Exercise 8.5 (a):

pf: $\forall \varepsilon > 0$, $\exists N_1 > 0$ s.t. $|a_n - s| < \varepsilon \quad \forall n > N_1$ (since $\lim_{n\to\infty} a_n = s$)

$\Rightarrow a_n > s - \varepsilon \quad \forall n > N_1$

$\exists N_2 > 0$ s.t. $|b_n - s| < \varepsilon \quad \forall n > N_2$ (since $\lim_{n\to\infty} b_n = s$)

$\Rightarrow b_n < s + \varepsilon \quad \forall n > N_2$.

Take $N := \max\{N_1, N_2\}$,

Then $\forall n > N$, we have $a_n > s - \varepsilon$ and $b_n < s + \varepsilon$,

therefore $s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon \quad \forall n > N$

$\Rightarrow |s_n - s| < \varepsilon \quad \forall n > N$.

Hence $\lim_{n\to\infty} s_n = s$. $\square$

Exercise 8.9 (a)

pf Since $s_n \geq a$ for all but finitely many n,

$\exists N > 0$ s.t. $s_n \geq a$ for any $n > N$.

Assume by contradiction that $s := \lim_{n\to\infty} s_n < a$,

We can choose $\varepsilon > 0$ small enough s.t. $\underbrace{\lim_{n\to\infty} s_n}_{= s} + \varepsilon < a$.

By definition of limit, $\exists N' > 0$ s.t. $|s_n - s| < \varepsilon \quad \forall n > N'$.

$\Rightarrow s_n < s + \varepsilon \quad \forall n > N'$

Then, for any $n > \max\{N, N'\}$, we have

$a \leq s_n < s + \varepsilon < a$, Contradiction. $\square$

Exercise 8.4

pf: Since $\lim_{n\to\infty} s_n = 0$, $\forall \varepsilon > 0$, $\exists N > 0$

s.t. $|s_n - 0| < \frac{\varepsilon}{M}$ $\forall n > N$.

$\Rightarrow |s_n t_n| = |s_n||t_n| < \frac{\varepsilon}{M} \cdot M = \varepsilon$ $\forall n > N$.

Hence $\lim_{n\to\infty} s_n t_n = 0$. $\square$

Exercise 9.12 (a)

pf: Choose a s.t. $L < a < 1$. Then $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| < a < 1$.

$\Rightarrow \exists N > 0$ s.t. $\left|\frac{s_{n+1}}{s_n}\right| < a$ $\forall n \geq N$.

Why? Think about this!!

$\Rightarrow |s_{n+1}| < a|s_n|$ $\forall n \geq N$.

$\Rightarrow |s_n| < a^{n-N}|s_N|$ $\forall n \geq N$.

(Theorem 9.7(b)).

Observe that $\lim_{n\to\infty} a^{n-N}|s_N| = 0$ $\forall\, 0 < a < 1$.

By squeeze lemma (Exercise 8.5(a)), we get $\lim_{n\to\infty} s_n = 0$. $\square$

Exercise 12.2

pf. Denote $v_N := \sup\{|s_n| : n > N\}$

We know that $v_1 \geq v_2 \geq \cdots \geq v_N \geq \cdots \geq \limsup_{n\to\infty} |s_n|$

$\limsup_{n\to\infty} |s_n| = 0 \iff \forall \varepsilon > 0,\ \exists N > 0$ s.t. $\sup\{|s_n| : n > N\} \leq \varepsilon$.

$\iff \forall \varepsilon > 0, \exists N > 0$ s.t. $|s_n| \leq \varepsilon \quad \forall n > N$.

$\iff \lim_{n\to\infty} s_n = 0 \quad \square$