FIRST MIDTERM PRACTICE PROBLEMS
MATH H54, FALL 2021

(1) (a) Let A be an $m \times n$ matrix, and $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ be the associated linear transformation. Suppose T_A is <u>injective</u>. What can you say about the relation between m and n? What can you say about the dimensions of the column space and the null space of A? Provide justifications for your answers.
(b) Same problems as in Part (a), but replace <u>injective</u> with <u>surjective</u>.

(2) Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^n$. Then $\vec{u}\vec{v}^T$ is an $n \times n$ matrix. Prove that

$$\det(\mathbb{I}_n + \vec{u}\vec{v}^T) = 1 + u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

(3) Let V be an n-dimensional vector space and $T \colon V \to V$ a linear transformation such that $\ker(T) = \mathrm{Im}(T)$.
(a) Prove that n is even.
(b) Give an example of such a linear transformation T.

(4) Let A and B be $m \times n$ matrices. Then $A + B$ also is an $m \times n$ matrix. Prove that

$$\mathrm{rank}(A + B) \leq \mathrm{rank}(A) + \mathrm{rank}(B).$$

(5) Let A be a square matrix. Suppose there exists a positive integer k such that $A^k = 0$ (here 0 denotes the zero matrix). Prove that the matrix $\mathbb{I} - A$ is invertible.

(6) Let V be the set consisting of 5×5 real matrices with the property that the entries in each row and column sum to zero. More concretely, a 5×5 matrix $A = [a_{ij}]$ belongs to the set V if and only if

$$a_{i1} + a_{i2} + \cdots + a_{i5} = 0 \quad \text{and} \quad a_{1j} + a_{2j} + \cdots + a_{5j} = 0 \quad \text{for any} \quad 1 \leq i, j \leq 5.$$

It is not hard to see that V is a vector space. Find the dimension of V, and prove your answer.

(7) Let V_1, V_2, V_3 be real vector spaces, and $T \colon V_1 \to V_2$, $S \colon V_2 \to V_3$ be linear transformations. Prove that the following two statements are equivalent to each other:
(a) $\mathrm{Im}(S \circ T) = \mathrm{Im}(S)$;
(b) $\mathrm{Ker}(S) + \mathrm{Im}(T) = V_2$.