1.

(a) Risk-neutral probability:

$$\begin{cases} q_1 = r_{11}\pi_1 + r_{21}\pi_2 + r_{31}\pi_3 \\ q_2 = r_{12}\pi_1 + r_{22}\pi_2 + r_{32}\pi_3 \\ 1 = \pi_1 + \pi_2 + \pi_3 \end{cases} \Rightarrow$$

$$\begin{cases} 30 = 50\pi_1 + 10\pi_2 + 5\pi_2 \\ 1 = 2\pi_1 + 0 + 0 \\ 1 = \pi_1 + \pi_2 + \pi_3 \end{cases} \Rightarrow.$$

$$\begin{cases} \pi_1 = \frac{1}{2} \\ \pi_2 = \frac{1}{2} \\ \pi_3 = 0 \end{cases}$$

Suppose a European call option

Payoff of state i:

$(r_{i1} - K, 0)+$.

$60\% \, r_{11} = \frac{3}{5} \times 50 = 30$.

If $K = 30$, by asset 1 with $30 per unit and get $50.

Payoff matrix: $(20, 0, 0)$.

$(50\pi_1 + 10\pi_2 + 5\pi_3) = 30$

$\Rightarrow \pi_1 = \frac{1}{2}$.

$q_c = 20\pi_1 + 0 = 10$.

(b). $60\% \, r_{21} = \frac{3}{5} \times 10 = 6$.

If $K = 6$, buy asset 1 with $6 per unit and get payoff matrix (if only asset 1 is included:)

$(46, 4, 0)$

$q_c = (46\pi_1 + 4\pi_2) = 25$.

(c). $60\% \, r_{12} = \frac{3}{5} \times 2 = 1.2$.

The payoff matrix:

$(48.8, 8.8, 3.8)$.

$q_c = \frac{1}{2} \cdot 48.8 + \frac{1}{2} \cdot 8.8$

$= 28.8$

2. (a).

From $t=0$ to $t=1$:

$q_0 = \frac{1}{1+r}(\lambda q_u + (1-\lambda) q_l)$

$q_0 = 100$

$1+r = 1+0.04 = 1.04$

$q_u = 110$, $q_l = 80$.

$\therefore 100 = \frac{1}{1.04}(110\lambda + 80(1-\lambda))$

$\Rightarrow 104 = 110\lambda + 80 - 80\lambda$

$\Rightarrow 104 = 30\lambda + 80$

$\Rightarrow 24 = 30\lambda$

$\Rightarrow \lambda = 0.8$.

$1-\lambda = 0.2$

Check:

If price goes up to 110:

$q_0 = 110$. $q_u = 121$, $q_l = 88$

$110 = \frac{1}{1.04}(121\lambda + 88(1-\lambda))$

$\Rightarrow \lambda = 0.8$.

The same in the case of $q_0 = 80$.

Thus, the risk-neutral probability measure is $(0.8, 0.2)$

(b).

At state 1,

the price of the bond is 121.

$(121-K)_+ = (121-90)_+ = 31$.

The price of the bond is 88 at state 2.

$(88-K)_+ = (88-90)_+ = 0$.

At state 3:

the price of the bond is 88.

The same as above.

At state 4:

the price of the bond is 64.

$(64-K)_+ = (64-90)_+ = 0$.

$q_c(110) = \frac{1}{1+r}(\lambda \cdot 31 + (1-\lambda)\cdot 0)$

$= \frac{1}{1.04} \cdot 0.8 \cdot 31$

≈ 23.85

$q_c(80) = 0$.

(c)

Suppose a portofolio of α shares of the stock and β shares of the bond can replicate the call option.

To find (α, β), we need to solve the following systems of equations:

$23.85 = 110\alpha + 1.04\beta$

$0 = 80\alpha + 1.04\beta$

$\Rightarrow 30\alpha = 23.85$

$\alpha = 0.795$.

$1.04\beta = -80\alpha = -63.6$

$\beta = \frac{-63.6}{1.04} \approx -61.15$.

Hence the portfolio consists of 0.795 shares of the stock and -61.15 shares of the bond

$q_{c0} = \alpha q_0 + \beta$

$\approx 0.795 \times 100 - 61.15$

$= 18.35$.

(d) Suppose that

$q_c > q_0\alpha + \beta = 18.35$

At $t=0$, the ~~investors~~

borrows \$61.15,

sells 1 call option on the stock and ~~buy~~ buys 0.795 shares of the stock.

$79.5 - 61.15 - q_c$

$= 18.35 - q_c < 0$

If at $t=1$, $q_1 = \$110$,

then the option is exercised: the investor sells 0.795 shares of the stock, get:

$0.795 \times \$110 = \$87.45$

The investor returns \$23.85 for the call option, and pays back the debt on the bond:

$1.04 \times 61.15 \approx 63.6$

Since $\$23.85 + \$63.6 = \$87.45$, the payoff on the portofolio is 0.

If at $t=1$, $q_1 = \$80$, the investor won't exercise the option, ~~pay~~ sells 0.795 shares of the ~~bond~~ stock for

$0.795 \times \$80 = \$63.6$.

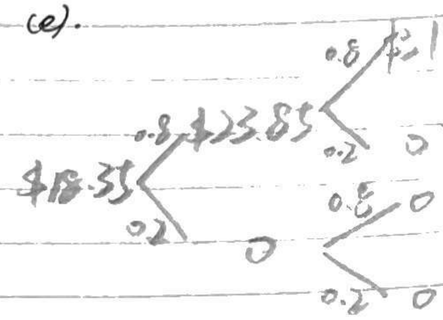
The debt on the bond this investor needs to pay back is

$\$1.04 \times 61.15 \approx \$63.6$.

The payoff on the portfolio is 0.

Hence if $q_c > 18.35$, it's possible to create an arbitrage portofolio.

(e).



3. (a)

If not invest:

when profit goes up,

$V(\omega_1) = 15 + \sum_{t=1}^{\infty} \beta^t \pi_t(\omega_1)$

$= 15 + \frac{1.1 \times 15}{r}$

$= 15 + \frac{16.5}{0.01}$

$= 15 + 1650 = 1665.$

when profit goes down,

$V(\omega_2) = 15 + \sum_{t=1}^{\infty} \beta^t \pi_t(\omega_2)$

$= 15 + \frac{0.9 \times 15}{r}$

$= 15 + \frac{13.5}{0.01}$

$= 15 + 1350 = 1365.$

$V_{0,0} = 0.6 \times 1665 + .4 \times 1365$

$= 1545.$

If invest:

when profit goes up,

$V_0^{\omega_1}(I) = 15 - I + \sum_{t=1}^{\infty} \beta^t \pi_t^I(\omega_1)$

$= 15 - I + \frac{1.15 \times 1.1 \times 15}{r}$

$= 15 - I + \frac{18.975}{.01}$

$= 1912.5 - I.$

when profit goes down:

$V_{0,0}^{w_2}(I) = 15 - I + \sum_{t=1}^{\infty} \beta^t \pi_t^I(w_2)$

$= 15 - I + \frac{1.15 \times 0.9 \times 15}{r}$

$= 15 - I + \frac{15.525}{0.01}$

$= 1567.5 - I$

$V_{0,0}^I(I) = 0.6 \times (1912.5 - I) + 0.4(1567.5 - I)$

$= 1774.5 - I$

if $I < 1774.5$.

The net gain is

$V_{0,0}^I - V_{0,0}^0$

$= 1774.5 - I - 1545$

$= 229.5 - I$ if $I < 229.5$

Thus,

$V_{0,0}(I) = \begin{cases} 229.5 - I, & \text{if } 0 < I \leq 229.5 \\ 0, & \text{if } I > 229.5 \end{cases}$

(b). At $t=1$,

if profit goes up,

$V_{1,1}^{w_1}(I) = 1.5 \times 1.1 - I + \sum_{t=1}^{\infty} \beta^t \pi_t^I(w_1)$

$= 16.5 - I + 1897.5$

$= 1914 - I$, if $I \leq 1914$

$\therefore V_{1,1}^{w_1}(I) = \begin{cases} 1914 - I, & \text{if } I \leq 1914 \\ 0, & \text{if } I > 1914. \end{cases}$

if profit goes down:

$V_{1,1}^{w_2}(I) = 15 \times 0.9 - I + \sum_{t=1}^{\infty} \beta^t \pi_t^I(w_2)$

$= 13.5 - I + 1552.5$

$= 1566 - I$ if $I < 1566$.

$\therefore V_{1,1}^{w_2}(I) = \begin{cases} 1566 - I & \text{if } 0 < I \leq 1566 \\ 0 & \text{if } I > 1566. \end{cases}$

Hence, if $I \leq 1566$.

$V_{0,1}(I) = (0.6(1914 - I) + 0.4(1566 - I)) \times \frac{1}{1.01}$

$= \frac{1774.8 - I}{1.01}$

if $1566 < I \leq 1914$:

$V_{0,1}(I) = \frac{0.6(1914 - I)}{1.01} = \frac{1148.4 - 0.6I}{1.01}$

if $I > 1914$, the NPV of investment is $V_{0,1}(I) = 0$.

If $I \leq 1566$,

the net gain is:

$\frac{1774.8 - I}{1.01} - 1545 = \frac{1774.8}{1.01} - \frac{I}{1.01} - 1545$

$\approx 212.23 - \frac{I}{1.01}$.

if $I \leq 214.35$.

If $1566 < I < 1914$:

the net gain is:

$\frac{0.6(1914 - I)}{1.01} - 1545$

$= \frac{1148.4}{1.01} - \frac{0.6I}{1.01} - 1545$

$\approx -407.97 - \frac{0.6}{1.01} I$.

Usually we assume $I > 0$.

$\therefore$ It's not optimal to invest in this case.

Thus

$V_{0,1}(I) = \begin{cases} 212.3 - \frac{I}{1.01}, & \text{if } 0 < I \leq 214.35 \\ 0, & \text{if } I > 214.35. \end{cases}$

(c)

If $0 < I < 214.35$ and

$229.5 - I < 212.23 - \frac{I}{1.01}$

$229.5 < 212.23 - \frac{I}{1.01} + I$

$229.5 - 212.23 < I(1 - \frac{1}{1.01})$

$I(\frac{1.01 - 1}{1.01}) > 17.27$

$I \times \frac{0.01}{1.01} > 17.27$.

$I > 1744.27 > 227.5 > 214.35$.

if profit goes up,
and not invest:

$$V_{1,1}^{w_1} = 15\times 1.1 + \sum_{t=1}^{\infty} \beta^t \pi_t(w_1)$$
$$= 15\times 1.1 + \frac{15\times 1.1}{r}$$
$$= 16.5 + \frac{16.5}{0.01}$$
$$= 1666.5$$

The net gain is:

$1914 - I - 1666.5 = 247.5 - I$

if $I \le 247.5$.

$$\therefore V_{1,1}^{w_1}(I) = \begin{cases} 247.5 - I & \text{if } 0 < I \le 247.5 \\ 0, & \text{if } I > 247.5 \end{cases}$$

if profit goes down,
and not invest:

$$V_{1,1}^{w_2} = 15\times 0.9 + \sum_{t=0}^{\infty} \beta^t \pi_t(w_2)$$
$$= 15\times 0.9 + \frac{15\times 0.9}{0.01}$$
$$= 13.5 + \frac{13.5}{0.01}$$
$$= 1363.5$$

The net gain is

$1566 - I - 1363.5 = 202.5 - I$

if $I \le 202.5$

$$\therefore V_{1,1}^{w_2}(I) = \begin{cases} 202.5 - I, & \text{if } 0 < I \le 202.5 \\ 0, & \text{if } I > 202.5 \end{cases}$$

c)

If $0 < I \le 202.5$,

$$V_{0,1}(I) = [0.6\times(247.5 - I) + 0.4\times(202.5 - I)] \times \frac{1}{1.01}$$
$$= \frac{229.5 - I}{1.01}$$

If $202.5 < I \le 247.5$

$$V_{0,1}(I) = \frac{0.6\times(247.5 - I)}{1.01} = \frac{148.5 - 0.6I}{1.01}$$

If $I > 247.5$, $V_{0,1}(I) = 0$

$$\therefore V_{0,1}(I) = \begin{cases} \frac{229.5 - I}{1.01} & 0 < I \le 202.5 \\ \frac{148.5 - 0.6I}{1.01} & 202.5 < I \le 247.5 \\ 0 & I > 247.5 \end{cases}$$

At $t=1$,

if $229.5 - I > 0.6(247.5 - I) \Rightarrow I < 202.5$.

At $t=0$:

if $0 < I \le 202.5$:

$229.5 - I > \frac{229.5 - I}{1.01}$

Best to invest at $t=0$.

If $202.5 < I < 229.5$

$$229.5 - I - \frac{148.5 - 0.6I}{1.01}$$
$$= 229.5 - I - 147.03 + \frac{0.6I}{1.01}$$
$$= 82.47 - \frac{0.4I}{1.01}$$
$$= 0.4(203.18 - I)$$

if $202.5 < I \le 203.18$,

$229.5 - I \geq \frac{148.5 - 0.6I}{1.01}$

Best to invest at $t=0$.

if $203.16 < I \le 229.5$,

$229.5 - I < \frac{148.5 - 0.6I}{1.01}$

Best to invest at $t=1$ when increasing profit is observed.

If $229.5 < I < 247.5$,

$\frac{0.6(247.5 - I)}{1.01} > 0$.

Best to invest at $t=1$ when increasing profit is observed.

If $I > 247.5$, not to invest in any cases.

In conclusion:

if $0 < I \le 203.18 \to 203.16$,

best to invest at $t=0$.

if $203.18 < I \le 247.5$,

best to invest at $t=1$.

if $I > 247.5$,

not to invest in any cases.

$\therefore\ \cancel{229.5 - I > 212.23 - \frac{I}{1.01}}$

~~if $I < 1744.27$~~

~~Best to invest at $t=0$~~

~~if $0 < I < 229.5$.~~

~~Not to invest in any cases~~

~~if $I > 229.5$.~~

4.

(a) For the production plan (y_0, y_1), the current dividend on equity $\delta_0 = -y_0$, and the future dividend is $\delta_1 = y_1$.

Let s be the number of shares of the firm and b be the number of shares of the bond that each consumer has in equilibrium.

The representative consumers solves the following problems:

$$\max_{(x_l, x_h, s, b)} \frac{1}{2}\ln x_l + \frac{1}{2}\ln x_h$$

$$\text{s.t. } b + e_0 s = 5 + \frac{1}{20}(e_0 - y_0)$$

$$x_l = y_l s + (1+r)b$$

$$x_h = y_h s + (1+r)b.$$

Let $\lambda_0 \geq 0$, $\lambda_l \geq 0$, and $\lambda_h \geq 0$ be langrange multipliers assigned, respectively, to budget constraints at $t=0$ and $t=1$.

Write down the Langrangian:

$$\mathcal{L}(x_l, x_h, s, b, \lambda_0, \lambda_l, \lambda_h) = \frac{1}{2}\ln x_l + \frac{1}{2}\ln x_h - \lambda_0(b + e_0 s - 5 - \frac{1}{20}(e_0 - y_0)) - \lambda_l(x_l - y_l s - (1+r)b) - \lambda_h(x_h - y_h s - (1+r)b)$$

The FOC's are:

$$\frac{\partial \mathcal{L}}{\partial x_l} = \frac{1}{2x_l} - \lambda_l = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_h} = \frac{1}{2x_h} - \lambda_h = 0$$

$$\frac{\partial \mathcal{L}}{\partial s} = -\lambda_0 e_0 + \lambda_l y_l + \lambda_h y_h = 0$$

$$\frac{\partial \mathcal{L}}{\partial b} = -\lambda_0 + (\lambda_l + \lambda_h)(1+r) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_0} = 5 + \frac{1}{20}(e_0 - y_0) - b - e_0 s = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_l} = y_l s + (1+r)b - x_l = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_h} = y_h s + (1+r)b - x_h = 0$$

(b) Set $\pi_l = \frac{\lambda_l}{\lambda_l + \lambda_h}$ $\pi_h = \frac{\lambda_h}{\lambda_l + \lambda_h}$

$$e_0 = \frac{\lambda_l y_l + \lambda_h y_h}{\lambda_0} = \frac{\lambda_l y_l + \lambda_h y_h}{(1+r)(\lambda_l + \lambda_h)} = \frac{1}{1+r}(\pi_l y_l + \pi_h y_h)$$

$$\frac{y_l}{1+r}s + b - \frac{x_l}{1+r} = 0$$

$$\frac{\pi_l y_l}{1+r}s + \pi_l b = \frac{\pi_l x_l}{1+r}$$

$$\frac{\pi_h y_h}{1+r}s + \pi_h b = \frac{\pi_h x_h}{1+r}$$

$$\frac{\pi_l y_l + \pi_h y_h}{1+r}s + \overset{1}{(\pi_l + \pi_h)}b = \frac{\pi_l x_l + \pi_h x_h}{1+r}$$

$$\Leftrightarrow e_0 s + b = \frac{\pi_l x_l + \pi_h x_h}{1+r}$$

$$\frac{\lambda_n x_n + \lambda_l x_l}{1+r} = 5 + \frac{1}{20}\left[(\lambda_l y_l + \lambda_n y_n)\frac{1}{1+r} - y_0\right]$$

If the EPV is positive, all consumers agree that the firm should choose y_0 to: $\max_{y_0} \frac{(18\lambda_l + 25.2\lambda_n)\sqrt{y_0}}{1+r} - y_0$

The FOC is

$$\frac{(18\lambda_l + 25\lambda_n)}{2(1+r)\sqrt{y_0}} = 1$$

$$\Rightarrow \frac{9\lambda_l + 12.6\lambda_n}{1+r} = \sqrt{y_0}$$

$$\Rightarrow y_0 = \left(\frac{9\lambda_l + 12.6\lambda_n}{1+r}\right)^2$$

hence:

$$y_l = \frac{18(9\lambda_l + 12.6\lambda_n)}{1+r}$$

$$y_n = \frac{25.2(9\lambda_l + 12.6\lambda_n)}{1+r} = \frac{25.2}{18} y_l = 1.4 y_l$$

Check the firm's EPV:

$$\frac{18\lambda_l + 25\lambda_n}{1+r} \cdot \frac{9\lambda_l + 12.6\lambda_n}{1+r} - \left(\frac{9\lambda_l + 12.6\lambda_n}{1+r}\right)^2$$

$$= 2\left(\frac{9\lambda_l + 12.6\lambda_n}{1+r}\right)^2 - \left(\frac{9\lambda_l + 12.6\lambda_n}{1+r}\right)^2$$

$$= \left(\frac{9\lambda_l + 12.6\lambda_n}{1+r}\right)^2$$

$$= y_0.$$

Therefore, all the consumers agree on the firm's objectives.

(d)

~~The~~

(c) The market clearing conditions:

$$y_0 = 20.5$$
$$20x_l = y_l$$
$$20x_n = y_n$$
$$20s = 1$$
$$20b = 0$$

$$\Rightarrow \frac{\lambda_l y_l + \lambda_n y_n}{1+r} = 20\left(\frac{\lambda_l x_l + \lambda_n x_n}{1+r}\right)$$

$$= 20\cdot\left(5 + \frac{y_0}{20}\right) = 100 + y_0$$

$$= 100 + 100 = 200$$

$$\left(\frac{x_n}{x_l} = \frac{2\lambda_n}{2\lambda_l} = \frac{\lambda_n}{\lambda_l} = \frac{\lambda_n}{\lambda_l}\right) X.$$

$$\frac{x_n}{x_l} = \frac{2\lambda_l}{2\lambda_n} = \frac{\lambda_l}{\lambda_n} = \frac{\lambda_l}{\lambda_n}.$$

$$\therefore \frac{x_n}{x_l} = \frac{y_n}{y_l} = \frac{25.2}{18} = 1.4$$

$$\therefore \frac{1-\lambda_n}{\lambda_n} = 1.4$$

$$\Rightarrow \lambda_n = \frac{1}{2.4} \approx 0.42$$

$$\lambda_l \approx 0.58$$

$$y_l = \frac{189}{1+r}$$

$$y_n = \frac{264.6}{1+r}.$$

$$\frac{(1-\frac{1}{2.4})\cdot\frac{189}{1+r} + \frac{1}{2.4}\cdot\frac{264.6}{1+r}}{1+r} = 200$$

$$\Rightarrow \frac{110.25 + 110.25}{(1+r)^2} = 200$$

$$\Rightarrow r = 0.05.$$

$$y_l = 180 \quad y_n = 252$$

$$x_l = 9 \quad x_n = 12.6$$

$$e_0 = \frac{1}{1.05}\left((1-\frac{1}{2.4})\cdot 180 + \frac{1}{2.4}\cdot 252\right)$$

$$= 200$$

$$s = \frac{1}{20}$$

$$b = 0.$$