On entropy of $\mathbb{P}$-twists

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Abstract

We show that the $\mathbb{P}$-twist associated with any $\mathbb{P}$-object of a smooth project variety is not conjugate to any standard autoequivalence. This result is obtained by computing the categorical entropy functions of $\mathbb{P}$-twists. We also determine the categorical polynomial entropy of spherical twists and $\mathbb{P}$-twists, under an additional assumption.

1 Introduction

Let X be a smooth projective variety over $\mathbb{C}$, and let $\mathrm{D}^{\mathrm{b}}(X)$ denote the bounded derived category of coherent sheaves on X. The group of autoequivalences $\operatorname{Aut}\mathrm{D}^{\mathrm{b}}(X)$ has been extensively studied in the literature. It always contains the subgroup of *standard* autoequivalences:

$$\operatorname{Aut}_{\mathrm{std}}\mathrm{D}^{\mathrm{b}}(X) \coloneqq (\operatorname{Pic}(X) \rtimes \operatorname{Aut}(X)) \times \mathbb{Z}[1],$$

which are generated by the automorphisms of X, tensoring line bundles, and shifts.

The existence of non-standard autoequivalences in the derived category can reveal geometric characteristics of the underlying algebraic variety X. For example, a classic result by Bondal and Orlov [BO01] states that if the canonical bundle (or its inverse) of X is ample, then its derived category has no autoequivalences other than the standard ones. On the other hand, for algebraic varieties with trivial canonical bundles, such as Calabi–Yau or hyperkähler varieties, their derived categories possesses richer symmetry elements like *spherical twists* [ST01] or $\mathbb{P}$-*twists* [HT06]. This leads to a natural question: given an autoequivalence $\Phi \in \operatorname{Aut}\mathrm{D}^{\mathrm{b}}(X)$, can we

determine whether Φ is conjugate to an element of $\operatorname{Aut}_{\mathrm{std}}\mathrm{D}^{\mathrm{b}}(X)$?

Spherical twists, as introduced in [ST01], are some of the simplest candidates for non-standard autoequivalences. Spherical twists can be seen as the *mirror* of Dehn twists along Lagrangian spheres under homological mirror symmetry, and thus have their origin in symplectic geometry, unlike standard autoequivalences. It was only recently proved in [LL23, Theorem A.2], using results on *categorical entropy functions* of spherical twists in [Ouc20, Theorem 1.4], that spherical twists are not conjugate to any standard autoequivalence.

The categorical entropy function $h_t(\Phi)\colon \mathbb{R} \to \mathbb{R}$, introduced in [DHKK14], is a *dynamical* invariant associated with an endofunctor of a triangulated category. Roughly speaking, it measures the exponential growth rate of certain quantities under large iterations Φ^n. Its precise definition is provided in Section 2.1. Note that the entropy function is conjugacy-invariant. Therefore, to prove that spherical twists are not conjugate to any standard autoequivalence, it suffices to show that their categorical entropy functions are different. The proof consists of the following steps:

(i) The categorical entropy function of the spherical twist T_E of a spherical object E is computed in [Ouc20, Theorem 1.4], under the condition that $E^\perp \neq \{0\}$ in $\mathrm{D^b}(X)$.

(ii) It is proved in [LL23, Proposition 1.1] that $E^\perp \neq \{0\}$ for any spherical object $E \in \mathrm{D^b}(X)$. Thus, the calculation of the categorical entropy function of T_E is complete.

(iii) It is not difficult to compute the categorical entropy functions of standard autoequivalences, and show that they differ from the categorical entropy functions of spherical twists obtained in Steps (i) and (ii).

In [HT06], the notion of $\mathbb{P}$-*twists* associated to $\mathbb{P}$-objects is introduced. Similar to spherical twists, $\mathbb{P}$-twists have their origins in symplectic geometry, where they can be regarded as the mirror of Dehn twists along Lagrangian complex projective space. One would expect that $\mathbb{P}$-twists also are not conjugate to any standard autoequivalence. Indeed, it is not difficult to obtain the analogous result of Step (i) for $\mathbb{P}$-twists, see Section 2.3. However, verifying the condition $E^\perp \neq \{0\}$ in $\mathrm{D^b}(X)$, as outlined in Step (ii), is quite challenging for a general $\mathbb{P}$-object.

In the present article, a novel approach is adopted. By leveraging recent advancements in dynamical invariants, specifically the *shifting numbers* of autoequivalences, as developed by Filip and the author [FF23, Fan24], the entropy functions of spherical twists and $\mathbb{P}$-twists are computed without the assumption $E^\perp \neq \{0\}$ in $\mathrm{D^b}(X)$. Consequently, the following theorem is established.

Theorem 1.1. *Consider a complex smooth projective variety X of dimension $2d$, and let $E \in \mathrm{D^b}(X)$ be a $\mathbb{P}^d$-object with the associated $\mathbb{P}^d$-twist $P_E \in \operatorname{Aut}\mathrm{D^b}(X)$. Let k be a nonzero integer, and Y be a smooth projective variety with an exact equivalence $\Psi\colon \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$.*

There does not exist a standard autoequivalence $\Phi \in \operatorname{Aut_{std}}\mathrm{D^b}(Y)$ such that $P_E^k = \Psi^{-1} \circ \Phi \circ \Psi$. In particular, P_E^k is not conjugate to any standard autoequivalence of X.

We now describe in more detail how to remove the assumption $E^\perp \neq \{0\}$. First, for any endofunctor Φ, its entropy function $h_t(\Phi)$ grows linearly as $t \to +\infty$ (resp. $t \to -\infty$), with slope $\tau^+(\Phi)$ (resp. $\tau^-(\Phi)$), called the *upper* (resp. *lower*) *shifting number* of Φ (see Proposition 2.2). Roughly speaking, shifting numbers measure the asymptotic amount by which Φ translates in the triangulated category [FF23, Fan24]. The shifting numbers are conjugacy invariants and always satisfy $\tau^+(\Phi) \geq \tau^-(\Phi)$. The proof of Theorem 1.1 consists of the following steps:

(i) For standard autoequivalences, the upper and lower shifting numbers coincide. Thus, if the strict inequality $\tau^+(P_E) > \tau^-(P_E)$ holds (which implies $\tau^+(P_E^k) > \tau^-(P_E^k)$ for any $k \neq 0$), then P_E^k is not conjugate to any standard autoequivalences.

(ii) It is not difficult to obtain $\tau^-(P_E) = -2d$ and $\tau^+(P_E) \leq 0$. The main difficulty lies in proving $\tau^+(P_E) = 0$.

(iii) $\tau^+(P_E) = 0$ can be ensured if $E^\perp \neq \{0\}$. However, without assuming $E^\perp \neq \{0\}$, we can use results from [Fan24] to show that $\tau^+(P_E) = 0$ if there exists a nonzero object $F \in \mathrm{D}^{\mathrm{b}}(X)$ and a heart $\mathscr{A} \subseteq \mathrm{D}^{\mathrm{b}}(X)$ such that

$$\lim_{n\to\infty} \frac{\phi^+_{\mathscr{A}}(P_E^n(F))}{n} = 0,$$

where $\phi^+_{\mathscr{A}}(P_E^n(F))$ denotes the maximal degree of cohomology objects of $P_E^n(F)$ with respect to the heart $\mathscr{A}$ (see Notation 2.6).

(iv) The desired limit is satisfied by $\mathcal{A} = \mathrm{Coh}(X)$ and $F = \mathcal{O}(-m)$ for sufficiently large m. This proves that $\tau^+(P_E) = 0 > -2d = \tau^-(P_E)$.

We also study the *categorical polynomial entropy functions* of spherical twists and $\mathbb{P}$-twists. The categorical polynomial entropy function, as defined in [FFO21], is a refined secondary invariant that measures the *polynomial growth*, rather than exponential growth, of certain quantities under large iterations of an endofunctor. It is proved in [FFO21, Section 6], that for a spherical twist (or a $\mathbb{P}$-twist) along a spherical object (or a $\mathbb{P}$-object) E, the polynomial entropy function h_t^{pol} is given by:

- $h_t^{\mathrm{pol}} = 0$ for $t < 0$.
- $h_t^{\mathrm{pol}} = 0$ for $t > 0$, under the assumption that $E^\perp \neq \{0\}$.
- $0 \leq h_0^{\mathrm{pol}} \leq 1$.

Using a similar approach as in the proof of Theorem 1.1, we remove the assumption $E^\perp \neq \{0\}$ for the case $t > 0$. Furthermore, we establish $h_0^{\mathrm{pol}} = 1$ under an additional assumption.

Theorem 1.2. *Let X be a complex smooth projective variety of dimension $d \geq 2$. Let $E \in \mathrm{D}^{\mathrm{b}}(X)$ be a spherical object (resp. $\mathbb{P}$-object) with the associated spherical twist $T_E \in \mathrm{Aut}\,\mathrm{D}^{\mathrm{b}}(X)$ (resp. $\mathbb{P}$-twist $P_E \in \mathrm{Aut}\,\mathrm{D}^{\mathrm{b}}(X)$). Then*

(i) For $t \neq 0$, $h_t^{\mathrm{pol}}(T_E) = 0$ (resp. $h_t^{\mathrm{pol}}(P_E) = 0$).

(ii) If there exists a Bridgeland stability condition $\sigma = (Z, \mathscr{P})$ on $\mathcal{D}$ such that $E \in \mathscr{P}(-1, 1]$, then $h_0^{\mathrm{pol}}(T_E) = 1$ (resp. $h_0^{\mathrm{pol}}(P_E) = 1$).

For instance, this fully determines the polynomial entropy function of spherical twists for K3 surfaces of Picard number one, cf. [BB17, Corollary 6.9].

Convention. Throughout this article, all triangulated categories (usually denoted by $\mathcal{D}$) are assumed to be $\mathbb{Z}$-graded, linear over a base field $\mathbf{k}$, saturated (i.e. admit a dg-enhancement which is smooth and proper), and of finite type (i.e. $\oplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(E, F[k])$ is finite-dimensional for any pair of objects E, F in $\mathcal{D}$). Furthermore, we assume that $\mathcal{D}$ admits a Serre functor $\mathbf{S}$, and contains a split generator $G \in \mathcal{D}$. Endofunctors of triangulated categories are assumed to be $\mathbf{k}$-linear, triangulated, and not virtually zero (i.e. any power is not the zero functor).

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2 Preliminaries

2.1 Dynamical invariants of endofunctors

In this subsection, we recall the definitions and basic properties of various dynamical invariants associated with endofunctors of triangulated categories.

Let $\Phi\colon \mathcal{D} \to \mathcal{D}$ be an endofunctor of a triangulated category $\mathcal{D}$. The *categorical entropy function* of Φ, introduced in [DHKK14, Definition 2.5], is a function $h_t(\Phi)\colon \mathbb{R} \to [-\infty, \infty)$ that depends on the variable t. According to [DHKK14, Theorem 2.7], it can be expressed as follows:

$$h_t(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{k\in\mathbb{Z}} \dim \operatorname{Hom}\left(G, \Phi^n G'[k]\right) e^{-kt} \right), \tag{2.1}$$

where G and G' are split generators of $\mathcal{D}$. For convenience, we denote

$$\epsilon_t(M, N) \coloneqq \sum_{k\in\mathbb{Z}} \dim \operatorname{Hom}(M, N[k]) e^{-kt}.$$

We collect some of the basic properties of the categorical entropy function $h_t(\Phi)$ in the following proposition.

Proposition 2.1. *Consider an endofunctor $\Phi\colon \mathcal{D} \to \mathcal{D}$ of a triangulated category $\mathcal{D}$.*

(i) The limit in (2.1) exists and is independent of the choice of the generators G, G'. See [DHKK14, Lemma 2.6 and Theorem 2.7].

(ii) The categorical entropy function $h_t(\Phi)\colon \mathbb{R} \to \mathbb{R}$ is real-valued and convex. See [FF23, Theorem 2.1.6].

(iii) $h_0(\Phi) \geq 0$. See [DHKK14, Definition 2.5 and Theorem 2.7].

(iv) Let $\Psi\colon \mathcal{D} \to \mathcal{D}'$ be an exact equivalence. Then $h_t(\Phi) = h_t(\Psi \circ \Phi \circ \Psi^{-1})$. See [Kik17, Lemma 2.9].

(v) $h_t(\Phi[\ell]) = h_t(\Phi) + \ell t$ for any integer ℓ.

(vi) $h_t(\Phi^k) = k h_t(\Phi)$ for $k \in \mathbb{Z}_{\geq 1}$.

(vii) $h_t(\Phi^{-1}) = h_{-t}(\Phi)$ for an autoequivalence Φ. See [FFO21, Lemma 2.11].

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There are also invariants called the *upper* and *lower shifting numbers* of Φ, which are related to the entropy function $h_t(\Phi)$ in the following way:

Proposition 2.2 ([EL21, Proposition 6.13], [FF23, Theorem 2.1.7])**.** *There exist real numbers $\tau^+(\Phi)$ and $\tau^-(\Phi)$, called the* upper *and* lower shifting numbers *of Φ, such that $h_t(\Phi)$ is bounded within the following ranges:*

$$\begin{aligned} t\cdot\tau^+(\Phi) \le h_t(\Phi) \le t\cdot\tau^+(\Phi) + h_0(\Phi) \qquad & \text{for } t\ge 0,\\ t\cdot\tau^-(\Phi) \le h_t(\Phi) \le t\cdot\tau^-(\Phi) + h_0(\Phi) \qquad & \text{for } t\le 0. \end{aligned}$$

Note that $\tau^+(\Phi) \ge \tau^-(\Phi)$.

Remark 2.3. By Proposition 2.2, when $h_0(\Phi)=0$ (as is the case for spherical twists and $\mathbb{P}$-twists, which we will discuss later), the entropy function $h_t(\Phi)$ is completely determined by the shifting numbers $\tau^\pm(\Phi)$.

Example 2.4. Let X be a smooth projective variety. A standard autoequivalence $\Phi \in \operatorname{Aut}_{\mathrm{std}} \mathrm{D}^{\mathrm{b}}(X)$ is of the form

$$\Phi = f^* \circ (-\otimes L)[\ell]$$

for some automorphism $f\in\operatorname{Aut}(X)$, line bundle L, and integer ℓ. By [DHKK14, Lemma 2.11], $h_t\left(f^*\circ(-\otimes L)\right)$ is a constant function of t. Therefore, the entropy function of Φ is

$$h_t(\Phi) = h_0\left(f^*\circ(-\otimes L)\right) + \ell t = h_0(\Phi) + \ell t.$$

In particular, the upper and lower shifting numbers coincide $\tau^+(\Phi)=\tau^-(\Phi)=\ell$.

Remark 2.5. Proposition 2.1(iv) and Proposition 2.2 imply that the shifting numbers $\tau^\pm(\Phi)$ are conjugacy invariants. Therefore, if an autoequivalence Φ' has $\tau^+(\Phi')\ne\tau^-(\Phi')$, then it cannot be conjugate to any standard autoequivalence, by Example 2.4. This will be used to show that spherical twists and $\mathbb{P}$-twists are not conjugate to any standard autoequivalence.

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Now, we recall a method for computing the shifting numbers, as developed in [Fan24], using bounded t-structures on $\mathcal{D}$. A full additive subcategory $\mathscr{A}\subseteq\mathcal{D}$ is the *heart* of a bounded t-structure on $\mathcal{D}$ if and only if:

(i) $\operatorname{Hom}(A_1[k_1], A_2[k_2]) = 0$ if $k_1>k_2$ and $A_1, A_2\in\mathscr{A}$,

(ii) for every nonzero object $E\in\mathcal{D}$, there is a (unique) sequence of exact triangles

$$\begin{tikzcd} 0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] & \cdots \arrow[r] & E_{\ell-1} \arrow[rr] & & E \arrow[dl] \\ & A_1[k_1] \arrow[ul, dashed] & & A_2[k_2] \arrow[ul, dashed] & & & & A_\ell[k_\ell] \arrow[ul, dashed] & \end{tikzcd}$$

with $k_1>\cdots>k_\ell$ and $A_1,\ldots,A_\ell\in\mathscr{A}\backslash\{0\}$.

The object $A_i\in\mathscr{A}\backslash\{0\}$ is called the cohomology object of E at degree k_i, with respect to the heart $\mathscr{A}$.

Throughout this article, the following notations will be used.

Notation 2.6. We denote the maximal (resp. minimal) degrees and cohomology objects of E with respect to the heart $\mathscr{A}$ as follows:

$$\phi^+_{\mathscr{A}}(E) \coloneqq k_1, \quad \phi^-_{\mathscr{A}}(E) \coloneqq k_\ell, \quad E^+_{\mathscr{A}} \coloneqq A_1, \quad E^-_{\mathscr{A}} \coloneqq A_\ell.$$

Additionally, the following "cut-off" notations will also be used: For a real number s, suppose $k_p \geq s > k_{p+1}$, then we define

$$E^{\geq s}_{\mathscr{A}} \coloneqq E_p \qquad \text{and} \qquad E^{<s}_{\mathscr{A}} \coloneqq \mathrm{Cone}\left(E_p \to E\right).$$

They satisfy $\phi^-_{\mathscr{A}}(E^{\geq s}_{\mathscr{A}}) = k_p \geq s > k_{p+1} = \phi^+_{\mathscr{A}}(E^{<s}_{\mathscr{A}})$. The notions $E^{>s}_{\mathscr{A}}$ and $E^{\leq s}_{\mathscr{A}}$ are defined similarly, and they satisfy $\phi^-_{\mathscr{A}}(E^{>s}_{\mathscr{A}}) > s \geq \phi^+_{\mathscr{A}}(E^{\leq s}_{\mathscr{A}})$.

The following proposition shows that the shifting numbers can be computed via the linear growth rates of the maximal (and minimal) degrees of cohomology objects of $\Phi^n(G)$ as $n \to \infty$.

Proposition 2.7 ([Fan24, Theorem 1.1])**.** *Consider an endofunctor $\Phi\colon \mathcal{D} \to \mathcal{D}$ of a triangulated category $\mathcal{D}$ with a split generator G. Let $\mathscr{A} \subseteq \mathcal{D}$ be the heart of a bounded t-structure.*

(i) The limit
$$\lim_{n\to\infty} \frac{\phi^+_{\mathscr{A}}(\Phi^n G)}{n}$$
exists, is independent of the choices of G and $\mathscr{A}$, and coincides with $\tau^+(\Phi)$.

(ii) The limit
$$\lim_{n\to\infty} \frac{\phi^-_{\mathscr{A}}(\Phi^n G)}{n}$$
exists, is independent of the choices of G and $\mathscr{A}$, and coincides with $\tau^-(\Phi)$.

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Next, we recall the notion of *categorical polynomial entropy function* $h^{\mathrm{pol}}_t(\Phi)$ [FFO21]. It can be regarded as a more refined invariant: While the entropy function measures the *exponential* growth rate of $\epsilon_t(G, \Phi^n G')$, the polynomial entropy function measures its *polynomial* growth rate. It can be expressed as follows [FFO21, Lemma 2.7]:

$$h^{\mathrm{pol}}_t(\Phi) = \limsup_{n\to\infty} \frac{\log \epsilon_t(G, \Phi^n G') - n h_t(\Phi)}{\log(n)},$$

and is independent of the choices of split generators G, G'.

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Finally, we recall that when a triangulated category $\mathcal{D}$ admits a *stability condition* [Bri07], the complexity of an endofunctor can be measured by the growth rate of *mass* with respect to the stability condition [DHKK14, Section 4.5], [Ike21]. Let $\sigma = (Z, \mathscr{P})$ be a stability condition on $\mathcal{D}$, where $Z\colon K_0(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism, and $\mathscr{P} = \{\mathscr{P}(\phi)\}_{\phi\in\mathbb{R}}$ is a collection of full additive subcategories of $\mathcal{D}$, satisfying various axioms. For a nonzero object E, its *mass function* with respect to σ is the following real function in t:

$$m_{\sigma,t}(E) = \sum_k |Z(A_k)| e^{\phi(A_k)t}$$

where the A_k's are the σ-semistable factors of E. The *mass growth function* of an endofunctor $\Phi\colon \mathcal{D} \to \mathcal{D}$ is defined as:

$$h_{\sigma,t}(\Phi) = \limsup_{n\to\infty} \frac{1}{n} \log m_{\sigma,t}(\Phi^n(G)),$$

and is independent of the choice of the split generator G [Ike21, Theorem 3.5(1)]. Similarly, one can define the *polynomial mass growth function* [FFO21, Definition 3.3]:

$$h^{\mathrm{pol}}_{\sigma,t}(\Phi) = \limsup_{n\to\infty} \frac{\log m_{\sigma,t}(\Phi^n(G)) - nh_{\sigma,t}(\Phi)}{\log(n)}.$$

The (polynomial) entropy functions and the (polynomial) mass growth functions are related as follows:

Lemma 2.8. *Let $\Phi\colon \mathcal{D} \to \mathcal{D}$ be an endofunctor of a triangulated category. Then*

(i) $h_t(\Phi) \geq h_{\sigma,t}(\Phi)$*. See* [Ike21, Theorem 3.5(2)].

(ii) For any $t \in \mathbb{R}$, if $h_t(\Phi) = h_{\sigma,t}(\Phi)$, then $h^{\mathrm{pol}}_t(\Phi) \geq h^{\mathrm{pol}}_{\sigma,t}(\Phi)$. See [FFO21, Lemma 3.6].

Remark 2.9. We do not provide the full definition of stability conditions here. However, note that by the *support property* of stability conditions, there exists a constant $C > 0$ such that $|Z(M)| > C$ for any semistable object $M \in \mathscr{P}_\sigma(\phi)$. This implies that if an object $E \in \mathcal{D}$ has ℓ nonzero cohomology objects with respect to the heart $\mathscr{A}_\sigma = \mathscr{P}_\sigma(0, 1]$, then we have a lower bound $m_{\sigma,0}(E) > \ell \cdot C$.

2.2 Spherical twists

In this subsection, we recall the notion of *spherical twists* [ST01], and previous results on the categorical entropy functions of spherical twists [Ouc20].

An object $E \in \mathcal{D}$ is called a *d-spherical object* if $\mathbf{S}(E) \cong E[d]$ and

$$\dim \mathrm{Hom}(E, E[k]) = \begin{cases} 1 & \text{if } k = 0 \text{ or } d, \\ 0 & \text{otherwise.} \end{cases}$$

Examples of spherical objects include line bundles in the derived categories of Calabi–Yau varieties and structure sheaves of rational (-2)-curves in projective surfaces, among others.

One of the most interesting features of spherical objects is that they induce autoequivalences of the triangulated category, known as *spherical twists* [ST01]. For a d-spherical object E, the associated spherical twist, denoted by $T_E\colon \mathcal{D} \to \mathcal{D}$, sends an object F to

$$T_E(F) = \mathrm{Cone}\left(\mathbf{R}\mathrm{Hom}(E, F) \otimes E \xrightarrow{\mathrm{ev}} F\right).$$

It follows that $T_E(E) \cong E[1-d]$, and $T_E(F) \cong F$ for any $F \in E^\perp = \{M \in \mathcal{D} : \mathbf{R}\mathrm{Hom}(E, M) = 0\}$.

Theorem 2.10 ([Ouc20, Theorem 3.1]). *Let $E \in \mathcal{D}$ be a d-spherical object, where $d \geq 2$. The categorical entropy function of the spherical twist T_E is given by:*

(i) $h_t(T_E) = (1-d)t$ *for* $t \leq 0$.

(ii) $h_t(T_E) \leq 0$ *for* $t > 0$.

(iii) If $E^{\perp} \neq \{0\}$, *then* $h_t(T_E) = 0$ *for* $t > 0$.

We refer to [Ouc20] for the proof. In Section 3.2.2, we will show that the condition $E^{\perp} \neq \{0\}$ can be replaced by certain conditions that are always satisfied when $\mathcal{D} = \mathrm{D^b}(X)$. The following lemma provides an example of such weaker conditions.

Lemma 2.11. *Let E be a d-spherical object, where $d \geq 2$. Suppose there exists a nonzero object $F \in \mathcal{D}$ and a heart $\mathscr{A} \subseteq \mathcal{D}$ of bounded t-structure such that*

$$\lim_{n\to\infty} \frac{\phi^+_{\mathscr{A}}(T^n_E(F))}{n} = 0.$$

Then $h_t(T_E) = 0$ *for* $t > 0$.

Proof. Let G be a split generator of $\mathcal{D}$. Then $G \oplus F$ is also a split generator. By Proposition 2.7,

$$\tau^+(T_E) = \lim_{n\to\infty} \frac{\phi^+_{\mathscr{A}}(T^n_E(G \oplus F))}{n} \geq \lim_{n\to\infty} \frac{\phi^+_{\mathscr{A}}(T^n_E(F))}{n} = 0.$$

From Theorem 2.10(i), we know that $h_0(T_E) = 0$. Thus, for $t > 0$, Proposition 2.2 gives

$$h_t(T_E) = t \cdot \tau^+(T_E) \geq 0.$$

Combining this with theorem 2.10(ii), we conclude that $h_t(T_E) = 0$ for all $t > 0$. $\square$

Remark 2.12. Note that if the condition $E^{\perp} \neq \{0\}$ holds, then any $0 \neq F \in E^{\perp}$ will satisfy $\lim_{n\to\infty} \frac{\phi^+_{\mathscr{A}}(T^n_E(F))}{n} = 0$, since $T^n_E(F) = F$ for all n. In Section 3.2.2, we will establish conditions that ensure the existence of such an object F and heart $\mathscr{A}$ which, while not necessarily meeting the condition as strong as $T^n_E(F) = F$, still satisfies $\lim_{n\to\infty} \frac{\phi^+_{\mathscr{A}}(T^n_E(F))}{n} = 0$.

2.3 $\mathbb{P}$-twists

In this subsection, we recall the notion of $\mathbb{P}$-*twists* [HT06]. We compute their categorical entropy functions, and obtain results analogous to Theorem 2.10 and Lemma 2.11.

An object $E \in \mathcal{D}$ is called a $\mathbb{P}^d$-*object* if $\mathbf{S}(E) \cong E[2d]$ and $\mathrm{Hom}(E, E[*]) \cong H^*(\mathbb{CP}^d, \mathbb{Z}) \otimes \mathbf{k}$ as $\mathbf{k}$-algebras. Examples of $\mathbb{P}$-objects include line bundles in the derived categories of hyperkähler manifolds and the structure sheaf of an embedded $\mathbb{P}^d$ in a $2d$-dimensional holomorphic symplectic variety, among others.

Similar to spherical objects, a $\mathbb{P}$-object also induces an autoequivalence of the triangulated category, known as the $\mathbb{P}$-*twist*. We now recall its definition following [HT06]. For a $\mathbb{P}$-object E, a generator $h \in \mathrm{Hom}(E, E[2])$ can be viewed as a morphism $h\colon E[-2] \to E$. Denote the image of h under the natural isomorphism $\mathrm{Hom}(E, E[2]) \cong \mathrm{Hom}(E^\vee, E^\vee[2])$ by $h^\vee$. The associated $\mathbb{P}$-twist $P_E \colon \mathcal{D} \to \mathcal{D}$ sends an object F to

$$P_E(F) = \mathrm{Cone}\left(\mathrm{Cone}\left(\mathrm{Hom}^{*-2}(E, F) \otimes E \xrightarrow{h^\vee \cdot \mathrm{id} - \mathrm{id} \cdot h} \mathrm{Hom}^*(E, F) \otimes E\right) \to F\right).$$

It follows that $P_E(E) \cong E[-2d]$, and $P_E(F) \cong F$ for any $F \in E^\perp$ (see [HT06, Lemma 2.5]).

Theorem 2.13. *Let $E \in \mathcal{D}$ be a $\mathbb{P}^d$-object. The categorical entropy function of the $\mathbb{P}$-twist P_E is given by:*

(i) $h_t(P_E) = -2dt$ *for* $t \leq 0$*.*

(ii) $h_t(P_E) \leq 0$ *for* $t > 0$*.*

(iii) If $E^\perp \neq \{0\}$*, then* $h_t(P_E) = 0$ *for* $t > 0$*.*

Proof. Let G be a split generator of $\mathcal{D}$. Define

$$M := \mathrm{Cone}\left(\mathrm{Hom}^{*-2}(E, G) \otimes E \xrightarrow{h^\vee \cdot \mathrm{id} - \mathrm{id} \cdot h} \mathrm{Hom}^*(E, G) \otimes E\right)[1].$$

Applying P_E^{n-1} to the exact triangle

$$G \to P_E(G) \to M \xrightarrow{+1},$$

one obtains

$$P_E^{n-1}(G) \to P_E^n(G) \to M[-2d(n-1)] \xrightarrow{+1}.$$

Then

$$\begin{aligned}
\epsilon_t(G, P_E^n(G)) &\leq \epsilon_t(G, M)e^{-2d(n-1)t} + \epsilon_t(G, P_E^{n-1}(G)) \\
&\leq \epsilon_t(G, M)e^{-2d(n-1)t} + \epsilon_t(G, M)e^{-2d(n-2)t} + \epsilon_t(G, P_E^{n-2}(G)) \\
&\leq \cdots \\
&\leq \epsilon_t(G, M)\left(\sum_{k=0}^{n-1} e^{-2dkt}\right) + \epsilon_t(G, G).
\end{aligned}$$

First, let us consider the case for $t \leq 0$.

$$\begin{aligned}
h_t(P_E) &= \lim_{n\to\infty} \frac{1}{n} \log \epsilon_t(G, P_E^n(G)) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log \left(\epsilon_t(G, M)\left(\sum_{k=0}^{n-1} e^{-2dkt}\right) + \epsilon_t(G, G)\right) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log \left(\epsilon_t(G, M) \cdot n \cdot e^{-2d(n-1)t} + \epsilon_t(G, G)\right) \leq -2dt.
\end{aligned}$$

On the other hand, since $G \oplus E$ is a split generator of $\mathcal{D}$,

$$\begin{aligned} h_t(P_E) &= \lim_{n\to\infty} \frac{1}{n} \log \epsilon_t(G \oplus E, P_E^n(G \oplus E)) \\ &\geq \lim_{n\to\infty} \frac{1}{n} \log \epsilon_t(G \oplus E, P_E^n(E)) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(\epsilon_t(G \oplus E, E) e^{-2dnt}\right) = -2dt. \end{aligned}$$

This proves $h_t(P_E) = -2dt$ for $t \leq 0$.

Next, we consider the case for $t > 0$.

$$\begin{aligned} h_t(P_E) &= \lim_{n\to\infty} \frac{1}{n} \log \epsilon_t(G, P_E^n(G)) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \log \left(\epsilon_t(G, M) \left(\sum_{k=0}^{n-1} e^{-2dkt} \right) + \epsilon_t(G, G) \right) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \log \left(\epsilon_t(G, M) \cdot n + \epsilon_t(G, G)\right) \leq 0. \end{aligned}$$

Moreover, assuming that $E^\perp \neq \{0\}$, say $0 \neq F \in E^\perp$. Consider the split generator $G \oplus F$,

$$\begin{aligned} h_t(P_E) &= \lim_{n\to\infty} \frac{1}{n} \log \epsilon_t(G \oplus F, P_E^n(G \oplus F)) \\ &\geq \lim_{n\to\infty} \frac{1}{n} \log \epsilon_t(G \oplus F, P_E^n(F)) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \epsilon_t(G \oplus F, F) = 0. \end{aligned}$$

□

In Section 3.2.3, we will show that, similar to spherical twists, the condition $E^\perp \neq \{0\}$ can be replaced by certain conditions that are always satisfied when $\mathcal{D} = \mathrm{D}^\mathrm{b}(X)$. The following lemma is the $\mathbb{P}$-twist analog of Lemma 2.11, and it can be proved using the same argument.

Lemma 2.14. *Let E be a $\mathbb{P}$-object. Suppose there exists a nonzero object $F \in \mathcal{D}$ and a heart $\mathscr{A} \subseteq \mathcal{D}$ of bounded t-structure such that*

$$\lim_{n\to\infty} \frac{\phi^+_{\mathscr{A}}(P_E^n(F))}{n} = 0.$$

Then $h_t(P_E) = 0$ for $t > 0$.

3 Proof of Main Theorems

In Section 3.1, we introduce various conditions that will be imposed for computing the categorical (polynomial) entropy functions of spherical twists and $\mathbb{P}$-twists. In Section 3.2, we study the categorical entropy functions and prove Theorem 1.1. In Section 3.3, we study the categorical polynomial entropy functions and prove Theorem 1.2.

3.1 Rigid objects and Conditions (c), (d), (e)

The conditions in the following definition are satisfied by both spherical and $\mathbb{P}$-objects.

Definition 3.1. Let $d \geq 2$ be a positive integer. An object $E \in \mathcal{D}$ is called *d-rigid* if it satisfies the following conditions:

(a) $\mathbf{S}(E) \cong E[d]$,

(b) $\mathrm{Hom}(E, E) = \mathbb{C}$, $\mathrm{Hom}(E, E[1]) = 0$, and $\mathrm{Hom}(E, E[k]) = 0$ for $k < 0$.

Note that a d-rigid object E also has the vanishing $\mathrm{Hom}(E, E[k]) = 0$ for $k > d$, since

$$\mathrm{Hom}(E, E[k]) \cong \mathrm{Hom}(E[k], \mathbf{S}(E))^\vee \cong \mathrm{Hom}(E[k], E[d])^\vee \cong \mathrm{Hom}(E, E[d-k])^\vee.$$

In the following, we gather some conditions on d-rigid objects that will be imposed at various stages of the computation of entropy functions.

Definition 3.2. Let $E \in \mathcal{D}$ be a d-rigid object. We say E satisfies *Condition (c)* if:

(c) $\mathcal{D}$ is indecomposable and $\langle E \rangle \subsetneq \mathcal{D}$. (Here, $\langle E \rangle$ denotes the smallest triangulated subcategory of $\mathcal{D}$ containing E.)

We say E satisfies *Condition (d)* if:

(d) there exists a heart $\mathscr{A} \subseteq \mathcal{D}$ of bounded t-structure such that (at least) one of the following holds:

- there exists a nonzero object $A \in \mathscr{A}$ such that $\mathrm{Hom}(E^+_{\mathscr{A}}, A) = 0$, or
- $\phi^+_{\mathscr{A}}(E) - \phi^-_{\mathscr{A}}(E) \leq 1$.

(The notations are defined in Notation 2.6.)

We say E satisfies *Condition (e)* if:

(e) there exists a Bridgeland stability condition $\sigma = (Z_\sigma, \mathscr{P}_\sigma)$ on $\mathcal{D}$ such that $E \in \mathscr{P}_\sigma(-1, 1]$.

Remark 3.3. Note that Condition (c) is satisfied for $\mathcal{D} = \mathrm{D^b}(X)$ when X is connected, which we always assume. Condition (d) is also satisfied for $\mathcal{D} = \mathrm{D^b}(X)$: In this case, we can simply take $\mathscr{A} = \mathrm{Coh}(X)$ and use the fact that for any coherent sheaf $E^+_{\mathscr{A}} \in \mathscr{A}$, we have $\mathrm{Hom}(E^+_{\mathscr{A}}, \mathcal{O}(-n)) = 0$ for sufficiently large n.

Condition (e) is more restrictive than Condition (d), and will be imposed only in the computation of categorical polynomial entropy $h_0^{\mathrm{pol}}(-)$ of spherical twists and $\mathbb{P}$-twists.

3.2 Entropy functions and Proof of Theorem 1.1

3.2.1 Lemmas

We begin with an elementary lemma concerning the extremal degrees $\phi^{\pm}_{\mathscr{A}}(-)$ and cohomology objects $E^{\pm}_{\mathscr{A}}$ of objects in an exact triangle. The relevant notations are defined in Notation 2.6.

Lemma 3.4. *Let $\mathscr{A} \subseteq \mathcal{D}$ be the heart of a bounded t-structure on a triangulated category $\mathcal{D}$. Suppose $M \to E \to N \xrightarrow{+1}$ is an exact triangle, where E, M, N are nonzero objects in $\mathcal{D}$. Then:*

(i) $\min\{\phi^-_{\mathscr{A}}(M), \phi^-_{\mathscr{A}}(N)\} \leq \phi^-_{\mathscr{A}}(E) \leq \phi^+_{\mathscr{A}}(E) \leq \max\{\phi^+_{\mathscr{A}}(M), \phi^+_{\mathscr{A}}(N)\}$.

(ii) If $\phi^-_{\mathscr{A}}(M) \geq \phi^+_{\mathscr{A}}(N)$, then E has nonzero cohomology objects at degrees $\phi^-_{\mathscr{A}}(M)$ and $\phi^+_{\mathscr{A}}(N)$.

(iii) If $\phi^+_{\mathscr{A}}(M) \geq \phi^+_{\mathscr{A}}(N)$, then $\phi^+_{\mathscr{A}}(E) = \phi^+_{\mathscr{A}}(M)$. Similarly, if $\phi^-_{\mathscr{A}}(N) \leq \phi^-_{\mathscr{A}}(M)$, then $\phi^-_{\mathscr{A}}(E) = \phi^-_{\mathscr{A}}(N)$.

(iv) If either of the following holds:

- $\phi^+_{\mathscr{A}}(N) \geq \phi^+_{\mathscr{A}}(M) + 2$, *or*
- $\phi^+_{\mathscr{A}}(N) = \phi^+_{\mathscr{A}}(M) + 1$ *and* $\mathrm{Hom}(N^+_{\mathscr{A}}, M^+_{\mathscr{A}}) = 0$,

then $\phi^+_{\mathscr{A}}(E) = \phi^+_{\mathscr{A}}(N)$ and $E^+_{\mathscr{A}} \cong N^+_{\mathscr{A}}$.

(v) Similarly, if either of the following holds:

- $\phi^-_{\mathscr{A}}(M) \leq \phi^-_{\mathscr{A}}(N) - 2$, *or*
- $\phi^-_{\mathscr{A}}(M) = \phi^-_{\mathscr{A}}(N) - 1$ *and* $\mathrm{Hom}(N^-_{\mathscr{A}}, M^-_{\mathscr{A}}) = 0$,

then $\phi^-_{\mathscr{A}}(E) = \phi^-_{\mathscr{A}}(M)$ and $E^-_{\mathscr{A}} \cong M^-_{\mathscr{A}}$.

Proof of (i). This follows directly from that

$$\mathrm{Hom}\left(E^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(E)], E\right) \neq 0 \quad \text{and} \quad \mathrm{Hom}\left(E, E^-_{\mathscr{A}}[\phi^-_{\mathscr{A}}(E)]\right) \neq 0.$$

Proof of (ii). First, if $\phi^-_{\mathscr{A}}(M) > \phi^+_{\mathscr{A}}(N)$, then the cohomology filtrations of M and N (with respect to $\mathscr{A}$) can be combined to form the cohomology filtration of E:

$$\begin{tikzcd}
0 \arrow[rr] & & * \arrow[r] \arrow[dl] & \cdots \arrow[r] & * \arrow[rr] & & * \arrow[rr] \arrow[dl] & & * \arrow[r] \arrow[dl] & \cdots \arrow[r] & * \arrow[rr] & & E \arrow[dl] \\
& M^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(M)] \arrow[ul, dashed] & & & & M^-_{\mathscr{A}}[\phi^-_{\mathscr{A}}(M)] \arrow[ul, dashed] & & N^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(N)] \arrow[ul, dashed] & & & & N^-_{\mathscr{A}}[\phi^-_{\mathscr{A}}(N)] \arrow[ul, dashed] &
\end{tikzcd}$$

Therefore, E has nonzero cohomology objects at both degrees $\phi^-_{\mathscr{A}}(M)$ and $\phi^+_{\mathscr{A}}(N)$.

Second, suppose $\phi^-_{\mathscr{A}}(M) = \phi^+_{\mathscr{A}}(N) = \phi$. Then we have

$$\begin{tikzcd}
0 \arrow[rr] & & * \arrow[rr] \arrow[dl] & & * \arrow[rr] \arrow[dl] & & * \arrow[rr] \arrow[dl] & & E \arrow[dl] \\
& M^{>\phi}_{\mathscr{A}} \arrow[ul, dashed] & & M^-_{\mathscr{A}}[\phi] \arrow[ul, dashed] & & N^+_{\mathscr{A}}[\phi] \arrow[ul, dashed] & & N^{<\phi}_{\mathscr{A}} \arrow[ul, dashed] &
\end{tikzcd}$$

Thus, E has a nonzero cohomology object at degree ϕ, which is an extension of $M^-_{\mathscr{A}}$ and $N^+_{\mathscr{A}}$ in $\mathscr{A}$.

Proof of (iii). The object M sits in the exact triangle

$$M^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(M)] \to M \to M^{<\phi^+_{\mathscr{A}}(M)}_{\mathscr{A}} \xrightarrow{+1} .$$

Thus, there exists an object X and exact triangles

$$M^{<\phi^+_{\mathscr{A}}(M)}_{\mathscr{A}} \to X \to N \xrightarrow{+1} \quad \text{and} \quad M^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(M)] \to E \to X \xrightarrow{+1} .$$

Applying (i) to the first exact triangle, we have

$$\phi^+_{\mathscr{A}}(X) \le \max\left\{\phi^+_{\mathscr{A}}\left(M^{<\phi^+_{\mathscr{A}}(M)}_{\mathscr{A}}\right), \phi^+_{\mathscr{A}}(N)\right\} \le \phi^+_{\mathscr{A}}(M).$$

Then, applying (ii) to the second exact triangle, we obtain $\phi^+_{\mathscr{A}}(E) = \phi^+_{\mathscr{A}}(M)$. The statement concerning $\phi^-_{\mathscr{A}}(-)$ can be proved similarly.

Proof of (iv). There are exact triangles

$$\begin{tikzcd}
M \arrow[rr] & & X \arrow[rr] \arrow[dl] & & E \arrow[dl] \\
& N^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(N)] \arrow[ul, dashed] & & N^{<\phi^+_{\mathscr{A}}(N)}_{\mathscr{A}} \arrow[ul, dashed] &
\end{tikzcd}$$

First, suppose $\phi^+_{\mathscr{A}}(N) \ge \phi^+_{\mathscr{A}}(M) + 2$. Then $\mathrm{Hom}\left(N^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(N)], M[1]\right) = 0$. Therefore, $X \cong M \oplus N^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(N)]$. Consequently, the order of M and $N^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(N)]$ can be exchanged, resulting in $\phi^+_{\mathscr{A}}(E) = \phi^+_{\mathscr{A}}(N)$ and $E^+_{\mathscr{A}} = N^+_{\mathscr{A}}$.

Second, suppose $\phi^+_{\mathscr{A}}(N) = \phi^+_{\mathscr{A}}(M) + 1$ and $\mathrm{Hom}(N^+_{\mathscr{A}}, M^+_{\mathscr{A}}) = 0$. There are exact triangles

$$\begin{tikzcd}
0 \arrow[rr] & & * \arrow[rr] \arrow[dl] & & * \arrow[rr] \arrow[dl] & & * \arrow[rr] \arrow[dl] & & E \arrow[dl] \\
& M^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(M)] \arrow[ul, dashed] & & M^{<\phi^+_{\mathscr{A}}(M)}_{\mathscr{A}} \arrow[ul, dashed] & & N^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(N)] \arrow[ul, dashed] & & N^{<\phi^+_{\mathscr{A}}(N)}_{\mathscr{A}} \arrow[ul, dashed] &
\end{tikzcd}$$

Consider the second and third objects from the left. Since $\mathrm{Hom}\left(N^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(N)], M^{<\phi^+_{\mathscr{A}}(M)}_{\mathscr{A}}[1]\right) = 0$, we can exchange their order, moving $N^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(N)]$ to the second position. Next, we have

$\operatorname{Hom}\left(N^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(N)], M^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(M)+1]\right) = 0$ by assumption. Therefore, we can also exchange their order, moving $N^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(N)]$ to the first position, and obtain

$$\begin{tikzcd}
0 \arrow[rr] & & * \arrow[rr] \arrow[dl] & & * \arrow[rr] \arrow[dl] & & * \arrow[rr] \arrow[dl] & & E \arrow[dl] \\
& N^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(N)] \arrow[ul, dashed] & & M^+_{\mathscr{A}}[\phi^+_{\mathscr{A}}(M)] \arrow[ul, dashed] & & M_{\mathscr{A}}^{<\phi^+_{\mathscr{A}}(M)} \arrow[ul, dashed] & & N_{\mathscr{A}}^{<\phi^+_{\mathscr{A}}(N)} \arrow[ul, dashed] &
\end{tikzcd}$$

Consequently, we have $\phi^+_{\mathscr{A}}(E) = \phi^+_{\mathscr{A}}(N)$ and $E^+_{\mathscr{A}} = N^+_{\mathscr{A}}$.

Finally, (v) can be proved similarly. $\square$

The following lemma will be useful for managing the degrees of nonzero cohomology objects.

Lemma 3.5. *Let $E \in \mathcal{D}$ be a d-rigid object satisfying Condition (c) for some $d \geq 2$. There exists $F \in \mathcal{D}$ such that:*

- $\operatorname{Hom}(E, F) \neq 0$, *and*
- $\operatorname{Hom}(E, F[k]) = 0$ *if $k < 0$ or $k > d-2$.*

Proof. First, note that Condition (c) implies that $\langle E, E^{\perp} \rangle \subsetneq \mathcal{D}$. Indeed, assume the contrary, that $\langle E, E^{\perp} \rangle = \mathcal{D}$. There are two possibilities:

(i) If $E^{\perp} = \{0\}$, then $\mathcal{D}$ is generated by E.

(ii) If $E^{\perp} \neq \{0\}$, then $\mathcal{D} \cong \langle E \rangle \oplus E^{\perp}$, since for any $A \in E^{\perp}$, we have

$$\operatorname{Hom}(A, E[k]) \cong \operatorname{Hom}(A, \mathbf{S}(E)[k-d]) \cong \operatorname{Hom}(E[k-d], A)^{\vee} = 0 \quad \text{for all } k.$$

Let $F_1 \in \mathcal{D}$ be an object such that $F_1 \notin \langle E, E^{\perp} \rangle$. By shifting, we may assume that $\operatorname{Hom}(E, F_1) \neq 0$ and $\operatorname{Hom}(E, F_1[k]) = 0$ for all $k < 0$. Denote

$$n = \max\{k : \operatorname{Hom}(E, F_1[k]) \neq 0\}.$$

If $n \leq d-2$, then we are done. Now suppose that $n \geq d-1$. Let

$$F_2 = \operatorname{Cone}\left(\operatorname{Hom}(E, F_1) \otimes E \xrightarrow{\mathrm{ev}} F_1\right)[1].$$

Applying $\operatorname{Hom}(E, -)$ to the exact triangle

$$\operatorname{Hom}(E, F_1) \otimes E \xrightarrow{\mathrm{ev}} F_1 \to F_2[-1] \xrightarrow{+1},$$

one obtains

$$\begin{array}{cccccc}
\to & 0 & \to & 0 & \to & \operatorname{Hom}(E, F_2[-2]) \\
\to & \operatorname{Hom}(E, F_1) \otimes \operatorname{Hom}(E, E) & \xrightarrow{\cong} & \operatorname{Hom}(E, F_1) & \to & \operatorname{Hom}(E.F_2[-1]) \\
\to & 0 & \to & \operatorname{Hom}(E, F_1[1]) & \to & \operatorname{Hom}(E, F_2) \\
\to & \operatorname{Hom}(E, F_1) \otimes \operatorname{Hom}(E, E[2]) & \to & \operatorname{Hom}(E, F_1[2]) & \to & \operatorname{Hom}(E.F_2[1]) \\
\to & \cdots & & & &
\end{array}$$

where the isomorphism in the second row and the zero in the third row both follow from Condition (b) of E being a d-rigid object. Therefore, $\operatorname{Hom}(E, F_2[k]) = 0$ for all $k < 0$ and $k > n-1$. Note that $F_2 \notin \langle E, E^{\perp} \rangle$; otherwise, we would have $F_1 \in \langle E, E^{\perp} \rangle$, which is a contradiction. Thus, we can find the desired object F inductively. $\square$

3.2.2 Entropy function of spherical twists

Proposition 3.6. *Let E be a d-spherical object in $\mathcal{D}$, where $d \geq 2$. Suppose E satisfies Conditions (c) and (d). Then, there exists a nonzero object $F \in \mathcal{D}$ and a heart $\mathscr{A} \subseteq \mathcal{D}$ such that*

$$\lim_{n\to\infty} \frac{\phi^+_{\mathscr{A}}(T^n_E(F))}{n} = 0.$$

In fact, there exists a nonzero object $\widetilde{F}$ and an integer $\ell > 0$ such that for any $n > \ell$, there is an exact triangle

$$\widetilde{F} \to T^n_E(F) \to C_n \xrightarrow{+1}$$

with $\phi^-_{\mathscr{A}}(\widetilde{F}) > \phi^+_{\mathscr{A}}(C_n)$. Therefore, $\phi^+_{\mathscr{A}}(T^n_E(F))$ stabilizes when $n > \ell$.

Proof. Let F be a nonzero object in $\mathcal{D}$. Applying T^{n-1}_E to the exact triangle

$$F \to T_E(F) \to \mathbf{R}\mathrm{Hom}(E, F) \otimes E[1] \xrightarrow{+1},$$

one obtains

$$T^{n-1}_E(F) \to T^n_E(F) \to \mathbf{R}\mathrm{Hom}(E, F) \otimes E[1 + (n-1)(1-d)] \xrightarrow{+1}.$$

Denote $M := \mathbf{R}\mathrm{Hom}(E, F) \otimes E[1]$. Then we have

$$\begin{tikzcd}
F \arrow[rr] & & * \arrow[dl] \arrow[rr] & & * \arrow[dl] \arrow[r] & \cdots \arrow[r] & * \arrow[rr] & & T^n_E(F) \arrow[dl] \\
& M \arrow[ul, dashed] & & M[1-d] \arrow[ul, dashed] & & & & M[(n-1)(1-d)] \arrow[ul, dashed] &
\end{tikzcd}$$

Since E satisfies Condition (d), there exists a heart $\mathscr{A} \subseteq \mathcal{D}$ such that (at least) one of the following statements holds:

- there exists a nonzero object $A \in \mathscr{A}$ such that $\mathrm{Hom}(E^+_{\mathscr{A}}, A) = 0$, or
- $\phi^+_{\mathscr{A}}(E) - \phi^-_{\mathscr{A}}(E) \leq 1$.

First, suppose there exists $A \in \mathscr{A}$ such that $\mathrm{Hom}(E^+_{\mathscr{A}}, A) = 0$. We claim that taking $F = A$ would satisfy the desired properties stated in the proposition. If $\mathbf{R}\mathrm{Hom}(E, F) = 0$, then $T^n_E(F) = F$ for all n, and we can simply take $\widetilde{F} = F$ and $C_n = 0$. Otherwise, we have $M \neq 0$ and $M^+_{\mathscr{A}} \cong (E^+_{\mathscr{A}})^{\oplus p}$ for some integer $p \geq 1$. Now, we compare $\phi^+_{\mathscr{A}}(F)(= \phi^-_{\mathscr{A}}(F) = 0)$ and $\phi^+_{\mathscr{A}}(M)$.

(i) If $\phi^+_{\mathscr{A}}(M) \leq 0$, then $\phi^+_{\mathscr{A}}(T^n_E(F)) = 0$ by Lemma 3.4(iii). Moreover, $(T^n_E(F))^+_{\mathscr{A}}$ is independent of n, since $\phi^+_{\mathscr{A}}(M[k(1-d)]) < 0$ for any $k > 0$. Therefore, taking $\widetilde{F} = (T_E(F))^+_{\mathscr{A}}$ and $\ell = 1$ would satisfy the desired property.

(ii) If $\phi^+_{\mathscr{A}}(M) \geq 2$, then $\phi^+_{\mathscr{A}}(T^n_E(F)) = \phi^+_{\mathscr{A}}(M)$ by Lemma 3.4(iv). Moreover, $(T^n_E(F))^+_{\mathscr{A}} \cong M^+_{\mathscr{A}}$ for all n. Therefore, taking $\widetilde{F} = M^+_{\mathscr{A}}$ and $\ell = 1$ would satisfy the desired property.

(iii) If $\phi^+_{\mathscr{A}}(M) = 1$, using $\mathrm{Hom}(M^+_{\mathscr{A}}, F) \cong \mathrm{Hom}\left((E^+_{\mathscr{A}})^{\oplus p}, A\right) = 0$, we still have $\phi^+_{\mathscr{A}}(T^n_E(F)) = \phi^+_{\mathscr{A}}(M)$ and $(T^n_E(F))^+_{\mathscr{A}} \cong M^+_{\mathscr{A}}$ for all n, by Lemma 3.4(iv).

Second, suppose $\phi^+_{\mathscr{A}}(E) - \phi^-_{\mathscr{A}}(E) \leq 1$. By Lemma 3.5, there exists $F \in \mathcal{D}$ such that $\mathrm{Hom}(E, F) \neq 0$, and $\mathrm{Hom}(E, F[k]) = 0$ if $k < 0$ or $k > d-2$. Therefore, $M = \mathbf{R}\mathrm{Hom}(E, F) \otimes E[1]$ is nonzero and satisfies

$$\phi^+_{\mathscr{A}}(M) - \phi^-_{\mathscr{A}}(M) \leq (d-2) + 1 = d - 1.$$

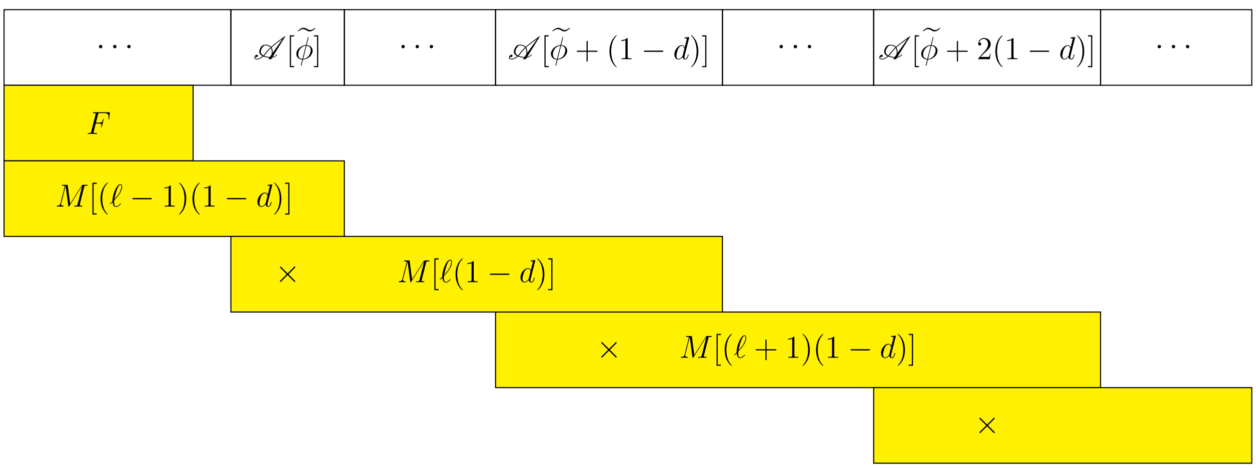
Choose $\ell > 0$ large enough so that

$$\phi^-_{\mathscr{A}}(F) > \phi^+_{\mathscr{A}}(M[\ell(1-d)]) = \phi^+_{\mathscr{A}}(M) + \ell(1-d).$$

We claim that $T^n_E(F)$ admits a nonzero cohomology object at degree $\phi^+_{\mathscr{A}}(M) + \ell(1-d)$ for any $n > \ell$. Moreover, $(T^n_E(F))^{\geq \phi^+_{\mathscr{A}}(M)+\ell(1-d)}_{\mathscr{A}}$ is independent of $n > \ell$. To visualize this, let

$$\widetilde{\phi} \coloneqq \phi^+_{\mathscr{A}}(M) + \ell(1-d),$$

and consider the following picture:



The yellow regions indicate the possible ranges of nonzero cohomological degrees (with respect to the heart $\mathscr{A}$) of the objects. The symbol "$\times$" indicates that the cohomology object at that particular degree must be nonzero.

Observe that

$$\min\left\{\phi^-_{\mathscr{A}}(F), \phi^-_{\mathscr{A}}(M), \phi^-_{\mathscr{A}}(M[1-d]), \ldots, \phi^-_{\mathscr{A}}(M[(\ell-1)(1-d)])\right\} \geq \widetilde{\phi} = \phi^+_{\mathscr{A}}(M[\ell(1-d)]),$$

and

$$\max\left\{M[(\ell+1)(1-d)], M[(\ell+2)(1-d)], \ldots\right\} < \widetilde{\phi}.$$

Thus, by Lemma 3.4(ii), $T^n_E(F)$ admits a nonzero cohomology object at degree $\widetilde{\phi} = \phi^+_{\mathscr{A}}(M) + \ell(1-d)$ for any $n > \ell$. Moreover, $(T^n_E(F))^{\geq \phi^+_{\mathscr{A}}(M)+\ell(1-d)}_{\mathscr{A}}$ is independent of $n > \ell$. Therefore, taking

$$\widetilde{F} = \left(T^{\ell+1}_E(F)\right)^{\geq \phi^+_{\mathscr{A}}(M)+\ell(1-d)}_{\mathscr{A}} \qquad \text{and} \qquad C_n = (T^n_E(F))^{<\phi^+_{\mathscr{A}}(M)+\ell(1-d)}_{\mathscr{A}}$$

would satisfy the desired property. $\square$

Corollary 3.7. *Let E be a d-spherical object in $\mathcal{D}$, where $d \geq 2$. Suppose E satisfies Conditions (c) and (d). Then, the categorical entropy function of the spherical twist T_E is given by:*

$$h_t(T_E) = \begin{cases} (1-d)t & \text{for } t \leq 0, \\ 0 & \text{for } t \geq 0. \end{cases}$$

Proof. This follows from Theorem 2.10, Lemma 2.11, and Proposition 3.6. □

Corollary 3.8 ([LL23, Theorem A.2])**.** *Consider a complex smooth projective variety X of dimension $d \geq 2$, and let $E \in \mathrm{D}^{\mathrm{b}}(X)$ be a d-spherical object with the associated spherical twist $T_E \in \operatorname{Aut} \mathrm{D}^{\mathrm{b}}(X)$. Let k be a nonzero integer, and Y be a smooth projective variety with an exact equivalence $\Psi\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$.*

There does not exist a standard autoequivalence $\Phi \in \operatorname{Aut}_{\mathrm{std}} \mathrm{D}^{\mathrm{b}}(Y)$ such that $T_E^k = \Psi^{-1} \circ \Phi \circ \Psi$. In particular, T_E^k is not conjugate to a standard autoequivalence of X.

Proof. Recall that Conditions (c) and (d) are always satisfied for $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(X)$, see Remark 3.3. Therefore, by Corollary 3.7, $\tau^+(T_E) = 0$ and $\tau^-(T_E) = 1 - d < 0$. Then, according to [FF23, Theorem 1.1(iii)], we have

$$\tau^+(T_E^k) = \begin{cases} 0 & \text{if } k > 0, \\ -k(d-1) & \text{if } k < 0, \end{cases} \qquad \tau^-(T_E^k) = \begin{cases} -k(d-1) & \text{if } k > 0, \\ 0 & \text{if } k < 0. \end{cases}$$

Thus, $\tau^+(T_E^k) > \tau^-(T_E^k)$ for any $k \neq 0$. The desired statement then follows from Proposition 2.1(iv) and Example 2.4. □

3.2.3 Entropy function of $\mathbb{P}$-twists

In the following, we prove the analogues of Proposition 3.6, Corollary 3.7, and Corollary 3.8 for $\mathbb{P}$-twists. The arguments are essentially identical to their spherical twist counterparts, so we will only highlight the differences for the $\mathbb{P}$-twists.

Proposition 3.9. *Let E be a $\mathbb{P}^d$-object satisfying Conditions (c) and (d). Then, there exists a nonzero object $F \in \mathcal{D}$ and a heart $\mathscr{A} \subseteq \mathcal{D}$ such that*

$$\lim_{n \to \infty} \frac{\phi^+_{\mathscr{A}}(P_E^n(F))}{n} = 0.$$

In fact, there exists a nonzero object $\widetilde{F}$ and an integer $\ell > 0$ such that for any $n > \ell$, there is an exact triangle

$$\widetilde{F} \to P_E^n(F) \to C_n \xrightarrow{+1}$$

with $\phi^-_{\mathscr{A}}(\widetilde{F}) > \phi^+_{\mathscr{A}}(C_n)$. Therefore, $\phi^+_{\mathscr{A}}(P_E^n(F))$ stabilizes when $n > \ell$.

Proof. Let F be a nonzero object in $\mathcal{D}$. Define

$$M := \operatorname{Cone}\left(\operatorname{Hom}^{*-2}(E,F) \otimes E \xrightarrow{h^\vee \cdot \mathrm{id} - \mathrm{id} \cdot h} \operatorname{Hom}^*(E,F) \otimes E\right)[1].$$

Applying P_E^{n-1} to the exact triangle

$$F \to P_E(F) \to M \xrightarrow{+1},$$

one obtains

$$P_E^{n-1}(F) \to P_E^n(F) \to M[-2d(n-1)] \xrightarrow{+1} .$$

Then we have

$$\begin{tikzcd}
F \arrow[rr] & & * \arrow[rr] \arrow[dl] & & * \arrow[r] \arrow[dl] & \cdots \arrow[r] & * \arrow[rr] & & P_E^n(F) \arrow[dl] \\
& M \arrow[ul, dashed] & & M[-2d] \arrow[ul, dashed] & & & & M[-2d(n-1)] \arrow[ul, dashed] &
\end{tikzcd}$$

Since E satisfies Condition (d), there exists a heart $\mathscr{A} \subseteq \mathcal{D}$ such that (at least) one of the following statements holds:

- there exists a nonzero object $A \in \mathscr{A}$ such that $\operatorname{Hom}(E^+_{\mathscr{A}}, A) = 0$, or
- $\phi^+_{\mathscr{A}}(E) - \phi^-_{\mathscr{A}}(E) \leq 1$.

First, suppose there exists $A \in \mathscr{A}$ such that $\operatorname{Hom}(E^+_{\mathscr{A}}, A) = 0$. Then, taking $F = A$ would satisfy the desired properties. The proof is identical to the case of spherical twists, as we also have $M^+_{\mathscr{A}} \cong (E^+_{\mathscr{A}})^{\oplus p}$ in this scenario.

Second, suppose $\phi^+_{\mathscr{A}}(E) - \phi^-_{\mathscr{A}}(E) \leq 1$. By Lemma 3.5, there exists $F \in \mathcal{D}$ such that $\operatorname{Hom}(E, F) \neq 0$, and $\operatorname{Hom}(E, F[k]) = 0$ if $k < 0$ or $k > 2d - 2$. Therefore, $N \coloneqq \mathbf{R}\operatorname{Hom}(E, F) \otimes E$ is nonzero and satisfies

$$\phi^+_{\mathscr{A}}(N) - \phi^-_{\mathscr{A}}(N) \leq 2d - 1.$$

By the definitions of M and N, there is an exact triangle

$$N[1] \to M \to N \xrightarrow{+1} .$$

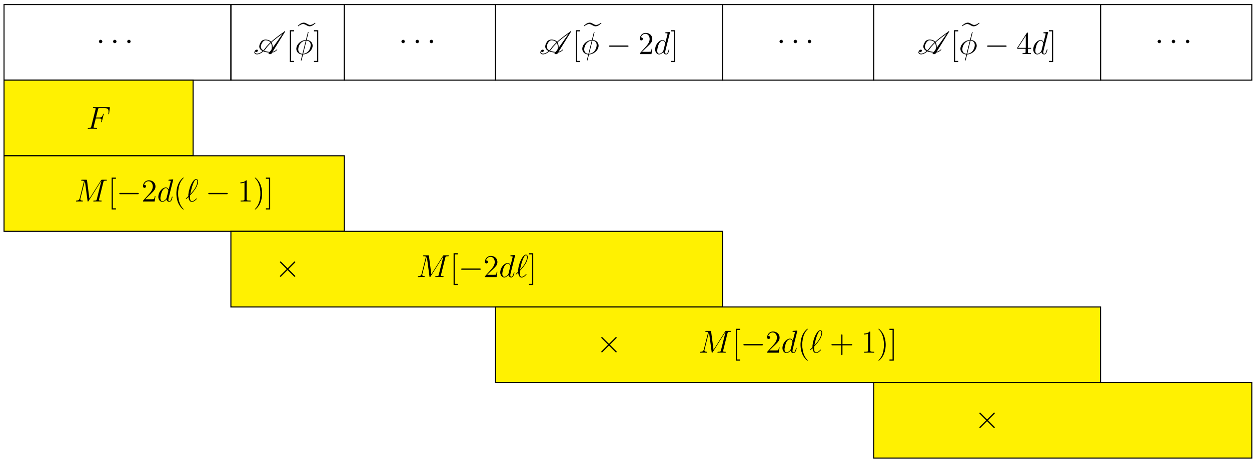
Thus, we have

$$\phi^+_{\mathscr{A}}(M) - \phi^-_{\mathscr{A}}(M) \leq 2d.$$

Choose $\ell > 0$ large enough so that

$$\phi^-_{\mathscr{A}}(F) > \phi^+_{\mathscr{A}}(M[-2d\ell]) = \phi^+_{\mathscr{A}}(M) - 2d\ell.$$

Let $\widetilde{\phi} \coloneqq \phi^+_{\mathscr{A}}(M) - 2d\ell$. Then we have the following picture, analogous to what we had in the case of spherical twists. The remainder of the proof follows identically.



$\square$

Corollary 3.10. *Let E be a $\mathbb{P}^d$-object in $\mathcal{D}$. Suppose E satisfies Conditions (c) and (d). Then, the categorical entropy function of the $\mathbb{P}$-twist P_E is given by:*

$$h_t(P_E) = \begin{cases} -2dt & \text{for } t \leq 0, \\ 0 & \text{for } t \geq 0. \end{cases}$$

Proof. This follows from Theorem 2.13, Lemma 2.14, and Proposition 3.9. $\square$

Corollary 3.11 (= Theorem 1.1)**.** *Consider a complex smooth projective variety X of dimension $2d$, and let $E \in \mathrm{D^b}(X)$ be a $\mathbb{P}^d$-object with the associated $\mathbb{P}^d$-twist $P_E \in \mathrm{Aut}\,\mathrm{D^b}(X)$. Let k be a nonzero integer, and Y be a smooth projective variety with an exact equivalence $\Psi\colon \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$.*

There does not exist a standard autoequivalence $\Phi \in \mathrm{Aut_{std}}\,\mathrm{D^b}(Y)$ such that $P_E^k = \Psi^{-1} \circ \Phi \circ \Psi$. In particular, P_E^k is not conjugate to a standard autoequivalence of X.

Proof. Recall that Conditions (c) and (d) are always satisfied for $\mathcal{D} = \mathrm{D^b}(X)$, see Remark 3.3. Therefore, by Corollary 3.10, $\tau^+(P_E) = 0$ and $\tau^-(P_E) = -2d < 0$. Then, according to [FF23, Theorem 1.1(iii)], we have

$$\tau^+(P_E^k) = \begin{cases} 0 & \text{if } k > 0, \\ -2dk & \text{if } k < 0, \end{cases} \qquad \tau^-(P_E^k) = \begin{cases} -2dk & \text{if } k > 0, \\ 0 & \text{if } k < 0. \end{cases}$$

Thus, $\tau^+(P_E^k) > \tau^-(P_E^k)$ for any $k \neq 0$. The desired statement then follows from Proposition 2.1(iv) and Example 2.4. $\square$

3.3 Polynomial entropy and Proof of Theorem 1.2

Proposition 3.12. *Let E be a d-spherical object in $\mathcal{D}$, where $d \geq 2$.*

(i) For $t < 0$, $h_t^{\mathrm{pol}}(T_E) = 0$.

(ii) If E satisfies Conditions (c) and (d), then $h_t^{\mathrm{pol}}(T_E) = 0$ for $t > 0$.

(iii) *If E satisfies Conditions (c) and (e), then $h_0^{\mathrm{pol}}(T_E) = 1$.*

The same statements hold for the $\mathbb{P}$-twists.

We will prove the proposition only in the case of spherical twists, as the proof for the $\mathbb{P}$-twists is essentially identical.

Proof of (i). This is proved in [FFO21], so we omit the proof here.

Proof of (ii). Let $M = \mathbf{R}\mathrm{Hom}(E, G) \otimes E[1]$. Applying T_E^{n-1} to the exact triangle

$$G \to T_E(G) \to M \xrightarrow{+1},$$

one obtains

$$T_E^{n-1}(G) \to T_E^n(G) \to M[(1-d)(n-1)] \xrightarrow{+1} .$$

Thus

$$\begin{aligned}
\epsilon_t(G, T_E^n(G)) &\leq \epsilon_t(G, M)e^{(1-d)(n-1)t} + \epsilon_t(G, T_E^{n-1}(G)) \\
&\leq \epsilon_t(G, M)e^{(1-d)(n-1)t} + \epsilon_t(G, M)e^{(1-d)(n-2)t} + \epsilon_t(G, T_E^{n-2}(G)) \\
&\leq \cdots \\
&\leq \epsilon_t(G, M)\left(\sum_{k=0}^{n-1} e^{(1-d)kt}\right) + \epsilon_t(G, G).
\end{aligned}$$

Let $t > 0$. By Corollary 3.7, $h_t(T_E) = 0$ when E satisfies Conditions (c) and (d). Therefore,

$$\begin{aligned}
h_t^{\mathrm{pol}}(T_E) &= \limsup_{n\to\infty} \frac{\log \epsilon_t(G, T_E^n(G))}{\log(n)} \\
&\leq \limsup_{n\to\infty} \frac{1}{\log(n)} \log\left(\epsilon_t(G, M)\left(\sum_{k=0}^{n-1} e^{(1-d)kt}\right) + \epsilon_t(G, G)\right) = 0.
\end{aligned}$$

To obtain the lower bound, recall from Proposition 3.6 that there exist nonzero objects $F, \widetilde{F}$ and an integer $\ell > 0$ such that for any $n > \ell$, there is an exact triangle

$$\widetilde{F} \to T_E^n(F) \to C_n \xrightarrow{+1}$$

with $\phi^-_{\mathscr{A}}(\widetilde{F}) > \phi^+_{\mathscr{A}}(C_n)$. Observe that $\mathrm{Hom}(\widetilde{F}, C_n) = \mathrm{Hom}(\widetilde{F}, C_n[-1]) = 0$ for all $n > \ell$. Thus

$$\mathrm{Hom}(\widetilde{F}, T_E^n(F)) \cong \mathrm{Hom}(\widetilde{F}, \widetilde{F}) \neq 0.$$

Let G be a split generator. Then both $G \oplus F$ and $G \oplus \widetilde{F}$ are also split generators. We have

$$\begin{aligned}
h_t^{\mathrm{pol}}(T_E) &= \limsup_{n\to\infty} \frac{\log \epsilon_t(G \oplus \widetilde{F}, T_E^n(G \oplus F))}{\log(n)} \\
&\geq \limsup_{n\to\infty} \frac{\log \epsilon_t(\widetilde{F}, T_E^n(F))}{\log(n)} \\
&\geq \limsup_{n\to\infty} \frac{\log(1)}{\log(n)} = 0.
\end{aligned}$$

This concludes the proof of (ii).

Proof of (iii). First, there is an upper bound

$$h_0^{\mathrm{pol}}(T_E) = \limsup_{n\to\infty} \frac{\log \epsilon_0(G, T_E^n(G))}{\log(n)}$$
$$\le \limsup_{n\to\infty} \frac{1}{\log(n)} \log\left(\epsilon_0(G,M)\cdot n + \epsilon_0(G,G)\right) \le 1.$$

Therefore, it suffices to show that $h_0^{\mathrm{pol}}(T_E) \ge 1$.

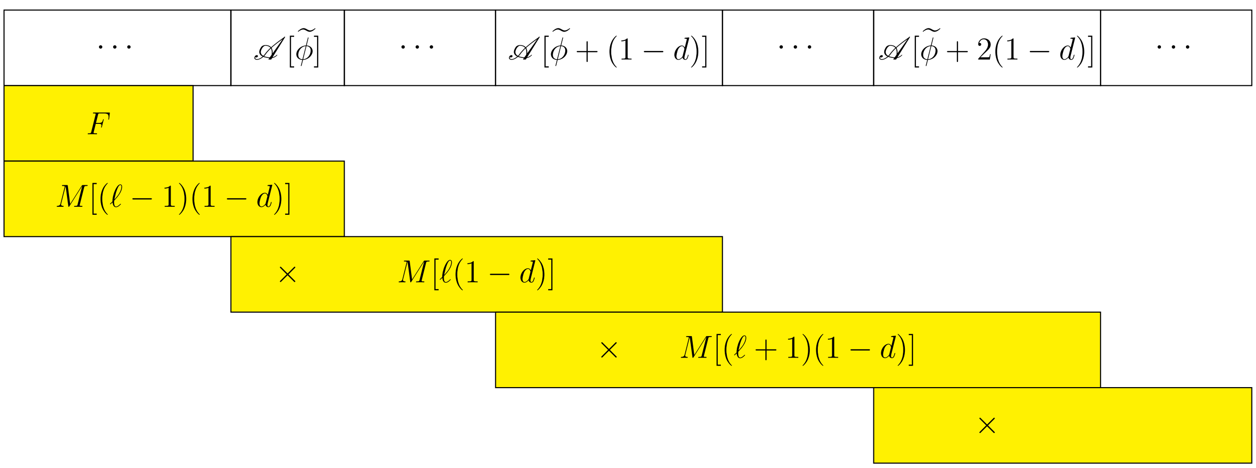
Let $\mathcal{A} = \mathscr{P}_\sigma(0,1]$ be the heart associated to the given Bridgeland stability condition on $\mathcal{D}$ in Condition (e). Since $E \in \mathscr{P}_\sigma(-1,1]$, we have $\phi^+_{\mathscr{A}}(E) - \phi^-_{\mathscr{A}}(E) \le 1$. By Lemma 3.5, there exists $F \in \mathcal{D}$ such that $\mathrm{Hom}(E,F) \ne 0$, and $\mathrm{Hom}(E, F[k]) = 0$ if $k < 0$ or $k > d-2$. Therefore, $M = \mathbf{R}\mathrm{Hom}(E,F)\otimes E[1]$ is nonzero and satisfies

$$\phi^+_{\mathscr{A}}(M) - \phi^-_{\mathscr{A}}(M) \le d-1.$$

Choose $\ell > 0$ large enough so that

$$\phi^-_{\mathscr{A}}(F) > \phi^+_{\mathscr{A}}(M[\ell(1-d)]) = \phi^+_{\mathscr{A}}(M) + \ell(1-d) =: \widetilde{\phi}.$$

Again, we have the following picture for $T_E^n(F)$.



Recall that $h_0(T_E) = 0$ by Theorem 2.10(i). We claim that $h_{\sigma,0}(T_E) = 0$. Indeed, by Lemma 2.8(i), $h_{\sigma,0}(T_E) \le h_0(T_E) = 0$. On the other hand, by Remark 2.9, there is a constant $C > 0$ (arising from the support property of stability conditions) such that

$$h_{\sigma,0}(T_E) = \limsup_{n\to\infty} \frac{1}{n}\log m_{\sigma,0}(\Phi^n G) \ge \limsup_{n\to\infty} \frac{\log C}{n} = 0.$$

This proves that $h_0(T_E) = h_{\sigma,0}(T_E) = 0$. By Lemma 2.8(ii), this implies that $h_0^{\mathrm{pol}}(T_E) \ge h_{\sigma,0}^{\mathrm{pol}}(T_E)$. Therefore, to obtain the desired lower bound for $h_0^{\mathrm{pol}}(T_E)$, it suffices to show that $h_{\sigma,0}^{\mathrm{pol}}(T_E) \ge 1$.

From the above picture and Lemma 3.4(ii), one observes that $T_E^n(F)$ admits at least $n-\ell$ nonzero cohomology objects (with respect to $\mathscr{A}$) when $n > \ell$, at degrees

$$\widetilde{\phi},\ \widetilde{\phi} + (1-d),\ \ldots,\ \widetilde{\phi} + (n-\ell-1)(1-d).$$

Therefore, by Remark 2.9, we have $m_{\sigma,0}(T_E^n(F)) > C(n-\ell)$. Thus,

$$h^{\mathrm{pol}}_{\sigma,0}(T_E) \geq \limsup_{n\to\infty} \frac{\log m_{\sigma,0}(T_E^n(F))}{\log(n)} \geq \limsup_{n\to\infty} \frac{\log(C(n-\ell))}{\log(n)} = 1.$$

This completes the proof. □

Corollary 3.13 (= Theorem 1.2)**.** *Let X be a complex smooth projective variety of dimension $d \geq 2$. Let $E \in \mathrm{D^b}(X)$ be a spherical object (resp. $\mathbb{P}$-object) with the associated spherical twist $T_E \in \operatorname{Aut} \mathrm{D^b}(X)$ (resp. $\mathbb{P}$-twist $P_E \in \operatorname{Aut} \mathrm{D^b}(X)$). Then*

(i) For $t \neq 0$, $h^{\mathrm{pol}}_t(T_E) = 0$ (resp. $h^{\mathrm{pol}}_t(P_E) = 0$).

(ii) If there exists a Bridgeland stability condition $\sigma = (Z, \mathscr{P})$ on $\mathcal{D}$ such that $E \in \mathscr{P}(-1, 1]$, then $h^{\mathrm{pol}}_0(T_E) = 1$ (resp. $h^{\mathrm{pol}}_0(P_E) = 1$).

Proof. This follows from Proposition 3.12, and the fact that Conditions (c) and (d) are always satisfied for $\mathcal{D} = \mathrm{D^b}(X)$, cf. Remark 3.3. □

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