3/3/2020

Last time: Three equiv. defns of conti. fcns $f: (X, d_X) \longrightarrow (Y, d_Y)$:

1) $\lim x_n = x_0 \Rightarrow \lim f(x_n) = f(x_0)$.
2) $\forall \varepsilon > 0, \forall x_0 \in X, \exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.
3) $U \subset Y$ open $\Rightarrow f^{-1}(U) \subset X$ open.

Rmk: 3) – conceptual, convenient for proving properties of conti. fcns.
1) 2) – hands on, check continuity of given fcn.

Thm $f: X \longrightarrow Y$ conti., $E \subset X$ cpt. Then $f(E) \subset Y$ cpt.

pf. $\forall$ open cover $\{U_\alpha\}$ of $f(E)$., $\Rightarrow$ $\{f^{-1}(U_\alpha)\}$ is an open cover of E.

E cpt $\Rightarrow$ there is finitely many $f^{-1}U_1, \dots, f^{-1}U_n$ that covers E.

$\Rightarrow f(E) \subset f(f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n)) \subset U_1 \cup \cdots \cup U_n$.

$f^{-1} f(F) \supset F$

So $\{U_\alpha\}$ admits a finite subcover of $f(E)$. □

Specialize to $Y = \mathbb{R}$: What are compact subsets in $\mathbb{R}$?

Heine-Borel thm: $K \subset \mathbb{R}^n$ compact $\Leftrightarrow$ closed and bounded.

HW # 1: $K \subset \mathbb{R}$ closed and bounded $\Rightarrow \sup K, \inf K \in K$

Corollary: (extreme value thm for compact ~~spaces~~ sets).

$f: (X, d) \longrightarrow \mathbb{R}$ conti., $E \subset X$ cpt. Then

1) f is bounded on E. (i.e. $\exists M > 0$ st. $|f(x)| < M \ \forall x \in E$).
2) f assumes its max and min on E. (i.e. $\exists x_1, x_2 \in E$ st. $f(x_1) \le f(x) \le f(x_2) \ \forall x \in E$).

pf By Thm, $f(E) \subset \mathbb{R}$ is cpt. $\Rightarrow$ closed and bounded. $\Rightarrow$ 1)

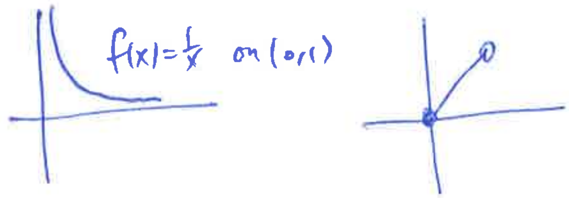
HW $\Rightarrow$ $\sup f(E) \in f(E)$, $\exists x_2 \in E$ st. $\sup f(E) = f(x_2)$
$\inf f(E) \in f(E)$, $\exists x_1 \in E$ st. $\inf f(E) = f(x_1)$ $\Rightarrow$ f attains max/min at x_2/x_1. □

Corollary (Ross 18.1) $f: [a,b] \longrightarrow \mathbb{R}$ conti. fcn. Then

1) f is bounded
2) f assumes its max. and min on $[a,b]$

pf: $[a,b]$ is a compact set. $\square$

Rmk. Not true for $f: (a,b) \longrightarrow \mathbb{R}$. non-compact domain



HW: The proof of Ross, Thm 18.1 is more complicated than ours.
Where's the hidden difficulties in our argument?

Def: $E \subset (X,d)$ is disconnected if $\exists\, U_1, U_2 \subset X$ open that "separates" E,
i.e.
1) $E \cap U_1 \neq \phi$, $E \cap U_2 \neq \phi$.
2) $E \subset U_1 \cup U_2$.
3) $(E \cap U_1) \cap (E \cap U_2) = \phi$.

Otherwise, $E \subset X$ is called connected.

e.g. $E = [0,1) \cup (1,2] \subset \mathbb{R}$ disconnected? $U_1 = (-1,1)$, $U_2 = (1,3)$.

~~e.g.~~ HW: $E \subset \mathbb{R}$ is connected $\iff$ E is an interval. (see Ross, 22 for proof)

Thm: $f: (X, d_X) \longrightarrow (Y, d_Y)$ conti, ~~connected.~~ $E \subset X$ conn. $\Rightarrow f(E) \subset Y$ conn.

pf Suppose $f(E) \subset Y$ disconn, $\exists\, U_1, U_2 \subset Y$ s.t.
1) $f(E) \subset U_1 \cup U_2$.
2) $f(E) \cap U_1 \neq \phi$, $f(E) \cap U_2 \neq \phi$.
3) $(f(E) \cap U_1) \cap (f(E) \cap U_2) = \phi$.

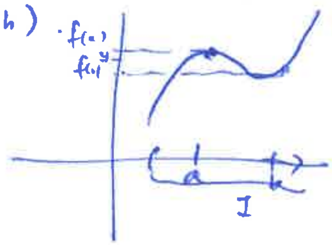
Let $V_1 = f^{-1}(U_1) \subset X$
$V_2 = f^{-1}(U_2) \subset X$

Check V_1, V_2 open, separates E. $\square$

Coro. (Intermediate Value Thm) $f: I \to \mathbb{R}$ conti., ($I \subset \mathbb{R}$ interval)

Then $\forall\, a, b \in I$, $a < b$, $\forall\, y$ between $f(a)$ and $f(b)$,

$\exists\, x \in [a,b]$ s.t. $f(x) = y$



pf $[a,b] \subset I$ is connected.

$\Rightarrow f([a,b])$ is connected. $\Rightarrow$ is an interval.

$f(a), f(b) \in f([a,b]) \Rightarrow y \in f([a,b])$.

$\Rightarrow \exists\, x \in [a,b]$ s.t. $f(x) = y$. $\square$

Rmk. IVT is very useful to prove the existence of zeros of a conti. fcn.

e.g. Any conti. fcn. on $[0,1]$ to itself has a fixed pt. i.e.,

if $f: [0,1] \to [0,1]$ is conti., then $\exists\, x_0 \in [0,1]$ s.t. $f(x_0) = x_0$.

pf Consider $g(x) = f(x) - x$. WTS: $g(x)$ has zero on $[0,1]$.

Suppose not. $0 \neq g(0) = f(0) - 0 \geq 0 \Rightarrow g(0) > 0$.

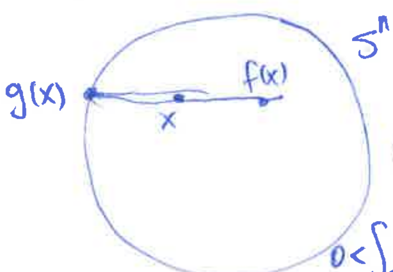
$0 \neq g(1) = f(1) - 1 \leq 0. \Rightarrow g(1) < 0$.

g is conti., by IVT, $\exists\, x_0 \in (0,1)$ s.t. $g(x_0) = 0$. $\square$

Brouwer fixed pt Thm: Any conti. fcn. on $D^n = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$ to itself has a fixed pt.

Suppose $f: D^n \to D^n$ conti. has no fixed pt. $\Rightarrow \exists\, g: D^n \to S^{n-1}$ conti. and $g(x) = x \ \forall x \in S^{n-1}$

$S^{n-1} = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$.



$$\Rightarrow \begin{tikzcd} S^{n-1} \arrow[r, hook, "i"] \arrow[rr, bend right, "id"'] & D^n \arrow[r, "g"] & S^{n-1} \end{tikzcd}$$

① Stokes thm:

ω: vol. form on S^{n-1}

$0 < \int_{S^{n-1}} \omega = \int_{S^{n-1}} i^* g^* \omega \underset{\text{Stokes}}{=} \int_{D^n} d(g^* \omega)$

$= \int_{D^n} g^*(d\omega) = 0$.

② Topology (homology gp).

$$\begin{tikzcd} H_{n-1}(S^{n-1}, \mathbb{Z}) \arrow[r, "i_*"] \arrow[rr, bend right] \arrow[d, equal] & H_{n-1}(D^n, \mathbb{Z}) \arrow[r, "g_*"] \arrow[d, equal] & H_{n-1}(S^{n-1}, \mathbb{Z}) \arrow[d, equal] \\ \mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$

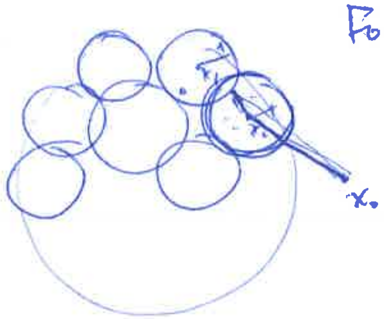
Thm $E \subset (X, d)$ compact. Then E is closed and bounded.

$\uparrow$ $\exists\, x_0 \in X$ and $R > 0$ s.t. $E \subset B_{x_0}(R)$.

pf ① Bounded:

$\forall x \in E$, Consider $\mathcal{U}_x = B_1(x) \subset X$. $\{\mathcal{U}_x\}_{x\in E}$ is an open cover of E.

E cpt. $\Rightarrow$ $E \subset (\mathcal{U}_{x_1} \cup \cdots \cup \mathcal{U}_{x_n})$ for some $x_1, \dots, x_n \in E$.



For any $x_0 \in X$. Consider

$$R = \max\{d(x_0, x_1), \dots, d(x_0, x_n)\} + 1.$$

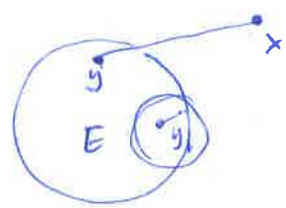
Claim: $E \subset B_{x_0}(R)$.

$\forall x \in E$, $\exists\, x_i$ $1 \le i \le n$ st. $E \in \mathcal{U}_{x_i}$

$\Rightarrow d(x, x_0) \le d(x, x_i) + d(x_i, x_0) < 1 + d(x_i, x_0) \le R$. □

② Closed: "$E^c \subset X$ is open"

$x \in E^c$:



$\forall y \in E$, let $r_y := \frac{1}{2} d(x, y) > 0$.

Then $B_{r_y}(x)$ doesn't contain y.

"If we take $r := \inf\{r_y : y \in E\}$. then $B_r(x) \cap E = \phi$. $\Rightarrow$ E^c is open."

What's wrong with this argument?

$\inf\{r_y : y \in E\}$ could be 0. e.g. $E = (0,1)$, $x = 1$

We need to use the compactness of E!!

$\{B_{r_y}(y)\}_{y\in E}$ is an open cover of E. E cpt $\Rightarrow$ $E \subset (B_{r_{y_1}}(y_1) \cup \cdots \cup B_{r_{y_n}}(y_n))$

Let $r = \min\{r_{y_1}, \dots, r_{y_n}\} > 0$. Claim: $B_r(x) \cap E = \phi$.

$\forall y \in E$, $y \in B_{r_{y_i}}(y_i)$ for some i,

$d(x,y) \ge d(x, y_i) - d(y_i, y) \ge 2 r_{y_i} - r_{y_i} = r_{y_i} \ge r$. $\Rightarrow y \notin B_r(x)$. □