1. (a)

The payoff of this derivative is

$r_3 = (1-75\%) r_1$
$= \frac{1}{4} r_1$
$= (16, 4, 1)$

By no arbitrage,

$q_3 = \frac{1}{4} q_1 = 8.$

(b)

The payoff of the derivative is

$r_3 = \frac{1}{4} r_1 - r_2$
$= (16, 4, 0)$

By no ~~a~~arbitrage,

$q_3 = \frac{1}{4} q_1 - q_2$
$= 8 - 1$
$= 7.$

(c)

Suppose the payoff is

$r_3 = (16, 0, 0).$

Assume the hedging portfolio consist of $\alpha_1$ shares of asset 1 & $\alpha_2$ shares of asset 2.

Then we have

$16 = 64\alpha_1 + 0 \Rightarrow \alpha_1 = \frac{1}{4}$

$0 = 16\alpha_1 + 0 \Rightarrow \alpha_1 = 0$

$0 = 4\alpha_1 + \alpha_2$

No solution.

Thus, the hedging portfolio doesn't exist, and this asset cannot be priced by no arbitrage.

2. (a)

$q_0 = 100 = \frac{1}{1.01}(110\lambda + 90(1-\lambda))$

$\Rightarrow 101 = 110\lambda + 90 - 90\lambda = 20\lambda + 90$

$\Rightarrow 11 = 20\lambda \Rightarrow \lambda = \frac{11}{20} = 0.55$

$1 - \lambda = 1 - \frac{11}{20} = \frac{9}{20} = 0.45$

It's easy to check that

$110 = \frac{1}{1.01}(121 \cdot \frac{11}{20} + 99 \cdot \frac{9}{20})$

$90 = \frac{1}{1.01}(99 \cdot \frac{11}{20} + 81 \cdot \frac{9}{20})$

(b)

$(K - q_2)_+ > 0$ iff the stock price goes down for 2 months in a row.

$95 - 81 = 14 > 0.$

The payoff of the put at $t=2$ is

$(0, 0, 0, 14).$

By no arbitrage pricing formula:

$q_{p_1}(110) = 0$

$q_{p_2}(90) = \frac{1}{1.01}(0.55 \cdot 0 + 0.45 \cdot 14)$
$\approx 6.24.$

(c)

Let $\alpha$ be the shares of the stock, $\beta$ be the shares of the bond in the replicating portfolio.

System of equations below is solved by $\alpha$ & $\beta$:

$$\begin{aligned} 0 &= 110\alpha + 1.01\beta \\ 6.24 &= 90\alpha + 1.01\beta \end{aligned} \Rightarrow \begin{cases} \alpha \approx -0.31 \\ \beta \approx 33.98 \end{cases}$$

By no arbitrage,

$q_{p_0} = 100\alpha + \beta \approx 2.78$

(Here $\alpha = -0.31188\cdots$ is used.)

(d)

Suppose that $q_{p0} < 1\alpha + \alpha\beta \approx 2.78$.

Consider a portfolio consisting of 1 share of put option, $-\alpha \approx +0.31$ shares of stock (i.e., ~~borrow~~ buy 0.31 shares), and $-\beta \approx 33.98$ shares of bond (i.e., ~~buy~~ borrow 33.98 shares of the bond).

The market value of the portfolio is:

$$q_{p0} - 1\alpha\alpha - \beta < 0.$$

If $q_1 = \$110$, the investor doesn't exercise the put, sells 0.31 shares of the stock, getting $0.31 \cdot \$110 \approx \$34.32$, and using the money to pay back the debt

$\$1.01 \times 33.98 \approx \$34.32$.

The net payoff on portfolio is 0.

Suppose $q_1 = \$90$, then the investor sells the put option, getting \$6.24, and also sells 0.31 shares of the stock, getting $0.31 \times \$90 \approx \$28.08$.

Use this money to pay back the debt of the bond

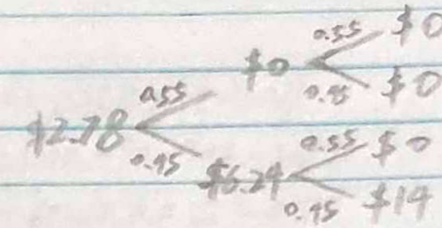
$\$1.01 \times 33.98 \approx \$34.32$

$\because \$28.08 + \$6.24 = \$34.32$

$\therefore$ The net payoff on the portfolio is ~~non~~ 0.

Hence an arbitrage portfolio is constructed.

(e)




3. (a)

If the entepreneur doesn't invest at $t=0$, the EPV is:

$$W_0(12) = 15 - 3 + 0.5\sum_{t=1}^{\infty}\frac{15-3\times1.2}{1.01^t} + 0.5\sum_{t=1}^{\infty}\frac{15-3\times0.8}{1.01^t}$$

$$= 12 + 0.5\left(\frac{15-3.6}{0.01} + \frac{15-2.4}{0.01}\right)$$

$$= 12 + 0.5\cdot\frac{30-6}{0.01} = 12 + 5\times\frac{24}{0.1}$$

$$= 12 + 5\times240 = 12 + 1200$$

$$= 1212.$$

If the entepreneur invests at $t=0$, the EPV is:

$$V_{0,0}(12;I) = 15 - 3 - I + 0.5\sum_{t=1}^{\infty}\frac{15-3\times1.2\times0.9}{1.01^t} \cancel{+0.5\sum_{t=1}^{\infty}(15-3\times1.2)}$$

$$+0.5\sum_{t=1}^{\infty}\frac{15-3\times0.8\times0.9}{1.01^t}$$

$$= 12 - I + 0.5\left(\frac{15-3.6\times0.9}{0.01} + \frac{15-2.4\times0.9}{0.01}\right)$$

$$= 12 - I + 0.5\left(\frac{30-(3.6+2.4)\times0.9)}{0.01}\right)$$

$$= 12 - I + \frac{5}{0.1}(30 - 6\times0.9)$$

$$= 12 - I + 50(30 - 5.4)$$

$$= 12 - I + 50(25 - 0.4) = 12 - I + 50\cdot24.6$$

$$= 12 - I + 5\times246 = 12 - I + 5\times240 + 5\times6$$

$$= 1212 + 30 - I$$

$$= 1242 - I.$$

EPV of net gain:

$$V_{0,0}(12;I) - W_0(12)$$

$$= (1242 - I) - 1212$$

$$= 30 - I.$$

If $I > 30$, it doesn't make sense to invest at $t=0$, the net gain is 0.

Hence $G_{0,0}(12;I) = \begin{cases} 30 - I, & \text{if } I \le 30 \\ 0 & \text{if } I > 30. \end{cases}$

(b)

Suppose the operating cost goes up,

$$15-C=15-3\times1.2=15-3.6=12-0.6$$
$$=11.4.$$

If not invest, the EPV is

$$W_1(11.4)=15-3\times1.2+\sum_{t=1}^{\infty}\frac{15-3\times1.2}{1.01^t}$$
$$=11.4+\frac{11.4}{0.01}=11.4\times101$$
$$=1140+11.4$$
$$=1151.4$$

If invest, the EPV is:

$$V_{1,1}(11.4;I)=11.4-I+\sum_{t=1}^{\infty}\frac{15-3\times1.2\times0.9}{1.01^t}$$
$$=11.4-I+\frac{15-3\times1.2\times0.9}{0.01}$$
$$=11.4-I+1500-36\times9$$
$$=11.4-I+1300-324$$
$$=11.4-I+1200-24$$
$$=11.4-I+1180-4$$
$$=11.4-I+1176$$
$$=1187.4-I.$$

The EPV of the net gain:

$$V_{1,1}(11.4;I)-W_1(11.4)$$
$$=(1187.4-I)-1151.4$$
$$=(87-51)-I=(86-50)-I$$
$$=36-I.$$

If $I>36$, it makes no sense to invest at $t=1$. The net gain is 0.

Hence, $G_{1,1}(11.4;I)=\begin{cases}36-I, & \text{if } I\le 36\\ 0 & \text{if } I>36.\end{cases}$

Suppose the operating cost goes down,

$$15-C=15-3\times0.8=15-2.4=13-0.4$$
$$=12.6.$$

If not invest, the EPV is:

~~$V_{1,1}(12.6$~~

$$W_1(12.6)=15-3\times0.8+\sum_{t=1}^{\infty}\frac{15-3\times0.8}{1.01^t}$$
$$=12.6+\frac{12.6}{0.01}=1260+12.6$$
$$=1272.6$$

If invest, the EPV is:

$$V_{1,1}(12.6;I)=15-3\times0.8-I+\sum_{t=1}^{\infty}\frac{15-3\times0.8\times0.9}{1.01^t}$$
$$=12.6-I+\frac{15-2.4\times0.9}{0.01}$$
$$=12.6+1500-24\times9-I$$
$$=12.6+1500-216-I$$
$$=12.6+1300-16-I$$
$$=2.6+1300-6-I$$
$$=2.6+1294-I$$
$$=1296.6-I.$$

The EPV of net gain:

$$V_{1,1}(12.6;I)-W_1(12.6)$$
$$=(1296.6-I)-1272.6$$
$$=24-I.$$

If $I>24$, it makes no sense to invest. The net gain is 0.

Hence $G_{1,1}(12.6;I)=\begin{cases}24-I, & \text{if } I\le 24\\ 0, & \text{if } I>0.\end{cases}$

At $t=0$:

if $I\le24$:

$$G_{0,1}(12;I)=\frac{0.5\times(36-I)+0.5\times(24-I)}{1.01}$$
$$=\frac{0.5(36+24)-I}{1.01}$$
$$=\frac{30-I}{1.01}$$

If $24<I\le36$:

$$G_{0,1}(12;I)=\frac{0.5(36-I)}{0.01}=\frac{18-0.5I}{1.01}$$

If $I>36$: $G_{0,1}(12)=0$

Hence $G_{0,1}(12;I)=\begin{cases}\frac{30-I}{1.01} & \text{if } I\le24\\ \frac{18-0.5I}{1.01} & \text{if } 24<I\le36\\ 0 & \text{if } I>36.\end{cases}$

(c)

$30-I > \frac{18-0.5I}{1.01} \Leftrightarrow 30\times1.01-1.01I > 18-0.5I$

$\Leftrightarrow 3\times10.1 > 18+1.01I-0.5I$

$\Leftrightarrow 30.3 > 18+0.51I \Leftrightarrow 18+0.51I < 30.3$

$\Leftrightarrow 0.51I < 30.3-18 = 20.3-8 = 12.3$

$\Leftrightarrow I < \frac{12.3}{0.51} \approx 24.12.$

If $I<24.12$, $G_{0,0}(12;I) > G_{0,1}(12;I)$.

If $I>24.12$, $G_{0,0}(12;I) < G_{0,1}(12;I)$.

The optimal strategy is following:

α. If $I\le 24.12$, invest immediately.

β. If $24.12<I\le 36$, do not invest at $t=0$.
Invest at $t=1$ only if cost goes up, otherwise, never invest.

γ. If $I>36$, never invest.

The gain from the invest is:

$$G(I)=\begin{cases} 30-I & \text{if } I\le 24.12 \\ \frac{18-0.5I}{1.01} & \text{if } 24.12<I\le 36 \\ 0 & \text{if } I>36 \end{cases}.$$

4. (a)

Since the consumers share the firm equally, the initial endowment in shares is $\frac{1}{16}$ for each consumer.
For the production plan $(y_0, y_1)$, the current dividend on equity $\delta_0=-y_0$, and the future dividend is $\delta_1=y_1$.
Let $s$ be the number of shares of the firm and $b$ be the number of shares of the bond that each consumer has in equilibrium.

The representative consumer solves the following utility maximization problem:

$$\max_{(x_l,x_h,s,b)} \ln x_l + 2\ln x_h$$

s.t.

$$b_0 + e_0 s = 4 + \frac{1}{16}(e_0-y_0),$$

$$x_l = y_l s + (1+r)b,$$

$$x_h = y_h s + (1+r)b$$

Let $\lambda_0\ge0$, $\lambda_l\ge0$ and $\lambda_h\ge0$ be Lagrange multipliers assigned, respectively, to budget constraints at $t=0$ & $t=1$.

The Lagrangian:

$$\mathcal{L}(x_l,x_h,s,b,\lambda_0,\lambda_l,\lambda_h)$$
$$=\ln x_l + 2\ln x_h$$
$$-\lambda_0\left(b+e_0 s-4-\frac{1}{16}(e_0-y_0)\right)$$
$$-\lambda_l(x_l-y_l s-(1+r)b)$$
$$-\lambda_h(x_h-y_h s-(1+r)b)$$

the FOCs are

$$\frac{\partial\mathcal{L}}{\partial x_l}=\frac{1}{x_l}-\lambda_l=0,$$

$$\frac{\partial\mathcal{L}}{\partial x_h}=\frac{2}{x_h}-\lambda_h=0,$$

$$\frac{\partial\mathcal{L}}{\partial s}=-\lambda_0 e_0+\lambda_l y_l+\lambda_h y_h=0,$$

$$\frac{\partial\mathcal{L}}{\partial b}=-\lambda_0+(\lambda_l+\lambda_h)(1+r)=0,$$

$$\cancel{\frac{\partial\mathcal{L}}{\partial\lambda_0}=4+\frac{1}{16}(e_0-}\,y_0$$

$$\frac{\partial\mathcal{L}}{\partial\lambda_0}=4+\frac{1}{16}(e_0-y_0)-b-se_0=0,$$

$$\frac{\partial\mathcal{L}}{\partial\lambda_l}=y_l s+(1+r)b-x_l=0,$$

$$\frac{\partial\mathcal{L}}{\partial\lambda_h}=y_h s+(1+r)b-x_h=0.$$

(b) Set $\bar\lambda_l = \frac{\lambda_l}{\lambda_l+\lambda_h}$, and $\bar\lambda_h = \frac{\lambda_h}{\lambda_l+\lambda_h}$.

$$e_0 = \frac{\lambda_l y_l + \lambda_h y_h}{\lambda_0} = \frac{1}{1+r}(\bar\lambda_l y_l + \bar\lambda_h y_h)$$

$$\frac{\bar\lambda_l \cancel{y_l} + \bar\lambda_h \cancel{y_h}}{1+r} s + b = \frac{\bar\lambda_l x_l + \bar\lambda_h x_h}{1+r}$$

$$\Leftrightarrow e_0 + s + b = \frac{\bar\lambda_l x_l + \bar\lambda_h x_h}{1+r}$$

$$\frac{\bar\lambda_l x_l + \bar\lambda_h x_h}{1+r} = 4 + \frac{1}{16}\left(\frac{\bar\lambda_l y_l + \bar\lambda_h y_h}{1+r} - y_0\right)$$

If the EPV is positive, all consumers agree that the firm should choose $y_0$ to

$$\max_{y_0} \frac{(12\bar\lambda_l + 19.2\bar\lambda_h)\sqrt{y_0}}{1+r} - y_0$$

The FOC is

$$\frac{12\bar\lambda_l + 19.2\bar\lambda_h}{1+r} \cdot \frac{1}{2\sqrt{y_0}} - 1 = 0$$

$$\Rightarrow \frac{12(\bar\lambda_l + 1.6\bar\lambda_h)}{2(1+r)} \cdot \frac{1}{\sqrt{y_0}} = 1$$

$$\Rightarrow \sqrt{y_0} = \frac{6(\bar\lambda_l + 1.6\bar\lambda_h)}{1+r}$$

$$\Rightarrow y_0 = \frac{36(\bar\lambda_l + 1.6\bar\lambda_h)^2}{(1+r)^2}$$

Hence $y_l = 12\sqrt{y_0} = \frac{72(\bar\lambda_l + 1.6\bar\lambda_h)^2}{(1+r)^2}$

$$y_h = 19.2\sqrt{y_0} = \frac{19.2 \cdot 6(\bar\lambda_l + 1.6\bar\lambda_h)^2}{1+r} = \frac{115.2(\bar\lambda_l + 1.6\bar\lambda_h)^2}{1+r} = 1.6 y_l.$$

Check firm's EPV:

$$\frac{\bar\lambda_l}{1+r} \cdot \frac{72(\bar\lambda_l + 1.6\bar\lambda_h)^2}{1+r} + \frac{\bar\lambda_h}{1+r} \cdot \frac{115.2(\bar\lambda_l + 1.6\bar\lambda_h)^2}{1+r} - y_0$$

$$= \frac{6(\bar\lambda_l + 1.6\bar\lambda_h)^2}{(1+r)^2}(12\bar\lambda_l + 19.2\bar\lambda_h) - y_0$$

$$= \frac{6(\bar\lambda_l + 1.6\bar\lambda_h)^2}{(1+r)^2} \cdot 12 \cdot (\bar\lambda_l + 1.6\bar\lambda_h) - y_0$$

$$= \frac{72(\bar\lambda_l + 1.6\bar\lambda_h)^2}{(1+r)^2} - y_0$$

$$= 2y_0 - y_0$$

$$= y_0 > 0.$$

Therefore, all the consumers agree on the firm's objectives. Now the intertemporal constraint becomes: ↗

~~$\bar\lambda_l x_l + \bar\lambda_h x_h$~~

$$\frac{\bar\lambda_l x_l + \bar\lambda_h x_h}{1+r} = 4 + \frac{y_0}{16}.$$

(c)

Market clearing conditions:

$y_0 = 4 \cdot 16 = 64$,

$16x_l = y_l$,

$16x_h = y_h$,

$16s = 1$,

$16b = 0$.

Thus

$y_l = 12\sqrt{64} = 12 \cdot 8 = 96$,

$y_h = 19.2\sqrt{64} = 153.6$,

$x_l = \frac{96}{16} = 6$

$x_h = \frac{153.6}{16} = 9.6$

$\frac{1}{x_l} = \lambda_l \Rightarrow x_l = \frac{1}{\lambda_l}$

$\frac{2}{x_h} = \lambda_h \Rightarrow x_h = \frac{2}{\lambda_h}$.

$$\frac{x_h}{x_l} = \frac{\frac{2}{\lambda_h}}{\frac{1}{\lambda_l}} = \frac{2\lambda_l}{\lambda_h} = \frac{2\bar\lambda_l}{\bar\lambda_h} = 1.6 \Rightarrow \bar\lambda_l = 0.8\bar\lambda_h.$$

$\because \bar\lambda_l + \bar\lambda_h = 1 \Rightarrow 1.8\bar\lambda_h = 1$

$\therefore \bar\lambda_h = \frac{1}{1.8} \approx 0.56$

$\bar\lambda_l = 1 - \frac{1}{1.8} \approx 0.44$.

$$\frac{(1 - \frac{1}{1.8}) \cdot 6 + \frac{1}{1.8} \cdot 9.6}{1+r} = 4 + \frac{1}{16} \cdot 64 = 4 + 4 = 8$$

$\Rightarrow r = 0$.

$e_0 = (1 - \frac{1}{1.8}) \cdot 96 + \frac{1}{1.8} \cdot 153.6 = 128$.

$s = \frac{1}{16}$

$b = 0$.