NOTE

1.

Question. Let X be a smooth projective complex variety. What geometric information can we obtain from $D^bCoh(X)$?

We will discuss its relation with mirror symmetry and birational geometry.

Remark. Even if we are only interested in studying holomorphic vector bundles, coherent sheaves naturally arise as the kernel or cokernel of morphisms among them.

Remark. Coherent sheaves form an abelian category; $D^bCoh(X)$ denotes its bounded derived category.

- Mirror symmetry (closely related to the development of stability conditions)
- Facts about slope stability
- Birational geometry

2.

Definition. Denote O_X the structure sheaf of X. It is a functor which sends open subsets U to $O_X(U)$, holomorphic functions on U, satisfying certain compatibility conditions. (In terms of geometric language, it is equivalent to the trivial line bundle over X.)

Definition. We say a sheaf M is an O_X-module if each $M(U)$ is an $O_X(U)$-module, and satisfy certain compatibility conditions.

Definition. We say an O_X-module E is a coherent sheaf if

- (finitely generated) for every $x \in X$, there exists an open neighborhood U and a surjective
$$O(U)^{\oplus n} \to E(U).$$
- the kernel of any
$$O(V)^{\oplus m} \to E(V)$$
is finitely generated.

Example.
- Structure sheaf.
- Skyscraper sheaf O_x, where $O_x(U) = C$ if $x \in U$, and is zero otherwise.
- Locally free sheaf (in geometric language: vector bundles). Locally, it is isomorphic to $O_X^{\oplus n}$. But globally can be nontrivial.

Theorem. A coherent sheaf F on a smooth projective variety of dimension n admits a resolution of locally free sheaves (i.e. vector bundles) of length at most n:

$$0 \to E^n \to E^{n-1} \to \cdots \to E^0 \to F \to 0.$$

3.

Homological mirror symmetry statement.

Proof of HMS for elliptic curves. (CatDyn 6)

Remark. GHKK canonical bases.

4.

Slope of coherent sheaves.

HN filtration. (CatDyn Hsueh-Yung)

Remark. On $Coh(X)$, the input of a Kähler class (a symplectic input) gives rise to the notion of slope, and therefore stability conditions and a nice refinement $Coh^{\omega}_{\mu}(X)$.

Explain the mirror side story.

Remark. Mirror dual between deformation space and stability space.

5.

Definition.
- An object is exceptional if $Hom(E, E[n]) = C$ if $n = 0$ and zero otherwise.
- A sequence of exceptional objects is called an exceptional collection if no morphism from right to left.
- It is called full is it generates the category.
- It is called strong if $Hom(E_i, E_j[n]) = 0$ whenever $n \neq 0$.

Theorem (Beilinson). $< O, O(1), \ldots, O(n) >$ is a strong full exceptional collection of $D^b(P^n)$.

In other words, $D^b(P^n)$ is formed by these building blocks.

Definition. A semiorthogonal decomposition of D is a collection $A_1, \ldots, A_n$ of full triangulated subcategories such that: no morphism from right to left; they generate the category.

Theorem (Orlov's blowup formula). Let $Y \subseteq X$ be a locally complete intersection subscheme of codimension c, and let $\tilde{X}$ be the blowup of X with center Y. Then $D^b(\tilde{X})$ admits a semiorthogonal decomposition: $D^b(X)$ followed by $c-1$ copies of $D^b(Y)$.

In particular, if we blowup P^n at, say a point, then the resulting variety still has an exceptional collection. This is a special case of a folklore conjecture by Orlov (the existence of full exceptional collection implies rationality).

Define spherical object. $(E \otimes K_X = E_X)$

$$T_E(F) = Cone(\bigoplus_i (Hom(E, F[i]) \otimes E[-i]) \xrightarrow{ev} F).$$

Remark. Bondal–Orlov: $Aut(D^b(X)) = Z \times (Aut(X) \ltimes Pic(X))$ if K_X is ample or anti-ample.

Calabi–Yau case: the spherical twists are mirror to Dehn twists.

6.

Connection with rationality.

- degree 1 hypersurface in Pn: obvious
- degree 2: projection
- degree 3: 1-dimension (elliptic curve), 2-dimension (cubic surface, blowup of P2 at 6 points), 3-dimension (irrational: Clemens-Griffiths: intermediate Jacobian)

Cubic fourfolds (my USTC talk)