FIRST MIDTERM PRACTICE PROBLEMS
MATH 185, SECTION 2

(1) Prove that
$$\left|\frac{5z-i}{5+iz}\right| = 1 \quad \text{if} \quad |z| = 1.$$
(2) Let $\Omega \subseteq \mathbb{C}$ be an open and connected subset of $\mathbb{C}$, and let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function. Suppose that there is a curve $\gamma \subseteq \Omega$ such that f is constant on γ. Prove that f is constant in Ω.

(3) Compute
$$\int_{\gamma_2(0)} \frac{\cos(\pi z)}{z^2-1},$$
where $\gamma_2(0)$ is the circle of radius two centered at $0 \in \mathbb{C}$, oriented positively.

(4) Compute
$$\int_{\gamma_1(0)} \frac{e^z}{z},$$
where $\gamma_1(0)$ is the circle of radius two centered at $0 \in \mathbb{C}$, oriented positively.

(5) Prove that the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by
$$f(z) = \frac{z}{1+|z|}$$
is not holomorphic at *any* point $z_0 \in \mathbb{C}$.

(6) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Assume that there exists a nonempty open subset $\Omega \subseteq \mathbb{C}$ such that $f(z) \notin \Omega$ for any $z \in \mathbb{C}$. Prove that f is a constant function.

(7) Let $f(z) = z^2$.
   (a) Calculate $\int_0^{2\pi} f(2+e^{it})dt$, and confirm that it is non-zero.
   (b) Does Cauchy's theorem imply $\int_{\gamma_1(2)} f(z)dz = 0$? (Here $\gamma_1(2)$ is the circle of radius one centered at $2 \in \mathbb{C}$, oriented positively.) Explain the seeming discrepancy with part (a).

(8) Let $f\colon \mathbb{D} \to \mathbb{C}$ be a holomorphic function on the unit disk. Suppose that
$$|f(z)| \le \frac{1}{1-|z|} \quad \text{for any} \quad |z| < 1.$$
Prove that
$$\left|f^{(n)}(0)\right| \le (n+1)!\left(1+\frac{1}{n}\right)^n \quad \text{for all} \quad n \ge 1.$$
(9) Prove that if a power series $\sum a_n z^n$ converges to some function $f\colon \mathbb{C} \to \mathbb{C}$ uniformly in $\mathbb{C}$, then $a_n = 0$ for all but finitely many n, hence f must be a polynomial.