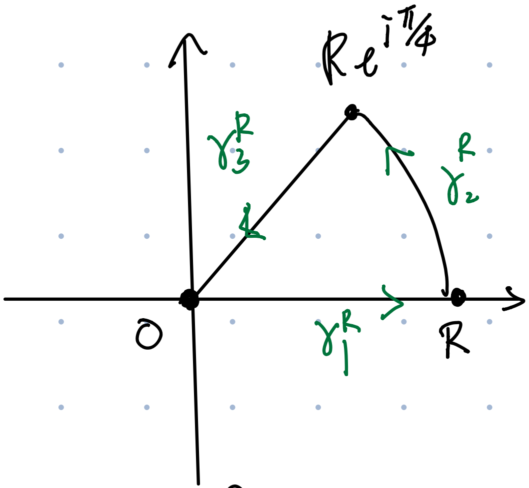
1. Prove that

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$



Consider the holo. fcn on $\mathbb{C}$:

$$f(z) = e^{-z^2}$$

By Cauchy's thm, $\int_{\gamma_1^R} f + \int_{\gamma_2^R} f + \int_{\gamma_3^R} f = 0$

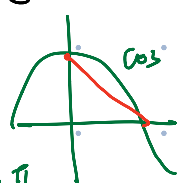
- $I_1^R := \int_{\gamma_1^R} e^{-z^2}\,dz = \int_0^R e^{-z^2}\,dz.$

Recall that $\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}$, hence $\lim_{R\to\infty} I_1^R = \frac{\sqrt{\pi}}{2}$

- $I_2^R := \int_{\gamma_2^R} e^{-z^2}\,dz = \int_0^{\pi/4} e^{-(Re^{i\theta})^2} (iRe^{i\theta})\,d\theta$

$$|I_2^R| \le \int_0^{\pi/4} \left|e^{-R^2 e^{2i\theta}}\right| \cdot R\,d\theta = R\int_0^{\pi/4} e^{-R^2\cos(2\theta)}\,d\theta$$

(since cosine fcn is convex in $[0, \frac{\pi}{2}]$, we have $\cos(2\theta) \ge 1 - \frac{4}{\pi}\theta \quad \forall\, 0 \le \theta \le \frac{\pi}{4}$)



$$\le R\cdot\int_0^{\pi/4} e^{-R^2(1-\frac{4}{\pi}\theta)}\,d\theta = R\cdot\int_0^{\pi/4} e^{-R^2\cdot\frac{4}{\pi}\phi}\,d\phi$$

$(\phi = \frac{\pi}{4} - \theta)$

$$< R\cdot\int_0^{\infty} e^{-R^2\cdot\frac{4}{\pi}\cdot\phi}\,d\phi = \frac{\pi}{4R}$$

Hence $\lim_{R\to\infty} I_2^R = 0.$

- $I_3^R := \int_{\gamma_3^R} e^{-z^2}\, dz = -\int_0^R e^{-(te^{i\pi/4})^2} \cdot e^{i\pi/4}\, dt$

$$= -e^{i\pi/4}\int_0^R e^{-it^2}\, dt$$

$$= -\frac{1+i}{\sqrt{2}}\int_0^R \cos(t^2) - i\sin(t^2)\, dt.$$

$$\Rightarrow \lim_{R\to\infty} I_3^R = -\frac{1+i}{\sqrt{2}}\left(\underbrace{\int_0^\infty \cos(x^2)\,dx}_{=J_1} - i\underbrace{\int_0^\infty \sin(x^2)\,dx}_{=J_2}\right)$$

$$= \left(-\frac{J_1}{\sqrt{2}} - \frac{J_2}{\sqrt{2}}\right) + i\left(-\frac{J_1}{\sqrt{2}} + \frac{J_2}{\sqrt{2}}\right)$$

Since $I_1^R + I_2^R + I_3^R = 0$ $\forall R$, Let $R \to +\infty$, we have:

$$\frac{\sqrt{\pi}}{2} + \left(-\frac{J_1}{\sqrt{2}} - \frac{J_2}{\sqrt{2}}\right) + i\left(-\frac{J_1}{\sqrt{2}} + \frac{J_2}{\sqrt{2}}\right) = 0$$

$$\Rightarrow \begin{cases} J_1/\sqrt{2} + J_2/\sqrt{2} = \sqrt{\pi}/2 \\ -J_1/\sqrt{2} + J_2/\sqrt{2} = 0. \end{cases}$$

$$\Rightarrow J_1 = J_2 = \sqrt{2\pi}/4. \quad \square$$

2. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_\varepsilon^R \frac{\sin x}{x}\,dx = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_\varepsilon^R \frac{e^{ix}-e^{-ix}}{2ix}\,dx$$

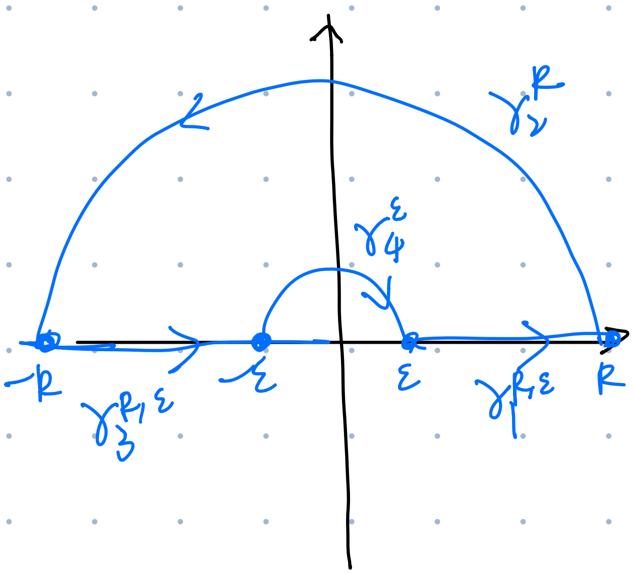
$$= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_\varepsilon^R \frac{e^{ix}}{2ix}\,dx + \int_{-R}^{-\varepsilon} \frac{e^{ix}}{2ix}\,dx \right)$$

$$= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_\varepsilon^R + \int_{-R}^{-\varepsilon} \right) \frac{e^{ix}}{2ix}\,dx.$$

Note that $\displaystyle\int_\varepsilon^R \frac{1}{x}\,dx = -\int_{-R}^{-\varepsilon} \frac{1}{x}\,dx.$

Hence $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_\varepsilon^R + \int_{-R}^{-\varepsilon} \right) \frac{e^{ix}-1}{2ix}\,dx$

as suggested by the hint.



Define $f(z) := \dfrac{e^{iz}-1}{2iz}$,

a holo. fun. on $\mathbb{C}\setminus\{0\}$.

By Cauchy's thm,

$$\left(\int_{\gamma_1^{R,\varepsilon}} + \int_{\gamma_2^R} + \int_{\gamma_3^{R,\varepsilon}} + \int_{\gamma_4^\varepsilon} \right) f = 0,$$

- Observe that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{\gamma_1^{R,\varepsilon}} + \int_{\gamma_3^{R,\varepsilon}}\right) f$.

- $\displaystyle\int_{\gamma_4^\varepsilon} \frac{e^{iz}-1}{2iz}\,dz = -\int_0^\pi \frac{e^{i(\varepsilon e^{i\theta})}-1}{2i(\varepsilon e^{i\theta})}\,(i\varepsilon e^{i\theta})\,d\theta$

Note that $\dfrac{e^{iz}-1}{2iz}$ is a bounded fcn near $z=0$,

since $\displaystyle\frac{e^{iz}-1}{2iz} = \frac{(1+iz+O(z^2))-1}{2iz} = \frac{1}{2} + O(z)$.

Say $\left|\dfrac{e^{iz}-1}{2iz}\right| \le M$ for any $|z|\le 1$.

Then $\forall\, 0<\varepsilon<1$, we have

$$\left|\int_{\gamma_4^\varepsilon} \frac{e^{iz}-1}{2iz}\,dz\right| \le M\cdot\varepsilon\cdot\pi.$$

Hence $\displaystyle\lim_{\varepsilon\to 0} \int_{\gamma_4^\varepsilon} \frac{e^{iz}-1}{2iz}\,dz = 0$.

- $\displaystyle\int_{\gamma_2^R} \frac{e^{iz}-1}{2iz}\,dz = \int_{\gamma_2^R} \frac{e^{iz}}{2iz}\,dz - \int_{\gamma_2^R} \frac{1}{2iz}\,dz$

By Jordan's lemma, $\displaystyle\lim_{R\to\infty} \int_{\gamma_2^R} \frac{e^{iz}}{2iz}\,dz = 0$,

$$\int_{\gamma_2^R} \frac{1}{2iz}\,dz = \frac{1}{2i}\int_0^\pi \frac{1}{Re^{i\theta}}\,(iRe^{i\theta})\,d\theta = \frac{\pi}{2}$$

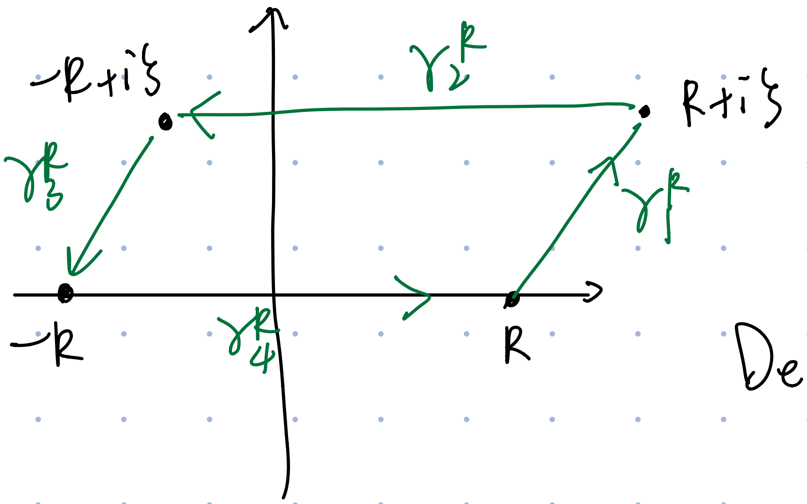
Hence $\displaystyle\lim_{R\to\infty}\int_{\gamma_1^R} \frac{e^{iz}-1}{2iz}\,dz = -\frac{\pi}{2}.$

Now by Cauchy's thm, $\displaystyle\int_{\gamma_1^{R,\varepsilon}} f + \int_{\gamma_2^R} f + \int_{\gamma_3^{R,\varepsilon}} f + \int_{\gamma_4^{\varepsilon}} f = 0.$

Let $\varepsilon\to 0$, $R\to\infty$, we have:

$$\int_0^\infty \frac{\sin x}{x}\,dx - \frac{\pi}{2} = 0. \quad \square$$

4. Prove that for all $\xi\in\mathbb{C}$ we have $e^{-\pi\xi^2} = \displaystyle\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi}\,dx.$



$$e^{-\pi z^2} e^{2\pi i z\cdot\xi} = e^{-\pi(z-i\xi)^2}\cdot e^{-\pi\xi^2}$$

Define $f(z) = e^{-\pi(z-i\xi)^2}$, a holo. fcn. on $\mathbb{C}$.

It suffices to show that $\displaystyle\lim_{R\to\infty}\int_{\gamma_4^R} f(z)\,dz = 1.$

- $\displaystyle\left|\int_{\gamma_1^R} f(z)\,dz\right| = \left|\int_0^1 e^{-\pi(R+i\xi t)^2}\cdot(-i\xi)\,dt\right|$

$$\leq |\xi| \max_{t\in[0,1]} e^{-\pi\cdot \mathrm{Re}((R+i\xi t)^2)}$$

$$\mathrm{Re}\left((R+i\xi t)^2\right) = \mathrm{Re}\left(R^2+2Ri\xi t - \xi^2 t^2\right)$$

for $t\in[0,1]$

$$\geq R^2 - 2R|\xi| - |\xi|^2.$$

Hence $\left|\int_{\gamma_1^R} f(z)\,dz\right| \leq |\xi|\cdot e^{-\pi\left(R^2-2R|\xi|-|\xi|^2\right)}$

$$\Rightarrow \lim_{R\to\infty}\int_{\gamma_1^R} f(z)\,dz = 0.$$

- Similarly, one can show that $\lim_{R\to\infty}\int_{\gamma_3^R} f(z)\,dz = 0.$
- $\int_{\gamma_2^R} f(z)\,dz = -\int_{-R}^{R} e^{-\pi(t+i\xi-i\xi)^2}\,dt$

$$= -\int_{-R}^{R} e^{-\pi t^2}\,dt$$

Hence $\lim_{R\to\infty}\int_{\gamma_2^R} f(z)\,dz = -1.$

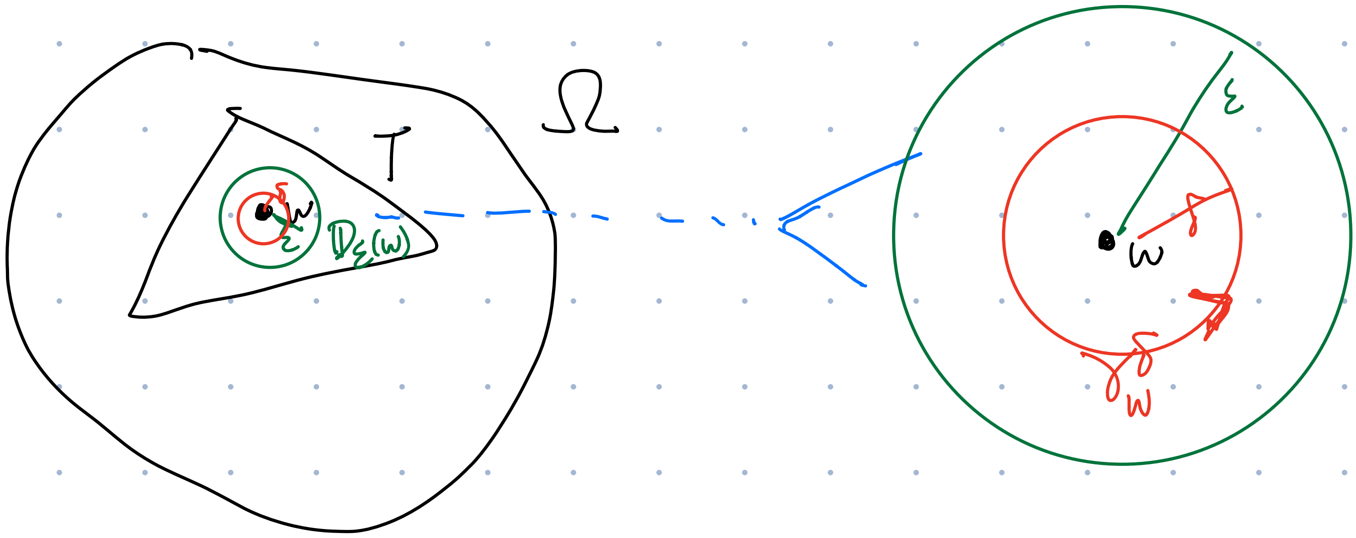
By Cauchy's thm, we conclude that

$$\lim_{R\to\infty}\int_{\gamma_4^R} f(z)\,dz = 1$$

$$\| \quad \lim_{R\to\infty}\int_{-R}^{R} e^{-\pi(x-i\xi)^2}\,dx. \qquad \square$$

6. Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then

$$\int_T f(z)\,dz = 0.$$



Choose $\varepsilon > 0$ small enough s.t. $\mathbb{D}_\varepsilon(w) \subseteq T$.

By assumption, $\exists\, M > 0$ s.t. $|f(z)| < M \ \forall z \in \mathbb{D}_\varepsilon(w)$.
(f is bounded near w)

Since f is holo. on $\Omega \setminus \{w\}$, we have

$$\int_T f(z)\,dz = \int_{\gamma_w^\delta} f(z)\,dz. \quad \text{for } \underline{\underline{\text{any}}} \ 0 < \delta < \varepsilon.$$

Observe that $\left| \int_{\gamma_w^\delta} f(z)\,dz \right| < M \cdot \text{length}(\gamma_w^\delta)$
$= M \cdot 2\pi\delta.$

Hence $\left| \int_T f(z)\,dz \right| \leq (2\pi M)\delta$ for $\underline{\underline{\text{any}}}$ $0 < \delta < \varepsilon$.

$\Rightarrow \int_T f(z)\,dz = 0.$ $\square$