HOMEWORK 3
MATH 104, SECTION 2

Some ground rules:

- You have to submit your homework via **Gradescope** to the corresponding assignment. The submission should be a **single PDF file**.
- Make sure the writing in your submission is clear enough! Answers which are illegible for the reader won't be given credit.
- Write your argument as clear as possible. Mastering mathematical writing is one of the goals of this course.
- Late homework will not be accepted under any circumstances.
- You're allowed to use any result that is proved in the lecture; but if you'd like to use other results, you have to prove them before using them.

Problem set (6 problems; due February 9 at 11am PT)

(1) Let (a_n) be a sequence with the property that its subsequences (a_{2n}), (a_{2n-1}), and (a_{3n}) are all convergent. Prove that (a_n) is convergent.

(2) Let (a_n) be a bounded sequence. Prove that

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n).$$

(3) Prove that $\limsup |a_n| = 0$ if and only if $\lim a_n = 0$.

(4) Let (a_n) be a sequence of nonzero real numbers. Assume that $\limsup \left|\frac{a_{n+1}}{a_n}\right| = L$ is finite. You'll prove $\limsup(|a_n|^{1/n}) \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$ in this problem. Using similar argument, you can show that

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le \liminf_{n\to\infty}(|a_n|^{1/n}) \le \limsup_{n\to\infty}(|a_n|^{1/n}) \le \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|,$$

which will be important for us later on in the course.

(a) Let L' be any number bigger than L. Prove that there exists $N > 0$ such that $\left|\frac{a_{n+1}}{a_n}\right| < L'$ for any $n > N$.

(b) Prove that for any $n > N$, we have $|a_n| < (L')^{n-N}|a_N|$.

(c) Prove that $\limsup(|a_n|^{1/n}) \le L'$. (You might need to use Ross, Theorem 9.7.)

(d) Finally, prove that $\limsup(|a_n|^{1/n}) \le L$.

(p.s.: think about why do we need to choose such $L' > L$ in step (a)?)

(5) Let (a_n) and (b_n) be bounded sequences.

(a) Prove that $(a_n + b_n)$ is bounded.

(b) Prove that

$$(\liminf_{n\to\infty} a_n)+(\liminf_{n\to\infty} b_n) \le \liminf_{n\to\infty}(a_n+b_n) \text{ and } (\limsup_{n\to\infty} a_n)+(\limsup_{n\to\infty} b_n) \ge \limsup_{n\to\infty}(a_n+b_n).$$

(c) Find an example of (a_n) and (b_n) such that

$$(\liminf_{n\to\infty} a_n) + (\liminf_{n\to\infty} b_n) < \liminf_{n\to\infty}(a_n + b_n).$$

(6) (a) Let (a_n) be a sequence such that $|a_{n+1} - a_n| < C^n$ for all n for some constant $0 < C < 1$. Prove that (a_n) is a Cauchy sequence, therefore is convergent.

(b) Let (a_n) be a sequence such that $|a_{n+1} - a_n| < \frac{1}{n}$ for all n. Is it true that such (a_n) is always convergent?