HW12 sol'n

#1.

(a) • $d_B(f,g) \geq 0$, and

$d_B(f,g) = 0 \iff f(x) = g(x) \ \forall x \in X$, i.e. $f = g$.

• $d_B(f,g) = d_B(g,f)$ is clear.

• Let $f_1, f_2, f_3 \in B(X)$. $\forall \varepsilon > 0$, $\exists x \in X$ s.t.

$$|f_1(x) - f_3(x)| > d_B(f_1, f_3) - \varepsilon.$$

Hence

$$\begin{aligned} d_B(f_1, f_3) - \varepsilon &< |f_1(x) - f_3(x)| \\ &\leq |f_1(x) - f_2(x)| + |f_2(x) - f_3(x)| \\ &\leq d_B(f_1, f_2) + d_B(f_2, f_3) \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$d_B(f_1, f_3) \leq d_B(f_1, f_2) + d_B(f_2, f_3). \quad \square$$

(b) Let $\{f_n\} \subset B(X)$ be a Cauchy seq., i.e. $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $d_B(f_n, f_m) < \varepsilon \quad \forall n, m > N$.

$$d_B(f_n, f_m) = \sup_{x \in X} |f_n(x) - f_m(x)|.$$

Hence $|f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m > N, \ \forall x \in X$.

Hence $\forall x \in X$, $\{f_n(x)\}$ is a Cauchy seq. in $\mathbb{R}$, therefore conv. to some $f(x) \in \mathbb{R}$.

$\boxed{\text{Claim:}}$

• $f \in B(X)$, and

• $f_n \to f$ in $B(X)$.

- <u>f is bounded:</u>

Say f_{N+1} is bounded by M, i.e. $|f_{N+1}(x)| < M \quad \forall x \in X$,

We have $|f_n(x) - f_{N+1}(x)| < \varepsilon \quad \forall n \geq N,\ x \in X$.

$\Rightarrow |f(x) - f_{N+1}(x)| = |\lim_{n\to\infty} f_n(x) - f_{N+1}(x)| \leq \varepsilon. \quad \forall x \in X.$

$\Rightarrow |f(x)| < M + \varepsilon \quad \forall x \in X$,

- <u>$f_n \to f$ in $B(X)$:</u>

$\forall \varepsilon > 0,\ \exists N > 0$ s.t. $\underset{\sup_{x\in X}|f_n(x) - f_m(x)|}{\underset{\|}{d_B(f_n, f_m)}} < \frac{\varepsilon}{2} \quad \forall n, m > N.$

$\Rightarrow |f_n(x) - f_m(x)| < \frac{\varepsilon}{2} \quad \forall n, m > N,\ \forall x \in X$

By taking limit $m \to \infty$, we have $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} \quad \forall n > N \ \forall x \in X$

$\Rightarrow d_B(f_n, f) \leq \frac{\varepsilon}{2} < \varepsilon \quad \forall n > N.$

Hence $\lim_{n\to\infty} f_n = f$ in $B(X)$. □

(c) <u>Claim:</u> If $\{f_n\}$ is a seq. of conti. fcn. on X such that $f_n \to f$ in $B(X)$, then f is also conti.

This follows from the fact that

$f_n \to f$ in $B(X) \iff f_n \to f$ uniformly,

and that the unif. limit of conti. fcns is conti. □

(d) X - complete metric space
$\cup$
E - closed subset.

Let $\{x_n\} \subset E$ be a Cauchy seq. and $\lim_{n\to\infty} x_n = x \in X$.

We need to show that $x \in E$.

But this is clear since a closed set contains all of its limit pts. $\square$

\#2: Let $z := \limsup b_n = \limsup \frac{a_n}{n}$. $\exists$ subseq. (b_{k_n}) s.t. $\lim b_{k_n} = z$.

Fix any $m \in \mathbb{N}$,

Write $k_n = l_n \cdot m + r_n$, where $0 \leq r_n < m$.

$\Rightarrow$ $a_{k_n} \leq l_n \cdot a_m + a_{r_n}$ by assumption.

$$\Rightarrow \quad b_{k_n} \leq \frac{l_n \cdot m}{k_n} b_m + \frac{a_{r_n}}{k_n}.$$

$$= \left(1 - \frac{r_n}{k_n}\right) b_m + \frac{a_{r_n}}{k_n}.$$

$$\Rightarrow \quad z = \lim_{n\to\infty} b_{k_n} = \liminf_{n\to\infty} b_{k_n} \leq \liminf_{n\to\infty} b_m$$

Since r_n is bounded.

$\Rightarrow$ b_m is convergent. $\square$

#3.

(a) • It's clear that $d(K_1, K_2) \geq 0$ and $d(K, K) = 0$.

• If $K_1 \neq K_2$, say $x \in K_2 \setminus K_1$.

Since $\mathbb{R}^2 \setminus K_1$ is open, $\exists\, r > 0$ s.t. $B_r(x) \cap K_1 = \phi$.

$\Rightarrow x \notin B_r(K_1)$.

$\Rightarrow d(K_1, K_2) \geq r > 0$.

• It's clear that $d(K_1, K_2) = d(K_2, K_1)$.

• If $d(K_1, K_2) < r_1$ and $d(K_2, K_3) < r_2$,

then $K_1 \subset B_{r_1}(K_2)$ and $K_2 \subset B_{r_2}(K_3) \Rightarrow K_1 \subset B_{r_1 + r_2}(K_3)$.

Similarly, $K_3 \subset B_{r_1 + r_2}(K_1)$. Hence $d(K_1, K_3) \leq r_1 + r_2$.

$\forall\, \varepsilon > 0$, we have $d(K_1, K_3) \leq d(K_1, K_2) + \varepsilon + d(K_2, K_3) + \varepsilon$.

$\Rightarrow d(K_1, K_3) \leq d(K_1, K_2) + d(K_2, K_3)$. $\square$

(b) $\forall\, K \in \mathcal{S}$ and $\varepsilon > 0$, we need to find a finite set K_0

s.t. $d(K, K_0) < \varepsilon$.

Consider the open cover $\{B_{\frac{\varepsilon}{2}}(x)\}_{x \in K}$ of K.

$\exists$ finite subcover $\{B_{\frac{\varepsilon}{2}}(x_1), \ldots, B_{\frac{\varepsilon}{2}}(x_n)\}$ since K is cpt.

Define $K_0 := \{x_1, \ldots, x_n\} \subset \mathbb{R}^2$.

It's easy to check that $d(K, K_0) \leq \frac{\varepsilon}{2} < \varepsilon$. $\square$

#4.

(a) Claim: $\lim_{x\to a^-} f(x) = \sup\{f(x) : x < a\} =: L_a$, i.e.

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < a - x < \delta \Rightarrow |f(x) - L_a| < \varepsilon$.

pf. It's clear that $\forall\, x < a$ we have $f(x) \leq L_a < L_a + \varepsilon$.

Since $L_a - \varepsilon < \sup\{f(x) : x < a\}$, $\exists\, y < a$

s.t. $f(y) > L_a - \varepsilon$.

Pick $\delta := a - y > 0$.

Then $\forall\, 0 < a - x < \delta \Leftrightarrow y < x < a$,

we have $\underset{\underset{L_a - \varepsilon.}{\vee}}{f(y)} \leq f(x)$ since f is increasing. $\square$

Similarly, one can show: $\lim_{x\to a^+} f(x) = \inf\{f(x) : x > a\}$.

(b). Observe that $A = \{a \in [0,1] : \lim_{x\to a^-} f(x) =: L_a < R_a = \lim_{x\to a^+} f(x)\}$

Claim: If $a_1 < a_2$ and $a_1, a_2 \in A$, then $R_{a_1} < L_{a_2}$.

pf $R_{a_1} = \inf\{f(x) : x > a_1\} \leq f\left(\frac{a_1 + a_2}{2}\right) \leq \sup\{f(x) : x < a_2\} = L_{a_2}$ $\square$

Hence the image of the map

$$\begin{aligned} A &\longrightarrow \{\text{open intervals in } \mathbb{R}\}. \\ a &\longmapsto (L_a, R_a) \end{aligned}$$

are disjoint open intervals.

By denseness of $\mathbb{Q}$, $\exists\, q_a \in \mathbb{Q}$ s.t. $q_a \in (L_a, R_a)$. $\forall a \in A$,

Since each (L_a, R_a) are disjoint, the map

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{Q} \\ a & \longmapsto & q_a \end{array}$$

is injective.

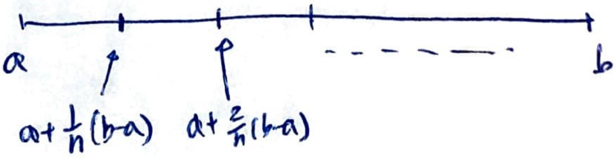
Hence A is either finite or countable. $\square$

#5:

f is uniformly conti. since $[a,b]$ is cpt.

$\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\underset{x,y \in [a,b]}{|x-y| < \delta} \Rightarrow |f(x) - f(y)| < \dfrac{\varepsilon}{(b-a)/4}$.

Consider a partition



Choose n large enough s.t. $\frac{2}{n}(b-a) < \delta$.

Then the graph of f above $\left(a + \frac{i}{n}(b-a),\ a + \frac{i+2}{n}(b-a)\right)$ can be covered by an open cube of size $\left(\frac{2}{n}(b-a) \times \dfrac{2\varepsilon}{(b-a)/4}\right)$.

Hence there exists
n open cubes, each with volume $\frac{2}{n}(b-a) \cdot \dfrac{2\varepsilon}{(b-a)/4}$
that covers Γ_f. $\Rightarrow$ total volume $= \varepsilon$.

$\square$

#6. WTS: $\forall x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\|x-y\| < \delta \Rightarrow \underbrace{\|T(x)-T(y)\|}_{\|T(x-y)\|} < \varepsilon.$$

T: linear.

$\Leftrightarrow$ WTS: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\|x\| < \delta \Rightarrow \|T(x)\| < \varepsilon.$$

One can represent T as a matrix (by fixing a basis of $\mathbb{R}^n$):

$$T(x) = Ax = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots \end{bmatrix}$$

$$\|T(x)\|^2 = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} x_j \right)^2 \leq n \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}^2 x_j^2 \right)$$

$$\leq \underbrace{\max_{i,j} \{a_{ij}^2\} \cdot n^2}_{C^2} \cdot \left(\sum_{j=1}^n x_j^2 \right) = C^2 \|x\|^2.$$

Choose $\delta := \frac{\varepsilon}{C}$. $\square$

#7. (a) Claim: The series conv. $\Leftrightarrow$ $x > 0$.

- $x < 0$: div. since $e^{-nx} \cos(nx) \not\to 0$.
- $x = 0$: obviously div.
- $x > 0$: conv. by Weierstrass M-test.

(b). False since $\sup\{|e^{-nx} \cos(nx)| : x > 0\} = 1$ $\forall n$. $\square$

#8: $|f(x)| \leq |\sin x| \quad \forall x \in \mathbb{R}.$

$$|f'(0)| = \left|\lim_{x\to 0} \frac{f(x)-f(0)}{x}\right| \leq \lim_{x\to 0} \frac{|\sin x|}{|x|} = 1. \quad \square$$

we know it exists since f is a combination of $\sin(ax)$.

#9. Let $c \in (a,b)$ and $|f'(c)| = N \geq 0$.

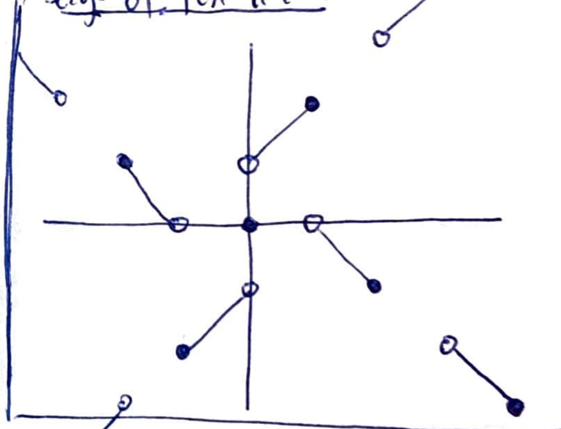
Then $\forall x \in (a,b)$, $\exists\, y$ between x and c s.t.

$$|f'(x)| = |f'(c) + f''(y)(x-c)| \leq N + M(b-a)$$

Hence $|f'|$ is unif. bdd on (a,b),

$\Rightarrow$ f is unif. conti. $\square$

e.g. of. fcn #10:



#10. Suppose f is conti. on $\mathbb{R}$.

Observe that f is injective:

If $f(x) = f(y)$, then $-x = f(f(x)) = f(f(y)) = -y \Rightarrow x = y$.

Use IVT, one can show that any inj. conti. fcn. is strictly monotone. (i.e. either strictly increasing or decreasing). cf. Ross. Thm. 18.6.

Suppose f is strictly increasing, then

$0 < 1 \Rightarrow f(0) < f(1) \Rightarrow 0 = f(f(0)) < f(f(1)) = -1$. Contradiction.

Similarly, there is a contraction for f strictly decreasing. $\square$