4/23/2020

Brouwer fixed point thm

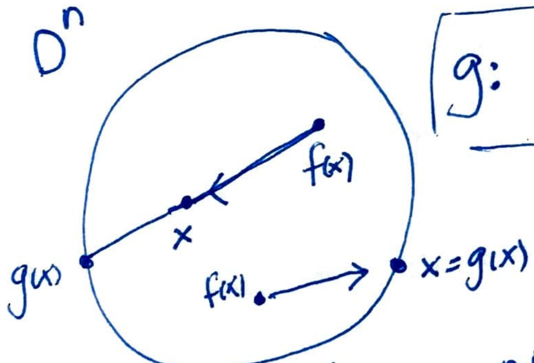
Any continuous map

$$f : D^n \longrightarrow D^n$$

$$D^n = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

has a fixed point.

Idea Suppose $\exists f : D^n \to D^n$ w/o fixed pt.



$$\boxed{g : D^n \longrightarrow S^{n-1}}$$

$$\partial D^n = S^{n-1} = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$$

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1) g is continuous.
(from f is continuous)

2) if $x \in \partial D^n \cong S^{n-1}$,
then $g(x) = x$.

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "i", "\text{inclusion}"'] \arrow[rr, bend right, "\mathrm{id}_{S^{n-1}}"'] & D^n \arrow[r, "g", "(\text{cont'l.})"'] & S^{n-1} \end{tikzcd}$$

<u>Claim</u>: $\nexists$ conti. $g: D^n \to S^{n-1}$ st.

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "i"] \arrow[rr, bend right, "\mathrm{id}_{S^{n-1}}"'] & D^n \arrow[r, "g"] & S^{n-1} \end{tikzcd}$$

<u>(general) Stokes thm:</u>

$$\underbrace{\partial B}_{(n-1)\text{-dim.}} \overset{i}{\hookrightarrow} \overbrace{B}^{n\text{-dim.}}$$

ω: "differential <u>(n-1)-form</u>" on B.

Then $$\underbrace{\int_B d\omega}_{n\text{-form.}} = \int_{\partial B} \underbrace{i^*\omega}_{\substack{(n-1)\text{-form} \\ \text{restriction to } \partial}}$$

<u>e.g.</u> $B = [a, b]$, $\partial B = \{a, b\}$

$\omega = f(x)$ on B.

$d\omega = f'(x)\,dx$

(FTC) $\displaystyle\int_{[a,b]} f'(x)\,dx =$ " " $f(b) - f(a)$

<u>e.g.</u> B: 2 dim. $\omega = F\,dx + G\,dy$, F, G fcns on B.
$\subset \mathbb{R}^2$

$$d\omega = \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dx\,dy$$

(Green) $$\iint_B \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dx\,dy = \oint_{\partial B} F\,dx + G\,dy$$

pf of Claim

M - cpt manifold of dim. K

$\exists^{\omega}$ K-form on M

s.t. $\int_M \omega = \mathrm{vol}(M) > 0$

$\omega_{S^{n-1}}$ the volume form on S^{n-1}

- $(n-1)$-form
- $\int_{S^{n-1}} \omega_{S^{n-1}} > 0$

$f^*\omega \quad \omega$

$f: M \to N$

$\boxed{\omega = id^*\omega}$

$$0 < \int_{S^{n-1}} \omega$$

$$= \int_{S^{n-1}} i^* g^* \omega$$

ω

$g: M \to N$

$$\overset{\text{Stokes}}{=} \int_{D^n} d(g^*\omega)$$

$$= \int_{D^n} g^*(d\omega)$$

n-form on S^{n-1}

$= 0$

$$= 0$$

Def A <u>topological space</u> is a set S, with a collection of subsets of S, s.t. $\mathcal{F}$ ← called open subset of S

1) $\phi, S \in \mathcal{F}$
2) if $U_\alpha \in \mathcal{F}$, $\alpha \in I$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{F}$
3) if $U_1, \dots, U_n \in \mathcal{F}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{F}$.

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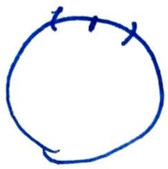
e.g. S = metric space
$\mathcal{F}$ = collection of open subsets of S

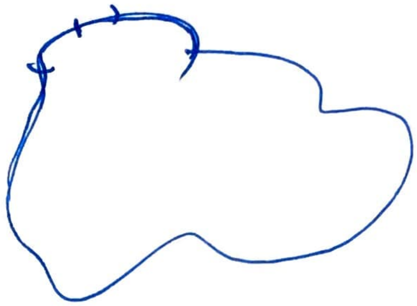
Def S_1, S_2 - top. space.
$f: S_1 \to S_2$ is <u>conti</u> if
$\forall U \underset{\text{open}}{\subset} S_2$, $f^{-1}(U) \subset S_1$ open.

Def S_1, S_2 are <u>homeomorphic</u> if
$\exists\ f: S_1 \underset{\text{conti.}}{\to} S_2$ and $g: S_2 \underset{\text{conti.}}{\to} S_1$
st $f \circ g = id_{S_2}$ and $g \circ f = id_{S_1}$

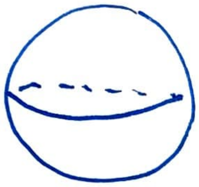
e.g.,

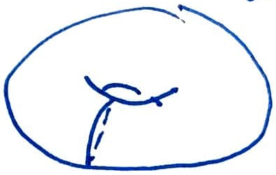


$\simeq$ homeom..



Q:





S^2 ...... T^2 cpt. subsets in $\mathbb{R}^3$

Associate certain invariants for each topological space.

Q We'll assign each topolgial space a certain abelian gps.

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e.g. $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{Q}\setminus\{0\}, \cdot)$

set G, w/ "$\cdot$"

- $g_1 g_2 = g_2 g_1$
- $\exists e \in G$, $eg = ge = g$
- $\forall g$, $\exists g^{-1}$

Def A gp homomorphism $f: G_1 \to G_2$

$$f(ab) = f(a) \cdot f(b)$$

$$f(e_1) = e_2 \ldots$$

$$H_K : \begin{Bmatrix} \text{top space} \\ \text{conti. maps} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{abel gp} \\ \text{gp homom.} \end{Bmatrix}$$

- each top. space $X \rightsquigarrow$ abel. gp. $\underline{H_K(X)}$
- each conti. map $f: X \to Y \rightsquigarrow$ gp hom.

$$X \xrightarrow{f} Y \xrightarrow{g} Z \qquad H_K(f): H_K(X) \to H_K(Y)$$

s.t. • $H_K(\underbrace{g \circ f}_{X \to Z}) = H_K(g) \circ H_K(f)$

• $H_K(\underbrace{id_X}_{X \to X}) = id_{H_K(X)}$

$X \cong Y$ i.e.

$f: X \to Y$, $g: Y \to X$

s.t. $g \circ f = id_X$, $f \circ g = id_Y$

$$H_K(g \circ f) = \underline{H_K(g)} \circ \underbrace{\underline{H_K(f)}}_{H_K(X) \to H_K(Y)}$$

$$\begin{aligned} &\| \\ &H_K(id_X) \\ &\| \\ &id_{H_K(X)} \end{aligned}$$

pf of Brouwer fixed pt

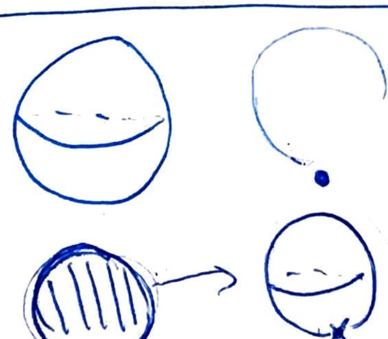
$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "i"] \arrow[rr, bend right, "id"'] & D^n \arrow[r, "g"] & S^{n-1} \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} H_{n-1}(S^{n-1}) \arrow[r, "i_*"] \arrow[rr, bend right, "id"'] \arrow[d, phantom, "\cong"] & H_{n-1}(D^n) \arrow[r, "g_*"] \arrow[d, phantom, "\cong"] & H_{n-1}(S^{n-1}) \arrow[d, phantom, "\cong"] \\ \mathbb{Z} \arrow[rr, bend right] & 0 & \mathbb{Z} \\ 1 \arrow[rr, mapsto] & & 1 \end{tikzcd}$$

$\square$

- $H_k(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } k=0, n-1 \\ 0 \end{cases}$
- $H_k(D^n) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ 0 \end{cases}$

S^2



$f_2:$ 

$f_0: \cdot \longmapsto \cdot$

"glue" the a 2-cell and a 0-cell to get S^2.

Def A <u>CW complex</u> is a top. space X

w/ $\{ f_\alpha : D^{n_\alpha} \to X \}$ s.t.

- $X = \cup f_\alpha(D^{n_\alpha})$
- $f_\alpha|_{(D^{n_\alpha})^\circ} : (D^{n_\alpha})^\circ \to X$

  homeom. onto its image

- $f_\alpha(\partial D^{n_\alpha})$ is the union of

  $\{ f_{\beta_1}(D^{n_{\beta_1}}), \dots, f_{\beta_K}(D^{n_{\beta_K}}) \}$

  for some $n_{\beta_1}, \dots, n_{\beta_K} < n_\alpha$.

e.g. S^2

$D^2 \xrightarrow{f_2^{(1)}} S^2$

$D^2 \xrightarrow{f_2^{(2)}} S^2$

equator

S^1

$D^1 \xrightarrow{f_1^{(1)}} S^2$

$D^1 \xrightarrow{f_1^{(2)}} S^2$

$D^0 \xrightarrow{f_0^{(1)}} S^2$

$\bullet \longmapsto \bullet$

$D^0 \xrightarrow{f_0^{(2)}} S^2$

$\bullet \longmapsto \bullet$

2-dim	1-dim	0-dim

$\mathbb{Z}^2 \to \mathbb{Z}^2 \to \mathbb{Z}^2$

$f_2^{(1)} \mapsto f_1^{(1)} + f_1^{(2)}$

$f_2^{(2)} \mapsto -(f_1^{(1)} + f_1^{(2)})$