Stokes matrices and character varieties

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Markoff numbers

A Markoff number is a positive integer x, y or z that satisfies the Diophantine equation

$$x^2 + y^2 + z^2 = xyz.$$

- 3 is a Markoff number since $(x, y, z) = (3, 3, 3)$ is a solution.
- Vieta involution: $(x, y, z) \to (yz - x, y, z)$.
- First few Markoff numbers: $3, 6, 15, 39, 87, \ldots$.

Markoff-type quantities show up in many fields of mathematics, e.g. in:

- 3×3 Stokes matrices;
- SL_2-character variety of the one-holed torus;
- $\cdots$

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Stokes matrices

- A Stokes matrix $s \in V(r)$ is an unipotent upper triangular matrix.
- The conjugacy class of $-s^{-1}s^T$ plays an important role in the study of Stokes matrices: it is...
  - linearization of Serre functor when s comes from exceptional collection;
  - monodromy data around infinity of certain (dual) Fuchsian system;
  - invariant under the natural B_r-action on $V(r)$:

$$B_r = \langle \sigma_1, \ldots, \sigma_{r-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2 \rangle .$$

$$s \xrightarrow{\sigma_i} \begin{bmatrix} \mathbb{I}_{i-1} & & & \\ & s_{i,i+1} & -1 & \\ & 1 & 0 & \\ & & & \mathbb{I}_{r-i-1} \end{bmatrix} s \begin{bmatrix} \mathbb{I}_{i-1} & & & \\ & s_{i,i+1} & 1 & \\ & -1 & 0 & \\ & & & \mathbb{I}_{r-i-1} \end{bmatrix} .$$

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3×3 Stokes matrices and Markoff-type quantity

$$s = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

- The characteristic polynomial of $-s^{-1}s^T$ is given by

$$p_k(\lambda) = (\lambda + 1)(\lambda^2 - k\lambda + 1),$$

where

$$k = x^2 + y^2 + z^2 - xyz - 2.$$

- The braid group B_3-action on $V(3)$ gives the Vieta involution,

$$\text{e.g.: } (x, y, z) \mapsto (x, z, xz - y).$$

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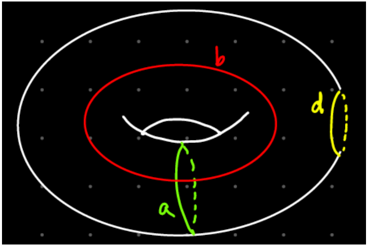
One-holed torus and Markoff-type quantity



- Vogt–Fricke: $\mathrm{Hom}(\pi_1(\Sigma_{1,1}), \mathrm{SL}_2(\mathbb{C})) /\!\!/ \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^3_{x,y,z}$, with coordinates given by

$$x = \mathrm{tr}(\rho(a)),\ \ y = \mathrm{tr}(\rho(b)),\ \ z = \mathrm{tr}(\rho(ab)).$$

- The boundary trace is

$$k := \mathrm{tr}(\rho(d)) = x^2 + y^2 + z^2 - xyz - 2.$$

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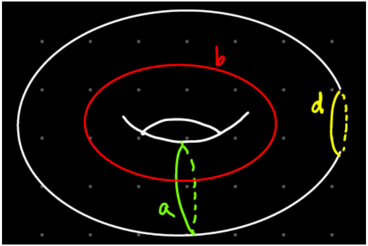

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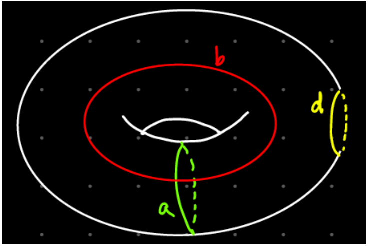

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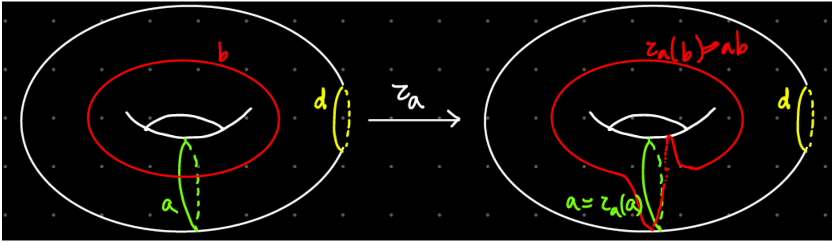
Vieta involutions and Dehn twists



- Dehn twist along a acts by $\tau_a^*\colon (x, y, z) \mapsto (x, z, xz - y)$, since $\mathrm{tr}(A^2B) = \mathrm{tr}(A)\mathrm{tr}(AB) - \mathrm{tr}(B)$ for any $A, B \in \mathrm{SL}_2$.
- $B_3 = \langle \tau_a, \tau_b | \tau_a\tau_b\tau_a = \tau_b\tau_a\tau_b \rangle$ form a braid group, and the boundary trace k is a B_3-invariant.

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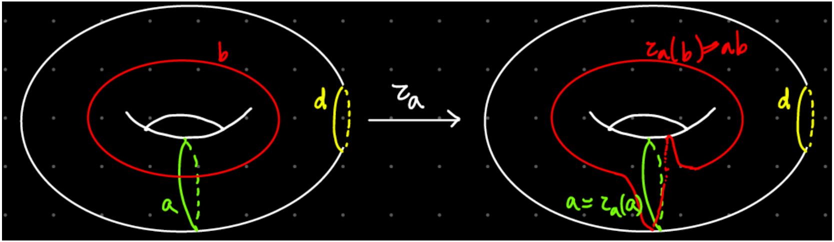

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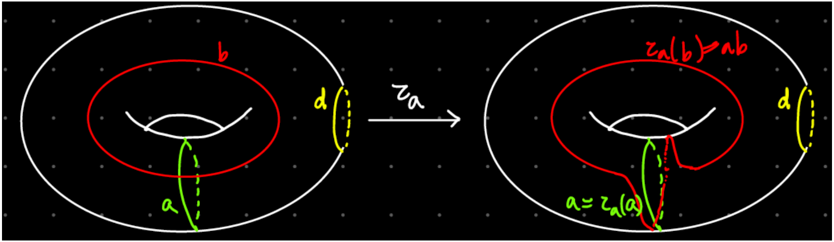

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We have seen a connection between

$$\begin{array}{rcl} \text{one-holed torus} & \longleftrightarrow & 3\times 3 \text{ Stokes matrix} \\ \text{Dehn twists} & \longleftrightarrow & \text{mutations } (B_3\text{-actions}) \\ \text{boundary trace} & \longleftrightarrow & -s^{-1}s^T \ (B_3\text{-invariants}) \\ X_k(\Sigma_{1,1}, \mathrm{SL}_2(\mathbb{C})) & \simeq & V_{p_k}(3) \end{array}$$

Here

$$X_k(\Sigma_{1,1}, \mathrm{SL}_2) = \{\rho \in X(\Sigma_{1,1}, \mathrm{SL}_2) \colon \mathrm{tr}\rho(d) = k\}$$

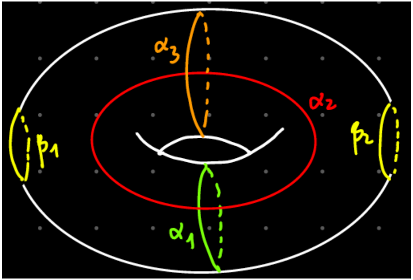
and

$$V_{p_k}(3) = \{s \in V(3) \colon \det(\lambda + s^{-1}s^T) = p_k(\lambda)\}$$

are B_3-invariant subvarieties.

Can we generalize these?

Two-holed torus



Let $\rho \in X = \mathrm{Hom}(\pi_1(\Sigma_{1,2}), \mathrm{SL}_2(\mathbb{C}))$, and define

$$\begin{aligned} a &= \mathrm{tr}\rho(\alpha_1), & b &= \mathrm{tr}\rho(\alpha_2), & c &= \mathrm{tr}\rho(\alpha_3), \\ d &= \mathrm{tr}\rho(\alpha_1\alpha_2\alpha_3), & e &= \mathrm{tr}\rho(\alpha_1\alpha_2), & f &= \mathrm{tr}\rho(\alpha_2\alpha_3). \end{aligned}$$

$X_{k_1,k_2}(\Sigma_{1,2}, \mathrm{SL}_2(\mathbb{C}))$ with fixed boundary traces $k_1 := \mathrm{tr}\rho(\beta_1)$ and $k_2 := \mathrm{tr}\rho(\beta_2)$ is a 4-dimensional subvariety of $\mathbb{C}^6_{a,b,c,d,e,f}$ defined by

$$ac + bd - ef = k_1 + k_2$$

$$a^2 + b^2 + \cdots + f^2 - abe - adf - bcf - cde + abcd - 4 = k_1 k_2$$

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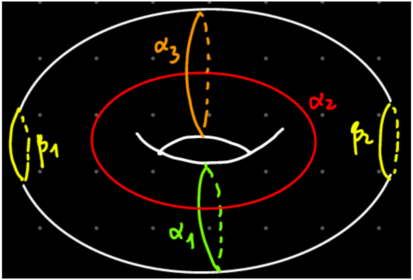

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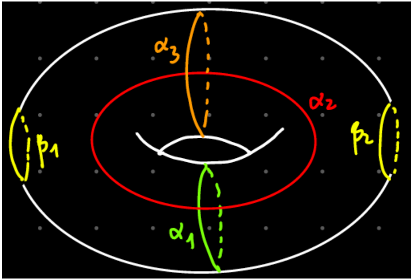

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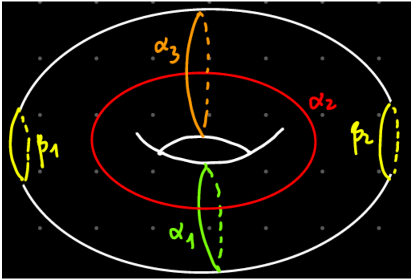

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4×4 Stokes matrices

$$s = \begin{bmatrix} 1 & a & e & d \\ 0 & 1 & b & f \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of $-s^{-1}s^T$ is given by

$$p_{k_1,k_2}(\lambda) = \lambda^4 - k_1 k_2 \lambda^3 + (k_1^2 + k_2^2 - 2)\lambda^2 - k_1 k_2 \lambda + 1$$

where

$$\begin{aligned} k_1 + k_2 &= ac + bd - ef, \\ k_1 k_2 &= a^2 + b^2 + \cdots + f^2 - abe - adf - bcf - cde + abcd - 4. \end{aligned}$$

Question:

Space	Braid invariants
$\mathrm{SL}_2(\mathbb{C})$-character varieties of surfaces	boundary monodromy

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbb{C})\text{-character varieties of surfaces} & & \text{boundary monodromy} \\ \big\updownarrow ?? & & \big\updownarrow ?? \\ \text{space of Stokes matrices} & & -s^{-1}s^{T} \end{array}$$

Space	Braid invariants
$\mathrm{SL}_2(\mathbb{C})$-character varieties of surfaces	boundary monodromy
$\|$	$\|$
moduli of points on complex affine 3-sphere	Coxeter invariants
$\hookdownarrow$	$\updownarrow$
space of Stokes matrices	$-s^{-1}s^T$

Moduli space of points on spheres

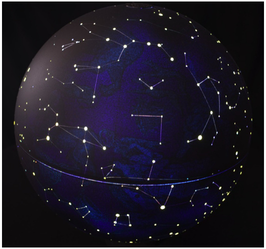
- $S(m) :=$ complex affine hypersurface in $\mathbb{C}^m$ defined by

$$\langle x, x \rangle = x_1^2 + \cdots + x_m^2 = 1.$$

- Define the moduli space of r points on $S(m)$ to be

$$A(r, m) := S(m)^r /\!\!/ \mathrm{SO}(m)$$

where $\mathrm{SO}(m)$ acts diagonally.



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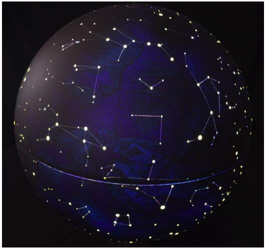

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Quandles and braid group actions

Definition

A quandle is a set X and a binary operation $\triangleleft: X \times X \to X$ such that:

- $x \triangleleft x = x$ and $x \triangleleft - : X \to X$ is a bijection for any $x \in X$.
- $x \triangleleft (y \triangleleft z) = (x \triangleleft y) \triangleleft (x \triangleleft z)$ for any $x, y, z \in X$.

If $(X, \triangleleft)$ is a quandle, then B_r acts on X^r by the following moves:

$$\sigma_i(x_1, \ldots, x_r) = (x_1, \ldots, x_{i-1}, x_i \triangleleft x_{i+1}, x_i, x_{i+2}, \ldots, x_r), \; 1 \le i \le r-1.$$

Example: $r = 3$.

$$\sigma_1\sigma_2\sigma_1(x_1, x_2, x_3) = ((x_1 \triangleleft x_2) \triangleleft (x_1 \triangleleft x_3), x_1 \triangleleft x_2, x_1),$$
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Examples of quandles

Example (Group quandle)

Let G be a group. Then $u \triangleleft v := uv^{-1}u$ gives a quandle structure on G.

Example (Sphere quandle)

For any $u \in S(m)$, define $s_u \in \mathrm{O}(m)$ by $s_u(v) := 2\langle u, v\rangle u - v$. Then $u \triangleleft v := s_u(v)$ gives a quandle structure on $S(m)$.

It's easy to check that the orthogonal group $\mathrm{O}(m)$ acts on $S(m)$ by quandle automorphisms. Therefore the B_r-action on $S(m)^r$ naturally descends to a B_r-action on the moduli space of points on spheres

$$A(r, m) = S(m)^r /\!\!/ \mathrm{SO}(m).$$

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$$\mathrm{Cl}(V,q) := \frac{\bigoplus_{i=0}^{\infty} V^{\otimes i}}{\langle v \otimes v - q(v) \colon v \in V \rangle}$$

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Coxeter invariants of $A(r, q)$

Consider the map

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Exceptional isomorphisms

Theorem (F.–Junho Peter Whang)

Write $r = 2g + n \geq 3$ where $n \in \{1, 2\}$. We have a B_r-equivariant isomorphism

$$A_P(r, 4) \cong X_k(\Sigma_{g,n}, \mathrm{SL}_2(\mathbb{C})),$$

where the Coxeter invariant P determine the boundary monodromy k, and vice versa.

Note: The definitions of $A(r, 4)$ and their Coxeter invariants have nothing to do with any Riemann surface!

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Braid actions and boundary curves of $\Sigma_{g,n}$

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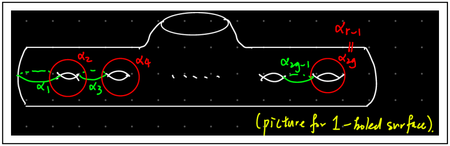

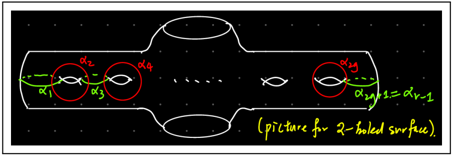

- $\pi_1(\Sigma_{g,n})$ is a free group of rank $2g + n - 1 = r - 1$.
- Dehn twists along $\alpha_1, \ldots, \alpha_{r-1}$ generate the braid group B_r.
- The boundary curve(s) are homotopic to:
  - r odd: $(\alpha_1\alpha_3 \cdots \alpha_{r-2})(\alpha_1\alpha_2 \cdots \alpha_{r-1})^{-1}(\alpha_2\alpha_4 \cdots \alpha_{r-1})$.
  - r even: $\alpha_1\alpha_3 \cdots \alpha_{r-1}$ and $(\alpha_1\alpha_2 \cdots \alpha_{r-1})^{-1}(\alpha_2\alpha_4 \cdots \alpha_{r-2})$.
- B_r acts on $X(\Sigma_{g,n}, G)$ via Dehn twists, and the conjugacy classes of the boundary curve(s) give B_r-invariants.

Sketch of proof

- Over $\mathbb{C}$, the quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2 \sim x_1x_4 - x_2x_3$.
- Define an isomorphism

$$X(\Sigma_{g,n}.\mathrm{SL}_2) \cong S(4)^r /\!\!/ \mathrm{SO}(4) = A(r,4);$$
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Connect with Stokes matrices

- Define $V(r,m) \subseteq V(r)$ to be the subset of Stokes matrices such that $\mathrm{rank}(s+s^T) \leq m$.
- By invariant theory for the orthogonal group, there is an isomorphism $S(m)^r /\!/ \mathrm{O}(m) \cong V(r,m)$ given by:

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Character varieties and Stokes matrices

We have a sequence of B_r-equivariant morphisms

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- When r is odd: Let k be the boundary trace of ρ, then

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The Markoff cubic

$$V_{p_k}(3) = \{x^2 + y^2 + z^2 - xyz - 2 - k = 0\} \subseteq \mathbb{C}^3 \cong V(3)$$

defines a log Calabi–Yau surface.

Problem: Study the integral points in $V_{p_k}(3)$.

By Markoff descent argument, one can show that $V_{p_k}(3)$ contains finitely many integral B_3-orbits except when $k = 2$, in which case $\mathrm{disc}(p_k) = 0$.

General problem: Study the integral B_r-orbits of $V_p(r)$.

For $r = 4$: given k_1, k_2, study the B_4-orbits of $(a, b, c, d, e, f) \in \mathbb{Z}^6$ with

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By Markoff descent argument, one can show that $V_{p_k}(3)$ contains finitely many integral B_3-orbits except when $k = 2$, in which case $\mathrm{disc}(p_k) = 0$.

General problem: Study the integral B_r-orbits of $V_p(r)$.

For $r = 4$: given k_1, k_2, study the B_4-orbits of $(a, b, c, d, e, f) \in \mathbb{Z}^6$ with

$$ac + bd - ef = k_1 + k_2,$$

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An arithmetic problem of Stokes matrices

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Case $r = 4$

Theorem (F.–Whang)

Let p be a degree 4 polynomial with $\mathrm{disc}(p) \neq 0$. Then $V_p(4)$ contains at most finitely many integral B_4-orbits.

Sketch of proof:

- By the main theorem, $V_p(4)$ is a finite disjoint union of varieties of the form $X_k(\Sigma_{1,2}, \mathrm{SL}_2)$. Their integral structures are compatible.
- Whang, 2020: The *non-degenerate* integral points of $X_k(\Sigma_{g,n}, \mathrm{SL}_2)$ consist of finitely many mapping class group orbits.
- B_4-equivalence coincides with $\Sigma_{1,2}$-equivalence, and analysis of the degenerate integral points.

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Thank you for your attention!