# HOMEWORK 2
# MATH H54

**Yu-Wei's Office Hours**: Sunday 1-2:30pm and Thursday 12-1:30pm (PDT)
**Michael's Office Hours**: Monday 12-3pm (PDT)

## Part I (no need to turn in)

This part of the homework provides some routine computational exercises. You don't have to turn in your solutions for this part, but being able to do the computations is vitally important for the learning process, so you definitely should do these practices before you start doing Part II of the homework.

The following exercises are from the corresponding sections of the UC Berkeley custom edition of Lay, Nagle, Saff, Snider, *Linear Algebra and Differential Equations*.

- **Exercise 2.1**: 5, 11, 17, 18
- **Exercise 2.2**: 29, 31, 33
- **Exercise 2.3**: 7, 13, 33
- **Exercise 3.1**: 1, 11, 13, 15, 25, 30

## Part II (Due September 15, 8am PDT)

Some ground rules:

- You have to submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW2**.
- The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own.
- **For True/False questions**: You have to prove the statement if your answer is "True"; otherwise, you have to provide an explicit counterexample.
- You're allowed to use any result that is proved in the lecture. But if you'd like to use other results, you have to prove it first before using it.

Problems:

(1) Let $A$ be an $m \times n$ matrix. Prove that the induced linear transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is injective if and only if there exists an $n \times m$ matrix $B$ such that $BA = \mathbb{I}_n$.

(2) Let $A$ be an $m \times n$ matrix. Prove that the induced linear transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is surjective if and only if there exists an $n \times m$ matrix $B$ such that $AB = \mathbb{I}_m$.

(3) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ and $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be two linear transformations. Prove that the composition $S \circ T\colon \mathbb{R}^3 \to \mathbb{R}^3$ can not be invertible.

(4) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let $\vec{v} \in \mathbb{R}^n$ be a vector. Suppose that $T^{k-1}(\vec{v}) \neq \vec{0}$ and $T^k(\vec{v}) = \vec{0}$ for some positive integer $k$. Prove that $\{\vec{v}, T(\vec{v}), \dots, T^{k-1}(\vec{v})\}$ is a linearly independent set. ($T^\ell(\vec{v}) \coloneqq T \circ \cdots \circ T(\vec{v}) = T(\cdots T(T(\vec{v}))\cdots)$ denotes $\ell$ times composition of $T$.)

(5) Let $A$ be an $n \times n$ matrix. If $AB = BA$ for all invertible matrices $B$, prove that $A = c\mathbb{I}_n$ for some scalar $c \in \mathbb{R}$.

(6) Define the *trace* of a square matrix to be the sum of its diagonal entries. More precisely, suppose $A = [a_{i,j}]_{1 \le i,j \le p}$ is a $p \times p$ matrix, then its trace is

$$\operatorname{tr}(A) \coloneqq a_{1,1} + a_{2,2} + \cdots + a_{p,p}.$$

Let $A$ be an $m \times n$ matrix and $B$ be an $n \times m$ matrix. Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(7) True/False: Let $A, B, C$ be $n \times n$ matrices. Then $\operatorname{tr}(ABC) = \operatorname{tr}(ACB)$.

(8) True/False: There are no $n \times n$ matrices $A, B$ such that $AB - BA = \mathbb{I}_n$. (Hint: Make use of one of the previous problems.)

(9) Prove that for any positive integer $n$, there exists a $2 \times 2$ matrix $A \neq \mathbb{I}_2$ such that $A^n = \mathbb{I}_2$. (Hint: Consider rotations in $\mathbb{R}^2$.)

(10) Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Show that the sum and product of two matrices $aI + bJ$ and $cI + dJ$ are again of the same form (i.e. can be written as $\star I + \star J$). Also, show that the formulas for the sum and product match those for *complex numbers* $a+bi$ and $c+di$. (Remark: This is called a *matrix representation* of complex numbers.)