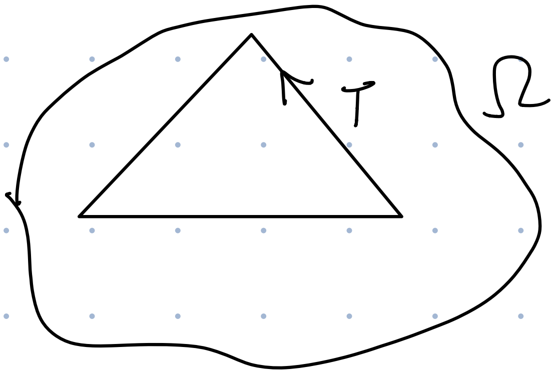
Thm (Goursat)

T: triangle in $\mathbb{C}$

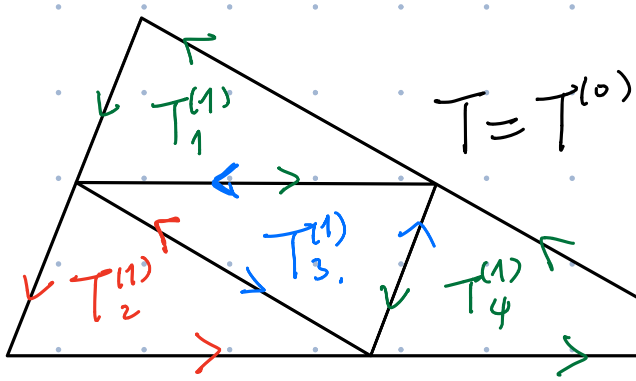
$\Omega \subseteq \mathbb{C}$ open, contains T & its interior

$f: \Omega \to \mathbb{C}$ holo. fn.



Then $\int_T f(z)\,dz = 0.$

pf:



$$\int_{T_1^{(1)}} + \int_{T_2^{(1)}} + \int_{T_3^{(1)}} + \int_{T_4^{(1)}} f = \int_{T^{(0)}} f$$

$$\Longrightarrow \left|\int_{T^{(0)}} f(z)\,dz\right| = \left|\sum_{i=1}^{4} \int_{T_i^{(1)}} f(z)\,dz\right|$$

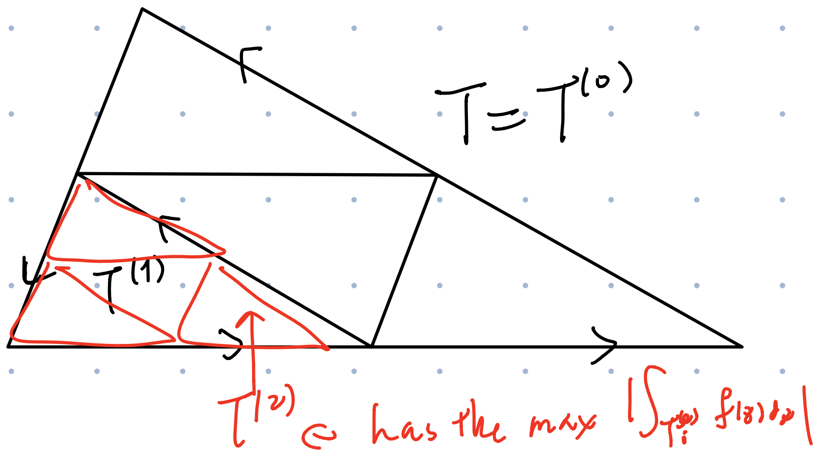
$$\leq \sum_{i=1}^{4} \left|\int_{T_i^{(1)}} f(z)\,dz\right|$$

$$\leq \max_{1\leq i\leq 4} \left|\int_{T_i^{(1)}} f(z)\,dz\right| \cdot 4$$

$T^{(1)}$

$\|$

Say $T_i^{(1)}$ achieves the $\uparrow$



$$\left|\int_{T^{(1)}} f(z)\,dz\right| \leq 4 \left|\int_{T^{(2)}} f(z)\,dz\right|$$

Continue this process indefinitely we obtain

- $T^{(0)}, T^{(1)}, T^{(2)}, \ldots\ldots$
- diameter of $T^{(n)} := \max\limits_{x,y \in T^{(n)} \cup \text{Interior of } T^{(n)}} |x-y| = d^{(n)} = \frac{1}{2^n} d^{(0)}$

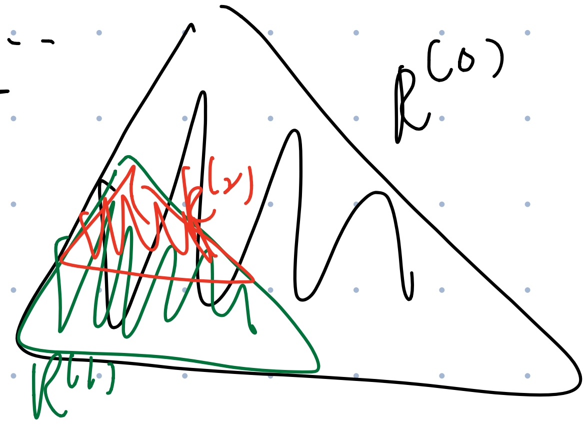
- perimeter of $T^{(n)} :=$ total length of $T^{(n)}$ $= p^{(n)} = \frac{1}{2^n} p^{(0)}$
- $\left|\int_{T^{(0)}} f(z)\,dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)\,dz\right| \quad \forall n$

Denote the $T^{(n)} \cup \text{Interior } T^{(n)} = R^{(n)}$

$R^{(0)} \supseteq R^{(1)} \supseteq R^{(2)} \supseteq \cdots$

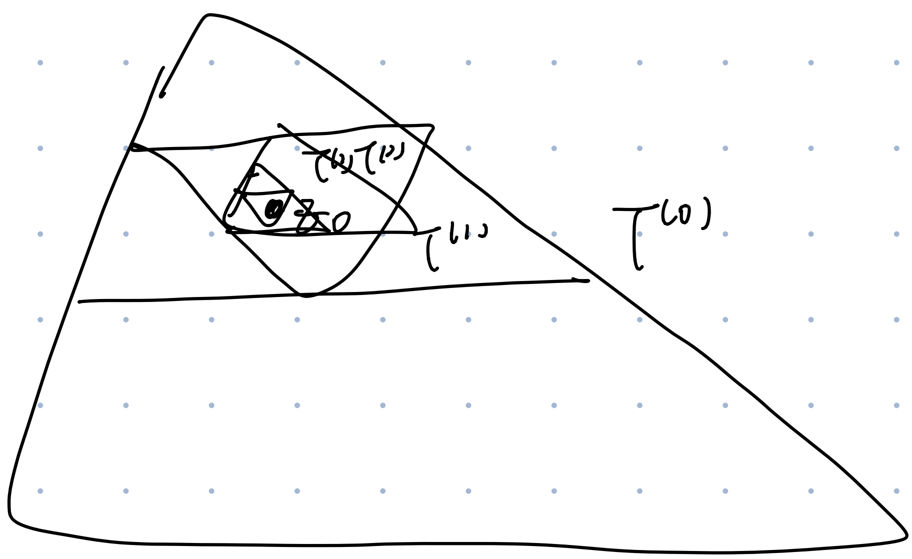
decreasing seq. of cpt subsets in $\mathbb{C}$,

where $\operatorname{diam}(R^{(n)}) \to 0$ as $n \to \infty$



$\Longrightarrow \exists!\ z_0 \in \mathbb{C}$ s.t. $z_0 \in R^{(n)}$ for any n.

(Idea: $\forall n$, pick any $a_n \in R^{(n)}$, we can show that (a_n) is a Cauchy seq.)



We'll use f is holo. (at z_0) to bound $\left|\int_{T^{(n)}} f(z)\,dz\right|$ for n large.

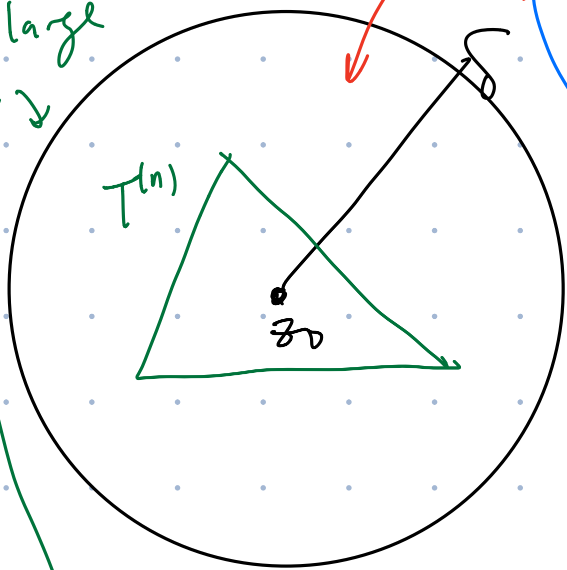
$\forall \varepsilon > 0, \exists \delta > 0$ s.t. for any $0 < |z - z_0| < \delta$, we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon.$$

$$\Rightarrow \left| f(z) - f(z_0) - (z-z_0) f'(z_0) \right| < \varepsilon |z - z_0|$$

satisfied by any pt in $\mathbb{D}_\delta(z_0)$

$\exists n$ large st.



$$\boxed{\varepsilon \cdot d^{(n)} \cdot p^{(n)}}$$
$$\geq$$
$$\varepsilon \cdot \sup_{z \in T^{(n)}} |z - z_0| \cdot \text{length}(T^{(n)})$$
$$\geq$$
$$\sup_{z \in T^{(n)}} \left| f(z) - f(z_0) - (z-z_0) f'(z_0) \right| \cdot \text{length}(T^{(n)})$$
$$\geq$$

$$\left| \int_{T^{(n)}} f(z)\,dz \right| \leq \left| \int_{T^{(n)}} f(z) - f(z_0) - (z-z_0) f'(z_0)\,dz \right| + \left| \int_{T^{(n)}} \boxed{\left(f(z_0) + (z-z_0) f'(z_0) \right)} dz \right|$$

why? $= 0$ $\quad$ is primitive $= \left(f(z_0) z + f'(z_0) \frac{1}{2} (z - z_0)^2 \right)'$

$$\left| \int_{T^{(0)}} f(z)\,dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z)\,dz \right|$$

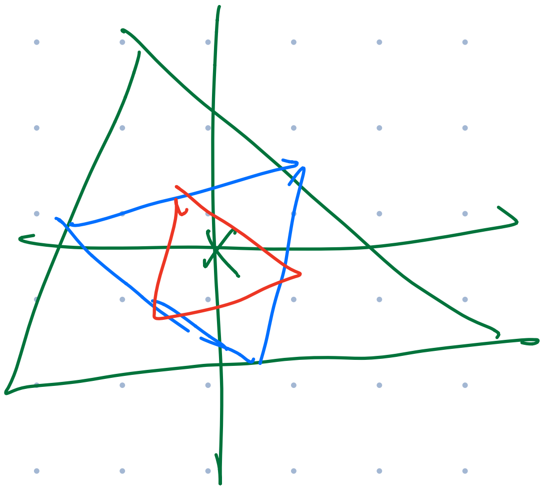
$$\leq 4^n \cdot \varepsilon \cdot \underbrace{d^{(n)}}_{= \frac{1}{2^n} d^{(0)}} \cdot \underbrace{p^{(n)}}_{= \frac{1}{2^n} p^{(0)}} = \varepsilon d^{(0)} p^{(0)}$$

n large st. $T^{(n)} \subseteq \mathbb{D}_\delta(z_0)$

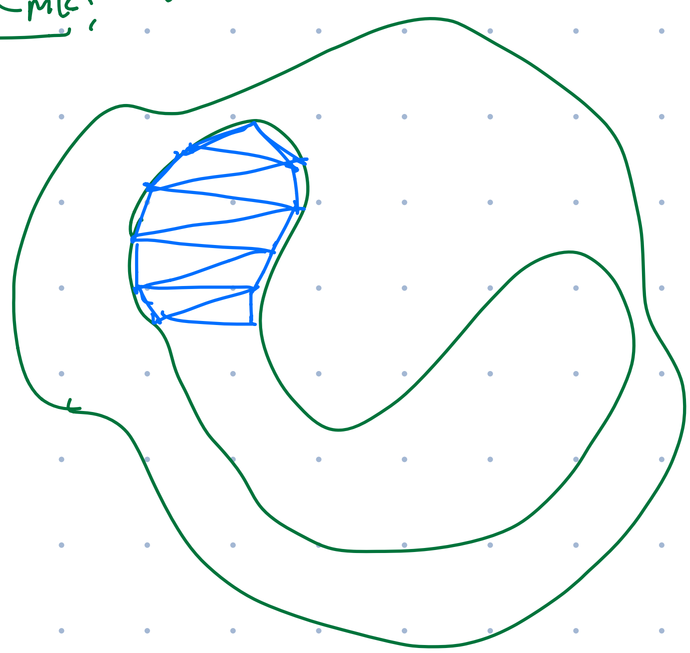
$$\Rightarrow \left|\int_{T^{(0)}} f(z)\,dz\right| \le \varepsilon\, d^{(0)} p^{(0)} \quad \forall \varepsilon>0$$

$$\Rightarrow \int_{T^{(0)}} f(z)\,dz = 0. \qquad \square$$

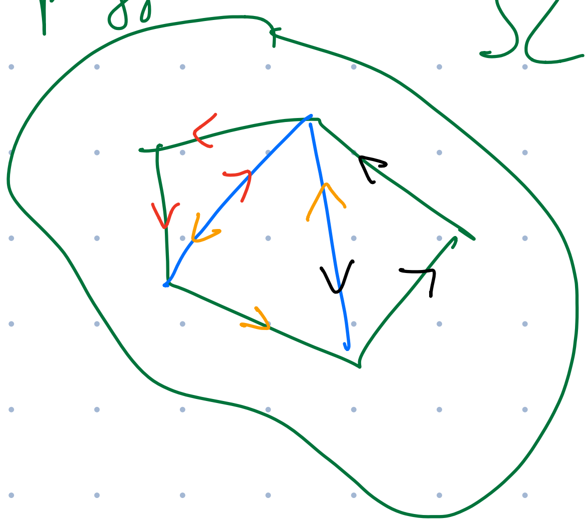
Rmk: $\frac{1}{z}$ $\quad \Omega = \mathbb{C}\setminus\{0\} \xrightarrow[\text{holo.}]{1/z} \mathbb{C}$



Rmk: general case. Ω

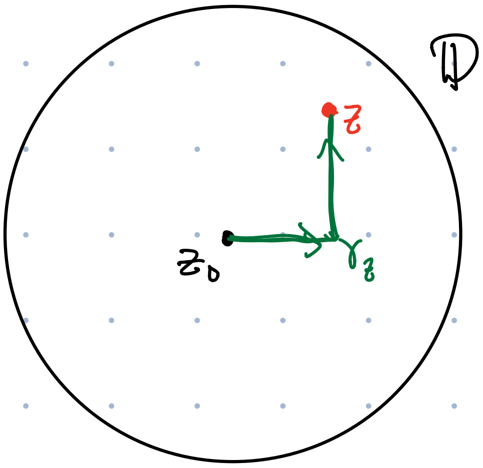


Rmk: polygon. Ω



Thm. f: holo. fcn. in $\mathbb{D}$ — an open disk.
then f has primitive in $\mathbb{D}$. (i.e. $\exists\, F: \mathbb{D} \to \mathbb{C}$ holo. s.t. $F' = f$)

pf:



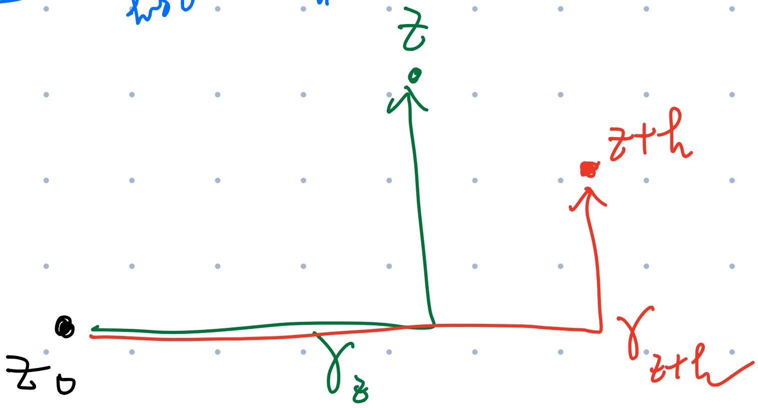
Construct F explicitly as follows:

Define
$$F: \mathbb{D} \longrightarrow \mathbb{C}$$
by
$$F(z) := \int_{\gamma_z} f(w)\, dw$$

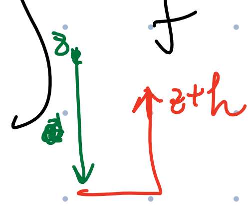
Claim: F is holo. & $F' = f$.

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\left(\int_{\gamma_{z+h}} f - \int_{\gamma_z} f\right)$$

Goal: $\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$

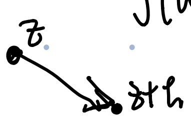


$$= \frac{1}{h}\left(\int f\right)$$



$$= \frac{1}{h}\left(\int f + \int f\right)$$

$$= \frac{1}{h}\int f(w)\, dw$$



$$[0,1] \longrightarrow \mathbb{C}$$
$$t \longmapsto z+th.$$

$$= \frac{1}{\cancel{h}} \int_0^1 f(z+th) \cdot \cancel{h} \, dt$$

$$= \int_0^1 f(z+th) \, dt.$$

Goal: $\boxed{\lim_{h \to 0} \int_0^1 f(z+th) \, dt = f(z)}$

Need: $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|h| < \delta$
$\Rightarrow \left| \int_0^1 f(z+th)\,dt - f(z) \right| < \varepsilon$

$$\left| \int_0^1 f(z+th)\,dt - f(z) \right| = \left| \int_0^1 \big(f(z+th) - f(z)\big)\,dt \right|$$

$$\leq \int_0^1 |f(z+th) - f(z)|\,dt$$

(Since f is conti. at z, so $\forall \varepsilon > 0$,
$\exists \delta > 0$ s.t. $|h| < \delta \Rightarrow |f(z+h) - f(z)| < \varepsilon$)

$\Rightarrow$ if $|h| < \delta$, then $\int_0^1 |f(z+th) - f(z)|\,dt$

$|th| < \delta \ \forall t \in [0,1]$

$$\leq \int_0^1 \varepsilon \, dt = \varepsilon. \quad \square$$

Rmk: In the construction of ~~the~~ primitive F on $\mathbb{D}$, we only used

- $\int_{\triangle} f = 0$
- f is conti.

Rmk: The statement is not true for an arbitrary open set Ω

e.g. $\Omega = \mathbb{C} \setminus \{0\}$, $f(z) = \frac{1}{z}$ holo. on Ω

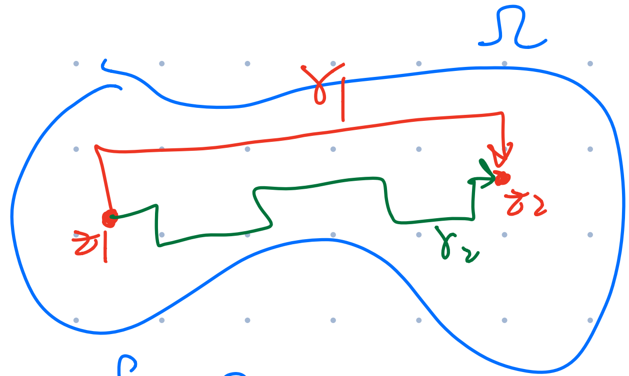
But f has no primitive on Ω.

"any holo. fcn. on Ω has a primitive" ← property of fcns on Ω

$\Updownarrow$

" $\forall z_1, z_2 \in \Omega$, $\forall$ holo. fcn f on Ω, $\forall$ curves γ_1, γ_2 connecting z_1, z_2 in Ω

we have $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ "



$\Updownarrow$

" $\forall$ closed curve γ in Ω, $\forall$ holo. fcn f on Ω,

$\int_{\gamma} f(z)\,dz = 0$ "

$\Updownarrow$

" $\forall$ closed curve γ in Ω, Ω contains the interior of γ "

$\Updownarrow$

" Ω is simply connected " ← property about topology of Ω

e.g. $\mathbb{C} \setminus \{0\}$ is NOT simply connected. b/c