HOMEWORK 2
MATH 104, SECTION 6

Office Hours: I'll be out of town for a conference next week. So I'll be holding office hours on Monday (1/27) 8-11am, and my office hours on Tuesday (1/28) and Wednesday (1/29) are cancelled.

Reading

There will be reading assigned for each lecture. You should come to the class having read the assigned sections of the textbook.

Due January 30: Ross, Section 8

Due February 4: Ross, Section 9

Problem set (10 problems; due January 30)

Submit your homework at the beginning of the lecture on Thursday. *Late homework will not be accepted under any circumstances.*

You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any.

Write clearly! Mastering mathematical writing is one of the goals of this course.

You have to staple your work if it is more than one page.

(1) Prove that $\sqrt{2}$ is not a rational number.

(2) Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$. (Read Ross, Section 1 if you're not familiar with mathematical induction.)

(3) For any ordered field F and any $a \in F$, one can define the notion of absolute value $|a| \in F$ of a (c.f. Ross, Definition 3.3). Prove that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

holds for any $a_1, a_2, \ldots, a_n \in F$.

(4) Formulate the definition of the greatest lower bound $\inf A$ of a set of real numbers. State a "greatest lower bound property" for $\mathbb{R}$ and show that it is equivalent to the least upper bound property of $\mathbb{R}$. (c.f. Ross, Section 4)

(5) Let $S \subset \mathbb{R}$ be a nonempty subset which is bounded above, and let $z = \sup S$. Prove that for any $\epsilon > 0$, there exists $a \in S$ such that $z - \epsilon < a \leq z$. Can $a \in S$ always be found so that $z - \epsilon < a < z$?

(6) Let $x, y \in \mathbb{R}$. Suppose that $x < y + \epsilon$ for any $\epsilon > 0$. Prove that $x \leq y$.

(7) Prove that $\sup\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 1$.

(8) Let

$$A = \{m + n\sqrt{2} | m, n \in \mathbb{Z} \text{ and } m + n\sqrt{2} > 0\}.$$

Prove that $\inf A = 0$.

(9) Define $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Prove that the sequence (a_n) diverges.

(10) (a) Suppose that (a_n) is a convergent sequence. Show that the subsequences $(a_{2n}) = (a_2, a_4, a_6, \cdots)$ and $(a_{2n-1}) = (a_1, a_3, a_5, \cdots)$ both converge.

(b) Let (a_n) be a sequence such that both (a_{2n}) and (a_{2n-1}) converge. Is it guaranteed that (a_n) converges?

(c) Let (a_n) be a sequence such that (a_{2n}), (a_{2n-1}), and (a_{3n}) converge. Prove that (a_n) is a convergent sequence.