Change of basis: V, $\dim V = n$

$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$, $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ bases of V

$$\begin{tikzcd} & \mathbb{R}^n \arrow[dd, bend right] \\ V \arrow[ur, "{[\ ]_{\mathcal{B}}}", "\cong"'] \arrow[dr, "{[\ ]_{\mathcal{C}}}"', "\cong"] & \\ & \mathbb{R}^n \end{tikzcd}$$

We'd like to understand the map

$$\mathbb{R}^n \xrightarrow{[\ ]_{\mathcal{B}}^{-1}} V \xrightarrow{[\ ]_{\mathcal{C}}} \mathbb{R}^n.$$

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{e}_1 \mapsto \vec{b}_1 \mapsto ??\,[\vec{b}_1]_{\mathcal{C}}$$

This is a linear map $\mathbb{R}^n \to \mathbb{R}^n$, so there is a unique associated matrix

$$\begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{bmatrix}_{n\times n}.$$

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e.g. $V = \mathbb{R}^n$ $\mathcal{B}$, $\mathcal{C}$ basis of V.

$T_{P_{\mathcal{B}}}^{-1} = [\ ]_{\mathcal{B}}$

$$\begin{tikzcd} & \mathbb{R}^n \arrow[dd, bend right] \\ V = \mathbb{R}^n \arrow[ur, "{[\ ]_{\mathcal{B}}}"] \arrow[dr, "{[\ ]_{\mathcal{C}}}"'] & \\ & \mathbb{R}^n \end{tikzcd}$$

$\to T_{P_C^{-1}} \circ T_{P_{\mathcal{B}}} = T_{P_C^{-1} P_{\mathcal{B}}}$

$$[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \iff \vec{b}_1 = x_1 \vec{c}_1 + \dots + x_n \vec{c}_n.$$

$$= \begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix}^{-1} \vec{b}_1$$

the associated matrix of this linear transf. is

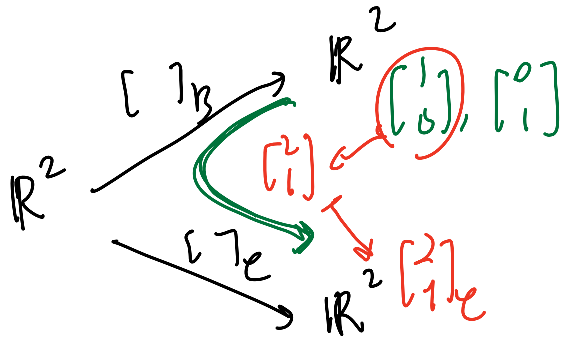
$$\begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix}^{-1} \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} = P_C^{-1} P_{\mathcal{B}}$$

$C^{-1}B$

$$\begin{bmatrix} C & | & B \end{bmatrix}_{n \times 2n} \xrightarrow{\text{row reductions}} \begin{bmatrix} I_n & | & C^{-1}B \end{bmatrix}$$

e.g. $\mathbb{R}^2$

$$\mathcal{B} = \left\{ \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}, \quad \mathcal{C} = \left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix} \right\}$$



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(2) Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^n$. Then $\vec{u}\vec{v}^T$ is an $n \times n$ matrix. Prove that
$$\det(\mathbb{I}_n + \vec{u}\vec{v}^T) = 1 + u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

$$\det\begin{bmatrix} 1+u_1v_1 & \cdots & u_1v_n \\ u_2v_1 & 1+u_2v_2 & \vdots \\ \vdots & & \ddots \\ u_nv_1 & \cdots & 1+u_nv_n \end{bmatrix}$$

$1+u_2v_2+\cdots+u_nv_n$ ← inductive hypothesis.

$$= (1+u_1v_1)\det\begin{bmatrix} 1+u_2v_2 & & \\ & \ddots & \\ \cdots & & 1+u_nv_n \end{bmatrix}$$

$$- \underset{v_2u_1}{u_2v_1} \det\begin{bmatrix} 1 & \cancel{v_3}\,0 & \cdots & \cancel{v_n}\,0 \\ u_3 & 1+\cancel{u_3v_3}\,0 & \cdots & \cancel{u_3v_n}\,0 \\ \vdots & \vdots & & \\ u_n & \cancel{u_nv_3}\,0 & \cdots & 1+\cancel{u_nv_n}\,0 \end{bmatrix}$$

$$+ (-1)\underset{v_3u_1}{u_3v_1}\det\begin{bmatrix} 1\ \cancel{u_1v_3} & \cancel{u_1v_2}\,0 & \cancel{u_1}v_4 & \cdots \\ u_2\cancel{v_3} & 1+\cancel{u_2v_2}\,0 & u_2v_4 & \cdots \\ u_4\cancel{v_3} & \cancel{u_4v_2}\,0 & 1+u_4v_4 & \\ & \vdots & & \ddots \\ & & & 1+u_nv_n \end{bmatrix}$$

$- \cdots\cdots$

$$\boxed{\det\begin{bmatrix}\vec{a}_1 & \cdots & \vec{a}_n\end{bmatrix} = \det\begin{bmatrix}\vec{a}_1 & \vec{a}_2+k\vec{a}_1 & \cdots\end{bmatrix}}$$

$\parallel$

$$= \begin{bmatrix} \vec{a}_1 \cdots \end{bmatrix} \begin{bmatrix} 1 & & k \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$$

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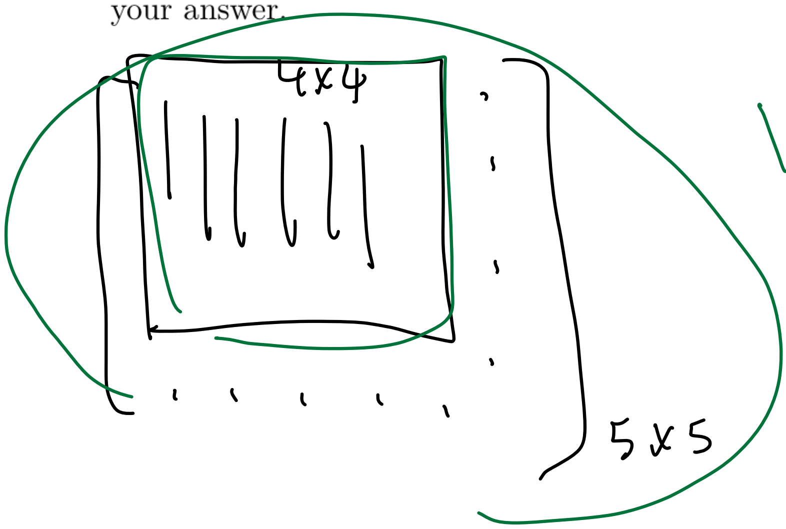
(6) Let V be the set consisting of 5×5 real matrices with the property that the entries in each row and column sum to zero. More concretely, a 5×5 matrix $A = [a_{ij}]$ belongs to the set V if and only if

$$a_{i1} + a_{i2} + \cdots + a_{i5} = 0 \quad \text{and} \quad a_{1j} + a_{2j} + \cdots + a_{5j} = 0 \quad \text{for any} \quad 1 \leq i, j \leq 5.$$

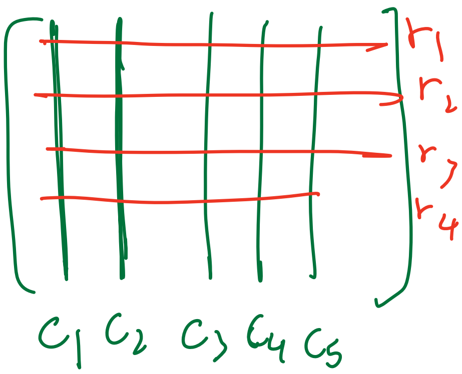
It is not hard to see that V is a vector space. Find the dimension of V, and prove your answer.



$V \longrightarrow M_{4\times 4}$

$\longmapsto \begin{bmatrix} | & | & | & | \end{bmatrix}_{4\times4}$

Show it's bijective.



$M_{5\times5} \xrightarrow{T} \mathbb{R}^9$

$\phi \longmapsto (c_1, c_2, c_3, c_4, c_5, r_1, r_2, r_3, r_4)$

$\ker(T) = V$

$\mathrm{Im}(T) = \mathbb{R}^9$

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(7) Let V_1, V_2, V_3 be real vector spaces, and $T\colon V_1 \to V_2$, $S\colon V_2 \to V_3$ be linear transformations. Prove that the following two statements are equivalent to each other:

(a) $\mathrm{Im}(S \circ T) = \mathrm{Im}(S)$;
(b) $\mathrm{Ker}(S) + \mathrm{Im}(T) = V_2$.

$V_1 \xrightarrow{T} V_2 \xrightarrow{S} V_3$.

$\mathrm{Im}(S \circ T) \subseteq \mathrm{Im}(S)$ is always true

(a) $\iff \quad \mathrm{Im}(S) \subseteq \mathrm{Im}(S \circ T)$

$\iff \quad \forall\, \vec{x} \in \mathrm{Im}(S)$, we have $\vec{x} \in \mathrm{Im}(S \circ T)$

$\iff \quad \forall\, \vec{y} \in V_2$, we have $S(\vec{y}) \in \mathrm{Im}(S \circ T)$

$\iff \quad \forall\, \vec{y} \in V_2,\ \exists\, \vec{z} \in V_1$

s.t. $S(\vec{y}) = S(T(\vec{z}))$

$\iff \quad \forall\, \vec{y} \in V_2,\ \exists\, \vec{z} \in V_1$

s.t. $S(\vec{y} - T(\vec{z})) = 0$

i.e. $\vec{y} - T(\vec{z}) \in \ker(S)$

$\iff \quad \forall\, \vec{y} \in V_2,\ \exists\, \vec{z} \in V_1$ and $\vec{w} \in \ker(S)$

s.t. $\vec{y} = T(\vec{z}) + \vec{w}$.

$\iff \quad \ker(S) + \mathrm{Im}(T) = V_2$.

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(5) Let A be a square matrix. Suppose there exists a positive integer k such that $A^k = 0$ (here 0 denotes the zero matrix). Prove that the matrix $\mathbb{I} - A$ is invertible.

$$(\mathbb{I} - A)(\mathbb{I} + A + A^2 + \cdots + A^{k-1})$$

$$= \mathbb{I}$$

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(3) Let V be an n-dimensional vector space and $T\colon V \to V$ a linear transformation such that $\ker(T) = \operatorname{Im}(T)$.

(a) Prove that n is even.

(b) Give an example of such a linear transformation T.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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(4) Let A and B be $m \times n$ matrices. Then $A + B$ also is an $m \times n$ matrix. Prove that

$$\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

$$\dim\big(\operatorname{Col}(A+B)\big) \leq \dim\big(\operatorname{Col}(A) + \operatorname{Col}(B)\big) \leq \dim \operatorname{Col}(A) + \dim \operatorname{Col}(B)$$

$$\begin{bmatrix} | & & | \\ \vec{a}_1+\vec{b}_1 & \cdots & \vec{a}_n+\vec{b}_n \\ | & & | \end{bmatrix} \qquad \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \qquad \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix}$$

$\dim(\operatorname{Col}(A+B)) = \operatorname{rank}(A+B)$

$\dim \operatorname{Col}(A) + \dim \operatorname{Col}(B) = \operatorname{rank}(A) + \operatorname{rank}(B).$

$A: m \times n, \quad B: n \times p$

$$rk(A) + rk(B) \leq rk(AB) + n$$

$$\begin{tikzcd} \operatorname{Col}(B) \arrow[r, hook] \arrow[rr, bend left, "S = T_A|_{\operatorname{Col}(B)}"] & \mathbb{R}^n \arrow[r, "T_A"] & \mathbb{R}^m \end{tikzcd}$$

$\operatorname{Im}(S) = \operatorname{Col}(AB)$

$\ker(S) = \ker(T_A) \cap \operatorname{Col}(B)$

$$\dim \operatorname{Col}(B) = \dim \ker(S) + \dim \operatorname{Im}(S)$$

$\dim \operatorname{Col}(B) = rk(B)$

$\ker(S) = \ker(T_A) \cap \operatorname{Col}(B)$

$\dim \operatorname{Im}(S) = rk(AB)$

$$\begin{aligned} rk(B) - rk(AB) &= \dim(\ker(T_A) \cap Col(B)) \\ &\leq \dim \ker(T_A) \\ &= n - rk(A) \end{aligned}$$