3/17/2020 Power series

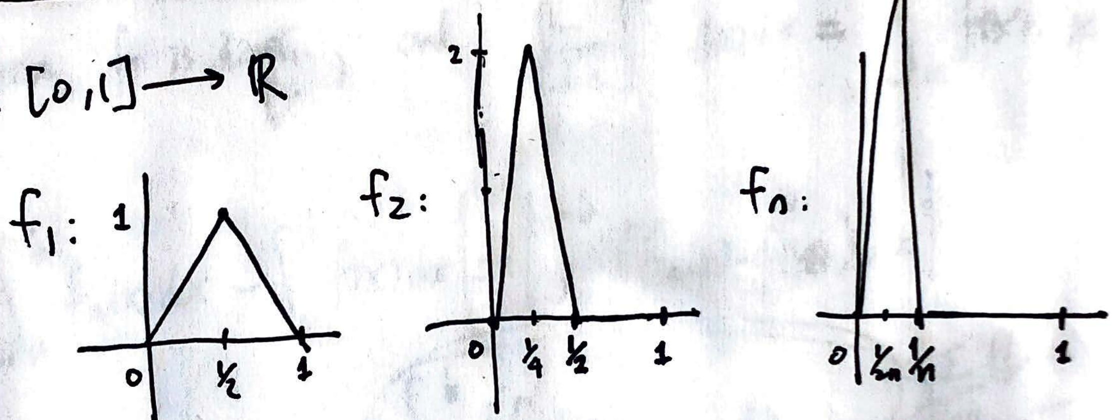
X : cpt metric space

$f_n: X \to \mathbb{R}$ (conti.)

$f_n \to f$ pointwisely $\quad f: X \to \mathbb{R}$

$\not\Rightarrow$ $f_n \to f$ uniformly

$f_n: [0,1] \to \mathbb{R}$

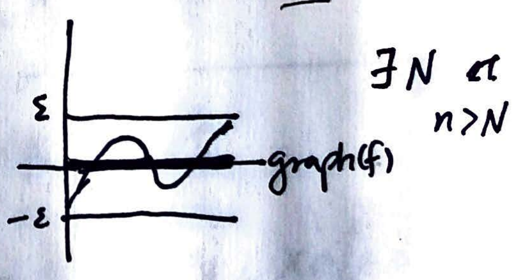


- $f_n \to f \equiv 0$ pointwise

  for $x=0$: $f_n(0) \equiv 0$

  for $x \neq 0$: $\exists\, n$ s.t. $x > \frac{1}{n} \Rightarrow f_n(x) = 0$

- $f_n \to 0$ not uniformly.



Thm $f_n: [a,b] \to \mathbb{R}$ integrable, $f_n \to f$ unif.

$\Rightarrow f$ is integrable, and $\lim_{n\to\infty} \int f_n = \int f$

②

$\int_0^1 f_1(x)\,dx = \frac{1}{2}$, $\quad \int_0^1 f_n(x)\,dx = \frac{1}{2}$

$$\int_0^1 f(x)\,dx = 0$$

$\Rightarrow f_n \to f$ not uniformly.

<u>Thm</u> $f_n : [a,b] \to \mathbb{R}$ conti. $f_n \to f$ unif.

Then f is conti., and $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

$\checkmark$

pf $\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| = \left| \int_a^b (f_n(x) - f(x))\,dx \right|$

$$\leq \int_a^b |f_n(x) - f(x)|\,dx < \varepsilon(b-a) \cdot \frac{1}{b-a} < \varepsilon$$

<u>WTS</u> $\forall \varepsilon > 0, \exists N > 0$ st.

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x) \right| < \varepsilon \quad \forall n > N$$

By $f_n \to f$ unif. $\exists N > 0$ st.

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \forall n > N,\ x \in [a,b].$$

Last time Given $(a_n) \subset \mathbb{R}$

- For what $x \in \mathbb{R}$,

  does the power series $\sum_{n=0}^{\infty} a_n x^n$ converge?

- What are some properties of the power series?

Thm $\beta = \limsup |a_n|^{1/n}$, $R = \frac{1}{\beta}$

(radius of convergence)

Then $\sum a_n x^n$ conv. if $|x| < R$

$\sum a_n x^n$ div. if $|x| > R$

Rmk $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{1/n} \leq \limsup |a_n|^{1/n} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$

So, if $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists, then $\beta = \lim \left|\frac{a_{n+1}}{a_n}\right|$

e.g. $\sum_{n=0}^{\infty} x^n$, i.e. $a_n \equiv 1 \ \forall n$, $R = 1$

- conv. if $|x| < 1$
- div. if $|x| > 1$
- for $x = 1$, $\rightarrow 1+1+1+\cdots$ div.
- for $x = -1$ $\rightarrow 1-1+1-1+\cdots$ div.

Interval of Convergence: $\underline{(-1,1)}$

e.g. $\sum_{n=1}^{\infty} \frac{x^n}{n}$, $a_n = \frac{1}{n}$, $R = 1$

- $x = 1 \to 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ div.
- $x = -1 \to -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \cdots$ conv.

Interval of conv. = $[-1, 1)$

e.g. $\sum_{n=1}^{\infty} \frac{x^n}{n^2} \to$ interval of conv. = $[-1, 1]$

Rmk The interval of convergence could be open/closed/half-open-half-closed interval, depending on the power series.

e.g. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $a_n = \frac{1}{n!}$

$\parallel$

$\exp(x)$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim \frac{1}{n+1} = 0$$

$R = +\infty$

$\Downarrow$

power series converges for any $x \in \mathbb{R}$

Rmk One can also consider $\sum a_n (x - x_0)^n$ (x_0: const.)

The interval of conv. centered at x_0

Properties of power series

Thm $\sum a_n x^n$ w/ radius of conv. $R > 0$

If $0 < R' < R$, then $\sum a_n x^n$ ~~conv. unif.~~

conv. unif. on $[-R', R']$.

pf (By Weierstrass M-test)

Recall W. M-test: $|f_n(x)| < M_n \ \forall x$, $\sum M_n < +\infty$
$\Rightarrow \sum f_n$ conv. unif.

$\forall x \in [-R', R']$,

$|a_n x^n| \leq |a_n| \cdot R'^n$

$\sum |a_n| \cdot R'^n$ conv.? By Root Test.

$\limsup (|a_n| \cdot R'^n)^{1/n}$

$= R' \cdot \limsup |a_n|^{1/n}$

$= R' \cdot \frac{1}{R} < 1$

$\Rightarrow \sum a_n x^n$ conv. unif on $[-R', R']$ by Weierstrass M-test. $\square$

Coro $\sum a_n x^n$. w/ radius of conv. $R>0$

$\Rightarrow \sum a_n x^n$ is a conti. fcn on $(-R,R)$

Pf We just proved : $\forall\, 0<R'<R$,

$\sum a_n x^n$ conv. unif. on $[-R', R']$

$\Longrightarrow \sum a_n x^n$ is conti. on $[-R', R']$

So $\forall x_0 \in (-R,R)$,

$\exists R'<R$ st. $x_0 \in \underbrace{[-R', R']}$

power series is conti. $\square$

Thm $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of conv. $R>0$

Then $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ also has r.o.c. $R>0$

and $\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad \forall |x|<R$

(power series can be integrated term-by-term inside $|x|<R$)

pf

① $\sum \frac{a_n}{n+1} x^{n+1}$ has r.o.c. $R > 0$

$$\limsup \left|\frac{a_n}{n+1}\right|^{1/n} = \left(\limsup |a_n|^{1/n}\right)\underbrace{\left(\lim \left(\frac{1}{n+1}\right)^{1/n}\right)}_{= 1}$$

$$\lim n^{1/n} = 1$$

$\Rightarrow \sum \frac{a_n}{n+1} x^{n+1}$ & $\sum a_n x^n$ have the same r.o.c.

② WTS: $\int_0^{x_0} f(x)\,dx = \boxed{\sum_{n=0}^{\infty} \frac{a_n}{n+1} x_0^{n+1}}$ $\forall |x_0| < R$

$0 < x_0 < R$

$\sum a_n x^n$ conv. unif. on $[0, x_0]$

$$\lim_{n\to\infty} \boxed{\int_0^{x_0} \left(\sum_{k=0}^{n} a_k x^k\right) dx} = \int_0^{x_0} f(x)\,dx$$

$$\downarrow$$

$$\sum_{k=0}^{n} \int_0^{x_0} a_k x^k \, dx = \sum_{k=0}^{n} a_k \frac{x_0^{k+1}}{k+1}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k \frac{x_0^{k+1}}{k+1} = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x_0^{n+1} \quad \square$$

e.g. $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$

$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \int_0^x \frac{1}{1-t}\,dt = -\log(1-x)$

for any $|x|<1$

$\|$

$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

$\Rightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \forall |x|<1$

(Taylor series of $\log(1+x)$ at $x=0$)

Q: Plug in $x=1$.

LHS $= \log 2$

RHS $= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

Are they the same?

(before, we only proved f conti. on $(-R,R)$)

Abel's thm $f(x) = \sum a_n x^n$ w/ radius $R>0$.

If f is convergent at $x=R$, then f is conti. at $x=R$

— — — — $x=-R$ — — — — — — — $x=-R$

($\Rightarrow$ power series is conti. on the interval of convergence)

Pf Suppose $f(x) = \sum a_n x^n$ has r.o.c. $= 1$,

and conv. at $x = 1$.

<u>WTS:</u> f is conti. on $[0, 1]$.

(root test doesn't give info on $x = 1$)

$\uparrow$

$\sum a_n x^n$ unif. conv. to f on $[0,1]$

$\Updownarrow$

<u>Unif Cauchy</u>: $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\left| \sum_{k=m}^{n} a_k x^k \right| < \varepsilon$ $\forall n \geq m > N$ $\forall x \in [0,1]$

<u>WLOG</u>, we can subtract f by a const. s.t. $f(1) = 0 = \sum a_n$

$$\sum_{k=m}^{n} a_k x^k = \sum_{k=m}^{n} (S_k - S_{k-1}) x^k \quad \Big| \quad S_k = \sum_{l=1}^{k} a_l$$

$$= \sum_{k=m}^{n} S_k x^k - \sum_{k=m}^{n} S_{k-1} x^k$$

$$= \sum_{k=m}^{n-1} (1-x) S_k x^k + S_n x^n - S_{m-1} x^m$$