Today: Proof of Riemann mapping thm.

Montel's thm: (proved last time): If $\mathcal{F}$ is a family of holo. fcns on Ω, and unif. bdd. on every cpt subset of Ω, then $\mathcal{F}$ is normal, i.e. $\forall$ seq. $\{f_n\}$ in $\mathcal{F}$, $\exists$ subseq. $\{f_{n_k}\}$ that unif. conv. on every cpt subset of Ω.

Thm $\{f_n : \Omega \to \mathbb{C}\}$ holo. injective.

Suppose $f_n \to f$ unif. on every cpt subset $K \subseteq \Omega$.

(Then f is holo.)

$\Longrightarrow$ f is either injective or a constant fcn.

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pf Assume f is not injective. $\boxed{\text{Want: } f \text{ is a const fcn.}}$

Say $z_1 \neq z_2$ in Ω where $f(z_1) = f(z_2)$.

$g_n(z) := f_n(z) - f_n(z_1)$

- $g_n(z) \longrightarrow \underbrace{f(z) - f(z_1)}_{\parallel}$ as $n \to \infty$

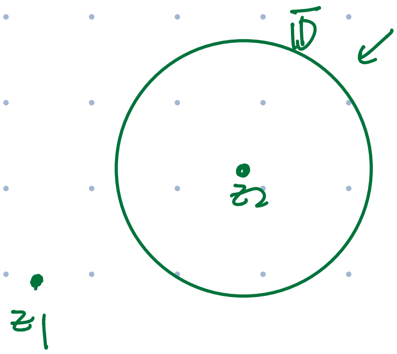
  $g(z)$ holo, has 2 zeros at z_1, z_2.

- $g_n(z_1) = 0 \quad \forall n$.

- Since f_n inj., so g_n doesn't have zeros other than z_1.

Suppose g is not const. fcn, then z_2 is an isolated zero.

$\overline{\mathbb{D}}$ ← $g(z) \neq 0$ for $z \in \overline{\mathbb{D}} \setminus \{z_2\}$.



the order of zero at z_2 of g

$$\boxed{\frac{1}{2\pi i}\int_{\partial \mathbb{D}} \frac{g'(z)}{g(z)}\,dz} \geq 1.$$

$$\boxed{\frac{1}{2\pi i}\int_{\partial \mathbb{D}} \frac{g_n'(z)}{g_n(z)}\,dz = 0} \quad \forall n$$

Since g_n doesn't have any zero in $\overline{\mathbb{D}}$.

Ex: $\frac{g_n'}{g_n} \to \frac{g'}{g}$ unif. on $\underset{\text{cpt}}{\partial \mathbb{D}}$ (by using $f_n \to f$ unif. on every cpt subset of Ω)

$$\Rightarrow \underbrace{\frac{1}{2\pi i}\int_{\partial \mathbb{D}} \frac{g_n'}{g_n}\,dz}_{\substack{\| \\ 0 \ \forall n}} \to \underbrace{\frac{1}{2\pi i}\int_{\partial \mathbb{D}} \frac{g'}{g}\,dz}_{\substack{\vee\!| \\ 1}} \quad \text{as } n \to \infty$$

contradiction! $\square$

Proof of Riemann Mapping Thm.:

Ω = open, simply connected, connected, $\subsetneq \mathbb{C}$

① $\exists$ injective holo. map $\Omega \xrightarrow{f} \mathbb{D}$, where 0 is in the image of f.

(then Ω and $f(\Omega)$ are biholo.)

(so, we may assume $\boxed{\Omega \subseteq \mathbb{D} \text{ and } 0 \in \Omega}$)

② Consider

$$\mathcal{F} = \{ f : \Omega \to \mathbb{D} \mid f \text{ inj. holo., } f(0)=0 \} \neq \phi.$$

use the theorems that we just proved

$$\exists F \in \mathcal{F} \quad \text{s.t.} \quad |F'(0)| = \sup_{f \in \mathcal{F}} |f'(0)|$$

③ Such F is surjective.

(then $F : \Omega \to \mathbb{D}$ bij. holo, therefore biholo.)

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① Ω = open, simply connected, connected, $\neq \mathbb{C}$

- $\exists z_1 \in \Omega$, $z_2 \notin \Omega$.

  By dilation & translation, can assume $\underline{1 \in \Omega,\ 0 \notin \Omega}$.

- Ω simply connected, $0 \notin \Omega$, so $\underline{\mathrm{Log}_\Omega}$ is well-defined. inj. holo. on Ω.

$$\boxed{\mathrm{Log}_\Omega(1) = 0}$$

$$\Omega \xrightarrow{\mathrm{Log}_\Omega} \mathbb{C}$$

- $\forall z \in \Omega$, $\log_\Omega(z) \neq 2\pi i$.

  If $\log_\Omega(z) = 2\pi i$ for some z,

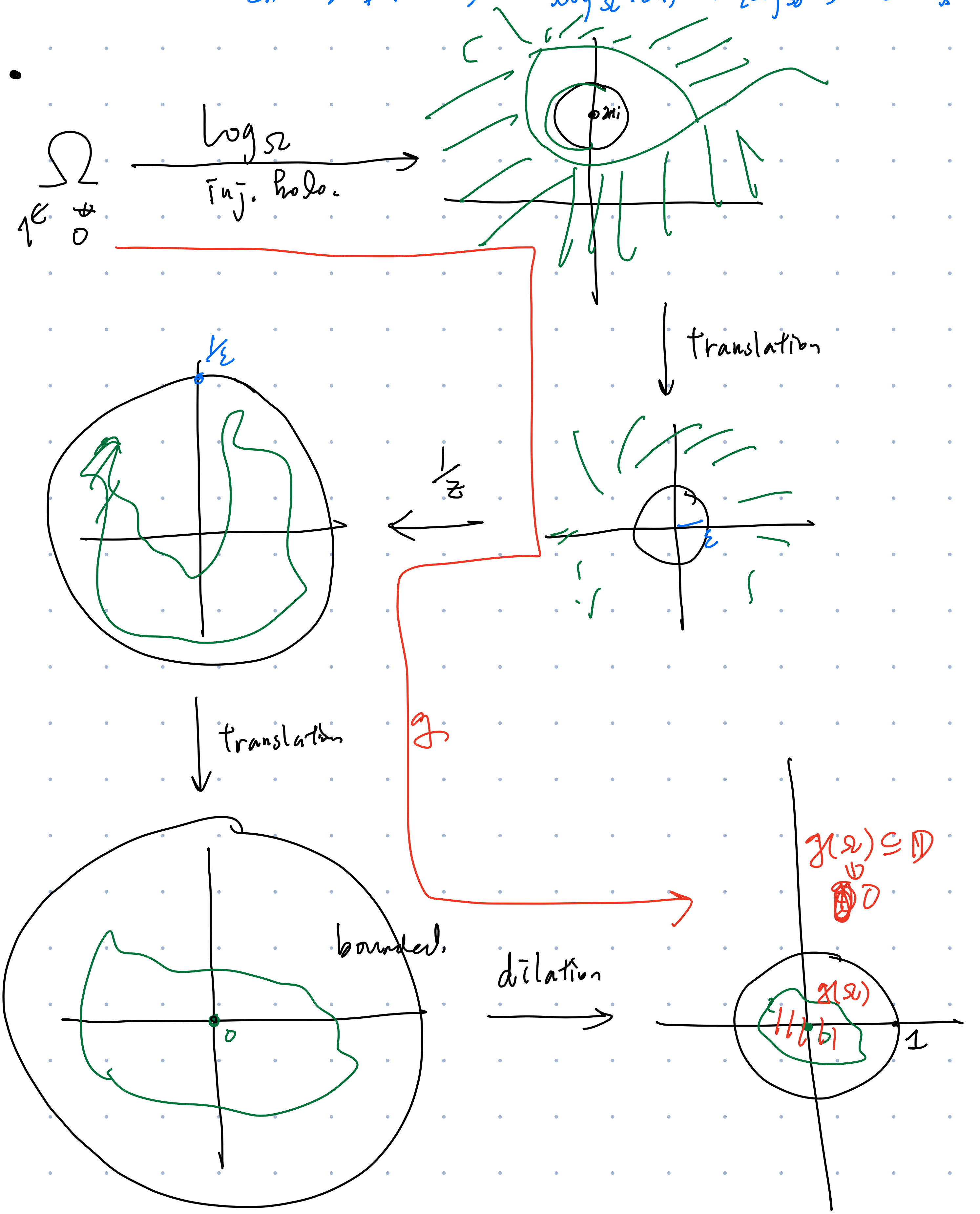
  then $z = e^{\log_\Omega(z)} = e^{2\pi i} = 1$,

  $\log_\Omega(1) = 0$. $\divideontimes$.

- In fact, $\exists \varepsilon > 0$ st. $\mathbb{D}_\varepsilon(2\pi i) \cap \log_\Omega(\Omega) = \phi$.

If not, $\exists \{z_n\} \subseteq \Omega_0$ s.t. $\log_\Omega(z_n) \to 2\pi i$.

$\Rightarrow z_n \to 1. \Rightarrow \log_\Omega(z_n) \to \log_\Omega(1) = 0.$ ↯



(2) $\Omega \subseteq \mathbb{D}$ and $0 \in \Omega$.

Consider

$$\mathcal{F} = \{ f : \Omega \to \mathbb{D} \mid f \text{ inj. holo., } f(0) = 0 \} \neq \phi.$$

b/c identity map on Ω is in $\mathcal{F}$.

$\exists F \in \mathcal{F}$ s.t. $|F'(0)| = \sup_{f \in \mathcal{F}} |f'(0)|$

- Let $s = \sup_{f \in \mathcal{F}} |f'(0)|$

  $\exists \{f_n\} \subseteq \mathcal{F}$ s.t. $|f_n'(0)| \to s$ as $n \to \infty$.

- Observe that $\mathcal{F}$ is a normal family, (since $\mathcal{F}$ is unif. bdd.)

  so, $\exists$ subseq. $\{f_{n_k}\}$ of $\{f_n\}$ s.t.

  (Montel's thm.) $f_{n_k} \to F$ conv. unif. on every cpt. subset of Ω.

  (We'll show that $F \in \mathcal{F}$, and $|F'(0)| = s$)

- $f_{n_k}' \to F'$ conv. unif. on every cpt. subset of Ω.

  $\Rightarrow |F'(0)| = \lim_{k \to \infty} |f_{n_k}'(0)| = s$.

- It remains to show that F is injective.

  By the thm we just proved, F is either injective or a const. fcn.

- $|F'(0)| = s = \sup_{f \in \mathcal{F}} |f'(0)| \geq 1$ since $id_\Omega : \Omega \to \Omega$, $z \mapsto z$ is in $\mathcal{F}$

  $id_\Omega'(0) = 1$.

$\Rightarrow$ F is not a const fcn.

$\Rightarrow$ F is injective. $\square$

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③ $\Omega \subseteq \mathbb{D}$ and $0 \in \Omega$.

Consider

$$\mathcal{F} = \{ f : \Omega \to \mathbb{D} \mid f \text{ inj. holo., } f(0) = 0 \}.$$

$\exists\, F \in \mathcal{F}$ s.t. $|F'(0)| = \sup_{f \in \mathcal{F}} |f'(0)|$

Claim F is surjective.

$F : \Omega \to \mathbb{D}$

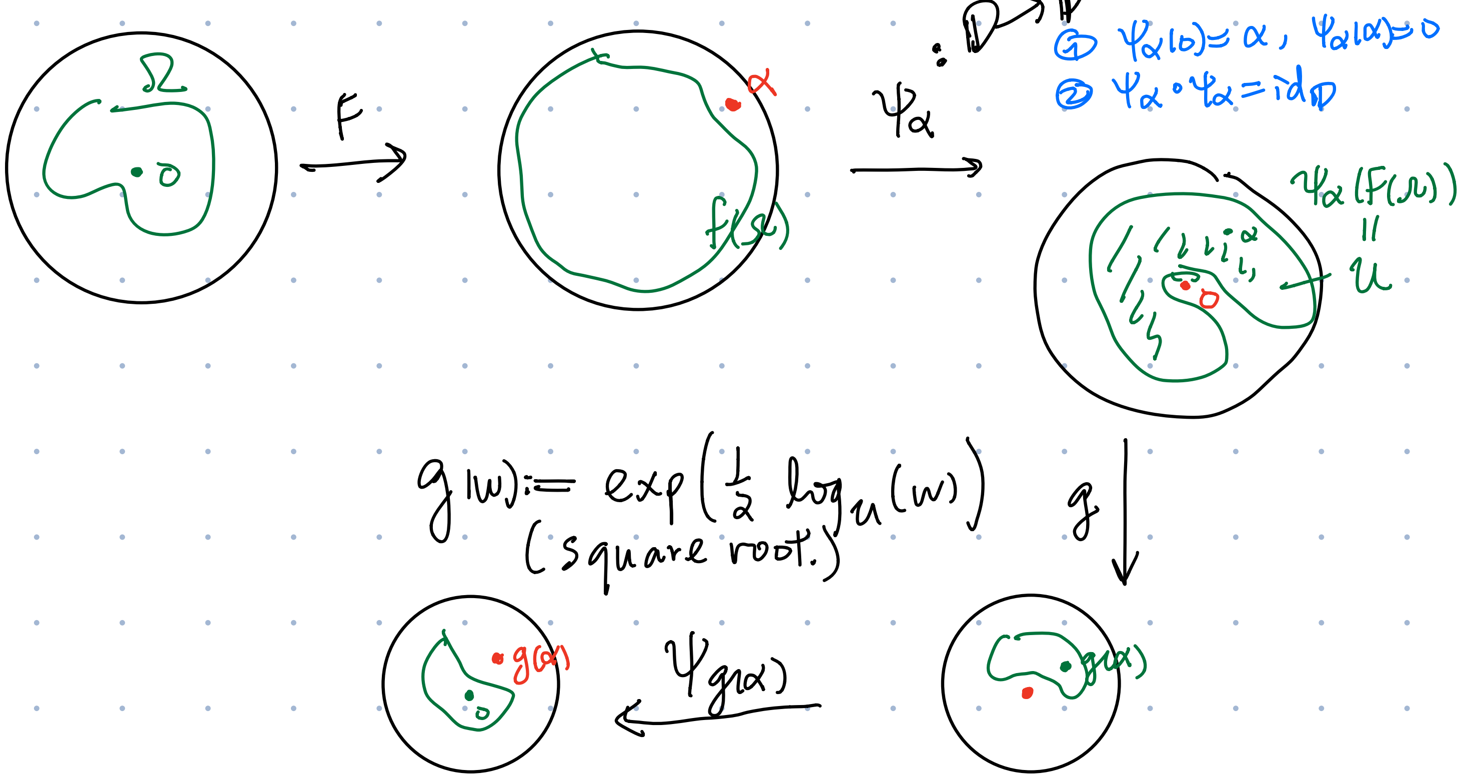
pf. Suppose such F is not surjective, i.e.,

- $\exists\, \alpha \in \mathbb{D}$ s.t. $\alpha \notin F(\Omega)$.

$\psi_\alpha : \mathbb{D} \to \mathbb{D}$ bi holo. $\frac{\alpha - z}{1 - \bar{\alpha} z}$

① $\psi_\alpha(0) = \alpha$, $\psi_\alpha(\alpha) = 0$

② $\psi_\alpha \circ \psi_\alpha = id_{\mathbb{D}}$



$g(w) := \exp\left(\frac{1}{2} \log_U(w)\right)$
(square root.)

$\tilde{F} := \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ F : \Omega \longrightarrow \mathbb{D}$

- $\tilde{F}(0) = 0 \Rightarrow \tilde{F} \in \mathcal{F}$

- Claim: $|\tilde{F}'(0)| > |F'(0)|$

pf: Consider $G := \psi_\alpha \circ (z^2) \circ \psi_{g(\alpha)} : \mathbb{D} \longrightarrow \mathbb{D}$

$\uparrow$ square.

Then $G \circ \tilde{F} = F$.

By Schwarz lemma (apply on G),

$|G'(0)| \leq 1$.

And If $|G'(0)| = 1$, then G is injective.

But G is not injective since (z^2) is not.

$\Rightarrow |G'(0)| < 1$.

$\Rightarrow |F'(0)| = |G'(0)||\tilde{F}'(0)| < |\tilde{F}'(0)|$.

Contradiction !!

$\square$

In the last 2 weeks, we'll discuss elliptic functions & modular functional forms.

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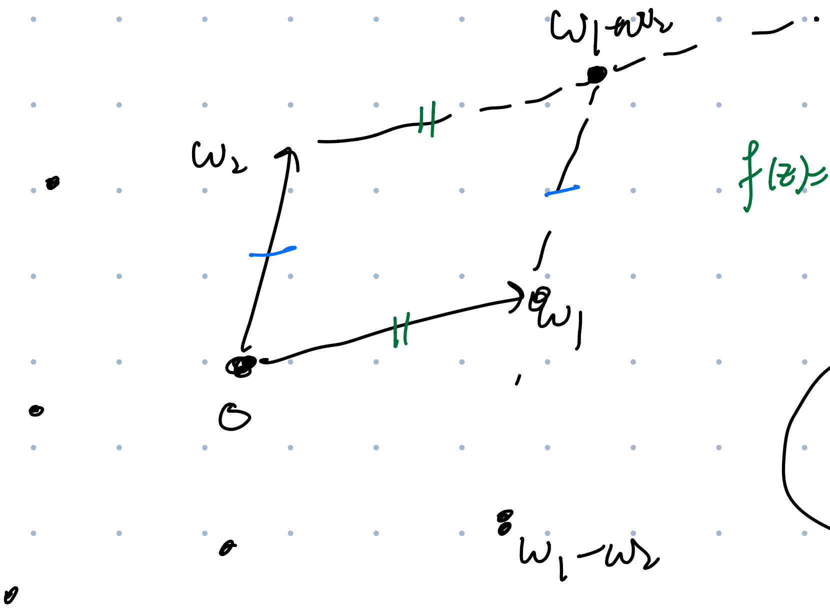
doubly periodic functions.

$\omega_1, \omega_2 \in \mathbb{C}$

$f(z) = f(z+\omega_1) = f(z+\omega_2) \quad \forall z \in \mathbb{C}$



elliptic curve.