Horner's Rule

Background

In this note, we consider the evaluation of *polynomials*.

$$\boxed{\mathcal{P}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n} \tag{1}$$

Polynomials are an engineer's friend. Polynomials offer an extremely rich universe of behavior, they are easy to differentiate, easy to integrate, and are fully characterized by a simple list of coefficients. Recalling *Taylor Series* expansions, we can approximate even very complex functions using polynomials.

Brute Force and Stupidity?

The evaluation of polynomials seems obvious. A polynomial of *degree* n is the sum of $n+1$ *monomial* terms:

$$\mathcal{P}(x) = (a_0) + (a_1 x) + \left(a_2 x^2\right) + \left(a_3 x^3\right) + \ldots + (a_n x^n) \tag{2}$$

An operations count is enlightening. The evaluation of the j'th monomial

$$a_j x^j = a_j \cdot x \cdot x \cdot \ldots \cdot x \tag{3}$$

requires j multiplications[1]. So, the total number of multiplications needed to evaluate (2) is

$$0 + 1 + 2 + 3 + \ldots + n = \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \tag{4}$$

Furthermore, the evaluation of (2) requires n additions. The total operation count, including both multiplications and additions, is

$$\text{operations count} = n + \frac{n(n+1)}{2} = \frac{n^2 + 3n}{2} \tag{5}$$

Enter Horner

Horner's Rule[2] offers an alternative approach to evaluating polynomials. Rather than viewing a polynomial as the sum of monomials, Horner's rule factors a order-n polynomial in a non-intuitive fashion:

$$\mathcal{P}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n \tag{6}$$

$$= (\cdots((a_n x + a_{n-1})x + a_{n-2})x + \ldots + a_1)x + a_0 \tag{7}$$

[1]You may argue that no one does all of these multiplications. Rather, you may suggest, that a sane person uses the y^x function. While this is true, the resulting operation count is even worse since the y^x is evaluated as

$$y^x = \exp\left(x \cdot \ln(y)\right)$$

and the exp and ln functions are evaluated using power series polynomials of their own.

[2]Named after William George Horner (1786-1837), a British mathematician. Interestingly, Horner's best known invention (for non-geeks) is the modern *zeotrope*, which creates the impression of a moving image using a rapid succession of static images – the precursor of moving pictures and animated gifs.

Perhaps a specific, and smaller, example would help understanding:

$$\begin{aligned} f(x) &= 5x^4 + 4x^3 + 3x^2 + 2x + 1 && (8) \\ &= (5x^3 + 4x^2 + 3x + 2)x + 1 && (9) \\ &= ((5x^2 + 4x + 3)x + 2)x + 1 && (10) \\ &= (((5x + 4)x + 3)x + 2)x + 1 && (11) \end{aligned}$$

Equation (8) is of the form (6), while (11) is of the form (7).

The operation count for the evaluation is telling. From (7) we see that the application of Horner's Rule to the evaluation of a order-n polynomial requires only n multiplications and n additions, or

$$\text{operations count} = 2n \tag{12}$$

Thus, we must compare $2n$ to $\frac{n^2+3n}{2}$. The "obvious" approach using the sum of monomials requires 2 times as many operations as Horner's Rule for $n = 5$, 3 times as many for $n = 9$, 4 times as many for $n = 13$, and 10 times as many for $n = 37$.

Modified Horner's Rule

Often, when dealing with power series expansions, we must evaluate a polynomial of the form

$$\mathcal{P}(x) = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \ldots + a_n \frac{x^n}{n!} \tag{13}$$

The Honer's rule factoring takes the following modified form in this case:

$$\mathcal{P}(x) = (\cdots((a_n \frac{x}{n} + a_{n-1}) \frac{x}{n-1} + a_{n-2}) \frac{x}{n-2} + \ldots + a_1) \frac{x}{1} + a_0 \tag{14}$$

A Bit of Code

The following code fragment implements (7).

```
function [P] = Horner(a,x)
m = length(a);
P = a(m);
for i = m-1:-1:1
    P = P*x + a(i);
end
```

The following code fragment implements (14).

```
function [P] = ModifiedHorner(a,x)
m = length(a);
P = a(m);
for i = m-1:-1:1
    P = P*x/i + a(i);
end
```