The Fourier Option

(Thoughts on Homework 12)

1 Background

As presented in the problem formulation, groundwater mechanics dictates that the leakage, $\gamma(x)$, is a smooth, continuous function. The heads in both aquifers, at both ends, equal the heads in the rivers. The difference in heads between the aquifers, at both ends, equal zero. Therefore, the leakage, at both ends, equal zero.

$$\gamma(0) = 0 \tag{1}$$

$$\gamma(w) = 0 \tag{2}$$

The sought after function, $\Gamma(x)$, is defined by

$$\gamma(x) = \frac{d^2\Gamma}{dx^2} \tag{3}$$

for $0 \leqslant x \leqslant w$. Since the second derivative of $\Gamma(x)$ is a smooth, continuous function, $\Gamma(x)$ must be a smooth, continuous function. We also know that, by definition, $\Gamma(x)$ is equal to zero at both ends.

$$\Gamma(0) = 0 \tag{4}$$

$$\Gamma(w) = 0 \tag{5}$$

Our goal in this note is to find a general representation for $\Gamma(x)$, and $\gamma(x)$, that satisfies all five of these conditions: (1) through (5).

2 Sinusoidal basis

2.1 The proposal

In class last week, Laina recommended that we represent $\Gamma(\chi)$ as a Fourier series[1].

$$\Gamma(x) = \sum_{n=0}^{\infty} \left[a_n \sin\left(\frac{n\pi x}{w}\right) + b_n \cos\left(\frac{n\pi x}{w}\right) \right] \tag{6}$$

A Fourier series is smooth, continuous, infinitely differentiable, easy to manipulate, and efficient to compute. Furthermore, the theory of Fourier series guarantees that we can get arbitrarily close to the true function.

2.2 Γ and its derivatives

The conditions given by (4) and (5) allow us to simplify (6). First, the b_n are all equal to 0, since $\cos(0)$ and $\cos(n\pi)$ equal 1. Also, the coefficient a_0 is unnecessary, since $\sin(0) = 0$. This leaves us with

$$\Gamma(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{w}\right) \tag{7}$$

[1] See, for example, `http://en.wikipedia.org/wiki/Fourier_series`.

Differentiating (7) with respect to x yields

$$\frac{d\Gamma}{dx} = \sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{w}\right) \cos\left(\frac{n\pi x}{w}\right) \tag{8}$$

Differentiating again yields

$$\frac{d^2\Gamma}{dx^2} = \sum_{n=1}^{\infty} -a_n \left(\frac{n\pi}{w}\right)^2 \sin\left(\frac{n\pi x}{w}\right) \tag{9}$$

2.3 The resulting γ

Substituting (9) into (2) we have

$$\gamma(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{w}\right) \tag{10}$$

where

$$c_n = -a_n \left(\frac{n\pi}{w}\right)^2 \tag{11}$$

Note that (10) satisfies (2) and (3) for any choice of c_n.

3 Computational considerations

3.1 Collation points

We consider a specific set of N evenly-spaced locations along the aquitard, $\{x_1, x_2, \ldots, x_N\}$, where

$$x_m = \frac{mw}{N+1} \tag{12}$$

for $m = 1, 2, \ldots, N$. We will call these the *collocation points*.

3.2 Evaluation at the collation points

Substituting (12) into (7) yields

$$\Gamma(x_m) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi x_m}{w}\right) \tag{13}$$

$$= \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi m}{N+1}\right) \tag{14}$$

This is exactly what is computed by the *discrete sine transform* command in `MATLAB`: see `dst(a)`. If N is selected[2] such that $N+1$ is a power of 2 the computation is very, very efficient, since it uses a a *fast Fourier transform* algorithm. The same efficient computation is available for $\gamma(x_m)$ as well.

[2] Choose $N = 2^k - 1$ for some integer k. For example, $N = 2^6 - 1 = 63$.