A Brief Introduction to Multiquadric Interpolation

Background

Given a set of n distinct, but otherwise arbitrary, observation locations in $\Re^2$,

$$\{(x_i, y_i) : i = 1, 2, ..., n\}$$

and given the values of a real function at these locations

$$\{z_i = z(x_i, y_i) : i = 1, 2, ..., n\}$$

the construction of a function $F(x, y)$ mapping $\Re^2 \rightarrow \Re$ and satisfying

$$F(x_i, y_i) = z_i \qquad \forall_{i=1,2,...n} \tag{1}$$

is called *scattered data interpolation*. An important class[1] of scattered data interpolation methods is known as *radial basis function interpolation* (e.g. Madych and Nelson [1988] and Powell [1990]).

The general class of radial basis function interpolation includes *kriging* (e.g. Isaaks and Srivastava [1989]), thin plate splines (Duchon [1978]), and multiquadric interpolation (Hardy [1990]), to name a few popular specific cases. Furthermore, radial basis function methods have been found to be pragmatically superior to many other published methods of scattered data interpolation (e.g. Franke [1982]).

Multiquadric Interpolation

Multiquadric interpolation is one of the more popular methods[2] of radial basis function interpolation. Multiquadric interpolants take the form

$$F(x, y) = \mu + \sum_{i=1}^{n} w_i \Phi(d_i(x, y)) \tag{2}$$

μ is an unknown coefficient[3] computed by applying (1). (This is discussed in more detail below.)

w_i is an unknown coefficient associated with observation i, which is computed by applying (1). (This is discussed in more detail below.)

[1] A recent *Google* search of the internet using the same keyword-phrase "radial basis function interpolation" resulted in about 721,000 hits.

[2] A recent *Google* search of the internet using the keyword-phrase "multiquadric" resulted in 24,700 hits.

[3] μ is actually a Lagrange multiplier.

Φ is, in general, a continuous real-valued function defined on $[0, \infty)$, known as the *basis function*. For multiquadric interpolation the particular basis function is

$$\Phi(d) = -\sqrt{d^2 + R^2} \tag{3}$$

where R is a user defined smoothing parameter.

$d_i(x, y)$ is the separation distance between observation i and the generic specified location (x, y).

$$d_i(x, y) = \sqrt{(x_i - x)^2 + (y_i - y)^2} \tag{4}$$

Solving for the Coefficients

The $n + 1$ unknown coefficients, $\{\mu, w_1, w_2, \ldots, w_n\}$, are computed by solving the following system of $n + 1$ linear equations.

$$\mu + \sum_{i=1}^{n} w_i \Phi(d_i(x_j, y_j)) = z_j \qquad \forall_{j=1,2,\ldots,n} \tag{5}$$

$$\sum_{i=1}^{n} w_i = 0 \tag{6}$$

These equations are merely the repeated application of (1) at the n observation locations (i.e. the interpolation function must equal the observed values at the observation locations), plus one regularity condition. These $n + 1$ linear equations are known as the *interpolation equations*.

Thus, the application of the multiquadric interpolation algorithm requires the construction and solution of a system of $n + 1$ linear equations in $n + 1$ unknowns. The equations may be written compactly in a partitioned matrix format as:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mu \end{bmatrix} = \begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix} \tag{7}$$

where

$$\mathbf{A} = \begin{bmatrix} \Phi(d_1(x_1, y_1)) & \Phi(d_1(x_2, y_2)) & \cdots & \Phi(d_1(x_n, y_n)) \\ \Phi(d_2(x_1, y_1)) & \Phi(d_2(x_2, y_2)) & \cdots & \Phi(d_2(x_n, y_n)) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(d_n(x_1, y_1)) & \Phi(d_n(x_2, y_2)) & \cdots & \Phi(d_n(x_n, y_n)) \end{bmatrix} \tag{8}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \tag{9}$$

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \tag{10}$$

and

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \tag{11}$$

The solution of these equations is facilitated by the fact that $\mathbf{A}$ is *conditionally positive definite* with respect to $\mathbf{B}$ (e.g. Barnes [1994]).

Recipe for Gridding Data

Compute the multiquadric weights and associated Lagrange multiplier

1. Create the $\mathbf{A}$ matrix as given by (8).
2. Create the $\mathbf{B}$ matrix as given by (9).
3. Create the $\mathbf{z}$ matrix as given by (11).
4. Solve[4] the system of linear equations given by (7).

$$\begin{bmatrix} \mathbf{w} \\ \mu \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix} \tag{12}$$

These multiquadric weights, $\mathbf{w}$, and the associated Lagrange multiplier, μ, are functions of the input data only. They only need to be computed once. They do not change as a function of the generic location at which we compute an interpolated value.

Interpolate the value at each grid node

For each node in the grid carry out the following steps.

1. Compute the coordinates of the current grid node, which we will denote (x, y).
2. Create the matrix $\mathbf{p}$ using (3) and (4), as

$$\mathbf{p} = \begin{bmatrix} \Phi(d_1(x,y)) \\ \Phi(d_2(x,y)) \\ \vdots \\ \Phi(d_n(x,y)) \end{bmatrix} \tag{13}$$

Note that the $\mathbf{p}$ matrix does change from grid node to grid node.

[4]In `MATLAB` we use the "backslash" operator to solve such a system of linear equations. We do not actually compute the matrix inverse.

3. Compute the interpolated value using (2), which can be conveniently calculated as the matrix product
$$F(x,y) = \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}^T \begin{bmatrix} \mathbf{w} \\ \mu \end{bmatrix} \tag{14}$$
4. Save the interpolated value in the grid, and move on to the next grid node.

References

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