(1) Let X be a set, and (f_n) be a sequence of functions $f_n\colon X \to \mathbb{R}$.

(a) Suppose that (f_n) converges to $f\colon X \to \mathbb{R}$ uniformly and each (f_n) is bounded. Prove that f is also bounded.

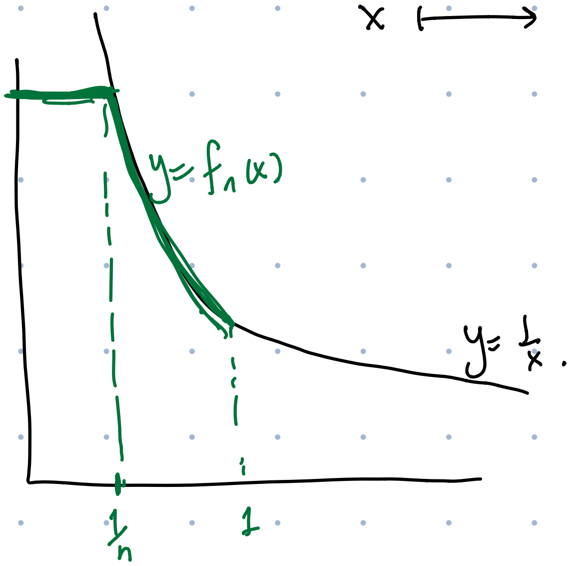
(b) Find an example of (f_n) converges to $f\colon X \to \mathbb{R}$ pointwisely and each (f_n) is bounded, but f is unbounded.

(a)
- $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $|f_n(x) - f(x)| < \varepsilon$ $\forall n > N, x \in X$.
- Take $\varepsilon = 1$, $\exists n > 0$ s.t. $|f_n(x) - f(x)| < 1$ $\forall x \in X$.
- f_n is bounded, i.e. $\exists M > 0$ s.t. $|f_n(x)| < M$ $\forall x \in X$.
- Hence $|f(x)| \le |f(x) - f_n(x)| + |f_n(x)| < M + 1$ $\forall x \in X$. $\square$

(b) Consider $X = (0,1)$.

$$f_n : (0,1) \longrightarrow \mathbb{R}$$

$$x \longmapsto \begin{cases} n & \text{if } x \in (0, \frac{1}{n}], \\ \frac{1}{x} & \text{if } x \in [\frac{1}{n}, 1) \end{cases}$$



Then $f_n \longrightarrow f(x) = 1/x$ pointwise on $(0,1)$

Each f_n is bounded on $(0,1)$.

but f is not. $\square$

(2) Let X be a set, and (f_n) be a sequence of functions $f_n\colon X \to \mathbb{R}$. Prove that if (f_n) converges to some function $f\colon X \to \mathbb{R}$ uniformly, then (f_n) is uniformly Cauchy.

- Since $f_n \longrightarrow f$ uniformly, $\forall \varepsilon > 0$, $\exists N > 0$ s.t.
$$|f_n(x) - f(x)| < \varepsilon/2 \quad \forall n > N,\ x \in X.$$
- Hence, $|f_n(x) - f_m(x)| \le |f_n(x) - f(x)| + |f_m(x) - f(x)| < \varepsilon$
$\forall n > N$, $x \in X$. $\square$

(3) Let X be a set. Consider the set $\mathcal{B}(X)$ consisting of real-valued *bounded* functions $f: X \to \mathbb{R}$. For $f_1, f_2 \in \mathcal{B}(X)$, define

$$d(f_1, f_2) := \sup_{x \in X} |f_1(x) - f_2(x)|.$$

Prove that $(\mathcal{B}(X), d)$ is a metric space.

- It's clear that $d(f_1, f_2) \geq 0$ and $d(f_1, f_2) = 0$ if and only if $f_1 = f_2$. It's also clear that $d(f_1, f_2) = d(f_2, f_1)$.

- Claim: $d(f_1, f_2) + d(f_2, f_3) \geq d(f_1, f_3)$. $\forall f_1, f_2, f_3 \in B(X)$.

pf: $\forall \varepsilon > 0$, $\exists x \in X$ s.t. $|f_1(x) - f_3(x)| \geq d(f_1, f_3) - \varepsilon$

Hence $d(f_1, f_3) - \varepsilon \leq |f_1(x) - f_3(x)| \leq |f_1(x) - f_2(x)| + |f_2(x) - f_3(x)|$
$\leq d(f_1, f_2) + d(f_2, f_3)$.

holds $\forall \varepsilon > 0$.

Hence $d(f_1, f_3) \leq d(f_1, f_2) + d(f_2, f_3)$. $\square$

(4) Consider the sequence of functions (f_n) defined by $f_n(x) = \frac{nx}{1+nx}$ for $x \geq 0$.
(a) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \geq 0$.
(b) Let $a > 0$. Prove or disprove: (f_n) converges uniformly to f on $[a, \infty)$.
(c) Prove or disprove: (f_n) converges uniformly to f on $[0, \infty)$.

(a) $f(x) = \begin{cases} 0 & \text{if } x = 0. \\ 1 & \text{if } x > 0. \end{cases}$

(b) Yes. $\forall \varepsilon > 0$, Take $N > 0$ large s.t. $\frac{1}{1+Na} < \varepsilon$.

Then $\forall x \in [a, \infty)$, $n > N$, we have:

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+nx} - 1\right| = \frac{1}{1+nx} \leq \frac{1}{1+na} < \varepsilon. \quad \square$$

(c) No. If $f_n \to f$ uniformly, then f should be continuous. $\square$

(5) Let X be a compact metric space, and (f_n) be a sequence of continuous functions $f_n \colon X \to \mathbb{R}$. Suppose that

- (f_n) converges pointwisely to a continuous function $f \colon X \to \mathbb{R}$.
- $f_{n+1}(x) \leq f_n(x)$ for any $x \in X$ and $n \in \mathbb{N}$.

Prove that (f_n) converges uniformly to f on X.
(Hint: Define $g_n := f_n - f$. Consider the set $E_n := \{x \in X : g_n(x) < \epsilon\}$. Show that $E_1 \subset E_2 \subset E_3 \subset \cdots$ and that $X = \cup E_n$.)

- $\forall x \in X$, $(f_n(x))$ is decreasing and conv. to $f(x)$, hence $f(x) = \inf_{n \geq 1} \{f_n(x)\}$.
- Define $g_n := \underbrace{f_n - f}_{\text{conti. fcn.}}$. Then $g_n(x) \geq g_{n+1}(x) \geq 0$. $\forall n, \forall x \in X$.
- $\forall \varepsilon > 0$, define

$$E_n := \{x \in X \mid g_n(x) < \varepsilon\} \subseteq X.$$

  - $\underline{E_n \text{ is open}}$: since g_n is conti, and $E_n = g_n^{-1}((-\infty, \varepsilon))$.
  - $\underline{X = \bigcup_{n=1}^{\infty} E_n}$: since $\lim_{n \to \infty} g_n(x) = 0 \quad \forall x \in X$.

- Since X is cpt., the open cover $\{E_n\}$ has a <u>finite subcover.</u>
- Since $g_n(x) \geq g_{n+1}(x) \geq 0 \quad \forall n, \forall x \in X$, we have:

$$\underline{E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots}$$

$\Longrightarrow$ $X = E_N$ for some $N > 0$.

$\Longrightarrow$ $|f_n(x) - f(x)| < \varepsilon \quad \forall x \in X, n \geq N.$ $\square$