#1: Find the QR decomp. of $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

via the Gram-Schmidt process.

Sol^{n}: ① normalize the 1st column: $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$

② $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ← a unit vector.

③ $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left\langle \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$

normalize $\rightsquigarrow \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \\ \sqrt{2/3} \end{bmatrix}$.

Hence $Q = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{6} \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2/3} \end{bmatrix}$, and

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & \sqrt{2/3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{3/2} \end{bmatrix}. \square$$

#2: Let $V = \mathcal{C}[-1,1]$ with $\langle f, g\rangle := \int_{-1}^{1} f(x)\, g(x)\, dx$.

Let $W = \mathrm{Span}\{1, x+1\} \subseteq V$.

Compute $\mathrm{proj}_W(x^3 + x^2)$

<u>Sol</u>^{n}: W has an <u>orthogonal</u> basis $\{1, x\}$.

Hence

$$\mathrm{proj}_W(x^3) = \frac{\cancelto{0}{\langle x^3, 1\rangle}}{\langle 1, 1\rangle} \cdot 1 + \frac{\langle x^3, x\rangle}{\langle x, x\rangle}\, x$$

$$= \left(\frac{\int_{-1}^{1} x^4\, dx}{\int_{-1}^{1} x^2\, dx} \right) x = \frac{3}{5} x.$$

$$\mathrm{proj}_W(x^2) = \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} \cdot 1 + \frac{\cancelto{0}{\langle x^2, x\rangle}}{\langle x, x\rangle}\, x$$

$$= \frac{\int_{-1}^{1} x^2 \, dx}{\int_{-1}^{1} 1 \, dx} = \frac{1}{3}.$$

Hence $\text{proj}_W(x^3+x^2) = \frac{3}{5}x + \frac{1}{3}$. $\square$

Note: $\{1, x+1\}$ is NOT orthogonal!!

So

$$\text{proj}_W(x^3+x^2) \neq \frac{\langle x^3+x^2, 1\rangle}{\|1\|^2} 1 + \frac{\langle x^3+x^2, x+1\rangle}{\|x+1\|^2}(x+1).$$