**HOMEWORK 10**
**MATH H54**

**Yu-Wei's Office Hours**: Sunday 1-2:30pm and Friday 12-1:30pm (PST)
**Michael's Office Hours**: Monday 12-3pm (PST)

Some ground rules:

- You have to submit your solutions via **Gradescope**, to the assignment **HW10**.
- The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own.
- You're allowed to use any result that is proved in the lecture. But if you'd like to use other results, you have to prove it first before using it.

**Problems:** (mostly taken from the textbook)
*You have to write down your computations, not just the final answers.*

In the following, boldface $\mathbf{x}$ or $\mathbf{x}(t)$ denotes a vector-valued function (we use $\vec{x}(t)$ in the lecture). Here we use a different notation because the derivative $\vec{x}'(t)$ may be hard to read, so we'll denote the derivative by $\mathbf{x}'(t)$ instead.

(1) Let $\mathbf{X}(t)$ be a fundamental matrix for the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Show that $\mathbf{x}(t) = \mathbf{X}(t)(\mathbf{X}(t_0))^{-1}\mathbf{x}_0$ is the solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(t_0) = \mathbf{x}_0$.

(2) Find a fundamental matrix for the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ for the given matrix $A$:

(a) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix}$.

(3) Solve the following initial value problems:

(a) $\mathbf{x}'(t) = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} \mathbf{x}(t); \ \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

(b) $\mathbf{x}'(t) = \begin{bmatrix} -2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \mathbf{x}(t); \ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(4) Find the general solutions of the system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 8\sin t \\ 0 \end{bmatrix}.$$

(5) Consider the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Show that $A$ has a unique eigenvalue $2$ with multiplicity $3$, and the corresponding eigenspace is of $1$-dimensional. Find a vector $\mathbf{u}_1$ such that $\mathbf{x}_1(t) = e^{2t}\mathbf{u}_1$ is a solution to the system.

(b) To obtain a second linearly independent solution, try $\mathbf{x}_2(t) = te^{2t}\mathbf{u}_1 + e^{2t}\mathbf{u}_2$. Find a vector $\mathbf{u}_2$ such that $\mathbf{x}_2(t)$ is a solution to the system.

(c) To obtain a third linearly independent solution, try $\mathbf{x}_3(t) = \frac{t^2}{2}e^{2t}\mathbf{u}_1 + te^{2t}\mathbf{u}_2 + e^{2t}\mathbf{u}_3$. Find a vector $\mathbf{u}_3$ such that $\mathbf{x}_3(t)$ is a solution to the system.

(6) Let $A = \begin{bmatrix} 5 & 2 & -4 \\ 0 & 3 & 0 \\ 4 & -5 & -5 \end{bmatrix}$.

(a) Find a fundamental matrix to the system $\mathbf{x}' = A\mathbf{x}$.

(b) Determine which initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ yield a solution $\mathbf{x}(t)$ that remains bounded for all $t \geq 0$, i.e. satisfies $||\mathbf{x}(t)|| \leq M$ for some constant $M$ and all $t \geq 0$.