≥ 2,

then
- $f(\tilde{z}) - f(z_0) - w = 0$ — doesn't satisfy this eq'n.
- $f'(\tilde{z}) = 0 \Rightarrow \tilde{z} = z_0$

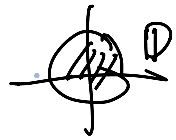
$\square$

$\Rightarrow \exists\, z_1 \neq z_2$ in $D_\varepsilon(z_0)$ s.t. $\begin{cases} f(z_1) - f(z_0) - w = 0 \\ f(z_2) - f(z_0) - w = 0 \end{cases}$

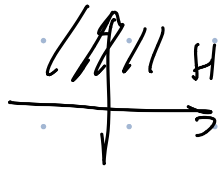
$\Rightarrow f(z_1) = f(z_0) + w = f(z_2)$. $\rightarrow$ contradicts w/ f inj. $\square$

§ examples of biholo. maps

$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$



$\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$



Thm $F : \mathbb{H} \longrightarrow \mathbb{D}$ is biholo.

$$z \longmapsto \frac{i - z}{i + z}$$

pf:
- F is clearly holo.
- F is injective: $\dfrac{i - z_1}{i + z_1} = \dfrac{i - z_2}{i + z_2}$

$$\Rightarrow (i - z_1)(i + z_2) = (i - z_2)(i + z_1)$$

$$-\cancel{1} + i z_2 - i z_1 - \cancel{z_1 z_2} \qquad\qquad \cancel{-1} + i z_1 - i z_2 - \cancel{z_1 z_2}$$

$$\Rightarrow z_1 = z_2.$$

- F is surjective:

$$W = \frac{i - z}{i + z}$$

$$\Leftrightarrow W(i + z) = i - z$$

$$\| \quad iW + Wz$$

$$\Leftrightarrow z(W + 1) = i(1 - W)$$

$$\Leftrightarrow z = i \, \frac{1 - W}{1 + W}.$$

$|F(z)| < 1$

$\| \quad \left| \dfrac{i - z}{i + z} \right|$

$\Updownarrow$

$|i - z|^2 < |i + z|^2$

$\| \quad (i - z)(\bar{i} - \bar{z}) \qquad \| \quad (i + z)(\bar{i} + \bar{z})$

$\| \quad \cancel{1} - i\bar{z} + \bar{i}z + \cancel{|z|^2} \qquad \| \quad \cancel{1} + i\bar{z} - \bar{i}z + \cancel{|z|^2}$

$\Updownarrow$

$i(\bar{z} - z) > 0$

$z = x + iy$

$\bar{z} = x - iy$

$\bar{z} - z = -2iy$

$i(\bar{z} - z) > 0$

$\| \quad i(-2iy)$

$\| \quad 2y$

$\Updownarrow \quad y > 0$

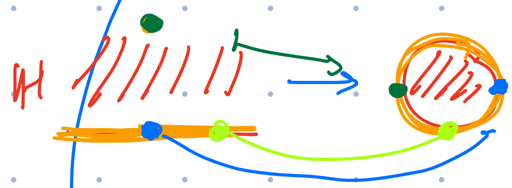
$\Updownarrow$

$\operatorname{Im}(z) > 0 \Leftrightarrow z \in \mathbb{H}$

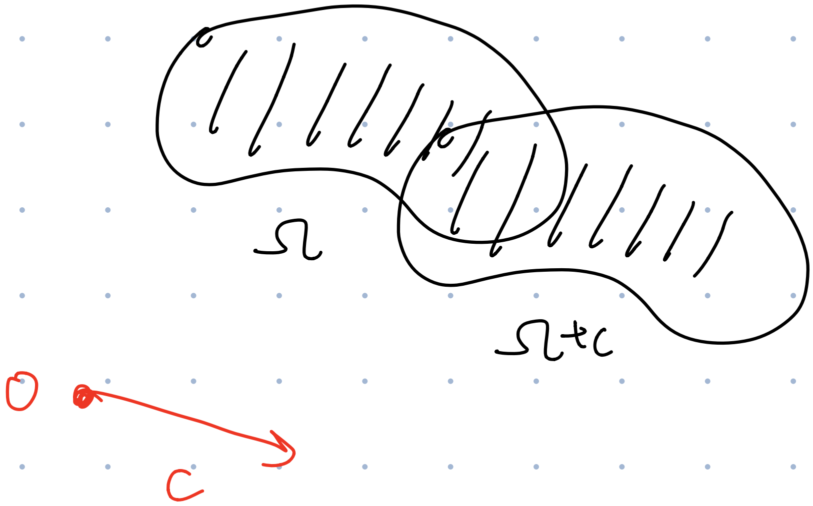
Rmk:

The map F extends continuous to the boundary:

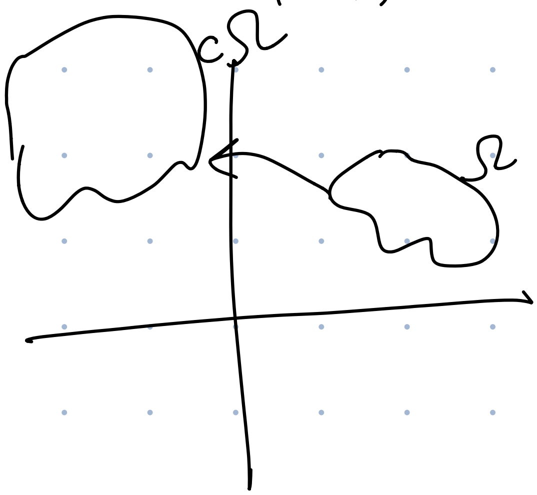
$$\bar{F} : \bar{\mathbb{H}} \to \bar{\mathbb{D}}$$



e.g. translation $c \in \mathbb{C}$, $z \mapsto z+c$

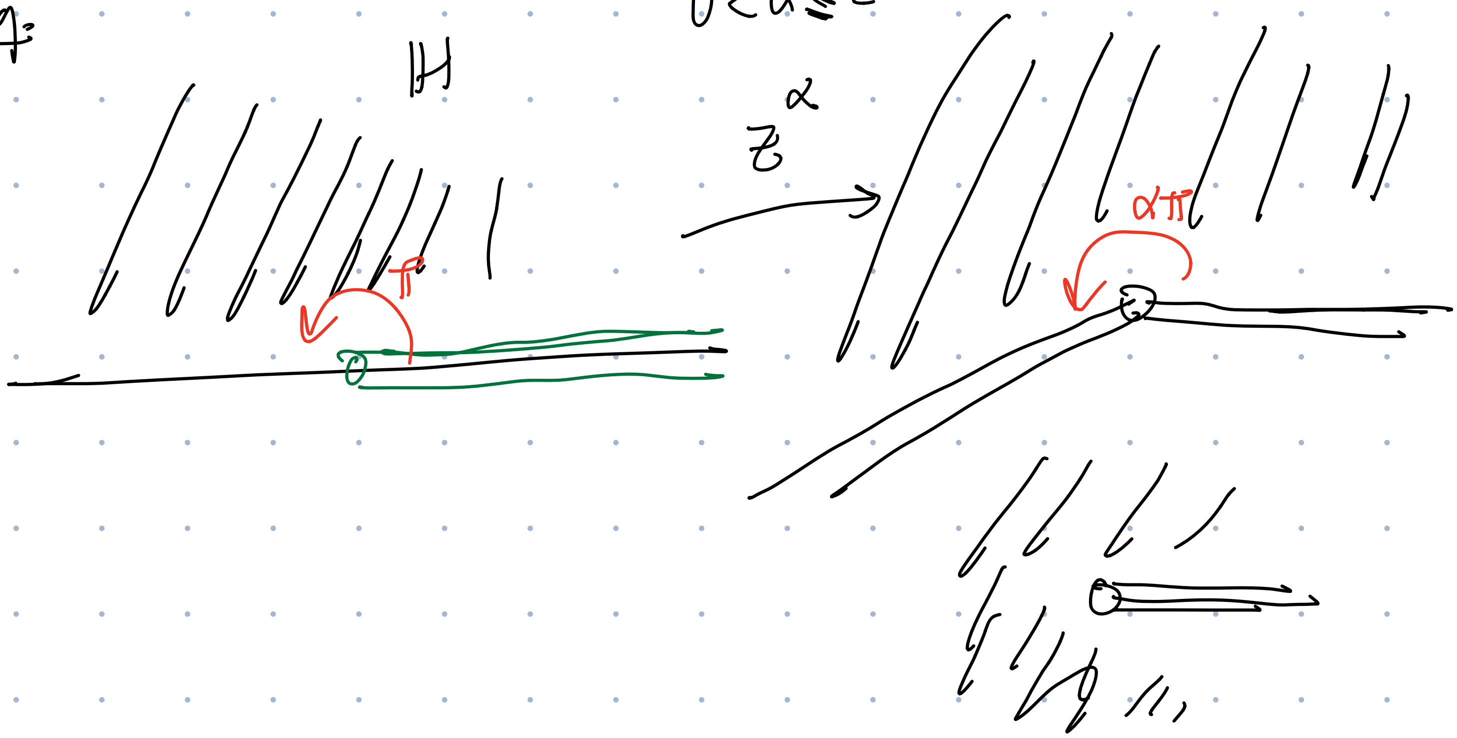


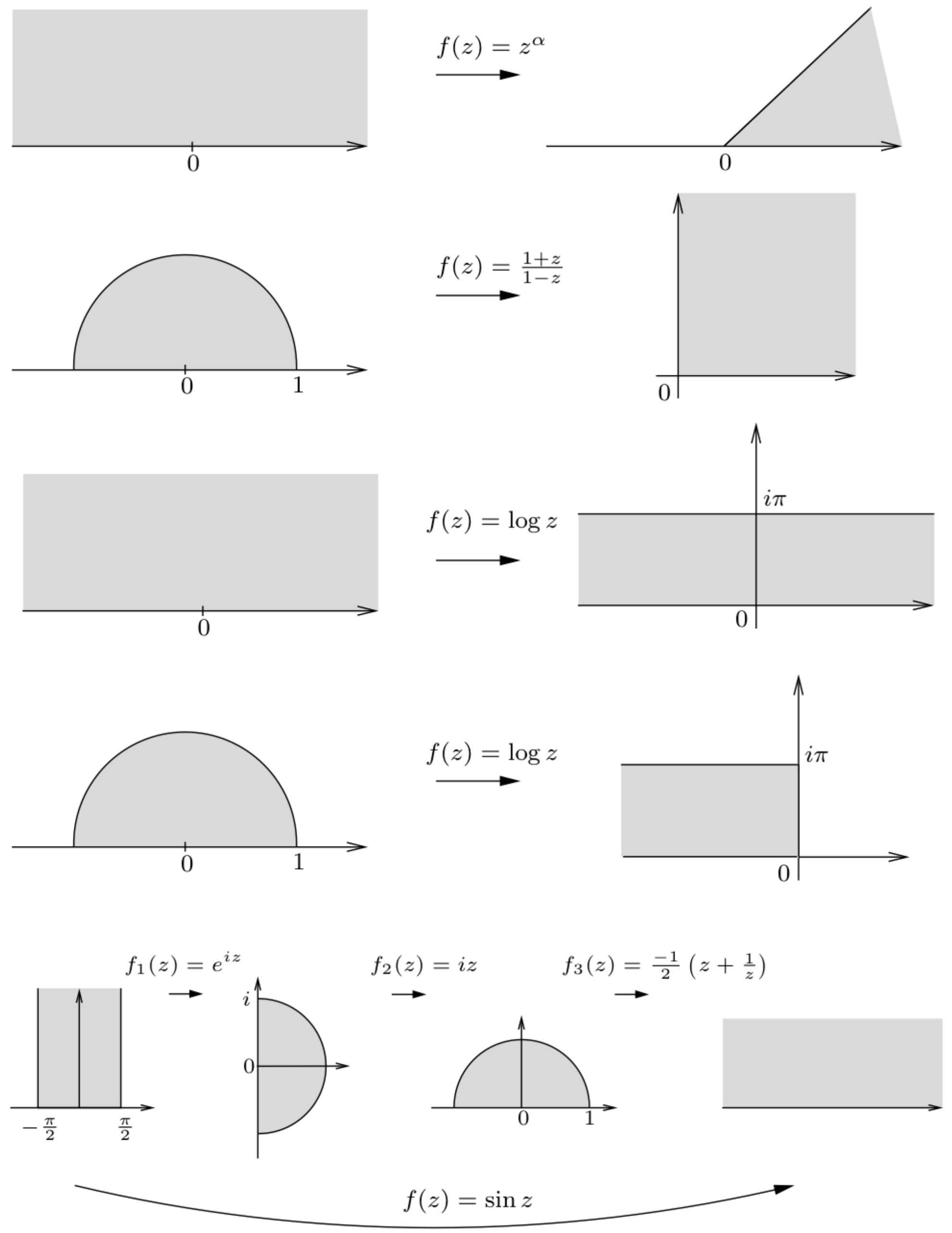
e.g. $c \in \mathbb{C}\setminus\{0\}$, $z \mapsto cz$



e.g.

$0 < \alpha \leq 2$





$f(z) = z^{\alpha}$
0
0
$f(z) = \frac{1+z}{1-z}$
0
1
0
$f(z) = \log z$
$i\pi$
0
0
$f(z) = \log z$
$i\pi$
0
1
0
$f_1(z) = e^{iz}$
$f_2(z) = iz$
$f_3(z) = \frac{-1}{2}\left(z + \frac{1}{z}\right)$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
i
0
0
1
$f(z) = \sin z$