Yu Xia's Answer for Problem Set 5

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(a)

$$\max_{(x_0, x_1)} \sqrt{x_0} + \sqrt{x_1}$$

$$\text{subject to}$$

$$x_0 = e_0 = 121,$$

$$x_1 = e_1 = 49,$$

$$x_0 \geqslant 0, x_1 \geqslant 0.$$

Plug in we have:

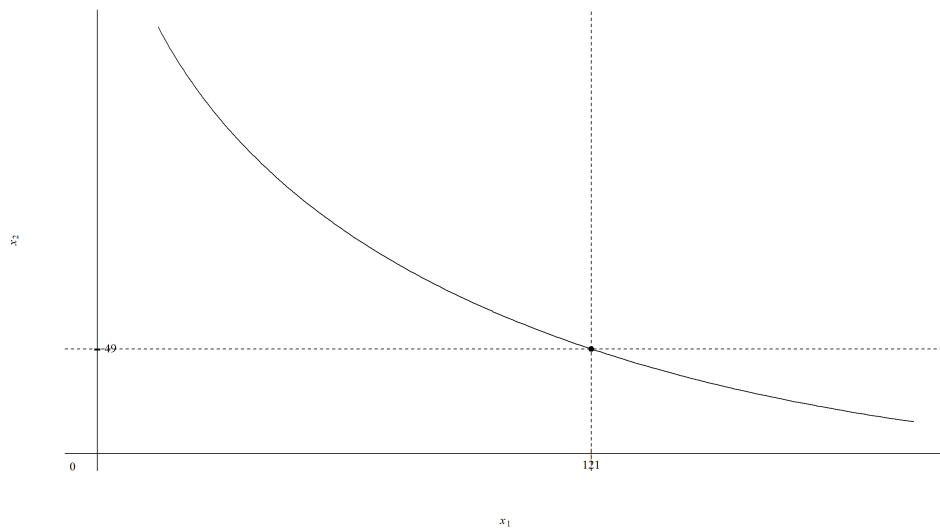$$V(121, 49) = \sqrt{121} + \sqrt{49} = 11 + 7 = 18$$

Figure 1: Consumer's indifference curve through the optimal consumption bundle of 1(a)

(b)

The payoff matrix is:

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} 1+r & 1 \\ 1+r & 1 \end{pmatrix}$$

The asset price is $(1, 1)$.

The consumer's problem is:

$$\max_{(x_0, x_1)} \sqrt{x_0} + \sqrt{x_1}$$

$$\text{subject to}$$

$$x_0 + b = 121,$$
$$x_1 = 49 + (1+r)\, b,$$
$$x_0 \geqslant 0, x_1 \geqslant 0.$$

$b = 121 - x_0$

$x_1 = 49 + (1+r)(121 - x_0) = 49 + 121(1+r) - x_0(1+r)$

$x_0(1+r) + x_1 = 49 + 121(1+r)$, the budget constraint.

$$MU_0 = \frac{1}{2}(x_0)^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x_0}}$$

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$$MU_1 = \frac{1}{2}\left(x_1\right)^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x_1}}$$

$$\text{MRS} = \frac{MU_0}{MU_1} = \frac{\dfrac{1}{2} \cdot \dfrac{1}{\sqrt{x_0}}}{\dfrac{1}{2} \cdot \dfrac{1}{\sqrt{x_1}}} = \frac{\dfrac{1}{\sqrt{x_0}}}{\dfrac{1}{\sqrt{x_1}}} = \frac{\dfrac{1}{\sqrt{x_0}}}{\dfrac{1}{\sqrt{x_1}}} = \frac{\sqrt{x_1}}{\sqrt{x_0}} = \sqrt{\frac{x_1}{x_0}}$$

On the other hand,

$\text{MRS} = 1 + r$

$$\therefore \sqrt{\frac{x_1}{x_0}} = 1 + r$$

$$\implies \frac{x_1}{x_0} = (1+r)^2$$

$$\implies x_1 = (1+r)^2 x_0$$

Plug into constraint:

$$x_0(1+r) + (1+r)^2 x_0 = 49 + 121(1+r)$$

$$\implies x_0 + x_0(1+r) = \frac{49}{1+r} + 121$$

$$\implies x_0(2+r) = \frac{49}{1+r} + 121$$

$$x_0 = \frac{49}{(1+r)(2+r)} + \frac{121}{2+r}$$

$$x_1 = (1+r)^2\left(\frac{49}{(1+r)(2+r)} + \frac{121}{2+r}\right) = \frac{49(1+r)}{2+r} + \frac{121(1+r)^2}{2+r} = \frac{49(1+r) + 121(1+r)^2}{2+r}$$

$\because r = 0.03$

$\therefore 1.03 x_0 + x_1 = 49 + 121 \times 1.03 = 173.63$

$$x_0 = \frac{49}{1.03 \times 2.03} + \frac{121}{2.03} \approx \boxed{83.0408}$$

$$x_1 = \frac{49 \times 1.03 + 121(1.03)^2}{2.03} \approx \boxed{88.09798}$$

$b = 121 - x_0 \approx \boxed{37.9592}$

The maximal utility level is:

$$V(x_0, x_1) = V\left(\frac{49}{(1+r)(2+r)} + \frac{121}{2+r}, \frac{49(1+r) + 121(1+r)^2}{2+r}\right)$$

$$= \sqrt{\frac{49}{(1+r)(2+r)} + \frac{121}{2+r}} + \sqrt{\frac{49(1+r) + 121(1+r)^2}{2+r}}$$

$$= \sqrt{\frac{49}{1.03 \times 2.03} + \frac{121}{2.03}} + \sqrt{\frac{49 \times 1.03 + 121(1.03)^2}{2.03}}$$

$$\approx \boxed{18.49872}$$

Check the corner solution:

If $x_0 = 0$, $0 + x_1 = 49 + 121(1+r) \implies x_1 = 49 + 121 \times 1.03 = 173.63$

$$V(0, 173.63) = \sqrt{173.63} \approx 13.17687 < V\left(\frac{49}{(1+r)(2+r)} + \frac{121}{2+r}, \frac{49(1+r) + 121(1+r)^2}{2+r}\right)$$

If $x_1 = 0$, $(1+r)x_0 + 0 = 49 + 121(1+r) \implies x_0 = \frac{49}{1+r} + 121 = \frac{49}{1.03} + 121 \approx 168.5728$

$$V\left(\frac{49}{1+r} + 121, 0\right) = \sqrt{\frac{49}{1+r} + 121} = \sqrt{\frac{49}{1.03} + 121} \approx 12.98356$$

$$< V\left(\frac{49}{(1+r)(2+r)} + \frac{121}{2+r}, \frac{49(1+r) + 121(1+r)^2}{2+r}\right)$$
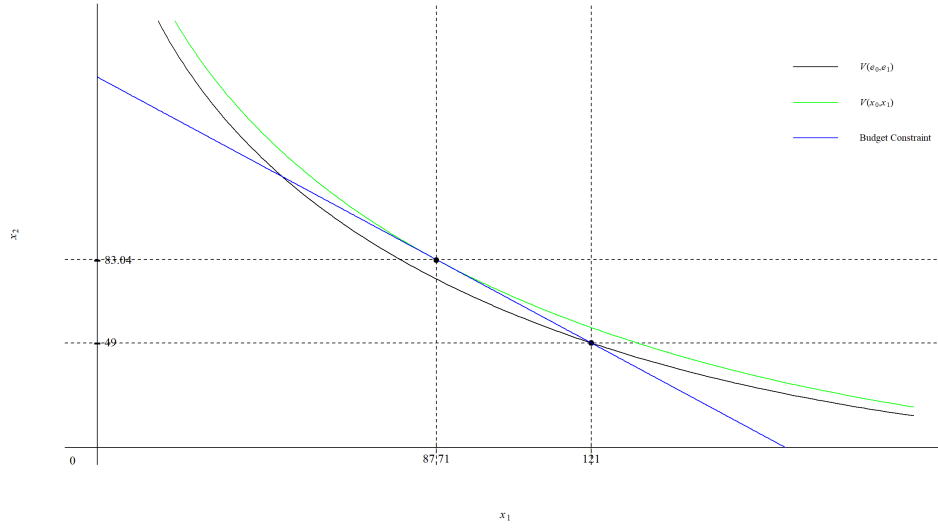
Figure 2: Consumer's indifference curve through the optimal consumption bundle of 1(b)

(c)

The consumer is better off. We can see this from the diagram.

We can also prove it by math:

$$V(121, 49) = 11 + 7 = 18 < V\left(\frac{49}{(1+r)(2+r)} + \frac{121}{2+r}, \frac{49(1+r) + 121(1+r)^2}{2+r}\right)$$

The consumer save part of his/her money, and invest on the risk-free bond at the same time, in order to maximize the utility.

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(a)

$$\max_{(x_1, x_2)} \frac{1}{4}\sqrt{x_1} + \frac{3}{4}\sqrt{x_2}$$

subject to

$$x_1 = e_1 = 121,$$
$$x_2 = e_2 = 49,$$
$$x_1 \geqslant 0, x_2 \geqslant 0.$$

Plug in we have:

$$V(121, 49) = \frac{1}{4}\sqrt{121} + \frac{3}{4}\sqrt{49} = \frac{1}{4} \times 11 + \frac{3}{4} \times 7 = \frac{11 + 21}{4} = \frac{32}{4} = \boxed{8}$$
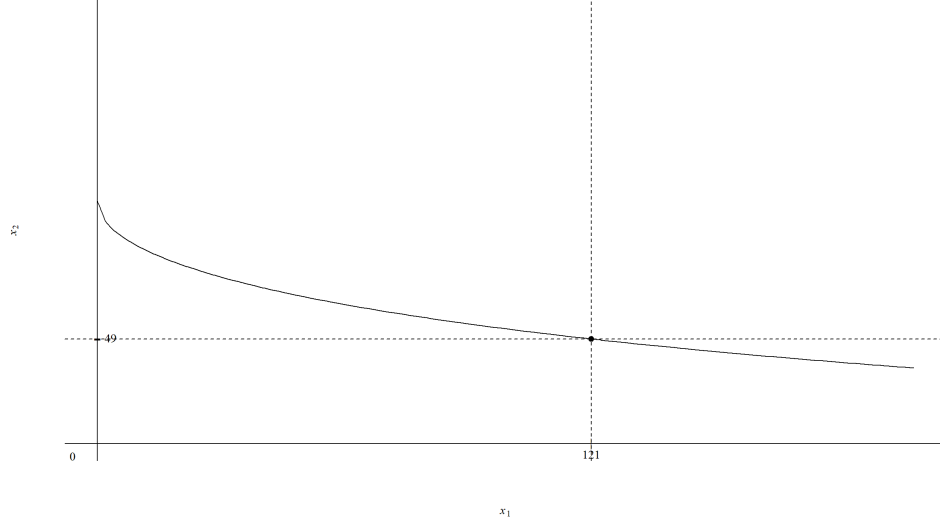

Figure 3: consumer's indifference curve through the optimal consumption bundle of 2(a)

(b)

The payoff matrix is:

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} 1 + r & 1 \\ 1 + r & 0 \end{pmatrix}$$

The asset price is $(1, q)$.

The consumer's problem is:

$$\max_{(x_1, x_2)} \frac{1}{4}\sqrt{x_1} + \frac{3}{4}\sqrt{x_2}$$

subject to

$$b + qk = 0,$$
$$x_1 = 121 + (1 + r) b + k,$$
$$x_2 = 49 + (1 + r) b,$$
$$x_1 \geqslant 0, x_2 \geqslant 0.$$

$$\implies b = -qk \implies x_1 = 121 + (1 + r)(-qk) + k = 121 - qk - rqk + k = 121 + (1 - q - rq) k$$

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It also implies:

$x_2 = 49 + (1 + r)(-qk) = 49 - qk - rqk \implies x_2 + qk + rqk = 49 \implies (1 + r) qk = 49 - x_2$

$\implies k = \dfrac{49 - x_2}{(1 + r) q}$

Thus,

$\implies x_1 = 121 + (1 - q - rq) \cdot \dfrac{49 - x_2}{(1 + r) q}$

$\implies (1 + r) qx_1 = 121(1 + r) q + (1 - q - rq)(49 - x_2) = 121(1 + r) q + (1 - q - rq) \cdot 49 + (1 - q - rq)(-x_2)$

$\implies (1 + r) qx_1 + (1 - q - rq) x_2 = 121(1 + r) q + 49(1 - q - rq)$

$\implies (1 + r) qx_1 + (1 - q - rq) x_2 = 121q + 121rq + 49 - 49q - 49rq$

$\implies (1 + r) qx_1 + (1 - q - rq) x_2 = 72(1 + r) q + 49$, the budget constraint.

$MU_1 = \dfrac{1}{4} \cdot \dfrac{1}{2}(x_1)^{-\frac{1}{2}} = \dfrac{1}{8} \cdot \dfrac{1}{\sqrt{x_1}}$

$MU_2 = \dfrac{3}{4} \cdot \dfrac{1}{2}(x_2)^{-\frac{1}{2}} = \dfrac{3}{8} \cdot \dfrac{1}{\sqrt{x_2}}$

$\text{MRS} = \dfrac{MU_1}{MU_2} = \dfrac{\dfrac{1}{8} \cdot \dfrac{1}{\sqrt{x_1}}}{\dfrac{3}{8} \cdot \dfrac{1}{\sqrt{x_2}}} = \dfrac{\dfrac{1}{\sqrt{x_1}}}{\dfrac{3}{\sqrt{x_2}}} = \dfrac{1}{\dfrac{3\sqrt{x_1}}{\sqrt{x_2}}} = \dfrac{\sqrt{x_2}}{3\sqrt{x_1}}$

On the other hand,

$\text{MRS} = \dfrac{(1 + r) q}{1 - q - rq}$

$\therefore \dfrac{\sqrt{x_2}}{3\sqrt{x_1}} = \dfrac{(1 + r) q}{1 - q - rq}$

In order to generate a positive supply and demand for the risky asset, the expected value of the consumer's portfolio should be 0, that is,

$pk - qk = 0$, where $p = 0.25 \implies \boxed{q = \dfrac{1}{4}}$

Rewrittng budget constraint:

$\dfrac{(1 + r) x_1}{4} + \left(1 - \dfrac{1}{4} - \dfrac{r}{4}\right) x_2 = 18(1 + r) + 49$

$\implies (1 + r) x_1 + (3 - r) x_2 = 72(1 + r) + 196$

$\implies (1 + r) x_1 + (3 - r) x_2 = 72 + 72r + 196$

$(1 + r) x_1 + (3 - r) x_2 = 268 + 72r$

Also,

$$\frac{\sqrt{x_2}}{3\sqrt{x_1}} = \frac{\frac{1}{4}(1+r)}{1 - \frac{1}{4} - \frac{1}{4}r} = \frac{1+r}{3-r} \implies \frac{x_2}{9x_1} = \frac{(1+r)^2}{(3-r)^2} \implies x_2 = \frac{(1+r)^2}{(3-r)^2} \cdot 9x_1$$

The budget constraint becomes:

$$\frac{1}{4}(1+r)x_1 + \left(1 - \frac{1}{4} - \frac{1}{4}r\right) \cdot \frac{(1+r)^2}{(3-r)^2} \cdot 9x_1 = 72(1+r) \cdot \frac{1}{4} + 49$$

$$\implies \frac{1}{4}(1+r)x_1 + \frac{3-r}{4} \cdot \frac{(1+r)^2}{(3-r)^2} \cdot 9x_1 = 72(1+r) \cdot \frac{1}{4} + \frac{4 \times 49}{4}$$

$$\implies (1+r)x_1 + \frac{(1+r)^2}{3-r} \cdot 9x_1 = 72(1+r) + 4 \times 49$$

$$\implies (1+r)x_1\left(1 + \frac{9(1+r)}{3-r}\right) = 72(1+r) + 196$$

$$\implies x_1\left(\frac{3-r}{3-r} + \frac{9+9r}{3-r}\right) = 72 + \frac{196}{1+r}$$

$$\implies x_1 \cdot \frac{12+8r}{3-r} = 72 + \frac{196}{1+r}$$

$$\implies x_1 \cdot \frac{3+2r}{3-r} = 18 + \frac{49}{1+r} = \frac{18+18r+49}{1+r} = \frac{67+18r}{1+r}$$

$$\implies x_1 = \frac{67+18r}{1+r} \cdot \frac{3-r}{3+2r}$$

$$\implies x_1 = \frac{(3-r)(67+18r)}{(1+r)(3+2r)}$$

$$x_2 = \frac{(1+r)^2}{(3-r)^2} \cdot 9x_1 = \frac{(1+r)^2}{(3-r)^2} \cdot 9 \cdot \frac{(67+18r)(3-r)}{(1+r)(3+2r)} = \frac{1+r}{3-r} \cdot 9 \cdot \frac{67+18r}{3+2r}$$

$$x_2 = \frac{9(1+r)(67+18r)}{(3-r)(3+2r)}$$

$$(1+r)b = x_2 - 49 = \frac{9(1+r)(67+18r)}{(3-r)(3+2r)} - 49$$

$$b = \frac{9(67+18r)}{(3-r)(3+2r)} - \frac{49}{1+r}$$

$$k = \frac{49 - \dfrac{9(1+r)(67+18r)}{(3-r)(3+2r)}}{(1+r) \cdot \dfrac{1}{4}} = \frac{196}{1+r} - \frac{36(67+18r)}{(3-r)(3+2r)}$$

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The maximal utility level is:

$$V(x_1, x_2) = V\left(\frac{(3-r)(67+18r)}{(1+r)(3+2r)}, \frac{9(1+r)(67+18r)}{(3-r)(3+2r)}\right)$$

$$= \frac{1}{4}\sqrt{\frac{(3-r)(67+18r)}{(1+r)(3+2r)}} + \frac{3}{4}\sqrt{\frac{9(1+r)(67+18r)}{(3-r)(3+2r)}}$$

$$= \frac{1}{4}\sqrt{\frac{(3-r)(67+18r)}{(1+r)(3+2r)}} + \frac{9}{4}\sqrt{\frac{(1+r)(67+18r)}{(3-r)(3+2r)}}$$

$$= \frac{1}{4}\sqrt{\frac{67+18r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}} + 9\sqrt{\frac{1+r}{3-r}}\right)$$

Check the corner solution:

If $x_1 = 0$, $0 + (3-r)x_2 = 268 + 72r \implies x_2 = \dfrac{268+72r}{3-r}$

$$V\left(0, \frac{268+72r}{3-r}\right) = \frac{3}{4}\sqrt{\frac{268+72r}{3-r}} = \frac{3}{4}\sqrt{\frac{4\cdot(67+18r)}{3-r}} = \frac{3}{2}\sqrt{\frac{67+18r}{3-r}}$$

If $x_2 = 0$, $(1+r)x_1 + 0 = 268 + 72r \implies x_1 = \dfrac{268+72r}{1+r}$

$$V\left(\frac{268+72r}{1+r}, 0\right) = \frac{1}{4}\sqrt{\frac{268+72r}{1+r}} = \frac{1}{2}\sqrt{\frac{67+18r}{1+r}}$$

$$\frac{V\left(\dfrac{(3-r)(67+18r)}{(1+r)(3+2r)}, \dfrac{9(1+r)(67+18r)}{(3-r)(3+2r)}\right)}{V\left(0, \dfrac{268+72r}{3-r}\right)} = \frac{\dfrac{1}{4}\sqrt{\dfrac{67+18r}{3+2r}}\left(\sqrt{\dfrac{3-r}{1+r}} + 9\sqrt{\dfrac{1+r}{3-r}}\right)}{\dfrac{3}{2}\sqrt{\dfrac{67+18r}{3-r}}}$$

$$= \frac{1}{4}\sqrt{\frac{67+18r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}} + 9\sqrt{\frac{1+r}{3-r}}\right) \cdot \frac{2}{3}\sqrt{\frac{3-r}{67+18r}}$$

$$= \frac{1}{4}\cdot\frac{2}{3}\sqrt{\frac{3-r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}} + 9\sqrt{\frac{1+r}{3-r}}\right)$$

$$= \frac{1}{6}\left(\frac{3-r}{\sqrt{(3+2r)(1+r)}} + 9\sqrt{\frac{1+r}{3+2r}}\right) > 1$$

In the interval $r \in (0, 1]$, the fraction above minimizes at $r = 1$ with value about 1.054093.

$$\frac{V\left(\dfrac{(3-r)(67+18r)}{(1+r)(3+2r)}, \dfrac{9(1+r)(67+18r)}{(3-r)(3+2r)}\right)}{V\left(\dfrac{268+72r}{1+r}, 0\right)} = \frac{\dfrac{1}{4}\sqrt{\dfrac{67+18r}{3+2r}}\left(\sqrt{\dfrac{3-r}{1+r}} + 9\sqrt{\dfrac{1+r}{3-r}}\right)}{\dfrac{1}{2}\sqrt{\dfrac{67+18r}{1+r}}}$$

$$= \frac{1}{4}\sqrt{\frac{67+18r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}} + 9\sqrt{\frac{1+r}{3-r}}\right) \cdot 2\sqrt{\frac{1+r}{67+18r}}$$

$$= \frac{1}{2}\sqrt{\frac{1+r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}} + 9\sqrt{\frac{1+r}{3-r}}\right)$$

$$= \frac{1}{2}\left(\sqrt{\frac{3-r}{3+2r}} + 9\frac{1+r}{\sqrt{(3+2r)(3-r)}}\right) \geqslant 2$$

In the interval $r \in (0, 1)$, the fraction above minimizes at about $r = 0$ with value approximates 2.

Hence,

$$\begin{cases} x_1 = \dfrac{(3-r)(67+18r)}{(1+r)(3+2r)} \\[2mm] x_2 = \dfrac{9(1+r)(67+18r)}{(3-r)(3+2r)} \\[2mm] b = \dfrac{9(67+18r)}{(3-r)(3+2r)} - \dfrac{49}{1+r} \\[2mm] k = \dfrac{196}{1+r} - \dfrac{36(67+18r)}{(3-r)(3+2r)} \\[2mm] \max V = \dfrac{1}{4}\sqrt{\dfrac{67+18r}{3+2r}}\left(\sqrt{\dfrac{3-r}{1+r}} + 9\sqrt{\dfrac{1+r}{3-r}}\right) \end{cases}$$

If $r = 0.03$,

$$x_1 = \frac{67+18\times0.03}{1.03}\cdot\frac{3-0.03}{3+2\times0.03} \approx 63.6442$$

$$x_2 = \frac{9\times1.03\,(67+18\times0.03)}{(3-0.03)(3+2\times0.03)} \approx 68.89107$$

$$b = \frac{9\,(67+18\times0.03)}{(3-0.03)(3+2\times0.03)} - \frac{49}{1.03} \approx 19.31172$$

$$k = \frac{196}{1.03} - \frac{36\,(67+18\times1.03)}{(3-0.03)(3+2\times0.03)} \approx -77.24686$$

$$V(x_1, x_2) = \frac{1}{4}\sqrt{\frac{67+18\times0.03}{3+2\times0.03}}\left(\sqrt{\frac{3-0.03}{1.03}} + 9\sqrt{\frac{1.03}{3-0.03}}\right) \approx 8.219481$$
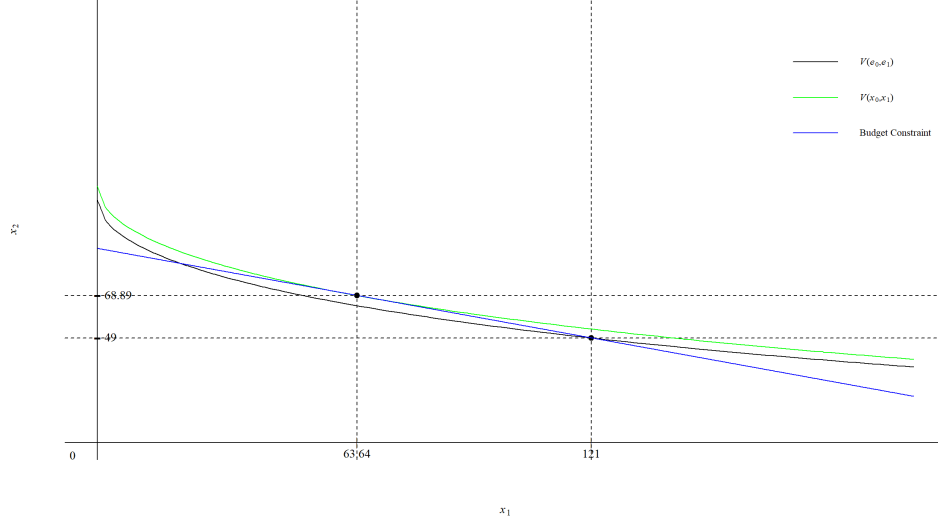
(c)

Figure 4: consumer's indifference curve through the optimal consumption bundle of 2(b)

(d)

The consumer is better off. His/her indifference curve is higher, as shown in the figure above. We can also prove it by math that

$$\frac{1}{4} \times 11 + \frac{3}{4} \times 7 = \frac{11}{4} + \frac{21}{4} = \frac{32}{4} = 8$$

While $\max V = \frac{1}{4}\sqrt{\frac{67+18r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}} + 9\sqrt{\frac{1+r}{3-r}}\right) > 8.$

$b = \dfrac{9\,(67+18r)}{(3-r)\,(3+2r)} - \dfrac{49}{1+r} > 0,$ which means the consumer has long position of bond, and short position on risky asset, in order to maximize his/her expected utility.

Also, $x_1 = x_2$ doesn't hold generally. It implies that the consumer doesn't hedge all his/her risk.

3

(a)

$$\max_{\left(x_1^A, x_2^A\right)} \frac{1}{2}\ln x_1^A + \frac{1}{2}\ln x_2^A$$

subject to

$$\hat{p}_1 x_1^A + \hat{p}_2 x_2^A = \hat{p}_1 + 2\hat{p}_2.$$

For Alice:

$$MU_1^A = \frac{1}{2}\cdot\frac{1}{x_1^A}$$

$$MU_2^A = \frac{1}{2}\cdot\frac{1}{x_2^A}$$

$$\mathrm{MRS}^A = \frac{MU_1^A}{MU_2^A} = MU_1^A\cdot\frac{1}{MU_2^A} = \frac{1}{2}\cdot\frac{1}{x_1^A}\cdot 2x_2^A = \frac{x_2^A}{x_1^A}$$

On the other hand,

$$\mathrm{MRS}^A = \frac{\hat{p}_1}{\hat{p}_2}$$

$$\therefore \frac{x_2^A}{x_1^A} = \frac{\hat{p}_1}{\hat{p}_2}$$

Substituting $x_2^A = \dfrac{\hat{p}_1}{\hat{p}_2} x_1^A$ into budget constraint for Alice:

$$\hat{p}_1 x_1^A + \hat{p}_2\frac{\hat{p}_1}{\hat{p}_2}x_1^A = \hat{p}_1 + 2\hat{p}_2$$

$$\implies \hat{p}_1 x_1^A + \hat{p}_1 x_1^A = \hat{p}_1 + 2\hat{p}_2$$

$$\implies 2\hat{p}_1 x_1^A = \hat{p}_1 + 2\hat{p}_2$$

$$\implies x_1^A = \frac{\hat{p}_1 + 2\hat{p}_2}{2\hat{p}_1} = \frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1}$$

(b)

$$\max_{\left(x_1^B, x_2^B\right)} \frac{1}{2}\ln x_1^B + \frac{1}{2}\ln x_2^B$$

subject to

$$\hat{p}_1 x_1^B + \hat{p}_2 x_2^B = 3\hat{p}_1 + \hat{p}_2.$$

For Bob:

$$MU_1^B = \frac{1}{2} \cdot \frac{1}{x_1^B}$$

$$MU_2^B = \frac{1}{2} \cdot \frac{1}{x_2^B}$$

$$\text{MRS}^B = \frac{MU_1^B}{MU_2^B} = MU_1^B \cdot \frac{1}{MU_2^B} = \frac{1}{2} \cdot \frac{1}{x_1^B} \cdot 2x_2^B = \frac{x_2^B}{x_1^B}$$

On the other hand,

$$\text{MRS}^B = \frac{\hat{p}_1}{\hat{p}_2}$$

$$\therefore \frac{x_2^B}{x_1^B} = \frac{\hat{p}_1}{\hat{p}_2}$$

Substituting $x_2^B = \dfrac{\hat{p}_1}{\hat{p}_2} x_1^B$ into budget constraint for Bob:

$$\hat{p}_1 x_1^B + \hat{p}_2 \frac{\hat{p}_1}{\hat{p}_2} x_1^B = 3\hat{p}_1 + \hat{p}_2$$

$$\implies \hat{p}_1 x_1^B + \hat{p}_1 x_1^B = 3\hat{p}_1 + \hat{p}_2$$

$$\implies 2\hat{p}_1 x_1^B = 3\hat{p}_1 + \hat{p}_2$$

$$\implies x_1^B = \frac{3\hat{p}_1 + \hat{p}_2}{2\hat{p}_1} = \frac{3}{2} + \frac{\hat{p}_2}{2\hat{p}_1}$$

(c)

The market for consumption in state 1:

$$x_1^A + x_1^B = e_1^A + e_1^B = 1 + 3 = 4$$

$$\iff \frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1} + \frac{3}{2} + \frac{\hat{p}_2}{2\hat{p}_1} = 4$$

$$\iff 2 + \frac{2\hat{p}_2}{2\hat{p}_1} + \frac{\hat{p}_2}{2\hat{p}_1} = 4$$

$$\iff \frac{3\hat{p}_2}{2\hat{p}_1} = 2$$

$$\iff \frac{\hat{p}_2}{\hat{p}_1} = 2 \times \frac{2}{3}$$

$$\iff \boxed{\frac{\hat{p}_1}{\hat{p}_2} = \frac{3}{4}}$$

$$x_1^A = \frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1} = \frac{1}{2} + \frac{4}{3} = \frac{3}{6} + \frac{8}{6} = \frac{11}{6}$$

$$x_1^B = \frac{3}{2} + \frac{\hat{p}_2}{2\hat{p}_1} = \frac{3}{2} + \frac{1}{2} \times \frac{4}{3} = \frac{3}{2} + \frac{2}{3} = \frac{9}{6} + \frac{4}{6} = \frac{13}{6}$$

$$x_2^A = \frac{\hat{p}_1}{\hat{p}_2} x_1^A = \frac{3}{4} \times \frac{11}{6} = \frac{1}{4} \times \frac{11}{2} = \frac{11}{8}$$

$$x_2^B = \frac{\hat{p}_1}{\hat{p}_2} x_1^B = \frac{3}{4} \times \frac{13}{6} = \frac{1}{4} \times \frac{13}{2} = \frac{13}{8}$$

Alice and Bob bear some risk:

$x_1^A \neq x_2^A$, $x_1^B \neq x_2^B$

They hedge some risks, but not completely, to maximize thier expected utility.
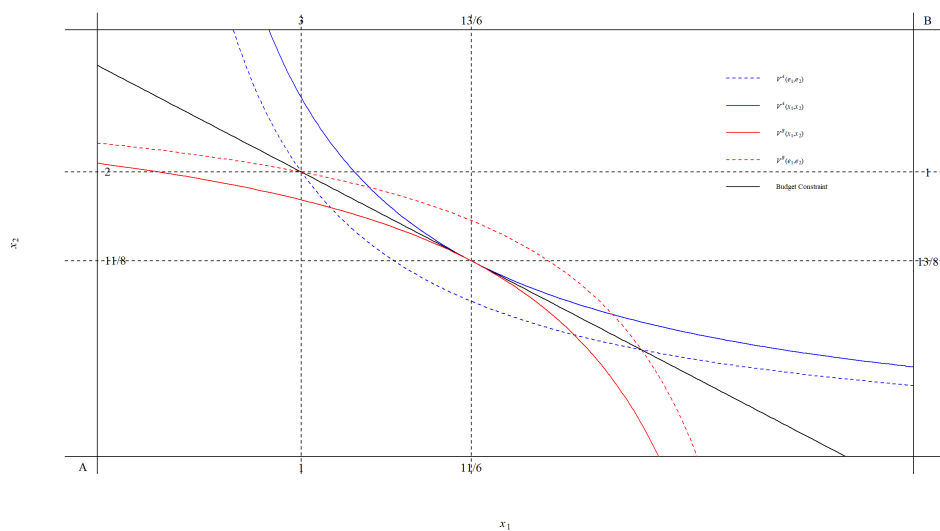
(d)

Figure 5: Edgeworth box diagram for Problem 3 equilibrium

It's impossible to increase one's utility without hurting the other one. Thus it is Pareto optimal.

4

(a)

$$\max_{\left(x_1^A, x_2^A\right)} \frac{1}{3} \ln x_1^A + \frac{2}{3} \ln x_2^A$$

subject to

$$\hat{p}_1 x_1^A + \hat{p}_2 x_2^A = 3\hat{p}_1 + 2\hat{p}_2.$$

For Alice:

$$MU_1^A = \frac{1}{3} \cdot \frac{1}{x_1^A}$$

$$MU_2^A = \frac{2}{3} \cdot \frac{1}{x_2^A}$$

$$\text{MRS}^A = \frac{MU_1^A}{MU_2^A} = MU_1^A \cdot \frac{1}{MU_2^A} = \frac{1}{3} \cdot \frac{1}{x_1^A} \cdot \frac{3}{2} x_2^A = \frac{x_2^A}{2x_1^A}$$

On the other hand,

$$\text{MRS}^A = \frac{\hat{p}_1}{\hat{p}_2}$$

$$\therefore \frac{x_2^A}{2x_1^A} = \frac{\hat{p}_1}{\hat{p}_2}$$

Substituting $x_2^A = \frac{2\hat{p}_1}{\hat{p}_2} x_1^A$ into budget constraint for Alice:

$$\hat{p}_1 x_1^A + \hat{p}_2 \frac{2\hat{p}_1}{\hat{p}_2} x_1^A = 3\hat{p}_1 + 2\hat{p}_2$$

$$\implies \hat{p}_1 x_1^A + 2\hat{p}_1 x_1^A = 3\hat{p}_1 + 2\hat{p}_2$$

$$\implies 3\hat{p}_1 x_1^A = 3\hat{p}_1 + 2\hat{p}_2$$

$$\implies x_1^A = \frac{3\hat{p}_1 + 2\hat{p}_2}{3\hat{p}_1} = 1 + \frac{2}{3} \cdot \frac{\hat{p}_2}{\hat{p}_1}$$

(b)

$$\max_{\left(x_1^B, x_2^B\right)} \frac{1}{2} \ln x_1^B + \frac{1}{2} \ln x_2^B$$

subject to

$$\hat{p}_1 x_1^B + \hat{p}_2 x_2^B = \hat{p}_1 + 2\hat{p}_2.$$

For Bob:

$$MU_1^B = \frac{1}{2} \cdot \frac{1}{x_1^B}$$

$$MU_2^B = \frac{1}{2} \cdot \frac{1}{x_2^B}$$

$$\mathrm{MRS}^B = \frac{MU_1^B}{MU_2^B} = MU_1^B \cdot \frac{1}{MU_2^B} = \frac{1}{2} \cdot \frac{1}{x_1^B} \cdot 2x_2^B = \frac{x_2^B}{x_1^B}$$

On the other hand,

$$\mathrm{MRS}^B = \frac{\hat{p}_1}{\hat{p}_2}$$

$$\therefore \frac{x_2^B}{x_1^B} = \frac{\hat{p}_1}{\hat{p}_2}$$

Substituting $x_2^B = \dfrac{\hat{p}_1}{\hat{p}_2} x_1^B$ into budget constraint for Bob:

$$\hat{p}_1 x_1^B + \hat{p}_2 \frac{\hat{p}_1}{\hat{p}_2} x_1^B = \hat{p}_1 + 2\hat{p}_2$$

$$\implies \hat{p}_1 x_1^B + \hat{p}_1 x_1^B = \hat{p}_1 + 2\hat{p}_2$$

$$\implies 2\hat{p}_1 x_1^B = \hat{p}_1 + 2\hat{p}_2$$

$$\implies x_1^B = \frac{\hat{p}_1 + 2\hat{p}_2}{2\hat{p}_1} = \frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1}$$

(c)

The market for consumption in state 1:

$$x_1^A + x_1^B = e_1^A + e_1^B = 3 + 1 = 4$$

$$\iff 1 + \frac{2}{3} \cdot \frac{\hat{p}_2}{\hat{p}_1} + \frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1} = 4$$

$$\iff \left(\frac{2}{3} + 1\right) \cdot \frac{\hat{p}_2}{\hat{p}_1} + \frac{1}{2} = 3$$

$$\iff \frac{5}{3} \cdot \frac{\hat{p}_2}{\hat{p}_1} = \frac{5}{2}$$

$$\iff \frac{\hat{p}_2}{\hat{p}_1} = \frac{5}{2} \times \frac{3}{5}$$

$$\iff \boxed{\frac{\hat{p}_1}{\hat{p}_2} = \frac{2}{3}}$$

$$x_1^A = 1 + \frac{2}{3} \cdot \frac{\hat{p}_2}{\hat{p}_1} = 1 + \frac{2}{3} \cdot \frac{3}{2} = 2$$

$$x_1^B = \frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1} = \frac{1}{2} + \frac{3}{2} = 2$$

$$x_2^A = \frac{2\hat{p}_1}{\hat{p}_2} x_1^A = 2 \times \frac{2}{3} \times 2 = \frac{8}{3}$$

$$x_2^B = \frac{\hat{p}_1}{\hat{p}_2} x_1^B = \frac{2}{3} \times 2 = \frac{4}{3}$$

Alice and Bob bear some risk:

$x_1^A \neq x_2^A,\ x_1^B \neq x_2^B$

They hedge some risks, but not completely, to maximize thier expected utility.
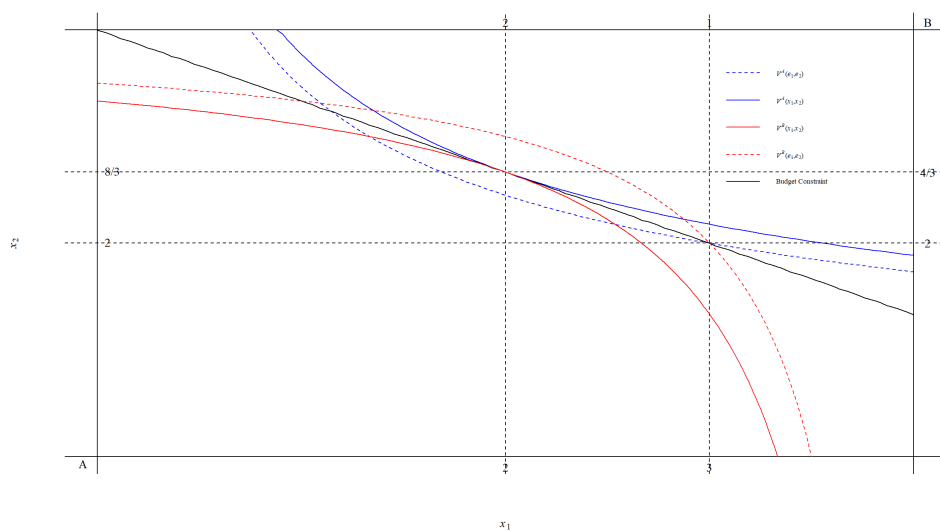
(d)

Figure 6: Edgeworth box diagram for Problem 4 equilibrium

It's impossible to increase one's utility without hurting the other one. Thus it is Pareto optimal.