Lecture 2
Forward Contracts on Financial Assets and Indices: Non-arbitrage and Replicate Arguments

Yicheng Zhu

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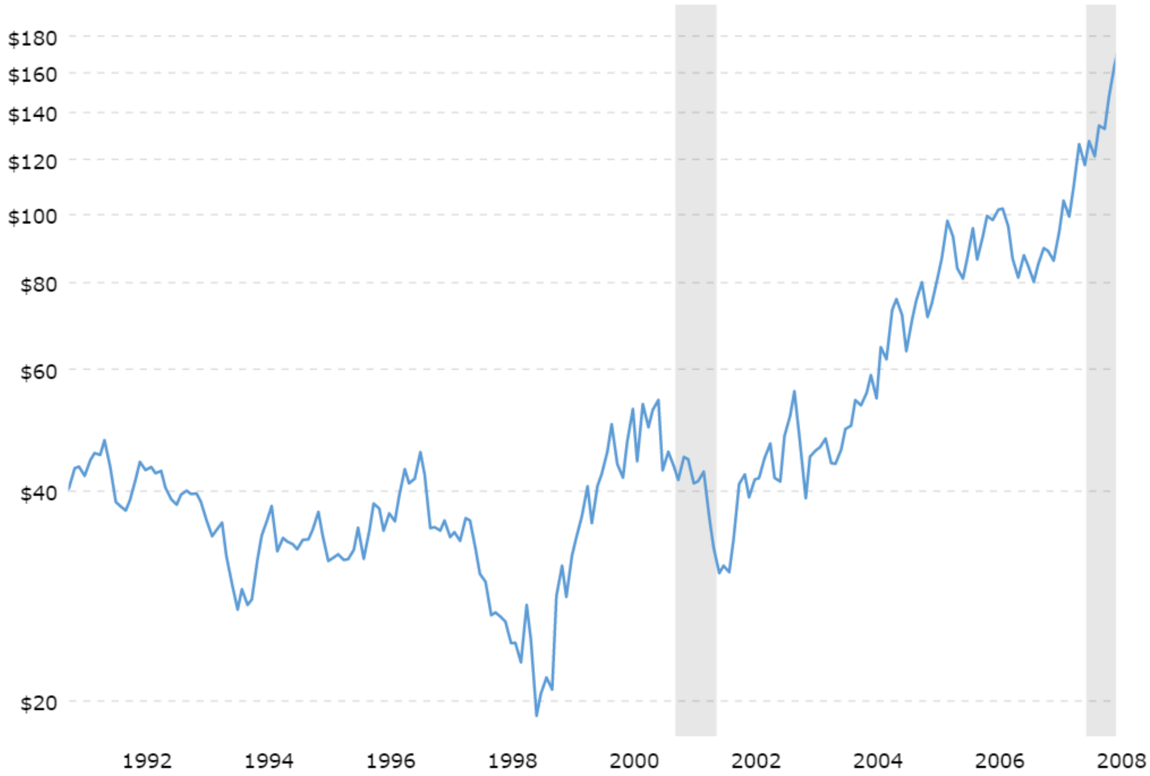
Outline

Forward Contracts: What Are They? 1

From early 2000's, the oil gas prices soared due to boom in global economy and traveling.



Forward Contracts: What Are They? 2

- Gas is a major source of cost for airlines.
- The rising oil price had become a major threat to airline's profitability.
- Cathay Pacific, Hong Kong's flag-carrying airline, entered some contracts with some major investment banks trying to lock in the prices of gas deliveries **in the future**.
- The contracts specified the delivery dates, the quantities, and the prices. The transaction would occur when the delivery actually happens.

Forward Contracts: What Are They? 3

Definition: A forward contract is an agreement between two counter-parties to buy or sell a pre-specified amount (N units) of the underlying asset on a pre-specified date T in the future for a pre-specified unit price K.

- A promise to trade: you are **obliged** to finish the transaction.
- **Expiration (maturity) date T:** Date when the purchase and payment are settled.
- Two counter-parties:
  - **long** position: the buyer of the underlying at expiration date T;
  - **short** position: the seller. receive cash (shorting)
- For simplicity, we always assume $N = 1$ throughout this course.

  Just multiply the quantity

Forward Contracts: Payoffs

($ I get)
實際未來價格

K
預測未來價格
($ I have to pay in future)

At the maturity T, let S_T be the underlying asset's unit price:

- The long position holder pays K, and receives one unit of asset which has value S_T.
- The long position holder receives **net payoff** $S_T - K$.
- The short position holder receives the opposite, with **net payoff** $K - S_T$.
- A zero-sum game:
  - If $S_T > K$, the buyer (long position) makes a profit, and the seller loses;
  - If $S_T < K$, the seller (short position) makes a profit.

Trading with Forward Contracts: Leverage

- The share of firm *Gamestop*'s stock was traded at $S = \$20$ at the end of 2020.
- The smart investors think the stock price was too low and it will rise soon.
- **Problem:** investors have limited funds for purchasing the stocks.
  - The investor can borrow money. However, regulation and banks are unwilling to allow for that.
- **Solution:** long a forward contract which allows the investors to buy one unit of Gamestop's stock at maturity, T. Suppose that the strike price is $K = \$20$.
- Suppose the investors need to pay 0 to enter the contract; however they need to put up \$2 cash as collateral.

Table: Payoff and return to hedge fund (10 times **leverage**)

	Scenarios	
Terminal price S_T	\$18	\$22
Stock return	-10%	10%
Payoff to long forward	$\$-2$	\$2
Return on long forward	-100%	100%

Cash vs. Physical Settlement

- So far, we assume (friction-less) physical settlement: delivery of goods does take place.
- In reality, it entails delivery costs (e.g. shipping costs, transaction costs, ...).
- Cash settlement is more efficient: counter-parties deliver **net profit / loss**.
- Cash settlement is feasible only when there is an accepted reference price upon which the settlement can be based on.
  - Can you do a cash settlement for small corporate bonds? Why and why not?

    No. Not actively traded. Can't find the fair price

Outline

Fundamentals of Derivative Pricing: Law of One Prices 1

- **Main Idea: "Law of One Prices"** two securities with identical payoffs should have the same prices.
  - Otherwise the investors can 'buy low and sell high' to take advantage of the opportunity.

Fundamentals of Derivative Pricing: Law of One Prices 2

- The two securities should be 'exactly the same', meaning that the payoff should be **identical all the time**.
  - Let's play a small gamble with a fair coin. After flipping the coin, I pay you \$10 if we get head and 0 otherwise.

    Same outcome?

  - If I do two separate trials, and charge you \$5 for the first trial, and \$4 for the second, have I violated Law of One Price?

Yes

give \$100 when everything is bad ($-\beta$)

good ($+\beta$)

Negative Beta for insurance firm

much more expensive than high risk firm

Pricing a Forward Contract: Replication Argument 1

Today is t, and an investor wants to long a forward contract with the underlying being stock ABC and strike price K with maturity T. How much should the investment bank charge the investor for the contract?

- We can create a portfolio with exactly the same payoff at time T. Law of One Price then suggest that the portfolio costs the same as the forward contract.
- This requires us to analyze the cash flow of the forward contract at time T.
  - For long position, the cash flow at time T is given by $S_T - K$.
  - S_T: the underlying stock ABC.
  - $-K$: the price the investor pays.

Pricing a Forward Contract: Replication Argument 2

$K - S_t$

long

- We now construct our portfolio **such that** the portfolio has the same payoff.
  - S_T: We can include one stock in our portfolio. This costs S_t today.
    sell ... earns
  - $-K$: We **borrow** $Ke^{-r_c(T-t)}$ today.
    lend
- The portfolio costs $S_t - Ke^{-r_c(T-t)}$ today. This must be the price of the forward contract as well.

value of the forward contract

$K - S_t$

Absence of Arbitrage 1

Why does Law of One Price hold? Because we assume there is no arbitrage opportunity in the economy. Formally,

- **An arbitrage opportunity** is a trading strategy that
  (1) costs nothing today;
  (2) generates non-negative payoff in the future;
  (3) generates positive payoffs with positive probability.

Absence of Arbitrage 2

- **Absence of Arbitrage:** There is no arbitrage opportunity in the market.
  - Why is this reasonable?
  - One implication: if a trading strategy pays 0 in all cases, it should cost 0.

Pricing a Forward Contract: Absence of Arbitrage Argument

- Let's revisit the example before. We can create a portfolio with

  (1) Long the forward contract;

  (2) short the underlying asset;

  (3) lend $Ke^{-r_c(T-t)}$.

  short

- The portfolio has cash flow zero at time T. It then must cost 0 at time t. We can find that the price of the forward contract must be $S_t - Ke^{-r_c(T-t)}$.

Outline

The Forward Price

- Suppose the delivery price for a forward contract is zero. How much do you want to pay to long the forward? What if the delivery price is positive infinity?
- **Formal Definition:** The Forward Price is the strike price for a forward contract such that the forward contract has value 0.
- **The problem:** How do we find the forward price, $F(S_t, t, T)$?
- **The solution:** By setting the price of a contract zero!

$$S_t - Ke^{-r_c(T-t)} = 0$$
$$K = S_t e^{r_c(T-t)}.$$

- As a result,

$$F(S_t, t, T) = S_t e^{r_c(T-t)}.$$

- For a non-dividend-paying asset S_t, the time-t forward price with maturity T is given by

$$F(S_t, t, T) = S_t e^{r_c(T-t)}.$$

  - r_c: continuously compounded interest rate at time-t.
  - The interest rate could vary overtime. It's also connected to the maturity. We will visit this issue later.

Forward Price vs. the Value of a Forward Contract

To summarize, we have found:

- The Forward Price – the strike price with which the value of a forward contract is zero.

$$F(S_t, t, T) = S_t \times e^{r_c(T-t)}.$$

- The value of an **existing** forward contract to buy the underlying at predetermined strike K (set at some time in the past) is the profit (cost) of closing the contract **immediately**.

$$S_t - Ke^{-r_c \times (T-t)}.$$

Value at initiation

Forward price vs. the value of a forward contract

Important Reminders:

- In practice, the values of most forward contracts at inception are zero.
- Most forward contracts start with strike fixed at corresponding forward prices.
- No exchange of money when two counter-parties enter into a forward contract
- But, as time passes, the value of a forward contract may not be zero.

Outline

The Forward Price of a Stock with Dividend Payment 1

So far we have assumed that the underlying asset pays nothing between now and the maturity of the forward contract.

- For stocks, however, they generally have some regular dividend payments.
- Consider a stock with price S_t which pays one known dividend D at time T_1 before the maturity T.
- What's the time t forward price of the stock such that $t < T_1 < T$?

The Forward Price of a Stock with One Dividend Payment 2

Result: The forward price is given by

$$F(S_t, t, T) = [S_t - PV_t(D)] \times e^{r_c(T-t)}.$$

forward only cares the share, no extra thing

- $PV_t(D)$ Present value of the dividend payment at **time** t.

- For replication argument, consider

  fundamental change?

  - Long the forward contract.

    S_T $- PV(D)$

  - (1) Long 1 unit of the underlying stock; (2) short-sell the claim to the dividend payment and (3) borrow $F(S_t, t, T)e^{-r_c(T-t)}$ at risk-free rate.

    $-K$.

- The two portfolios yield the same payoffs. This can help to find the value of the forward price.

PV(S) = Price of dividend + Stock after the dividend before the maturity

One share in forward contract is different from one share

The Forward Price of a Stock with Constant Dividend Yield 3

- Continuously-compounded dividend yield q (i.e. interest rate on your stock shares)
  - If you have 1 share at t, then you will have $1 \times e^{q(T-t)}$ shares at T.
  - This is stock dividend: dividends are paid with stocks.
  - q is the "implied" continuously-compounded dividend yield; It does not mean that the company is paying dividend continuously.
- How many units of the underlying stock do you need to replicate the payoff of the forward contract with **one unit** of delivery?

$$e^{-q(T-t)}.$$

- The corresponding forward price is $F(S_t, t, T) = S_t e^{(r_c - q)\times(T-t)}$.

$$K^{-r(T-t)} = e^{-q(T-t)}$$

Outline

Taking Advantage of an Arbitrage Opportunity 1

- So far we have found **what the forward price is supposed to be**.
- However, when people trade forward contracts, they negotiate and the forward price might deviate from what the formula says.
- What should you do if

We are short side

Forward price — Supposed Forward price

$$F(S_t, t, T) > [S_t - PV_t(D)] \times e^{r_c(T-t)}?$$

different for different parties e.g hedge fund / bank

  - The forward price is higher than what it is supposed to be.
  - It means that we should try to **sell** the stock in the future with the forward price.

can derive the Rf rate by quoted forward price

r_c = risk free
because it is risk free product

Taking Advantage of an Arbitrage Opportunity 2

- At time t: sell at K .
  (a) **Short** with forward price $F(S_t, t, T)$.
  (b) Buy one unit of the stock with price S_t.
  (c) Sell the claim to the dividend with price $PV_t(D)$.
  (d) Borrow $S_t - PV_t(D)$ with riskfree rate.
- At time T_1
  (a) Receive dividend D from stock, and deliver it to the buyer of the claim to it.
- At time T:
  (a) Receive $F(S_t, t, T)$ from sale of stock.
  (b) Repay the loan $[S_t - PV_t(D)] \times e^{r_c(T-t)}$

$$\text{Payoff at time } T = F(S_t, t, T) - [S_t - PV_t(D)] \times e^{r_c(T-t)} > 0$$

Summary

- Definition and concepts of forward contracts.
- Derivation of forward prices.
- Derivation of the value of forward contracts.
- Forward contracts on stocks with dividends.