#1:

($\Leftarrow$): If $A\vec{x}=\vec{0}$ for some $\vec{x}\in\mathbb{R}^n$,

if part.

then $\vec{0}=B(A\vec{x})=(BA)\vec{x}=I_n\vec{x}=\vec{x}$.

This shows T_A is injective.

($\Rightarrow$): T_A injective $\Rightarrow$ the reduced echelon form of A is:

only if part.

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \\ & 0 & & \end{bmatrix} \quad \text{and } m \geq n.$$

We showed in class that elementary row operations can be realized as left multiplicat^{n} by invertible matrices.

Hence $\exists$ C: $m\times m$ invertible matrix s.t.

$$CA=\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \\ & 0 & & \end{bmatrix}$$

Define an $n\times m$ matrix D as:

$$D=\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}=\left[\, I_n \;\middle|\; 0_{n\times(m-n)} \,\right]$$

One can easily check that

$$D\cdot\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \\ & 0 & & \end{bmatrix}=I_n.$$

Hence we can take $B:=DC$. and get: $BA=DCA=I_n$. $\square$

#2:

($\Rightarrow$) For each $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ith component in $\mathbb{R}^m$, $\exists\, \vec{b}_i \in \mathbb{R}^n$ s.t. $A\vec{b}_i = \vec{e}_i$.

Take $B = \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_m \\ | & | & & | \end{bmatrix}$, then $AB = I_m$.

($\Leftarrow$) $\forall\, \vec{x} \in \mathbb{R}^m$, we have $\vec{x} = I_m \vec{x} = (AB)\vec{x} = A(B\vec{x})$.

Hence T_A is surjective. $\square$

#3: S can't be surjective: Let A be the unique 3×2 matrix such that $T_A = S$. It's obvious that A can't have pivot in each row.

Then $S \circ T$ can't be surjective, therefore can't be invertible. $\square$

(one can also show that T is NOT injective and get the same conclusion.)

#4:

Suppose that $a_0 \vec{v} + a_1 T(\vec{v}) + \cdots + a_{k-1} T^{k-1}(\vec{v}) = \vec{0}$.

Apply T^{k-1} on both sides, get:

$$\vec{0} = T^{k-1}(\vec{0}) = T^{k-1}(a_0 \vec{v} + a_1 T(\vec{v}) + \cdots + a_{k-1} T^{k-1}(\vec{v}))$$

$$= a_0 T^{k-1}(\vec{v}) + \underbrace{a_1 T^k(\vec{v}) + \cdots + a_{k-1} T^{2k-2}(\vec{v})}_{= 0 \text{ since } T^k(\vec{v}) = 0}$$

$$= a_0 \underbrace{T^{k-1}(\vec{v})}_{\neq \vec{0}}$$

$\Rightarrow a_0 = 0$.

$\Rightarrow \quad a_1 T(\vec{v}) + a_2 T^2(\vec{v}) + \cdots + a_{k-1} T^{k-1}(\vec{v}) = \vec{0}.$

Now apply T^{k-2} on both sides, by the same argument, one gets $a_1 = 0$.

One can iterate this argument and get $a_0 = a_1 = \cdots = a_{k-1} = 0$. $\square$

#5: Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

- Take $B = \begin{bmatrix} c & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$. Then $c \neq 0$

$$AB = BA$$

$$A \begin{bmatrix} c & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = AB, \qquad BA = \begin{bmatrix} c & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} A$$

$$A B = \begin{bmatrix} c a_{11} & a_{12} & \cdots & a_{1n} \\ c a_{21} & & & \\ \vdots & & & \\ c a_{n1} & & & \end{bmatrix}, \qquad BA = \begin{bmatrix} c a_{11} & c a_{12} & \cdots & c a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{1n} & & & \end{bmatrix}$$

Take any $c \neq 1$.

$$\Rightarrow a_{12} = a_{13} = \cdots = a_{1n} = 0,$$
$$a_{21} = a_{31} = \cdots = a_{n1} = 0.$$

By the same argument, one can show that any off-diagonal entry of A is zero. i.e.

$$A = \begin{bmatrix} a_{11} & & & O \\ & a_{22} & & \\ & & \ddots & \\ O & & & a_{nn} \end{bmatrix}$$

- Take $B = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$. then

$$AB = BA$$

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & \end{bmatrix} \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & \end{bmatrix}$$

$$\begin{bmatrix} 0 & a_{11} & & & \\ a_{22} & 0 & & & \\ & & a_{33} & & \\ & & & \ddots & \\ & & & & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} 0 & a_{22} & & & \\ a_{11} & 0 & & & \\ & & a_{33} & & \\ & & & \ddots & \\ & & & & \end{bmatrix}$$

$$\Rightarrow a_{11} = a_{22}.$$

By the same argument, one sees that $a_{11} = a_{22} = \cdots = a_{nn}$. $\square$

#6: The (i,j)-th entry of AB is:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Hence
$$\operatorname{tr}(AB) = \sum_{l=1}^{m} \sum_{k=1}^{n} a_{lk} b_{kl}.$$

Similarly, the (i,j)-th entry of BA is:

$$(BA)_{ij} = \sum_{l=1}^{m} b_{il} a_{lj}$$

Hence
$$\operatorname{tr}(BA) = \sum_{k=1}^{n} \sum_{l=1}^{m} b_{kl} a_{lk}.$$

Therefore $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. $\square$

#7: False.

e.g.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 4 & 1 \end{bmatrix} \quad \operatorname{tr} = 9$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix} \quad \operatorname{tr} = 6.$$

#8: True.

If there were such A, B, then

$$\begin{aligned} &\operatorname{tr}(AB - BA) = \operatorname{tr}(I_n) = n \\ &\quad\| \\ &\operatorname{tr}(AB) - \operatorname{tr}(BA) = 0 \quad (\text{by } \#6). \quad \square \end{aligned}$$

#9:

$$A = \begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{bmatrix}$$

#10: These are straightforward computations.

e.g.:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}$$

and $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$. $\square$