HOMEWORK 10
MATH 104, SECTION 2

Some ground rules:

- You have to submit your homework via **Gradescope** to the corresponding assignment. The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You're allowed to use any result that is proved in the lecture; but if you'd like to use other results, you have to prove them before using them.

Problem set (5 problems; due April 20 at 11am PT)

(1) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that the derivative $f'(x)$ exists for any $x \in \mathbb{R}$, but $f'\colon \mathbb{R} \to \mathbb{R}$ is not a continuous function.

(2) We say a function $f\colon (a,b) \to \mathbb{R}$ is *strictly increasing* if $f(x) < f(y)$ for any $a < x < y < b$. Suppose f is differentiable on (a,b).

(a) Prove or disprove: If f is strictly increasing, then $f'(x) > 0$ for any $x \in (a,b)$.

(b) Prove or disprove: If $f'(x) > 0$ for any $x \in (a,b)$, then f is strictly increasing.

(Hint: Mean value theorem.)

(3) Prove that the equation $e^x = 1 - x$ has a unique solution in $\mathbb{R}$.

(4) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying $|f(x) - f(y)| \leq |x - y|^2$ for any $x, y \in \mathbb{R}$. Prove that f is a constant function.

(5) Let $f\colon (a,b) \to \mathbb{R}$ be an unbounded differentiable function. Prove that the derivative $f'\colon (a,b) \to \mathbb{R}$ is also unbounded.