

HOMEWORK 7
MATH H54, FALL 2021

PART I (DUE OCTOBER 26, 11AM)

Some ground rules:

- Please submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW7**.
- The submission should be a **single PDF file**.
- Late homework will not be accepted/graded under any circumstances.
- Make sure the writing in your submission is clear enough. Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own, and acknowledge the students with whom you worked.
- **For True/False questions:** You have to prove the statement if your answer is "True"; otherwise, you have to provide an explicit counterexample and justification.
- You are allowed to use any result that is proved in the lecture. But if you would like to use other results, you have to prove it first before using it.

Problems:

- (1) Let V be a finite dimensional inner product space, and let $W \subseteq V$ be a subspace. Prove that the projection map

$$\text{proj}_W: V \rightarrow W, \quad \vec{v} \mapsto \text{proj}_W \vec{v}$$

is linear.

- (2) Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthonormal basis of an inner product space V . Let \vec{x} be any vector in V , and let θ_i be the angle between \vec{x} and \vec{v}_i for each i . Prove that

$$\cos^2 \theta_1 + \dots + \cos^2 \theta_n = 1.$$

- (3) Prove the uniqueness of QR decomposition: Suppose A is an $m \times n$ matrix with linearly independent columns. Prove that there exists a *unique* pair of matrices Q and R such that

- $A = QR$,
- Q is an $m \times n$ matrix with orthonormal columns,
- R is an $n \times n$ upper triangular matrix with positive entries on the diagonal.

- (4) Let A be an $m \times n$ matrix. Prove that its transpose A^T is the *unique* $n \times m$ matrix such that

$$\langle A\vec{x}, \vec{y} \rangle_{\mathbb{R}^m} = \langle \vec{x}, A^T \vec{y} \rangle_{\mathbb{R}^n} \quad \text{holds for any } \vec{x} \in \mathbb{R}^n \text{ and } \vec{y} \in \mathbb{R}^m.$$

(In fact, this is one of the main reasons that the transpose of a matrix is important.)

- (5) Let A be a real $n \times n$ matrix. Prove that A is anti-symmetric (i.e. $A = -A^T$) if and only if $\langle A\vec{v}, \vec{v} \rangle = 0$ for all $\vec{v} \in \mathbb{R}^n$.
- (6) Let $\vec{v}_1, \dots, \vec{v}_n$ be vectors in an inner product space V . Define the *Gram determinant* to be $G(\vec{v}_1, \dots, \vec{v}_n) := \det(\langle \vec{v}_i, \vec{v}_j \rangle)$, i.e. the determinant of the matrix with entry $\langle \vec{v}_i, \vec{v}_j \rangle$ in the (i, j) -th component. Prove that $G(\vec{v}_1, \dots, \vec{v}_n) \neq 0$ if and only if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a linearly independent set.
- (7) Describe all least-square solutions of the linear system

$$x + y = 1,$$

$$x + y = -1.$$

- (8) Apply the Cauchy–Schwarz inequality (Problem 7 in the last homework) to show that for any continuous function $f: [a, b] \rightarrow \mathbb{R}$, we have

$$\left(\frac{1}{b-a} \int_a^b f(x) dx \right)^2 \leq \frac{1}{b-a} \int_a^b f(x)^2 dx.$$

In other words, the square of the average value of $f(x)$ on $[a, b]$ does not exceed the average value of $f(x)^2$. (Hint: Plug in f and 1 into the Cauchy–Schwarz inequality. Consider a suitable inner product on $\mathcal{C}[a, b]$.)

PART II (EXTRA CREDIT PROBLEMS, DUE NOVEMBER 2, 11AM)

The following problem worths up to 2 extra points in total (out of 10), so you can potentially get 12/10 for this homework.

Please submit your solutions to this part of the homework via **Gradescope**, to the assignment **HW7_extra_credit**, which is separated from Part I.

- (1) Let $V = \mathcal{C}[-1, 1]$ be the vector space consisting of real-valued continuous functions on $[-1, 1]$, with inner product given by $\langle f, g \rangle := \int_{-1}^1 f(x)g(x)dx$. Consider the subspace

$$W = \left\{ f \in V : f(-x) = -f(x) \text{ for any } x \in [-1, 1] \right\} \subseteq V.$$

Find (with proof) the orthogonal complement of W in V .

- (2) Continue the notations in Problem (6) of Part I. Suppose $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of W , where W is a subspace of V . Let \vec{x} be any vector in V . Prove that

$$\|\vec{x} - \text{proj}_W \vec{x}\|^2 = \frac{G(\vec{x}, \vec{v}_1, \dots, \vec{v}_n)}{G(\vec{v}_1, \dots, \vec{v}_n)}.$$