绸带

不要只抄书，要用自己的语言叙述出来

Group theory: to describe symmetries (structures) (and some actions)

要不去简单瞄一眼范畴论?

期末考也得简单复习一下群的内容

Notice

fixed $g \in G$, $a \mapsto ag$ 是一个双射（因为有左右逆元 g^{-1} : $a \mapsto ag^{-1}$），$a \mapsto ga$ 也一样，所以 $|A| = |gA| = |Ag|$.

无限和有限的情况不一样！！！需要多加注意！！！

比如说部分可以"等于"整体。。。

Note that $H = \{g^2 \mid g \in G\}$ may equal to G. (e.g. $G = \mathbb{C}$).

感觉很多题都是构造 bijective homo 证同构或构造 bijective 证相等 或 证两边互相包含从而相等。

（证明大小相等也一样）

数数/研究结构：构造一个 homo，把kernel求出来

Chapter 0 Preliminaries

0.1 Basics

0.2 Properties of the Integers

0.3 $\mathbb{Z}/n\mathbb{Z}$: The Integers Modulo n

Part 1 Group Theory

Chapter 1 Introduction to Groups

1.1 Basic Axioms and Examples

Definition.

1. A **group** is an order pair $(G, \star)$ where G is a set and $\star$ is a binary operation on G satisfying the following axioms:
    1. $(a \star b) \star c = a \star (b \star c)$, for all $a, b, c \in G$, i.e., $\star$ is associative,
    2. there exists an element e in G, called an identity of G, such that for all $a \in G$, we have $a \star e = e \star a = a$,
    3. for each $a \in G$, there is an element $a^{-1} \in G$, called an inverse of a, such that $a \star a^{-1} = a^{-1} \star a = e$.
2. The group G is called Abelian (or commutative) if $a \star b = b \star a$ for all $a, b \in G$.

Proposition 1. If G is a group under the operation $\star$, then

1. the identity of G is unique
2. for each $a \in G$, a^{-1} is uniquely determined

3. $(a^{-1})^{-1} = a$ for all $a \in G$
4. $(a \star b)^{-1} = (b^{-1}) \star (a^{-1})$
5. for any $a_1, a_2, \ldots, a_n \in G$, the value of $a_1 \star a_2 \star \cdots \star a_n$ is independent of how the expression is bracketed (this is called the general associative law)

> Proof:
>
> 1. $e_1 = e_1 e_2 = e_2$.
> 2. If $ab = ca = 1$, then $c = ce = c(ab) = (ca)b = b$.
> 3. Since $(a^{-1})a = 1$, $(a^{-1})^{-1} = a$.
> 4. $abb^{-1}a^{-1} = 1$, so $(ab)^{-1} = b^{-1}a^{-1}$.
> 5. proof by mathematical induction

Proposition 2. Let G by any group and let $a, b \in G$, then the equations $ax = b$ and $ya = b$ have unique solutions for $x, y \in G$. In particular, the left and right cancellation law holds in G:

1. if $au = av$, then $u = v$, and
2. if $ub = vb$, then $u = v$.

> Proof:
>
> (1) 两边同时左乘 a^{-1} 即可
>
> (2) similarly.

Definition. For G a group and $x \in G$ define the **order** of x to be the smallest positive integer n such that $x^n = 1$, and denote this integer by $|x|$. If no positive power of x is 1, the order of x is defined to be infinity.

Definition. Let $G = \{g_1, g_2, \ldots g_n\}$ be a finite group of $g_1 = 1$, the **multiplication table** or **group table** is the $n \times n$ matrix whose i, j entry is the element $g_i g_j$.

1.2 Dihedral Groups

For each $n \in \mathbb{Z}^+$, $n \geq 3$, let D_{2n} be the set of symmetries of a regular n-gon. $|D_{2n}| = 2n$.

Fix a regular n-gon centered at the origin in an x, y plane and label the vertices consecutively from 1 to n in a clockwise manner. Let r be the rotation clockwise about the origin through $\frac{2\pi}{n}$ radian. Let s be the reflection about the line of symmetry through vertex 1 and the origin.

1. $1, r, \ldots, r^{n-1}$ are all distinct and $r^n = 1$, so $|r| = n$.
2. $|s| = 2$.
3. $s \neq r^i$ for any i.
4. $sr^i \neq sr^j$, so $D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$. i.e., each element can be written uniquely in the form $s^k r^i$ for some $k = 0$ or 1 and $0 \leq i \leq n-1$.
5. $rs = sr^{-1}$.
6. $r^i s = sr^{-i}$, for all $0 \leq i \leq n$.

A subset S of elements of a group G with the property that every element of G can be written as a (finite) product of elements of S and their inverses is called a set of **generators** of G. (if G is finite, then it is not necessary to include the inverses of elements of S as well.)

Any equations in a general group G that the generators satisfy are called **relations**.

Representation: generators and their relations.

1.3 Symmetric Groups

Permutation: a bijection from a nonempty set Ω to itself.

S_Ω: the set of all permutations of Ω. S_Ω is a group under function composition (called the **symmetric group** on the set Ω).

It is important to notice that the elements of S_Ω are permutations of Ω, not the elements of Ω itself.

S_n: the symmetric group on $\{1, 2, \ldots, n\}$, called the symmetric group of degree n. $|S_\Omega| = n!$.

A **cycle** is a string of integers which represents the element of S_n which cyclically permutes these integers (and fixs all other integers).

A cycle of length t is called a t-cycle.

Disjoint cycles commute.

1.4 Matrix Groups

1.5 The Quaternion Group

The **quaternion group** Q_8 is defined by

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

where

$$\begin{gathered} 1 \cdot a = a \cdot 1 = a \\ (-1) \cdot (-1) = 1 \\ i \cdot i = j \cdot j = k \cdot k = -1 \\ i \cdot j = k \\ j \cdot k = i \\ k \cdot i = j \end{gathered}$$

No that Q_8 is a non-abelian group of order 8.

1.6 Homomorphisms and Isomorphisms

Definition. Let $(G, \star)$ and $(H, \diamond)$ be groups. A map $\varphi : G \to H$ such that

$$\varphi(x \star y) = \varphi(x) \diamond \varphi(y) \ \forall x, y \in G$$

is called a **homomorphism**. (Note that φ does not need to be surjective.)

Definition. The map $\varphi : G \to H$ is called a **isomorphism**, if

1. φ is a homomorphism
2. φ is a bijection.

Then the groups G and H are called isomorphic, written $G \cong H$.

If $\varphi : G \to H$ is an isomorphism, then

1. $|G| = |H|$,
2. G is abelian if and only if H is abelian,
3. $|x| = |\varphi(x)|, \forall x \in G$.

1.7 Group Actions

Definition. A group action of G on a set A is a **map (function)** from $G \times A$ to A (written as $g \cdot a$, for $g \in G$ and $a \in A$) satisfying:

1. $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$, for all $g_1, g_2 \in G, a \in A$
2. $1 \cdot a = a$, for all $a \in A$.

Two important facts:

1. for each fixed $g \in G$, $\sigma_g : A \to A$ is a permutation of A,
2. the map from G to S_A defined by $g \mapsto \sigma_g$ is a homomorphism. (is called the **permutation representation**)

> Proof:
>
> (1) σ_g has inverse $\sigma_{g^{-1}}$, so it is a bijection.
>
> (2) it is easy to verify the axioms of homomorphism.

Chapter 2 Subgroups

2.1 Definition and Examples

Definition. Let G be a group. A subset H of G is a **subgroup** of G if H is a nonempty set and H is closed under products and inverses (i.e. $xy \in H$ and $x^{-1} \in H$ for $\forall x, y \in H$.). If H is a subgroup of G, we shall write $H \leq G$.

Langrange's Theorem: if $H \leq G$, then $|H| \mid |G|$.

Proposition 1. A subset H is a subgroup of G if and only if

1. $H \neq \varnothing$. **Don't forget this!!!**
2. $xy^{-1} \in H$ for all $x, y \in H$.

> Proof:
>
> If $xy^{-1} \in H$, then $y^{-1} = 1y^{-1} \in H$, $xy = x(y^{-1})^{-1} \in H$.
>
> The other direction is straightforward.

2.2 Centralizers and Normalizers, Stabilizers and Kernels

Let A be a nonempty subset of G.

Definition. $C_G(A) = \{g \in G \mid gag^{-1} = a, \forall a \in A\}$ is called the **centralizer** of A in G. ($C_G(A)$ is the set of elements of G which commute with every element in A.)

We can prove that $C_G(A) \leq G$.

Definition. Define $Z(G) = \{g \in G \mid gx = xg, \forall g \in G\}$ as the **center** of G.

$Z(G) \leq G$.

Definition. Define $gAg^{-1} = \{gag^{-1} \mid a \in A\}$. Define the **normalizer** of A in G to be the set $N_G(A) = \{g \in G, gAg^{-1} = A\}$.

We can prove that $C_G(A) \leq N_G(A) \leq G$.

Definition. if G is a group action on a set S and $s \in S$, the **stabilizer** of s in G is the set $G_s = \{g \in G \mid g \cdot s = s\}$.
We can prove that $G_s \leq G$.

Definition. The kernel of an action of G on S is defined as $\{g \in G \mid g \cdot s = s, \forall s \in S\}$.

We can prove that the kernel is also a subgroup of G.

2.3 Cyclic Groups and Cyclic Subgroups

Definition. A group H is cyclic if $H = \{x^n \mid n \in \mathbb{Z}\}$, written as $H = \langle x \rangle$ and say H is generated by x.

It is easy to find out that cyclic groups are abelian.

Proposition 2. If $H = \langle x \rangle$, then $|H| = |x|$. More specifically,

1. if $|H| = n < \infty$, then $x^n = 1$ and $1, x, x^2, \ldots, x^{n-1}$ are all the distinct elements in H, and
2. if $|H| = \infty$, then $x^n \neq 1$ for all $n \neq 0$ and $x^a \neq x^b$ for all $a \neq b$ in $\mathbb{Z}$.

> Proof:
>
> (1) If $x^a = x^b$, then $x^{b-a} = 0$, so $\forall 0 \leq i < j \leq n-1$, $x^i \neq x^j$.
>
> (2) Similar.

Proposition 3. Let G be an arbitary group, $x \in G$ and $m, n \in \mathbb{Z}$. If $x^n = 1$ and $x^m = 1$, then $x^d = 1$ where $d = (n, m)$. In particular, if $x^m = 1$, then $|x| \mid m$.

> Proof:
>
> Let $d = an + bm$, then $x^d = (x^n)^a (x^m)^b = 1$.

Theorem 4. Any two groups of the same order are isomorphic. More specifically,

1. if $n \in \mathbb{Z}^+$ and $\langle x \rangle$, $\langle y \rangle$ are both cyclic groups of order n, then the map
$$\varphi : \langle x \rangle \to \langle y \rangle$$
$$x^k \mapsto y^k$$
is well defined and is an isomorphism.
2. if $\langle x \rangle$ is an infinite cyclic group, the map
$$\varphi : \langle \mathbb{Z} \rangle \to \langle x \rangle$$
$$k \mapsto x^k$$
is well defined and is an isomorphism.

> Proof: straighforward.

Notation: for each $n \in \mathbb{Z}^+$, let Z_n be the cyclic group of order n. Up to isomorphism, Z_n is the unique cyclic group of order n and $Z_n \cong \mathbb{Z}/n\mathbb{Z}$.

Proposition 5. Let G be a group, $x \in G$ and $a \in \mathbb{Z} - \{0\}$.

1. If $|x| = \infty$, then $|x^a| = \infty$.
2. If $|x| = n < \infty$, then $|x^a| = \frac{n}{(a,n)}$.
3. In particular, if $|x| = n < \infty$, and a is a positive integer dividing n, then $|x^a| = \frac{n}{a}$.

> Proof:
>
> (1) straightforward
>
> (2) Let $y = a^x, d = (a, n), n = db, a = dc$. Then
>
> $y^b = x^{ab} = x^{dcb} = (x^{db})^c = 1^c = 1$.
>
> If $|y| = k$, by proposition 3, $n \mid ak$, i.e., $db \mid dck$. Thus $b \mid k$.
>
> Therefore, $b = k$.

Proposition 6. Let $H = \langle x \rangle$.

1. Assume $|x| = \infty$, then $H = \langle x^a \rangle$ iff $a = \pm 1$.
2. Assume $|x| = n < \infty$, then $H = \langle x^a \rangle$ iff $(a, n) = 1$. In particular, the number of generators of H is $\varphi(n)$.

> Proof:
>
> 1. Otherwise, $x \notin \langle x^a \rangle$.
> 2. $\langle x^a \rangle = H = \langle x \rangle$, so $|x^a| = |x|$, if and only if $\frac{n}{(a,n)} = n$ (by proposition 5), i.e., $(a, n) = 1$.

Theorem 7. Let $H = \langle x \rangle$ be a cyclic group.

1. Every subgroup of H is cyclic.
2. If $|H| = \infty$, then for any distinct nonnegative integer a and b, $\langle x^a \rangle \neq \langle x^b \rangle$. Furthermore, for every integer m, $\langle x^m \rangle = \langle x^{|m|} \rangle$.
3. If $|H| = n < \infty$, then for each positive integer a dividing n, there is a unique subgroup of H of order a. This subgroup is the cyclic group $\langle x^d \rangle$ where $d = \frac{n}{a}$. Furthermore, for every integer m, $\langle x^m \rangle = \langle x^{(n,m)} \rangle$, so that the subgroups of H correspond bijectively with the positive divisors of n.

> Proof:

> 1. Consider the element x^k with smallest k. If $x^l \in \langle x \rangle$ with $k \nmid l$, then $x^{l \bmod k} \in \langle x \rangle$, a contradiction.
> 2. obviouslly $\langle x^m \rangle \subseteq \left\langle x^{(n,m)} \right\rangle$. Also, let $d = (n, m) = an + bm$, then $x^d = (x^m)^b \in \langle x^m \rangle$, so $\left\langle x^{(n,m)} \right\rangle \subseteq \langle x^m \rangle$. The other parts are trivial.

2.4 Subgroups Generated by Subsets of a Group

Proposition 8. If $\mathcal{A}$ is any nonempty set of subgroups of G, then the intersection of all members of $\mathcal{A}$ is also a subgroup of G.

> Proof: verify the axioms or proposition 1.

Definition. If A is any subset of the group G, define

$$\langle A \rangle = \bigcap_{A \subseteq H H \leq G} H.$$

This is called the subgroup of G generated by A.

Proposition 9. $\langle A \rangle = \{a_1^{\epsilon_1} a_2^{\epsilon_2} \dots a_n^{\epsilon_n}\}$.

> Proof: omitted.

2.5 The Lattice of Subgroups of a Group

... 看图说话.jpg

Chapter 3 Quotient Groups and Homomorphisms

3.1 Definitions and examples

Definition. If φ is a homomofphism $\varphi : G \to H$, the **kernel** of φ is the set

$$\{g \in G \mid \varphi(g) = 1\}$$

and will be denoted by $\ker \varphi$ (here 1 is the identity of H).

Proposition 1. Let G and H be groups and let $\varphi : G \to H$ be a homomorphism.

1. $\varphi(1_G) = 1_H$.
2. $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$.
3. $\varphi(g^n) = \varphi(g)^n$ for all $n \in \mathbb{Z}$.
4. ker φ is a subgroup of G.
5. $im(\varphi)$, the image of G under φ, is a subgroup of H.

> Proof: straightforward.

Definition. Let $\varphi : G \to H$ be a homomorphism with kernel K. The **quotient group** or **factor group**, G/K, is the group whose elements are fibers of φ with group operation defined above: namely if X the fiber above a and Y is the fiber above b, then the product of X with Y is defined to be the fiber above the product ab.

quotient group 应该是 well defined 的，~~因为 $im(\varphi)$ is subgroup of H, 所以对于任意的 a, b 都能找到 ab 的 fiber.~~ 注意这里的 product 和 G 中集合的 product 的定义不同。 quotient group 中的 product 在下面的 theorem 3 中才有正式的定义。

注意一下 quotient group 中的元素和 G 中的一个 fiber，它们俩一个是元素另一个是集合，实际上却是相同的（本质上都是集合）。

所以说 G/N 是集合组成的集合。

Proposition 2. Let $\varphi : G \to H$ be a homomorphism of groups with kernel K. Let $X \in G/K$ be the fiber above a, i.e., $X = \varphi^{-1}(a)$. Then

1. For any $u \in X$, $X = \{uk \mid k \in K\}$.
2. For any $u \in X$, $X = \{ku \mid k \in K\}$.

(That is, the left coset of any element is equal to the right coset)

> Proof:
>
> (1) $\forall uk, \varphi(uk) = \varphi(u) = a$, so $\{uk\} \subseteq X$.
>
> Conversely, let $g \in X$ and $k = u^{-1}g$. Then $\varphi(k) = \varphi(u)^{-1}\varphi(g) = 1$, so $k \in K$. Hence $X \subseteq \{uk\}$.
>
> (2) Similar.

Definition. For any $N \leq G$ and any $g \in G$, let $gN = \{gn \mid n \in N\}$ and $Ng = \{ng \mid n \in N\}$ called respectively a **left coset** and a **right coset** of N in G. Any element of a coset is called a **representative** of the coset.

quotient group 中的元素和 left (right) coset 一一对应，所以 quotient group 中的元素的乘法也可以对应到 cosets 的乘法（见下面这个 theorem）

(而且 $gN \cong N$. $g = g1 \in gN$.

Theorem 3. Let G be a group and let K be the kernel of some homomorphism from G to another group. Then the set whose elements are the left cosets of K in G with operation defined by

$$uK \circ vK = (uv)K$$

forms a group, G/K. In particular, this operation is well defined in the sense that if u_1 is any element in uK and v_1 is any element in vK, then $u_1v_1 \in uvK$, i.e., $u_1v_1K = uvK$ so that the multiplication does not depend on the choice of representatives for the cosets. The same statement is true with "right cosets" inplace of "left cosets".

Notice: if K is not the kernel, the multiplication above is not well defined.

> Proof:
>
> $\forall x \in uK, y \in vK, \varphi(xy) = \varphi(x)\varphi(y)$, so $xy \in uvK$.
>
> Conversely, let x be any element in uK, z be any element in uvK then $\varphi(x^{-1}z) = \varphi(v)$, so $\exists y \in vK$ s.t. $xy = z$.

Proposition 4. Let N be any subgroup of the group G. The set of left cosets of N in G form a partition of G, Furthermore, for all $u, v \in G$, $uN = vN$ if and only if $v^{-1}u \in N$ and in particular, $uN = vN$ if and only if u and v are the representatives of the same coset.

> Proof:
>
> If $x = un = vm$, then $u = xn^{-1} = vmn^{-1}$.
>
> $\forall ut \in uN, ut = vmn^{-1}t = v(mn^{-1}t) \in vN$.

(两个 cosets 要么不交要么相等，所以证两个 cosets 相等就只需要证明它们的交集非空了。

(note: if $n \in N$, then $(gn)N = gN$. since N is subgroup, so n has an inverse in N.

($uN = vN \Leftrightarrow uN \cap vN \neq \varnothing \Leftrightarrow \{u1\} \cap vN \neq \varnothing \Leftrightarrow u \in vN \Leftrightarrow u = vn \Leftrightarrow v^{-1}u = n \Leftrightarrow v^{-1}u \in N$. ~~类似的，$u1 = vn \Leftrightarrow uN = vN$ for some n.~~

Proposition 5. Let G by a group and let N be a subgroup of G.

1. The operation on the set of left cosets of N in G described by

   $$uN \cdot vN = (uv)N$$

   is well defined if and only if $gng^{-1} \in N$ for all $g \in G$ and all $n \in N$.
2. If the above operation is well defined, then it makes the set of left cosets of N in G into a group. In particular, the identity of this group is the coset $1N$ and the inverse of gN is the coset $g^{-1}N$, i.e., $(gN)^{-1} = g^{-1}N$.

> Proof:
>
> 1. Let $u = 1, u_1 = n \in uN, v = v_1 = g^{-1} \in vN$. Since $1g^{-1}N = ng^{-1}N$, $g^{-1}n_1 = ng^{-1}$, which implies $gng^{-1} = n_1 \in N$.
>
>   Conversely, let $u_1 = un, v_1 = vm$, then $u_1v_1 = unvm = uvv^{-1}nvm = uv(v^{-1}nv)m = (uv)n_1m \in uvN$.
> 2. It is straightforward to verify the axioms.

Definition. The element gng^{-1} is called the **conjugate** of $n \in N$ by g. The set $gNg^{-1} = \{gng^{-1} \mid n \in N\}$ is called the **conjugate** of N by g. The element g is said to **normalize** N if $gNg^{-1} = N$. A subgroup N of a group G is called normal if every element in G normalize N, i.e., if $gNg^{-1} = N$ for all $g \in G$. If N is a normal subgroup of G, we shall write $N \trianglelefteq G$.

Theorem 6. Let N be a subgroup of the group G. The following are equivalent:

1. $N \trianglelefteq G$.
2. $N_G(N) = G$.
3. $gN = Ng$ for all $g \in G$.
4. the operation on left cosets of N in G described in Proposition 5 makes the set of left cosets into a group.
5. $gNg^{-1} \subseteq N$ for all $g \in G$.

> Proof: omitted.

（注意到当 N 是 finite 的时候，fixed g, gng^{-1} 是一个双射 $gng^{-1} \in N \Leftrightarrow gNg^{-1} = N$.；当 N 是 infinite group 的时候呢? 不成立：https://math.stackexchange.com/a/994779

（这时候证明 $5 \to 1$ 就要用别的方法了。fixed g, then $gNg^{-1} \subseteq N$. 考虑 g^{-1}, then $g^{-1}Ng \subseteq N$. 两边同时左乘 g 且右乘 g^{-1}，得 $N \subseteq gNg^{-1}$. 因此 $gNg^{-1} = N$.

> Also, if one has a set of generators for N, it suffices to check that all conjugates of these generators lie in N to prove that N is a normal subgroup. (this is because the conjugate of a product is the
> product of the conjugates and the conjugate of the inverse is the inverse of the conjugate)
>
> Similarly, if generators for G are also known, then it suffices to check that these generators for G normalize N.
>
> In particular, if generators for both N and G are known, this reduces the calculations to a small number of conjugations to check.
>
> Finally, it is often possible to prove directly that $N_G(N) = G$ without excessive computations.

Proposition 7. A subgroup N of the subgroup G is normal if and only if it is the kernel of some homomorphism.

> Proof:
>
> If N is the kernel of the homomorphism φ, then proposition 2 shows that the left cosets are the same as the right cosets. By proposition 6, N is normal.
>
> Conversely, let $H = G/N$ and define $\pi : G \to G/N$ by
>
> $$\pi(g) = gN.$$
>
> By the definition, $\pi(g_1g_2) = g_1g_2N = g_1Ng_2N = \pi(g_1)\pi(g_2)$, so it is a homomorphism.
>
> Now
>
> $$\begin{aligned} \ker \pi &= \{g \in G \mid \pi(g) = 1N\} \\ &= \{g \in G \mid gN = 1N\} \\ &= \{g \in G \mid g \in N\} \\ &= N. \end{aligned}$$

Definition. Let $N \trianglelefteq G$. The homomorphism $\pi : G \to G/N$ defined by $\pi(g) = gN$ is called the **natural projection (homomorphism)** of G into G/N. If $\overline{H} \leq G/N$ is a subgroup of G/N, the **complete image** of $\overline{H}$ in G is the preimage of $\overline{H}$ under the natural projection homomorphism.

quotient group $\Leftrightarrow$ homomorphism

quotient groups of a cyclic group are cyclic

3.2 More on Cosets and Lagrange's Theorem

Theorem 8. (Lagrange's Theorem) If G is a finite group and H is a subgroup of G, then $|H| \mid |G|$, and the number of left cosets of H in G equals $\frac{|G|}{|H|}$.

> Proof: the left cosets partition G and each coset has the same size $|H|$, so $|G| = k\,|H|$ where k is the number of cosets.

Definition. If G is a group (possibly infinite) and $H \leq G$, the number of left cosets of H in G is called the index of H in G and is denoted by $|G : H|$.

(Lagrange's Theorem: $|G : 1| = |G : H|\,|H : 1|$)

Corollary 9. If G is a finite group and $x \in G$, then the order of x divides the order of G. In particular $x^{|G|} = 1$ for all $x \in G$.

> Proof: $|x| = |\langle x \rangle|$. The rest parts are straightforward.

Corollary 10. If G is a group of prime order p, then G is cyclic, hence $G \cong Z_p$.

> Proof: Let $x \in G$ with $x \neq 1$, then $|x| = |G|$, so $G = \langle x \rangle$.

Theorem 11. (Cauchy's Theorem) If G is a finite group and p is a prime dividing $|G|$, then G has an element of order p.

> Proof: omitted.

Theorem 12. (Sylow) If G is a finite group of order $p^\alpha m$ where p is a prime and $p \nmid m$, then G has a subgroup of order p^α.

> Proof: omitted.

Definition. Let H and K be subgroups of a group and define HK be the set

$$HK = \{hk \mid h \in H, k \in K\}.$$

Proposition 13. If H and K are finite subgroups of a group then

$$|HK| = \frac{|H|\,|K|}{|H \cap K|}.$$

> Proof:
>
> Write $HK = \cap_{h \in H} hK$.
>
> now we want to find out all duplicated cosets: $h_1K = h_2K \Leftrightarrow h_1^{-1}h_2 \in K$ and also $h_1^{-1}h_2 \in H$, so $h_1^{-1}h_2 \in H \cap K$.
>
> Hence, for each cosets, there are $|H \cap K|$ duplicated ones, so the number of cosets is $\frac{|H|}{|H \cap K|}$.
>
> Since cosets are disjoints, $|HK| = \frac{|H||K|}{|H \cap K|}$.

Proposition 14. If H and K are subgroups of a group, HK is a subgroup if and only if $HK = KH$.

> Proof:
>
> If $HK = KH$, let $a = h_1k_1, b = h_2k_2$. Then $ab^{-1} = h_1(k_1k_2^{-1})h_2^{-1} = h_1h_3k_3 = (h_1h_3)k_3 \in HK$, so $HK \leq G$.
>
> Conversely, since $K \leq HK, H \leq HK, KH \subseteq HK$. $\forall hk = (h_1k_1)^{-1} \in HK$, $hk = k_1^{-1}h_1^{-1} \in KH$.

Corollary 15. If H and K are subgroups of G and $H \leq N_G(K)$, then HK is a subgroup of G. In particular, if $K \trianglelefteq G$ then $HK \leq G$ for any $H \leq G$. (and also $KH \leq G$.)

> Proof:

> $\forall hk \in HK, hk = (hkh^{-1})h \in KH$. Similarly $kh = h(h^{-1}kh) \in HK$, so $HK = KH$.

Definition. If A is any subset of $N_G(K)$ (or $C_G(K)$), we shall say A normalizes K (centralizes K, respectively.)

With this terminology, Corollary 15 states that HK is a subgroup if H normalizes K (similarly, HK is a subgroup if K normalizes H).

3.3 The Isomorphism Theorems

Theorem 16. (The first Isomorphism Theorem / Fundamental Theorem of Homomorphisms) If $\varphi : G \to H$ is a homomorphism of groups, then $\ker \varphi \trianglelefteq G$ and $G/\ker \varphi \cong \varphi(G)$.

> 证明同构前请弄清楚两个集合分别是啥，包含哪些元素
>
> Proof:
>
> Let $N = \ker \varphi$. By prop. 2, $\forall g \in G, gN = Ng$. By theorem. 6, $N \trianglelefteq G$.
>
> $G/N = \{gN \mid g \in G\}, \varphi(G) : \{\varphi(g) \mid g \in G\}$
>
> Let $\psi : G/N \to \varphi(G)$ be $\psi : gN \mapsto \varphi(g)$, we will show that ψ is an isomorphism.
>
> 1. Homomorphism: $\forall g_1N, g_2N \in G/N, \psi((g_1N)(g_2N)) = \psi((g_1g_2)N) = g_1g_2 = \psi(g_1N)\psi(g_2N)$.
> 2. Injective: If $\psi(g_1N) = \psi(g_2N) : \varphi(g_1) = \varphi(g_2)$, then
> $\varphi(g_1g_2^{-1}) = \varphi(g_1)\varphi(g_2)^{-1} = 1 \Rightarrow g_1g_2^{-1} \in N \Rightarrow g_1N = g_2N$.
> 3. Surjective: If $\varphi(g) \in \varphi(G)$, then $\psi(gN) = \varphi(g)$.
>
> Therefore $G/N \cong \varphi(G)$.

Corollary 17. Let $\varphi : G \to H$ be a homomorphism of groups.

1. φ is injective if and only if $\ker \varphi = 1$.
2. $|G : \ker \varphi| = |\varphi(G)|$.

> Proof:
>
> 1. If φ is injective, then $\forall g \neq 1, \varphi(g) \neq \varphi(1) = 1$, so $\ker \varphi = 1$.
>
>   Conversely, $\forall g_1 \neq g_2, \varphi(g_1)\varphi(g_2)^{-1} = \varphi(g_1g_2^{-1}) \neq 1$.
> 2. $|G : \ker \varphi| = |G/\ker \varphi| = |\varphi(G)|$.

Theorem 18. (The Second or Diamond Isomorphism Theorem) Let G be a group, let A and B be subgroups of G and assume $A \leq N_G(B)$. Then AB is a subgroup of G, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and $AB/B \cong A/A \cap B$.

> 大概就是构造了一个 surjective 的 homomorphism $\varphi : A \to AB/B$，再用 the first homomorphism theorem 去证明 $A/\ker \varphi = AB/B$ 并求出 $\ker \varphi = A \cap B$.
>
> Proof:
>
> By Corollary 15, $AB \leq G$.
>
> Since $A \leq N_G(B)$ (by assumption) and $B \leq N_G(B)$ (trivially),
> $\forall ab \in AB, n \in B, (ab)n(ab)^{-1} = a(bnb^{-1})a^{-1} = an_1a^{-1} = n_2 \in B$, so $AB \leq N_G(B)$ and $B \trianglelefteq AB$.
>
> Define $\varphi : A \to AB/B$ by $\varphi(a) = aB$.
>
> It is easy show that φ is an homomorphism: $\varphi(a_1a_2) = (a_1a_2)B = (a_1B)(a_2B) = \varphi(a_1)\varphi(a_2)$.
>
> Next, it is clear that φ is surjective, i.e., $\varphi(A) = AB/B$.
>
> $\ker \varphi : \{a : a \in A, \varphi(a) = 1B\} = \{a : a \in A, aB = 1B\} = \{a : a \in A, a \in B\} = A \cap B$.
>
> By the First isomorphism theorem, $A \cap B \trianglelefteq A$ and $A/A \cap B \cong AB/B$, completing the proof.

fig2($|AB : A| = |B : A \cap B|$:

> 还是构造一个映射，然后证明是双射
>
> Proof:
>
> First, $AB = BA$.

> Let $S_1 = \{bA : b \in B\}, S_2 = \{b(A \cap B) : b \in B\}$.
>
> Let $\varphi : S_2 \to S_1$ by $\varphi(b(A \cap B)) = bA$.
>
> 1. Injective: $\forall b_1(A \cap B) \neq b_2(A \cap B), b_1^{-1}b_2 \notin A \cap B$. But $b_1^{-1}b_2 \in B$, so $b_1^{-1}b_2 \notin A$. Hence $b_1A \neq b_2A$.
> 2. Surjective: $\forall bA \in S_1, \varphi(b(A \cap B)) = bA$.
>
> Therefore φ is bijective, and hence $|AB : A| = |B : A \cap B|$.

Theorem 19. (The Third Isomorphism Theorem) Let G be a group and let H and K be normal subgroups of G with $H \leq K$. Then $K/H \trianglelefteq G/H$ and

$$(G/H)/(K/H) \cong G/K.$$

> 思路与上一个定理的证明类似，还是构造 surjective 的 homomorphism
>
> Proof:
>
> First we prove that $K/H \trianglelefteq G/H$.
>
> $K/H = \{kH \mid k \in K\}, G/H = \{gH \mid g \in G\}$.
>
> $\forall gH \in G/H, kH \in K/H, (gH)kH(gH)^{-1} = (gkg^{-1})H = k_1H \in K/H$, so $K/H \trianglelefteq G/H$.
>
> Next, we define $\varphi : G/H \to G/K$ by $gH \mapsto gK$.
>
> 1. We must prove that φ is well defined: Suppose $g_1H = g_2H$, then $g_1^{-1}g_2 \in H \subseteq K$, so $g_1K = g_2K$.
> 2. Homomorphism: $\varphi((g_1H)(g_2H)) = \varphi((g_1g_2H)) = g_1g_2K = (g_1K)(g_2K) = \varphi(g_1H)\varphi(g_2H)$.
> 3. Surjective: Obvious.
>
> Finally,
>
> $$\begin{aligned}\ker\varphi &= \{\varphi(gH) = 1 \mid gH \in G/H\} \\ &= \{gK = 1 \mid gH \in G/H\} \\ &= \{gK = K \mid gH \in G/H\} \\ &= \{g \in K \mid gH \in G/H\} \\ &= K/H\end{aligned}$$
>
> Therefore, by the First Isomorphism Theorem, $(G/H)/(K/H) \cong G/K$

Theorem 20. (The Fourth or Lattice Isomorphism Theorem) Let G be a group and let N be a normal subgroup of G. Then there is a bijection from the set of subgroups A of G which contain N onto the set of subgroups A/N of G/N. In particular, every subgroup of G/N is of the form A/N for some subgroup A of G containing N. This bijection has the following properties: for all $A, B \leq G$ with $N \leq A$ and $N \leq B$,

1. $A \leq B$ if and only $A/N \leq B/N$.
2. if $A \leq B$, then $|B : A| = |B/N : A/N|$.
3. $\langle A, B\rangle/N = \langle A/N, B/N\rangle$.
4. $(A \cap B)/N = (A/N) \cap (B/N)$.
5. $A \trianglelefteq G$ if and only if $A/N \trianglelefteq G/N$.

> 没有说怎么证存在这样的 bijection。。。
>
> 这几条都挺 trivial 的。。。
>
> Proof:
>
> > Lemma. Let $A \leq G, N \trianglelefteq A$, then $\forall g \in G, gN \in A/N$ if and only if $g \in A$.
> >
> > Proof:
> >
> > If $gN \in A/N$, let $gN = aN$ where $a \in A$. Then $a^{-1}g \in N \subseteq A$. Hence $g \in A$.
> >
> > Conversely, it is obviously true.
>
> 1. If $A \leq B$, then $\forall aN \in A/N (a \in A)$. Since $a \in B$, aN is also in B/N
>
>   Conversely, $\forall a \in A$, $aN \in A/N$. Let $aN = bN \in B/N$, then $b^{-1}a \in N \Rightarrow b^{-1}a \in B \Rightarrow a \in B$.
>
>   Therefore $A \leq B$ if and only $A/N \leq B/N$.
> 2. If $A \leq B$, let S_1 be all cosets of A/N in B/N, and S_2 be all cosets of A in B.

Let $\varphi : S_1 \to S_2$ be $\varphi(bN(A/N)) = bA$. We will show that φ is a bijection.

- Injective: $\forall b_1N(A/N), b_2N(A/N)$, if they are not the same element, then $(b_1N)^{-1}b_2N = b_1^{-1}b_2N \notin A/N$. Hence $b_1^{-1}b_2 \notin A$, so $b_1A \neq b_2A$.
- Surjective: $\forall bA \in S_2$, $\exists bN(A/N) \in S_1$ s.t. $\varphi(bN(A/N)) = bA$.

Therefore φ is a bijection between two cosets, so $|B : A| = |B/N : A/N|$.

3. First we prove that $\langle A, B\rangle/N \subseteq \langle A/N, B/N\rangle$.

   $\forall g = \left(\prod_i a_i^{c_i} b_i^{d_i}\right) \in \langle A, B\rangle/N$, $g = \prod_i (a_i^{c_i}N)\left(b_i^{d_i}N\right) \in \langle A/N, B/N\rangle$ by proposition 5.

   Second, we prove that $\langle A/N, B/N\rangle \subseteq \langle A, B\rangle/N$.

   $\forall g = \prod_i (a_i^{c_i}N)\left(b_i^{d_i}N\right) \in \langle A/N, B/N\rangle$, $g = \left(\prod_i a_i^{c_i} b_i^{d_i}\right) \in \langle A, B\rangle/N$ by proposition 5.

   Therefore, $\langle A, B\rangle/N = \langle A/N, B/N\rangle$.

4. First we prove that $(A \cap B)/N \subseteq (A/N) \cap (B/N)$.

   $\forall aN \in (A \cap B)/N$, we have $a \in A \cap B$. Then $a \in A$ and $a \in B$, so $aN \in A/N$ and $aN \in B/N$.

   Conversely, if $aN \in A/N$ and $aN \in B/N$, then $a \in A \wedge a \in B \Rightarrow a \in A \cap B$, so $aN \in (A \cap B)/N$.

   Therefore $(A \cap B)/N = (A/N) \cap (B/N)$.

5. If $A \trianglelefteq G$, then $\forall gN \in G/N, aN \in A/N, (gN)(aN)(gN)^{-1} = (gag^{-1})N = a_1N \in A/N$ for some $a_1 \in A$.

   Conversely, $\forall g \in G, a \in A$, we have $gN \in G/N, aN \in A/N$. Then $(gN)(aN)(gN)^{-1} = (gag^{-1})N \in A/N \Rightarrow gag^{-1} \in A$.

   Therefore $A \trianglelefteq G$ if and only if $A/N \trianglelefteq G/N$.

3.4 Composition Series and the Holder Program

Proposition 21. If G is a finite abelian group and p is a prime dividing $|G|$, then G contains an element of order p.

Proof:

Perform mathematical induction on $|G|$.

Suppose the statement is true for order smaller than G.

Choose an element $x \neq 1$.

(If $|G| = p$, then $|x| > 1$. By the Lagrange's theorem, $|x| = \frac{|G|}{|G:\langle x\rangle|} = p$. 这一步可能不需要?

If $p \mid |x|$, i.e., $|x| = np$, then $|x^n| = p$.

Otherwise, let $N = \langle x\rangle$. Since G is abelian, $N \trianglelefteq G$. By Lagrange's Theorem, $|G/N| = \frac{|G|}{|N|}$. Since $|N| \neq 1$, $|G/N| < |G|$. Since $p \nmid |N|$, $p \mid |G/N|$.

We can now apply the induction hypothesis to G/N to conclude that it has an element $\overline{y} = yN$ of order p. Since $y \notin N (\overline{y} \neq \overline{1})$, but $y^p \in N (\overline{y}^p = \overline{1})$, $\langle y^p\rangle \neq \langle y\rangle$, so $|y^p| < |y|$. By proposition 2.5(2), $|y^p| = \frac{|y|}{(|y|,p)}$, so $p \mid |y|$. Choosing $y^{\frac{|y|}{p}}$, we can conclude that is has order p.

The induction is complete.

为什么要在这里证明一个 Cauchy's Theorem 的弱化版呢？因为需要让我们观察到 G 的性质和 正规子群的N 以及 G/N 的性质的关系。

the full isomorphism type of G cannot be determined from the isomorphism types of N and G/N alone.

这里指的应该是多个不同的 G 除以同一个正规子群 N 仍能得到相同的 G / N：

$V_4/\langle a\rangle = Z_2$;

$Z_4/\langle x^2\rangle = Z_2$;

$\langle a\rangle \cong \langle x^2\rangle$;

but $V_4 \ncong Z_4$.

Definition. A (finite or infinite) group G is called **simple** if $|G| > 1$ and the only normal subgroups of G are 1 and G.

Definition. In a group G, a sequence of subgroups

$$1 = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_{k-1} \leq N_k = G$$

is called a **composition series** if $N_i \trianglelefteq N_{i+1}$ and N_{i+1}/N_i is a simple group, $0 \leq i \leq k-1$. If the above sequence is a composition series, the quotient groups N_{i+1}/N_i are called **composition factors** of G.

(Keep in mind that it is not assumed that each $N_i \trianglelefteq G$.

Theorem 22. (Jordan-Holder) Let G be a finite group with $G \neq 1$. Then

1. G has a composition series and
2. The composition factors in a composition series are unique, namely, if $1 = N_0 \leq N_1 \leq \cdots \leq N_r = G$ and $1 = M_0 \leq M_1 \leq \cdots \leq M_s = G$ are two composition series of G, then $r = s$ and there is some permutation π of $\{1, 2, \ldots, r\}$, such that

$$M_{\pi(i)}/M_{\pi(i)-1} \cong N_i/N_{i-1}$$

3.

Theorem. There is a list consisting of 18 (infinite) families of simple groups and 26 simple groups not belonging to these families (the *sporadic* simple groups) such that every finite simple group is isomorphic to one of the groups in this list.

Theorem. (Feit-Thompson) If G is a simple group of odd order, then $G \cong Z_p$ for some prime p.

> Proof. omitted.

Definition. A group G is **solvable** if there is a chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_s = G$$

such that G_{i+1}/G_i is abelian for $i = 0, \ldots, s-1$.

Theorem. The finite group G is solvable if and only if for every divisor n of $|G|$ such that $\left(n, \frac{|G|}{n}\right) = 1$, G has a subgroup of order n.

Theorem. If N and G/N are solvable, then G is solvable.

> Proof.
>
> let $\overline{G} = G/N$.
>
> By the assumption, let $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_n = N$ and $\overline{1} = \overline{G_0} \trianglelefteq \overline{G_1} \trianglelefteq \cdots \trianglelefteq \overline{G_m} = \overline{G}$.
>
> By the Lattice Isomorphism Theorem, there are subgroups G_i of G with $N \leq G_i$ such that $G_i/N = \overline{G_i}$ and $G_i \trianglelefteq G_{i+1}$.
>
> By the Third Isomorphism Theorem, $\overline{G_{i+1}}/\overline{G_i} = (G_{i+1}/N)/(G_i/N) \cong G_{i+1}/G_i$.
>
> Thus $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_n = N = G_0 \trianglelefteq \cdots \trianglelefteq G_m = G$ is a chain...

3.5 Transpositions and the Alternating Group

Transpositions and Generation of S_n

Definition. A 2-cycle is called a **transposition**.

$$(a_1\ a_2\ \dots\ a_m) = (a_1\ a_m)(a_1\ a_{m-1})\cdots(a_1\ a_2)$$

Every element of S_n may be written as a product of transpositions, or equivalently,

$$S_n = \langle T \rangle \ where\ T = \{(i\ j) \mid 1 \le i < j \le n\}.$$

The Alternating Group

Definition.

1. $\epsilon(\sigma)$ is called the **sign** of σ.
2. σ is called an even permutation if $\epsilon(\sigma) = 1$ and an **odd permutation** if $\epsilon(\sigma) = -1$.

Proposition 23. The map $\epsilon : S_n \to \{\pm 1\}$ is a homomorphism (where $\{\pm\}$ is a multiplicative version of the cyclic group of order 2).

> Proof: we first apply σ, calculate the factors of form $x_{\sigma(j)} - s_{\sigma(i)}$ where $i < j$, extract them to the outside. Then repeat this process by applying τ.

Proposition 24. Transpositions are all odd permutations and ϵ is a surjective homomorphism.

> Proof:
>
> It is clear that applying $(1\ 2)$ to any polynomial will flip exactly one factor $(x_1 - x_2)$, so $\epsilon((1,2)) = -1$.
>
> $\forall i, j$, let $\lambda = (1\ i)(2\ j)$, then
>
> $$\begin{aligned}\epsilon((i\ j)) &= \epsilon(\lambda(1\ 2)\lambda) \\ &= \epsilon(1\ 2)\epsilon(\lambda)^2 \\ &= -1\end{aligned}$$

Definition. The **alternating group of degree** n, denoted by A_n, is the kernel of the homomorphism ϵ (i.e., the set of even permutations).

(by the definition, $S_n/A_n \cong \epsilon(S_n) = \{\pm 1\}$

an m-cycle is an odd permutation if and only if m is even.

Proposition 25. The permutation σ is odd if and only if the number of cycles of even length in its cycle decomposition is odd.

> Proof: obvious

A_n is a non-abelian simple group for all $n \ge 5$.

Chapter 4 Group Actions

4.1 Group Actions and Permutation Representations

Definition.

1. The **kernel** of the action is the set of elements of G that act trivially on every element of A: $\{g \in G \mid g \cdot a = a \text{ for all } a \in A\}$.
2. For each $a \in A$, the **stabilizer** of a in G is the set of elements of G that fix element a: $\{g \in G \mid g \cdot a = a\}$ and is denoted by G_a.
3. An action is **faithful** if its kernel is the identity.

Proposition 1. For any group G and any nonempty set A there is a bijection between the actions of G on A and the homomorphisms of G into S_A.

> Proof: for any homomorphism φ, let $g \cdot a = \varphi(g)(a)$.
>
> For any action, let $\varphi(g)(a) = g \cdot a$.

Definition. If G is a group, a **permutation representation** of G is any homomorphism of G into the symmetric group S_A for some nonempty set A. We shall say a given action of G on A **affords** or **induces** the associated permutation representation of G.

> Any group action and only a group action can afford a permutation representation.
>
> That is, the φ generated in proposition 1. by the group action.
>
> .
>
> A permutation representation: $\varphi : G \to S_A$.

Proposition 2. Let G be a group acting on the nonempty set A. The relation on A defined by

$$a \sim b \text{ if and only if } a = g \cdot b \text{ for some } g \in G$$

is a equivalence relation. For each $a \in A$, the number of elements in the equivalence class containing a is $|G : G_a|$, the index of the stabilizer of a.

> Proof:
>
> - Reflexive: $a = 1 \cdot a$.
> - Symmetric: $a = g \cdot b \Rightarrow b = g^{-1} \cdot a$.
> - Transitive: $a = g \cdot b \wedge b = h \cdot c \Rightarrow a = (gh) \cdot c$.
>
> Next, construct a map from the equivalence class of a to the cosets of G_a by $g \cdot a \mapsto gG_a$.
>
> - Surjective: $\forall g \in G, g \cdot a \mapsto gG_a$.
> - Well defined & injective: $g \cdot a = h \cdot a \Leftrightarrow g^{-1}h \cdot a = a \Leftrightarrow g^{-1}h \in G_a \Leftrightarrow gG_a = hG_a$.
>
> Hence the map is a bijection.

Definition. Let G be a group acting on the nonempty set A.

1. The equivalence class $\{g \cdot a \mid g \in G\}$ is called the **orbit** of G containing a.
2. The action G on A is called **transitive** if there is only one orbit, i.e., given any two elements $a, b \in A$, there is some $g \in G$ such that $a = g \cdot b$.

(Each orbit is a subset of A.

Cycle Decompositions

Let σ be a permutation. For each orbit $\mathcal{O}$ of size d, σ acts as a d-cycle. This prove the existence of a cycle decomposition for each $\sigma \in S_n$.

We can also rearrange the elements in $\mathcal{O}$, it follows that the cycle decomposition above is unique up to a rearrangement of the cycles and up to a cyclic permutation of the integers within each cycle.

Subgroups of symmetric groups are called **permutation groups**.

4.2 Groups Acting on Themselves by Left Multiplication -- Cayley's Theorem

In this section G is any group and we first consider G **acting on itself** (i.e., $A = G$) **by left multiplication**:

$$g \cdot a = ga \text{ for all } g \in G, a \in G$$

Theorem 3. Let G be a group, let H be a subgroup of G and let H act by left multiplication on the set A of left cosets of H in G. Let π_H be the associated permutation representation afforded by this action. Then

1. G acts transitively on A.
2. The stabilizer in G of the point $1H \in A$ is the subgroup H.
3. The kernel of the action (i.e., the kernel of π_H) is $\bigcap_{x \in G} xHx^{-1}$, and $\ker \pi_H$ is the largest normal subgroup of G contained in H.

> Proof:
>
> 1. $\forall aH, bH \in A, (ba^{-1}) \cdot aH = bH$.
> 2. the stabilizer is $\{g \in G \mid g \cdot 1H = 1H\} = \{g \in G \mid g \in H\} = H$.
> 3. By the definition of π_H, we have
>
> $$\begin{aligned} \ker \pi_H &= \{g \in G \mid gxH = xH \ \forall x \in G\} \\ &= \{g \in G \mid x^{-1}gxH = H \ \forall x \in G\} \\ &= \{g \in G \mid x^{-1}gx \in H \ \forall x \in G\} \\ &= \{g \in G \mid g \in xHx^{-1} \ \forall x \in G\} \\ &= \bigcap_{x \in G} xHx^{-1} \end{aligned}$$
>
> If $N \trianglelefteq G$ and $N \leq H$, then $\forall x \in G, N = xNx^{-1} \leq xHx^{-1}$. Hence $N \leq \bigcap_{x \in G} xHx^{-1} = \ker \pi_H$.

Corollary 4. (Cayley's Theorem) Every group is isomorphic to a subgroup of some symmetric group. If G is a group of order n, then G is isomorphic to a subgroup of S_n.

> Proof: Choose $H = \{1\}$. Apply the Theorem 3 to obtain a homomorphism from G to $S_G \cong S_n$, then the kernel is $\{1\}$. Hence the homomorphism is an injective, so G is isomorphic to its image in S_G.

Left regular representation of G: $\sigma_g(x) = gx$ for all $g \in G$.

Right regular representation: $\tau_h(x) = xh^{-1}$.

Corollary 5. If G is a finite group of order n and p is the smallest prime dividing $|G|$, then any subgroup of index p is normal.

> Proof. Suppose $H \leq G$ and $|G : H| = p$. Let π_H be the permutation representation afforded by multiplication on the set of left cosets of H in G. Let $K = \ker \pi$ and let $|H : K| = k$. Then $|G : K| = kp$.
>
> By the First Isomorphic Theorem, $G/K = G/\pi_H \cong \pi_H(G)$, and since each element in $\pi_H(g)$ is a permutation of length p (since H has p cosets), $\pi_H(G)$ is a subset of S_p.
>
> By the Lagrange's Theorem, $pk = |G/K| \mid p!$, so $k \mid (p-1)!$. Since p is the smallest prime dividing $|G|$ and k divides $|G|$, every prime factor of k is not less than p. Hence $k = 1$, so $H = K \trianglelefteq G$, completing the proof.

####

4.3 Groups Acting on Themselves by Conjugation --- the Class Equation

In this section, G is any group and we first consider G **acting on itself** (i.e., $A = G$) **by conjugation**

$$g \cdot a = gag^{-1}.$$

Definition. Two elements a and b of G are said to be **conjugate** in G if there is some $g \in G$ such that $b = gag^{-1}$ (i.e., if and only if there are in the same orbit of G acting on itself by conjugation). The orbits of G acting on itself by conjugation are called the **conjugacy classes** of G.

Definition. Two subsets S and T are said to be **conjugate** in G if there is some $g \in G$ such that $T = gSg^{-1}$ (i.e., if and only if there are in the same orbit of G acting on its subsets by conjugation).

Proposition 6. The number of conjugates of a subset S in a group G is the index of the normalizer of S, $|G : N_G(S)|$. In particular, the number of conjugates of an element s of G is the index of the centralizer of s, $|G : C_G(S)|$.

> Proof.

> $G_S = \{gSg^{-1} = S \mid g \in G\} = N_G(S)$. Then apply proposition 2.
>
> $N_G(\{s\}) = C_G(s)$.

Theorem 7. (The Class Equation) Let G be a finite group and let $g_1, g_2, \ldots, g_r$ be the representatives of the distinct conjugacy classes of G not in the center $Z(G)$ of G. Then

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

> Proof: Apply proposition 6. to $g_1, \ldots, g_r$ to get the relation between the size of the conjugacy class and the number of cosets.

Theorem 8. If p is a prime and P is a group of prime power order p^α for some $\alpha \geq 1$, then P has a nontrivial center: $Z(P) \neq 1$.

> Proof:
>
> By the Class Equation, and the definition, $C_P(g_i) \neq P$, so $p \mid |P : C_P(g_i)|$. Since $p \mid |G|$, $p \mid |Z(G)|$ hence the center must be nontrivial.

Corollary 9. Omitted.

Conjugacy in S_n.

Proposition 10. Let σ, τ be elements of the symmetric group S_n and suppose σ has cycle decomposition

$$(a_1\ a_2\ \ldots\ a_{k_1})(b_1\ b_2\ \ldots\ b_{k_2})\ldots.$$

Then $\tau\sigma\tau^{-1}$ has cycle decomposition

$$(\tau(a_1)\ \tau(a_2)\ \ldots\ \tau(a_{k_1}))(\tau(b_1)\ \tau(b_2)\ \ldots\ \tau(b_{k_2}))\ldots,$$

that is, $\tau\sigma\tau^{-1}$ is obtained from σ by replacing each entry i in the cycle decomposition from σ by the entry $\tau(i)$.

> Proof:
>
> Observe that if $\sigma(i) = j$, then
>
> $$\tau\sigma\tau^{-1}(\tau(i)) = \tau(j).$$

Definition.

1. If $\sigma \in S_n$ is the product of disjoint cycles of lengths $n_1, n_2, \ldots, n_r$ with $n_1 \leq n_2 \leq \cdots \leq n_r$ (including its 1-cycles) then the integers $n_1, n_2, \ldots, n_r$ are called the **cycle type** of σ.
2. If $n \in \mathbb{Z}^+$, a **partition** of n is any nondecreasing sequence of positive integers whose sum is n.

Proposition 11. Two elements of S_n are conjugate in S_n if and only if they have the same cycle type. The number of conjugacy classes of S_n equals the number of partitions of n.

> Proof.
>
> Necessity: by Proposition 10.
>
> Sufficiency: Construct a τ such that $\tau\sigma_1\tau^{-1} = \sigma_2$.

4.4 Autumorphisms

Definition. Let G be a group. An isomorphism from G onto itself is called an **automorphism** of G. The set of all automorphisms of G is denoted by $\mathrm{Aut}(G)$.

(and $\mathrm{Aut}(G)$ is a group under composition

Proposition 13. Let H be a normal subgroup of the group G. Then G acts by conjugation on H as automorphisms of H. More specifically, the action of G on H by conjugation is defined for each $g \in G$ by

$$h \mapsto ghg^{-1} \text{ for each } h \in H.$$

For each $g \in G$, conjugation by g is an automorphism of H. The permutation representation afforded by this action is a homomorphism of G into $\mathrm{Aut}(H)$ with kernel $C_G(H)$. In particular, $G/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

> Proof.
>
> Let φ_g be conjugation by g. Note that because g normalizes H, φ_g maps H to itself.
>
> We can prove that φ_g is a bijection from H to H.
>
> Also, φ_g is a homomorphism, so it is an isomorphism (automorphism).
>
> Hence, the permutation representation $\psi : G \to S_H$ has image contained in the subgroup $Aut(H)$ of S_H.
>
> Finally,
>
> $$\begin{aligned} \ker \psi &= \{g \in G \mid \varphi(g) = 1\} \\ &= \{g \in G \mid ghg^{-1} = h \ , \forall h\} \\ &= C_G(H). \end{aligned}$$
>
> The First Isomorphic Theorem implies the final statement of the proposition.

Proposition 13 shows that a group acts by conjugation on a normal subgroup as structure preserving permutations, i.e., as automorphisms.

Corollary 14. If K is any subgroup of the group G and $g \in G$, then $K \cong gKg^{-1}$. Conjugate elements and conjugate subgroups have the same order.

> Proof:
>
> Letting $G = H$ in the proposition 13 shows that conjugation by $g \in G$ is an automorphism of G.
>
> Note that the order of an element is equal to the order of the cyclic group generated by that element.

Corollary 15. For any subgroup H of a group G, the quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$. In particular, $G/Z(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G)$.

> Proof.
>
> 1. Apply corollary 13 by $H = H, G = N_G(H)$.
> 2. Apply (1) by $H = G$, where $N_G(H) = G$ and $C_G(H) = Z(G)$.

Definition. Let G be a group and let $g \in G$. Conjugation by g is called an **inner automorphism** of G and the subgroup of $\mathrm{Aut}(G)$ consisting of all inner automorphisms is denoted by $Inn(G)$.

(Note that $Inn(G) \leq \mathrm{Aut}(G)$. Also, $\mathrm{Inn}(G) = \psi(G) \cong G/Z(G)$.

(Note also that if H is a normal subgroup of G, conjugation by an element of G when restricted to H is an automorphism of H but need not be an inner automorphism of H (as we shall see).)

(Although the preceding example was fairly trivial, it illustrates that the action of G by conjugation on a normal subgroup H can be restricted by knowledge of the automorphism group of H. This in tum can be used to investigate the structure of G and will lead to some classification theorems when we consider semidirect products in Section 5.5.)

Definition. A subgroup H of a group G is called **characteristic** in G, denoted $H \text{ char } G$, if for every automorphism of G maps H to itself, i.e., $\sigma(H) = H$ for all $\sigma \in \text{Aut}(G)$.

(propositions:

1. Characteristic subgroups are normal,

   > consider the action of G on H by conjugation).

2. If H is the unique subgroup of G of a given order, then H is characteristic in G,

   > $\forall \varphi \in \text{Aut}(G)$, $\varphi(H)$ also has order $|H|$, so $\varphi(H) = H$.

3. If $K \text{ char } H$ and $H \trianglelefteq G$, then $K \trianglelefteq G$.

   > consider the action of G by conjugation, then $\varphi(H) = H$, so φ restricted on H is a automorphism of H, hence $\varphi(K) = K$, hence $K \trianglelefteq G$.

4. (addition) If $K \text{ char } H$ and $H \text{ char } G$, then $K \text{ char } G$.

   > Similar to (3).

(Thus we may think of characteristic subgroups as "strongly normal" subgroups.

Proposition 16. The automorphism group of the cyclic group of order n is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$, an abelian group of order $\varphi(n)$.

> Proof:
>
> $\forall \psi \in \text{Aut}(Z_n)$, if $\psi(x) = x^a$, then ψ is uniquely determined by a.
>
> Since $|x| = |\psi_a(x)|$, $(a, n) = 1$.
>
> Furthermore, for every a relatively prime to n, we can prove that the map $x \mapsto x^a$ is an automorphism of Z_n.

4.5 Sylow's Theorem

Definition. Let G be a group and p be a prime.

1. A group of order p^α for some $\alpha \geq 1$ is called a **p-group**. Subgroup of G which are p-groups are called **p-subgroups**.
2. If G is a group of order $p^\alpha m$ where $p \nmid m$, then a subgroup of order p^α is called a **Sylow p-subgroup** of G.
3. The set of Sylow p-subgroups of G will be denoted by $Syl_p(G)$ and the number of Sylow p-subgroups of G will be denoted by $n_p(G)$.

Theorem 18. (Sylow's Theorem) Let G be a group of order $p^\alpha m$, where p is a prime not dividing m.

1. Sylow p-subgroups of G exist, i.e., $Syl_p(G) \neq \varnothing$.
2. If P is a Sylow p-subgroup of G and Q is any p-subgroup of G, then there exists $g \in G$ such that $Q \leq gPg^{-1}$, i.e., Q is contained in some conjugate of P. In particular, any two Sylow p-subgroups of G are conjugate in G.
3. The number of Sylow p-subgroups of G is of the form $1 + kp$, i.e.,

   $$n_p \equiv 1 \pmod{p}.$$

   Further, n_p is the index in G of the normalizer $N_G(P)$ for any Sylow p-subgroup P, hence n_p divides m.

> Proof: Omitted.

Corollary 20. Let P be a Sylow p-subgroup of G. Then the following are equivalent:

1. P is the unique Sylow p-subgroup of G, i.e., $n_p = 1$.
2. P is normal in G.
3. P is characteristic in G.
4. All subgroups generated by elements of p-power order are p-groups, i.e., if X is any subset of G such that $|x|$ is a power of p for all $x \in X$, then $\langle X \rangle$ is a p-group.

> Proof:
>
> - $1 \to 2$: $\forall g \in G, gPg^{-1} = P$ since $\left|gPg^{-1}\right| = |P|$.
> - $2 \to 1$: If P and Q are Sylow p-subgroups of G, then since G is normal, $Q = gPg^{-1} = P$.
> - $1 \to 3$: $\forall \varphi \in \mathrm{Aut}(G),\ |\varphi(P)| = |P|$, so $\varphi(P) = P$, hence P char G.
> - $3 \to 2$: obvious.
> - $1 \to 4$: Let X be a subset described in the statement. Since $\forall x \in X$, $\langle x \rangle$ is a p-subgroup, there is some g that $x \in gPg^{-1} = P$. Thus $X \subseteq P$, and so $\langle X \rangle \leq P$. By the Lagrange's Theorem, $|\langle X \rangle| \mid |P|$, so $\langle X \rangle$ is a p-group.
> - $4 \to 1$: Let X be the union of all Sylow p-subgroups of G (each element has order $|x| \mid p^\alpha$). If P is any Sylow p-subgroup, P is a subgroup of the p-group $\langle X \rangle$. Since P is a p-subgroup of G of maximal order, we must have $P = \langle X \rangle$, so (1) holds.

4.6 Simplicity of A_n

Theorem 24. A_n is simple for all $n \geq 5$.

> Proof: Omitted.

Chapter 5 Direct and Semidirect Products and Abelian Groups

5.1 Direct Products

Definition.

1. The **direct product** $G_1 \times G_2 \times \cdots \times G_n$ of the groups $G_1, G_2, \ldots, G_n$ with operators $\star_1, \star_2, \ldots, \star_n$, respectively, is the set of n-tuples $(g_1, g_2, \ldots, g_n)$ where $g_i \in G_i$ with operation defined componentwise:
$$(g_1, g_2, \ldots, g_n) \star (h_1, h_2, \ldots, h_n) = (g_1 \star_1 h_1, g_2 \star_2, h_2, \ldots, g_n \star_n h_n).$$
2. Similarly, the **direct product** $G_1 \times G_2 \times \cdots$ of the groups $G_1, G_2, \ldots$ with operators $\star_1, \star_2, \ldots$, respectively, is the set of sequences $(g_1, g_2, \ldots)$ where $g_i \in G_i$ with operation defined componentwise:
$$(g_1, g_2, \ldots) \star (h_1, h_2, \ldots) = (g_1 \star_1 h_1, g_2 \star_2, h_2, \ldots).$$

Proposition 1. If $G_1, \ldots, G_n$ are groups, their direct product is a group of order $|G_1|\,|G_2| \cdots |G_n|$ (if any G_i is infinite, so is the direct product).

> Proof:
>
> prove the group axioms (obvious)
>
> the formula of the order of G is clear.

Proposition 2. Let $G_1, G_2, \ldots, G_n$ be groups and let $G = G_1 \times \cdots \times G_n$ be their direct product.

1. For each fixed i, the set of elements of G which have the identity of G_j in the j^{th} position for all $j \neq i$ and arbitrary elements of G_i in position i is a subgroup of G isomorphic to G_i:
$$G_i \cong \{(1, 1, \ldots, 1, g_i, 1, \ldots, 1 \mid g_i \in G_i\}.$$
If we identity G_i with this subgroup, then $G_i \trianglelefteq G$ and
$$G/G_i \cong G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n.$$
2. For each fixed i, define $\pi_i : G \to G_i$ by
$$\pi_i((g_1, g_2, \ldots, g_n)) = g_i.$$
Then π_i is a surjective homomorphism with
$$\begin{aligned}\ker \pi_i &= \{(g_1, g_2, \ldots, g_{i-1}, 1, g_{i+1}, \ldots, g_n) \mid g_j \in G_j \text{ for all } j \neq i\} \\ &\cong G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n.\end{aligned}$$
3. Under the identifications if part (1), if $x \in G_i$ and $y \in G_j$ for some $i \neq j$, then $xy = yx$.

> Proof: trivial.

5.2 The Fundamental Theorem of Finitely Generated Abelian Groups

Definition.

1. A group G is **finitely generated** if there is a finite set A of G such that $G = \langle A \rangle$.
2. For each $r \in \mathbb{Z}$ with $r \geq 0$, let $\mathbb{Z}^r = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ be the direct product of r copies of the group $\mathbb{Z}$, where $\mathbb{Z}^0 = 1$. The group $\mathbb{Z}^r$ is called the **free abelian group of rank** r.

Theorem 3. *(Fundamental Theorem of Finitely Generated Abelian Groups)* Let G be a finitely generated abelian group. Then

(1)

$$G \cong \mathbb{Z}^r \times Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_s},$$

for some integers $r, n_1, n_2, \ldots, n_s$ satisfying the following conditions:

(a) $r \geq 0$ and $n_j \geq 2$ for all j, and
(b) $n_{i+1} \mid n_i$ for $1 \leq i \leq s-1$

(2) the expression in (1) is unique: if $G \cong \mathbb{Z}^t \times Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_u}$, where t and $m_1, m_2, \ldots, m_u$ satisfy (a) and (b) (i.e., $t \geq 0$, $m_j \geq 2$ for all j and $m_{i+1} \mid m_i$ for $1 \leq i \leq u-1$), then $t = r$, $u = s$ and $m_i = n_i$ for all i.

Definition. The integer r in Theorem 3 is called the *free rank* or *Betti number* of G and the integers $n_1, n_2, \ldots, n_s$ are called the *invariant factors* of G. The description of G in Theorem 3(1) is called the *invariant factor decomposition* of G.

The order of a finite abelian group is just the product of its invariant factors (by Proposition 1). If G is a finite abelian group with invariant factors $n_1, n_2, \ldots, n_s$, where $n_{i+1} \mid n_i$, $1 \leq i \leq s-1$, then G is said to be of *type* $(n_1, n_2, \ldots, n_s)$.

Corollary 4. If n is the product of distinct primes, then up to isomorphism the only abelian group of order n is the cyclic group of order n, Z_n.

Theorem 5. Let G be an abelian group of order $n > 1$ and let the unique factorization of n into distinct prime powers be

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}.$$

Then

(1) $G \cong A_1 \times A_2 \times \cdots \times A_k$, where $|A_i| = p_i^{\alpha_i}$
(2) for each $A \in \{A_1, A_2, \ldots, A_k\}$ with $|A| = p^{\alpha}$,

$$A \cong Z_{p^{\beta_1}} \times Z_{p^{\beta_2}} \times \cdots \times Z_{p^{\beta_t}}$$

with $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_t \geq 1$ and $\beta_1 + \beta_2 + \cdots + \beta_t = \alpha$ (where t and $\beta_1, \ldots, \beta_t$ depend on i)

(3) the decomposition in (1) and (2) is unique, i.e., if $G \cong B_1 \times B_2 \times \cdots \times B_m$, with $|B_i| = p_i^{\alpha_i}$ for all i, then $B_i \cong A_i$ and B_i and A_i have the same invariant factors.

Definition. The integers p^{β_j} described in the preceding theorem are called the *elementary divisors* of G. The description of G in Theorem 5(1) and 5(2) is called the *elementary divisor decomposition* of G.

The subgroups A_i described in part (1) of the theorem are the Sylow p_i-subgroups of G. Thus (1) says that G is isomorphic to the direct product of its Sylow subgroups (note that they are normal — since G is abelian — hence unique). Part 1 is often referred to as *The Primary Decomposition Theorem* for finite abelian groups.[2] As with Theorem 3, we shall prove this theorem later.

Proposition 6. Let $m, n \in \mathbb{Z}^+$.

1. $Z_m \times Z_n \cong Z_{mn}$ if and only if $(m, n) = 1$.
2. If $n = \prod p_i^{\alpha_i}$, then $Z_n \cong Z_{p_1}^{\alpha_1} \times \cdots \times Z_{p_k}^{\alpha_k}$.

> Proof:
>
> Let $Z_m = \langle x \rangle, Z_n = \langle y \rangle$.
>
> Let $l = lcm(m, n)$. If $(m, n) \neq 1$, then $(x^a y^b)^l = 1$, where $l < mn$, so $Z_m \times Z_n$ is not a cyclic group of order mn.
>
> Conversely, $|xy| = lcm(m, n) = mn$, so $Z_m \times Z_n \cong Z_{mn}$.

Definition.

(1) If G is a finite abelian group of type $(n_1, n_2, \ldots, n_t)$, the integer t is called the *rank* of G (the free rank of G is 0 so there will be no confusion).

(2) If G is any group, the *exponent* of G is the smallest positive integer n such that $x^n = 1$ for all $x \in G$ (if no such integer exists the exponent of G is ∞).

5.4 Recognizing Direct Products

Definition. Let G be a group, let $x, y \in G$ and let A, B be nonempty subsets of G.

1. Define $[x, y] = x^{-1}y^{-1}xy$, called the **commutator** of x and y.
2. Define $[A, B] = \{[a, b] \mid a \in A, b \in B\}$, the group generated by commutators of elements from A and from B.
3. Define $G' = \{[x, y] \mid x, y \in G\}$, the subgroup of G generated by commutators of elements from G, called the **commutator subgroup** of G.

Proposition 7. Let G be a group, let $x, y \in G$ and let $H \leq G$. Then

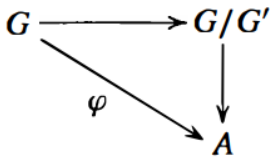
(1) $xy = yx[x, y]$ (in particular, $xy = yx$ if and only if $[x, y] = 1$).

(2) $H \trianglelefteq G$ if and only if $[H, G] \leq H$.

(3) $\sigma[x, y] = [\sigma(x), \sigma(y)]$ for any automorphism σ of G, G' char G and G/G' is abelian.

(4) G/G' is the largest abelian quotient of G in the sense that if $H \trianglelefteq G$ and G/H is abelian, then $G' \leq H$. Conversely, if $G' \leq H$, then $H \trianglelefteq G$ and G/H is abelian.

(5) If $\varphi : G \to A$ is any homomorphism of G into an abelian group A, then φ factors through G' i.e., $G' \leq \ker \varphi$ and the following diagram commutes:



Proof:

1. by definition
2. $ghg^{-1} \in H \Leftrightarrow h^{-1}g^{-1}hg \in H \Leftrightarrow [H, G] \leq H$.
3. $\sigma[x, y] = \sigma(x^{-1}y^{-1}) = \sigma(x)^{-1}\sigma(y)^{-1}\sigma(x)\sigma(y) = [\sigma(x), \sigma(y)]$.

   We can see that σ maps a commutator to another commutator. Also σ has a two side inverse, so it maps the set of commutators bijectively onto itself. Since the commutators are a generating set of G', $\sigma(G') = G'$, so $G' \operatorname{char} G$.

   Finally, $(xG')(yG') = (xy)G' = (yx[x, y])G' = (yx)G' = (yG')(xG')$, so G/G' is abelian.
4. Suppose $H \trianglelefteq G$ and G/H is abelian. Then

$$\begin{aligned}(yH)(xH) &= (xH)(yH)\\ 1H &= (xH)^{-1}(yH)^{-1}(xH)(yH)\\ &= x^{-1}y^{-1}xyH\\ &= [x, y]H,\end{aligned}$$

   so $[x, y] \in H$. Hence $G' \leq H$.

   Conversely, $H/G' \leq G/G'$. We can use the isomorphism theorems to prove that $H \trianglelefteq G$ and $G/H \cong (G/G')/(H/G')$, so G/H is abelian.

Proposition 8. Let H and K be subgroups of the group G. The number of distinct ways of writing each element of the set HK in the form hk, for some $h \in H$ and $k \in K$ is $|H \cap K|$. In particular, if $H \cap K = 1$, then each element of HK can be written uniquely as a product hk, for some $h \in H$ and $k \in K$.

Proof:

If $a = hk$, then $a = (ht)(t^{-1}k)$ where $t \in H \cap K$, hence the number of ways is at least $|H \cap K|$.

Since $|HK| = \frac{|H||K|}{|H \cap K|}$, the number of ways must be exactly $|H \cap K|$.

Theorem 9. Suppose G is a group with subgroups H and K such that

(1) H and K are normal in G, and
(2) $H \cap K = 1$.

Then $HK \cong H \times K$.

Proof:

By corollary 3.15, $HK \leq G$.

Since $H \trianglelefteq G$, $\forall h \in H, k^{-1}hk \in H$, so $h^{-1}k^{-1}hk \in H$. Similarly, $h^{-1}k^{-1}hk \in K$. Hence $[h, k] = 1$, and $hk = kh$.

Now, since all elements from h and k are commutative, we can construct an isomorphism: let

$$\varphi : (h, k) \mapsto hk$$

then

$$\begin{aligned}\varphi((h_1, k_1)(h_2, k_2)) &= \varphi(h_1h_2, k_1k_2)\\ &= h_1h_2k_1k_2\\ &= h_1k_1h_2k_2\\ &= \varphi(h_1, k_1)\varphi(h_2, k_2).\end{aligned}$$

so it is a homomorphism. (it is easy to show that φ is bijective).

Hence $HK \cong H \times K$.

Definition. If G is a group and H and K are normal subgroups of G with $H \cap K = 1$, we call HK the *internal direct product* of H and K. We shall (when emphasis is called for) call $H \times K$ the *external direct product* of H and K.

5.5 Semidirect Products

Theorem 10. Let H and K be groups and let φ be a homomorphism from K into $\mathrm{Aut}(H)$. Let $\cdot$ denote the (left) action of K on H determined by φ. Let G be the set of ordered pairs (h, k) with $h \in H$ and $k \in K$ and define the following multiplication on G:

$$(h_1, k_1)(h_2, k_2) = (h_1\, k_1 \cdot h_2\,,\; k_1 k_2).$$

(1) This multiplication makes G into a group of order $|G| = |H||K|$.

(2) The sets $\{(h, 1) \mid h \in H\}$ and $\{(1, k) \mid k \in K\}$ are subgroups of G and the maps $h \mapsto (h, 1)$ for $h \in H$ and $k \mapsto (1, k)$ for $k \in K$ are isomorphisms of these subgroups with the groups H and K respectively:

$$H \cong \{(h, 1) \mid h \in H\} \quad \text{and} \quad K \cong \{(1, k) \mid k \in K\}.$$

Identifying H and K with their isomorphic copies in G described in (2) we have

(3) $H \trianglelefteq G$

(4) $H \cap K = 1$

(5) for all $h \in H$ and $k \in K$, $khk^{-1} = k \cdot h = \varphi(k)(h)$.

> Proof:
>
> (1) it is straightforward to check that G is a group, and $|G| = |H|\,|K|$. Note that $(h, k)^{-1} = (k^{-1} \cdot h^{-1}, k^{-1})$.
>
> (2) also straightforward
>
> (4) it is clear.
>
> (5)
>
> $$\begin{aligned}(1, k)(h, 1)(1, k)^{-1} &= (k \cdot h, k)(1, k^{-1}) \\ &= (k \cdot h, 1),\end{aligned}$$
>
> so $khk^{-1} = k \cdot h = \varphi(k)(h)$.
>
> (3) By (5), $K \leq N_G(H)$. Since $G = HK$ and $H \leq N_G(H)$, we have $G = N_G(H)$ (verify each element by writting $g = hk$). Hence $H \trianglelefteq G$.

Definition. Let H and K be groups and let φ be a homomorphism from K into $\mathrm{Aut}(H)$. The group described in Theorem 10 is called the *semidirect product* of H and K with respect to φ and will be denoted by $H \rtimes_\varphi K$ (when there is no danger of confusion we shall simply write $H \rtimes K$).

Proposition 11. Let H and K be groups and let $\varphi : K \to \mathrm{Aut}(H)$ be a homomorphism. Then the following are equivalent:

(1) the identity (set) map between $H \rtimes K$ and $H \times K$ is a group homomorphism (hence an isomorphism)

(2) φ is the trivial homomorphism from K into $\mathrm{Aut}(H)$

(3) $K \trianglelefteq H \rtimes K$.

> Proof:
>
> $1 \to 2$: in multiplication, $k \cdot h = h$, so φ is trivial.

> $2 \to 3$: by theorem 10(5), $khk^{-1} = \varphi(k)(h) = h$, so H normalize K. moreover, K normalize K, so $H \rtimes K = HK$ normalize K.
>
> $3 \to 1$: as in the proof of theorem 9, $\forall h \in H, k \in K, [h, k] \in H \cap K = 1$. hence $khk^{-1} = h$, so the product is the same as the direct product.

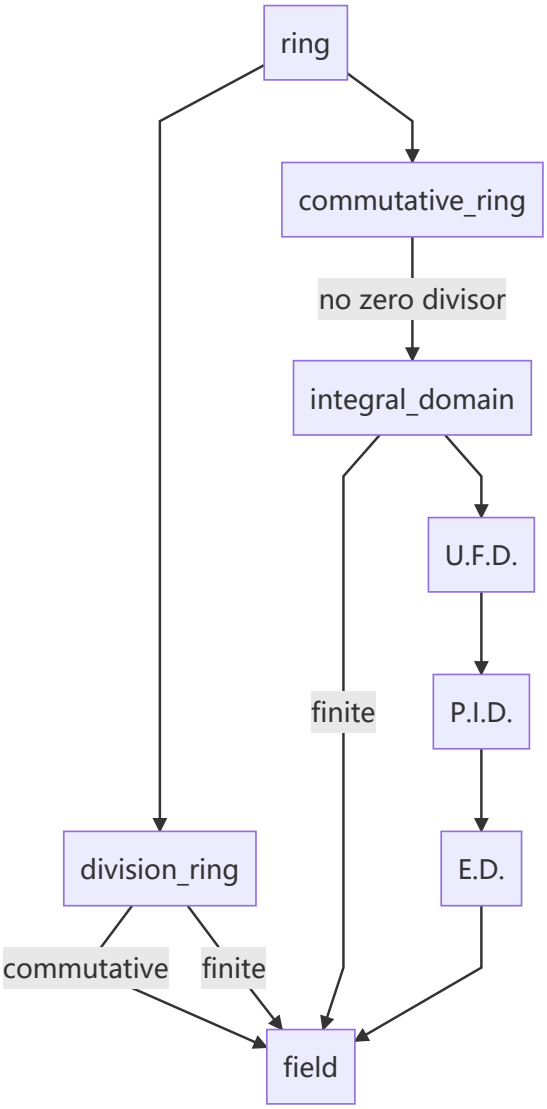
Theorem 12. Suppose G is a group with subgroups H and K such that
(1) $H \trianglelefteq G$, and
(2) $H \cap K = 1$.
Let $\varphi : K \to \mathrm{Aut}(H)$ be the homomorphism defined by mapping $k \in K$ to the automorphism of left conjugation by k on H. Then $HK \cong H \rtimes K$. In particular, if $G = HK$ with H and K satisfying (1) and (2), then G is the semidirect product of H and K.

> Proof:
>
> It is still true that HK is a subgroup of G (Corollary 3.15) and, by Proposition 8, every element of HK can be written uniquely as a product hk, for some $h \in H$ and $k \in K$, i.e., there is a bijection between HK and the collection of ordered pairs (h, k), given by $hk \mapsto (h, k)$ (so the group H appears as the set of elements $(h, 1)$ and K appears as the set of elements $(1, k)$). Given two elements h_1k_1 and h_2k_2 of HK, we first see how to write their product (in G) in the same form:
>
> $$\begin{aligned}(h_1k_1)(h_2k_2) &= h_1k_1h_2(k_1^{-1}k_1)k_2 \\ &= h_1(k_1h_2k_1^{-1})k_1k_2 \\ &= h_3k_3,\end{aligned} \tag{5.1}$$
>
> where $h_3 = h_1(k_1h_2k_1^{-1})$ and $k_3 = k_1k_2$. Note that since $H \trianglelefteq G$, $k_1h_2k_1^{-1} \in H$, so $h_3 \in H$ and $k_3 \in K$.

Definition. Let H be a subgroup of the group G. A subgroup K of G is called a *complement* for H in G if $G = HK$ and $H \cap K = 1$.

semidirect product 在干什么？（应该也是一种组装 groups 的方式，并且得到的结构和 direct product 不一样。

Part 2 Ring Theory



R is an integral domain $\rightarrow R[x]$ is an integral domain.

F is a field $\rightarrow F[x]$ is a E.D.

R is a U.F.D. $\leftrightarrow R[x]$ is a U.F.D.

Chapter 7 Introduction to Rings

7.1 Basic Definitions and Examples

Definition.

(1) A *ring* R is a set together with two binary operations $+$ and $\times$ (called addition and multiplication) satisfying the following axioms:

(i) $(R, +)$ is an *abelian* group,

(ii) $\times$ is associative : $(a \times b) \times c = a \times (b \times c)$ for all $a, b, c \in R$,

(iii) the *distributive laws* hold in R : for all $a, b, c \in R$

$$(a+b)\times c = (a\times c)+(b\times c) \quad \text{and} \quad a\times(b+c) = (a\times b)+(a\times c).$$

(2) The ring R is *commutative* if multiplication is commutative.

(3) The ring R is said to have an *identity* (or *contain a* 1) if there is an element $1 \in R$ with

$$1 \times a = a \times 1 = a \qquad \text{for all } a \in R.$$

（群（以及半群，幺半群）啥的都只有一个运算，而环、域等结构都有两个运算）。

An example of a ring without an identity is $2\mathbb{Z}$.

One motivation that $(R, +)$ is commutative is that , if $1 \in R$, then

$$(1+1)(a+b) = 1(a+b) + 1(a+b) = a+b+a+b$$
$$(1+1)(a+b) = (1+1)a + (1+1)b = a+a+b+b$$

, so $a+b = b+a$.

Definition. A ring R with identity 1, where $1 \neq 0$, is called a *division ring* (or *skew field*) if every nonzero element $a \in R$ has a multiplicative inverse, i.e., there exists $b \in R$ such that $ab = ba = 1$. A commutative division ring is called a *field*.

Examples

(1) The simplest examples of rings are the *trivial rings* obtained by taking R to be any commutative group (denoting the group operation by $+$) and defining the multiplication $\times$ on R by $a \times b = 0$ for all $a, b \in R$. It is easy to see that this multiplication defines a commutative ring. In particular, if $R = \{0\}$ is the trivial group, the resulting ring R is called the *zero ring*, denoted $R = 0$. Except for the zero ring, a trivial ring does not contain an identity ($R = 0$ is the only ring where $1 = 0$; we shall often exclude this ring by imposing the condition $1 \neq 0$). Although trivial rings have two binary

$\mathbb{Z}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}/n\mathbb{Z}$ are rings

$\mathbb{H}$ (the (real) *Hamilton Quaternions*) is also a ring. (the *rational* Hamilton Quaternions) is also a ring.

Proposition 1. Let R be a ring. Then

(1) $0a = a0 = 0$ for all $a \in R$.

(2) $(-a)b = a(-b) = -(ab)$ for all $a, b \in R$ (recall $-a$ is the additive inverse of a).

(3) $(-a)(-b) = ab$ for all $a, b \in R$.

(4) if R has an identity 1, then the identity is unique and $-a = (-1)a$.

> Proof:
>
> (1): $0a = (0+0)a = 0a + 0a$, so $0a = 0$.
>
> (2): $(-a)b + ab = (-a+a)b = 0$, so $(-a)b = -ab$.
>
> (3): apply (2) by $a = a', b = -b'$.
>
> (4): $1 = 1 \cdot 1' = 1'$.

Definition. Let R be a ring.

(1) A nonzero element a of R is called a *zero divisor* if there is a nonzero element b in R such that either $ab = 0$ or $ba = 0$.

(2) Assume R has an identity $1 \neq 0$. An element u of R is called a *unit* in R if there is some v in R such that $uv = vu = 1$. The set of units in R is denoted $R^\times$.

(note that $0a = 0$, so (2) is meaningless when $1 = 0$.

It is easy to see that the units in a ring R form a group under multiplication so $R^\times$ will be referred to as the *group of units* of R. In this terminology a *field is a commutative ring F with identity $1 \neq 0$ in which every nonzero element is a unit, i.e., $F^\times = F - \{0\}$.*

Observe that **a zero divisor can never be a unit**.

Example: $\mathbb{Q}(\sqrt{D})$ is a field.

Definition. A commutative ring with identity $1 \neq 0$ is called an *integral domain* if it has no zero divisors.

Proposition 2. Assume a, b and c are elements of any ring with a not a zero divisor. If $ab = ac$, then either $a = 0$ or $b = c$ (i.e., if $a \neq 0$ we can cancel the a's). In particular, if a, b, c are any elements in an integral domain and $ab = ac$, then either $a = 0$ or $b = c$.

> Proof:
>
> $ab = ac \Rightarrow a(b-c) = 0 \Rightarrow a = 0$ or $b - c = 0$. (since a is not a zero divisor, then if $a \neq 0$ and $b - c \neq 0$, then $a(b-c) \neq 0$).

Corollary 3. Any finite integral domain is a field.

> Proof:
>
> Let a be a nonzero element. By the cancellation law, $\varphi : x \mapsto ax$ is injective. Since R is finite, φ is also surjective. Thus $\exists b \in R$ s.t. $ab = 1$, so a is a unit in R. Hence R is a field.

A remarkable result of Wedderburn is that a finite division ring is necessarily commutative, i.e., is a field.

Definition. A *subring* of the ring R is a subgroup of R that is closed under multiplication.

multiplication in R when restricted to S give S the structure of a ring. To show that a subset of a ring R is a subring it suffices to check that it is *nonempty* and *closed under subtraction and under multiplication.*

That is,

1. $S \neq \varnothing$.
2. $\forall a, b \in S, a - b \in S$ and $ab \in S$.

(子环 S 的 0 是 R 的 0， S 的 1 不一定是 R 的 1. e.g. $2\mathbb{Z}/10\mathbb{Z} \subseteq \mathbb{Z}/10\mathbb{Z}$ has identity 6, but identity of $\mathbb{Z}/10\mathbb{Z}$ is 1.

Examples:

The *integral* Quaternions, form a subring of either the real or the rational Quaternions.

(it is not directly defined in this way, but comes from other things)

7.2 Examples: Polynomial Rings, Matrix Rings, and Group Rings

Polynomial Rings

Fix a commutative ring R with identity. We define the ring of polynomials in a form which may already be familiar, at least for polynomials with real coefficients. A definition in terms of Cartesian products is given in Appendix I. Let x be an indeterminate. The formal sum

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with $n \geq 0$ and each $a_i \in R$ is called a *polynomial* in x with coefficients a_i in R. If $a_n \neq 0$, then the polynomial is said to be of *degree* n, $a_n x^n$ is called the *leading term*, and a_n is called the *leading coefficient* (where the leading coefficient of the zero polynomial is taken to be 0). The polynomial is *monic* if $a_n = 1$. The set of all such polynomials is called the ring of *polynomials in the variable x with coefficients in R* and will be denoted $R[x]$.

The operations of addition and multiplication which make $R[x]$ into a ring are the same operations familiar from elementary algebra: addition is "componentwise"

$$\begin{aligned}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) + (b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0)\\ = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0)\end{aligned}$$

(here a_n or b_n may be zero in order for addition of polynomials of different degrees to be defined). Multiplication is performed by first defining $(ax^i)(bx^j) = abx^{i+j}$ for polynomials with only one nonzero term and then extending to all polynomials by the distributive laws (usually referred to as "expanding out and collecting like terms"):

$$\begin{aligned}&(a_0 + a_1 x + a_2 x^2 + \ldots) \times (b_0 + b_1 x + b_2 x^2 + \ldots)\\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \ldots\end{aligned}$$

(in general, the coefficient of x^k in the product will be $\sum_{i=0}^{k} a_i b_{k-i}$). These operations make sense since R is a ring so the sums and products of the coefficients are defined. An easy verification proves that $R[x]$ is indeed a ring with these definitions of addition and multiplication.

The ring R appears in $R[x]$ as the *constant polynomials*. Note that by definition of the multiplication, $R[x]$ is a *commutative ring with identity* (the identity 1 from R).

(the definition of addition and multiplication are still valid if R is not commutative.)

Proposition 4. Let R be an integral domain and let $p(x), q(x)$ be nonzero elements of $R[x]$. Then

(1) degree $p(x)q(x) =$ degree $p(x) +$ degree $q(x)$,
(2) the units of $R[x]$ are just the units of R,
(3) $R[x]$ is an integral domain.

> Proof:
>
> (1)(3) if $p(x)$ and $q(x)$ are polynomials with leading terms $a_n x^n$ and $b_m x^m$, then $p(x)q(x)$ has leading term $a_n b_m x^{n+m}$. Since $a_n b_m \neq 0$, $p(x)q(x) \neq 0$, and it is a polynomial with degree $n + m$.
>
> (2) if $p(x)q(x) = 1$, then $n = m = 0$ and $a_n b_m = 1$, so the units must be in R. Hence the units are the units of R.

Matrix Rings

Fix an arbitrary ring R and let n be a positive integer. Let $M_n(R)$ be the set of all $n \times n$ *matrices with entries from* R. The element (a_{ij}) of $M_n(R)$ is an $n \times n$ square array of elements of R whose entry in row i and column j is $a_{ij} \in R$. The set of matrices becomes a ring under the usual rules by which matrices of real numbers are added and multiplied. Addition is componentwise: the i, j entry of the matrix $(a_{ij}) + (b_{ij})$ is $a_{ij} + b_{ij}$. The i, j entry of the matrix product $(a_{ij}) \times (b_{ij})$ is $\sum_{k=1}^{n} a_{ik}b_{kj}$ (note that these matrices need to be square in order that multiplication of any two elements be defined). It is a straightforward calculation to check that these operations make $M_n(R)$ into a ring. When R is a field we shall prove that $M_n(R)$ is a ring by less computational means in Part III.

If R is a nontrivial ring and $n \geq 2$, then $M_n(R)$ is not commutative.

$M_n(R)$ has zero divisors for all nonzero rings R whenever $n \geq 2$ (指的是存在 zero divisor，而不是每个元素都有 zero divisor)

Scalar matrix: $A = aI$. (informal notation)

$GL_n(R)$: the **general linear group** of degree n over R (the invertible $n \times n$ matrices)

Group Rings

Fix a commutative ring R with identity $1 \neq 0$ and let $G = \{g_1, g_2, \ldots, g_n\}$ be any finite group with group operation written multiplicatively. Define the *group ring*, RG, of G with coefficients in R to be the set of all formal sums

$$a_1g_1 + a_2g_2 + \cdots + a_ng_n \qquad a_i \in R, \quad 1 \leq i \leq n.$$

If g_1 is the identity of G we shall write a_1g_1 simply as a_1. Similarly, we shall write the element $1g$ for $g \in G$ simply as g.

Addition is defined "componentwise"

$$\begin{aligned}(a_1g_1 + a_2g_2 + \cdots + a_ng_n) + (b_1g_1 + b_2g_2 + \cdots + b_ng_n)\\ = (a_1 + b_1)g_1 + (a_2 + b_2)g_2 + \cdots + (a_n + b_n)g_n.\end{aligned}$$

multiplication:

$$\left(\sum a_ig_i\right)\left(\sum b_jg_j\right) = \sum_{i,j}(a_ib_j)g_ig_j$$

(Again, commutativity of R is not needed.)

RG is commutative if and only if G is a commutative group.

G is a *subgroup of the group of units of* RG.

7.3 Ring Homomorphisms and Quotient Rings

Definition. Let R and S be rings.

(1) A *ring homomorphism* is a map $\varphi : R \to S$ satisfying

- **(i)** $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in R$ (so φ is a group homomorphism on the additive groups) and
- **(ii)** $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$.

(2) The *kernel* of the ring homomorphism φ, denoted $\ker \varphi$, is the set of elements of R that map to 0 in S (i.e., the kernel of φ viewed as a homomorphism of additive groups).

(3) A bijective ring homomorphism is called an *isomorphism*.

(since $\varphi(b) + \varphi(-b) = \varphi(b + (-b)) = \varphi(0) = 0, \varphi(a - b) = \varphi(a + (-b)) = \varphi(a) + \varphi(-b) = \varphi(a) - \varphi(b)$).

Proposition 5. Let R and S be rings and let $\varphi : R \to S$ be a homomorphism.

(1) The image of φ is a subring of S.

(2) The kernel of φ is a subring of R. Furthermore, if $\alpha \in \ker\varphi$ then $r\alpha$ and $\alpha r \in \ker\varphi$ for every $r \in R$, i.e., $\ker\varphi$ is closed under multiplication by elements from R.

> Proof:
>
> (1) $\forall\varphi(a), \varphi(b) \in \varphi(R), \varphi(a) - \varphi(b) = \varphi(a-b) \in \varphi(R), \varphi(a)\varphi(b) = \varphi(ab) \in \varphi(R)$.
>
> (2) $\forall a, b \in \ker\varphi, \varphi(a-b) = \varphi(a) - \varphi(b) = 0$, so $a - b \in \ker\varphi$. $\varphi(ab) = \varphi(a)\varphi(b) = 0$, so $ab \in \ker\varphi$.
>
> If $\alpha \in \ker\varphi$, then $\varphi(\alpha r) = \varphi(\alpha)\varphi(r) = 0$, so $\alpha r \in \ker\varphi$. Similarly, $r\alpha \in \ker\varphi$.

P241 XY is the fiber over ab: 直接用 $(r+I)\times(s+I) = (rs)+I$ 来证. ($sI = I$ by prop 5.)

Let $\varphi : R \to S$ be a homomorphism. Let $I = \ker\varphi$, then we have

$$(r+I)+(s+I) = (r+s)+I$$
$$(r+I)\times(s+I) = (rs)+I.$$

Now, for any $I \leq (R,+)$. Since $(R,+)$ is abelian, the first property holds (because every subgroup is normal). We consider when the second property holds.

If we we want the multiplication in (2) well defined, that the multiplication is independent of the particular representatives r and s chosen. If we use representation $r+\alpha$ and $s+\beta$ instead of r and s, we have

$$(r+\alpha)(s+\beta)+I = rs+I.$$

for all $r, s \in R$ and $\alpha, \beta \in I$.

First, by letting $s = 0$ and r be arbitrary, we have $r\beta \in I$ for every $r \in R$ and $\beta \in I$. Similarly, $\alpha s \in I$.

Conversely, if $r\beta \in I$ and $\alpha s \in I$, then $(r+\alpha)(s+\beta)+I = rs+\alpha s+r\beta+\alpha\beta+I = rs+I$. Hence these are the necessary and sufficient condition.

Finally, if the multiplication property (2) is well defined, then we can verify the quotient follows directly from the axioms of rings. For example, distributive laws is verified as follows:

$$\begin{aligned}(r+I)[(s+I)+(t+I)] &= (r+I)[(s+t)+I] \\ &= r(s+t)+I \\ &= (rs+rt)+I \\ &= (rs+I)+(rt+I) \\ &= [(r+I)(s+I)]+[(r+I)(t+I)].\end{aligned}$$

It shows that the quotient R/I of the ring R by I has a natural ring structure if and only if .... (so it must be a subring of R since it is closed under multiplication.)

Then we define these subsets as ideals:

Definition. Let R be a ring, let I be a subset of R and let $r \in R$.

(1) $rI = \{ra \mid a \in I\}$ and $Ir = \{ar \mid a \in I\}$.

(2) A subset I of R is a *left ideal* of R if

- **(i)** I is a subring of R, and
- **(ii)** I is closed under left multiplication by elements from R, i.e., $rI \subseteq I$ for all $r \in R$.

Similarly I is a *right ideal* if (i) holds and in place of (ii) one has

- **(ii)′** I is closed under right multiplication by elements from R, i.e., $Ir \subseteq I$ for all $r \in R$.

(3) A subset I that is both a left ideal and a right ideal is called an *ideal* (or, for added emphasis, a *two-sided ideal*) of R.

For commutative rings the notions of left, right and two-sided ideal coincide.

We emphasize that to prove a subset I of a ring R is an ideal it is necessary to prove that I is nonempty, closed under subtraction and closed under multiplication by all the elements of R (and not just by elements of I).

If R has a 1 then (-1)a = -a so in this case I is an ideal if it is nonempty, closed under addition and closed under multiplication by all the elements of R.

Proposition 6. Let R be a ring and let I be an ideal of R. Then the (additive) quotient group R/I is a ring under the binary operations:

$$(r+I)+(s+I)=(r+s)+I \qquad \text{and} \qquad (r+I)\times(s+I)=(rs)+I$$

for all $r, s \in R$. Conversely, if I is any subgroup such that the above operations are well defined, then I is an ideal of R.

注意这是 additive quotient group，所以元素是 $\overline{r}=r+I$.

> Proof:
>
> The first statement: by the discussion above.
>
> Conversely, by proposition 5, $\forall \alpha \in I, r \in R, r\alpha \in I$ and $\alpha r \in I$, so I is an ideal.

Definition. When I is an ideal of R the ring R/I with the operations in the previous proposition is called the *quotient ring* of R by I.

R/I 中的 0 为 $0+I=I$, 1 为 $1+I$ （如果有的话）.

Theorem 7.

(1) *(The First Isomorphism Theorem for Rings)* If $\varphi : R \to S$ is a homomorphism of rings, then the kernel of φ is an ideal of R, the image of φ is a subring of S and $R/\ker\varphi$ is isomorphic as a ring to $\varphi(R)$.

(2) If I is any ideal of R, then the map

$$R \to R/I \qquad \text{defined by} \qquad r \mapsto r+I$$

is a surjective ring homomorphism with kernel I (this homomorphism is called the *natural projection* of R onto R/I). Thus every ideal is the kernel of a ring homomorphism and vice versa.

> Proof:
>
> (1) By the previous discussion, $\varphi(R)$ is a subring of S and $\ker\varphi$ is a substring of R.
>
> Since $\forall \alpha \in I, r \in R, r\alpha \in I$ and $\alpha r \in I$, I is an ideal of R.
>
> Let $\psi : r+I \mapsto \varphi(r)$. Clearly ψ is surjective. Then $\varphi(r)=\varphi(s) \Leftrightarrow \varphi(r-s)=0 \Leftrightarrow r-s \in I \Leftrightarrow r+I=s+I$, so it is well defined and surjective, hence it is a ring isomorphism.
>
> (2) If I is an ideal, then R/I is a ring and the map $\pi : r \to r+I$ is a group homomorphism with kernel I. Then, since
>
> $$\pi(rs)=(rs)+I=(r+I)(s+I)=\pi(r)\pi(s),$$
>
> it is a ring homomorphism.

Theorem 8. Let R be a ring.

(1) *(The Second Isomorphism Theorem for Rings)* Let A be a subring and let B be an ideal of R. Then $A+B=\{a+b \mid a \in A,\ b \in B\}$ is a subring of R, $A \cap B$ is an ideal of A and $(A+B)/B \cong A/(A \cap B)$.

(2) *(The Third Isomorphism Theorem for Rings)* Let I and J be ideals of R with $I \subseteq J$. Then J/I is an ideal of R/I and $(R/I)/(J/I) \cong R/J$.

(3) *(The Fourth or Lattice Isomorphism Theorem for Rings)* Let I be an ideal of R. The correspondence $A \leftrightarrow A/I$ is an inclusion preserving bijection between the set of subrings A of R that contain I and the set of subrings of R/I. Furthermore, A (a subring containing I) is an ideal of R if and only if A/I is an ideal of R/I.

> Sketch of proof:

> For each theorem, (for example, (2)), let $\varphi : R/I \to R/J$ by $r + I \mapsto r + J$ be the corresponding homomorphism. This map is multiplicative since
> $\varphi((r_1 + I)(r_2 + I)) = \varphi((r_1 r_2) + I = (r_1 r_2) + J = (r_1 + J)(r_2 + J) = \varphi(r_1 + J)\varphi(r_2 + J)$.

Definition. Let I and J be ideals of R.

(1) Define the *sum* of I and J by $I + J = \{a + b \mid a \in I,\ b \in J\}$.

(2) Define the *product* of I and J, denoted by IJ, to be the set of all finite sums of elements of the form ab with $a \in I$ and $b \in J$.

(3) For any $n \geq 1$, define the n^{th} *power* of I, denoted by I^n, to be the set consisting of all finite sums of elements of the form $a_1 a_2 \cdots a_n$ with $a_i \in I$ for all i. Equivalently, I^n is defined inductively by defining $I^1 = I$, and $I^n = II^{n-1}$ for $n = 2, 3, \ldots$.

$I + J$ is the smallest ideal containing both I and J.

IJ is an ideal contained in $I \cap J$. ($IJ \subseteq I, IJ \subseteq J$.) Note that it may that $I \nsubseteq IJ$ since it is not necessary to have $1 \notin J$. Also, in general, $IJ \neq \{ab \mid a \in I, b \in J\}$.

Example. Let $I = 6\mathbb{Z}$ and $J = 10\mathbb{Z}$, then $I + J = 2\mathbb{Z}$ and $IJ = 60\mathbb{Z} \neq 30\mathbb{Z} = I \cap J$.

7.4 Properties of Ideals

Through out this section R is a ring with identity $1 \neq 0$.

Definition. Let A be any subset of the ring R.

(1) Let (A) denote the smallest ideal of R containing A, called *the ideal generated by* A.

(2) Let RA denote the set of all finite sums of elements of the form ra with $r \in R$ and $a \in A$ i.e., $RA = \{r_1 a_1 + r_2 a_2 + \cdots + r_n a_n \mid r_i \in R,\ a_i \in A,\ n \in \mathbb{Z}^+\}$ (where the convention is $RA = 0$ if $A = \emptyset$).
Similarly, $AR = \{a_1 r_1 + a_2 r_2 + \cdots + a_n r_n \mid r_i \in R,\ a_i \in A,\ n \in \mathbb{Z}^+\}$ and $RAR = \{r_1 a_1 r_1' + r_2 a_2 r_2' + \cdots + r_n a_n r_n' \mid r_i, r_i' \in R,\ a_i \in A,\ n \in \mathbb{Z}^+\}$.

(3) An ideal generated by a single element is called a *principal ideal.*

(4) An ideal generated by a finite set is called a *finitely generated ideal.*

RA is a left ideal that contains A, and if I is the smallest left ideal containing A, then $RA \subseteq I$, (here we use RA but not rA because the left idea must be closed under addition), so RA is the smallest left ideal containing A. Similarly, AR is the smallest right ideal containing A, and RAR is the smallest ideal containing A.

Also note that $(a) + (b)$ is the smallest ideal containing (a) and (b), so it is the smallest ideal containing $\{a, b\}$.

When R is commutative, (a) is just the set of all R-multiples of a.

The formation of principal ideals in a commutative ring is a particularly simple way of creating ideals, similar to generating cyclic subgroups of a group. Notice that the element $b \in R$ belongs to the ideal (a) if and only if $b = ra$ for some $r \in R$, i.e., if and only if *b is a multiple of a* or, put another way, *a divides b in R*. Also, $b \in (a)$ if and only if $(b) \subseteq (a)$. Thus containment relations between ideals, in particular between principal ideals, is seen to capture some of the arithmetic of general commutative rings. Commutative rings in which all ideals are principal are among the easiest to study and these will play an important role in Chapters 8 and 9.

The ideal $(2, x)$ in $\mathbb{Z}[x]$ is *not* a principle ideal.

Proposition 9. Let I be an ideal of R.

(1) $I = R$ if and only if I contains a unit.

(2) Assume R is commutative. Then R is a field if and only if its only ideals are 0 and R.

> Proof:
>
> (1) If I contains a unit u that $uv = 1$, then $\forall r \in R, r = 1r = uvr = u(vr) \in I$.
>
> Conversely, if $I = R$, then I contains the unit 1.
>
> (2) If R is a field, then every non-zero element is a unit, so if $I \neq \{0\}$, then I contains a unit, so $I = R$.
>
> Conversely, if the only ideals are 0 and R, let u be any nonzero element in R. Then $(u) = R$, so there is some v that $uv = 1$, that is, u is a unit. Hence every nonzero element in R is a unit, so R is a field.

Corollary 10. If R is a field then any nonzero ring homomorphism from R into another ring is an injection.

> Proof:
>
> If R is a field, then $\ker \varphi \neq R$ since there is some element u that $\varphi(u) \neq 0$. Then $\ker \varphi = 0$ since $\ker \varphi$ is an ideal but R only has two ideals 0 and R. Hence R is injective.

Definition. An ideal M in an arbitrary ring S is called a *maximal ideal* if $M \neq S$ and the only ideals containing M are M and S.

Proposition 11. In a ring with identity every proper ideal is contained in a maximal ideal.

> Proof: omitted.

Proposition 12. Assume R is commutative. The ideal M is a maximal ideal if and only if the quotient ring R/M is a field.

> Proof:
>
> By the fourth isomorphism theorem, if there is an ideal $M \subset I \subset R$, then I/M is an ideal of R/M. However R/M is a field, so has only two ideals 0 and R/M, so either $I = M$ or $I = R$. Thus M is a maximal ideal.
>
> Conversely, if M is a maximal ideal, then R/M has only two ideals 0 and R, then R/M is a field.

Definition. Assume R is commutative. An ideal P is called a *prime ideal* if $P \neq R$ and whenever the product ab of two elements $a, b \in R$ is an element of P, then at least one of a and b is an element of P.

Proposition 13. Assume R is commutative. Then the ideal P is a prime ideal in R if and only if the quotient ring R/P is an integral domain.

> Proof:
>
> Let if P is a prime ideal, then $\forall a, b \in R, ab \in P \Rightarrow a \in P$ or $b \in P$. Then $\overline{a}\overline{b} = \overline{ab} \in \overline{0} \Rightarrow \overline{a} \in \overline{0}$ or $\overline{b} \in \overline{0}$, so there is no zero divisors.

Conversely, if R/P is an integral domain, then $\forall a, b \in R$, is $ab \in P$, then $\overline{a}\overline{b} \in \overline{0} \Rightarrow \overline{a} = \overline{0}$ or $\overline{b} = \overline{0}$, that is, $a \in P$ or $b \in P$.

Corollary 14. Assume R is commutative. Every maximal ideal of R is a prime ideal.

Proof:

It is straightforward that if I is a maximal ideal, than R/I is a field (so it is a integral domain), so I is a prime ideal.

7.5 Rings of Fractions

Throughout this section R is a commutative ring.

Theorem 15. Let R be a commutative ring. Let D be any nonempty subset of R that does not contain 0, does not contain any zero divisors and is closed under multiplication (i.e., $ab \in D$ for all $a, b \in D$). Then there is a commutative ring Q with 1 such that Q contains R as a subring and every element of D is a unit in Q. The ring Q has the following additional properties.

(1) every element of Q is of the form rd^{-1} for some $r \in R$ and $d \in D$. In particular, if $D = R - \{0\}$ then Q is a field.

(2) (uniqueness of Q) The ring Q is the "*smallest*" ring containing R in which all elements of D become units, in the following sense. Let S be any commutative ring with identity and let $\varphi : R \to S$ be any injective ring homomorphism such that $\varphi(d)$ is a unit in S for every $d \in D$. Then there is an injective homomorphism $\Phi : Q \to S$ such that $\Phi|_R = \varphi$. In other words, any ring containing an isomorphic copy of R in which all the elements of D become units must also contain an isomorphic copy of Q.

Proof:

(1)

Let $\mathcal{F} = \{(r, d) \mid r \in R,\ d \in D\}$ and define the relation $\sim$ on $\mathcal{F}$ by

$$(r, d) \sim (s, e) \qquad \text{if and only if} \qquad re = sd.$$

Then $\sim$ is a equivalence relation. Let the equivalence classes be

$$\frac{r}{d} = \{(a, b) \mid a \in R,\ b \in D \text{ and } rb = ad\}.$$

and define Q be the set of equivalence classes, and define addition and multiplication.

At last, we can verify that

(1) these operations are well defined (i.e., do not depend on the choice of representatives for the equivalence classes),

(2) Q is an abelian group under addition, where the additive identity is $\frac{0}{d}$ for any $d \in D$ and the additive inverse of $\frac{a}{d}$ is $\frac{-a}{d}$,

(3) multiplication is associative, distributive and commutative, and

(4) Q has an identity ($= \frac{d}{d}$ for any $d \in D$).

Hence Q is a field.

(2)

> Extend $\varphi : R \to S$ to $\Phi : Q \to S$ by $\Phi(rd^{-1}) = \varphi(r)\varphi(d)^{-1}$. (we can prove that this map is well defined. and Φ is a ring homomorphism). Finally, $rd^{-1} \in \ker \Phi \Rightarrow r \in \ker \Phi \cap R = \ker \varphi$. Since φ is injective, $\ker \varphi = 0$, so $\ker \Phi = 0$ and Φ is injective.

(another motivation: turn polynomial ring into polynomial field.

Definition. Let R, D and Q be as in Theorem 15.

(1) The ring Q is called the *ring of fractions* of D with respect to R and is denoted $D^{-1}R$.

(2) If R is an integral domain and $D = R - \{0\}$, Q is called the *field of fractions* or *quotient field* of R.

If A is a subset of a field F (for example, if A is a subring of F), then the intersection of all the subfields of F containing A is a subfield of F and is called the subfield *generated* by A. This subfield is the smallest subfield of F containing A (namely, any subfield of F containing A contains the subfield generated by A).

Corollary 16. Let R be an integral domain and let Q be the field of fractions of R. If a field F contains a subring R' isomorphic to R then the subfield of F generated by R' is isomorphic to Q.

> Proof: omitted.

7.6 The Chinese Remainder Theorem

Throughout this section we shall assume unless otherwise stated that all rings are commutative with an identity $1 \neq 0$.

Given an arbitrary collection of rings (not necessarily satisfying the conventions above), their *(ring) direct product* is defined to be their direct product as (abelian) groups made into a ring by defining multiplication componentwise. In particular, if R_1 and R_2 are two rings, we shall denote by $R_1 \times R_2$ their direct product (as rings), that is, the set of ordered pairs (r_1, r_2) with $r_1 \in R_1$ and $r_2 \in R_2$ where addition and multiplication are performed componentwise:

$$(r_1, r_2) + (s_1, s_2) = (r_1 + s_1, r_2 + s_2) \qquad \text{and} \qquad (r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2).$$

We note that a map φ from a ring R into a direct product ring is a homomorphism if and only if the induced maps into each of the components are homomorphisms.

Definition. The ideals A and B of the ring R are said to be *comaximal* if $A + B = R$.

Theorem 17. *(Chinese Remainder Theorem)* Let $A_1, A_2, \ldots, A_k$ be ideals in R. The map

$$R \to R/A_1 \times R/A_2 \times \cdots \times R/A_k \qquad \text{defined by} \qquad r \mapsto (r + A_1, r + A_2, \ldots, r + A_k)$$

is a ring homomorphism with kernel $A_1 \cap A_2 \cap \cdots \cap A_k$. If for each $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$ the ideals A_i and A_j are comaximal, then this map is surjective and $A_1 \cap A_2 \cap \cdots \cap A_k = A_1 A_2 \cdots A_k$, so

$$R/(A_1 A_2 \cdots A_k) = R/(A_1 \cap A_2 \cap \cdots \cap A_k) \cong R/A_1 \times R/A_2 \times \cdots \times R/A_k.$$

> Proof: we prove this theorem by mathematical induction.
>
> If $k = 2$, let $A = A_1$ and $B = A_2$.

> Define $\varphi : R \to R/A \times R/B$ by $\varphi(r) = (r + A, r + B)$. We can verify that φ is a ring homomorphism.
>
> .
>
> Since $A + B = R$, there are elements $x \in A$ and $y \in B$ that $x + y = 1$. Then $\varphi(x) = (0, 1)$ and $\varphi(y) = (1, 0)$ since $x + B = 1 - y + B = 1 + B$. If $r_1A + r_2B \in R/A \times R/B$, then we can verify that $\varphi(r_1x + r_2y) = r_1A + r_2B$. This shows that φ is surjective.
>
> Finally, if $A = A_1$ and $B = A_2 \cdots A_k$, by hypothesis, $\forall 2 \leq i \leq k, \exists x_i \in A_1, y_i \in A_i$ s.t. $x_i + y_i = 1$. Then $1 = (x_2 + y_2) \cdots (x_k + y_k) \in (A_1 + y_2) \cdots (A_1 + y_k) = A + (y_2 \cdots y_k) \subseteq A_1 + (A_2 \cdots A_k)$.
>
> (and then the induction part can use the above conclusion).

Corollary 18. Let n be a positive integer and let $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ be its factorization into powers of distinct primes. Then

$$\mathbb{Z}/n\mathbb{Z} \cong (\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z}) \times (\mathbb{Z}/p_2^{\alpha_2}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_k^{\alpha_k}\mathbb{Z}),$$

as rings, so in particular we have the following isomorphism of multiplicative groups:

$$(\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z})^\times \times (\mathbb{Z}/p_2^{\alpha_2}\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_k^{\alpha_k}\mathbb{Z})^\times.$$

> Proof:
>
> Coordinates of each unit in $\mathbb{Z}/n\mathbb{Z}$ is a unit in $\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z}$. (consider the inverse of each element / coordinate).

If we compare orders on the two sides of this last isomorphism, we obtain the formula

$$\varphi(n) = \varphi(p_1^{\alpha_1})\varphi(p_2^{\alpha_2}) \dots \varphi(p_k^{\alpha_k})$$

Corollary 18 is also a step toward a determination of the decomposition of the abelian group $(\mathbb{Z}/n\mathbb{Z})^\times$ into a direct product of cyclic groups. The complete structure is derived at the end of Section 9.5.

Chapter 8 Euclidean Domains, Principal Ideal Domains, and Unique Factorization Domains

All rings in this chapter are commutative.

8.1 Euclidean Domains

We first define the notion of a *norm* on an integral domain R. This is essentially no more than a measure of "size" in R.

Definition. Any function $N : R \to \mathbb{Z}^+ \cup \{0\}$ with $N(0) = 0$ is called a *norm* on the integral domain R. If $N(a) > 0$ for $a \neq 0$ define N to be a *positive norm*.

We observe that this notion of a norm is fairly weak and that it is possible for the same integral domain R to possess several different norms.

Definition. The integral domain R is said to be a *Euclidean Domain* (or possess a *Division Algorithm*) if there is a norm N on R such that for any two elements a and b of R with $b \neq 0$ there exist elements q and r in R with

$$a = qb + r \qquad \text{with } r = 0 \text{ or } N(r) < N(b).$$

The element q is called the *quotient* and the element r the *remainder* of the division.

The importance of the existence of a Division Algorithm on an integral domain R is that it allows a *Euclidean Algorithm* for two elements a and b of R.

The quotient and the remainder are not unique.

这个 norm 并不是 E.D. 的一部分.

Every field is a E.D.

$\mathbb{Z}$ is a E.D.

If F is a field, then $F[x]$ is a E.D. Otherwise not.

$\mathbb{Z}[i]$ is a E.D.

Proposition 1. Every ideal in a Euclidean Domain is principal. More precisely, if I is any nonzero ideal in the Euclidean Domain R then $I = (d)$, where d is any nonzero element of I of minimum norm.

> Proof:
>
> If $I = 0$, then there is nothing to prove.
>
> Otherwise, let d be any nonzero element of I of minimum norm. (such a d exists, because we can first choose an element $r \in I$, and continuously checking whether there is an element with norm $< N(r)$ for $N(r) - 1$ times.) Clearly $(d) \subseteq I$.
>
> Now for each element $a \in I$, use the division algorithm to obtain $a = qd + r$ with $r = 0$ or $N(r) < N(d)$. Since $N(d)$ is the minimum, r must be 0, so $a = qd \in (d)$, showing $I = (d)$.

Proposition 1 can also be used to prove that some integral domains R are not Euclidean Domains (with respect to any norm) by proving the existence of ideals of R that are not principal.

Note that $(2, x)$ is not principal in $\mathbb{Z}[x]$. (page 252, example (3) of section 7.4)

$(3, 2 + \sqrt{-5})$ is not principal in $\mathbb{Z}[-5]$.

Definition. Let R be a commutative ring and let $a, b \in R$ with $b \neq 0$.

(1) a is said to be a *multiple* of b if there exists an element $x \in R$ with $a = bx$. In this case b is said to *divide* a or be a *divisor* of a, written $b \mid a$.

(2) A *greatest common divisor* of a and b is a nonzero element d such that

(i) $d \mid a$ and $d \mid b$, and

(ii) if $d' \mid a$ and $d' \mid b$ then $d' \mid d$.

A greatest common divisor of a and b will be denoted by g.c.d.(a, b), or (abusing the notation) simply (a, b).

.

Note that $b \mid a$ in a ring R if and only if $a \in (b)$ if and only if $(a) \subseteq (b)$. In particular, if d is any divisor of both a and b then (d) must contain both a and b and hence must contain the ideal generated by a and b.

and

if I is the ideal of R generated by a and b, then d is a greatest common divisor of a and b if

(i) I is contained in the principal ideal (d), and

(ii) if (d') is any principal ideal containing I then $(d) \subseteq (d')$.

and $(d) = (a, b)$.

Proposition 2. If a and b are nonzero elements in the commutative ring R such that the ideal generated by a and b is a principal ideal (d), then d is a greatest common divisor of a and b.

> Proof:
>
> by the previous discussion.

Proposition 3. Let R be an integral domain. If two elements d and d' of R generate the same principal ideal, i.e., $(d) = (d')$, then $d' = ud$ for some unit u in R. In particular, if d and d' are both greatest common divisors of a and b, then $d' = ud$ for some unit u.

> Proof:
>
> It is clear if d or d' is zero. Now we assume both of them are nonzero.
>
> Since $d \in (d')$, $d = xd'$ for some x. Similarly $d' = yd$ for xome d. Thus $d = xyd$ and so $d(1 - xy) = 0$. Since $d \neq 0, xy = 1$, that is, both xy are units. (note that R is commutative.)
>
> The second assertion follows from $(d) = (a, b) = (d')$.

The converse is obvious: If $(d) = I$, then $(ud) = (d) = I$.

Theorem 4. Let R be a Euclidean Domain and let a and b be nonzero elements of R. Let $d = r_n$ be the last nonzero remainder in the Euclidean Algorithm for a and b described at the beginning of this chapter. Then

(1) d is a greatest common divisor of a and b, and

(2) the principal ideal (d) is the ideal generated by a and b. In particular, d can be written as an *R-linear combination* of a and b, i.e., there are elements x and y in R such that

$$d = ax + by.$$

> Proof:

> By proposition 1, every ideal is principle and so is (a, b).
>
> Let $d = r_n$, we will show that
>
> 1. $d \mid a$ and $d \mid b$, so $(a, b) \subseteq (d)$.
> 2. $d = ax + by$, so $(a, b) \subseteq (d)$.
>
> For (1), we can observe that every element r_i is a multiple of r_n, and so is a and b.
>
> For (2), we can use mathematical induction on i to construct $d = r_i x_i + r_{i+1} y_i$.

当 R 只是 P.I.D 而不是 E.D. 时候，只能保证 Theorem 4 也成立，但用不了 Euclidean Algorithm 了.

Proposition 5 omitted.

8.2 Principal Ideal Domains (P.I.D.s)

Definition. A *Principal Ideal Domain* (P.I.D.) is an integral domain in which every ideal is principal.

Proposition 1 proved that every E.D. is a P.I.D.. However, not every P.I.D. is a E.D..

Proposition 6. Let R be a Principal Ideal Domain and let a and b be nonzero elements of R. Let d be a generator for the principal ideal generated by a and b. Then

(1) d is a greatest common divisor of a and b

(2) d can be written as an *R-linear combination* of a and b, i.e., there are elements x and y in R with

$$d = ax + by$$

(3) d is unique up to multiplication by a unit of R.

This is just proposition 2 and 3.

Proposition 7. Every nonzero prime ideal in a Principal Ideal Domain is a maximal ideal.

> Proof:
>
> Let (p) be a nonzero prime ideal. Let $I = (m)$ be a ideal contatining (p). We will show that $I = (p)$ or $I = R$.
>
> Since $p \in (m)$, $p = rm$ for some r. Since (p) is a prime ideal and $rm \in (p)$, either $r \in (p)$ or $m \in (p)$.
>
> If $r \in (p)$, that is, $r = ps$, then $p = rm = psm$, so $sm = 1$ which implies m is a unit. Hence $I = (m) = R$.
>
> If $m \in (p)$, then $(m) \subseteq (p)$, so $I = (p)$.
>
> Hence (p) is maximal.

关键之处在于 $p \in (m)$ 能够写成 $p = rm$ 的形式.

Corollary 8. If R is any commutative ring such that the polynomial ring $R[x]$ is a Principal Ideal Domain (or a Euclidean Domain), then R is necessarily a field.

> Proof:
>
> Assume $R[x]$ is a P.I.D.. Then R is a integral domain.
>
> The ideal (x) is a nonzero prime ideal in $R[x]$ because $R[x]/(x) \cong R$.
>
> By proposition 7, (x) is a maximal, hence R is a field by proposition 7.4.12.

8.3 Unique Factorization Domains (U.F.D.s)

Definition. Let R be an integral domain.

(1) Suppose $r \in R$ is nonzero and is not a unit. Then r is called *irreducible* in R if whenever $r = ab$ with $a, b \in R$, at least one of a or b must be a unit in R. Otherwise r is said to be *reducible*.

(2) The nonzero element $p \in R$ is called *prime* in R if the ideal (p) generated by p is a prime ideal. In other words, a nonzero element p is a prime if it is not a unit and whenever $p \mid ab$ for any $a, b \in R$, then either $p \mid a$ or $p \mid b$.

(3) Two elements a and b of R differing by a unit are said to be *associate* in R (i.e., $a = ub$ for some unit u in R).

Proposition 10. In an integral domain a prime element is always irreducible.

> Proof:
>
> Suppose (p) is a nonzero prime ideal and $p = ab$. Then $ab = p \in (p)$. Suppose $a = pr \in (p)$. Then $p = ab = prb$, so $rb = 1$ and b is a unit. Hence p is irriducible.

Proposition 11. In a Principal Ideal Domain a nonzero element is a prime if and only if it is irreducible.

> Proof:
>
> By proposition 10, every prime is irriducible, and we only need to prove irriducibles are primes.
>
> If $M = (m)$ is a ideal containing (p), then $p = rm \in (m)$. Since p is irriducible, either r or m is a unit. This means either $(p) = (m)$ or $(m) = (1)$, respectively. Thus (p) is a maximal ideal, and therefore is a prime ideal.

efinition. A *Unique Factorization Domain* (U.F.D.) is an integral domain R in which 'ery nonzero element $r \in R$ which is not a unit has the following two properties:

(i) r can be written as a finite product of irreducibles p_i of R (not necessarily distinct): $r = p_1 p_2 \cdots p_n$ and

(ii) the decomposition in (i) is *unique up to associates*: namely, if $r = q_1 q_2 \cdots q_m$ is another factorization of r into irreducibles, then $m = n$ and there is some renumbering of the factors so that p_i is associate to q_i for $i = 1, 2, \ldots, n$.

(U.F.D. implies 不仅可以这样分解，而且分解还是唯一的.

Proposition 12. In a Unique Factorization Domain a nonzero element is a prime if and only if it is irreducible.

> Proof:
>
> By proposition 10, every prime is irriducible.
>
> Conversely, let p be an irriducible in R and assume $p \mid ab$, i.e., $ab = pc$.
>
> Write down the factorizatino of ab, since p is irriducible, there must be a factor in ab that is associate to p. Suppose $a = (up)p_2 \cdots p_n$, then $p \mid a$, completing the proof.

Proposition 13. Let a and b be two nonzero elements of the Unique Factorization Domain R and suppose

$$a = u p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n} \qquad \text{and} \qquad b = v p_1^{f_1} p_2^{f_2} \cdots p_n^{f_n}$$

are prime factorizations for a and b, where u and v are units, the primes $p_1, p_2, \ldots, p_n$ are *distinct* and the exponents e_i and f_i are ≥ 0. Then the element

$$d = p_1^{\min(e_1, f_1)} p_2^{\min(e_2, f_2)} \cdots p_n^{\min(e_n, f_n)}$$

(where $d = 1$ if all the exponents are 0) is a greatest common divisor of a and b.

> Proof: straightforward.

Theorem 14. Every Principal Ideal Domain is a Unique Factorization Domain. In particular, every Euclidean Domain is a Unique Factorization Domain.

> Proof: omitted.

Corollary 15. *(Fundamental Theorem of Arithmetic)* The integers $\mathbb{Z}$ are a Unique Factorization Domain.

> Proof:
>
> $\mathbb{Z}$ is a E.D., so

Factorization in the Gaussian Integers

Summary

In summary, we have the following inclusions among classes of commutative rings with identity:

$$\textit{fields} \subset \textit{Euclidean Domains} \subset \textit{P.I.D.s} \subset \textit{U.F.D.s} \subset \textit{integral domains}$$

All containments are proper:

$\mathbb{Z}$ is a Euclidean Domain, but not a field

$\mathbb{Z}[(1+\sqrt{-19})/2]$ is a P.I.D., but not a Euclidean Domain

$\mathbb{Z}[x]$ is a U.F.D., but not a P.I.D.

$\mathbb{Z}[\sqrt{-5}]$ is an integral domain, but not a U.F.D.

Chapter 9 Polynomial Rings

For convenience, the ring R will always be a commutative ring with identity $1 \neq 0$.

9.1 Definitions and Basic Properties

In this way, $R[x]$ is a commutative ring with identity (the identity 1 from R) in which we identify R with the subring of constant polynomials.

Proposition 1. Let R be an integral domain. Then
(1) degree $p(x)q(x) =$ degree $p(x) +$ degree $q(x)$ if $p(x), q(x)$ are nonzero
(2) the units of $R[x]$ are just the units of R
(3) $R[x]$ is an integral domain.

> this is proposition 7.2.4

Proposition 2. Let I be an ideal of the ring R and let $(I) = I[x]$ denote the ideal of $R[x]$ generated by I (the set of polynomials with coefficients in I). Then

$$R[x]/(I) \cong (R/I)[x].$$

In particular, if I is a prime ideal of R then (I) is a prime ideal of $R[x]$.

> Proof:
>
> There is a natural map given by reducing each of the coefficients of a polynomial module I. It is easy to prove that such map φ is a homomorphism.
>
> The kernel is precisely the set of polynomials each of whose coefficients is an element of I, which is to say that $\ker \varphi = I[x] = (I)$.
>
> Since I is a prime ideal, then R/I is a integral domain. Hence also $(R/I)[x]$ is an integral domain follows from proposition 1. It shows that $R[x]/(I)$ is an integral domain and (I) is a prime ideal.

Note that it is not true that if I is a maximal ideal of R then (I) is a maximal ideal of $R[x]$. However, if I is maximal in R then the ideal of $R[x]$ generated by I and x is maximal in $R[x]$.

($R[x]/(I, x) \cong R/I$, where the right one is a field since I is maximum.

Definition. The *polynomial ring in the variables* $x_1, x_2, \ldots, x_n$ *with coefficients in* R, denoted $R[x_1, x_2, \ldots, x_n]$, is defined inductively by

$$R[x_1, x_2, \ldots, x_n] = R[x_1, x_2, \ldots, x_{n-1}][x_n]$$

A polynomial is called *homogeneous* or aform if all its terms have the same degree.

9.2 Polynomial Rings over Fields I

We now consider more carefully the situation where the coefficient ring is a *field* F. We can define a *norm* on $F[x]$ by defining $N(p(x)) =$ degree of $p(x)$ (where we set

Theorem 3. Let F be a field. The polynomial ring $F[x]$ is a Euclidean Domain. Specifically, if $a(x)$ and $b(x)$ are two polynomials in $F[x]$ with $b(x)$ nonzero, then there are *unique* $q(x)$ and $r(x)$ in $F[x]$ such that

$$a(x) = q(x)b(x) + r(x) \qquad \text{with } r(x) = 0 \text{ or degree } r(x) < \text{degree } b(x) .$$

> Proof:
>
> If $a(x) = 0$, then we set $q(x) = r(x) = 0$.
>
> Otherwise, let $n = degree\ a(x), m = degree\ b(x)$. We may apply mathematical induction on n.

If $n < m$, then we set $q(x) = 0$ and $r(x) = a(x)$.

Otherwise, let $a_1(x) = a(x) - \frac{a_n}{b_m}x^{n-m}b(x)$, then $degree\ a_1(x) < n$. By the inducition hypothesis, $a_1(x) = q_1(x)b(x) + r(x)$. Letting $q(x) = q_1(x) + \frac{a_n}{b_m}x^{n-m}$, we have $a(x) = q(x)b(x) + r(x)$.

For the uniqueness, if $a(x) = q_1(x)b(x) + r_1(x) = q_2(x)b(x) + r_2(x)$, then $q_1(x)b(x) - q_2(x)b(x)$ has degree $< m$ if it is nonzero since F is an integral domain. However it is a multiple of $b(x)$, so it must be zero. Therefore $q_1(x) = q_2(x)$ and $r_1(x) = r_2(x)$.

Corollary 4. If F is a field, then $F[x]$ is a Principal Ideal Domain and a Unique Factorization Domain.

Proof: this is immediate from proposition 3.

We note that the quotient and remainder in the Division Algorithm applied to $a(x), b(x) \in F[x]$ are independent offield extensions in the following sense.

(mark: the g.c.d. of two polynomials is unique if we specify it to be monic.

9.3 Polynomial Rings that are Unique Factorization Domains

Proposition 5. *(Gauss' Lemma)* Let R be a Unique Factorization Domain with field of fractions F and let $p(x) \in R[x]$. If $p(x)$ is reducible in $F[x]$ then $p(x)$ is reducible in $R[x]$. More precisely, if $p(x) = A(x)B(x)$ for some nonconstant polynomials $A(x), B(x) \in F[x]$, then there are nonzero elements $r, s \in F$ such that $rA(x) = a(x)$ and $sB(x) = b(x)$ both lie in $R[x]$ and $p(x) = a(x)b(x)$ is a factorization in $R[x]$.

Proof:

Let $p(x) = A(x)B(x)$, multiplying both sides by the dominators, we obtain an equation $dp(x) = A_1(x)B_1(x)$ where each polynomial is in $R[x]$. Since $F[x]$ is a U.F.D., we can factorize all polynomials, and divide $A_1(x)$ and $B_1(x)$ by some irreducibles to obtain $p(x) = a(x)b(x)$.

More precisely, if $d = p_1p_2 \cdots p_n$, since p_1 is irreducible in R, p_1 is a prime, then by proposition 2, $p_1R[x]$ is prime and $(R/p_1R)[x] \cong R[x]/p_1[x]$ is an integral domain. Then one of the two polynomials, say $A_1(x)$, is a multiple of p_1. Hence $\frac{1}{p_1}A_1(x) \in R$ and we can cancel each irreducibles in this way to obtain $p(x) = a(x)b(x)$.

Corollary 6. Let R be a Unique Factorization Domain, let F be its field of fractions and let $p(x) \in R[x]$. Suppose the greatest common divisor of the coefficients of $p(x)$ is 1. Then $p(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. In particular, if $p(x)$ is a monic polynomial that is irreducible in $R[x]$, then $p(x)$ is irreducible in $F[x]$.

Proof:

By Gauss' Lemma, if $p(x)$ is reducible in $F[x]$ then it is reducible in $R[x]$.

Conversely, if $p(x) = a(x)b(x)$ in $R[x]$, then neither $a(x)$ nor $b(x)$ are constant polynomials (otherwise, let's say a is constant that is not a unit, then the g.c.d. of the coefficients will not be 1.), so the same factorization shows that $p(x)$ is reducible in $F[x]$.

Theorem 7. R is a Unique Factorization Domain if and only if $R[x]$ is a Unique Factorization Domain.

Proof:

If $R[x]$ is a U.F.D., then obviously R is a U.F.D..

Conversely, let $p(x) = dp_1(x)$, where d is the g.c.d. of the coefficients of $p(x)$, then we can uniquely factorizes d, so if we can uniquely factorize $p_1(x)$ then we can uniquely factorizes $p(x)$. Thus we may assume the g.c.d. of $p(x)$ is 1. We may further assume $degree\ p(x) > 0$.

Let F be the field of fractions of F.

By Gauss's Lemma, $p(x)$ can be factorizes into irreducibles in $F[x]$ and also in $R[x]$. Since the g.c.d. of coefficients of $p(x)$ is 1, the g.c.d. of coefficients of each factor is also 1. By Corollary 6, each factor is irriducible in $R[x]$. This shows that $p(x)$ can be factorized as a finite product of irreducibles in $R[x]$.

The uniqueness of the factorization follows from the uniqueness in $F[x]$. If there are two factorization in $F[x]$, say $p(x) = p_1(x) \cdots = q_1(x) \cdots$ where $p_1(x) = \frac{a}{b} q_1(x)$ and all coefficients of each polynomial has g.c.d. 1. In this case $bp_1(x) = aq_1(x)$, so consider the coefficients (dividing both sides with the g.c.d. of the coefficients), we have $a = ub$ for some unit $u \in R$. Hence $p_1(x)$ and $q_1(x)$ are associate in $R[x]$. This completing the proof.

Corollary 8. If R is a Unique Factorization Domain, then a polynomial ring in an arbitrary number of variables with coefficients in R is also a Unique Factorization Domain.

Proof: It is straightforward using mathematical induction on the number of variables.

9.4 Irreducibility Criteria

Proposition 9. Let F be a field and let $p(x) \in F[x]$. Then $p(x)$ has a factor of degree one if and only if $p(x)$ has a root in F, i.e., there is an $\alpha \in F$ with $p(\alpha) = 0$.

Proof:

If $p(x)$ has a factor of degree 1, then since F is a field, we may assume the factor is monic, i.e., is of the form $(x - \alpha)$, so $p(\alpha) = 0$.

Conversely, let $p(x) = q(x)(x - \alpha) + r$. Since $p(\alpha) = 0$, r must be 0, so $p(x)$ has $(x - \alpha)$ as a factor.

Proposition 10. A polynomial of degree two or three over a field F is reducible if and only if it has a root in F.

This follows from proposition 9, since a polynomial of degree two or three is reducible if and only if it has at least one linear factor.

Proposition 11. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a polynomial of degree n with integer coefficients. If $r/s \in \mathbb{Q}$ is in lowest terms (i.e., r and s are relatively prime integers) and r/s is a root of $p(x)$, then r divides the constant term and s divides the leading coefficient of $p(x)$: $r \mid a_0$ and $s \mid a_n$. In particular, if $p(x)$ is a *monic* polynomial with integer coefficients and $p(d) \neq 0$ for all integers d dividing the constant term of $p(x)$, then $p(x)$ has no roots in $\mathbb{Q}$.

Proof:

By hypothesis, $p(r/s) = 0 = a_n(r/s)^n + \cdots + a_0$. Multiplying through by s^n obtain $0 = a_n r^n + \cdots + a_0 s^n$. Thus $a_n r^n = s(-a_{n-1} r^{n-1} - \cdots - a_0 s^{n-1})$, so s divides $a_n r^n$. Since $\gcd(s, r) = 1$, it follows that $s \mid a_n$. Similarly, $r \mid a_0$.

Proposition 12. Let I be a proper ideal in the integral domain R and let $p(x)$ be a nonconstant monic polynomial in $R[x]$. If the image of $p(x)$ in $(R/I)[x]$ cannot be factored in $(R/I)[x]$ into two polynomials of smaller degree, then $p(x)$ is irreducible in $R[x]$.

Proof:

> If $p(x) = a(x)b(x)$ where $a(x)$ and $b(x)$ are monic, nonconstant polynomials in $R[x]$, then $\overline{p(x)} = \overline{a(x)}\,\overline{b(x)}$, so $\overline{p(x)}$ is reducible in $(R/I)[x]$.

(it can also be used in several variables, but some care must be exercised. For example, a non-unit element may be a unit in the quotient.

Proposition 13. *(Eisenstein's Criterion)* Let P be a prime ideal of the integral domain R and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial in $R[x]$ (here $n \geq 1$). Suppose $a_{n-1}, \ldots, a_1, a_0$ are all elements of P and suppose a_0 is not an element of P^2. Then $f(x)$ is irreducible in $R[x]$.

> Proof:
>
> Suppose $f(x) = a(x)b(x)$, where $a(x), b(x)$ are non constant polynomials.
>
> Reducing this equation modulo P, then $x^n = \overline{a(x)}\,\overline{b(x)}$. Since P is a prime ideal, R/P is a integral domain, so both $\overline{a(x)}$ and $\overline{b(x)}$ will have 0 constant term (otherwise we can find out that the left side will have at least two terms). Hence the constant terms of $a(x)$ and $b(x)$ are elements of P, and a_0 is an element of P^2, a contradiction.

Corollary 14. *(Eisenstein's Criterion for $\mathbb{Z}[x]$)* Let p be a prime in $\mathbb{Z}$ and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{Z}[x]$, $n \geq 1$. Suppose p divides a_i for all $i \in \{0, 1, \ldots, n-1\}$ but that p^2 does not divide a_0. Then $f(x)$ is irreducible in both $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$.

> Proof: p is a prime in $\mathbb{Z}$, so $f(x)$ is irreducible in $\mathbb{Z}[x]$. By Gauss' Lemma, $f(x)$ is also irreducible in $\mathbb{Q}[x]$.

9.5 Polynomial Rings over Fields II

Proposition 15. The maximal ideals in $F[x]$ are the ideals $(f(x))$ generated by irreducible polynomials $f(x)$. In particular, $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible.

> This follows from Proposition 8.2.7 and Proposition 8.3.11 that $F[x]$ is a P.I.D., and $(f(x))$ is a prime ideal.
>
> In particular, $f(x)$ is irreducible $\Leftrightarrow$ $(f(x))$ is prime $\Leftrightarrow$ $(f(x))$ is maximal $\Leftrightarrow$ $F[x]/(f(x))$ is a field.

Proposition 16. Let $g(x)$ be a nonconstant element of $F[x]$ and let

$$g(x) = f_1(x)^{n_1} f_2(x)^{n_2} \cdots f_k(x)^{n_k}$$

be its factorization into irreducibles, where the $f_i(x)$ are distinct. Then we have the following isomorphism of rings:

$$F[x]/(g(x)) \cong F[x]/(f_1(x)^{n_1}) \times F[x]/(f_2(x)^{n_2}) \times \cdots \times F[x]/(f_k(x)^{n_k}).$$

> Proof:
>
> if $f_i(x)$ and $f_j(x)$ are distinct, then they are coprime and the g.c.d. is 1. Hence they are comaximal.
>
> Then apply the CRT to get the conclusion.

The next result concerns the number of roots of a polynomial over a field F. By Proposition 9, a root α corresponds to a linear factor $(x-\alpha)$ of $f(x)$. If $f(x)$ is divisible by $(x-\alpha)^m$ but not by $(x-\alpha)^{m+1}$, then α is said to be a root of *multiplicity* m.

Proposition 17. If the polynomial $f(x)$ has roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ in F (not necessarily distinct), then $f(x)$ has $(x-\alpha_1)\cdots(x-\alpha_k)$ as a factor. In particular, a polynomial of degree n in one variable over a field F has at most n roots in F, even counted with multiplicity.

> Proof:
>
> The first statement follows by induction from Proposition 9.
>
> Since each linear factor is irreducible, the second statement follows since $F[x]$ is a U.F.D.. (if it has more roots, then the degree of $f(x)$ will be greater than n.)

Proposition 18. A finite subgroup of the multiplicative group of a field is cyclic. In particular, if F is a finite field, then the multiplicative group $F^\times$ of nonzero elements of F is a cyclic group.

> Proof:
>
> By the Fundamental Theorem,
>
> $$G \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \mathbb{Z}/n_k\mathbb{Z}.$$
>
> where $n_k \mid n_{k-1} \mid \cdots \mid n_2 \mid n_1$.
>
> In general, if H is a cyclic group and $d \mid |H|$, then H contains precisely d elements of order dividing d: $x^{\frac{|H|}{d}}, \ldots, x^{\frac{|H|}{d}(d-1)}$.
>
> Since $n_k \mid n_i$, it follows that each direct factor contains n_k elements of order dividing n_k. If $k > 1$, then the number of such elements will be more than n_k, but then there will be more than n_k roots or the polynomial $x^{n_k} - 1 = 0$ in the field F, contradicting Proposition 17.
>
> Hence $k = 1$ and the group is cyclic.

Corollary 19. Let p be a prime. The multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ of nonzero residue classes mod p is cyclic.

> This is the multiplicative group of the finite field $\mathbb{Z}/p\mathbb{Z}$.

Corollary 20. Let $n \geq 2$ be an integer with factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ in $\mathbb{Z}$, where $p_1, \ldots, p_r$ are distinct primes. We have the following isomorphisms of (multiplicative) groups:

(1) $(\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z})^\times \times (\mathbb{Z}/p_2^{\alpha_2}\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_r^{\alpha_r}\mathbb{Z})^\times$

(2) $(\mathbb{Z}/2^\alpha\mathbb{Z})^\times$ is the direct product of a cyclic group of order 2 and a cyclic group of order $2^{\alpha-2}$, for all $\alpha \geq 2$

(3) $(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ is a cyclic group of order $p^{\alpha-1}(p-1)$, for all odd primes p.

> Proof: omitted.

Part 3 Modules and Vector Spaces

modules are "similar" to rings

Chapter 10 Introduction to Module Theory

10.1 Basic Definitions and Examples

Definition. Let R be a ring (not necessarily commutative nor with 1). A *left R-module* or a *left module over R* is a set M together with

(1) a binary operation $+$ on M under which M is an abelian group, and

(2) an action of R on M (that is, a map $R \times M \to M$) denoted by rm, for all $r \in R$ and for all $m \in M$ which satisfies

(a) $(r+s)m = rm + sm$, for all $r, s \in R, m \in M$,

(b) $(rs)m = r(sm)$, for all $r, s \in R, m \in M$, and

(c) $r(m+n) = rm + rn$, for all $r \in R, m, n \in M$.

If the ring R has a 1 we impose the additional axiom:

(d) $1m = m$, for all $m \in M$.

Modules satisfying axiom 2(d) are called unital modules and in this book all our modules will be *unital* (this is to avoid "pathologies" such as having $rm = 0$ for all $r \in R$ and $m \in M$).

Modules over a field F and vector spaces (linear spaces) over F are the same.

(The module concept is a generalization of vector space.

Definition. Let R be a ring and let M be an R-module. An *R-submodule* of M is a subgroup N of M which is closed under the action of ring elements, i.e., $rn \in N$, for all $r \in R, n \in N$.

Example: $\mathbb{Z}$-modules are the same as abelian groups, and $\mathbb{Z}$-submodules are the same as subgroups.

Example: $F[x]$-modules.

Proposition 1. *(The Submodule Criterion)* Let R be a ring and let M be an R-module. A subset N of M is a submodule of M if and only if

(1) $N \neq \emptyset$, and

(2) $x + ry \in N$ for all $r \in R$ and for all $x, y \in N$.

> Proof:
>
> If N is a submodule, then $ry \in N$ and so $x + ry \in N$.
>
> Conversely, taking $r = -1$ shows that $x - y \in N$, so N is a subgroup of M.
>
> Taking $x = 0$ shows that $ry \in N$, so N is a submodule.

Definition. Let R be a commutative ring with identity. An *R-algebra* is a ring A with identity together with a ring homomorphism $f : R \to A$ mapping 1_R to 1_A such that the subring $f(R)$ of A is contained in the center of A.

(A has a natural left (and right) (unital) R-module structure defined by $r \cdot a = f(r)a$. (and we assume the action is this one.

Definition. If A and B are two R-algebras, an *R-algebra homomorphism* (or isomorphism) is a ring homomorphism (isomorphism, respectively) $\varphi : A \to B$ mapping 1_A to 1_B such that $\varphi(r \cdot a) = r \cdot \varphi(a)$ for all $r \in R$ and $a \in A$.

Eg. any ring with identity is a $\mathbb{Z}$-algebra (let $f(n) = 1 + 1 + \cdots + 1$).

Suppose that A is an R-algebra. Then A is a ring with identity that is a (unital) left R-module satisfying $r \cdot (ab) = (r \cdot a)b = a(r \cdot b)$ for all $r \in R$ and $a, b \in A$ (these are all equal to the product $f(r)ab$ in the ring A—recall that $f(R)$ is contained in the center of A). Conversely, these conditions on a ring A define an R-algebra, and are sometimes used as the definition of an R-algebra (cf. Exercise 22).

(here $f(r) = r \cdot 1_A$.

10.2 Quotient Modules and Module Homomorphisms

Definition. Let R be a ring and let M and N be R-modules.

(1) A map $\varphi : M \to N$ is an *R-module homomorphism* if it respects the R-module structures of M and N, i.e.,
- **(a)** $\varphi(x + y) = \varphi(x) + \varphi(y)$, for all $x, y \in M$ and
- **(b)** $\varphi(rx) = r\varphi(x)$, for all $r \in R$, $x \in M$.

(2) An R-module homomorphism is an *isomorphism (of R-modules)* if it is both injective and surjective. The modules M and N are said to be *isomorphic*, denoted $M \cong N$, if there is some R-module isomorphism $\varphi : M \to N$.

(3) If $\varphi : M \to N$ is an R-module homomorphism, let $\ker \varphi = \{m \in M \mid \varphi(m) = 0\}$ (the *kernel* of φ) and let $\varphi(M) = \{n \in N \mid n = \varphi(m) \text{ for some } m \in M\}$ (the *image* of φ, as usual).

(4) Let M and N be R-modules and define $\mathrm{Hom}_R(M, N)$ to be the set of all R-module homomorphisms from M into N.

there is a tiny difference between module homomorphism and the ring homomorphism: (1)(b).

also, a R-module homomorphism need not to be a ring homomorphism, and vice versa.

$\mathbb{Z}$-module homomorphisms are the same as abelian group homomorphisms.

Proposition 2. Let M, N and L be R-modules.

(1) A map $\varphi : M \to N$ is an R-module homomorphism if and only if $\varphi(rx + y) = r\varphi(x) + \varphi(y)$ for all $x, y \in M$ and all $r \in R$.

(2) Let φ, ψ be elements of $\mathrm{Hom}_R(M, N)$. Define $\varphi + \psi$ by

$$(\varphi + \psi)(m) = \varphi(m) + \psi(m) \qquad \text{for all } m \in M.$$

Then $\varphi + \psi \in \mathrm{Hom}_R(M, N)$ and with this operation $\mathrm{Hom}_R(M, N)$ is an abelian group. If R is a commutative ring then for $r \in R$ define $r\varphi$ by

$$(r\varphi)(m) = r(\varphi(m)) \qquad \text{for all } m \in M.$$

Then $r\varphi \in \mathrm{Hom}_R(M, N)$ and with this action of the commutative ring R the abelian group $\mathrm{Hom}_R(M, N)$ is an R-module.

(3) If $\varphi \in \mathrm{Hom}_R(L, M)$ and $\psi \in \mathrm{Hom}_R(M, N)$ then $\psi \circ \varphi \in \mathrm{Hom}_R(L, N)$.

(4) With addition as above and multiplication defined as function composition, $\mathrm{Hom}_R(M, M)$ is a ring with 1. When R is commutative $\mathrm{Hom}_R(M, M)$ is an R-algebra.

> Proof:
>
> (1) Take $r = 1$ to see that $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $y = 0$ to verify that $\varphi(rx) = r\varphi(x)$.
>
> The converse is obvious.
>
> (2) It is straightforward to check.
>
> The commutativity of R is used to verify the second axiom of R-module homomorphism, namely

> $(r_1\varphi)(r_2m) = r_1\varphi(r_2m) = r_1r_2\varphi(m) = r_2r_1\varphi(m) = r_2(r_1\varphi(m)) = r_2((r_1\varphi)m).$
>
> (3) it is straightforward to check.
>
> (4) It is straightforward to check.
>
> If R is commutative, then (2) shows that the ring $Hom_R(M, M)$ is an R-module.
>
> (Define $\varphi r = r\varphi$, 不需要这个吧? 直接按照 R-algebra 的定义也可以证明) by exercise 10.1.22, $Hom_R(M, M)$ is an R-algebra.

In (2), The domain M could in fact be any set - it does not have to be an R-module nor an abelian group

Definition. The ring $\mathrm{Hom}_R(M, M)$ is called the *endomorphism ring of* M and will often be denoted by $\mathrm{End}_R(M)$, or just $\mathrm{End}(M)$ when the ring R is clear from the context. Elements of $\mathrm{End}(M)$ are called *endomorphisms*.

Proposition 3. Let R be a ring, let M be an R-module and let N be a submodule of M. The (additive, abelian) quotient group M/N can be made into an R-module by defining an action of elements of R by

$$r(x + N) = (rx) + N, \qquad \text{for all } r \in R,\ x + N \in M/N.$$

The natural projection map $\pi : M \to M/N$ defined by $\pi(x) = x + N$ is an R-module homomorphism with kernel N.

> Proof:
>
> Since M is an abelian group, the quotient group M/N is defined and it is an abelian group.
>
> To see that the action is well defined, suppose $x + N = y + N$, then $(x - y) \in N$. Since N is a (left) R-submodule, $r(x - y) \in N$, and thus $(rx) + N = (ry) + N$.
>
> We can check that M/N follows the R-module axioms by definition.
>
> The kernel of π is, in the same sense as quotient groups, N.
>
> It remains only to show π is an R-module homomorphism. This can be done by verify the axioms. For example,
>
> $$\pi(rm) = rm + N = r(m + N) = r\pi(m).$$

All the isomorphism theorems stated for groups also hold for R-modules. The proofs are similar to aboves that we can show the group homomorphism are also R-module homomorphisms.

Definition. Let A, B be submodules of the R-module M. The *sum* of A and B is the set

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

One can easily check that the sum oftwo submodules A and B is a submodule and is the smallest submodule which contains both A and B.

Theorem 4. (Isomorphism Theorems)

(1) *(The First Isomorphism Theorem for Modules)* Let M, N be R-modules and let $\varphi : M \to N$ be an R-module homomorphism. Then $\ker \varphi$ is a submodule of M and $M/\ker \varphi \cong \varphi(M)$.

(2) *(The Second Isomorphism Theorem)* Let A, B be submodules of the R-module M. Then $(A + B)/B \cong A/(A \cap B)$.

(3) *(The Third Isomorphism Theorem)* Let M be an R-module, and let A and B be submodules of M with $A \subseteq B$. Then $(M/A)/(B/A) \cong M/B$.

(4) *(The Fourth or Lattice Isomorphism Theorem)* Let N be a submodule of the R-module M. There is a bijection between the submodules of M which contain N and the submodules of M/N. The correspondence is given by $A \leftrightarrow A/N$, for all $A \supseteq N$. This correspondence commutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N).

10.3 Generation of Modules, Direct Sums, and Free Modules

Let R be a ring with 1. As in the preceding sections the term "module" will mean "left module."

Definition. Let M be an R-module and let $N_1, \ldots, N_n$ be submodules of M.

(1) The *sum* of $N_1, \ldots, N_n$ is the set of all finite sums of elements from the sets N_i: $\{a_1 + a_2 + \cdots + a_n \mid a_i \in N_i \text{ for all } i\}$. Denote this sum by $N_1 + \cdots + N_n$.

(2) For any subset A of M let

$$RA = \{r_1a_1 + r_2a_2 + \cdots + r_ma_m \mid r_1, \ldots, r_m \in R,\ a_1, \ldots, a_m \in A,\ m \in \mathbb{Z}^+\}$$

(where by convention $RA = \{0\}$ if $A = \emptyset$). If A is the finite set $\{a_1, a_2, \ldots, a_n\}$ we shall write $Ra_1 + Ra_2 + \cdots + Ra_n$ for RA. Call RA the *submodule of M generated by A*. If N is a submodule of M (possibly $N = M$) and $N = RA$, for some subset A of M, we call A a *set of generators* or *generating set* for N, and we say N *is generated by* A.

(3) A submodule N of M (possibly $N = M$) is *finitely generated* if there is some finite subset A of M such that $N = RA$, that is, if N is generated by some finite subset.

(4) A submodule N of M (possibly $N = M$) is *cyclic* if there exists an element $a \in M$ such that $N = Ra$, that is, if N is generated by one element:

$$N = Ra = \{ra \mid r \in R\}.$$

Definition. Let $M_1, \ldots, M_k$ be a collection of R-modules. The collection of k-tuples $(m_1, m_2, \ldots, m_k)$ where $m_i \in M_i$ with addition and action of R defined componentwise is called the *direct product* of $M_1, \ldots, M_k$, denoted $M_1 \times \cdots \times M_k$.

It is evident that the direct product of a collection of R-modules is again an R-module. The direct product of $M_1, \ldots, M_k$ is also referred to as the *(external) direct sum* of $M_1, \ldots, M_k$ and denoted $M_1 \oplus \cdots \oplus M_k$. The direct product and direct sum of an infinite number of modules (which are different in general) are defined in Exercise 20.

Proposition 5. Let $N_1, N_2, \ldots, N_k$ be submodules of the R-module M. Then the following are equivalent:

(1) The map $\pi : N_1 \times N_2 \times \cdots \times N_k \to N_1 + N_2 + \cdots + N_k$ defined by

$$\pi(a_1, a_2, \ldots, a_k) = a_1 + a_2 + \cdots + a_k$$

is an isomorphism (of R-modules): $N_1 + N_2 + \cdots + N_k \cong N_1 \times N_2 \times \cdots \times N_k$.

(2) $N_j \cap (N_1 + N_2 + \cdots + N_{j-1} + N_{j+1} + \cdots + N_k) = 0$ for all $j \in \{1, 2, \ldots, k\}$.

(3) Every $x \in N_1 + \cdots + N_k$ can be written *uniquely* in the form $a_1 + a_2 + \cdots + a_k$ with $a_i \in N_i$.

> Proof:
>
> 1 -> 2:
>
> Suppose for some j that (2) fails to hold, and let $a_j \in N_j \cap (N_1 + \cdots + N_{j-1} + N_{j+1} + N_k)$ with $a_j \neq 0$.
>
> Then $a_j = a_1 + \cdots + a_{j-1} + a_{j+1} + \cdots + a_k$ for some $a_i \in N_i$.
>
> Hence $\pi(a_1, \ldots, a_{j-1}, -a_j, a_{j+1}, \ldots, a_k) = 0$, but the preimage is nonzero, a contradiciton.
>
> .
>
> 2 -> 3:
>
> If $a_1 + \cdots + a_k = b_1 + \cdots + b_k$, for each $1 \leq j \leq k$,
> $(a_j - b_j) = -(a_1 - b_1 + \cdots + a_{j-1} - b_{j-1} + a_{j+1} - b_{j+1} + \cdots + a_k - b_k) \in N_j \cap (N_1 + \cdots + N_{j-1} + N_{j+1} + \cdots + N_k)$
> ., so $a_j - b_j = 0$. Hence the representation is unique.
>
> .
>
> 3 -> 1: straightforward.

If an R-module $M = N_1 + N_2 + \cdots + N_k$ is the sum of submodules $N_1, N_2, \ldots, N_k$ of M satisfying the equivalent conditions of the proposition above, then M is said to be the *(internal) direct sum* of $N_1, N_2, \ldots, N_k$, written

$$M = N_1 \oplus N_2 \oplus \cdots \oplus N_k.$$

Part 4 Field Theory and Galois Theory

Chapter 13 Field Theory

13.1 Basic Properties of Field Extensions

Recall that a field F is a commutative ring with identity in which every nonzero element has an inverse. Equivalently, the set $F^\times = F - \{0\}$ of nonzero elements of F is an abelian group under multiplication.

Definition. The *characteristic* of a field F, denoted $\mathrm{ch}(F)$, is defined to be the smallest positive integer p such that $p \cdot 1_F = 0$ if such a p exists and is defined to be 0 otherwise.

This notation makes sense also for any integral domain (and its characteristic will be the same as for its field of fractions).

It is easy to see that

$$n \cdot 1_F + m \cdot 1_F = (m+n) \cdot 1_F \qquad \text{and that}$$
$$(n \cdot 1_F)(m \cdot 1_F) = mn \cdot 1_F$$

Hence, if $n = ab$ is composite with $n \cdot 1_F = 0$, then $ab \cdot 1_F = (a \cdot 1_F)(b \cdot 1_F) = 0$. Since there is no zero divisor, one of $a \cdot 1_F$ or $b \cdot 1_F$ is 0, so the smallest such integer is necessarily a prime. It also follows that if $n \cdot 1_F = 0$, then n is divisible by p.

Proposition 1. The characteristic of a field F, $\mathrm{ch}(F)$, is either 0 or a prime p. If $\mathrm{ch}(F) = p$ then for any $\alpha \in F$,

$$p \cdot \alpha = \underbrace{\alpha + \alpha + \cdots + \alpha}_{p \text{ times}} = 0.$$

> Proof:
>
> The first statement is from the above discussion.
>
> For the second statement, $p \cdot \alpha = p \cdot (1_F \alpha) = (p \cdot 1_F)\alpha = 0$.

Examples:

- $ch(\mathbb{Q}) = ch(\mathbb{R}) = ch(\mathbb{Z}) = 0$.
- $ch(\mathbb{F}_p) = p$.
- The integral domain $\mathbb{F}_p[x]$ of polynomials in the variable x with coefficients in the field $\mathbb{F}_p$ has characteristic p, as does its field of fractions $\mathbb{F}_p(x)$ (the field of rational functions in x with coefficients in $\mathbb{F}_p$).

If we define $(-n) \cdot 1_F = -(n \cdot 1_F)$ for positive n and $0 \cdot 1_F = 0$, then we have a natural ring homomorphism (by equation (1))

$$\varphi : \mathbb{Z} \longrightarrow F$$
$$n \longmapsto n \cdot 1_F$$

and we can interpret the characteristic of F by noting that $\ker(\varphi) = \mathrm{ch}(F)\mathbb{Z}$. Taking the quotient by the kernel gives us an *injection* of either $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$ into F (depending on whether $\mathrm{ch}(F) = 0$ or $\mathrm{ch}(F) = p$). Since F is a field, we see that F contains a subfield isomorphic either to $\mathbb{Q}$ (the field of fractions of $\mathbb{Z}$) or to $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (the field of fractions of $\mathbb{Z}/p\mathbb{Z}$) depending on the characteristic of F, and in either case is the smallest subfield of F containing 1_F (the field *generated* by 1_F in F).

(field generated by a subset: addition, substraction, multiplication and division.

Definition. The *prime subfield* of a field F is the subfield of F generated by the multiplicative identity 1_F of F. It is (isomorphic to) either $\mathbb{Q}$ (if $\mathrm{ch}(F) = 0$) or $\mathbb{F}_p$ (if $\mathrm{ch}(F) = p$).

Examples:

(1) The prime subfield of both $\mathbb{Q}$ and $\mathbb{R}$ is $\mathbb{Q}$.
(2) The prime subfield of the field $\mathbb{F}_p(x)$ is isomorphic to $\mathbb{F}_p$, given by the constant polynomials.

Definition. If K is a field containing the subfield F, then K is said to be an *extension field* (or simply an *extension*) of F, denoted K/F or by the diagram

$$\begin{array}{c} K \\ | \\ F \end{array}$$

In particular, every field F is an extension of its prime subfield. The field F is sometimes called the *base field* of the extension.

The notation K/F for a field extension is a shorthand for "K over F" and is not the quotient of K by F.

If K/F is any extension of fields, then the multiplication defined in K makes K into a *vector space* over F. In particular every field F can be considered as a vector space over its prime field.

Definition. The *degree* (or *relative degree* or *index*) of a field extension K/F, denoted $[K : F]$, is the dimension of K as a vector space over F (i.e., $[K : F] = \dim_F K$). The extension is said to be *finite* if $[K : F]$ is finite and is said to be *infinite* otherwise.

(是把整个 K 当成一个 vector space 而不是商群！！！

Proposition 2. Let $\varphi : F \to F'$ be a homomorphism of fields. Then φ is either identically 0 or is injective, so that the image of φ is either 0 or isomorphic to F.

> This is Corollary 10 of Chapter 7.

Theorem 3. Let F be a field and let $p(x) \in F[x]$ be an irreducible polynomial. Then there exists a field K containing an isomorphic copy of F in which $p(x)$ has a root. Identifying F with this isomorphic copy shows that there exists an extension of F in which $p(x)$ has a root.

> Proof:
>
> Consider the quotient
>
> $$K = F[x]/(p(x))$$
>
> of the polynomial ring $F[x]$ by the ideal generated by p(x). Since $p(x)$ is irreducible in the P.I.D. $F[x]$, K is maximal and hence is a field.
>
> Taking the canonical projection π of $F[x]$ to the quotient $F[x]/(p(x))$ restricted to $F \subset F[x]$ gives a homomorphism $\varphi = \pi_F : F \to K$ which is not identically 0 since $\varphi(1) = \pi(1) = 1$. Hence by the proposition above, $\varphi(F) \cong F$ is an isomorphic copy of F contained in K.
>
> Then
>
> $$p(\pi(x)) = \pi(p(x)) = p(x) \bmod p(x) = 0,$$
>
> so K contains a root of $p(x)$, completing the proof.

If $F = \mathbb{R}, p(x) = x^2 + 1$, then $K = \mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$, where the root of $p(x)$ is $\pi(x) \mapsto i$.

Theorem 4. Let $p(x) \in F[x]$ be an irreducible polynomial of degree n over the field F and let K be the field $F[x]/(p(x))$. Let $\theta = x \bmod (p(x)) \in K$. Then the elements

$$1, \theta, \theta^2, \ldots, \theta^{n-1}$$

are a basis for K as a vector space over F, so the degree of the extension is n, i.e., $[K : F] = n$. Hence

$$K = \{a_0 + a_1\theta + a_2\theta^2 + \cdots + a_{n-1}\theta^{n-1} \mid a_0, a_1, \ldots, a_{n-1} \in F\}$$

consists of all polynomials of degree $< n$ in θ.

> Proof: For any $a(x) \in F[x]$, let $a(x) = q(x)p(x) + r(x)$, so $a(x) \equiv r(x) \pmod{p(x)}$. Hence every polynomial in $F[x]$ is represented by a polynomial of degree less than n. Hence the images $1, \theta, \ldots, \theta^{n-1}$ of $1, x, \ldots, x^{n-1}$ span the quotient as a vector space over F.
>
> To see this elements are linearly independent, suppose $\sum_{i=0}^{n-1} b_i\theta^i = 0$. Then in $F[x]$, $p(x)$ divides $\sum_{i=0}^{n-1} b_i\theta^i$. However the right part has degree less than n, so it is impossible (for not all $b_i = 0$.
>
> Hence $[K : F] = n$ by definition.

Note that $p(x)$ should be irreducible. (important!!!

Corollary 5. Let K be as in Theorem 4, and let $a(\theta), b(\theta) \in K$ be two polynomials of degree $< n$ in θ. Then addition in K is defined simply by usual polynomial addition and multiplication in K is defined by

$$a(\theta)b(\theta) = r(\theta)$$

where $r(x)$ is the remainder (of degree $< n$) obtained after dividing the polynomial $a(x)b(x)$ by $p(x)$ in $F[x]$.

By the results above, all polynomials $q(x)$ of degree $< n$ can divide by nonzero elements as well using the Euclidean Algorithm (i.e., multiply their inverses module p(x)), since $p(x)$ is invertible, so $\gcd(p(x), q(x)) = 1$. (e.g. 计应数 HW8 Problem 2(3).)

Definition. Let K be an extension of the field F and let $\alpha, \beta, \cdots \in K$ be a collection of elements of K. Then the smallest subfield of K containing both F and the elements $\alpha, \beta, \ldots$, denoted $F(\alpha, \beta, \ldots)$ is called the field *generated by* $\alpha, \beta, \ldots$ *over* F.

Definition. If the field K is generated by a single element α over F, $K = F(\alpha)$, then K is said to be a *simple* extension of F and the element α is called a *primitive element* for the extension.

$F(\alpha)$ 应该是 $\sum_i a_i\alpha^i$ 了.

Theorem 6. Let F be a field and let $p(x) \in F[x]$ be an irreducible polynomial. Suppose K is an extension field of F containing a root α of $p(x)$: $p(\alpha) = 0$. Let $F(\alpha)$ denote the subfield of K generated over F by α. Then

$$F(\alpha) \cong F[x]/(p(x)).$$

> Proof:
>
> In the case of $p(x)$ has degree 1, it is obviously true, since both sides are F.
>
> Otherwise:

> There is a natural homomorphism $\varphi : F[x] \to F(\alpha)$ by $\varphi(a(x)) = a(\alpha)$ (by mapping F to F and x to α then extend it.)
>
> Since $p(\alpha) = 0$, $(p(x)) \subseteq \ker \varphi$, so we can construct another mapping $\psi : F[x]/(p(x)) \to F(\alpha)$ induced by the previous one.
>
> (it is easy to verify that ψ is well defined: $\psi(a(x) + q(x)p(x)) = \psi(a(x))$.
>
> Since $p(x)$ is irreducible (so $F[x]/(p(x))$ is a field), and ψ is not identically zero, so $\ker \psi = \{0\}$ and ψ is injective. Also $\psi(x) = \alpha$ and $\psi(F) = F$ are in the image of ψ, so ψ is surjective.
>
> Hence ψ is an isomorphism.

(那如果两个不同的 $p_1(x)$和 $p_2(x)$ 有同一个根 α 呢？这里是先扩展出 K 再选 α 的，前面这种情况会认为是不同的根.

(以及方程的另一个根可能不在 $F(\alpha)$ 内？好像也没有另一个根这一说。。。根的个数都是不知道的。。。

(例如，方程 $x^3 - 2 = 0$ 在 $\mathbb{Q}$ 内没有解，在 $\mathbb{R}$ 内有一个解，在 $\mathbb{C}$ 内有三个解. 但它们的 $F(\alpha)$ 都同构于 $\mathbb{Q}[x]/(x^3 - 2)$.

但本质上这三个解无法区分，只是我们人为地去把它们区分开了. (换句话说，是我们先用某种方式定义了 $\mathbb{R}$，再用 $\mathbb{R}$ 去构造一个解，表现为第一个解.

Different roots of the same irreducible polynomial have the same algebraic properties

Corollary 7. Suppose in Theorem 6 that $p(x)$ is of degree n. Then

$$F(\alpha) = \{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1} \mid a_0, a_1, \ldots, a_{n-1} \in F\} \subseteq K.$$

> Proof: conbine Theorem 6 and Corollary 5.

Theorem 8. Let $\varphi : F \xrightarrow{\sim} F'$ be an isomorphism of fields. Let $p(x) \in F[x]$ be an irreducible polynomial and let $p'(x) \in F'[x]$ be the irreducible polynomial obtained by applying the map φ to the coefficients of $p(x)$. Let α be a root of $p(x)$ (in some extension of F) and let β be a root of $p'(x)$ (in some extension of F'). Then there is an isomorphism

$$\begin{aligned} \sigma : F(\alpha) &\xrightarrow{\sim} F'(\beta) \\ \alpha &\longmapsto \beta \end{aligned}$$

mapping α to β and extending φ, i.e., such that σ restricted to F is the isomorphism φ.

> Proof:
>
> It is straightforward (to verify axioms and use propositions) to find out that
>
> $$F(\alpha) \cong F[x]/(p(x)) \cong F'[x]/(p'(x)) \cong F'(\beta).$$

This extension theorem will be of considerable use when we consider Galois Theory later. It can be represented pictorially by the diagram

$$\begin{array}{ccc} \sigma: & F(\alpha) & \xrightarrow{\sim} & F'(\beta) \\ & | & & | \\ \varphi: & F & \xrightarrow{\sim} & F' \end{array}$$

13.2 Algecraic Extensions

Let F be a field and let K be an extension of F. (remind that $F \subseteq K$).

Definition. The element $\alpha \in K$ is said to be *algebraic* over F if α is a root of some nonzero polynomial $f(x) \in F[x]$. If α is not algebraic over F (i.e., is not the root of any nonzero polynomial with coefficients in F) then α is said to be *transcendental* over F. The extension K/F is said to be *algebraic* if every element of K is algebraic over F.

Note that if α is algebraic over a field F then it is algebraic over any extension field L of F (if $f(x)$ having α as a root has coefficients in F then it also has coefficients in L).

Proposition 9. Let α be algebraic over F. Then there is a unique monic irreducible polynomial $m_{\alpha,F}(x) \in F[x]$ which has α as a root. A polynomial $f(x) \in F[x]$ has α as a root if and only if $m_{\alpha,F}(x)$ divides $f(x)$ in $F[x]$.

> Proof:
>
> Let $g(x) \in F[x]$ be a polynomial of minimal degree having α as a root. We may suppose $g(x)$ is monic.
>
> Suppose $g(x) = a(x)b(x)$ is reducible, then since $g(\alpha) = a(\alpha)b(\alpha) = 0$, one of $a(\alpha)$ or $b(\alpha)$ is 0, contradict to $g(x)$ is of minimial degree. Thus $g(x)$ is irreducible.
>
> If $f(x)$ is another polynomial having α as a root, by the Euclidean algorithm, $f(x) = q(x)g(x) + r(x)$, so $f(\alpha) = q(\alpha)g(\alpha) + r(\alpha) = 0$, which implies $r(\alpha) = 0$. Since $g(x)$ is of minimial degree, $r(x)$ must be 0, so $g(x) \mid f(x)$.
>
> Letting $m_{\alpha,F} = g(x)$ completes the proof.

为什么 $g(x)$ 一定存在呢？（因为 α is algebraic. Then by definition....

另外我们也可以证明 different monic irreducible polynomials have no common roots：如果有相同的 root α，那么这两个多项式都是 α 的最小多项式，矛盾.

Corollary 10. If L/F is an extension of fields and α is algebraic over both F and L, then $m_{\alpha,L}(x)$ divides $m_{\alpha,F}(x)$ in $L[x]$.

> Proof:
>
> $m_{\alpha,F}(\alpha) = 0$ in L, so by Proposition 9, $m_{\alpha,L}(x) \mid m_{\alpha,F}(x)$.
>
> However, $m_{\alpha,L}(\alpha)$ may not equal to 0.

E.g. $F = \mathbb{Q}, L = \mathbb{R}, \alpha = \sqrt{2}, m_{\alpha,F} = x^2 - 2, m_{\alpha,L} = x - \sqrt{2}$, so $(x - \sqrt{2}) \mid (x^2 - 2)$.

Definition. The polynomial $m_{\alpha,F}(x)$ (or just $m_\alpha(x)$ if the field F is understood) in Proposition 9 is called the *minimal polynomial* for α over F. The *degree* of $m_\alpha(x)$ is called the *degree* of α.

Proposition 11. Let α be algebraic over the field F and let $F(\alpha)$ be the field generated by α over F. Then

$$F(\alpha) \cong F[x]/(m_\alpha(x))$$

so that in particular

$$[F(\alpha) : F] = \deg\ m_\alpha(x) = \deg \alpha,$$

i.e., the degree of α over F is the degree of the extension it generates over F.

> Proof:

> This follows immediately from theorem 6.

为什么只有 $p(x) = m_\alpha(x)$ 的情况下刚好能除出来 $F(\alpha)$? 如果更大的话，$m_\alpha(x) \mid p(x)$，$p(x)$ 就不是 irreducible 的了.

Proposition 12. The element α is algebraic over F if and only if the simple extension $F(\alpha)/F$ is finite. More precisely, if α is an element of an extension of degree n over F then α satisfies a polynomial of degree at most n over F and if α satisfies a polynomial of degree n over F then the degree of $F(\alpha)$ over F is at most n.

> Proof:
>
> If α is algebraic over F, then the extension $F(\alpha) \cong F[x]/(m_\alpha(x))$ has degree deg $m_\alpha(x)$ which is finite.
>
> Conversely, suppose α is an element of an extension of degree n over F, then the $n+1$ elements $1, \alpha, \ldots, \alpha^n$ are linearly dependent over F. Hence α is the root of some equation $b_0 + b_1 x + \cdots + b_n x^n = 0$ with $b_i \in F$.
>
> The following two statements are obvious.

Corollary 13. If the extension K/F is finite, then it is algebraic.

> Proof:
>
> For $\alpha \in K$, $F(\alpha)$ is a subfield of K, so it is finite. By Proposition 12, α is algebraic. Hence K is algebraic.

The converse is not true (see below).

example: quadratic extensions

Theorem 14. Let $F \subseteq K \subseteq L$ be fields. Then

$$[L : F] = [L : K][K : F],$$

i.e. extension degrees are multiplicative, where if one side of the equation is infinite, the other side is also infinite. Pictorially,

> Proof:
>
> If $\alpha_1, \ldots, \alpha_m$ is a basis for L over K and $\beta_1, \ldots, \beta_n$ is a basis for K over F, then for each element $l \in L$, $l = \sum_{i=1}^m a_i \alpha_i = \sum_{i=1}^m (\sum_{j=1}^n b_j \beta_j)\alpha_i$. Hence $\{\alpha_i \beta_j\}$ spans L over F.
>
> Next, for $l = 0$, a_i must be 0 since $\{\alpha_i\}$ is a basis, ans similarly b_j must be 0 for all i and j.

Corollary 15. Suppose L/F is a finite extension and let K be any subfield of L containing F, $F \subseteq K \subseteq L$. Then $[K : F]$ divides $[L : F]$.

> Proof:
>
> This is immediate from Theorem 14.

Example: taking $F = \mathbb{Q}, K = \mathbb{Q}(\sqrt{2}), L = \mathbb{Q}(\sqrt[6]{2})$, then $[L : F] = 6, [L : K] = 2, [K : F] = 3$. Note that the minimal polynomial for $\sqrt[6]{2}$ in L/K is $x^3 - \sqrt{2}$, so it is dependent on K (in the same way? as the number of cosets)

Definition. An extension K/F is *finitely generated* if there are elements $\alpha_1, \alpha_2, \ldots, \alpha_k$ in K such that $K = F(\alpha_1, \alpha_2, \ldots, \alpha_k)$.

Lemma 16. $F(\alpha, \beta) = (F(\alpha))(\beta)$, i.e., the field generated over F by α and β is the field generated by β over the field $F(\alpha)$ generated by α.

> Proof:
>
> The field $F(\alpha, \beta)$ contains F, α, so it contains $F(\alpha)$. Also it contains β, so it contains $(F(\alpha))(\beta)$.
>
> Since $(F(\alpha))(\beta)$ contains F, α, β, so it contains $F(\alpha, \beta)$.
>
> Hence $F(\alpha, \beta) = (F(\alpha))(\beta)$.

By this lemma, we can generated the field $F(\alpha_1, \ldots)$ recursively.

Theorem 17. The extension K/F is finite if and only if K is generated by a finite number of algebraic elements over F. More precisely, a field generated over F by a finite number of algebraic elements of degrees $n_1, n_2, \ldots, n_k$ is algebraic of degree $\leq n_1 n_2 \cdots n_k$.

> Proof:
>
> If K/F is finite, let $\alpha_1, \ldots, \alpha_k$ be a basis for K. Then K is generated by $\alpha_1, \ldots, \alpha_k$.
>
> Conversely, $[F(\alpha_1, \ldots, \alpha_{i+1}) : F(\alpha_1, \ldots, \alpha_i)] = [(F(\alpha_1, \ldots, \alpha_i))(\alpha_{i+1}) : F(\alpha_1, \ldots, \alpha_i)] \leq [F(\alpha_{i+1}) : F]$, so $[F(\alpha_1, \ldots, \alpha_k) : F] \leq n_1 n_2 \cdots n_k$.

Corollary 18. Suppose α and β are algebraic over F. Then $\alpha \pm \beta, \alpha\beta, \alpha/\beta$ (for $\beta \neq 0$), (in particular α^{-1} for $\alpha \neq 0$) are all algebraic.

> Proof:
>
> All the elements are in the extension $F(\alpha, \beta)$, which is finite by Theorem 17, so by Corollary 13, they are all algebraic.

Corollary 19. Let L/F be an arbitrary extension. Then the collection of elements of L that are algebraic over F form a subfield K of L.

> Proof:
>
> This is immediately from Corollary 18.

Theorem 20. If K is algebraic over F and L is algebraic over K, then L is algebraic over F.

> Proof:
>
> Let α be any element over L. Then $\sum_{i=0}^{n} a_i \alpha^i = 0$ for some $a_i \in K$.
>
> Consider
>
> $$[F(\alpha, a_0, \ldots, a_n) : F] = [F(\alpha, a_0, \ldots, a_n) : F(a_0, \ldots, a_n)][F(a_0, \ldots, a_n) : F].$$
>
> First first term is at most n since α is algebraic over $F(a_0, \ldots, a_n)$, since the minimal polynomial of α is a divisor of the polynomial above, and $[F(\alpha, a_0, \ldots, a_n) : F(a_0, \ldots, a_n)] = \deg(m_\alpha(x))$ by corollary 7.
>
> For the second term, a_i is algebraic over F, so a_i has finite degree. By Theorem 17, $[F(a_0, \ldots, a_n) : F] \leq$ the product of degrees of a_i's is also finite.
>
> Hence $F(\alpha, a_0, \ldots, a_n)/F$ is finite, so α has finite degree. Therefore α is algebraic over F.
>
> Hence L is algebraic over F.

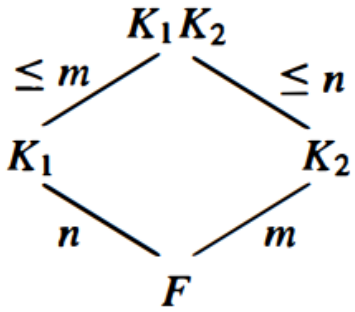
Definition. Let K_1 and K_2 be two subfields of a field K. Then the *composite field* of K_1 and K_2, denoted K_1K_2, is the smallest subfield of K containing both K_1 and K_2. Similarly, the composite of any collection of subfields of K is the smallest subfield containing all the subfields.

Proposition 21. Let K_1 and K_2 be two finite extensions of a field F contained in K. Then

$$[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$$

with equality if and only if an F-basis for one of the fields remains linearly independent over the other field. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_m$ are bases for K_1 and K_2 over F, respectively, then the elements $\alpha_i\beta_j$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ span K_1K_2 over F.

> Proof:
>
> $\{\alpha_i\beta_j\}$ 能够 generates 所有 K_1 中的元素（考虑 1 怎么用 K_2 中的元素表示出来的，再乘上这个元素在 K_1 中的表示）和 K_2 中的元素，并且这个集合 generates 出的元素对乘法和加法封闭. 所以就能够 generate 所有元素.
>
> From $K_1K_2 = F(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) = K_1(\beta_1, \ldots, \beta_m)$ we see that $\beta_1, \ldots, \beta_m$ spans K_1K_2 over K_1. Hence $[K_1K_2 : K_1] \leq m = [K_2 : F]$ with equality if these elements are linearly independent over K_1.
>
> Then $[K_1K_2 : F] = [K_1K_2 : K_1][K_1 : F] \leq nm$.



Corollary 22. Suppose that $[K_1 : F] = n$, $[K_2 : F] = m$ in Proposition 21, where n and m are relatively prime: $(n, m) = 1$. Then $[K_1K_2 : F] = [K_1 : F][K_2 : F] = nm$.

> Proof:
>
> Since K_1 and K_2 are subfields of K_1K_2, $[K_1K_2 : F]$ is divisible by both n, m, so it is divisible by $lcm(n, m) = nm$. By proposition 11, $[K_1K_2 : F] \leq nm$, so it is equal to nm.

13.4 Splitting Fields and Algebraic Closures

Definition. The extension field K of F is called a *splitting field* for the polynomial $f(x) \in F[x]$ if $f(x)$ factors completely into linear factors (or *splits completely*) in $K[x]$ and $f(x)$ does not factor completely into linear factors over any proper subfield of K containing F.

Theorem 25. For any field F, if $f(x) \in F[x]$ then there exists an extension K of F which is a splitting field for $f(x)$.

> Proof:
>
> 考虑对 $n = deg\ f$ 归纳：每次选择一个 degree >=2 的 irreducible factor $g(x)$，扩充出一个域 E/F 使其包含 $g(x)$ 的一个 root α. 不断重复该过程.

书上说最后要取所有 E 中包含 F 和所有 α_i 的子域的交，有这个必要吗?

这个的意思是说对于一个多项式 $f(x)$ 的 splitting field.

Definition. If K is an algebraic extension of F which is the splitting field over F for a collection of polynomials $f(x) \in F[x]$ then K is called a *normal* extension of F.

We shall generally use the term "splitting field" rather than "normal extension" (cf. also Section 14.9).

这个是指一些多项式的 splitting field.

为什么要强调 aigebraic extension?

这个条件等价于

Let K be a finite extension of F. Prove that K is a splitting field over F if and only if every irreducible polynomial in $F[x]$ that has a root in K splits completely in $K[x]$. [Use Theorems 8 and 27.]

Proof:

1

Let K be a finite extension of F. Suppose K is a splitting field over F for $f(x)$.

Let $p(x)$ be an irreducible polynomial with a root α in K. Let β also be a root of $p(x)$. By theorem 8, we have an isomorphism:

$$\sigma : F(\alpha) \to F(\beta)$$

Recall that $K = K(\alpha)$. Now by theorem 27, the isomorphism σ above extends to an isomorphism φ such that

$$\varphi : K(\alpha) \longrightarrow K(\beta) \iff \varphi : K \longrightarrow K(\beta)$$

Since φ is an isomorphism, the fields $K(\beta)/F$ and K/F have the same degree over F. By the product formula

$$[K(\beta) : K][K : F] = [K(\beta) : F]$$

showing that $[K(\beta) : K] = 1$ and so $K(\beta) = K$.

In other words, if one root α of an irreducible polynomial is in K, then so is every other root β i.e. $p(x)$ splits completely, which is exactly what we wanted.

$K(\alpha)$ 是 $f(x)$ over $F(\alpha)$ 的 splitting field：如果 $f(x)$ 在 $K(\alpha)$ 的一个 contains $F(\alpha)$ 的 proper subfield H 中也能 splits，那么 $f(x)$ 在 $K - H$ 中也会有至少一个根（否则会在 $K \cap H$ 中 splits，与 K 是 over F 的 splitting field 矛盾），加上 H 中的 n 个根就有 $> n$ 个根了，与 n 次多项式最多有 n 个根矛盾.（这个性质好像经常用得到）

$K(\beta)$ 类似.

2

Let K be a finite extension of F and suppose that every irreducible polynomial with a root in K splits completely.

This means that it is generated with finitely many (algebraic) elements over F, call them $g_1, \ldots, g_n$. Let $m_i(x)$ be the minimal polynomial for g_i. The minimal polynomial for g_i will split completely in K (since it is irreducible and has a root in K). Considering the product $m_1(x) \ldots m_n(x)$, we see that K contains the splitting field of this polynomial. On the other hand, the splitting field will contain the generators of K, and therefore K.

We conclude that K is a splitting field.

Proposition 26. A splitting field of a polynomial of degree n over F is of degree at most $n!$ over F.

> Proof:
>
> 每次把一个 root a_i 加进来，degree 最多会变大 $deg\ f_i$. 并且加入一个 root 之后 f 的度会变小. 乘起来就 $\leq n(n-1)\cdots 1 = n!$.

Definition. A generator of the cyclic group of all n^{th} roots of unity is called a *primitive* n^{th} root of unity.

Definition. The field $\mathbb{Q}(\zeta_n)$ is called the *cyclotomic field of* n^{th} *roots of unity.*

Theorem 27. Let $\varphi : F \xrightarrow{\sim} F'$ be an isomorphism of fields. Let $f(x) \in F[x]$ be a polynomial and let $f'(x) \in F'[x]$ be the polynomial obtained by applying φ to the coefficients of $f(x)$. Let E be a splitting field for $f(x)$ over F and let E' be a splitting field for $f'(x)$ over F'. Then the isomorphism φ extends to an isomorphism $\sigma : E \xrightarrow{\sim} E'$, i.e., σ restricted to F is the isomorphism φ:

$$\begin{array}{ccc} \sigma: & E & \xrightarrow{\sim} & E' \\ & | & & | \\ \varphi: & F & \xrightarrow{\sim} & F' \end{array}$$

> Proof:
>
> 每次选出一个 degree ≥ 2 的 irreducible polynomial，以及它的一个 root α，然后把 F 扩展成 $F(\alpha)$ 并把 F' 扩展成 $F'(\beta)$（β 为这个 polynomial 对应到 F' 后的一个 root）.
>
> 然后令 $f_1(x) = f(x)/(x-\alpha) \in F(\alpha)[x]$，$f_1'$ 也类似.
>
> Then apply the induction hypothesis 就能得到一个 $E \cong E'$ 和对应的 σ.
>
> 并且，$f_1(x)$ 的各个 roots 都是 $f(x)$ 的 roots，所以会在 E 当中. 如果有 E 的 proper subfield K 包含了 $F(\alpha)$ 和 $f_1(x)$ 的所有 roots，那么 K 也会包含所有 $f(x)$ 的 roots，与 E 是最小的满足要求的 field 矛盾.
>
> $$\begin{array}{ccc} \sigma: & E & \xrightarrow{\sim} & E' \\ & | & & | \\ \sigma': & F_1 & \xrightarrow{\sim} & F_1' \\ & | & & | \\ \varphi: & F & \xrightarrow{\sim} & F'. \end{array}$$

φ 只 apply to coefficients.

Corollary 28. *(Uniqueness of Splitting Fields)* Any two splitting fields for a polynomial $f(x) \in F[x]$ over a field F are isomorphic.

Definition. The field $\overline{F}$ is called an *algebraic closure* of F if $\overline{F}$ is algebraic over F and if every polynomial $f(x) \in F[x]$ splits completely over $\overline{F}$ (so that $\overline{F}$ can be said to contain all the elements algebraic over F).

每个 $\overline{F}$ 里的元素都是某个多项式 $f(x) \in F[x]$ 的根，且每个多项式 $f(x) \in F[x]$ 的根都在 $\overline{F}$ 当中。

$\overline{F}$ 也是最小的、满足每个 $f(x) \in F[x]$ splits 的域：如果 $K \supset \overline{F}$ 也满足这两点，那么令 $g \in K - \overline{F}$，g 的 minimal polynomial $m_{g,F}$ 就不在 $\overline{F}$ 中 split，矛盾. Convserse is also true: 如果 K 是 F 的 algebraic closure，那么 K 和 $\overline{F}$ 都是最小的、满足每个 $f(x) \in F[x]$ splits 的域，所以它们相等.（如果只是同构而不相等，那么 either 它们的交也满足 $f(x)$ splits，与 K 最小矛盾，or 它们的交不包含某个 $f(x)$ 的所有根，那么 $K - K \cap \overline{F}$ 和 $\overline{F} - K \cap \overline{F}$ 都会有一些根，与 $f(x)$ 最多有 n 个根矛盾.

$\overline{F}$ 也是最大的、F 的 algebraic extension：对于一个 F 的 algebraic extension L，那么 L 中每个元素 α 的 minimal polynomial 的所有根都在 K 中（也包含 α），所以 $L \subseteq K$. Converse 和上面的类似：注意两个 algebraic extension 的并仍然"是" algebraic extension （即使它不是 field）.

Definition. A field K is said to be *algebraically closed* if every polynomial with coefficients in K has a root in K.

对于一个 $f(x)$，可以每次除掉 $f(x)$ 的一个 root α 的 linear factor $(x - \alpha)$，所以 $f(x)$ 就能 split 了.

并且 K is algebraically closed 当且仅当 $K = \overline{K}$.

Proposition 29. Let $\overline{F}$ be an algebraic closure of F. Then $\overline{F}$ is algebraically closed.

> Proof:
>
> Let $f(x)$ be a polynomial in $\overline{F}[x]$ and let α be a root of $f(x)$. Then α generates $\overline{F}(\alpha)$ which is algebraic over $\overline{F}$, and $\overline{F}$ is algebraic over α, so α is algebraic over α by theorem 20. Then α is a root of its minimal polynomial $m_{\alpha,F}(x) \in F[x]$, so its root α is in $\overline{F}$. Hence $\overline{F}$ is algebraically closed.

Proposition 30. For any field F there exists an algebraically closed field K containing F.

> Proof: omitted.

Proposition 31. Let K be an algebraically closed field and let F be a subfield of K. Then the collection of elements $\overline{F}$ of K that are algebraic over F is an algebraic closure of F. An algebraic closure of F is unique up to isomorphism.

> Proof:
>
> by definition, $\overline{F}$ is an algebraic extension over F.
>
> Every $f(x) \in F[x]$ splits over K, and each root α is algebraic over F, so $\overline{F}$ contains all the roots.

Theorem. *(Fundamental Theorem of Algebra)* The field $\mathbb{C}$ is algebraically closed.

> Proof: omitted.

Corollary 32. The field $\mathbb{C}$ contains an algebraic closure for any of its subfields. In particular, $\overline{\mathbb{Q}}$, the collection of complex numbers algebraic over $\mathbb{Q}$, is an algebraic closure of $\mathbb{Q}$.

> Proof: directly by proposition 31.

Let F be a field and let $f(x) \in F[x]$ be a polynomial. Over a splitting field for $f(x)$ we have the factorization

$$f(x) = (x-\alpha_1)^{n_1}(x-\alpha_2)^{n_2}\cdots(x-\alpha_k)^{n_k}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are distinct elements of the splitting field and $n_i \geq 1$ for all i. Recall that α_i is called a *multiple* root if $n_i > 1$ and is called a *simple* root if $n_i = 1$. The integer n_i is called the *multiplicity* of the root α_i.

Definition. A polynomial over F is called *separable* if it has no multiple roots (i.e., all its roots are distinct). A polynomial which is not separable is called *inseparable*.

Definition. The *derivative* of the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in F[x]$$

is defined to be the polynomial

$$D_x f(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1 \in F[x].$$

$$D_x(f(x)+g(x)) = D_x f(x) + D_x g(x)$$
$$D_x(f(x)g(x)) = f(x)D_x g(x) + (D_x f(x))g(x).$$

Proposition 33. A polynomial $f(x)$ has a multiple root α if and only if α is also a root of $D_x f(x)$, i.e., $f(x)$ and $D_x f(x)$ are both divisible by the minimal polynomial for α. In particular, $f(x)$ is separable if and only if it is relatively prime to its derivative: $(f(x), D_x f(x)) = 1$.

Proof:

If $f(x) = (x-a)^n g(x)$ with $n \geq 2$, then $D_x f(x) = n(x-a)^{n-1}g(x) + (x-a)^n D_x g(x)$, which is a multiple of $x-a$, so a is a root of $D_x f(x)$.

Conversely, if a is a root of $f(x)$, write $f(x) = (x-a)h(x)$. Then $D_x f(x) = h(x) + (x-a)D_x h(x)$. Substituting $x = a$ yields $h(a) = 0$, so $h(x) = (x-a)h_1(x)$ and $f(x)$ has a multiple root a.

The equivalence with divisibility by the minimal polynomial for a follows from Proposition 9: if a is a root of a polynomial $g(x)$, then $m_a(x) \mid g(x)$.

The last statement is then clear (let a denote any root of a common factor of $f(x)$ and $D_x f(x)$).

Corollary 34. Every irreducible polynomial over a field of characteristic 0 (for example, $\mathbb{Q}$) is separable. A polynomial over such a field is separable if and only if it is the product of distinct irreducible polynomials.

Proof:

For any irreducible $p(x) \in F[x]$ of degree n, the derivative $D_x p(x)$ is of degree $n-1$. However $p(x)$ only has factors 1 and $p(x)$, so $(p(x), D_x p(x)) = 1$. Hence $p(x)$ is separable.

The second statement follows from proposition 9 that different irreducible polynomials do not have zeros in common.

The proof fails if F has characteristic p that $D_x p(x)$ may be 0. In this case, every term in $p(x)$ must be the form $a_i x^{pi}$, so $p(x) = q(x^p)$ for some q. (**)

Proposition 35. Let F be a field of characteristic p. Then for any $a, b \in F$,

$$(a+b)^p = a^p + b^p, \quad \text{and} \quad (ab)^p = a^p b^p.$$

Put another way, the p^{th}-power map defined by $\varphi(a) = a^p$ is an injective field homomorphism from F to F.

> Proof:
>
> The first statement follows from the Binomial Theorem.
>
> The second statement is trivial (follows from the associativity).
>
> Also, since F is a field, then any homomorphism (including φ) should be identically zero (obviously not) or injective.

Definition. The map in Proposition 35 is called the *Frobenius endomorphism* of F.

Corollary 36. Suppose that $\mathbb{F}$ is a finite field of characteristic p. Then every element of $\mathbb{F}$ is a p^{th} power in $\mathbb{F}$ (notationally, $\mathbb{F} = \mathbb{F}^p$).

> Proof: The Frobenius endomorphism is injective and the domain equals codomain, so it is surjective.

Proposition 37. Every irreducible polynomial over a finite field $\mathbb{F}$ is separable. A polynomial in $\mathbb{F}[x]$ is separable if and only if it is the product of distinct irreducible polynomials in $\mathbb{F}[x]$.

> Proof:
>
> If $p(x)$ is irreducible but inseparable, by discussion after corollary 34, $p(x) = q(x^p)$ for some $q(x)$.
>
> Also $a_i = b_i^p$ for some b_i. Then
>
> $$\begin{aligned} p(x) &= q(x^p) \\ &= a_m (x^p)^m + \cdots + a_1 x^p + a_0 \\ &= b_m^p (x^p)^m + \cdots + b_1^m x^p + b_0^m \\ &= (b_m x^m + \cdots + b_1 x + b_0)^p, \end{aligned}$$
>
> which contradicts to $p(x)$ is irreducible.
>
> Hence $p(x)$ is separable.
>
> The second statement is similar to corollary 34.

The important part of the proof is the fact that every element in the characteristic p field $\mathbb{F}$ is a p^{th} power in $\mathbb{F}$. This suggest the following definition:

Definition. A field K of characteristic p is called *perfect* if every element of K is a p^{th} power in K, i.e., $K = K^p$. Any field of characteristic 0 is also called perfect.

With this definition, we see that we have proved that every irreducible polynomial over a perfect field is separable.

Definition. The field K is said to be *separable* (or *separably algebraic*) over F if every element of K is the root of a separable polynomial over F (equivalently, the minimal polynomial over F of every element of K is separable). A field which is not separable is *inseparable*.

Corollary 39. Every finite extension of a perfect field is separable. In particular, every finite extension of either $\mathbb{Q}$ or a finite field is separable.

> Proof:
>
> Finite extension K/F => K is algebraic => minimal polynomial of any $a \in K$ is irreducible => minimal polynomial is separable since F is perfect.

此外，algebraic extension of a perfect field is also separable.

13.6 Cyclotomic Polynomials and Extensions

The purpose of this section is to prove that the cyclotomic extension

$$\mathbb{Q}(\zeta_n)/\mathbb{Q}$$

generated by the n^{th} roots of unity over $\mathbb{Q}$ introduced in Section 4 is of degree $\varphi(n)$ where φ denotes Euler's phi-function (= the number of integers a, $1 \le a < n$ relatively prime to n = the order of the group $(\mathbb{Z}/n\mathbb{Z})^\times$).

Definition. Let μ_n denote the *group of n^{th} roots of unity over* $\mathbb{Q}$.

Definition. Define the n^{th} *cyclotomic polynomial* $\Phi_n(x)$ to be the polynomial whose roots are the primitive n^{th} roots of unity:

$$\Phi_n(x) = \prod_{\zeta \text{ primitive } \in \mu_n} (x - \zeta) = \prod_{\substack{1 \le a < n \\ (a,n)=1}} (x - \zeta_n{}^a)$$

(which is of degree $\varphi(n)$).

The roots of the polynomial $x^n - 1$ are precisely the n^{th} roots of unity so we have the factorization

$$x^n - 1 = \prod_{\substack{\zeta^n = 1 \\ \text{i.e. } \zeta \in \mu_n}} (x - \zeta).$$

If we group together the factors $(x - \zeta)$ where ζ is an element of order d in μ_n (i.e., ζ is a primitive d^{th} root of unity) we obtain

$$x^n - 1 = \prod_{d|n} \prod_{\substack{\zeta \in \mu_d \\ \zeta \text{ primitive}}} (x - \zeta).$$

The inner product is $\Phi_d(x)$ by definition so we have the factorization

$$x^n - 1 = \prod_{d|n} \Phi_d(x). \tag{13.4}$$

This factorization allows us to compute $\Phi_n(x)$ for any n recursively: clearly $\Phi_1(x) = x - 1$ and $\Phi_2(x) = x + 1$. Then

$$x^3 - 1 = \Phi_1(x)\Phi_3(x) = (x - 1)\Phi_3(x)$$

which gives

$$\Phi_3(x) = x^2 + x + 1.$$

Similarly

$$x^4 - 1 = \Phi_1(x)\Phi_2(x)\Phi_4(x) = (x - 1)(x + 1)\Phi_4(x)$$

gives

$$\Phi_4(x) = x^2 + 1$$

(in these cases these could also be obtained directly from the explicit roots of unity). Continuing in this fashion we can compute $\Phi_n(x)$ for any n. Note also that for p a prime we recover our polynomial

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1.$$

Lemma 40. The cyclotomic polynomial $\Phi_n(x)$ is a monic polynomial in $\mathbb{Z}[x]$ of degree $\varphi(n)$.

> Proof:
>
> It is clear that $\Phi_n(x)$ is monic and has degree $\varphi(n)$.
>
> To shows that $\Phi_n(x) \in \mathbb{Z}[x]$, we want to use mathematical induction.
>
> Let $x^n - 1 = f(x)\Phi_n(x)$, where $f(x) = \prod_{d|n, d<n} \Phi_d(x)$. By Euclidean algorithm, $\Phi_n(x) = (x^n - 1) \bmod f(x)$ which lies in $\mathbb{Q}[x]$ since both $x^n - 1$ and $f(x)$ are in $\mathbb{Z}[x]$ and $\mathbb{Z}[x]$ is an E.d..
>
> Then by Gauss' Lemma, $\Phi_n(x) \in \mathbb{Z}[x]$.

Theorem 41. The cyclotomic polynomial $\Phi_n(x)$ is an irreducible monic polynomial in $\mathbb{Z}[x]$ of degree $\varphi(n)$.

Corollary 42. The degree over $\mathbb{Q}$ of the cyclotomic field of n^{th} roots of unity is $\varphi(n)$:

$$[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n).$$

> Proof:
>
> By Theorem 41, $\Phi_n(x)$ is the minimal polynomial for any n^{th} root of unity ξ_n.

Chapter 14 Galois Theory

14.1 Basic Definitions

Let K be a field.

Definition.

(1) An isomorphism σ of K with itself is called an *automorphism* of K. The collection of automorphisms of K is denoted $\mathrm{Aut}(K)$. If $\alpha \in K$ we shall write $\sigma\alpha$ for $\sigma(\alpha)$.

(2) An automorphism $\sigma \in \mathrm{Aut}(K)$ is said to *fix* an element $\alpha \in K$ if $\sigma\alpha = \alpha$. If F is a subset of K (for example, a subfield), then an automorphism σ is said to *fix* F if it fixes all the elements of F, i.e., $\sigma a = a$ for all $a \in F$.

note that σ fix F is not equivalent to $\sigma(F) = F$.

note that any field has at least one automorphism -- the identity map 1.

The prime field of K is generated by $\bar{1} \in K$ and since any automorphism σ takes 1 to 1 (and 0 to 0), i.e., $\sigma(1) = 1$, it follows that $\sigma a = a$ for all a in the prime field. Hence any automorphism of a field K fixes its prime subfield. In particular we see that $\mathbb{Q}$ and $\mathbb{F}_p$ have only the trivial automorphism: $\mathrm{Aut}(\mathbb{Q}) = \{1\}$ and $\mathrm{Aut}(\mathbb{F}_p) = \{1\}$.

$\sigma(a) = \sigma(1a) = \sigma(1)\sigma(a)$, so $\sigma(1) = 1$. Similar for $\sigma(0) = 0$.

Definition. Let K/F be an extension of fields. Let $\mathrm{Aut}(K/F)$ be the collection of automorphisms of K which fix F.

Note that if F is the prime subfield of K then $\mathrm{Aut}(K) = \mathrm{Aut}(K/F)$, since every automorphism of K automatically fixes F.

Proposition 1. $\mathrm{Aut}(K)$ is a group under composition and $\mathrm{Aut}(K/F)$ is a subgroup.

> Proof:
>
> it is clear that $\mathrm{Aut}(K)$ is a group.
>
> If σ, τ fix F, then also $\sigma\tau, \sigma^{-1}$ fix F, so $\mathrm{Aut}(K/F) \leq \mathrm{Aut}(K)$.

Proposition 2. Let K/F be a field extension and let $\alpha \in K$ be algebraic over F. Then for any $\sigma \in \mathrm{Aut}(K/F)$, $\sigma\alpha$ is a root of the minimal polynomial for α over F i.e., $\mathrm{Aut}(K/F)$ permutes the roots of irreducible polynomials. Equivalently, any polynomial with coefficients in F having α as a root also has $\sigma\alpha$ as a root.

> Proof:
>
> Suppose α satisfies the equation
>
> $$\begin{aligned} \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 &= 0 \\ \sigma(\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0) &= \sigma(0) = 0 \\ (\sigma\alpha)^n + \sigma(a_{n-1})(\sigma a)^{n-1} + \cdots + \sigma(a_1)(\sigma a) + \sigma(a_0) &= 0 \\ (\sigma\alpha)^n + a_{n-1}(\sigma a)^{n-1} + \cdots + a_1(\sigma a) + \sigma(a_0) &= 0 \end{aligned}$$
>
> so σa is also a root.

The above proposition is extremely useful for determining the automorphisms of algebraic extensions

Examples:

- $\mathrm{Aut}(\mathbb{Q}(\sqrt{2})) = \{1, \sigma\}$.
- $\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2})) = \{1\}$.

In general, if K is generated over F by some collection of elements, then any automorphism $\sigma \in \mathrm{Aut}(K/F)$ is completely determined by what it does to the generators.

Field extension $\Leftrightarrow$ a group.

$\Rightarrow$: $\mathrm{Aut}(K/F)$.

$\Leftarrow$: fixed field.

Proposition 3. Let $H \leq \mathrm{Aut}(K)$ be a subgroup of the group of automorphisms of K. Then the collection F of elements of K fixed by all the elements of H is a subfield of K.

> Proof:
>
> It is easy to verify that F is closed.

it is not important that H actually be a subgroup of $\mathrm{Aut}(K)$. It is also true when H is just a subset.

Definition. If H is a subgroup of the group of automorphisms of K, the subfield of K fixed by all the elements of H is called the *fixed field* of H.

Proposition 4. The association of groups to fields and fields to groups defined above is inclusion reversing, namely

(1) if $F_1 \subseteq F_2 \subseteq K$ are two subfields of K then $\mathrm{Aut}(K/F_2) \leq \mathrm{Aut}(K/F_1)$, and

(2) if $H_1 \leq H_2 \leq \mathrm{Aut}(K)$ are two subgroups of automorphisms with associated fixed fields F_1 and F_2, respectively, then $F_2 \subseteq F_1$.

> Proof:
>
> (1) if $h \in \mathrm{Aut}(K/F_2)$, then h also fixes F_1. Hence $\mathrm{Aut}(K/F_2) \leq \mathrm{Aut}(K/F_1)$.
>
> (2) if $x \in F_2, h \in H_1$, then h also fixes x since $x \in F_1$. Hence $F_2 \subseteq F_1$.

Proposition 5. Let E be the splitting field over F of the polynomial $f(x) \in F[x]$. Then

$$|\mathrm{Aut}(E/F)| \leq [E : F]$$

with equality if $f(x)$ is separable over F.

> Proof:
>
> We first prove for arbitrary F, φ, E, F', E':
>
> $$\begin{array}{ccccc} \tau: & F(\alpha) & \xrightarrow{\sim} & F'(\beta) \\ & | & & | \\ \varphi: & F & \xrightarrow{\sim} & F' \end{array}$$
>
> Let $p(x)$ be a irreducible factor of $f(x)$, α be a root of $p(x)$.
>
> Here $\beta = \tau\alpha$, so the number of extensions of φ to τ is equal to the number of different roots β of $p'(x)$, since α and β and their powers are the generators.
>
> Hence the number of exte nsions is at most $deg\ p(x) = [F(\alpha) : F]$, with equality if the roots of $p(x)$ are distinct.
>
> .

> Then we can use mathematical induction to prove that the number of extensions is at most $[E:F]$, with equality if all the roots of $f(x)$ are distinct (since different irreducible factors have different roots).
>
> .
>
> In the particular case, let $F = F'$ and $\varphi = 1$ and $E = E'$, so the number of automorphisms $\mathrm{Aut}(E/F)$ is the number of extensions of φ.

Definition. Let K/F be a finite extension. Then K is said to be *Galois* over F and K/F is a *Galois* extension if $|\mathrm{Aut}(K/F)| = [K : F]$. If K/F is Galois the group of automorphisms $\mathrm{Aut}(K/F)$ is called the *Galois group of* K/F, denoted $\mathrm{Gal}(K/F)$.

Galois 本质上是个啥？之后可能会讲到吧。。。

注意 K/F 要 finite.

(so $\mathrm{Aut}(K/F)$ is a finite group)

Corollary 6. If K is the splitting field over F of a separable polynomial $f(x)$ then K/F is Galois.

> Proof:
>
> That is just a restatement.

Note also that the splitting field of any polynomial over $\mathbb{Q}$ is Galois, since the splitting field of $f(x)$ is clearly the same as the splitting field of the product of the irreducible factors of $f(x)$ (i.e., the polynomial obtained by removing multiple factors), which is separable (Corollary 13.34). This is also true for any finite field $\mathbb{F}_p$.

Definition. If $f(x)$ is a separable polynomial over F, then the *Galois group of* $f(x)$ *over* F is the Galois group of the splitting field of $f(x)$ over F.

example:

- $Gal(E/F)$ is a subgroup of $S_{deg\ f(x)}$ since it acts as permutations of the roots of $f(x)$, and the roots are the generators of the splitting field.
- Galois extension of a Galois extension is not necessarily Galois.
- The extension of finite fields $\mathbb{F}_{p^n}/\mathbb{F}_p$ constructed after Proposition 13.37 is Galois by Corollary 6 since $\mathbb{F}_{p^n}$ is the splitting field over $\mathbb{F}_p$ of the separable polynomial $x^{p^n} - x$. It follows that the group of automorphisms for this extension is of order n.
  . $Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong Z_n$, with the Frobenius automorphism $\sigma_p : \alpha \mapsto \alpha^p$ as generator. （考虑 $x \in \mathbb{F}_p$, 则 $x^p = x$.）（需要注意 $\mathbb{F}_{p^n}$ 并不是 $\{0, 1, \ldots, 8\}$ 这样的 $\mathbb{Z}_{p^n}$.
-

notes: Exercise: the isomorphism is unique determined by the action on the generators of the extension.

Hence we can find the elements that are not only roots of irreducible polynomials, but also generators. By proposition 2, we must map these elements to other roots.

See more on the example of the splitting field of $x^8 - 2$: the extension is generated by $\theta = \sqrt[8]{2}$ and i, where θ is the root of $x^8 - 2$ and i is the root of $x^2 + 1$, so we must map θ to $\theta\xi_8^j$ and i to $\pm i$ that are the other roots of these polynomials and this uniquely determines the isomorphism. (note that some combination may be illegal.)

Definition. A *character*[1] χ of a group G with values in a field L is a homomorphism from G to the multiplicative group of L:

$$\chi : G \to L^\times$$

i.e., $\chi(g_1g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$ and $\chi(g)$ is a nonzero element of L for all $g \in G$.

Definition. The characters $\chi_1, \chi_2, \ldots, \chi_n$ of G are said to be *linearly independent* over L if they are linearly independent as functions on G, i.e., if there is no nontrivial relation

$$a_1\chi_1 + a_2\chi_2 + \cdots + a_n\chi_n = 0 \qquad (a_1, \ldots, a_n \in L \text{ not all } 0) \tag{14.2}$$

as a function on G (that is, $a_1\chi_1(g) + a_2\chi_2(g) + \cdots + a_n\chi_n(g) = 0$ for all $g \in G$).

Theorem 7. *(Linear Independence of Characters)* If $\chi_1, \chi_2, \ldots, \chi_n$ are distinct characters of G with values in L then they are linearly independent over L.

> proof:
>
> Assume they are linearly dependent, and m is the smallest integer possible:
>
> $$a_1\chi_1 + \cdots + a_m\chi_m = 0.$$
>
> Let g be an element that $\chi_1(g_0) \neq \chi_m(g_0)$, then
>
> $$a_1\chi_1(g_0g) + \cdots + a_m\chi_m(g_0g) = 0$$
> $$a_1\chi_1(g_0)\chi_1(g) + \cdots + a_m\chi_m(g_0)\chi_m(g) = 0$$
>
> and
>
> $$a_1\chi_m(g_0)\chi_1(g) + \cdots + a_m\chi_m(g_0)\chi_m(g) = 0.$$
>
> _Substracting the above two equations,
>
> $$(\chi_1(g_0) - \chi_m(g_0))a_1\chi_1(g) + \cdots + = 0,$$
>
> with not all zero coefficients, contradicts m is the smallest integer possible. Hence they are linearly independent.

Consider now an injective homomorphism σ of a field K into a field L, called an *embedding* of K into L. Then in particular σ is a homomorphism of the multiplicative group $G = K^\times$ into the multiplicative group $L^\times$, so σ may be viewed as a character of $K^\times$ with values in L. Note also that this character contains all of the useful information about the values of σ viewed simply as a *function* on K, since the only point of K not considered in $K^\times$ is 0, and we know σ maps 0 to 0.

Corollary 8. If $\sigma_1, \sigma_2, \ldots, \sigma_n$ are distinct embeddings of a field K into a field L, then they are linearly independent as functions on K. In particular distinct automorphisms of a field K are linearly independent as functions on K.

Theorem 9. Let $G = \{\sigma_1 = 1, \sigma_2, \ldots, \sigma_n\}$ be a subgroup of automorphisms of a field K and let F be the fixed field. Then

$$[K : F] = n = |G|.$$

> Proof: omitted.

Corollary 10. Let K/F be any finite extension. Then

$$|\mathrm{Aut}(K/F)| \leq [K : F]$$

with equality if and only if F is the fixed field of $\mathrm{Aut}(K/F)$. Put another way, K/F is Galois if and only if F is the fixed field of $\mathrm{Aut}(K/F)$.

> Proof:
>
> Let F_1 be the fixed field of $\mathrm{Aut}(K/F)$, so $F \subseteq F_1 \subseteq K$.
>
> Then by theorem 9, $[K : F] = [K : F_1][F_1 : F] = |\mathrm{Aut}(K/F)|[F_1 : F] \geq |\mathrm{Aut}(K/F)|$, with equality if and only F is the fixed field of $\mathrm{Aut}(K/F)$.

Corollary 11. Let G be a finite subgroup of automorphisms of a field K and let F be the fixed field. Then every automorphism of K fixing F is contained in G, i.e., $\mathrm{Aut}(K/F) = G$, so that K/F is Galois, with Galois group G.

> Proof:
>
> By definition, $G \leq \mathrm{Aut}(K/F)$.
>
> By theorem 9, $|G| = [K : F]$ and then by corollary 10,
>
> $$[K : F] = |G| \leq |\mathrm{Aut}(K/F)| \leq [K : F],$$
>
> so we must have equality and $\mathrm{Aut}(K/F) = G = [K : F]$ and K/F is Galois.

Corollary 12. If $G_1 \neq G_2$ are distinct finite subgroups of automorphisms of a field K then their fixed fields are also distinct.

> Proof:
>
> If the fixed fields are the same, then by corollary 11, G_1 and G_2 must be the same ($= \mathrm{Aut}(K/F_1)$), a contradiction.

Theorem 13. The extension K/F is Galois if and only if K is the splitting field of some separable polynomial over F. Furthermore, if this is the case then every irreducible polynomial with coefficients in F which has a root in K is separable and has all its roots in K (so in particular K/F is a separable extension).

> Proof:
>
> Proposition 5 proves that the splitting field of a separable polynomial is Galois.
>
> .
>
> Conversely, we first shows that if K/F is Galois then every irreducible polynomial $p(x)$ in $F[x]$ having a root in K splits completely in K.
>
> Let $G = \mathrm{Gal}(K/F)$, $\alpha \in K$ be a root of $p(x)$.
>
> Let $\alpha, \alpha_2, \ldots, \alpha_r$ be the different elements of $\{\sigma\alpha \mid \sigma \in G\}$, so they are roots of $p(x)$.
>
> Consider the polynomial $f(x) = \prod_{i=1}^{r}(x - \alpha_i)$.

Since each $\sigma \in G$ permutates the factors to different (because it is an isomorphism) factors of $f(x)$, it keeps $f(x)$ unchanged, so the coefficients of $f(x)$ lie in the fixed field of G, which is F by corollary 10. Since $p(x)$ is irreducible, it is the minimal polynomial of α, so $p(x) \mid f(x)$.

Besides, since all α_i are roots of $p(x)$, $f(x) \mid p(x)$.

Hence $f(x) = p(x)$ and all roots of $p(x)$ are in K.

.

Finally, let $\omega_1, \ldots, \omega_n$ be a basis of K/F. For all i, let $p_i(x)$ be the minimal polynomial of ω_i. Consider the polynomial $g(x) = \prod_{i=1}^{m} p_i(x)$ where $p_1, \ldots, p_m$ are all the different polynomials. All roots of $g(x)$ is the roots of $p_i(x)$, and they lie in K by the assumption that if a root lies in K, then all roots lie in K.

Also $\omega_1, \ldots \omega_n$ are the basis, so the splitting field contains the basis, i.e., it also contains K.

Hence K is the splitting field of some separable polynomial $g(x)$.

Definition. Let K/F be a Galois extension. If $\alpha \in K$ the elements $\sigma\alpha$ for σ in $\mathrm{Gal}(K/F)$ are called the *conjugates* (or *Galois conjugates*) of α over F. If E is a subfield of K containing F, the field $\sigma(E)$ is called the *conjugate field* of E over F.

The proof of the theorem shows that in a Galois extension K/F the other roots of the minimal polynomial over F of any element $\alpha \in K$ are precisely the distinct conjugates of α under the Galois group of K/F.

The second statement in this theorem also shows that K is not Galois over F if we can find even one irreducible polynomial over F having a root in K but not having *all* its roots in K. This justifies in a very strong sense the intuition from earlier examples that Galois extensions are extensions with "enough" distinct roots of irreducible polynomials (namely, if it contains one root then it contains all the roots).

Finally, notice that we now have 4 characterizations of Galois extensions K/F:

(1) splitting fields of separable polynomials over F
(2) fields where F is precisely the set of elements fixed by $\mathrm{Aut}(K/F)$ (in general, the fixed field may be larger than F)
(3) fields with $[K : F] = |\mathrm{Aut}(K/F)|$ (the original definition)
(4) finite, normal and separable extensions.

Theorem 14. *(Fundamental Theorem of Galois Theory)* Let K/F be a Galois extension and set $G = \mathrm{Gal}(K/F)$. Then there is a bijection

$$\left\{ \begin{array}{c} \text{subfields } E \\ \text{of } K \\ \text{containing } F \end{array} \begin{array}{c} K \\ | \\ E \\ | \\ F \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{subgroups } H \\ \text{of } G \end{array} \begin{array}{c} 1 \\ | \\ H \\ | \\ G \end{array} \right\}$$

given by the correspondences

$$E \longrightarrow \left\{ \begin{array}{c} \text{the elements of } G \\ \text{fixing } E \end{array} \right\}$$

$$\left\{ \begin{array}{c} \text{the fixed field} \\ \text{of } H \end{array} \right\} \longleftarrow H$$

which are inverse to each other. Under this correspondence,

(1) (inclusion reversing) If E_1, E_2 correspond to H_1, H_2, respectively, then $E_1 \subseteq E_2$ if and only if $H_2 \leq H_1$

(2) $[K : E] = |H|$ and $[E : F] = |G : H|$, the index of H in G:

$$\begin{array}{lll} K & & \\ | & \} & |H| \\ E & & \\ | & \} & |G : H| \\ F & & \end{array}$$

(3) K/E is always Galois, with Galois group $\mathrm{Gal}(K/E) = H$:

$$\begin{array}{ll} K & \\ | & H \\ E & \end{array}$$

(4) E is Galois over F if and only if H is a normal subgroup in G. If this is the case, then the Galois group is isomorphic to the quotient group

$$\mathrm{Gal}(E/F) \cong G/H.$$

More generally, even if H is not necessarily normal in G, the isomorphisms of E (into a fixed algebraic closure of F containing K) which fix F are in one to one correspondence with the cosets $\{\sigma H\}$ of H in G.

(5) If E_1, E_2 correspond to H_1, H_2, respectively, then the intersection $E_1 \cap E_2$ corresponds to the group $\langle H_1, H_2 \rangle$ generated by H_1 and H_2 and the composite field E_1E_2 corresponds to the intersection $H_1 \cap H_2$. Hence the lattice of subfields of K containing F and the lattice of subgroups of G are "dual" (the lattice diagram for one is the lattice diagram for the other turned upside down).

> Proof: omitted.

这一类题的时候可以把上面这几个图画出来，会好观察很多.

14.3 Finite Fields

Proposition 15. Any finite field is isomorphic to $\mathbb{F}_{p^n}$ for some prime p and some integer $n \geq 1$. The field $\mathbb{F}_{p^n}$ is the splitting field over $\mathbb{F}_p$ of the polynomial $x^{p^n} - x$, with cyclic Galois group of order n generated by the Frobenius automorphism σ_p. The subfields of $\mathbb{F}_{p^n}$ are all Galois over $\mathbb{F}_p$ and are in one to one correspondence with the divisors d of n. They are the fields $\mathbb{F}_{p^d}$, the fixed fields of ${\sigma_p}^d$.

> proof:
>
> The only finite field (unique up to isomorphism) is the field of order p^n by $\mathbb{F}_{p^n}$.
>
> We have seen that $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong Z_n$.
>
> It has a unique subfield of order p^d for $d \mid n$, since the Galois group has a unique subgroup of order d. By the fundamental theorem, there is a one to one correspondence between them.

The degree of $\mathbb{F}_{p^n}/\mathbb{F}_p$ is n.

Corollary 16. The irreducible polynomial $x^4 + 1 \in \mathbb{Z}[x]$ is reducible modulo every prime p.

Proof:

For $p = 2$, $x^4 + 1 = (x + 1)^4$.

For odd prime p, $8 \mid (p^2 - 1)$, so $x^4 + 1 \mid x^8 - 1 \mid x^{p^2-1} - 1 \mid x^{p^2} - x$.

Since the roots of $x^{p^2} - x$ are in the field $\mathbb{F}_{p^2}/\mathbb{F}_p$ of degree 2, any root of $x^4 + 1$ is at most of degree 2 over $\mathbb{F}$, so $x^4 + 1$ cannot be irreducible (otherwise the degree of a root = the degree of minimal polynomial = 4).

Proposition 17. The finite field $\mathbb{F}_{p^n}$ is simple. In particular, there exists an irreducible polynomial of degree n over $\mathbb{F}_p$ for every $n \geq 1$.

Proof:

By proposition 9.18, the multiplicative group $\mathbb{F}_{p^n}$ is cyclic (of order $p^n - 1$) . Let θ be a generator, then the powers of θ enumerates all the elements except 0, so $\mathbb{F}_{p^n} = \mathbb{F}_p(\theta)$. Also, since the extension is of degree n, the minimal polynomial of θ is of degree n over $\mathbb{F}_p$.

Proposition 18. The polynomial $x^{p^n} - x$ is precisely the product of all the distinct irreducible polynomials in $\mathbb{F}_p[x]$ of degree d where d runs through all divisors of n.

Proof:

Each root of $x^{p^n} - x$ generates a subfield $\mathbb{F}(\alpha)$ of $\mathbb{F}_{p^n}$, so its minimal polynomial is of degree $d \mid n$.

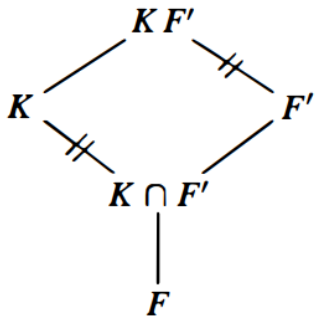
Conversely, let α be any root of $f(x)$, then $[\mathbb{F}_p(\alpha) : \mathbb{F}_p] = d$, so $\mathbb{F}_p(\alpha) \cong \mathbb{F}_{p^d}$. Then $\alpha^{p^d} - \alpha = 0$, so $\alpha^{p^n} - \alpha = 0$. Hence $f(x) \mid (x^{p^n} - x)$.

14.4 Composite Extensions and Simple Extensions

Proposition 19. Suppose K/F is a Galois extension and F'/F is any extension. Then KF'/F' is a Galois extension, with Galois group

$$\mathrm{Gal}(KF'/F') \cong \mathrm{Gal}(K/K \cap F')$$

isomorphic to a subgroup of $\mathrm{Gal}(K/F)$. Pictorially,



Proof:

If K/F is Galois, then K is the splitting field of some separable polynomial $f(x) \in F[x]$.

Hence KF'/F' is the splitting field of $f(x) \in F'[x]$, so this extension is Galois.

Then $\varphi : \mathrm{Gal}(KF'/F') \to \mathrm{Gal}(K/F)$ by $\sigma \mapsto \sigma|_K$ is well defined and clearly it is a homomorphism.

Consider the kernel $\ker \varphi = \{\sigma \in \mathrm{Gal}(KF'/F') \mid \sigma|_K = 1\}$. Since σ fixes F' and σ fixes K, σ fixes KF', so $\ker \varphi = \{1\}$. Hence φ is injective. （用得到吗?

Let H be the image of φ and K_H denote the corresponding fixed subfield of K containing F of H. Since every element in H fixes F', K_H contains $K \cap F'$.

On the other hand, the composite field $K_H F'$ is fixed by $\mathrm{Gal}(KF'/F')$ (any σ fixes F', and it fixes K_H by definition of K_H.) By the Fundamental Theorem, $K_H F' = F'$ (by the one-to-one correspondence of Galois group and fixed field). Hence $K_H \subseteq F'$, so $K_H \subseteq K \cap F'$.

Hence $K_H = K \cap F'$. By the Fundamental Theorem, $H = \mathrm{Gal}(K/K \cap F')$.

Hence $\mathrm{Gal}(KF'/F') \cong \mathrm{Gal}(K/K \cap F')$.

Corollary 20. Suppose K/F is a Galois extension and F'/F is any finite extension. Then

$$[KF' : F] = \frac{[K : F][F' : F]}{[K \cap F' : F]}.$$

Proof:

By proposition 19,

$$[KF' : F] = [KF' : F'][F' : F] = [K : K \cap F'][F' : F] = \frac{[K : F][F' : F]}{[K \cap F' : F]}$$

.

This does not hold in general if neither of the two extensions is Galois.

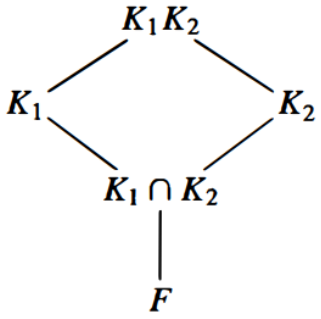
Proposition 21. Let K_1 and K_2 be Galois extensions of a field F. Then

(1) The intersection $K_1 \cap K_2$ is Galois over F.

(2) The composite K_1K_2 is Galois over F. The Galois group is isomorphic to the subgroup

$$H = \{(\sigma, \tau) \mid \sigma|_{K_1 \cap K_2} = \tau|_{K_1 \cap K_2}\}$$

of the direct product $\mathrm{Gal}(K_1/F) \times \mathrm{Gal}(K_2/F)$ consisting of elements whose restrictions to the intersection $K_1 \cap K_2$ are equal.



Proof:

(1) Let $p(x)$ be an irreducible polynomail in $F[x]$ with a root in $K_1 \cap K_2$. Since all its roots are in K_1 and all its roots are in K_2, all the roots lie in $K_1 \cap K_2$, so $K_1 \cap K_2$ is Galois.

(2) If K_1 is the splitting field of the separable polynomial $f_1(x)$ and K_2 is the splitting field of the saparable polynomial $f_2(x)$, then K_1K_2 is the splitting field of the squarefree part of $f_1(x)f_2(x)$, hence is Galois over F.

The map

$$\varphi : \mathrm{Gal}(K_1K_2/F) \to \mathrm{Gal}(K_1/F) \times \mathrm{Gal}(K_2/F)$$
$$\sigma \mapsto (\sigma|_{K_1}, \sigma|_{K_2})$$

is clearly a homomorphism. The kernel consists of the elements σ which are trivial on both K_1 and K_2, hence trivial on the composite, so the map is injective.

The image lies in the subgroup H, since $(\sigma_{K_1})|_{K_1 \cap K_2} = \sigma|_{K_1 \cap K_2} = (\sigma_{K_2})|_{K_1 \cap K_2}$.

> The order of H is (enumerate σ, and count the number of τ satisfy the condition)
>
> $$\begin{aligned} |H| &= |\mathrm{Gal}(K_1/F)|\,|\mathrm{Gal}(K_2/K_1\cap K_2)| \\ &= |\mathrm{Gal}(K_1/F)|\frac{|\mathrm{Gal}(K_2/F)|}{|\mathrm{Gal}(K_1\cap K_2/F)|} \\ &= |\mathrm{Gal}(K_1K_2/F)| \end{aligned}$$
>
> by corollary 20, so φ is an isomorphism and $\mathrm{Gal}(K_1K_2/F)\cong H$.

Corollary 22. Let K_1 and K_2 be Galois extensions of a field F with $K_1 \cap K_2 = F$. Then

$$\mathrm{Gal}(K_1K_2/F) \cong \mathrm{Gal}(K_1/F) \times \mathrm{Gal}(K_2/F).$$

Conversely, if K is Galois over F and $G = \mathrm{Gal}(K/F) = G_1 \times G_2$ is the direct product of two subgroups G_1 and G_2, then K is the composite of two Galois extensions K_1 and K_2 of F with $K_1 \cap K_2 = F$.

> Proof:
>
> the first statement is obvious by theorem 21.
>
> Conversely, let G_1 and G_2 be the two groups. Let K_1 and K_2 be the fixed field of G_1 and G_2, respectively.
>
> By the fundamental theorem, $K_1 \cap K_2$ is the fixed field of $G_1G_2 = G$, so $K_1 \cap K_2 = F$. K_1K_2 is the fixed field of $G_1 \cap G_2 = \{1\}$, so $K_1K_2 = K$.

Corollary 23. Let E/F be any finite separable extension. Then E is contained in an extension K which is Galois over F and is minimal in the sense that in a fixed algebraic closure of K any other Galois extension of F containing E contains K.

> Proof:
>
> There exists a Galois extension of F containing E, for example the composite of the splitting fields of the minimal polynomials for a basis for E over F (which are all separable since E is separable over F).
>
> By the fundamental theorem, the Galois group of the intersection of all the Galois extensions of F containing E will permutes the roots of the minimal polynomials, so it will contains K (and also is K because K is also such polynoamial).

Definition. The Galois extension K of F containing E in the previous corollary is called the *Galois closure* of E over F.

要求 E separable.

Proposition 24. Let K/F be a finite extension. Then $K = F(\theta)$ if and only if there exist only finitely many subfields of K containing F.

> Proof: omitted.

The primitive generator of $F(a, b, c, \ldots)$ is simply $a + b + c + \cdots$.

Theorem 25. *(The Primitive Element Theorem)* If K/F is finite and separable, then K/F is simple. In particular, any finite extension of fields of characteristic 0 is simple.

> Proof:

> Let L be the Galois closure of K over F. Then any subfield of K containing F corresponds to a subgroup of the Galois group $\mathrm{Gal}(L/F)$ by the Fundamental Theorem. Since there are only finitely many such subgroups (note that $|\mathrm{Gal}(L/F)| = [L : F]$ which is finite), the previous proposition shows that K/F is simple.
>
> The last statement follows since any finite extension of fields in characteristic 0 is simple (proposition 13.34).

14.6 Galois Groups of Polynomials

If K is a Galois extension of F, then K is the splitting field of some separable polynomial $f(x) \in F[x]$ of degree n. Then any automorphism $\sigma \in \mathrm{Gal}(K/F)$ maps a root of $f(x)$ to another root, so $\mathrm{Gal}(K/F) \subseteq S_n$. In general, if $f(x)$ is the product of some irreducibles $f_1(x) \cdots f_k(x)$, then $\mathrm{Gal}(K/F) \leq S_{n_1} \times \cdots \times S_{n_k}$.

Definition. Let $x_1, x_2, \ldots, x_n$ be indeterminates. The *elementary symmetric functions* $s_1, s_2, \ldots, s_n$ are defined by

$$\begin{aligned} s_1 &= x_1 + x_2 + \cdots + x_n \\ s_2 &= x_1x_2 + x_1x_3 + \cdots + x_2x_3 + x_2x_4 + \cdots + x_{n-1}x_n \\ &\vdots \\ s_n &= x_1x_2 \cdots x_n \end{aligned}$$

i.e., the i^{th} symmetric function s_i of $x_1, x_2, \ldots, x_n$ is the sum of all products of the x_j's taken i at a time.

这里可以把 s_i 理解成一个 function，也可以理解成一个元素.

Definition. The *general polynomial of degree n* is the polynomial

$$(x - x_1)(x - x_2) \cdots (x - x_n)$$

whose roots are the indeterminates $x_1, x_2, \ldots, x_n$.

$$(x - x_1)(x - x_2) \cdots (x - x_n) = x^n - s_1x^{n-1} + s_2x^{n-2} + \cdots + (-1)^n s_n. \quad (14.13)$$

For any field F, $F(x_1, \ldots, x_n)$ is the splitting field of the general polynomial, so it is a Galois extension of $F(s_1, \ldots, s_n)$.

(这里 $F(x_1, \ldots, x_n)$ 既可以指 the field of rational functions，也可以指由这些元素 generate 出来的一个 field. 这两者是同一个东西.

那么对于 $\sigma \in S_n$，把 σ 作用在 $x_1, \ldots, x_n$ 上就可以得到一个 $F(x_1, \ldots, x_n)$ 的 automorphism.因此 $S_n \leq \mathrm{Aut}(F(x_1, \ldots, x_n))$.

Proposition 30. The fixed field of the symmetric group S_n acting on the field of rational functions in n variables $F(x_1, x_2, \ldots, x_n)$ is the field of rational functions in the elementary symmetric functions $F(s_1, s_2, \ldots, s_n)$.

> Proof:
>
> the elementary symmetric functions are fied under any permutation of S_n on their subscipts, so the subfield $F(s_1, \ldots, s_n)$ is contained the fixed field of S_n. By the fundamental theorm, the fixed field of S_n has index precisely $n!$ in $F(x_1, \ldots, x_n)$.
>
> Also, since $F(x_1, \ldots, x_n)$ is the splitting field over $F(s_1, \ldots, s_n)$ of the polynomial (14.13), the extension has degree at most $n!$.
>
> Hence $[F(x_1, \ldots, x_n) : F(s_1, \ldots, s_n)] = n!$ and S_n is precisely the fixed field of $F(s_1, \ldots, s_n)$.

Definition. A rational function $f(x_1, x_2, \ldots, x_n)$ is called *symmetric* if it is not changed by any permutation of the variables $x_1, x_2, \ldots, x_n$.

Corollary 31. *(Fundamental Theorem on Symmetric Functions)* Any symmetric function in the variables $x_1, x_2, \ldots, x_n$ is a rational function in the elementary symmetric functions $s_1, s_2, \ldots, s_n$.

> Proof: any symmetric function lies in the fixed field of S_n, so it is in the field $F(s_1, \ldots, s_n)$.

Theorem 32. The general polynomial

$$x^n - s_1 x^{n-1} + s_2 x^{n-2} + \cdots + (-1)^n s_n$$

over the field $F(s_1, s_2, \ldots, s_n)$ is separable with Galois group S_n.

> The statement on the book (p622) shows that $x_1, \ldots, x_n$ are independent if and only if $s_1, \ldots, s_n$ are independent.

Over finite fields, tere is no such "generic" polynomial because each every polynomial has a cyclic Galios group (all extensions of finite fields are cyclic).

Over $\mathbb{Q}$, most polynomials have the full symmetric group as Galois group.

Definition. Define the *discriminant* D of $x_1, x_2, \ldots, x_n$ by the formula

$$D = \prod_{i<j} (x_i - x_j)^2.$$

Define the discriminant of a polynomial to be the discriminant of the roots of the polynomial.

The discriminant is a symmetric function on $x_1, \ldots, x_n$, so it is in the field $F(s_1, \ldots, s_n)$.

Proposition 33. If $\mathrm{ch}(F) \neq 2$ then the permutation $\sigma \in S_n$ is an element of A_n if and only if it fixes the square root of the discriminant D.

> Proof: by the definition of A_n.
>
> If $ch(F) = 2$, then $x_i - x_j = x_j - x_i$, so σ fixes every square root of D.

Since D is symmetric on F, $D \in F$, so it can be written as a polynomial over F.

Proposition 34. The Galois group of $f(x) \in F[x]$ is a subgroup of A_n if and only if the discriminant $D \in F$ is the square of an element of F.

> Proof:
>
> by proposition 33, the Galois group is contained in A_n if and only if every element of the Galois group fixes $\sqrt{D}$, if and only if $D \in F$. (这里 F 就是前面的 $F(s_1, \ldots, s_n)$.

- Polynomials of Degree 2
- Polynomials of Degree 3

- Polynomials of Degree 4

Theorem 35. *(Fundamental Theorem of Algebra)* Every polynomial $f(x) \in \mathbb{C}[x]$ of degree n has precisely n roots in $\mathbb{C}$ (counted with multiplicity). Equivalently, $\mathbb{C}$ is algebraically closed.

> Proof: omitted.

14.7 Solvable and Radical Extensions: Insolvability of the Quintic

Solve for the roots of a polynomial by *radicals*: the algebraic operations of addition, subtraction, multiplication, division and the extraction of n^{th} roots.

The symbol $\sqrt[n]{a}$ for $a \in F$ will be used to denote any root of the polynomial $x^n - a \in F[x]$.

Definition. The extension K/F is said to be *cyclic* if it is Galois with a cyclic Galois group.

Proposition 36. Let F be a field of characteristic not dividing n which contains the n^{th} roots of unity. Then the extension $F(\sqrt[n]{a})$ for $a \in F$ is cyclic over F of degree dividing n.

> proof:
>
> The extension $F(\sqrt[n]{a})$ is the splitting field of the separable polynomial $x^n - a$ which has roots $\sqrt[n]{a}\xi_n^j$, so it is Galois.
>
> For any $\sigma \in \mathrm{Gal}(K/F)$, $\sigma(\sqrt[n]{a}) = \xi_\sigma \sqrt[n]{a}$ is another root of the polynomial. This gives a map
>
> $$\begin{aligned} \mathrm{Gal}(K/F) &\to \mu_n \\ \sigma &\mapsto \xi_\sigma \end{aligned}$$
>
> Since F contains μ_n, every n^{th} root of unity is fixed by $\mathrm{Gal}(K/F)$, so
>
> $$\begin{aligned} \xi_{\sigma\tau}(\sqrt[n]{a}) &= \sigma\tau(\sqrt[n]{a}) \\ &= \sigma(\xi_\tau \sqrt[n]{a}) \\ &= \xi_\tau \sigma(\sqrt[n]{a}) \\ &= \xi_\tau \xi_\sigma(\sqrt[n]{a}) \\ &= \xi_\sigma \xi_\tau(\sqrt[n]{a}), \end{aligned}$$
>
> so the map is a homomorphism.
>
> The kernal fixes F and $\sqrt[n]{a}$, so it fixed all the elements in $F(\sqrt[n]{a})$, and it can only contain the identity.
>
> Since the groups are finite, the map is also surjective
>
> Therefore $\mathrm{Gal}(K/F) \cong \mu_n$, and it is cyclic over F of degree n.

好多这种求结构的例子都是通过证明同构（或homo）来完成的...

Let now K be any cyclic extension of degree n over a field F of characteristic not dividing n which contains the n^{th} roots of unity. Let σ be a generator for the cyclic group $\mathrm{Gal}(K/F)$.

Definition. For $\alpha \in K$ and any n^{th} root of unity ζ, define the *Lagrange resolvent* $(\alpha, \zeta) \in K$ by

$$(\alpha, \zeta) = \alpha + \zeta\sigma(\alpha) + \zeta^2\sigma^2(\alpha) + \cdots + \zeta^{n-1}\sigma^{n-1}(\alpha).$$

If we apply σ to (α, ξ), we will obtain $\xi^{-1}(\alpha, \xi)$,

It follows that

$$\sigma(\alpha, \zeta)^n = (\zeta^{-1})^n(\alpha, \zeta)^n = (\alpha, \zeta)^n$$

so that $(\alpha, \zeta)^n$ is fixed by $\mathrm{Gal}(K/F)$, hence is an element of F for any $\alpha \in K$.

Proposition 37. Any cyclic extension of degree n over a field F of characteristic not dividing n which contains the n^{th} roots of unity is of the form $F(\sqrt[n]{a})$ for some $a \in F$.

Proof:

Let ζ be a primitive n^{th} root of unity. By the linear independence of the automorphisms $1, \sigma, \ldots, \sigma^{n-1}$ (Theorem 7), there is an element $\alpha \in K$ with $(\alpha, \zeta) \neq 0$. Iterating (19) we have

$$\sigma^i(\alpha, \zeta) = \zeta^{-i}(\alpha, \zeta), \qquad i = 0, 1, \ldots,$$

and it follows that σ^i does not fix (α, ζ) for any $i < n$. Hence this element cannot lie in any proper subfield of K, so $K = F((\alpha, \zeta))$. Since we proved $(\alpha, \zeta)^n = a \in F$ above, we have $F(\sqrt[n]{a}) = F((\alpha, \zeta)) = K$. This proves the following converse of Proposition 36.

this element cannot lie in any proper subfield of K because ...? **why**

For simplicity we now consider the situation of a base field F of characteristic 0.

As in the previous propositions the results are valid over fields whose characteristics do not divide any of the orders of the roots that will be taken.

Definition.

(1) An element α which is algebraic over F can be *expressed by radicals* or *solved for in terms of radicals* if α is an element of a field K which can be obtained by a succession of simple radical extensions

$$F = K_0 \subset K_1 \subset \cdots \subset K_i \subset K_{i+1} \subset \cdots \subset K_s = K \qquad (14.21)$$

where $K_{i+1} = K_i(\sqrt[n_i]{a_i})$ for some $a_i \in K_i$, $i = 0, 1, \ldots, s-1$. Here $\sqrt[n_i]{a_i}$ denotes some root of the polynomial $x^{n_i} - a_i$. Such a field K will be called a *root extension* of F.

(2) A polynomial $f(x) \in F[x]$ can be *solved by radicals* if all its roots can be solved for in terms of radicals.

(this gives a precise meaning of expressed by the operations

the following example can be found in the book.

$$-1 + \sqrt{17} + \sqrt{2(17 - \sqrt{17})} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{2(17 - \sqrt{17})} - 2\sqrt{2(17 + \sqrt{17})}}$$

Lemma 38. If α is contained in a root extension K as in (21) above, then α is contained in a root extension which is Galois over F and where each extension K_{i+1}/K_i is cyclic.

> *Proof:* Let L be the Galois closure of K over F. For any $\sigma \in \mathrm{Gal}(L/F)$ we have the chain of subfields
>
> $$F = \sigma K_0 \subset \sigma K_1 \subset \cdots \subset \sigma K_i \subset \sigma K_{i+1} \subset \cdots \subset \sigma K_s = \sigma K$$
>
> where $\sigma K_{i+1}/\sigma K_i$ is again a simple radical extension (since it is generated by the element $\sigma(\sqrt[n_i]{a_i})$, which is a root of the equation $x^{n_i} - a_i$.)
>
> Also, the composite of two root extensions is again a root extension (we can add the subfields one by one: a_i, n_i).
>
> Hence the composite of all of the conjugate fields σK for $\sigma \in \mathrm{Gal}(L/F)$ is a again a root extension.
>
> Let the composite field be C. Let $K = F(\theta)$ by the Primitive Element Theorem, and $f(x)$ be the minimal polynomial of θ. Then C is the splitting field of $f(x)$ because $\mathrm{Gal}(L/F)$ acts transitively on the roots of $f(x)$, so the composite is minimal that contains all the roots of $f(x)$. Also the $f(x)$ is irreducible, so it is separable, and then C/F is Galois. Hence $C = L$.
>
> we see that α is contained in a Galois root extension.
>
> Then, let $F = L_0 \subset L_1 \subset \cdots \subset L_s = L$ be the chain of Galois root extension. We now adjoin to F the $n_i - th$ roots of unity in the chain, obtaining the field F'. Then
>
> $$F \subseteq F' = F'L_0 \subseteq F'L_1 \subseteq \cdots \subseteq F'L_s = F'L.$$
>
> The extension from F to F' can be given as a chain of subfields with each individual extension cyclic ($\mathrm{Gal}(F(\xi)/F)$ is cyclic, because we can only map one root of $x^n - 1$ to another root, and this is uniquely determined by $\sigma(\xi_n)$ that the map of one primitive root of unity.)
>
> The field $F'L$ is a Galois extension over F because it is the composite of two Galois extensions.
>
> Each $F'L_{i+1}/F'L_i$ is cyclic by proposition 36. Hence $F'L$ is a Galois root extension with cyclic intermediate extensions.

Recall from Section 3.4 (cf. also Section 6.1) that a finite group G is *solvable* if there exists a chain of subgroups

$$1 = G_s \leq G_{s-1} \leq \cdots \leq G_{i+1} \leq G_i \leq \cdots \leq G_0 = G \qquad (14.22)$$

with G_i/G_{i+1} cyclic, $i = 0, 1, \ldots, s-1$. We have proved that subgroups and quotient groups of solvable groups are solvable and that if $H \leq G$ and G/H are both solvable, then G is solvable.

(好像没有学过 subgroups and quotient groups of solvable groups are solvable?)

Theorem 39. The polynomial $f(x)$ can be solved by radicals if and only if its Galois group is a solvable group.

> Proof:
>
> First we suppose $f(x)$ can be solved by radicals, then each root of $f(x)$ is contained in an extension as in lemma 38. Let L be the composite of all such extensions, then L is also an extension of the same type.
>
> Let G_i be the subgroups of the galois group corresponding to subfield K_i, then $\mathrm{Gal}(K_{i+1}/K_i) = G_i/G_{i+1}$. Hence the Galois group $G = \mathrm{Gal}(L/F)$ is a solvable group.
>
> Since L contains the splitting field of $f(x)$, the Galois group of $f(x)$ is a quotient group of the solvable group G (by the Fundamental Theorem of Galois Theory), hence is solvable.
>
> .
>
> Conversely, let G be the Galois group of $f(x)$, and K be the splitting field of $f(x)$.
>
> Taking the fixed fields of the subgroups in the chain (14.22) for G gives a chain

$$F = K_0 \subset K_1 \subset \cdots \subset K_s = K.$$

where K_{i+1}/K_i is a cyclic extension of degree n_i.

Let F' be the cyclotomic field over F of all roots of unity of order n_i. Then

$$F \subseteq F' = F'K_0 \subseteq F'K_1 \subseteq \cdots \subseteq F'K_s = F'K.$$

The extension F'/F is cyclic by the proof of lemma 38.

The extension $F'K_{i+1}/F'K$ is cyclic by the proof of lemma 38, also it has degree dividing n_i by proposition 19. By proposition 37, each such extension is a simple radical extension.

Each root of $f(x)$ is therefore contained in the splitting field $K \subseteq$ the root extension $F'K$ so that $f(x)$ can be solved by radicals.

Corollary 40. The general equation of degree n cannot be solved by radicals for $n \geq 5$.

For $n \geq 5$ the group S_n is not solvable as we showed in Chapter 4. The corollary follows immediately from Theorems 32 and 39.

Orbit-stabilier Theorem

Consider the map $f_x : G \to X$ by $f_x(g) = g \cdot x$. Then
$f(g) = f(h) \Leftrightarrow g \cdot x = h \cdot x \Leftrightarrow g^{-1}h \cdot x = x \Leftrightarrow g^{-1}h \in G_x \Leftrightarrow h \in gG_x$. Thus $|G \cdot x| = |G : G_x| = |G| / |G_x|$.

This is proposition 4.1.2.

Burnside's Lemma

First $\sum_{g \in G} |X^g| = \sum_{g \in G} \sum_{x \in X} [g \cdot x = x] = \sum_{x \in X} |G_x|$.

Then

$$\begin{aligned} \sum_{x \in X} |G_x| &= \sum_{x \in X} \frac{|G|}{|G \cdot x|} \\ &= |G| \sum_{O \in X/G} \sum_{x \in O} \frac{1}{|O|} \\ &= |G|\,|X/G|. \end{aligned}$$

Q.E.D..

Polya Theorem

额。。。算了这不重要。。。

EOF