<u>3/31/2020</u>

**<u>Recall</u>** $f: I \longrightarrow \mathbb{R}$ is <u>differentiable at $a \in I$</u>, if

$I$: open interval containing $a \in \mathbb{R}$

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} \text{ exists and finite.}$$

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} =: f'(a)$$

**<u>Def</u>** $X$: metric space, $f: X \to \mathbb{R}$.

Say $x_0 \in X$ is a <u>local min of $f$</u> if

$\exists\, r > 0$ s.t. $f(x) \geq f(x_0)$ $\forall\, x \in B_r(x_0)$

Similarly, local max.

**<u>Thm</u>** $f: I \to \mathbb{R}$, $f$ is diff'ble at $a \in I$.

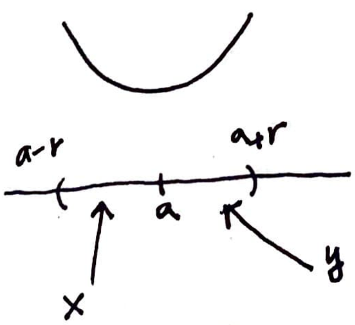
Suppose $a$ is a local min/max of $f$.

Then $f'(a) = 0$

**<u>pf</u>** Say $a$ is a local min. of $f$, i.e.

$\exists\, r > 0$ s.t.

$f(x) \geq f(a)$ $\forall\, |x-a| < r$




$$\frac{\overbrace{f(y)-f(a)}^{\geq 0}}{\underbrace{y-a}_{>0}} \geq 0$$

$$\frac{\overbrace{f(x)-f(a)}^{\geq 0}}{\underbrace{x-a}_{<0}} \leq 0$$

So if
$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \text{ exists,}$$

then $f'(a) = 0$. $\square$

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Recall $\forall (x_n)$ s.t. $\lim x_n = a$, $x_n \neq a \ \forall n$

we have $\displaystyle\lim_{n\to\infty} \frac{f(x_n)-f(a)}{x_n - a} = f'(a)$

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So, if we take $(x_n) \subset (a-r, a)$

that conv. to $a$,

$\Rightarrow f'(a) \leq 0$

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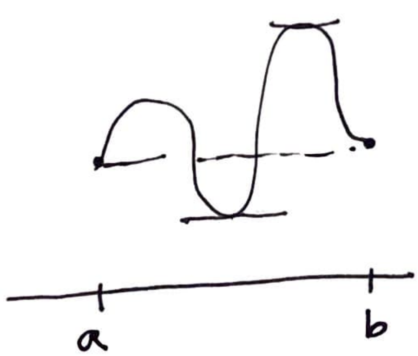
Thm (Rolle) $f : [a,b] \longrightarrow \mathbb{R}$ conti,

& diff'ble on $(a,b)$

& $f(a) = f(b)$.

$\Rightarrow \exists\, c \in (a,b)$ st. $f'(c) = 0$



pf Let $y = f(a) = f(b)$

① $\exists c \in (a,b)$ s.t. $f(c) > y$.

By extremum value thm (conti. fcn on cpt set).

max. of $f$ on $[a,b]$ is attained

at $d \in [a,b]$.

And $d \neq a, b$, $\rightsquigarrow$ $d \in (a,b)$

$d$ is a local max. of $f$. $\Rightarrow f'(d) = 0$

② $\exists c \in (a,b)$ s.t. $f(c) < y$.

$\rightsquigarrow$ again by extreme value thm

to get "global min" of $f$.

③ $f(x) = y \quad \forall x \in [a,b]$

$f'(x) = 0 \quad \forall x \in (a,b)$ □

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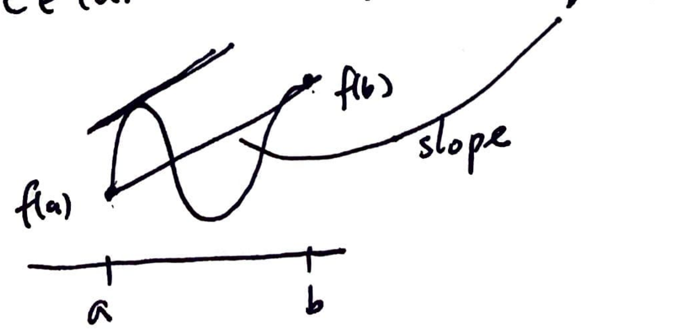
<u>Thm</u> (Mean value thm) $f : [a,b] \to \mathbb{R}$ conti,

& diff'ble on $(a,b)$

$\Rightarrow \exists c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$



<u>Idea</u> "tilt" the graph of f

s.t. the values at the

end pts are the same

<u>pf</u> $L(x) := (x-a)\cdot\dfrac{f(b)-f(a)}{b-a}$

$L(a)=0, \quad L(b)=f(b)-f(a)$

$g(x) := f(x)-L(x)$

$g(a)=f(a), \quad g(b)=f(a)$

By Rolle's Thm,

$\exists\, c\in(a,b)$ s.t. $\underline{\underline{g'(c)=0}}$

$g'(x)=f'(x)-L'(x)$

$\quad = f'(x)-\dfrac{f(b)-f(a)}{b-a}$

$0=g'(c)=f'(c)-\dfrac{f(b)-f(a)}{b-a}$

$\Rightarrow f'(c)=\dfrac{f(b)-f(a)}{b-a}.$ $\square$

# Appliations

**Prop** $f: I \longrightarrow \mathbb{R}$ diff$^{ble}$

& $f'(x) = 0 \quad \forall x \in I$

$\Longrightarrow$ $f$ is a const fcn.

**pf** Take any $x \neq y$ in $I$.

By MVT, $\exists c$ btw $x$ & $y$

s.t. $f'(c) = \dfrac{f(y) - f(x)}{y - x}$

$\quad\;\; \| $

$\quad\;\; 0$

$\Rightarrow$ $f(y) = f(x)$. $\square$

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**Recall** $f: (a,b) \longrightarrow \mathbb{R}$ is Lipschitz conti

iff $\exists K > 0$ s.t.

$|f(x) - f(y)| < K \cdot |x - y| \quad \forall x, y \in (a,b)$

($\Rightarrow$ unif. conti.)

**Prop** $f: (a,b) \to \mathbb{R}$ diff$^{ble}$. Then

$f$ is Lip conti. $\Leftrightarrow$ $f': (a,b) \longrightarrow \mathbb{R}$ is bounded.

pf ($\Leftarrow$) $f'$ bounded $\to$ by $M$

WTS: $f$ Lip. conti.

$x, y \in (a,b)$

MVT $\Rightarrow \exists\, z \in (a,b)$

s.t. $f'(z) = \dfrac{f(y)-f(x)}{y-x}$

$\Rightarrow \left|\dfrac{f(y)-f(x)}{y-x}\right| = |f'(z)| < M$

$\Rightarrow |f(y)-f(x)| < M \cdot |y-x|$ $\square$

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Thm (Chain Rule)

$f: I \longrightarrow \mathbb{R}$, $g: J \longrightarrow \mathbb{R}$

Suppose $f$ is diff$^{ble}$ at $a \in I$,

$g$ is diff$^{ble}$ at $f(a) \in J$.

$\Rightarrow g \circ f$ is diff$^{ble}$ at $a$ and

$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

Idea (which doesn't work)

$$\frac{g(f(x))-g(f(a))}{x-a} = \frac{g(f(x))-g(f(a))}{f(x)-f(a)} \cdot \frac{f(x)-f(a)}{x-a}$$

$x \to a$ and $x \neq a$ | may be zero for $x \neq a$ | $x \to a$ and $x \neq a$

<u>WTS</u>: $\forall$ seq. $(x_n)$ s.t. $\lim x_n = a$, $x_n \neq a$ $\forall n$,

we have $\displaystyle\lim_{n\to\infty} \frac{g(f(x_n)) - g(f(a))}{x_n - a} = g'(f(a)) \cdot f'(a)$

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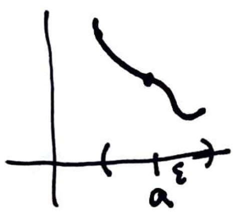
<u>Case 1</u> $\exists \varepsilon > 0$ s.t. $f(x) \neq f(a)$ $\forall$ $0 < |x-a| < \varepsilon$




Since $x_n \to a$,

$\exists N > 0$

s.t. $0 < |x_n - a| < \varepsilon$ $\forall n > N$

$\Rightarrow$ $\forall n > N$, $f(x_n) \neq f(a)$

$$\frac{g(f(x_n)) - g(f(a))}{x_n - a} = \frac{g(f(x_n)) - g(f(a))}{f(x_n) - f(a)} \cdot \frac{f(x_n) - f(a)}{x_n - a}$$

As $n \to \infty$.

$$\frac{g(f(x_n)) - g(f(a))}{f(x_n) - f(a)} \to g'(f(a)) \qquad \frac{f(x_n) - f(a)}{x_n - a} \to f'(a)$$

since $f(x_n) \to f(a)$

($f$ diff$^{ble}$ at $a$ $\Rightarrow$ conti. at $a$)

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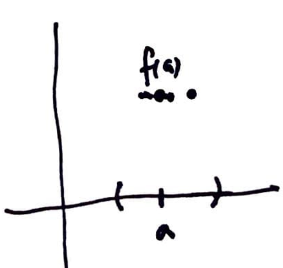
<u>Case 2</u> $\forall \varepsilon > 0$, $\exists$ $0 < |x-a| < \varepsilon$ s.t. $f(x) = f(a)$



$\forall n$, take $\varepsilon = \frac{1}{n}$, $\exists z_n$

s.t. $0 < |z_n - a| < \frac{1}{n}$, $f(z_n) = f(a)$

$(z_n) \to a$, $z_n \neq a \ \forall n$

$$f'(a) = \lim_{n\to\infty} \frac{f(z_n) - f(a)}{z_n - a} = 0$$

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__WTS__ $\forall$ seq. $(x_n)$ s.t. $\lim x_n = a$, $x_n \neq a$

$$\boxed{\lim_{n\to\infty} \frac{g(f(x_n)) - g(f(a))}{x_n - a} =} \ g'(f(a))\, f'(a) = \boxed{0}$$

__in Case 2__

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Since $g$ is diff$^{ble}$ at $f(a)$,

$\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $\left| \frac{g(y) - g(f(a))}{y - f(a)} - g'(f(a)) \right| < \cancel{\varepsilon}\ 1$

$\forall\, 0 < |y - f(a)| < \delta$

$$\Rightarrow \left| \frac{g(y) - g(f(a))}{y - f(a)} \right| < C \quad \forall\, 0 < |y - f(a)| < \delta$$

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$\lim f(x_n) = f(a) \Rightarrow \exists N > 0$ s.t. $|f(x_n) - f(a)| < \delta$ $\forall n > N$

$$\boxed{\text{Claim } \left| \frac{g(f(x_n)) - g(f(a))}{x_n - a} \right| \le \left| \frac{f(x_n) - f(a)}{x_n - a} \right| \cdot C \quad \forall n > N}$$

① $f(x_n) \neq f(a)$

② $f(x_n) = f(a)$

0 0 as $n \to \infty$ b/c $f'(a) = 0$