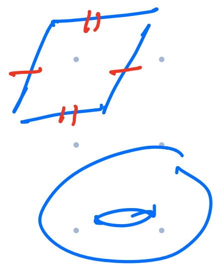
Today: Modular functions, modular forms.

last week: ell. fcns w.r.t. a lattice $\Lambda \subseteq \mathbb{C}$

$\Leftrightarrow$ fcns on the complex torus $\mathbb{C}/\Lambda$



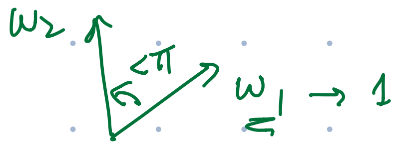
This week: fcns on the space of complex torus

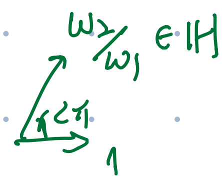
$\Updownarrow$

the space of lattices in $\mathbb{C}$ up to equivalence

$\Lambda_1 \sim \Lambda_2$ if $\exists\, c \in \mathbb{C}\setminus\{0\}$ s.t. $c\Lambda_1 = \Lambda_2$

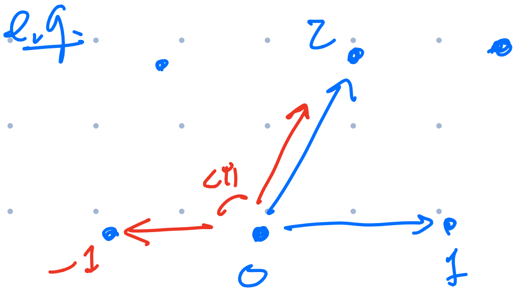
- any Λ is equiv. to $\Lambda_z = \{m + nz \mid m,n \in \mathbb{Z}\}$ for some $z \in \mathbb{H}$

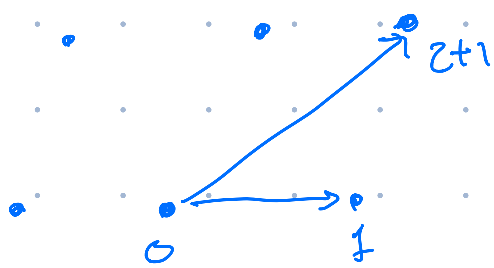




- sometimes $\Lambda_z \sim \Lambda_{z'}$ for some $z, z' \in \mathbb{H}$

e.g.



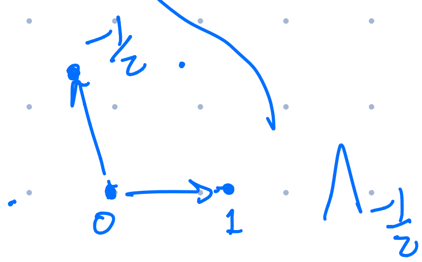


$\Lambda_z = \Lambda_{z+1}$

$\times \frac{1}{z}$

$z \mapsto \frac{az+b}{cz+d}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$



$\Lambda_{-\frac{1}{z}}$

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{H} \\ \tau &\longmapsto \tau+1 \end{aligned} \quad \text{--- given by } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \longmapsto \frac{-1}{\tau} \quad \text{--- given by } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

<u>Fact</u>: $SL_2(\mathbb{Z}) = \langle \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{= T}, \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{= S} \rangle.$

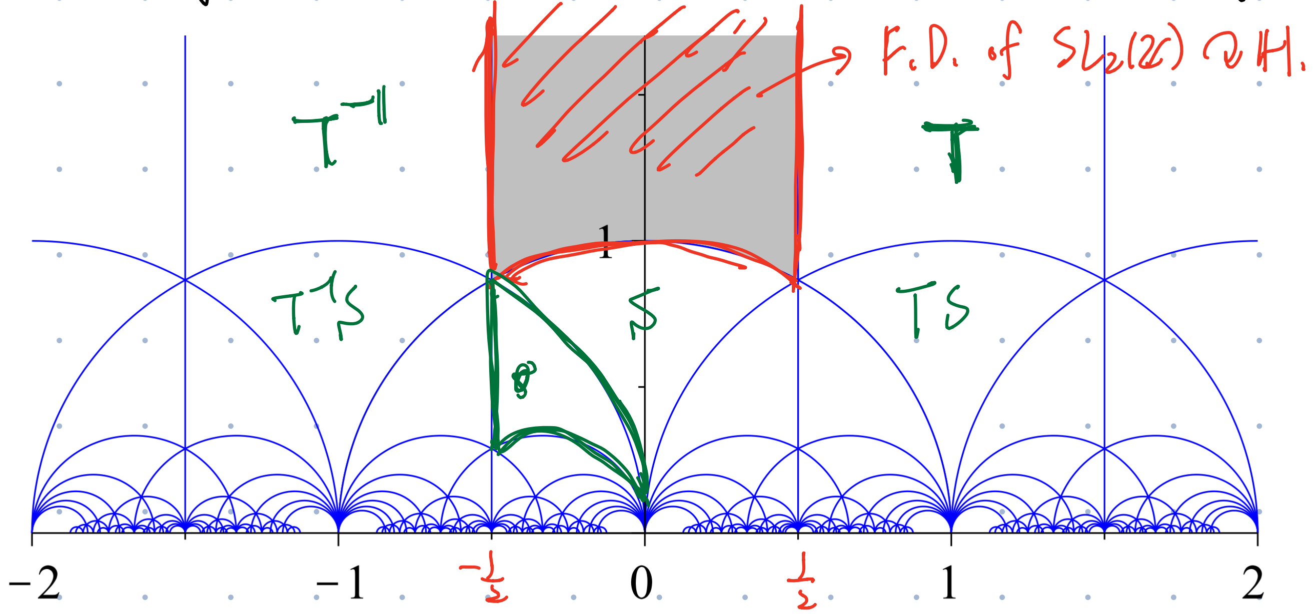
integral 2x2 matrices w/ det. = 1.

$$\boxed{\text{Fact: } \Lambda_\tau \sim \Lambda_{\tau'} \text{ if and only if } \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ s.t. } \tau' = \frac{a\tau+b}{c\tau+d}.}$$

<u>Def</u>: (modular function). $f: \mathbb{H} \longrightarrow \mathbb{C}$ is <u>modular</u> if

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau) \quad \forall \tau \in \mathbb{H}, \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

(i.e. f is invariant under the $SL_2(\mathbb{Z})$-action on $\mathbb{H}$).



Rmk: $f: \mathbb{H} \to \mathbb{C}$ is modular $\iff$ $f(z) = f(z+1) = f(\frac{-1}{z})$ $\forall z \in \mathbb{H}$.

Q: Find a nonconst. holo. modular fcn.?

A:
$$j(z) = \frac{\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right)^3}{q \prod_{n=1}^{\infty} (1-q^n)^{24}}, \text{ where } q = e^{2\pi i z}$$
$$\sigma_3(n) = \sum_{d \mid n} d^3$$

Note: This is the simplest modular fcn!!!

Def: $f: \mathbb{H} \to \mathbb{C}$ is a modular form of weight k

if $$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \forall z \in \mathbb{H},\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Rmk: If we can find 2 modular forms f_1, f_2 of the same weight k, then f_1/f_2 would be a modular fcn.

$$f_1\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f_1(z)$$
$$f_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f_2(z)$$
$$\Rightarrow \frac{f_1}{f_2}\left(\frac{az+b}{cz+d}\right) = \frac{f_1}{f_2}(z)$$

Rmk: $\boxed{f(z)\,dz}$ Ask: Is $f(z)dz$ invari. under $SL_2(\mathbb{Z})$-actn?

$\|$??

$$f\left(\frac{az+b}{cz+d}\right) d\left(\frac{az+b}{cz+d}\right) = f\left(\frac{az+b}{cz+d}\right) \frac{a(\cancel{cz}+d) - c(\cancel{az}+b)}{(cz+d)^2} dz = f\left(\frac{az+b}{cz+d}\right) \frac{dz}{(cz+d)^2}$$

$$\Longleftrightarrow \quad f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z)$$

$$\Longleftrightarrow \quad f(z) \text{ is a modular form of wt } 2.$$

$\boxed{f(z)(dz)^{k/2}}$ invar. under $SL_2\mathbb{Z}$ $\Longleftrightarrow$ $f(z)$ is modular form of wt k.

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e.g. $$\wp(z, \tau) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z-m-n\tau)^2} - \frac{1}{(m+n\tau)^2} \right).$$

$$\boxed{\wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 \wp(z,\tau)}$$

- $\wp(z,\tau) = \frac{1}{z^2} + a_2(\tau) z^2 + a_4(\tau) z^4 + \cdots\cdots$

  where $a_{2k}(\tau) = (2k+1) E_{2k+2}(\tau)$,

  where $E_{2k+2}(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k+2}}$

- $$\wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \frac{(c\tau+d)^2}{z^2} + a_2\left(\frac{a\tau+b}{c\tau+d}\right)\left(\frac{z}{c\tau+d}\right)^2 + a_4\left(\frac{a\tau+b}{c\tau+d}\right)\left(\frac{z}{c\tau+d}\right)^4 + \cdots$$

$$= (c\tau+d)^2 \left[\frac{1}{z^2} + \frac{a_2\left(\frac{a\tau+b}{c\tau+d}\right)}{(c\tau+d)^4} z^2 + \frac{a_4\left(\frac{a\tau+b}{c\tau+d}\right)}{(c\tau+d)^6} z^4 + \cdots \right]$$

$$\Rightarrow \quad a_2(\tau) \cdot (c\tau+d)^4 = a_2\left(\frac{a\tau+b}{c\tau+d}\right)$$

$$a_4(\tau)(c\tau+d)^6 = a_4\left(\frac{a\tau+b}{c\tau+d}\right).$$

$\vdots$

$\Rightarrow$ $a_2(\tau)$ is modular form of wt 4

$a_4(\tau)$ — — — — . 6

$\vdots$

$a_{2n}(\tau)$ $\vdots$ $2n+2$

$\Rightarrow$ $E_4(\tau)$ is modular form of wt 4

$\vdots$

$$\boxed{E_{2n}(\tau) \text{ is modular form of wt } 2n \quad \forall n \geq 2}$$

<u>Rmk</u>: Modular forms are very rare, in fact, for each k, the space of modular form of wt k is <u>finite dim'l</u>.

Moreover, all modular forms are generated by <u>E_4</u>, <u>E_6</u>

i.e. any modular form of wt k can be written as:

$$* \underbrace{E_4^{*} E_6^{*}} + * E_4^{*} E_6^{*} + \cdots$$

Theorem. *The zeros of* $\wp(z,\tau)$ $(\tau \in \mathfrak{H},\ z \in \mathbb{C})$ *are given by*

$$z = m + \frac{1}{2} + n\tau \pm \left(\frac{\log(5+2\sqrt{6})}{2\pi i} + 144\pi i \sqrt{6} \int_{\tau}^{i\infty} (t-\tau) \boxed{\frac{\Delta(t)}{E_6(t)^{3/2}}} dt \right)$$

$(m, n \in \mathbb{Z})$, *where* $E_6(t)$ *and* $\Delta(t)$ $(t \in \mathfrak{H})$ *denote the normalized Eisenstein series of weight* 6 *and unique normalized cusp form of weight* 12 *on* $\mathrm{SL}_2(\mathbb{Z})$, *respectively, and the integral is to be taken over the vertical line* $t = \tau + i\mathbb{R}_+$ *in* $\mathfrak{H}$.

Idea: $\tau \in \mathbb{H} \rightsquigarrow \Lambda_\tau \rightsquigarrow \wp(z,\tau) \rightsquigarrow$ zeros of $\wp(z,\tau)$??

$z_0(\tau)$ - a zero of $\wp(z,\tau)$

$\downarrow$

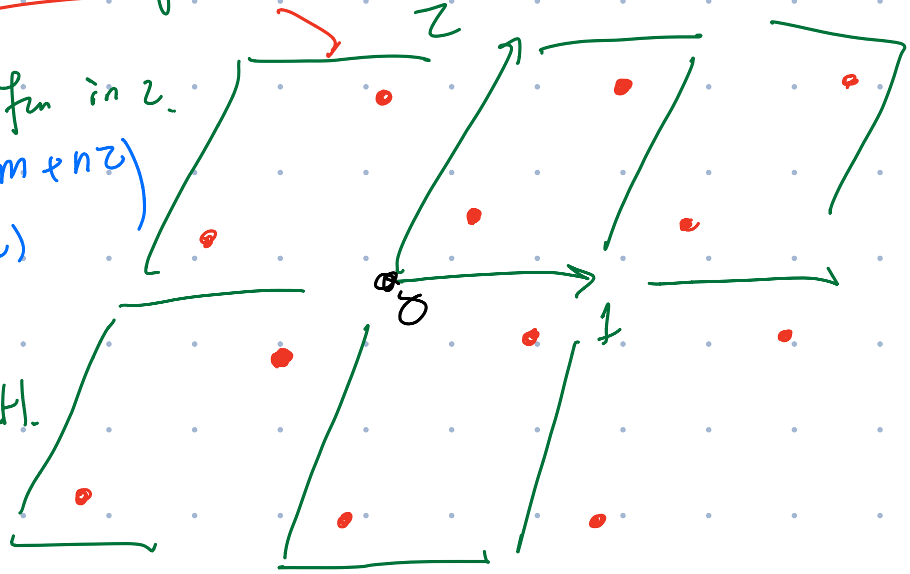
not a single-valued fun in τ.

$\begin{pmatrix} z_0(\tau)+m+n\tau \\ -z_0(\tau) \end{pmatrix}$

But $\boxed{z_0''(\tau)^2}$

$\downarrow$

well-defined fun in $\tau \in \mathbb{H}$.



$$\boxed{\wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 \, \wp(z,\tau)}$$

So, if $z_0(\tau)$ is a zero of $\wp(z,\tau)$,

then $\boxed{\frac{z_0(\tau)}{c\tau+d}}$ is a zero of $\wp\left(z, \frac{a\tau+b}{c\tau+d}\right)$.

$$\| \ \pm z_0\left(\frac{a\tau+b}{c\tau+d}\right) + \Lambda$$

$$\Longrightarrow \quad \boxed{z_0''\left(\frac{a\tau+b}{c\tau+d}\right)^2 = (c\tau+d)^6 \, z_0''(\tau)^2}$$

- ~~But~~ $(z_0'')^2$ is not holo., But $\underbrace{E_6(\tau)^3 \, z_0''(\tau)^2}_{\text{modular form of 24}}$ holo.
- use classific^{n} of modular form of wt 24,

$$\Rightarrow \quad E_6(z)^3\, z_0''(z)^2 = \Delta(z)^2$$

$$\Rightarrow \quad z_0''(z)^2 = \frac{\Delta(z)^2}{E_6(z)^3}$$

$$\boxed{f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)} \quad \forall z \in \mathbb{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$$

$$= \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle$$

$$f\left(\frac{z+1}{0z+1}\right) = (0z+1)^k f(z)$$

Plug in $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2\mathbb{Z}$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$f\Big(\underbrace{\frac{-z+0}{0z-1}}_{=z}\Big) = \underbrace{(0z-1)^k}_{=(-1)^k} f(z)$$

$$f\left(\frac{0z-1}{z+0}\right) = (1z+0)^k f(z)$$

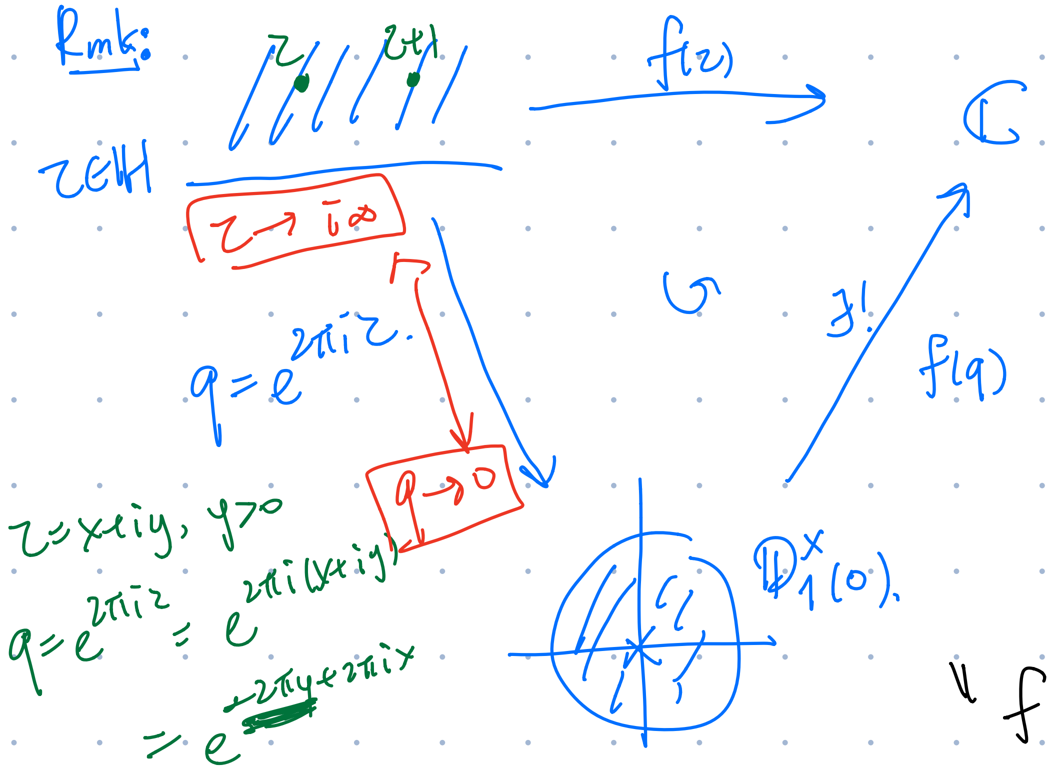
$$\Rightarrow \quad f(z) = (-1)^k f(z) \quad \forall z.$$

k odd $\Rightarrow$ modular form of wt k $\equiv 0$.

Def (precise version) $k \in \mathbb{Z}$, $f : \mathbb{H} \to \mathbb{C}$ is a <u>modular form of wt k</u> if:

- f is holo. in $\mathbb{H}$.
- $f(z) = f(z+1)$, $f\left(\frac{-1}{z}\right) = z^k f(z)$ $\forall z \in \mathbb{H}$.
- $\boxed{f(z) \text{ is bounded as } \operatorname{Im} z \to +\infty.}$

Rmk:



$f(q)$ is bounded near $q=0$

$\Updownarrow$

$f(q)$ has removable sing. at $q=0$

$\Updownarrow$

"f is holo. in $\mathbb{H} \cup \{i\infty\}$"

Ex: $E_{2k}(\tau) := \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}} : \mathbb{H} \longrightarrow \mathbb{C}$

is a modular form of wt $2k$, if $k \geq 2$.