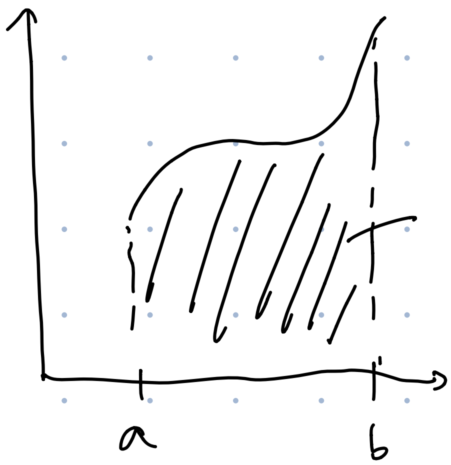
Today: Integrable functions.

Integration:

Setting: $f: [a,b] \longrightarrow \mathbb{R}$ bounded.

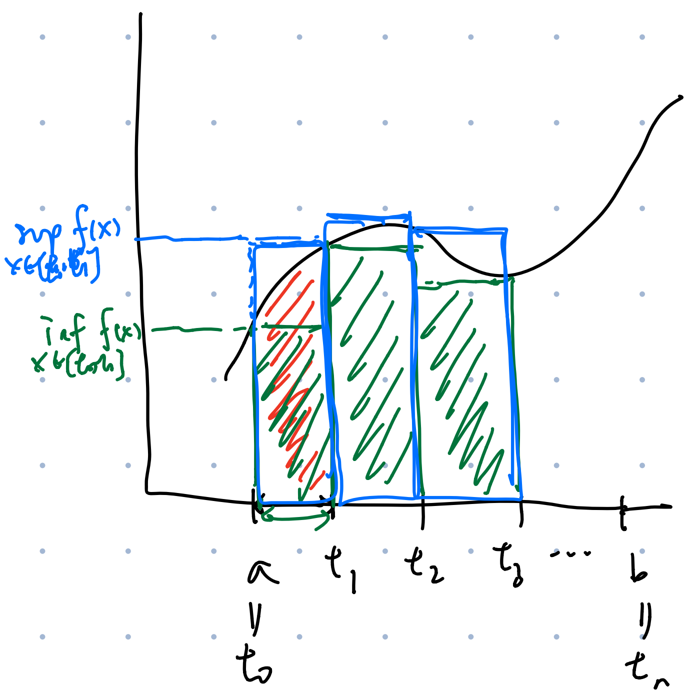
Q: Define "integration" of f. (if f is "integrable")



Area = " $\int_a^b f(x)\,dx$ "

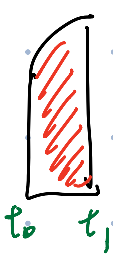
Def: A partition of $[a,b]$ is a set of numbers.

$$P = \{ a = t_0 < t_1 < t_2 < \cdots < t_n = b \}.$$



$$\left(\inf_{x \in [t_0, t_1]} f(x) \right) \cdot (t_1 - t_0)$$

$$\wedge\!|$$

Area of 

$$\wedge\!|$$

$$\left(\sup_{x \in [t_0, t_1]} f(x) \right) \cdot (t_1 - t_0)$$

Def: Lower sum (upper) of f with respect to (w.r.t.) P:

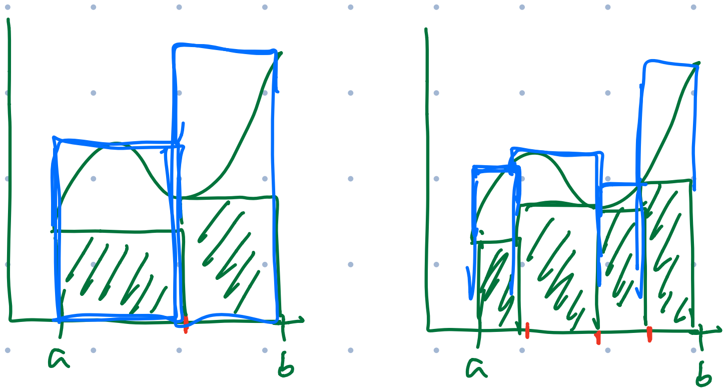
$$L(f,P) := \sum_{k=1}^{n} (t_k - t_{k-1}) \cdot \inf_{x \in [t_{k-1}, t_k]} f(x)$$

$(U(f,P))$ (sup)

Idea: If "$\int_a^b f(x)\,dx$" is defined (i.e. f is integrable), then we should have

$$\text{``}\int_a^b f(x)\,dx\text{''} \geq L(f,P) \quad \forall P$$

$$\text{``}\int_a^b f(x)\,dx\text{''} \leq U(f,P) \quad \forall P.$$



Idea: "Finer partition $\rightsquigarrow$ $L(f,P) \nearrow$ & $U(f,P) \searrow$

If they approach the same value, then f is integrable"

Def Lower integral of f: (upper)

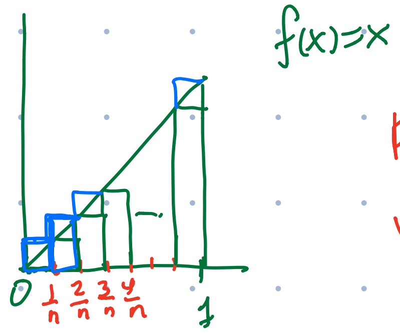
$$L(f) := \sup \{ L(f,P) \mid P \text{ partition of } [a,b] \}$$

$$U(f) := \inf \{ U(f,P) \mid P \text{ partition of } [a,b] \}$$

Rmk: We'll show that $\boxed{L(f) \leq U(f)}$.

Def: Say f is <u>integrable</u> on $[a,b]$ if $L(f) = U(f)$.
In this case, the integral $\int_a^b f(x)\,dx := L(f) = U(f)$.

<u>e.g.</u>



For the partition $P_n = \{0 < \frac{1}{n} < \frac{2}{n} < \cdots < 1\}$.

we have:

$$L(f, P_n) = \left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n}\right)\frac{1}{n}$$

$$= \frac{1}{n} \cdot \frac{n(n-1)}{2} \cdot \frac{1}{n}$$

$$= \frac{n-1}{2n}$$

$$U(f, P_n) = \left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}\right) \cdot \frac{1}{n}$$

$$= \frac{1}{n} \cdot \frac{n(n+1)}{2} \cdot \frac{1}{n}$$

$$= \frac{n+1}{2n}.$$

Def: Say Q is a <u>refinement</u> of P (we'll denote this by $P \subseteq Q$)

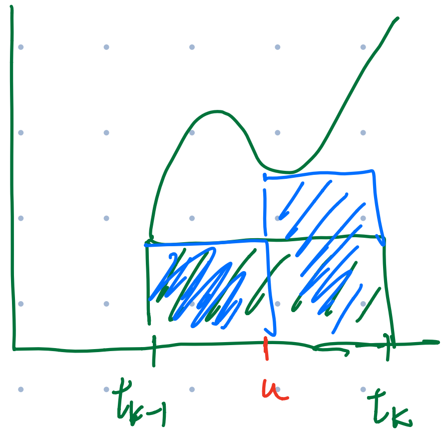
$Q = \{a = s_0 < s_1 < \cdots < s_m = b\}$ $\qquad$ $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$

If $\{t_0, \ldots, t_n\} \subseteq \{s_0, \ldots, s_m\}$.

Lemma: If $P \subseteq Q$, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Idea:



$\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$

$\mathcal{Q} = \cdots < \underline{t_{k-1}} < u < \underline{t_k} < \cdots$

$$\inf_{x\in[t_{k-1},t_k]} f(x) \le \inf_{x\in[t_{k-1},u]} f(x)$$

$$\inf_{x\in[t_{k-1},t_k]} f(x) \le \inf_{x\in[u,t_k]} f(x)$$

$$\cdots + \boxed{\overbrace{(t_k - t_{k-1}) \inf_{x\in[t_{k-1},t_k]} f(x)}^{\mathcal{P}}} + \cdots$$

$$\le$$

$$\cdots + \boxed{\overbrace{(u - t_{k-1}) \inf_{x\in[t_{k-1},u]} f(x) + (t_k - u) \inf_{x\in[u,t_k]} f(x)}^{\mathcal{Q}}} + \cdots$$

$\square$

Prop: $L(f) \le U(f)$:

$$L(f) = \sup_{\mathcal{P}} \{L(f,\mathcal{P})\}, \qquad U(f) = \inf_{\mathcal{P}} \{U(f,\mathcal{P})\}$$

pf: • Claim: $L(f,\mathcal{P}) \le U(f,\mathcal{Q})$ $\forall\, \mathcal{P},\mathcal{Q}$ partitions of $[a,b]$

pf Consider the common refinement $\underline{\mathcal{P}\cup\mathcal{Q}}$ of $\mathcal{P}$ and $\mathcal{Q}$.

$$L(f,\mathcal{P}) \le L(f, \underline{\mathcal{P}\cup\mathcal{Q}})$$
$$\le U(f, \underline{\mathcal{P}\cup\mathcal{Q}}) \le U(f,\mathcal{Q}) \quad \square$$

• Since $L(f,P) \leq U(f,Q) \quad \forall P, Q.$

$$\Rightarrow \quad L(f) = \sup_P L(f,P) \leq U(f,Q) \quad \forall Q.$$

$$\Rightarrow \quad L(f) \leq U(f) = \inf_Q U(f,Q). \quad \square$$

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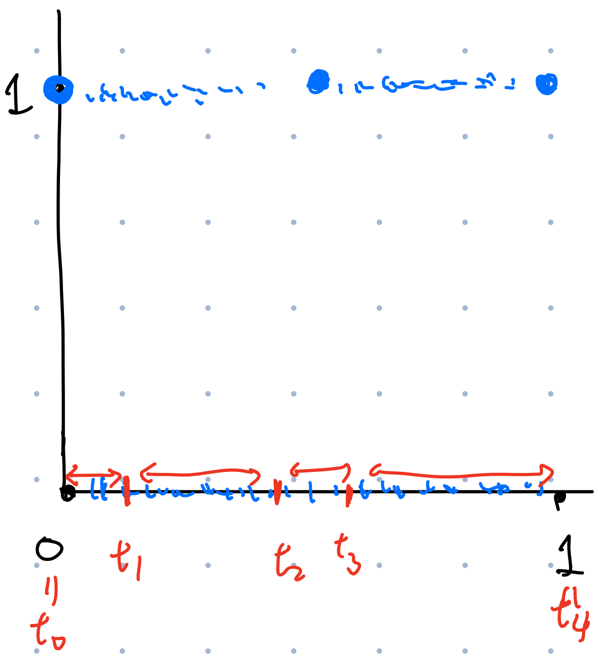
e.g. $f : [0,1] \longrightarrow \mathbb{R}$

$$x \longmapsto \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$



For any partition $P = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ we know

$$\inf_{x \in [t_{k-1}, t_k]} f(x) = 0$$

$$\Rightarrow \quad L(f,P) = 0 \quad \forall P$$

$$\left(\Rightarrow L(f) = \sup_P L(f,P) = 0 \right).$$

$$\sup_{x \in [t_{k-1}, t_k]} f(x) = 1$$

$$\Rightarrow \quad U(f,P) = \sum (t_k - t_{k-1}) \cdot \overbrace{\sup_{x \in [t_{k-1}, t_k]} f(x)}^{1}$$

$$= \sum (t_k - t_{k-1})$$

$$= 1 \quad \forall P$$

$$\left(\Rightarrow U(f) = \inf_P U(f,P) = 1 \right)$$

$$L(f) \neq U(f)$$

$$\Downarrow$$

f is NOT integrable.

Riemann-Lebesgue thm.: $f: [a,b] \longrightarrow \mathbb{R}$ bounded.

f is integrable if and only if

the set $\{ x \in [a,b] \mid f \text{ is not continuous at } x \}$

has "measure zero".

Thm A bounded fcn $f: [a,b] \longrightarrow \mathbb{R}$ is integrable

if and only if

"$\forall \varepsilon > 0$, $\exists$ partition P of $[a,b]$

s.t. $U(f,P) - L(f,P) < \varepsilon$."

pf: $(\Rightarrow)$

$$\underbrace{L(f)}_{\sup_P L(f,P)} = \underbrace{U(f)}_{\inf_P U(f,P)}$$

$\forall \varepsilon > 0$,

- $\exists P_1$ s.t. $0 \le L(f) - L(f, P_1) < \frac{\varepsilon}{2}$
- $\exists P_2$ s.t. $0 \le U(f, P_2) - U(f) < \frac{\varepsilon}{2}$.

$P = P_1 \cup P_2$.

Then
$$\begin{aligned}
& U(f,P) - L(f,P) \\
&\le U(f,P_2) - L(f,P_1) \\
&< \left(U(f) + \tfrac{\varepsilon}{2}\right) + \left(-L(f) + \tfrac{\varepsilon}{2}\right) \\
&= U(f) - L(f) + \varepsilon = \varepsilon. \quad \square
\end{aligned}$$

($\Leftarrow$) $\forall \varepsilon>0$, $\exists P$ s.t. $U(f,P)-L(f,P)<\varepsilon$.

$$\begin{aligned} L(f) \le U(f) &= \inf U(f,P) \\ &\le U(f,P) \\ &< L(f,P)+\varepsilon \le L(f)+\varepsilon. \end{aligned}$$

$\Rightarrow$ $L(f) \le U(f) < L(f)+\varepsilon \quad \forall \varepsilon>0$.

$\Rightarrow$ $L(f)=U(f)$, so f is integrable. $\square$

Thm Any continuous function $f:[a,b]\to\mathbb{R}$ is integrable.

<u>Idea</u>: $U(f,P)-L(f,P) = \sum (t_k - t_{k-1})\cdot \underbrace{\left(\sup_{x\in[t_{k-1},t_k]} f(x) - \inf_{x\in[t_{k-1},t_k]} f(x)\right)}_{<\frac{\varepsilon}{b-a}}$

If we can choose a partition s.t.

Then $U(f,P)-L(f,P) < \frac{\varepsilon}{b-a}\cdot \underbrace{\sum(t_k-t_{k-1})}_{b-a} = \varepsilon$

we can find such partition b/c f is unif conti. on $[a,b]$

pf: • f is conti. on the cpt set $[a,b]$ $\Rightarrow$ unif. conti. on $[a,b]$

• $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{b-a}$

- Therefore, if we choose P s.t. each subintervals of the partition P has length $< \delta$,

then $\sup_{x \in [t_{k-1}, t_k]} f(x) - \inf_{x \in [t_{k-1}, t_k]} f(x) < \frac{\varepsilon}{b-a}$.

$$\Rightarrow \quad U(f,P) - L(f,P)$$

$$= \sum (t_k - t_{k-1}) \left(\sup_{x \in [t_{k-1}, t_k]} f(x) - \inf_{x \in [t_{k-1}, t_k]} f(x) \right)$$

$$< \frac{\varepsilon}{b-a} \underbrace{\sum (t_k - t_{k-1})}_{= b-a}$$

$$= \varepsilon.$$

$\square$