Sand and Gravel

1 Problem

Your employer, Shakespeare Construction, is a large and profitable heavy construction contractor. You are planning a job that will require a large amount of sand and gravel. It is estimated that the job will need

Table 1: Material demands for the large, upcoming, job.

Material	Demands [$\mathfrak{m}^3$]
Coarse gravel	20,000
Fine gravel	20,000
Sand	20,000

There are three pits from which your employer can obtain material: the Antony[1] Pit (call it pit A) and the Brutus[2] Pit (call it pit B), and the Caesar[3] Pit (call it pit C). The plan is to haul unwashed material from these three pits and separate (by screening and washing) the needed sand and gravel at the job site. Analysis shows that the material at each pit has the following compositions: Not surprisingly, the raw material from each pit has a different

Table 2: Material composition from the three pits.

Material	Pit A	Pit B	Pit C
Coarse gravel	20%	30%	10%
Fine gravel	15%	40%	30%
Sand	25%	20%	30%
Clay (waste)	40%	10%	30%

cost and availability.

Table 3: Material cost (including the cost of hauling and the cost of waste disposal), and the availability from the three pits during the construction season of interest.

Material	Pit A	Pit B	Pit C
Cost [$/$\mathfrak{m}^3$]	20	30	25
Available [$\mathfrak{m}^3$]	35,000	25,000	30,000

How much material should be purchased and hauled from each of the three pits to minimize the total cost while satisfying all of the constraints?

[1] "For Brutus is an honorable man, so are they all, all honorable men."
[2] "There is a tide in the affairs of men, which, taken at the flood, leads on to fortune."
[3] "Et tu, Brutè? Then fall, Caesar!"

2 Background - Linear Programming

The name *linear programming*[4] is often misconstrued today. It is not a computer language, or a computer science paradigm like "object oriented programming". In fact, the name pre-dates digital computers and computer science[5]. *Linear programming* is an class of optimization problems, and associated solution algorithms, that were originally created in secret during the Second World War. The optimization involves a linear cost function and linear constraints, thus the name.

The general form of the *linear programming problem* that we will consider looks like:

$$\underset{\mathbf{x}}{\text{minimize}} \ \mathbf{f}^T\mathbf{x} \tag{1}$$

Subject to:

$$\mathbf{A}\mathbf{x} \leqslant \mathbf{b} \tag{2}$$

$$\mathbf{C}\mathbf{x} = \mathbf{d} \tag{3}$$

$$\mathbf{lb} \leqslant \mathbf{x} \leqslant \mathbf{ub} \tag{4}$$

where

- `x` is the $(m \times 1)$ vector of decision variables (what we are looking for).
- `f` is the $(m \times 1)$ vector of known unit costs.
- `A` is the $(p \times m)$ matrix of known inequality constraint coefficients.
- `b` is the $(p \times 1)$ vector of known inequality constraint right-hand-side coefficients.
- `C` is the $(n \times m)$ matrix of known equality constraint coefficients.
- `d` is the $(n \times 1)$ vector of known equality constraint right-hand-side coefficients.
- `lb` is the $(m \times 1)$ vector of known decision variable lower bounds.
- `ub` is the $(m \times 1)$ vector of known decision variable upper bounds.

The available MATLAB function to solve this problem is `linprog`, which takes the basic form

```
[x] = linprog( f, A, b, C, d, lb, ub )        (5)
```

There are a number of additional, optional arguments and returns values, but this is the basic form.

[4]See, for example, `http://en.wikipedia.org/wiki/Linear_programming`.

[5]According to Wikipedia, "the first computer science degree program in the United States was formed at Purdue University in 1962." See `http://en.wikipedia.org/wiki/Computer_science`.

3 Formulation

Our *decision variables* will be the volume of raw material hauled from each of the three pits. Let

- x_A be the volume, $[m^3]$, from Pit A.
- x_B be the volume, $[m^3]$, from Pit B.
- x_C be the volume, $[m^3]$, from Pit C.

Our objective is to minimize the total cost:

$$\underset{x_A, x_B, x_C}{\text{minimize}}\, \text{Total Cost} = 20x_A + 30x_B + 25x_C \tag{6}$$

Subject to a set of operational constraints.

- Required coarse gravel

$$0.20x_A + 0.30x_B + 0.10x_C \geqslant 20,000 \tag{7}$$

- Required fine gravel

$$0.15x_A + 0.40x_B + 0.30x_C \geqslant 20,000 \tag{8}$$

- Required sand

$$0.25x_A + 0.20x_B + 0.30x_C \geqslant 20,000 \tag{9}$$

- Material availability

$$0 \leqslant x_A \leqslant 25,000 \tag{10}$$
$$0 \leqslant x_B \leqslant 50,000 \tag{11}$$
$$0 \leqslant x_C \leqslant 30,000 \tag{12}$$

Our objective function is linear in the decision variables, as are all of the constraints. Thus, our problem is a *linear programming* problem.

4 Solution

We let

$$\mathbf{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} 20 \\ 30 \\ 25 \end{bmatrix} \tag{13}$$

$$\mathbf{A} = \begin{bmatrix} 0.20 & 0.30 & 0.10 \\ 0.15 & 0.40 & 0.30 \\ 0.25 & 0.20 & 0.30 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 20,000 \\ 20,000 \\ 20,000 \end{bmatrix} \tag{14}$$

$$\mathbf{lb} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{ub} = \begin{bmatrix} 25,000 \\ 50,000 \\ 30,000 \end{bmatrix} \tag{15}$$

Then the call to MATLAB reads

```
[x, fval, exitflag, output, lambda] = linprog( f, -A, -b, [], [], lb, ub )
```
(16)

This requires a small bit of explanation. Note the "minus signs" in front of the `A` and `b` entries. These are necessary since the physical constraints are "$\geqslant$", while `linprog` expects "$\leqslant$". To keep `linprog` happy, we rewrite (7) through (9) as

$$-0.20x_A - 0.30x_B - 0.10x_C \leqslant -20,000 \tag{17}$$
$$-0.15x_A - 0.40x_B - 0.30x_C \leqslant -20,000 \tag{18}$$
$$-0.25x_A - 0.20x_B - 0.30x_C \leqslant -20,000 \tag{19}$$

which is accomplished by the addition of two "minus signs". Furthermore, as there are no equality constraints, `C` and `d` are null matrices.

The resulting optimal solution is

$$x_A = 25,000 \text{ [m}^3\text{]} \tag{20}$$
$$x_B = 44,643 \text{ [m}^3\text{]} \tag{21}$$
$$x_C = 16,071 \text{ [m}^3\text{]} \tag{22}$$

with an associated total cost of approximately $\$2,241,071$. Note that we use all of the available resource from Pit A (the least expensive pit), and then mix-and-blend from the other two pits to produce the required quantities. Multiplying `A` and `x` we find

$$\mathbf{Ax} = \begin{bmatrix} 20,000 \\ 26,429 \\ 20,000 \end{bmatrix} \tag{23}$$

Thus, we are producing just enough coarse gravel and sand, but excess fine gravel.