Today: Power series.

- they're important examples of holo. fcns.
- we'll prove later that ANY holo. fcn. is locally a power series fcn. (this is NOT true for smooth real-valued fcns)

Review of series:

Let $\{a_0, a_1, a_2, \cdots\} \subseteq \mathbb{C}$ be a seq. of cpx #.

- partial sums: $S_n := a_0 + a_1 + \cdots + a_n \in \mathbb{C}$
- say the series $\sum_{n=0}^{\infty} a_n$ converges if $\lim_{n\to\infty} S_n$ exists,

i.e. there exists $s \in \mathbb{C}$ s.t.

$$\forall \varepsilon > 0, \exists N > 0$$

$$\text{s.t. } n > N \Rightarrow |S_n - s| < \varepsilon.$$

Recall (Cauchy criterion) $\lim S_n$ exists $\Leftrightarrow$ $\forall \varepsilon > 0, \exists N > 0$ s.t. $|S_n - S_m| < \varepsilon$ $\forall n, m > N$.

Now, suppose S_n is the partial sum $a_0 + \cdots + a_n$.

$$|S_n - S_m| = |a_{n+1} + \cdots + a_m| = \left| \sum_{k=n+1}^{m} a_k \right|.$$

(if $n < m$)

Rmk: In particular, if $\sum a_n$ conv., then $\lim |a_n| = 0$

Rmk: Usually, we use this to show a series is divergent.

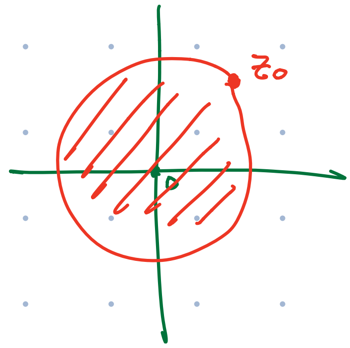
Rmk: $\lim |a_n| = 0 \not\Rightarrow \sum a_n$ conv. (e.g. $1 + \frac{1}{2} + \frac{1}{3} + \cdots$)

Def Say $\sum a_n$ <u>converges absolutely</u> if $\sum |a_n|$ converges

($\sum a_n$ conv. absolutely $\Rightarrow$ $\sum a_n$ converges)
easy to show:

Def: Say a <u>power series</u> $\sum_{n=0}^{\infty} a_n z^n$ is <u>convergent at $z = z_0$</u>

if $\sum_{n=0}^{\infty} a_n z_0^n$ is convergent.



Lemma: If $\sum a_n z^n$ conv. abs. at z_0, then

It also conv. abso. for any $|z| \le |z_0|$.

pf For any $|z| \le |z_0|$, we want to show: $\sum a_n z^n$ conv. abs.

$$\Updownarrow$$

$$\sum \underbrace{|a_n z^n|}_{\substack{\| \\ |a_n| \cdot |z|^n \le |a_n| \cdot |z_0|^n}} \text{ conv.}$$

Since $\sum a_n z^n$ conv. abs. at z_0,

we have $\sum |a_n| |z_0|^n$ conv.

Since $|z| \le |z_0|$, so $|a_n||z|^n \le |a_n||z_0|^n$

So By Comparison test, $\sum |a_n||z|^n$ conv.

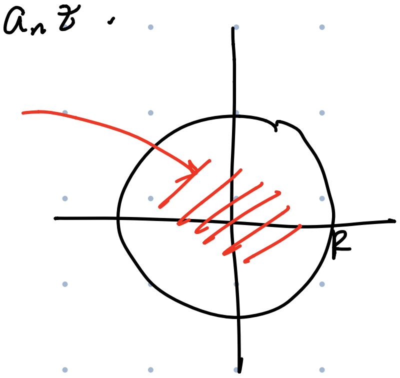
$\Rightarrow$ $\sum a_n z^n$ conv.. $\square$

Def: The <u>radius of convergent</u> of $\sum a_n z^n$ is:

$$\left(\limsup |a_n|^{1/n} \right)^{-1} \in [0, \infty]$$

Thm Let R be the radius of conv. of $\sum a_n z^n$.

Then $\begin{cases} |z|<R \Rightarrow \sum a_n z^n \text{ conv. abs.} \\ |z|>R \Rightarrow \sum a_n z^n \text{ div.} \end{cases}$



pf Let $L = \limsup |a_n|^{1/n}$ and $R = 1/L$

Assume for simplicity that $L \neq 0, \infty$

① $\boxed{|z|<R}$ (want: $\sum |a_n||z|^n$ conv.)

$\Downarrow$

$|z| \cdot L < 1$

$\exists\, \varepsilon>0$, s.t. $\overbrace{|z| \cdot (L+\varepsilon)}^{=\alpha} < 1$

⊕ $\exists N>0$ s.t. $|a_n|^{1/n} < L+\varepsilon \quad \forall\, n>N$.

(since $\limsup |a_n|^{1/n} = L$)

$$\Rightarrow \sum |a_n||z|^n = \underbrace{\sum_{n=0}^{N} |a_n||z|^n}_{\text{finite}} + \boxed{\sum_{n=N+1}^{\infty} \boxed{|a_n|}\,|z|^n}$$

$|a_n| < (L+\varepsilon)^n$ since $n>N$.

⊕ $$\sum_{n=N+1}^{\infty} |a_n||z|^n < \sum_{n=N+1}^{\infty} \underbrace{(L+\varepsilon)^n |z|^n}_{\alpha^n}$$ Converges since $0<\alpha<1$

② $\boxed{|z|>R}$ (want: $\sum a_n z^n$ div.)

$\Updownarrow$

$|z| \cdot L > 1$

$\exists \varepsilon > 0$ s.t. $|z| \cdot (L - \varepsilon) > 1$

$\limsup |a_n|^{1/n} = L \Rightarrow \exists$ subseq. $\{a_{n_k}\}$ of $\{a_n\}$

s.t. $\lim |a_{n_k}|^{1/n_k} = L$

$\Rightarrow \exists K > 0$ s.t. if $k > K$

then $|a_{n_k}|^{1/n_k} > L - \varepsilon$

$$\Rightarrow |a_{n_k}| > (L-\varepsilon)^{n_k}$$

$$|a_{n_k} z^{n_k}| > (L-\varepsilon)^{n_k} |z|^{n_k} > 1.$$

$\Rightarrow \sum a_n z^n$ div. $\Leftarrow$ $\lim |a_n| \neq 0$ $\square$

HW: (ratio test) If $\lim \left|\frac{a_{n+1}}{a_n}\right| = L$, then $\lim |a_n|^{1/n} = L$

Hint: $\forall \varepsilon > 0, \exists N > 0$ s.t. $L - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon \;\; \forall n > N$

For any $n > N$, $\frac{a_n}{a_N} = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N}$

$$|a_N|^{1/n} (L-\varepsilon)^{\frac{n-N}{n}} < \left|\frac{a_n}{a_N}\right|^{1/n} < (L+\varepsilon)^{\frac{n-N}{n}} |a_N|^{\frac{1}{n}}$$

e.g. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{1}{2} z^2 + \frac{1}{3!} z^3 + \cdots$ conv. abs. $\forall z \in \mathbb{C}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1/(n+1)!}{1/n!}\right| = \frac{1}{n+1}$ $\quad \lim \left|\frac{a_{n+1}}{a_n}\right| = 0 \Rightarrow \lim |a_n|^{1/n} = 0$

$\Rightarrow R = +\infty$

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$ Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

<u>Thm</u> If $\sum a_n z^n$ has radius of conv. $R > 0$.

and define $f(z) := \sum_{n=0}^{\infty} a_n z^n$ on $|z| < R$.

(this is a well-defined fcn by previous thm).

Then: 1) f is holo. on $\{|z| < R\}$.

Define $g(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$

Then 2) g also has radius of conv. $= R$.

3) $g(z) = f'(z)$ $\forall\, |z| < R$.

<u>Coro</u>: Any power series $f(z) = \sum a_n z^n$ is infinitely differentiable in its radius of conv. $\{|z| < R\}$, and all derivatives of f are also holo., and

$$f^{(n)}(z) = \sum_{n=0}^{\infty} a_n \left(\frac{d}{dz}\right)^k z^n$$

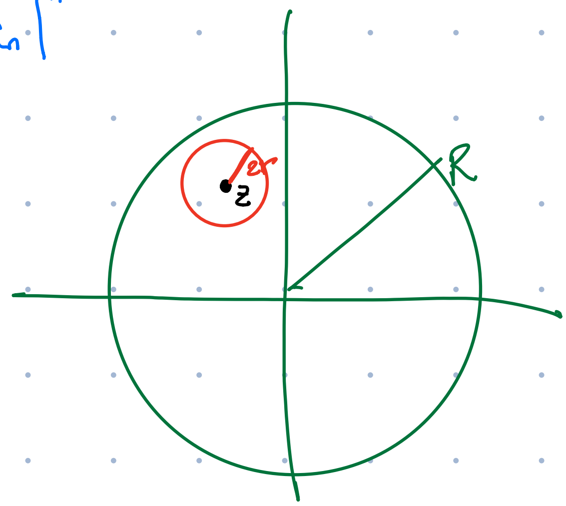
$$= k!\, a_k + (k+1)\cdot k\cdot(k-1)\cdot\dots\cdot 2\, a_{k+1} z$$

$$+ (k+2)(k+1)\cdot\dots\cdot 3\cdot a_{k+2} z^2$$

$$+ \dots$$

Pf 2): $\overline{\lim}\sup |n a_n|^{1/n} \overset{??}{=} \overline{\lim}\sup |a_n|^{1/n}$

$$\underbrace{\overline{\lim}\sup |n a_n|^{1/n}}_{\displaystyle = \left(\underbrace{\lim n^{1/n}}_{=1}\right)\left(\overline{\lim}\sup |a_n|^{1/n}\right)} = \overline{\lim}\sup |a_n|^{1/n}$$



pf 1)3): "$\forall \varepsilon > 0, \exists \delta > 0$

Fix $z \in \mathbb{D}_R(0)$ s.t. $0 < |h| < \delta \Rightarrow \left| \frac{f(z+h) - f(z)}{h} - g(z) \right| < \varepsilon$"

(Given z, ε, goal: find δ s.t. $\nearrow$)

Choose some $r > 0$ small enough. s.t. $\mathbb{D}_{2r}(z) \subseteq \mathbb{D}_R(0)$

$$\frac{f(z+h) - f(z)}{h} - g(z) = \frac{\sum_{n=0}^{\infty} a_n (z+h)^n - \sum_{n=0}^{\infty} a_n z^n}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1}$$

$$= \underbrace{\left(\sum_{n=0}^{N} \frac{a_n (z+h)^n - a_n z^n}{h} - \sum_{n=1}^{N} n a_n z^{n-1} \right)}_{I_N} + \underbrace{\frac{\sum_{n=N+1}^{\infty} a_n (z+h)^n - \sum_{n=N+1}^{\infty} a_n z^n}{h}}_{II_N} - \underbrace{\sum_{n=N+1}^{\infty} n a_n z^{n-1}}_{III_N}$$

I_N:
$$\sum_{n=0}^{N} \frac{a_n (z+h)^n - a_n z^n}{h} - \sum_{n=1}^{N} n a_n z^{n-1}$$

$$= \sum_{n=0}^{N} \frac{a_n}{\cancel{h}} \left[\cancel{z^n} + \cancel{\binom{n}{1} z^{n-1}} \cancel{h} + \cdots + h^{n-1} - \cancel{z^n} \right] - \sum_{n=1}^{N} \cancel{n a_n z^{n-1}}$$

$$= \sum_{n=0}^{N} a_n h \left(\binom{n}{2} z^{n-2} + \binom{n}{3} z^{n-3} h + \cdots + h^{n-2} \right)$$

$$|I_N| \leq \sum_{n=2}^{N} \left| a_n h \left(\binom{n}{2} z^{n-2} + \binom{n}{3} z^{n-3} h + \cdots + h^{n-2} \right) \right|$$

$$\leq |h| \cdot \sum_{n=2}^{N} |a_n| \left(\binom{n}{2} |z|^{n-2} + \binom{n}{3} |z|^{n-3} |h| + \cdots + |h|^{n-2} \right)$$

(If $|h| < R - r$)

$$\leq |h| \cdot \underbrace{\sum_{n=2}^{N} |a_n| \left(\binom{n}{2} (R-r)^{n-2} + \binom{n}{3} (R-r)^{n-2} + \cdots + (R-r)^{n-2} \right)}_{= G_N}$$

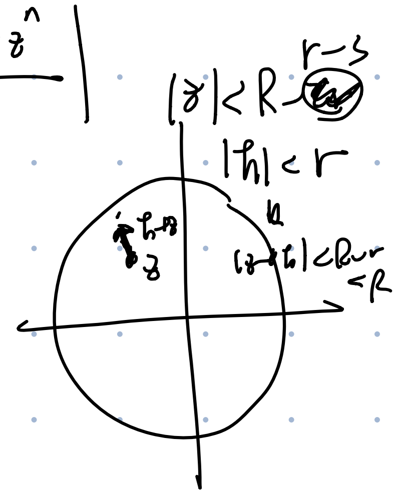
(G_N positive number depends only on N)

Summarize: If $0 < |h| < R - r$.

then $|I_N| \leq |h| \cdot G_N$,

where $G_N > 0$ depends only on N.

$$|II_N| = \left| \frac{\sum_{n=N+1}^{\infty} a_n (z+h)^n - \sum_{n=N+1}^{\infty} a_n z^n}{h} \right| \leq \sum_{n=N+1}^{\infty} |a_n| \cdot \left| \frac{(z+h)^n - z^n}{h} \right|$$



$$= \sum_{n=N+1}^{\infty} |a_n| \cdot \left| \frac{\cancel{h} \left((z+h)^{n-1} + (z+h)^{n-2} h + \cdots + h^{n-1} \right)}{\cancel{h}} \right|$$

$$= \sum_{n=N+1}^{\infty} |a_n| \left| (z+h)^{n-1} + (z+h)^{n-2} h + \cdots + h^{n-1} \right|$$

(If $|h| < R - r$, $|h| < r$

$\Rightarrow |z+h| < R - r$)

$$\leq \sum_{n=N+1}^{\infty} |a_n| \cdot n \cdot (R-r)^{n-1} \quad \text{conv.}$$

$$\Uparrow$$

$$\left(\text{Since } \sum_{n=1}^{\infty} n a_n z^{n-1} \text{ conv. absolutely in } |z|<R. \right)$$

$$|III_N| = \left| \sum_{n=N+1}^{\infty} n a_n z^{n-1} \right| \leq \sum_{n=N+1}^{\infty} n |a_n| \cdot (R-r)^{n-1}$$

$$(|z| < R-r)$$

<u>Summary</u>: If $|h| < \min\{r, R-r\}$,

then: $|I_N| \leq G_N \cdot |h|$

$$|II_N|, |III_N| \leq \sum_{n=N+1}^{\infty} |a_n| \cdot n \cdot (R-r)^{n-1}$$

Since $\sum n a_n z^{n-1}$ conv. abso.. in $|z|<R$

$\forall \varepsilon > 0$, $\exists \underline{N} > 0$ s.t.

$$\left| \sum_{n=N+1}^{\infty} |a_n| \cdot n \cdot (R-r)^{n-1} \right| < \varepsilon/3.$$

For this N, define

$$\delta := \min\{r, R-r, \frac{\varepsilon}{3G_N}\} > 0.$$

Then $\forall\ 0<|h|<\delta$,

we have

$$\left|\frac{f(t+h)-f(t)}{h}-g(t)\right| \le |\mathrm{I}_N|+|\mathrm{II}_N|+|\mathrm{III}_N|$$

$$< G_N \cdot |h| + \frac{\varepsilon}{3}+\frac{\varepsilon}{3}$$

$$\le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon. \qquad \square$$