Today: uncountable sets, measure zero sets, Cantor sets.

Recall, say a set A is countable if $\exists$ bijection $f: \mathbb{N} \to A$

- $|\mathbb{N}| < |\underbrace{P(\mathbb{N})}_{\uparrow \text{ uncountable.}}|$
- $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$

Thm: $\mathbb{R}$ is uncountable. (i.e. $|\mathbb{R}| > |\mathbb{N}|$)

pf. (Cantor's diagonal argument).

Want to show: any $\boxed{f: \mathbb{N} \longrightarrow \mathbb{R}}$ is not surjective.

$a_{ij} \in \{0, 1, \cdots, 9\}$

decimal expansion of a real number $f(1)$

$1 \longmapsto f(1) = \text{integer} . a_{11} a_{12} a_{13} a_{14} \cdots$

$2 \longmapsto f(2) = \cdots\cdots . a_{21} a_{22} a_{23} \cdots\cdots$

$3 \longmapsto f(3) \quad \cdots . a_{31} a_{32} a_{33} \cdots$

$4 \longmapsto f(4) \quad \cdots$

$\vdots$

Choose a real number

$r = 0 . b_1 b_2 b_3 \cdots\cdots$

s.t. $b_i \neq a_{ii} \quad \forall i$.

Then $r \neq f(i)$ for any i.

$0 . b_1 b_2 \cdots b_i \cdots\cdots \neq \cdots . a_{i1} a_{i2} \cdots a_{ii} \cdots$

$\square$

Thm: $|P(\mathbb{N})| = |\mathbb{R}|$.

pf: By Schröder-Berstein thm, it's enough to construct

injective maps from $\mathbb{P}(\mathbb{N})$ to $\mathbb{R}$
and from $\mathbb{R}$ to $\mathbb{P}(\mathbb{N})$

$(|\mathbb{P}(\mathbb{Q})| = |\mathbb{P}(\mathbb{N})|)$

① Construct an injective map from $\mathbb{R}$ to $\mathbb{P}(\mathbb{Q})$:

$$f: \mathbb{R} \longrightarrow \mathbb{P}(\mathbb{Q})$$
$$\underline{r} \longmapsto \{q \in \mathbb{Q} \mid q < r\}.$$

If $r \neq s$ in $\mathbb{R}$, say $r < s$.

need: $f(r) = \{q \in \mathbb{Q} \mid q < r\} \neq \{q \in \mathbb{Q} \mid q < s\} = f(s)$.

By denseness of $\mathbb{Q}$, $\exists\, \tilde{q}$ s.t. $r < \tilde{q} < s$

Then $\tilde{q} \in f(s)$, but $\tilde{q} \notin f(r)$

So $f(r) \neq f(s)$. □

② Construct an injective map from $\mathbb{P}(\mathbb{N})$ to $\mathbb{R}$:

$$g: \mathbb{P}(\mathbb{N}) \longrightarrow \mathbb{R}$$
$$\underset{\text{subset of } \mathbb{N}.}{S} \longmapsto \sum_{i=1}^{\infty} \frac{a_i}{3^i} = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots$$

where $a_i = \begin{cases} 2 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$

It's easy to check g is injective. □

e.g. $\underline{S = \{1,2,3\}} \longmapsto \boxed{\frac{2}{3} + \underline{\frac{2}{3^2} + \frac{2}{3^3}}} + \frac{0}{3^4} + \frac{0}{3^5} + \cdots$

Rmk: $|\mathbb{N}| < |\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$

Q: Is there any set S s.t. $|\mathbb{N}| < |S| < |\mathbb{R}|$?

A: The existence of S can not be proved using "standard" set theory axioms. (Gödel, Cohen)

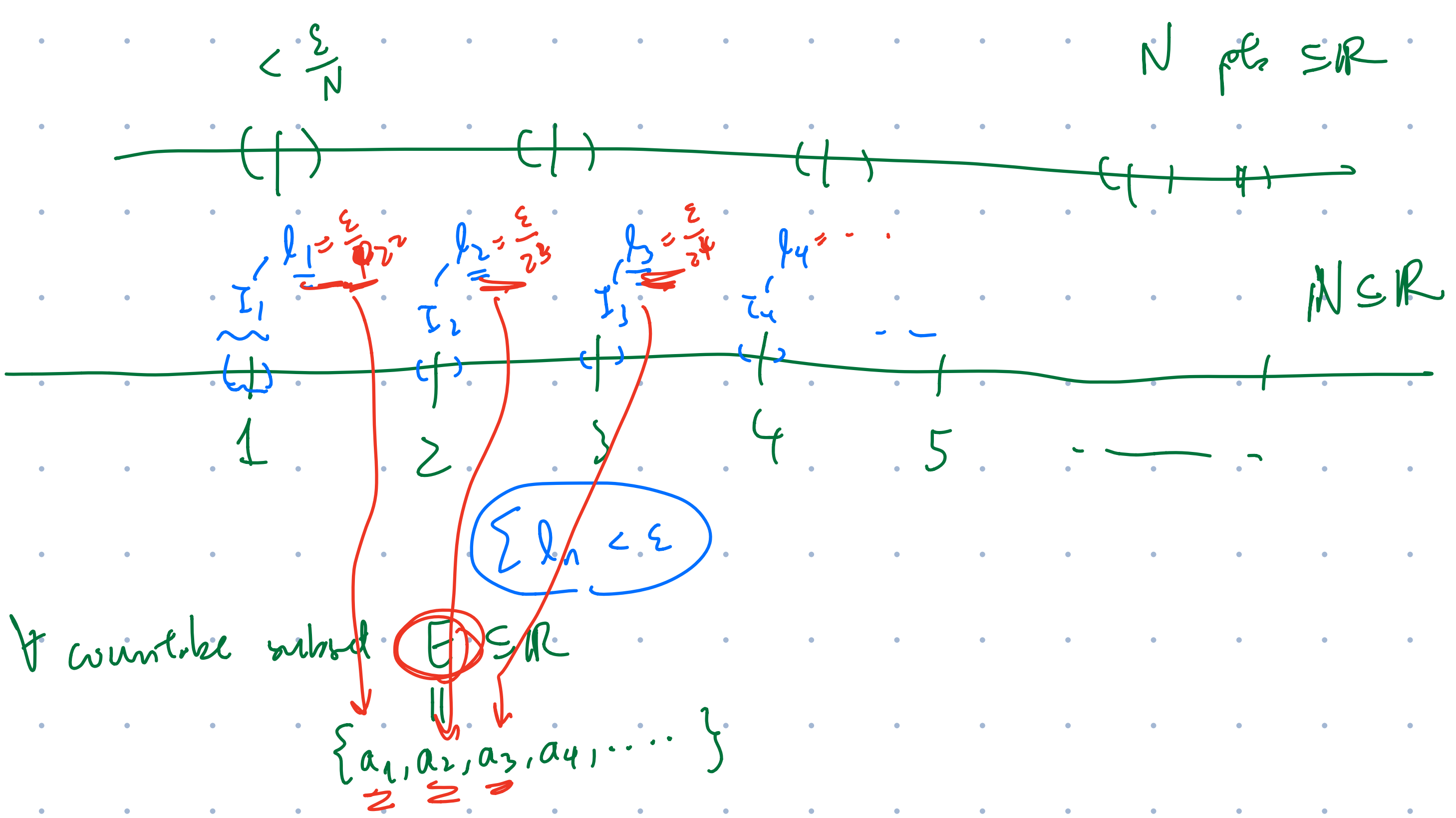
Def: A subset $E \subseteq \mathbb{R}$ <u>has measure zero</u> if:

$\forall \varepsilon > 0$,

$\exists$ finite or countably many open intervals $\{I_1, I_2, \cdots\}$.

s.t.

- $E \subseteq \left(\bigcup_{n=1}^{\infty} I_n \right)$
- $\sum_{n=1}^{\infty} \text{length}(I_n) < \varepsilon$.



$\Rightarrow$ <u>Countable subsets in $\mathbb{R}$ are of measure zero.</u>

Rmk: $E \subseteq \mathbb{R}^n$ has measure zero if

$\forall \varepsilon > 0$

$\exists$ finite or countably many open "cubes" $\{C_1, C_2, \dots\}$

st

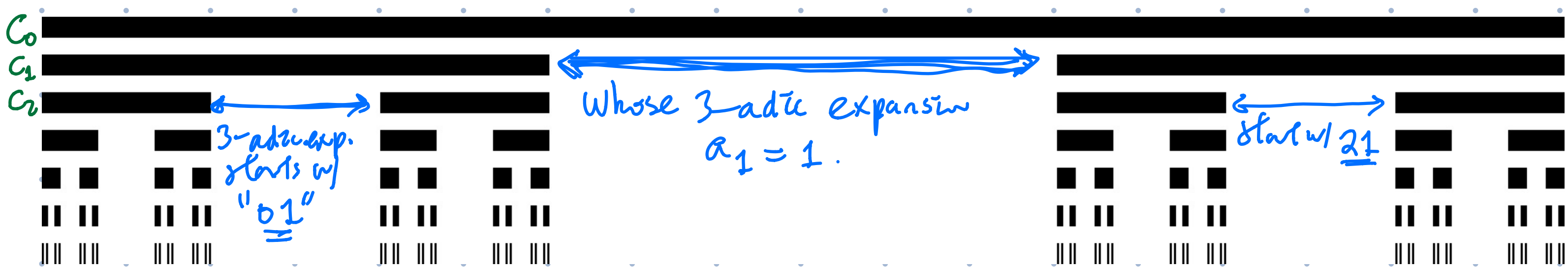
- $E \subseteq \left(\bigcup_{n=1}^{\infty} C_n \right)$
- $\sum_{n=1}^{\infty} \mathrm{Vol}(C_n) < \varepsilon$.

$\mathrm{Vol} = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$

$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$

Ex:
- any subset of a measure zero set has measure zero.
- any countable union of measure zero sets has measure zero.
  (i.e. If $A_1, A_2, A_3, \dots \subseteq \mathbb{R}$ are of measure zero,
  then $\bigcup_{n=1}^{\infty} A_n \subseteq \mathbb{R}$ is of measure zero).

Cantor set: (uncountable subset in $\mathbb{R}$, has measure zero).



$C_0 = [0, 1] \subseteq \mathbb{R}$

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

Consider 3-adic exp. of pts $\in [0, 1]$.

$\frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \frac{a_4}{3^4} + \cdots$

$a_i \in \{0, 1, 2\}$

Cantor set $\mathcal{C} := \bigcap_{n=0}^{\infty} C_n$ = $\{ x \in [0, 1] \mid$ ~~the digits in its 3-adic exp. $\neq 1$~~

i.e. $\exists\, a_i \in \{0, 2\}$ st.

$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots \}$

- $\mathcal{C}$ is of measure zero:

$\forall \varepsilon > 0$, choose n large s.t. $\left(\frac{2}{3}\right)^n < \varepsilon$.

Then one can find finite collection of open intervals of total length $< \varepsilon$ that covers C_n.

which also covers $\mathcal{C}$. $\square$

- $\mathcal{C}$ is uncountable:

$\forall \; f : \mathbb{N} \longrightarrow \mathcal{C}$, Want to show: f is NOT surjective.

$1 \longmapsto f(1) = \frac{a_{11}}{3} + \frac{a_{12}}{3^2} + \frac{a_{13}}{3^3} + \cdots$ $a_{ij} \in \{0,2\}$

$2 \longmapsto f(2) = \frac{a_{21}}{3} + \frac{a_{22}}{3^2} + \frac{a_{23}}{3^3} + \cdots$

$\vdots$

Define $b := \frac{b_1}{3} + \frac{b_2}{3^2} + \frac{b_3}{3^3} + \cdots$

s.t. $b_i \neq a_{ii} \quad \forall i$

$b_i \in \{0,2\} \Rightarrow b \in \mathcal{C}$

$b \neq f(i) \quad \forall i$.

$\square$