HW 8 sol'n

#1: Define $f(x) := \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Exercise for you: prove that f is discontinuous at every real number.

#2: f is continuous at $x \in (0,1) \setminus \mathbb{Q}$, discontinuous at $x \in (0,1) \cap \mathbb{Q}$.

pf: • For $x \in (0,1) \setminus \mathbb{Q}$, $f(x) = 0$

$\forall \varepsilon > 0$, let n be any integer s.t. $n > \frac{1}{\varepsilon}$.

Consider the set $S_n := \left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \cdots, \frac{1}{n}, \frac{2}{n}, \cdots \frac{n-1}{n} \right\}$.

$= \{$all rational number in $(0,1)$ s.t. denominator $\leq n\}$.

Since x is irrational, $x \notin S_n$.

Since S_n is a finite set, $\exists \delta > 0$ s.t. $(x-\delta, x+\delta) \cap S_n = \phi$.

Then $\forall y \in (x-\delta, x+\delta)$, we have $f(y) \leq \frac{1}{n+1} < \varepsilon$.

So we have $|y-x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$.

Hence f is continuous at x. $\square$

• For $x \in (0,1) \cap \mathbb{Q}$. $x = \frac{p}{q}$, where $p, q > 0$, $\gcd(p,q) = 1$. $f(x) = \frac{1}{q}$.

There exists a seq. of irrat'l numbers $(x_n) \longrightarrow x$.

$$\underset{\substack{\| \\ 0}}{(f(x_n))} \not\longrightarrow f(x) = \frac{1}{q}$$

Hence f is discontinuous at x. $\square$

#3 (a) • $\forall \varepsilon>0$, $\exists \delta>0$ st. $z_1, z_2 \in X$, $|z_1-z_2|<\delta \Rightarrow |f(z_1)-f(z_2)|<\varepsilon$

(f is unif. conti.)

• $\exists N>0$ st. $|x_n-x_m|<\delta$ $\forall n, m>N$. ((x_n) is Cauchy in X)

$\Rightarrow |f(x_n)-f(x_m)|<\varepsilon$ $\forall n, m>N$.

Hence $(f(x_n))$ is Cauchy in Y. □

(b) $f: (0,1) \longrightarrow \mathbb{R}$ conti. fcn.
$x \longmapsto \frac{1}{x}$

$(x_n=\frac{1}{n})$ Cauchy sequence., $(f(x_n)=n)$ is not Cauchy.

#4: (a) No

$(x_n=e^{-n}) \subset (0,1)$ Cauchy, but $(A(x_n)=-n)$ not Cauchy.

(b) No

Observe that $|B(2n\pi)-B(2n\pi+\frac{1}{n})|=(2n\pi+\frac{1}{n})\sin(\frac{1}{n})$.

So $\lim_{n\to\infty}|B(2n\pi)-B(2n\pi+\frac{1}{n})|=\lim_{n\to\infty}(2n\pi+\frac{1}{n})\sin(\frac{1}{n})$

$= \lim_{x\to 0}(\frac{2\pi}{x}+x)\sin x$

$= 2\pi$.

(Since $\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to 0} x\sin x=0$.)

$\Rightarrow \exists N>0$ st. $|B(2n\pi)-B(2n\pi+\frac{1}{n})|>\pi$ $\forall n>N$.

Let $\varepsilon=\pi$. We want to show: $\forall \delta>0$ $\exists x,y \in [0,\infty)$ st. $|x-y|<\delta$ but $|B(x)-B(y)|\geq \varepsilon=\pi$.

Choose any n s.t. $n>\max\{\frac{1}{\delta}, N\}$.

Let $x=2n\pi$ and $y=2n\pi+\frac{1}{n}$. Then $|x-y|=\frac{1}{n}<\delta$

but $|B(x)-B(y)|\geq\pi$. □

(c) Yes

- If $x, y \in [1, \infty)$, then

$$|C(x)-C(y)| = \left|\frac{1}{x^2+1} - \frac{1}{y^2+1}\right| = \frac{|x-y|(x+y)}{(x^2+1)(y^2+1)} < |x-y|.$$

Similarly, if $x, y \in (-\infty, -1]$, then $|C(x)-C(y)| < |x-y|$.

- $C(x)$ is conti. on $[-1,1]$, therefore unif. conti. ($[-1,1]$ is compact).

So $\forall \varepsilon > 0$, $\exists \delta' > 0$ s.t. $\begin{matrix} x, y \in [-1,1] \\ |x-y| < \delta' \end{matrix} \Rightarrow |C(x)-C(y)| < \varepsilon/2$

Define $\delta := \min\{\delta', \frac{\varepsilon}{2}, 1\}$.

Then $\forall x, y \in \mathbb{R}$ with $|x-y| < \delta$,

- If x, y both ≥ 1 or both ≤ -1,
  then $|C(x)-C(y)| < |x-y| < \delta \leq \frac{\varepsilon}{2} < \varepsilon$.
- If x, y both in $[-1,1]$,
  then $|C(x)-C(y)| < \frac{\varepsilon}{2}$ since $|x-y| < \delta \leq \delta'$.
- If $x > 1$ and $y < 1$. then

$$|C(x)-C(y)| \leq \underbrace{|C(x)-C(1)|}_{\text{both } < \frac{\varepsilon}{2} \text{ by previous argument}} + \underbrace{|C(1)-C(y)|} < \varepsilon.$$

Similarly, $x < -1$, $y > -1$. case also thus: $|C(x)-C(y)| < \varepsilon$. $\square$

(d) No

Observe that $\lim_{n\to\infty} |\log n - \log(n+1)| = \lim_{n\to\infty} |\log(\frac{n+1}{n})| = 0$.

Let $\varepsilon = \frac{1}{2}$. WTS: $\forall \delta > 0$ $\exists x, y \in [0, \infty)$ s.t. $|x-y| < \delta$ but $|D(x)-D(y)| \geq \varepsilon$.

$\forall \delta > 0$, $\exists n > 0$ s.t. $|\log n - \log(n+1)| < \delta$.

Take $x = \log n$, $y = \log(n+1)$. Then $|D(x)-D(y)| = 1 > \frac{1}{2}$. $\square$

(e) $\left(x_n = \frac{2}{(2n-1)\pi}\right) \subset (0,\infty)$ is a Cauchy seq.

No

but $\left(E(x_n) = \sin\left(\frac{(2n-1)\pi}{2}\right) = \begin{cases} 1, & n:\text{odd} \\ -1, & n:\text{even} \end{cases} \right)$ is not Cauchy. $\square$

#5: (a) Suppose such $k>0$ exists.

Let $x = \frac{1}{n^2}$ and $y = \frac{1}{(n+1)^2}$.

$$\Rightarrow \left| \frac{1}{n} - \frac{1}{n+1} \right| \le k \left| \frac{1}{n^2} - \frac{1}{(n+1)^2} \right| \quad \forall n.$$

$$\Rightarrow \frac{1}{n(n+1)} \le k \cdot \frac{2n+1}{n^2(n+1)^2} \quad \forall n$$

$$\Rightarrow \frac{n(n+1)}{2n+1} \le k \quad \forall n. \quad \text{Contradiction.} \ \square$$

(b) $\forall \varepsilon>0$, let $\delta = \varepsilon^2$, then $\forall\, |x-y| < \delta = \varepsilon^2$, we have

$$|f(x)-f(y)| = |\sqrt{x}-\sqrt{y}| \le \sqrt{|x-y|} < \sqrt{\delta} = \varepsilon. \ \square$$

#6: (a) f restricts on $[0,L] \longrightarrow \mathbb{R}$ attains its supremum and infimum by extreme value thm. (f is conti. and $[0,L]$ is cpt.)

i.e. $\exists\, x, y \in [0,L]$ s.t. $f(x) \le f(z) \le f(y) \quad \forall z \in [0,L]$.

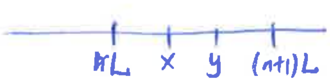
Since $f(x+L) = f(x) \ \forall x \in \mathbb{R}$, we have $f(x) \le f(z) \le f(y) \quad \forall z \in \mathbb{R}$.

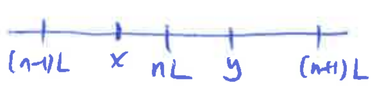
Hence f attains its sup. and inf. on $\mathbb{R}$. $\square$

(b) f is unif. conti. on $[0,L]$, (Since $[0,L]$ is cpt.) So

$\forall \varepsilon>0$, $\exists \delta'>0$ s.t. $x,y \in [0,L]$, $|x-y|<\delta' \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{2}$.

Define $\delta = \min\{\delta', L\}$. Then $\forall x,y \in \mathbb{R}$ with $|x-y| < \delta$,

-  $|f(x)-f(y)| = |f(x-nL) - f(y-nL)| < \frac{\varepsilon}{2}$

-  $|f(x)-f(y)| \le |f(x)-f(nL)| + |f(nL)-f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$ $\square$

#7: • Define $\tilde{f}: \mathbb{R} \longrightarrow \mathbb{R}$

- For $x \in \mathbb{Q}$, define $\tilde{f}(x) := f(x)$
- For $x \in \mathbb{R} \setminus \mathbb{Q}$,

Let $(a_n) \subset \mathbb{Q}$ be a seq. conv. to x.

Since $f: \mathbb{Q} \longrightarrow \mathbb{R}$ is unif. conti. and (a_n) is Cauchy,

by #3(a), $(f(a_n))$ is Cauchy.

Define: $\tilde{f}(x) := \lim f(a_n)$.

Claim: This is well-defined. i.e. If $(a_n) \subset \mathbb{Q}$ and $(b_n) \subset \mathbb{Q}$ both conv. to x,

then $\lim f(a_n) = \lim f(b_n)$.

pf: Consider $(c_n) := (a_1, b_1, a_2, b_2, a_3, b_3, \cdots)$

One can check that $\lim c_n = x$.

By #3(a), $(f(c_n))$ converges in $\mathbb{R}$.

Since $(f(a_n))$ and $(f(b_n))$ are both subseq. of $(f(c_n))$,

we have $\lim f(a_n) = \lim f(c_n) = \lim f(b_n)$. $\square$

• $\tilde{f}$ is unif. conti.:

Since f is unif. conti, $\forall \varepsilon > 0$, $\exists \delta' > 0$ s.t. $x', y' \in \mathbb{Q}$, $|x'-y'| < \delta' \Rightarrow |f(x') - f(y')| < \frac{\varepsilon}{3}$

Define $\delta = \frac{\delta'}{3} > 0$.

Claim: $\forall x, y \in \mathbb{R}$, $|x-y| < \delta \Rightarrow |\tilde{f}(x) - \tilde{f}(y)| < \varepsilon$.

pf: Let $(x_n) \subset \mathbb{Q}$ be a seq. conv. to x, $(y_n) \subset \mathbb{Q}$ conv. to y.

Then $\tilde{f}(x) = \lim f(x_n)$, $\tilde{f}(y) = \lim f(y_n)$.

- $\exists N_1 > 0$ s.t. $|\tilde{f}(x) - f(x_n)| < \frac{\varepsilon}{3}$ $\forall n > N_1$, $\exists N_2 > 0$ s.t. $|\tilde{f}(y) - f(y_n)| < \frac{\varepsilon}{3}$ $\forall n > N_2$
- $\exists N_3 > 0$ s.t. $|x_n - x| < \delta$ $\forall n > N_3$, $\exists N_4 > 0$ s.t. $|y_n - y| < \delta$ $\forall n > N_4$.

Take any $n > \max\{N_1, N_2, N_3, N_4\}$. Then $|x_n - y_n| \leq |x_n - x| + |x - y| + |y_n - y| < \delta'$

$\Rightarrow |\tilde{f}(x) - \tilde{f}(y)| \leq |\tilde{f}(x) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - \tilde{f}(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. $\square$

#8. (a) Let $|x_2-x_1|=M$. Then $|x_3-x_2|\le kM$, $|x_4-x_3|\le k^2M, \cdots$

$\forall n>m$, we have

$$|x_n-x_m|\le |x_{m+1}-x_m|+|x_{m+2}-x_{m+1}|+\cdots\cdots$$
$$= k^{m-1}\cdot M + k^{m}\cdot M + k^{m+1}\cdot M + \cdots$$
$$= \frac{k^{m-1}\cdot M}{1-k}.$$

$\forall \varepsilon>0$. ~~Choose N~~ Choose $N>0$ s.t. $\dfrac{k^{N-1}\cdot M}{1-k}<\varepsilon$.

$\Rightarrow |x_n-x_m|<\varepsilon \quad \forall n>m\ge N.$ $\square$

(b) $x_{n+1}=f(x_n)$. $\forall n$ Let $\lim x_n = x^*$

Since f is continuous, we have $\lim f(x_n) = f(\lim x_n) = f(x^*)$

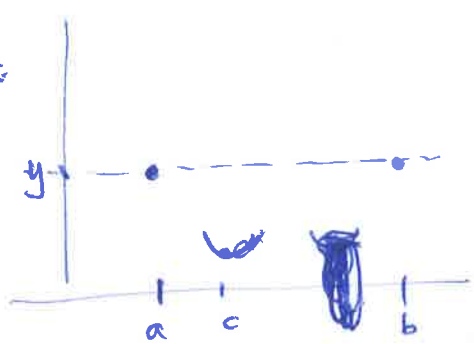
$\lim f(x_n) = \lim x_{n+1} = x^*$. $\square$

(c) If x^* and y^* both satisfy $f(x^*)=x^*$, $f(y^*)=y^*$, then

$$|f(x^*)-f(y^*)| \le k\,|x^*-y^*| < |x^*-y^*|$$

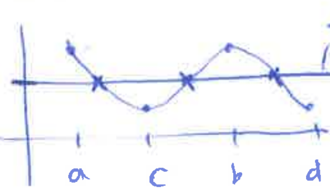
$|f(x^*)-f(y^*)| = |x^*-y^*|$ contradiction, unless $x^*=y^*$. $\square$

#9.



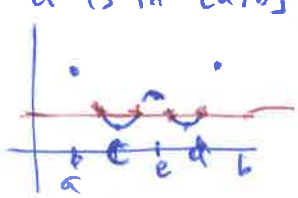
- Choose any $y\in\mathbb{R}$, $\exists$ a&b s.t. $f(a)=f(b)=y$.
- ~~f attains min on [a,b], twice say at c<d.~~
- ~~f attains max. on [c,d], twice say at e<f.~~
- f attains min on [a,b], say at c.
- $f(c)=f(d)$ for some other d.

1) d is not in [a,b].



this value take ≥ 3 times by IVT. $\times$

2) d is in [a,b], f attains max. on [c,d], say at e.



this value takes ≥ 4 times by IVT. $\times$. $\square$

Same statement isn't true for "thrice".

e.g.