Entropy of an autoequivalence on Calabi–Yau manifolds

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We'll see...

- Counterexamples of a conjecture on categorical entropy.
- Homological mirror symmetry $\Longrightarrow$ expect counterexamples.
- An interesting(?) entropy formula of an autoequivalence.
- Related project: Dynamics on space of Bridgeland stability conditions motivated from Teichmüller theory.

Examples of autoequivalences

Standard autoequivalences

- $\mathcal{D} = \mathcal{D}^b(X)$.
- Standard autoequivalences: $\otimes\mathcal{L}$, $\mathrm{Aut}(X)$, $[n]$.
- Bondal–Orlov '01: When K_X is (anti-)ample, the group of autoequivalences is generated by the standard ones.

Examples of autoequivalences

Spherical twists (Seidel–Thomas '01)

- Let X be a Calabi–Yau manifold of dimension d.
- $E \in \mathcal{D}^b(X)$ is spherical if

$$\mathrm{Hom}(E, E[*]) \cong H^*(S^d; \mathbb{C}).$$

e.g. Lagrangian sphere in derived Fukaya category.

e.g. $\mathcal{O}_X$ is spherical iff $H^i(\mathcal{O}_X) = 0$ for $0 < i < d$, i.e. X is *strict* CY.

Examples of autoequivalences

Spherical twists (Seidel–Thomas '01)

- Let X be a Calabi–Yau manifold of dimension d.
- $E \in \mathcal{D}^b(X)$ is <u>spherical</u> if

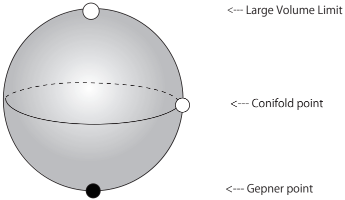
$$\mathrm{Hom}(E, E[*]) \cong H^*(S^d; \mathbb{C}).$$

- <u>Spherical twist</u> $\mathrm{T}_E : F \mapsto \mathrm{Cone}(\mathrm{Hom}^\bullet(E, F) \otimes E \to F)$.

  e.g. Dehn twist along Lagrangian sphere.

Autoequivalences from monodromies

- Kähler moduli of CY hypersurface $X \subset \mathbb{CP}^{d+1}$:



- Monodromies $\rightsquigarrow$ Autoequivalences on $\mathcal{D}^b(X)$
- Kontsevich '96, Horja '99:
  LVL $\rightsquigarrow \otimes\mathcal{O}(1)$, Conifold $\rightsquigarrow \mathrm{T}_{\mathcal{O}_X}$, Gepner $\rightsquigarrow \mathrm{T}_{\mathcal{O}_X} \circ \otimes\mathcal{O}(1)$.
- Ballard–Favero–Katzarkov '12: $(\mathrm{T}_{\mathcal{O}_X} \circ \otimes\mathcal{O}(1))^{d+2} = [2]$.

- Entropy: Measures "complexity" of an autoequivalence.
  e.g. $\otimes\mathcal{O}(1)$, $\mathrm{T}_{\mathcal{O}_X}$, $\mathrm{T}_{\mathcal{O}_X} \circ \otimes\mathcal{O}(1)$ all have zero entropy.

Theorem ($d \geq 3$)

$\mathrm{T}_{\mathcal{O}_X} \circ \otimes\mathcal{O}(-1)$ *has positive entropy.*
Its exponential is the unique $\lambda > 1$ *satisfying*

$$\sum_{k \geq 1} \frac{\chi(\mathcal{O}(k))}{\lambda^k} = 1.$$

(e.g. quintic CY3: $\lambda^4 - 9\lambda^3 + 11\lambda^2 - 9\lambda + 1 = 0$.)

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$\Longrightarrow$ Counterexamples of Kikuta–Takahashi conjecture.

Plan

- Topological entropy and Gromov–Yomdin theorem.
- Categorical entropy and Kikuta–Takahashi conjecture.
- Reason to expect counterexamples via HMS.
- Counterexamples.

- (X, d) compact, $f : X \to X$ continuous surjective.
- Topological entropy $h_{\rm top}(f)$ measures "how fast points spread out when iterate f".
- $N(n, \epsilon) := \max\{\#F : F \subset X, \max_{0 \le i \le n-1} d(f^i(x), f^i(y)) \ge \epsilon$ for any $x, y \in F\}$.

Definition

$$h_{\rm top}(f) := \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\log N(n, \epsilon)}{n} \in [0, \infty].$$

- $h_{\mathrm{top}}(f)$ is an topological invariant: If $(X, d) \cong (X, d')$, then one gets the same topological entropy.
- $f^n = \mathrm{id}_X \implies h_{\mathrm{top}}(f) = 0$.

Theorem (Gromov–Yomdin)

X compact Kähler manifold, $f : X \to X$ holomorphic surjective.

$$h_{\mathrm{top}}(f) = \log \rho(f^*).$$

Here ρ is the spectral radius of $f^* : H^*(X; \mathbb{C}) \to H^*(X; \mathbb{C})$.

Categorical entropy

Definition (Dimitrov–Haiden–Katzarkov–Kontsevich '13)

Definition

For $E, F \in \mathcal{D}$, the *complexity* of F relative to E is

$$\delta(E,F) := \inf\left\{ k \;\middle|\; \begin{tikzcd}[column sep=small] 0 \arrow[rr] & & A_1 \arrow[dl] & \cdots & A_{k-1} \arrow[rr] & & F \oplus F' \arrow[dl] \\ & E[n_1] \arrow[ul, dashed] & & \cdots & & E[n_k] \arrow[ul, dashed] & \end{tikzcd} \right\}$$

Definition

If $\mathcal{D}$ has a split generator G, then the categorical entropy of an autoequivalence Φ is

$$h_{\mathrm{cat}}(\Phi) := \lim_{n\to\infty} \frac{\log \delta(G, \Phi^n G)}{n} \in [-\infty, \infty).$$

- The limit exists. And is independent of the choice of G.
- $\Phi^n = [m] \implies h_{\mathrm{cat}}(\Phi) = 0$.

Conjecture (Kikuta–Takahashi)

For $\mathcal{D} = \mathcal{D}^b(X)$ and Φ an autoequivalence on $\mathcal{D}$,

$$h_{\mathrm{cat}}(\Phi) = \log \rho(\Phi_{H^*}).$$

<u>Proved</u>: $\dim X = 1$; standard autoequivalences.

Reason to expect counterexamples

- Thurston: examples of pseudo-Anosov maps on Riemann surface S $(g > 1)$ that act trivially on H^*. These maps are symplectomorphisms, but not holomorphic. Gromov–Yomdin fails in these cases: $h_{\mathrm{top}}(f) = \log \lambda > 0 = \log \rho(f^*)$.
- DHKK: $h_{\mathrm{cat}}(f_*) = \log \lambda > 0$. Here f_* is the induced autoequivalence on $\mathrm{Fuk}(S)$.
- Idea: If there are autoequivalences on $\mathrm{Fuk}(X)$ with $h_{\mathrm{cat}}(\Phi) > \log \rho(\mathrm{HH}_\bullet(\Phi))$ for some Calabi–Yau X, then by homological mirror symmetry, one may expect to find counterexamples of the conjecture on the mirror.

Counterexamples

Theorem ($d \geq 3$)

$\Phi := \mathrm{T}_{\mathcal{O}_X} \circ \otimes \mathcal{O}(-1)$ *has positive categorical entropy. Its exponential is the unique* $\lambda > 1$ *satisfying*

$$\sum_{k \geq 1} \frac{\chi(\mathcal{O}(k))}{\lambda^k} = 1.$$

Claim

$d \geq 4$ even. $X \subset \mathbb{CP}^{d+1}$ CY hypersurface of degree $d+2$. Then

$$\rho(\Phi_{H^*}) = 1.$$

Hence $h_{\mathrm{cat}}(\Phi) > 0 = \log \rho(\Phi_{H^*})$. So Kikuta–Takahashi conjecture fails in this case.

Proof of Claim

- $d \geq 4$ even. $X \subset \mathbb{CP}^{d+1}$ CY hypersurface of degree $d + 2$.
- Recall that $(\mathrm{T}_{\mathcal{O}_X} \circ \otimes\mathcal{O}(1))^{d+2} = [2]$.

  $\implies (\mathrm{T}_{\mathcal{O}_X} \circ \otimes\mathcal{O}(1))_{H^*}^{d+2} = \mathrm{id}_{H^*}$.
- Fact: $(\mathrm{T}_S^2)_{H^*} = \mathrm{id}_{H^*}$ when X is of even dimension.

  $\implies \Phi_{H^*}^{d+2} = (\mathrm{T}_{\mathcal{O}_X} \circ \otimes\mathcal{O}(-1))_{H^*}^{d+2} = \mathrm{id}_{H^*}$.

  $\implies \rho(\Phi_{H^*}) = 1$.

Sketch of proof of Theorem

- DHKK: If G and G' are both split generators of $\mathcal{D}^b(X)$, then

$$h_{\mathrm{cat}}(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{a\in\mathbb{Z}} \dim \mathrm{Hom}(G, \Phi^n G'[a]).$$

- Orlov: $G = \oplus_{i=1}^{d+1} \mathcal{O}(i)$ and $G' = \oplus_{i=1}^{d+1} \mathcal{O}(-i)$ are split generators.

- Lemma: Recursive formula for the dimension of $\mathrm{Hom}(\mathcal{O}, \Phi^n(G') \otimes \mathcal{O}(-k)[a])$ via Kodaira vanishing.

- $+$ some combinatorics $\implies$ Theorem.

Thank you!!