Dynamical aspects of categories

Yu-Wei Fan (Tsinghua U.)

Outline:

- Background and examples of $\mathrm{Stab}(\Delta$ category$)$
  (relations with Teichmüller theory and Calabi–Yau geometry)
- Study $\mathrm{Aut}(D)$ via its action on $\mathrm{Stab}(D)$
  (dynamical invariants, classifications)
- Quasi-convergent paths in $\mathrm{Stab}(D)$ and SOD
  (Daniel Halpern-Leistner: relating Stab, QH*, birational geometry)

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Analogy between Teichmüller theory and stability data:

Riemann surface S	Triangulated category D
curve C	object E
$C_1 \cap C_2$	$\mathrm{Hom}(E_1, E_2)$
metric g	stability σ
geodesic	semistable object
$C_1 \# \cdots \# C_n$	HN filtration $\mathrm{gr}_i(E) = E_i$
length $\sum \ell_g(C_i)$	mass $\sum \lvert Z_\sigma(E_i)\rvert$
$\mathrm{MCG}(S) \curvearrowright \mathrm{Teich}(S)$	$\mathrm{Aut}(D) \curvearrowright \mathrm{Stab}(D)$
Dehn twist	spherical twist

Other things that admit categorical analogues: Topological entropy, pseudo-Anosov maps, systoles and systolic inequalities, $\mathrm{SL}(2,\mathbb{R})$-action , counting of saddle connections, etc.

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- $\sigma \in \mathrm{Stab}(D^b(X))$ is analogous to a Kähler/symplectic structure on X.
- It is conjectured that for a Calabi–Yau manifold X with Kähler class ω, there is a stability condition on $D^b(X)$ with central charge

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  and semistable objects given by special Lagrangians.
- There are deep connections among $\mathrm{Stab}(D^b(X))$, quantum cohomology of X, birational geometry of X, which we'll discuss later.

Examples of stability conditions

Example: Let X be an elliptic curve. Then any $\sigma \in \mathrm{Stab}(D^b(X))$ is (up to a natural group action) equivalent to the slope stability:

- $Z(E) = -\deg(E) + i \cdot \mathrm{rank}(E)$.
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Its mirror is FS $\left(W \colon \mathbb{C}^* \xrightarrow{z+1/z} \mathbb{C}\right)$. By Haiden–Katzarkov–Kontsevich, its stability space can be parametrized by 1-forms $\phi_{a,b} = \exp\left(z + a + \frac{b}{z}\right) \frac{\mathrm{d}z}{z}$:

- $Z_{a,b} = \int_L \phi_{a,b}$.
- L is $\sigma_{a,b}$-semistable if and only if it is a finite length sLag w.r.t. $\phi_{a,b}$.

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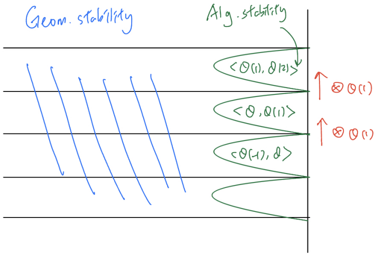

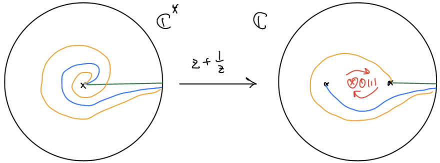

Consider the orbit of $\Phi \in \mathrm{Aut}(D)$ acting on $\sigma \in \mathrm{Stab}(D)$

$$\left\{\dots, \Phi^{-1}\sigma,\ \sigma,\ \Phi\sigma, \dots\right\}.$$

We can use their mass functions $m_\sigma(\bullet)$ and phase functions $\phi_\sigma^\pm(\bullet)$ to extract dynamical invariants of Φ.

- The growth rate of $m_{\Phi^n\sigma}(\bullet)$ gives the categorical generalization of topological entropy.
- The growth rate of $\phi^\pm_{\Phi^n\sigma}(\bullet)$ gives the categorical generalization of Poincaré translation number.

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Using the analogy between $\mathsf{MCG}(S) \curvearrowright \mathsf{Teich}(S)$ and $\mathrm{Aut}(D) \curvearrowright \mathrm{Stab}(D)$, we can also study:

- trichotomy like finite order/reducible/pseudo-Anosov;
- properties of finite order autoequivalences;
- various categorical generalizations of pseudo-Anosov maps.

Topological entropy

Let (X, d) be a compact metric space and $f\colon X \to X$ continuous. Consider

$$N(n, \epsilon) := \max\left\{\ell\colon \exists x_1, \ldots, x_\ell \text{ s.t. } \max_{0 \le k \le n}\{d(f^k(x_i), f^k(x_j))\} \ge \epsilon \ \forall x_i, x_j\right\}$$

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The topological entropy of f is defined to be

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- (Gromov) Moreover, if X is Kähler and f is holomorphic, then $h_{\mathrm{top}}(f) = \log\rho(f^*_{H^*(X,\mathbb{C})})$.
- (Cantat) If a compact complex surface X admits $h_{\mathrm{top}}(f) > 0$, then X is either a torus, a K3 surface, an Enriques surface, or a rational surface.

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Remark: Existence of manifold with corner structure of the compactification of Stab in D. H.-L.'s program $\implies$ $\hom(G, E) \leq C \cdot m_\sigma(E)$ for all E $\implies$

- $h_{\mathrm{cat}}(\Phi) = h_\sigma(\Phi)$.
- $\mathcal{M}_\sigma^{ss}(v)$ admits a proper good moduli space.

Categorical polynomial entropy

For $h_{\mathrm{cat}}(\Phi) = 0$, one can consider a more refined invariant, its polynomial entropy

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- Let L be a line bundle on X. Then $h_{\mathrm{cat}}(-\otimes L) = 0$, and

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  where $\nu(L) = \max\{m \mid c_1(L)^m \not\equiv 0\}$ the numerical dimension of L.
- Let X be an elliptic curve, and $\Phi \in \mathrm{Aut}(D^b(X))$. Then
  - $h_{\mathrm{cat}}(\Phi) = h_{\mathrm{poly}}(\Phi) = 0$ iff $[\Phi] \in \mathrm{SL}(2,\mathbb{Z})$ is elliptic.
  - $h_{\mathrm{cat}}(\Phi) = 0$ and $h_{\mathrm{poly}}(\Phi) = 1$ iff $[\Phi] \in \mathrm{SL}(2,\mathbb{Z})$ is parabolic.
  - $h_{\mathrm{cat}}(\Phi) > 0$ iff $[\Phi] \in \mathrm{SL}(2,\mathbb{Z})$ is hyperbolic.

Shifting numbers

(F.–Filip) The limits $\tau^{\pm}(\Phi) := \lim_{n\to\infty} \frac{1}{n}\phi_{\sigma}^{\pm}(\Phi^n G)$ always exist, and are independent of the choices of G and σ.

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$f \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$	$\Phi \in \mathrm{Aut}(\mathcal{D})$
$x_0 \in \mathbb{R}$	$G \in \mathcal{D}$
amount of translation	phases $\phi_\sigma^{\pm} \colon \mathrm{Ob}(\mathcal{D}) \to \mathbb{R}$
$f^{(n)}(x_0) - x_0$	$\phi_\sigma^{\pm}(\Phi^n G) - \phi_\sigma^{\pm}(G)$
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Note: $\tau^{\pm}(\mathrm{Serre}_D)$ is the upper/lower Serre dimension of D.

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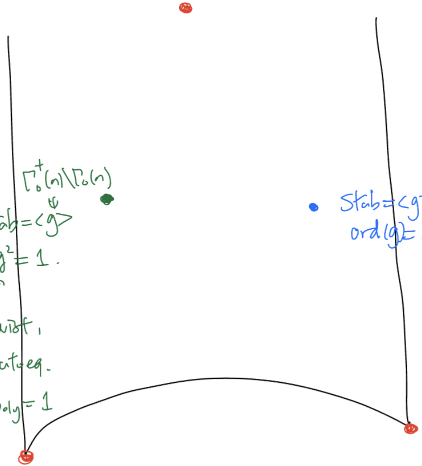
- Moreover, this is the main theorem that we use to fully classify finite subgroups of $\mathrm{Aut}(D)/[1]$ for K3 surfaces of $\rho = 1$.
- Corollary: $\mathrm{Aut}(D^b(X))/[1]$ contains an order 3 element if and only if X admits an associated cubic fourfold.

X : K3 surface, $g = 1$, $H^2 = 2n$,

$$\mathbb{H} / \Gamma_0^+(n)$$
$$\Gamma_0^+(n) = \left\langle \Gamma_0(n), \begin{bmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{bmatrix} \right\rangle$$



$\Gamma_0^+(n) \setminus \Gamma_0(n)$
$\Downarrow$
$\mathrm{Stab} = \langle g \rangle$
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$\mathrm{Stab} = \langle g \rangle$. $g \in \Gamma_0(n)$
$\mathrm{ord}(g) = 2$ or 3. $\longleftrightarrow$

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cusps $\longleftrightarrow$ $\otimes \mathcal{O}_{X'}(1)$ for some $D^b(X) \simeq D^b(X')$

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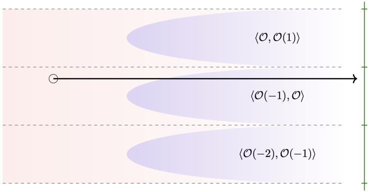
Daniel Halpern-Leistner's NMMP proposal



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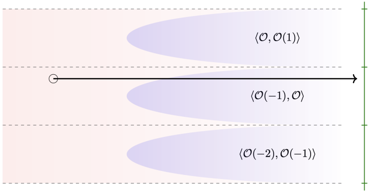

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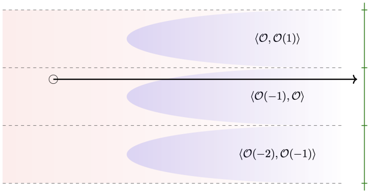

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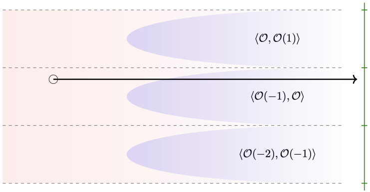

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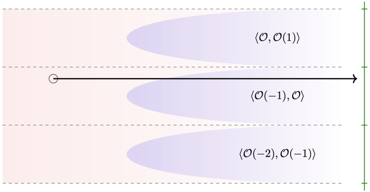

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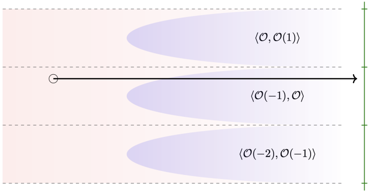

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- Corollary: Existence of noncommutative minimal model, D-equivalence conjecture, (one side of) Dubrovin conjecture.

Thank you for your attention!