Math H54 Final Exam (Practice Exam) December 17, 2019

Name: Solution

- You have 170 minutes to complete the exam (3:10pm – 6:00pm).
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- For the proofs, make sure your arguments are as clear as possible. If you want to use theorems, you must write the name of the theorem or state the precise result you are using. Exception: if you are asked to prove a theorem, you are not allowed to use it!
- Do not detach pages from this exam packet or unstaple the packet.
- In case of an emergency, please follow the instructions of the instructor. In any situation, you are not allowed to leave the room with your exam packet.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. (2 points each) Determine if each statement is TRUE or FALSE. Give a short justification for your answer.

(a) $\det(A+B) = \det(A) + \det(B)$ for square matrices A, B of the same size.

$\boxed{\text{False}}$

$$\det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

(b) There is an orthogonal matrix A has 2 as an eigenvalue.

$\boxed{\text{False}}$ $\langle Av, Av\rangle = \langle v, v\rangle$ for any v. Since A is orthogonal.

Suppose $Av = \lambda v$, then $\lambda^2 \|v\|^2 = \|v\|^2$.

Hence 2 can't be an eigenvalue.

(c) Let A be an $m \times n$ matrix and suppose there is a matrix C such that $AC = I_m$. Then $A\vec{x} = \vec{b}$ is consistent for any $\vec{b} \in \mathbb{R}^m$.

$\boxed{\text{True}}$ $\forall \vec{b} \in \mathbb{R}^m$, take $\vec{x} = C\vec{b}$.

Then $A\vec{x} = AC\vec{b} = I_m\vec{b} = \vec{b}$.

(d) For any second order homogeneous linear differential equation $y'' + by' + cy = 0$ (b, c are constant), there is a unique solution $y(t)$ satisfying $y(0) = y(1) = 0$.

$\boxed{\text{False}}$ Consider $y'' + \pi^2 y = 0$.

Both $y(t) \equiv 0$ and $y(t) = \sin(\pi t)$ are sol'ns satisfying $y(0) = y(1) = 0$.

(e) For any second order homogeneous linear differential equation $y'' + by' + cy = 0$ (b, c are constant), there is a unique solution $y(t)$ satisfying $y(1) = y'(1) = 0$.

$\boxed{\text{True}}$ This is the uniqueness theorem.

2. (10 points) Find a basis for the row space, column space, and the null space of

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{bmatrix}.$$

Row reduction:

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot

$$\mathrm{Row}(A) = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\rangle$$

$$\mathrm{Col}(A) = \left\langle \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} \right\rangle$$

$$\mathrm{Nul}(A) = \left\langle \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\rangle.$$

3. (10 points) Show that if λ is an eigenvalue of A, then it is an eigenvalue of A^T.

λ is an eigenvalue of A $\Leftrightarrow$ $\det(A - \lambda I) = 0$.

$\Updownarrow$ since $\det(B) = \det(B^T)$ $\forall B$

λ is an eigenvalue of A^T. $\Leftrightarrow$ $\det(A^T - \lambda I) = 0$ $\square$

4. (10 points) Suppose that A is an $m \times n$ matrix (not necessarily a square matrix) such that $\mathrm{Nul}(A) = \{\vec{0}\}$. Prove that $A^T A$ is invertible.

$A^T A$ is a square matrix. It's invertible iff $\mathrm{Nul}(A^T A) = \{\vec{0}\}$.

$$A^T A \vec{v} = \vec{0} \Rightarrow \underbrace{\vec{v}^T A^T A \vec{v}}_{= \langle A\vec{v}, A\vec{v} \rangle} = 0$$

$$\Rightarrow A\vec{v} = \vec{0}$$

$$\Rightarrow \vec{v} = \vec{0} \quad \text{since } \mathrm{Nul}(A) = \{\vec{0}\}. \qquad \square$$

5. (10 points) Let n be an odd number. Let A be an $n \times n$ matrix such that $a_{ij} = -a_{ji}$ for all $i, j = 1, \ldots, n$. Show that $\det(A) = 0$.

$$A = -A^T.$$

$$\Rightarrow \det(A) = (-1)^n \det(A^T) = -\det(A)$$

$$\Rightarrow \det(A) = 0. \quad \square$$

6. (10 points) Let $T : V \to V$ be a linear map and $\vec{v} \in V$ with the property that $T^{k-1}(\vec{v}) \neq 0$ but $T^k(\vec{v}) = 0$. Show that $\{\vec{v}, T(\vec{v}), \ldots, T^{k-1}(\vec{v})\}$ is a linearly independent set.

Suppose there is $a_0 \vec{v} + a_1 T(\vec{v}) + \cdots + a_{k-1} T^{k-1}(\vec{v}) = \vec{0}$.

$$\Rightarrow \vec{0} = T^{k-1}\left(a_0 \vec{v} + a_1 T(\vec{v}) + \cdots + a_{k-1} T^{k-1}(\vec{v})\right)$$

$$= a_0 T^{k-1}(\vec{v}). \quad \Rightarrow \underline{a_0 = 0}.$$

Similarly, we have

$$\vec{0} = T^{k-2}\left(a_1 T(\vec{v}) + \cdots + a_{k-1} T^{k-1}(\vec{v})\right)$$

$$= a_1 T^{k-1}(\vec{v}) \quad \Rightarrow \underline{a_1 = 0}.$$

Do this inductively, one gets $a_0 = \cdots = a_{k-1} = 0$. $\square$

7. (10 points) Suppose that A, B are $n \times n$ matrices that commute, i.e. $AB = BA$, and suppose that B has n distinct eigenvalues.

(a) Show that if $B\vec{v} = \lambda\vec{v}$, then $BA\vec{v} = \lambda A\vec{v}$.

(b) Show that every eigenvector of B is also an eigenvector of A.

(c) Show that A is diagonalizable.

(d) Show that AB is diagonalizable.

(a) $B\vec{v} = \lambda\vec{v} \Rightarrow BA\vec{v} = AB\vec{v} = A(\lambda\vec{v}) = \lambda A\vec{v}$. $\square$

(b) Since B has n distinct eigenvalues, each eigenspace of B is 1-dim'l.

Let $\vec{v}$ be an eigenvector of B. $B\vec{v} = \lambda\vec{v}$.

$\Rightarrow B(A\vec{v}) = \lambda(A\vec{v})$. $\Rightarrow$ $A\vec{v} \in \text{Span}\{\vec{v}\}$.

$\Rightarrow$ $\vec{v}$ is an eigenvector of A. $\square$

(c) The eigenbasis of B also gives an eigenbasis of A.

Hence A is diagonalizable. $\square$

(d) AB also shares the same eigenvectors as A and B.

Hence also diagonalizable. $\square$

Note: These statements are not true without the condition: $AB = BA$.

8. (10 points) Find a function $y(t)$ satisfying

$$y'' + 2y' + 2y = 0; \quad y(0) = 0, \quad y'(0) = 2.$$

Auxilliary eq'n: $r^2 + 2r + 2 = 0$. roots: $-1 \pm i$.

$\Rightarrow$ general sol'n: $y(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$

$0 = y(0) = c_1$

$\Rightarrow$ $y(t) = c_2 e^{-t} \sin t$, $y'(t) = c_2 \left(-e^{-t} \sin t + e^{-t} \cos t\right)$

$2 = y'(0) = c_2$.

Hence the sol'n is $y(t) = 2e^{-t} \sin t$. $\square$

9. (10 points) Find a vector-valued function $\vec{x}(t)$ satisfying

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \vec{x}(t); \quad \vec{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

The matrix has eigenvalues: 3 (mult. 2) and -3 (mult. 1)

$\underline{\lambda = 3}$: eigenspace $= \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\rangle$.

$\underline{\lambda = -3}$: eigen space $= \left\langle \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\rangle$.

$\Rightarrow$ general sol'n: $\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$\Rightarrow$ $c_1 = c_2 = c_3 = 1$.

Hence the sol'n is $\vec{x}(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -e^{-3t} \\ e^{3t} - e^{-3t} \\ e^{3t} + e^{-3t} \end{bmatrix}. \quad \square$$

10. (10 points) Find a function $u(x,t)$ satisfying

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \quad t > 0,$$
$$u(x,0) = x^2, \quad 0 < x < \pi.$$

By the method of separation of variables,

If $x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$. is the Fourier cosine series of x^2 on $(0,\pi)$

then the sol'n is $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-3n^2 t} \cos(nx)$

By computing $\int_0^{\pi} x^2 \cos(nx)\,dx$, we get:

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) \quad \text{for } 0 < x < \pi.$$

Hence $u(x,t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} e^{-3n^2 t} \cos(nx)$ is the sol'n $\square$