Lecture 3
Forward Contracts on Interest Rate

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Outline

1. Zero Coupon Bonds

2. Forward Contract on Interest Rates

3. Forward Rate Agreements

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2. Forward Contract on Interest Rates

3. Forward Rate Agreements

Zero Coupon Bonds

- Zero coupon bonds are bonds that pays no coupons (dividends) before the maturity.
- They are the simplest assets (no risk, no uncertainty, one piece of cash).
- We use $Z(t, T)$ to denote **time** t price of a zero-coupon bond that pays \$1 at time T.
  - Suppose the continuously-compounded interest rate is r_c, $Z(t, T) = e^{-r_c(T-t)}$.
  - If we invest \$1 in riskfree rate, we receive $e^{r_c(T-t)}$ at maturity;
  - Adjust proportionally, and we know that at time t we only need to invest $e^{-r_c(T-t)}$ in zero-coupon bonds to get \$1 at time T.

Why do we need zero coupon bonds?

- In real financial market, there is no such thing as zero coupon bonds.
- In practice, bonds are issued with very complicated terms (interest payment, maturity, ...), making it very hard to make comparison.
  - For example, it is very hard to say a 5-year bond with 5% annual interest payment should be more expensive than a 3-year bond with 6% semi-annual interest payment.
- People can, however, bundle different bonds and create 'artificial' zero coupon bonds.
- The artificial bonds can be used to price complex bonds (in practice), and provide direct understanding of economic outlooks.

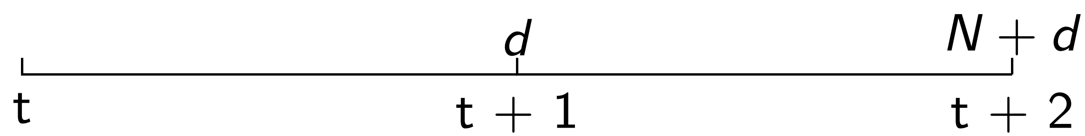
Pricing Complex Bonds with Zero Coupon Bonds 1

- In practice, bonds have complex cash flow dynamics.
- There are regularly-scheduled dividend payments, and the payment of principle at maturity.
- We can use zero coupon bonds to price the bonds.

Pricing Complex Bonds with Zero Coupon Bonds 2

Suppose a two-year bond delivers interest d every one year, and then face value N at maturity.

- The cash flow is illustrated in the following figure.



- The bond can be considered as a combination of different zero coupon bonds.

Cash flow	d at $t+1$	$N+d$ at $t+2$
time-t price	$dZ(t,t+1)$	$(d+N)Z(t,t+2)$

- The time-t price of the bond should be

$e^{-r(T-t)}$

$$dZ(t,t+1)+(N+d)Z(t,t+2).$$

- Note: Here I do not replace $Z(t,T)$'s with $e^{-r_c(T-t)}$ as r_c could vary with t and T.

Finding the Prices of Zero Coupon Bonds

- In practice, we are quoted with prices of bonds with multiple cash flow, rather than zero coupon bonds.

- We can use exactly the opposite argument to extract the prices of zero coupon bonds.

Finding the Prices of Zero Coupon Bonds

- Suppose we have two bonds:
  - One with maturity 6 months. No dividend payment. Face value: \$100, traded at \$95.
  - One with maturity 1 year, two dividend payments of \$5 after 6 months and 1 year, respectively. Face value: \$100, traded at \$100.
- The prices then must meet the following equations:

$$\begin{aligned} 95 &= 100Z(t, t + 0.5) \\ 100 &= 5Z(t, t + 0.5) + 105Z(t, t + 1). \end{aligned}$$

- We can then solve for $Z(t, t + 0.5)$ and $Z(t, t + 1)$.

$$\begin{aligned} Z(t, t + 0.5) &= 0.95 \\ Z(t, t + 1) &= 0.907. \end{aligned}$$

- We can further calculate the continuously-compounded interest rates (annualized) for 6-month and 1-year deposits, which are 10.26% and 9.75%, respectively.

Yield Curve

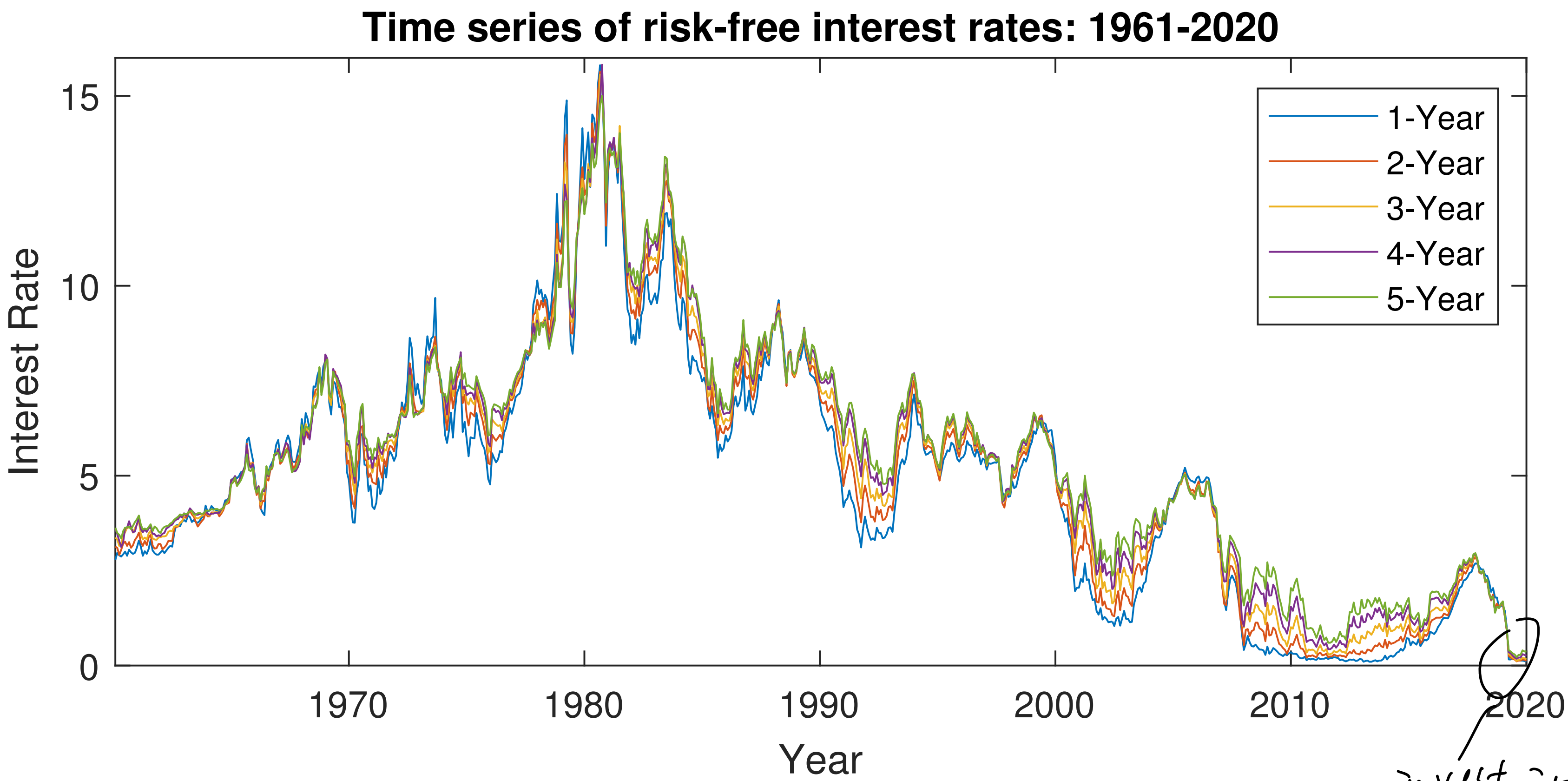
- In the previous slide, we solved the interest rates associated to 6-month and 1-year zero coupon bonds, respectively.
- In practice, people use more complicated products traded in the market and ‘solve’ for prices and then corresponding interest rates for zero coupon bonds of different maturities.
- The curve that plots (zero coupon bond) interest rates against maturity is called yield curve. This is core for fixed income asset pricing.

Outline

Forward Contract on Interest Rates 1

So far, we have assumed that interest rate is constant.

- What is the reality?



invest in
safe asset

Forward Contract on Interest Rates 2

- Suppose you are a trade company and is scheduled to receive 1 million HKD after six months. The current interest rate for deposits is very high.
- You want to lock in the interest rates. However, interest rates might fluctuate.
- You want to construct a contract **such that** you can lock in the interest rates for future deposits.

Forward Contract on Interest Rates 3

Now let's think in the investment bank's position:

- A client want a contract to lock in the interest rate for deposits from T_1 to T_2 in the future.
- At T_1, the client deposit the cash.
- At T_2, the client get back the cash, with market interest rate at T_1. Instead, the client wants you to pay a pre-agreed amount, and pays you the market interest.

Forward Rate 1

long = buy fixed, pay mkt

short = buy mkt, pay fixed.

The timeline for the investment bank goes like:

- t: initiation of the contract.
- T_1: Market interest rate r_{mkt} is known, but no cash flow.
- T_2: Receives r_{mkt} from client, and pay a pre-specified rate r_K to the client.

Forward Rate 2

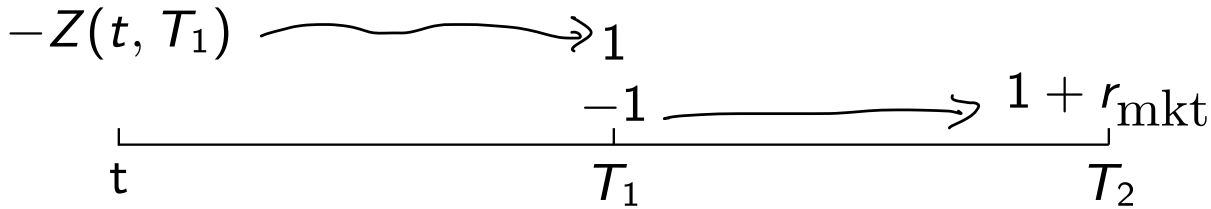
I bank (short) cash flow = $-r$ fix + r mkt

How do we price this contract for the bank? replicate $-r_K$.

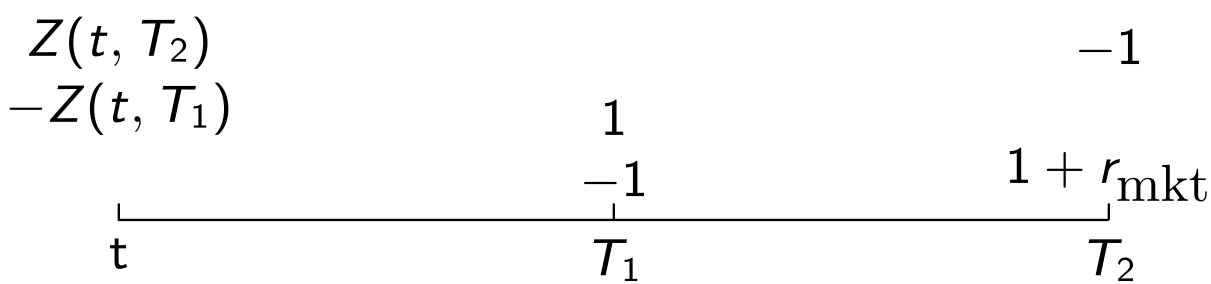
- $-r_K$ at time T_2 is easy: it's a fixed cash flow, and it can be priced as bonds. $-r_K Z(t, T_2)$.

Forward Rate 3

- r_{mkt} at time T_2 is tricky: we don't know it at time t.
- However, we can **deposit $\$1$ at time T_1**! We then can get $1 + r_{\text{mkt}}$ at T_2. This requires $Z(t, T_1)$ at time t.



- We now have $\$1$ too much at T_2. We get rid of it by short-selling and get $Z(t, T_2)$ at time t.



- The net cost of replicating r_{mkt} at time T_2 is then $Z(t, T_1) - Z(t, T_2)$.

- The value of the contract for the investor is then

$$-r_K Z(t, T_2) + Z(t, T_1) - Z(t, T_2)$$

.

$(r_K + 1)(-Z(t, t+1))$

- We further convert r_K to continuous compounding by using

$$e^{r_{c,K}(T_2 - T_1)} = 1 + r_K.$$

- Set the value zero. With some algebra, we get

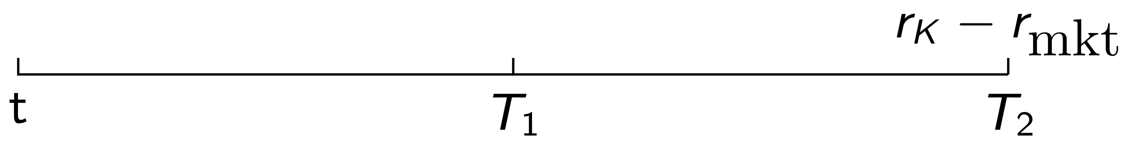
$$e^{r_{c,K}(T_2 - T_1)} Z(t, T_2) = Z(t, T_1)$$

$$r_{c,K} = \frac{1}{T_2 - T_1} \ln\left(\frac{Z(t, T_1)}{Z(t, T_2)}\right).$$

- This is the continously-compounded interest rate the investment bank is willing to agree with at time t for a deposit from time T_1 to T_2. We call this **forward rate**.

Outline

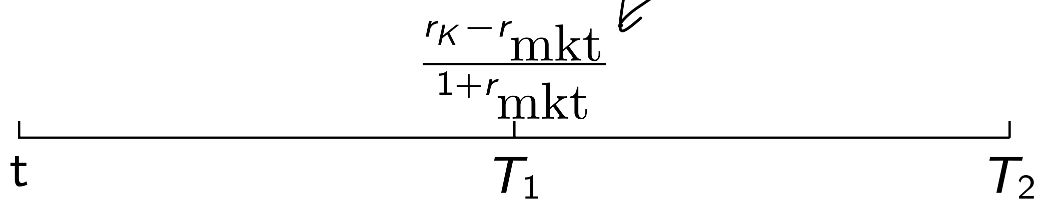
- Now let's revisit the forward contract we just created, the **net** cash flow for the investor is



- However, r_{mkt} is revealed at T_1. As a result, the cash flow can be moved to T_1 as it's determined, with a discount, $1 + r_{\mathrm{mkt}}$.

- The new cash flow is given by

everything is certain



- This is the cash flow for forward contracts on interest rates in practice. Such contracts are called **Forward Rate Agreements (FRA)**.

Forward Rate Agreements (FRA) 2

- FRAs are OTC contracts that guarantee borrowing or lending rate
  - on given principal amount;
  - starting in future date;
  - for a specified period.

Cash Flow of FRA

For an investor who tries to borrow notional principle N for a fixed rate

- At time t, he enters a FRA contract with cost 0, with the 'fixed' leg given by the **simple** interest rate r^{FRA}_{t,T_1,T_2}. The corresponding borrowing period is from T_1 to T_2.
  - Here *simple interest rate* is the annualized return of holding a corresponding zero coupon bond.
- At time T_1, the interest from T_1 to T_2 is revealed. If the investor does not enter the FRA, we will need to pay $1 + r^{\text{simple}}_{T_1,T_2}(T_2 - T_1)$ at T_2 for each 1 dollar he borrows at T_1. The contract implies that the investor needs to pay $N(1 + r^{FRA}_{t,T_1,T_2}(T_2 - T_1))$ and receives $N(1 + r^{\text{simple}}_{T_1,T_2}(T_2 - T_1))$ at time T_2.
- The profit and loss are settled at time T_1. As a result, the investor needs to discount the cash flow with the market simple interest rate. As a result, the payoff at T_1 is

$$\frac{(r^{\text{simple}}_{T_1,T_2} - r^{\text{FRA}}_{T_1,T_2}) \times (T_2 - T_1)}{1 + r^{\text{simple}}_{T_1,T_2}(T_2 - T_1)} \times N.$$