Today: finish the proof of Chain Rule, Taylor Series

Chain rule: $f: I \to \mathbb{R}$, $\quad g: \underset{\displaystyle f(I)\subset}{J} \to \mathbb{R}$

Suppose $\boxed{f \text{ is differentiable at } a \in I,}$

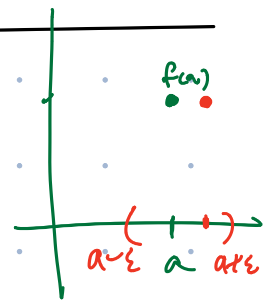
and g is —— $f(a) \in J$.

Then $g \circ f : I \to \mathbb{R}$ is diff at $a \in I$,

and $(g\circ f)'(a) = g'(f(a)) \cdot f'(a)$

The case that we haven't proved yet:

$$\text{"}\boxed{\forall \varepsilon > 0,\ \exists\ 0<|x-a|<\varepsilon \ \text{ s.t. } f(x)=f(a).}\text{"}$$



Q: What does this tell us about $\boxed{f'(a)}$?

$\exists (x_n)$, $x_n \neq a$, $\lim x_n = a$, $f(x_n) = f(a)$. $\quad \overset{\shortparallel}{f'(a)} = \lim\limits_{n\to\infty} \dfrac{\overset{0\ \forall n}{f(x_n)-f(a)}}{x_n - a} = 0$

It remains to prove the Chain Rule in the case where $\boxed{f'(a)=0.}$

Need to show: If $f'(a)=0$, then $(g\circ f)'(a) = 0$

i.e. $\lim\limits_{x\to a} \dfrac{g(f(x)) - g(f(a))}{x-a} = 0.$

Claim: ~~If $f'(a)=0$,~~ ~~then~~ $\exists\ C>0, \delta>0$ s.t.

$$\left|\frac{g(f(x))-g(f(a))}{x-a}\right| \le C \cdot \left|\frac{f(x)-f(a)}{x-a}\right| \quad \forall\ 0<|x-a|<\delta.$$

(Claim $\Rightarrow$ If $f'(a)=0$ then $(g\circ f)'(a)=0$).

pf of Claim:

- g is diff'ble at $f(a)$, so $\forall \underset{1}{\varepsilon}>0$, $\exists \eta>0$ s.t.

$$0<|y-f(a)|<\eta \Rightarrow \left|\frac{g(y)-g(f(a))}{y-f(a)}-g'(f(a))\right|<\varepsilon.$$

$$\Rightarrow \left|\frac{g(y)-g(f(a))}{y-f(a)}\right|<\underbrace{|g'(f(a))|+\underset{1}{\varepsilon}}_{=C}$$

- f is diff'ble at a, so it's conti. at a.

so, $\exists \delta>0$ s.t.

$$|x-a|<\delta \Rightarrow |f(x)-f(a)|<\eta.$$

- Let's prove that $\left|\frac{g(f(x))-g(f(a))}{x-a}\right|\leq C\left|\frac{f(x)-f(a)}{x-a}\right|$ $\forall 0<|x-a|<\delta$

① If $f(x)\neq f(a)$, then $\left|\frac{g(f(x))-g(f(a))}{f(x)-f(a)}\right|<C$

$$\Rightarrow \left|\frac{g(f(x))-g(f(a))}{x-a}\right|<C\left|\frac{f(x)-f(a)}{x-a}\right|.$$

② if $f(x)=f(a)$. Then LHS=RHS=0. $\square$

§ Taylor series

Def: $f: I \to \mathbb{R}$, $c \in I$. Suppose f has derivative of all orders at c.

i.e.
- $f'(c)$ exists,
  and $f'(x)$ is well-defined in some open neighborhood of c
- $f''(c)$ exists,
  and $f''(x)$
- $\vdots$

The Taylor series for f about c:

$$\boxed{\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k}$$

→ k-th derivative of f at c.

Rmk: 1st partial sum: $\frac{f^{(0)}(c)}{0!}(x-c)^0 = f(c)$

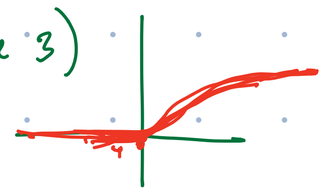
2nd partial sum: $f(c) + f'(c)(x-c)$ → gives the linear approx. of f at c.

$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2$ → gives the quadratic approx. of f at c.

$\vdots$

Q:
- For which $x \in \mathbb{R}$ does the series converge?
- When $\sum \frac{f^{(k)}(c)}{k!}(x-c)^k$ conv., is it same as $f(x)$?

(No. in general, see Textbook. p. 257, example 3)



Def:

$$R_n(x) := f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Then $\lim_{n\to\infty} R_n(x) = 0$ for some $x \in \mathbb{R}$,

$$\Updownarrow$$

The Taylor series of f about c $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ converges at x,

and $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(x)$.

Taylor's thm: Suppose f has derivatives of all orders on $I \ni c$,

$\forall n \in \mathbb{N}$ $\forall x_0 \in I$, $x_0 \neq c$, $\exists$ y between x_0 and c s.t. $y \leftarrow$ depends on n.

$$R_n(x_0) = \frac{f^{(n)}(y)}{n!}(x_0-c)^n$$

i.e. $f(x_0) = f(c) + \frac{f'(c)}{1!}(x_0-c) + \cdots + \frac{f^{(n-1)}(c)}{(n-1)!}(x_0-c)^{n-1} + \frac{f^{(n)}(y)}{n!}(x_0-c)^n$.

Rmk: If we take $n=1$, $\exists$ y_1 between x_0 and c,

s.t. $f(x_0) = f(c) + f'(y_1)(x_0-c)$. (MVT)

If we take $n=2$, $\exists$ y_2 between x_0 and c,

s.t. $f(x_0) = f(c) + f'(c)(x_0-c) + \frac{f''(y_2)}{2}(x_0-c)^2$

---

pf: Let M be the unique real number s.t.

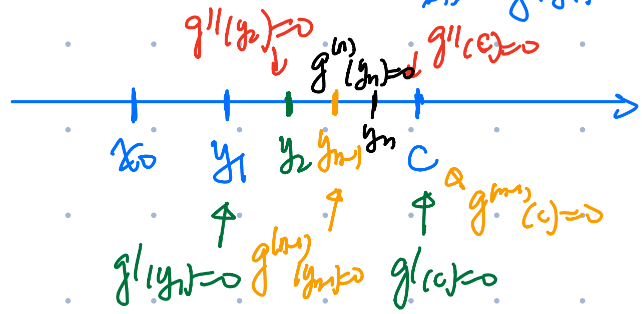
$$f(x_0) = f(c) + \frac{f'(c)}{1!}(x_0-c) + \cdots + \frac{f^{(n-1)}(c)}{(n-1)!}(x_0-c)^{n-1} + \frac{M}{n!}(x_0-c)^n$$

Goal: Prove $\exists\, y$ btw x_0 & c s.t. $f^{(n)}(y) = M$.

$$g(x) = f(x) - \Big(\underbrace{f(c) + \frac{f'(c)}{1!}(x-c) + \cdots + \frac{f^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1} + \frac{M}{n!}(x-c)^n}_{\text{deg } n \text{ poly}} \Big)$$

- $g(x_0) = 0$
- $g(c) = 0$
- $g'(c) = 0 = \cdots = g^{(n-1)}(c)$
- $g^{(n)}(x) = f^{(n)}(x) - M$.

By Rolle's thm $\Rightarrow$ $\exists\, y_1$ btw x_0 & c s.t. $g'(y_1) = 0$



$\Rightarrow$ Goal $\Leftrightarrow$ " $\exists\, y$ btw x_0 & c s.t. $g^{(n)}(y) = 0$ " $\square$

Coro: $f : I \to \mathbb{R}$ has derivative of all orders. ($I \hat{C}$)

Suppose $\exists D > 0$, $|f^{(n)}(x)| < D \quad \forall n, \ \forall x \in I$,

Then $\lim_{n \to \infty} R_n(x) = 0 \quad \forall x \in I$.

pf: $\forall x \in I$, $\forall n$ $\exists\, y_n$ btw x & c s.t.

$$|R_n(x)| = \left| \frac{f^{(n)}(y_n)}{n!}(x-c)^n \right| < \frac{D}{n!} |x-c|^n$$

$$\lim_{n \to \infty} \frac{A^n}{n!} = 0 \Rightarrow \lim_{n \to \infty} R_n(x) = 0 \quad \forall x \in I. \quad \square$$

e.g. $f(x) = e^x$, $f^{(n)}(x) = e^x \quad \forall n$.

$\forall R > 0$.

Claim: $\boxed{\sum_{k=0}^{\infty} \frac{x^k}{k!}}$ conv. unif. to $f(x)$ on $[-R, R]$

Taylor exp. of f about 0

$x \in [-R, R]$,

$$|R_n(x)| = \left| \frac{f^{(n)}(y_n)}{n!} x^n \right| = \left| \frac{e^{y_n}}{n!} x^n \right| \leq \frac{e^R \cdot R^n}{n!} \to 0$$

unif.
(indep. of $x \in [-R, R]$)

$$\Rightarrow \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \quad \forall x \in \mathbb{R}.$$