2nd midterm: tomorrow 1pm PST — Thursday 1pm PST.

More info on the course website.

My OH: Today 12-1:30 pm PST.

Linear algebra ≈ the study of the category of vector spaces

Objects: vector spaces.

Morphisms: linear transformations between vector spaces.

- study the kernel and image of morphisms.
- composition of morphisms. ($\leftrightarrow$ matrix multiplication).

To better understand the morphisms:

- represent by matrices.
- diagonalization.

Additional structure: Inner product space.

$\longrightarrow$ length, orthogonal, projection, ...

$\longrightarrow$ adjoint transformation: $\langle Tv, w\rangle = \langle v, T^*w\rangle$.

$\longrightarrow$ symmetric matrices: $T^* = T$.

- $\Leftrightarrow$ orthogonally diagonalizable
- eigenvalues are real.

A, B square.

$AB - BA = A$

WTS: $A^k = 0$ for some $k \geq 1$

Claim: $A^k B - BA^k = kA^k$

Claim: $A^3 B - BA^3 = 3A^3$

$AB - A = BA$

$A(B - I)$

~~det A~~

$\det(B - I) = \det(B)$

or

$\det(A) = 0$

$(-)A^2$

Claim: $A^2 B - BA^2 = 2A^2$

$A^2 B - ABA = A^2$

$ABA - BA^2 = A^2$

$A(-)$

$A^3 B - ABA^2 = 2A^3$

$\cancel{A^2 BA - BA^3 = 2A^3}$

$ABA^2 - BA^3 = A^3$

$T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$

$X \mapsto XB - BX$

$A^k \mapsto kA^k \quad \forall k \geq 1$

$A^{n^2+1} = 0$

If $A^k \neq 0$ $\forall k$, then k is an eigenvalue of T $\forall k \geq 1$.

A: $m \times n$ $\quad$ rk r

$m\times n \quad n \times n$

U, V orthogonal,

SVD $\quad A = U \Sigma V$

$$\Sigma = \left[\begin{array}{c|c} \begin{matrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right] \quad m \times n$$

rank $(A^T A)$ $\quad$ rank $\begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ & & & \lambda_{r+1} \ddots \lambda_n \end{pmatrix}^T$

$v_1, \ldots, v_r, v_{r+1}, \ldots, v_n$ orthonormal eigen basis of $\mathbb{R}^n$

$\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n$ $\quad$ $\lambda_r \neq 0$, $\lambda_{r+1}, \ldots, \lambda_n = 0$

$\mathrm{Nul}(A^T A) = \mathrm{Nul}(A)$

r nonzero λ_i's

Claim: $A = [u_1 \cdots u_m] \begin{bmatrix} \|Av_1\| & & & 0 \\ & \ddots & & \\ & & \|Av_r\| & \\ 0 & & & \ddots \end{bmatrix} [v_1 \cdots v_n]^T$

$\Rightarrow$ $\{Av_1, \ldots, Av_r\}$ are orthogonal basis of $\mathrm{Col}(A)$ $\quad (Av_1 \neq 0, \ldots, Av_r \neq 0)$

$$\langle Av_1, Av_2 \rangle = \langle v_1, A^T A v_2 \rangle$$
$$= \langle v_1, \lambda_2 v_2 \rangle = 0$$

$\{u_1, \ldots, u_r\}$ o.n. basis of $\mathrm{Col}(A)$

$u_1 = \dfrac{Av_1}{\|Av_1\|}$, ...

extend to o.n. basis of $\mathbb{R}^m$ $\quad \{u_1, \ldots, u_{r+1}, \ldots, u_m\}$

$$A\begin{bmatrix} v_1 \cdots v_n \end{bmatrix} = \begin{bmatrix} \|Av_1\|u_1 \cdots \|Av_r\|u_r \; 0 \cdots 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \cdots u_r \; u_{r+1} \cdots u_m \end{bmatrix} \begin{bmatrix} \|Av_1\| & & & \\ & \ddots & & \\ & & \|Av_r\| & \\ & 0 & & 0 \ddots 0 \end{bmatrix}$$

$m \times n$

$$A = \underbrace{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix}}_{\text{orthogonal}} \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ & 0 & & \end{bmatrix} \underbrace{\begin{bmatrix} — & v_1^T & — \\ & \vdots & \\ — & v_n^T & — \end{bmatrix}}_{\text{orthogonal}}$$

$a_1 = (v_1^T x)$

$X = a_1v_1 + \cdots + a_rv_r + \cdots + a_nv_n$

$Ax = a_1\sigma_1u_1 + \cdots + a_r\sigma_ru_r$

$= (v_1^T x)\sigma_1u_1 + \cdots$

$Av_1 = \sigma_1u_1$

$\vdots$

$Av_r = \sigma_ru_r$

$Av_{r+1} = 0$

$\vdots$