

Finite subgroups of derived automorphisms of generic K3 surfaces

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Joint work with Kuan-Wen Lai (Bonn)

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Slogan of main results

Let X be a Calabi–Yau manifold. It is expected that

$$D := D^b\mathrm{Coh}(X) \cong D^\pi\mathrm{Fuk}(X^\vee).$$

This triangulated category has autoequivalences arising from both complex geometry of X and symplectic geometry of X^\vee , for instance:

$$\mathrm{Aut}(X), \quad L \otimes -; \quad \mathrm{Symp}(X^\vee).$$

Slogan: When X is a K3 surface of Picard number one,

- there is no finite order autoequivalence arising from either $\mathrm{Aut}(X)$, $L \otimes -$, or $\mathrm{Symp}(X^\vee)$;
- there are interesting finite order autoequivalences given by **mixings** of complex and symplectic geometric autoequivalences;
- we give full **classification** and **counting** of finite subgroups of $\mathrm{Aut}(D)$ and $\mathrm{Aut}(D)/[2]$ up to conjugations.

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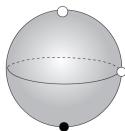
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Finite order autoequivalences and Gepner points

Consider an one-parameter family of Calabi–Yau manifolds:

$$\{x_0^{n+1} + x_1^{n+1} + \cdots + x_n^{n+1} + t \cdot x_0 x_1 \cdots x_n = 0\} \subseteq \mathbb{CP}^n.$$



The monodromies correspond to autoequivalences of $D^b\mathrm{Coh}(X)$:

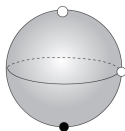
- large complex structure limit point: $\mathcal{O}(1) \otimes -$;
- conifold point: $T_{\mathcal{O}_X}(-)$;
- Gepner point: $\Phi := T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}(1))$, where $\Phi^{n+1} = [2]$.

In particular, when X is a K3 surface, we have $\Phi^4 = [2]$, therefore $\Phi^2[-1]$ is of order two in $\mathrm{Aut}(D^b(X))$.

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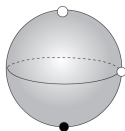
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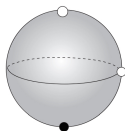
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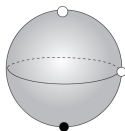
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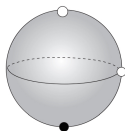
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Automorphisms of K3 surfaces

As a warm-up, let us sketch the proof of the following result.

Theorem (Nikulin)

Let X be a complex projective K3 surface with $NS(X) \cong \langle H \rangle$. Then

$$\mathrm{Aut}(X) = \begin{cases} \{id\} & \text{if } H^2 > 2 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } H^2 = 2 \end{cases}$$

- There is an injective map $\mathrm{Aut}(X) \hookrightarrow O(H^2(X, \mathbb{Z}))$.
- $f \in \mathrm{Aut}(X)$ acts trivially on $NS(X)$: pullback of H is still ample.
- f acts as $\pm id$ on $T(X) := NS(X)^\perp$: true for any odd Picard number.
- Its induced actions on the discriminant groups $T(X)^*/T(X)$ and $NS(X)^*/NS(X) \cong \mathbb{Z}/(H^2)\mathbb{Z}$ coincide $\implies f = id$ unless $H^2 = 2$.
- When $H^2 = 2$, X is a double cover over \mathbb{P}^2 and the covering involution gives the nontrivial automorphism.

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Some difficulties of generalizing to $\text{Aut}(D^b(X))$

- If $\Phi \in \text{Aut}(D^b(X))$ is of finite order, then its induced actions on

$$T(X) \quad \text{and} \quad N(X) = H^0(X) \oplus \text{NS}(X) \oplus H^4(X) \cong \mathbb{Z}^3$$

are of finite order.

- Φ still acts as $\pm \text{id}$ on $T(X)$; but its action on $N(X)$ can be more complicated.

Strategy: We show that any finite order Φ fixes a **Bridgeland stability condition** on $D^b(X)$. In particular, it fixes a 2-plane in $N(X)_{\mathbb{R}}$ pointwisely, therefore the action of Φ on $N(X)$ is either id or a reflection.

- In order to fully classify the finite subgroups of $\text{Aut}(D^b(X))$, one needs to determine the conditions on finite subgroups of $T(X)$ and $N(X)$ of which there exists finite order lifts in $\text{Aut}(D^b(X))$.

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Bridgeland stability conditions

A **Bridgeland stability condition** on D is a pair $\sigma = (Z, P)$:

- $Z: N(D) \rightarrow \mathbb{C}$ group homomorphism (central charge)
- $P = \{P(\phi)\}_{\phi \in \mathbb{R}}$ additive subcategories (semistable of phase ϕ)

satisfying several axioms, including the Harder–Narasimhan property:

- for any $E \in D$, there exists a unique sequence of exact triangles

$$\begin{array}{ccccccc} 0 & \xrightarrow{\quad} & * & \xrightarrow{\quad} & * & \longrightarrow \dots \longrightarrow & * & \xrightarrow{\quad} & E \\ & \nwarrow \text{dashed} & \swarrow & & \nwarrow \text{dashed} & & \swarrow & & \nwarrow \text{dashed} \\ & & A_1 & & A_2 & & & & A_n \end{array}$$

where $A_i \in P(\phi_i)$ and $\phi_1 > \dots > \phi_n$.

(Analogy in SG: $P \leftrightarrow$ special Lagrangian submanifolds, $Z \leftrightarrow \int_L \Omega$)

(Analogy in AG: $P \leftrightarrow$ slope semistable sheaves, $Z \leftrightarrow -\deg + \sqrt{-1} \cdot \text{rank}$)

(Analogy in flat surface: $P \leftrightarrow$ straight lines, $\phi \leftrightarrow$ slope, $|Z| \leftrightarrow$ length)

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Autoequivalences act on stability conditions

A natural way of studying a group is via considering its actions on various spaces. The group of autoequivalences $\text{Aut}(D)$ of a triangulated category admits a natural action on the space of its **Bridgeland stability conditions**.

This action ($\text{Aut}(D) \curvearrowright \text{Stab}(D)$) can be used to:

- define complexity (e.g. categorical entropy) of autoequivalences;
- provide classifications of autoequivalences (e.g. finite order, “reducible”, “pseudo-Anosov”, etc.);
- provide analogy with Teichmüller theory –

One of our key results, any finite order $\Phi \in \text{Aut}(D^b(X))$ fixes a stability condition on $D^b(X)$ (when X is a K3 surface of Picard number one), actually is motivated from the analogy with Teichmüller theory.

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This action ($\text{Aut}(D) \curvearrowright \text{Stab}(D)$) can be used to:

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A dictionary of analogy

(after Gaiotto, Moore, Neitzke; Bridgeland, Smith; Dimitrov, Haiden, Katzarkov, Kontsevich, etc.)

Riemann surface Σ	Triangulated category \mathcal{D}
curve C	object E
$C_1 \cap C_2$	$\mathrm{Hom}(E_1, E_2)$
metric g	Bridgeland stability condition σ
geodesics	semistable objects
length $\ell_g(C)$	mass $m_\sigma(E)$
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- Nielsen asked (1923): Let $G \subseteq \text{MCG}(\Sigma)$ be a finite subgroup. Does there always exist a lifting $G \subseteq \text{Diff}(\Sigma)$? (Recall that $\text{MCG}(\Sigma) = \text{Diff}(\Sigma)/\text{isotopy}$).
- Kerckhoff (1983): Yes! Moreover, there exists a metric g such that $G \subseteq \text{Isom}(\Sigma, g)$. Or equivalently, G fixes a point in $\text{Teich}(\Sigma)$. (There is a natural action of $\text{MCG}(\Sigma)$ on $\text{Teich}(\Sigma)$, e.g. $\text{MCG}(T^2) = \text{SL}(2, \mathbb{Z})$ acts on $\text{Teich}(T^2) = \mathbb{H}$.) (Rephrase: any finite subgroup of $\text{MCG}(\Sigma)$ can be realized as symmetries with respect to a metric on Σ .)
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Main theorems (F.–Lai, 2023)

- The answers to both problems are yes, for $D = D^b\mathrm{Coh}(X)$ where X is a curve, a (twisted) abelian surface, a generic twisted K3 surface, or a K3 surface of Picard number $\rho = 1$.

For K3 surfaces of $\rho = 1$, we obtain:

- every finite subgroup of $\mathrm{Aut}(D)$ is of **order 2**, and is generated by an **anti-symplectic involution**;
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 - ▶ “0-reducible”: which fixes a class $w \in N(D)$ with $w^2 = 0$ (e.g. $\otimes \mathcal{O}(1)$)
- hyperbolic: $\rho([\Phi]_{N(D)}) > 1$

Modulo certain conjectures regarding the (polynomial) entropy of the reducible autoequivalences, we expect the following trichotomy:

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For K3 or abelian surfaces, Bridgeland (2008) showed that there is an $\text{Aut}(D)$ -equivariant covering map

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where $Q_0^+(D) = \{v \in \mathbb{P}(N(D) \otimes \mathbb{C}) \mid v^2 = 0, v\bar{v} > 0\} \setminus \bigcup_{\delta^2 = -2} \delta^{\perp}$.

- For abelian surfaces, there is no spherical objects in D , so:
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Avoiding δ^\perp

Suppose X is a K3 surface of $\rho = 1$ and degree $2n$.

- We have $Q_0^+(D) \cong \mathbb{H} \setminus \text{"}(-2)\text{-points"}$.
- By Dolgachev (1996) and Kawatani (2014), the action of $\text{Aut}(D)$ on $Q_0^+(D)$ factors through $\text{Im}(\text{Aut}(D) \xrightarrow{f} \text{PSL}(2, \mathbb{R})) = \Gamma_0^+(n)$ the Fricke modular group, where $\Gamma_0^+(n) = \left\langle \Gamma_0(n), \begin{bmatrix} & -1/\sqrt{n} \\ \sqrt{n} & \end{bmatrix} =: \omega_n \right\rangle$.

We showed that the following statements are equivalent:

- $f(\Phi)$ fixes a (-2) -point in \mathbb{H}
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- $\Phi = T_S \Psi$ for some spherical object S and some $\Psi \in \text{Deck}(\pi)$

Moreover, we showed that autoequivalences of the form $T_S \Psi$ must be of infinite order in $\text{Aut}(D)/[1]$. This resolves the first issue.

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- Let $\Phi \in \text{Aut}(D)/[1]$ be of finite order, then the previous discussion shows that it fixes a point in $Q_0^+(D)$.
- Kawatani (2019): $\pi_1(Q_0^+(D)) \cong \star_{\text{free}} T_S^2$.
- Bayer–Bridgeland (2017): $\text{Stab}_{\text{red}}^\dagger(D)/\mathbb{C}$ is contractible.
- Combining these two results, we have $\text{Deck}(\pi) \cong \star_{\text{free}} T_S^2$.
- We showed that this implies the fixed point of Φ in $Q_0^+(D)$ can be lifted to a fixed point in $\text{Stab}_{\text{red}}^\dagger(D)/\mathbb{C}$, which proves the realization problem for cyclic subgroups of $\text{Aut}(D)/[1]$.
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A few further problems

- Find good representatives of 0-reducible autoequivalences.
- Do 0-reducible autoequivalences have zero entropy? ($h(\otimes \mathcal{O}(1)) = 0$)
- Generalize the realization results to:
 - ▶ general special cubic fourfolds $\mathrm{Ku}(Y)$
 - ▶ K3 surfaces of Picard number $\rho \geq 2$
 - ▶ ...?

Thank you for your attention!

Reference: F.–Lai, *Nielsen realization problem for Bridgeland stability conditions on generic K3 surfaces*, arXiv:2302.12663