Yu Xia's Answer for Problem Set 5

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(a)

If $F_0 \leqslant 45$, investor enters the forward contract at $t = 0$. The market value of investor's portfolio is 0. At $t = 1$, he/she pays F_0 and sells ethanol on the spot.

Then in state 1, investor's payoff would be:

$65 - F_0 > 0$

In state 2:

$45 - F_0 \geqslant 0$

which is an arbitrage portfolio, a contradiction.

If $F_0 \geqslant 65$, investor enters the forward contract at $t = 0$. The market value of investor's portfolio is 0. At $t = 1$, he/she buys ethanol on the spot and sells at F_0 according to forward contract.

Then in state 1, investor's payoff would be:

$F_0 - 65 \geqslant 0$

In state 2:

$F_0 - 45 > 0$

which is an arbitrage portfolio, a contradiction.

In conclusion, $\boxed{F_0 \in (45, 65)}$

If there exist a risk neutral probability measure $\pi = \{\pi_1, \pi_2\}$ such that

$$0 = \frac{1}{1+r}\left(\pi_1 \left(65 - F_0\right) + \pi_2 \left(45 - F_0\right)\right)$$

$$1 = \frac{1}{1+r}\left(\pi_1 \left(1+r\right) + \pi_2 \left(1+r\right)\right) \iff \pi_1 + \pi_2 = 1$$

$$\implies 0 = \frac{1}{1+r}\left(\pi_1 \left(65 - F_0\right) + \left(1 - \pi_1\right)\left(45 - F_0\right)\right)$$

$\pi_1\,(65 - F_0) + (1 - \pi_1)\,(45 - F_0) = 0$

$\pi_1\,(65 - F_0) + (45 - F_0) - \pi_1\,(45 - F_0) = 0$

$\pi_1\,[(65 - F_0) - (45 - F_0)] = F_0 - 45$

$\pi_1\,(65 - F_0 - 45 + F_0) = F_0 - 45$

$\pi_1 = \boxed{\dfrac{F_0 - 45}{20}}$

$\pi_2 = 1 - \pi_1 = 1 - \dfrac{F_0 - 45}{20} = \dfrac{20}{20} - \dfrac{F_0 - 45}{20} = \dfrac{20 - (F_0 - 45)}{20} = \dfrac{20 - F_0 + 45}{20} = \boxed{\dfrac{65 - F_0}{20}}$

(b)

If $F_0 \leqslant 45$, investor enters the forward contract at $t = 0$. The market value of investor's portfolio is 0. At $t = 1$, he/she pays F_0 and sells ethanol on the spot.

Then in state 1, investor's pay off would be:

$65 - F_0 > 0$

In state 2:

$45 - F_0 \geqslant 0$

No matter which state investor is in, he/she makes money out of nothing.

Similarly, if $F_0 \geqslant 65$, investor enters the forward contract at $t = 0$. The market value of investor's portfolio is 0. At $t = 1$, he/she buys ethanol on the spot and sells at F_0 according to forward contract.

Then in state 1, investor's payoff would be:

$F_0 - 65 \geqslant 0$

In state 2:

$F_0 - 45 > 0$

No matter which state investor is in, he/she makes money out of nothing.

(c)

The payoff matrix is:

$$R = \begin{pmatrix} \rho_1 & 1+r \\ \rho_2 & 1+r \\ \rho_3 & 1+r \end{pmatrix} = \begin{pmatrix} 65 - F_0 & 1+r \\ 45 - F_0 & 1+r \\ 55 - F_0 & 1+r \end{pmatrix}$$

Let the set of risk neutral probability measure $\pi = \{\pi_1, \pi_2, \pi_3\}$, <u>where π_i stands for the probability of state i.</u>

$0 = \pi_1\rho_1 + \pi_2\rho_2 + \pi_3\rho_3$

Substitude $1 - \pi_1 - \pi_2$ for π_3, we have:

$0 = \pi_1\rho_1 + \pi_2\rho_2 + (1 - \pi_1 - \pi_2)\,\rho_3$

$0 = \pi_1\rho_1 + \pi_2\rho_2 + \rho_3 - \pi_1\rho_3 - \pi_2\rho_3$

$0 = \pi_1\,(\rho_1 - \rho_3) + \pi_2\,(\rho_2 - \rho_3) + \rho_3$

$-\pi_2\,(\rho_2 - \rho_3) - \rho_3 = \pi_1\,(\rho_1 - \rho_3)$

$\pi_1\,(\rho_1 - \rho_3) = -\pi_2\,(\rho_2 - \rho_3) - \rho_3$

$\pi_1 = \dfrac{-\pi_2\,(\rho_2 - \rho_3) - \rho_3}{\rho_1 - \rho_3}$

$\pi_1 = \dfrac{\pi_2\,(\rho_3 - \rho_2) - \rho_3}{\rho_1 - \rho_3}$

$\pi_3 = 1 - \pi_1 - \pi_2 = 1 - \dfrac{-\rho_3 + \pi_2\,(\rho_3 - \rho_2)}{\rho_1 - \rho_3} - \pi_2$

$\pi_3 = \dfrac{\rho_1 - \rho_3}{\rho_1 - \rho_3} + \dfrac{\rho_3 - \pi_2\,(\rho_3 - \rho_2)}{\rho_1 - \rho_3} - \dfrac{\pi_2\,(\rho_1 - \rho_3)}{\rho_1 - \rho_3}$

$\pi_3 = \dfrac{\rho_1 - \rho_3 + \rho_3 - (\pi_2\rho_3 - \pi_2\rho_2) - (\pi_2\rho_1 - \pi_2\rho_3)}{\rho_1 - \rho_3}$

$\pi_3 = \dfrac{\rho_1 - \pi_2\rho_3 + \pi_2\rho_2 - \pi_2\rho_1 + \pi_2\rho_3}{\rho_1 - \rho_3}$

$\pi_3 = \dfrac{\rho_1 + \pi_2\rho_2 - \pi_2\rho_1}{\rho_1 - \rho_3}$

$\pi_3 = \dfrac{\rho_1 + \pi_2\,(\rho_2 - \rho_1)}{\rho_1 - \rho_3}$

$\pi_3 = \dfrac{\rho_1 - \pi_2\,(\rho_1 - \rho_2)}{\rho_1 - \rho_3}$

$\because 0 < \pi_1 = \dfrac{-\rho_3 + \pi_2\,(\rho_3 - \rho_2)}{\rho_1 - \rho_3} < 1$ and $\rho_1 > \rho_3 > \rho_2$

$\therefore 0 < -\rho_3 + \pi_2\,(\rho_3 - \rho_2) < \rho_1 - \rho_3$

$\rho_3 < \pi_2\,(\rho_3 - \rho_2) < \rho_1 - \rho_3 + \rho_3 = \rho_1$

$\dfrac{\rho_3}{\rho_3 - \rho_2} < \pi_2 < \dfrac{\rho_1}{\rho_3 - \rho_2}$

$\because 0 < \pi_3 = \dfrac{\rho_1 - \pi_2\,(\rho_1 - \rho_2)}{\rho_1 - \rho_3} < 1$

$\therefore 0 < \rho_1 - \pi_2\,(\rho_1 - \rho_2) < \rho_1 - \rho_3$

$\rho_3 - \rho_1 < -\rho_1 + \pi_2\,(\rho_1 - \rho_2) < 0$

$\rho_3 - \rho_1 + \rho_1 = \rho_3 < \pi_2\,(\rho_1 - \rho_2) < \rho_1$

$$\frac{\rho_3}{\rho_1-\rho_2} < \pi_2 < \frac{\rho_1}{\rho_1-\rho_2}$$

$\because \rho_1 > \rho_3 > \rho_2$

$\therefore \rho_1 - \rho_2 > \rho_3 - \rho_2$

$$\frac{\rho_1}{\rho_1-\rho_2} < \frac{\rho_1}{\rho_3-\rho_2}$$

$$\frac{\rho_3}{\rho_3-\rho_2} > \frac{\rho_3}{\rho_1-\rho_2}$$

Thus,

$$\frac{\rho_3}{\rho_3-\rho_2} < \pi_2 < \frac{\rho_1}{\rho_1-\rho_2}$$

$\because \pi_2 \in (0,1)$

$\therefore \dfrac{\rho_3}{\rho_3-\rho_2} < 1$

$\rho_3 < \rho_3 - \rho_2$

$\rho_2 < 0$

On the other hand,

$$\frac{\rho_1}{\rho_1-\rho_2} > 0$$

$\rho_1 > 0$

Plug in we have:

$\rho_3 - \rho_2 = 10$

$\rho_1 - \rho_3 = 10$

$\rho_1 - \rho_2 = 20$

$$\pi_1 = \frac{10\pi_2 - (55 - F_0)}{10} = \frac{10\pi_2 - 55 + F_0}{10}$$

$$\pi_3 = \frac{65 - F_0 - 20\pi_2}{10}$$

$$\frac{55 - F_0}{10} < \pi_2 < \frac{65 - F_0}{20}$$

$\rho_2 < 0 \implies F_0 > 45$

$\rho_1 > 0 \implies F_0 < 65$

$$\boxed{\begin{array}{l} \pi_1 = \dfrac{10\pi_2 - 55 + F_0}{10} \\ \pi_3 = \dfrac{65 - F_0 - 20\pi_2}{10} \\ \dfrac{55 - F_0}{10} < \pi_2 < \dfrac{65 - F_0}{20} \\ 45 < F_0 < 65 \end{array}}$$

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(a)

The arbitrage free forward price per pound, denote as f_0, is given by the equation:

$$0 = \frac{1}{(1+0.4\%)^2}\left((0.5)^2(1.28 - f_0) + 2(0.5)^2(1.24 - f_0) + (0.5)^2(1.20 - f_0)\right)$$

$$0 = 0.25(1.28 - f_0) + 0.5(1.24 - f_0) + 0.25(1.20 - f_0)$$

$$0 = (1.28 - f_0) + 2(1.24 - f_0) + (1.20 - f_0)$$

$$(1 + 2 + 1)f_0 = 1.28 + 2 \times 1.24 + 1.20$$

$$4f_0 = 1.28 + 2.48 + 1.20 = 4.96$$

$$f_0 = 1.24$$

$$\therefore F_0 = 50,000 f_0 = 50,000 \times 1.24 = \boxed{62,000}$$

(b)

By the no-arbitrage pricing formula, we have:

$$\begin{aligned} 0 = \frac{1}{1+r}&\left(\pi\frac{\Phi_{1u} - \Phi_0}{1000} + (1-\pi)\frac{\Phi_{1d} - \Phi_0}{1000}\right) \\ &+ \frac{\pi}{(1+r)^2}\left(\pi\left(1.28 - \frac{\Phi_{1u}}{1000}\right) + (1-\pi)\left(1.24 - \frac{\Phi_{1u}}{1000}\right)\right) \\ &+ \frac{1-\pi}{(1+r)^2}\left(\pi\left(1.24 - \frac{\Phi_{1d}}{1000}\right) + (1-\pi)\left(1.20 - \frac{\Phi_{1d}}{1000}\right)\right) \end{aligned}$$

and

$$0 = \frac{1}{1+r}\left(\pi\left(1.28 - \frac{\Phi_{1u}}{1000}\right) + (1-\pi)\left(1.24 - \frac{\Phi_{1u}}{1000}\right)\right)$$

$$\implies 0 = \pi(1280 - \Phi_{1u}) + (1-\pi)(1240 - \Phi_{1u})$$

$$\implies 0 = 1280\pi - \Phi_{1u}\pi + 1240(1-\pi) - \Phi_{1u}(1-\pi)$$

$$\implies 0 = 1280\pi + 1240 - 1240\pi - \Phi_{1u}$$

$\implies \Phi_{1u} = 1280\pi + 1240\,(1-\pi) = \boxed{40\pi + 1240}$

also

$0 = \frac{1}{1+r}\left(\pi\left(1.24 - \frac{\Phi_{1d}}{1000}\right) + (1-\pi)\left(1.20 - \frac{\Phi_{1d}}{1000}\right)\right)$

$\implies 0 = \pi\,(1240 - \Phi_{1d}) + (1-\pi)\,(1200 - \Phi_{1d})$

$\implies \Phi_{1d} = 1240\pi + 1200\,(1-\pi) = \boxed{1200 + 40\pi}$

$\therefore 1.28 - \frac{\Phi_{1u}}{1000} = 1.28 - (0.04\pi + 1.24) = 1.28 - 0.04\pi - 1.24 = 0.04 - 0.04\pi$

$1280 - \Phi_{1u} = 40 - 40\pi$

$1.24 - \frac{\Phi_{1u}}{1000} = -0.04\pi$

$1240 - \Phi_{1u} = -40\pi$

$1.24 - \frac{\Phi_{1d}}{1000} = 1.24 - (1.20 + 0.04\pi) = 1.24 - 1.20 - 0.04\pi = 0.04 - 0.04\pi$

$1240 - \Phi_{1d} = 40 - 40\pi$

$1.20 - \frac{\Phi_{1d}}{1000} = -0.04\pi$

$1200 - \Phi_{1d} = -40\pi$

Thus

$$
\begin{aligned}
0 = \frac{1}{1+r}&\left(\pi\left(0.04\pi + 1.24 - \frac{\Phi_0}{1000}\right) + (1-\pi)\left(1.20 + 0.04\pi - \frac{\Phi_0}{1000}\right)\right) \\
&+ \frac{\pi}{(1+r)^2}\left(\pi\,(0.04 - 0.04\pi) + (1-\pi)\,(-0.04\pi)\right) \\
&+ \frac{1-\pi}{(1+r)^2}\left(\pi\,(0.04 - 0.04\pi) + (1-\pi)\,(-0.04\pi)\right)
\end{aligned}
$$

$\implies$

$$
\begin{aligned}
0 = \frac{1}{1+r}&\left(\pi\left(0.04\pi + 1.24 - \frac{\Phi_0}{1000}\right) + (1-\pi)\left(1.20 + 0.04\pi - \frac{\Phi_0}{1000}\right)\right) \\
&+ \frac{1}{(1+r)^2}\left(\pi\,(0.04 - 0.04\pi) + (1-\pi)\,(-0.04\pi)\right)
\end{aligned}
$$

$\implies$

$$
\begin{aligned}
0 = &\left(\pi\left(0.04\pi + 1.24 - \frac{\Phi_0}{1000}\right) + (1-\pi)\left(1.20 + 0.04\pi - \frac{\Phi_0}{1000}\right)\right) \\
&+ \frac{1}{1+r}\left(\pi\,(0.04 - 0.04\pi) + (1-\pi)\,(-0.04\pi)\right)
\end{aligned}
$$

$\Longrightarrow$

$$0 = \pi\left(0.04\pi + 1.24 - \frac{\Phi_0}{1000}\right) + (1-\pi)\left(1.20 + 0.04\pi - \frac{\Phi_0}{1000}\right) - \frac{1}{1+r}\left(\pi\left(0.04\pi - 0.04\right) + (1-\pi)\left(0.04\pi\right)\right)$$

$\Longrightarrow$

$$\frac{1}{1+r}\left(\pi\left(0.04\pi - 0.04\right) + (1-\pi)\left(0.04\pi\right)\right) = \pi\left(0.04\pi + 1.24 - \frac{\Phi_0}{1000}\right) + (1-\pi)\left(1.20 + 0.04\pi - \frac{\Phi_0}{1000}\right)$$

$\Longrightarrow$

$$\frac{1}{1+r}\left(\left(0.04\pi^2 - 0.04\pi\right) + \left(0.04\pi - 0.04\pi^2\right)\right) = \pi\left(1.24 + 0.04\pi - \frac{\Phi_0}{1000}\right) + (1-\pi)\left(1.20 + 0.04\pi - \frac{\Phi_0}{1000}\right)$$

$\Longrightarrow$

$$\frac{1}{1+r}\left(0.04\pi^2 - 0.04\pi + 0.04\pi - 0.04\pi^2\right) = 1.24\pi + \pi\left(0.04\pi - \frac{\Phi_0}{1000}\right) + 1.20\left(1-\pi\right) + (1-\pi)\left(0.04\pi - \frac{\Phi_0}{1000}\right)$$

$\Longrightarrow \; 0 = 1.24\pi + 1.20 - 1.20\pi + 0.04\pi - \dfrac{\Phi_0}{1000}$

$\Longrightarrow \; \Phi_0 = 40\pi + 1200 + 40\pi$

$\Longrightarrow \; \boxed{\Phi_0 = 80\pi + 1200}$

$\Phi_{1u} - \Phi_0 = 40\pi + 1240 - (80\pi + 1200) = 40 - 40\pi$

$\Phi_{1d} - \Phi_0 = 1200 + 40\pi - (80\pi + 1200) = -40\pi$

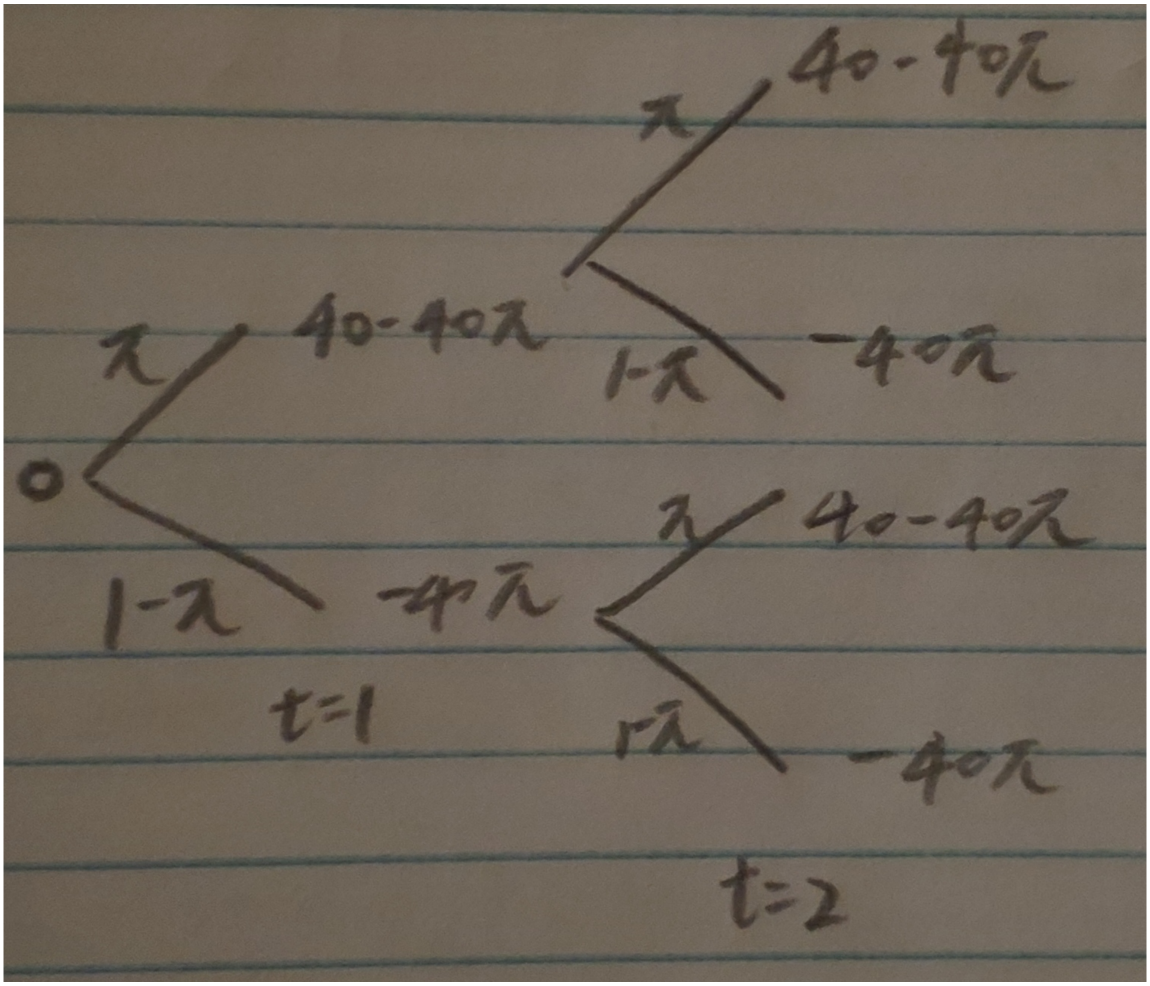

Figure 1: Time-decision tree of (b)

(c)

Similarly,

$$
\begin{aligned}
0 = \frac{1}{1+r}\left(\pi\left(1.26 - X\right) + \left(1-\pi\right)\left(1.22 - X\right)\right) \\
+ \frac{\pi}{\left(1+r\right)^2}\left(\pi\left(1.28 - X\right) + \left(1-\pi\right)\left(1.24 - X\right)\right) \\
+ \frac{1-\pi}{\left(1+r\right)^2}\left(\pi\left(1.24 - X\right) + \left(1-\pi\right)\left(1.20 - X\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
0 = \frac{1}{1+r}\left(\left(1.26\pi - X\pi\right) + \left(1.22\left(1-\pi\right) - X\left(1-\pi\right)\right)\right) \\
+ \frac{\pi}{\left(1+r\right)^2}\left(\left(1.28\pi - X\pi\right) + \left(1.24\left(1-\pi\right) - X\left(1-\pi\right)\right)\right) \\
+ \frac{1-\pi}{\left(1+r\right)^2}\left(\left(1.24\pi - X\pi\right) + \left(1.20\left(1-\pi\right) - X\left(1-\pi\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
0 = \frac{1}{1+r}\left(\left(1.26\pi - X\pi\right) + \left(\left(1.22 - 1.22\pi\right) - \left(X - X\pi\right)\right)\right) \\
+ \frac{\pi}{\left(1+r\right)^2}\left(\left(1.28\pi - X\pi\right) + \left(\left(1.24 - 1.24\pi\right) - \left(X - X\pi\right)\right)\right) \\
+ \frac{1-\pi}{\left(1+r\right)^2}\left(\left(1.24\pi - X\pi\right) + \left(\left(1.20 - 1.20\pi\right) - \left(X - X\pi\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
0 = \frac{1}{1+r}\left(1.26\pi - X\pi + 1.22 - 1.22\pi - X + X\pi\right) \\
+ \frac{\pi}{\left(1+r\right)^2}\left(1.28\pi - X\pi + 1.24 - 1.24\pi - X + X\pi\right) \\
+ \frac{1-\pi}{\left(1+r\right)^2}\left(1.24\pi - X\pi + 1.20 - 1.20\pi - X + X\pi\right)
\end{aligned}
$$

$$
\begin{aligned}
0 = \frac{1}{1+r}\left(1.26\pi + 1.22 - 1.22\pi - X\right) \\
+ \frac{\pi}{\left(1+r\right)^2}\left(1.28\pi + 1.24 - 1.24\pi - X\right) \\
+ \frac{1-\pi}{\left(1+r\right)^2}\left(1.24\pi + 1.20 - 1.20\pi - X\right)
\end{aligned}
$$

$$
0 = \frac{1}{1+r}\left(0.04\pi + 1.22 - X\right) + \frac{\pi}{\left(1+r\right)^2}\left(0.04\pi + 1.24 - X\right) + \frac{1-\pi}{\left(1+r\right)^2}\left(0.04\pi + 1.20 - X\right)
$$

$$
0 = \left(0.04\pi + 1.22 - X\right) + \frac{\pi}{1+r}\left(0.04\pi + 1.24 - X\right) + \frac{1-\pi}{1+r}\left(0.04\pi + 1.20 - X\right)
$$

$$
\begin{aligned}
0 = \left(0.04\pi + 1.22 - X\right) + \frac{1}{1+r}\left(0.04\pi^2 + 1.24\pi - X\pi\right) \\
+ \frac{1}{1+r}\left(0.04\pi\left(1-\pi\right) + 1.20\left(1-\pi\right) - X\left(1-\pi\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
0 = (1+r)\left(0.04\pi + 1.22 - X\right) + \big(0.04\pi^2 + 1.24\pi - X\pi\big) \\
+ \big(0.04\pi - 0.04\pi^2 + 1.20 - 1.20\pi - (X - X\pi)\big)
\end{aligned}
$$

$$
\begin{aligned}
0 = \left(0.04\pi\left(1+r\right) + 1.22\left(1+r\right) - X\left(1+r\right)\right) + \big(0.04\pi^2 + 1.24\pi - X\pi\big) \\
+ \big(0.04\pi - 0.04\pi^2 + 1.20 - 1.20\pi - X + X\pi\big)
\end{aligned}
$$

$$
\begin{aligned}
0 = 0.04\pi + 0.04\pi r + 1.22 + 1.22r - (X + Xr) + 0.04\pi^2 + 1.24\pi - X\pi \\
+ 0.04\pi - 0.04\pi^2 + 1.20 - 1.20\pi - X + X\pi
\end{aligned}
$$

$$0 = 0.04\pi + 0.04\pi r + 1.22 + 1.22r - X - Xr + 1.24\pi + 0.04\pi + 1.20 - 1.20\pi - X$$

$$0 = 1.22 + 1.20 + 0.04\pi + 1.24\pi + 0.04\pi - 1.20\pi + 0.04\pi r + 1.22r - Xr - X - X$$

$$X\left(2+r\right) = 2.42 + 0.12\pi + 0.04\pi r + 1.22r$$

$$\boxed{X = \frac{2.42 + 0.12\pi + 0.04\pi r + 1.22r}{2+r}}$$

$$25,000X = \frac{60,500 + 3,000\pi + 1,000\pi r + 30,500r}{2+r}$$

$$
\begin{aligned}
25,000\left(1.26 - X\right) &= 25,000\left(\frac{1.26\left(2+r\right)}{2+r} - \frac{2.42 + 0.12\pi + 0.04\pi r + 1.22r}{2+r}\right) \\
&= 25,000 \times \frac{2.52 + 1.26r - 2.42 - 0.12\pi - 0.04\pi r - 1.22r}{2+r} \\
&= 25,000 \times \frac{0.1 + 0.04r - 0.12\pi - 0.04\pi r}{2+r} \\
&= \frac{2,500 + 1,000r - 3,000\pi - 1,000\pi r}{2+r}
\end{aligned}
$$

$$
\begin{aligned}
25,000\,(1.22 - X) &= 25,000\left(\frac{1.22\,(2+r)}{2+r} - \frac{2.42 + 0.12\pi + 0.04\pi r + 1.22r}{2+r}\right) \\
&= 25,000 \times \frac{2.44 + 1.26r - 2.42 - 0.12\pi - 0.04\pi r - 1.22r}{2+r} \\
&= 25,000 \times \frac{0.02 + 0.04r - 0.12\pi - 0.04\pi r}{2+r} \\
&= \frac{500 + 1,000r - 3,000\pi - 1,000\pi r}{2+r}
\end{aligned}
$$

$$
\begin{aligned}
25,000\,(1.28 - X) &= 25,000\left(\frac{1.28\,(2+r)}{2+r} - \frac{2.42 + 0.12\pi + 0.04\pi r + 1.22r}{2+r}\right) \\
&= 25,000 \times \frac{2.56 + 1.26r - 2.42 - 0.12\pi - 0.04\pi r - 1.22r}{2+r} \\
&= 25,000 \times \frac{0.14 + 0.04r - 0.12\pi - 0.04\pi r}{2+r} \\
&= \frac{3,500 + 1,000r - 3,000\pi - 1,000\pi r}{2+r}
\end{aligned}
$$

$$
\begin{aligned}
25,000\,(1.24 - X) &= 25,000\left(\frac{1.24\,(2+r)}{2+r} - \frac{2.42 + 0.12\pi + 0.04\pi r + 1.22r}{2+r}\right) \\
&= 25,000 \times \frac{2.48 + 1.26r - 2.42 - 0.12\pi - 0.04\pi r - 1.22r}{2+r} \\
&= 25,000 \times \frac{0.06 + 0.04r - 0.12\pi - 0.04\pi r}{2+r} \\
&= \frac{1,500 + 1,000r - 3,000\pi - 1,000\pi r}{2+r}
\end{aligned}
$$

$$
\begin{aligned}
25,000\,(1.20 - X) &= 25,000\left(\frac{1.2\,(2+r)}{2+r} - \frac{2.42 + 0.12\pi + 0.04\pi r + 1.22r}{2+r}\right) \\
&= 25,000 \times \frac{2.4 + 1.26r - 2.42 - 0.12\pi - 0.04\pi r - 1.22r}{2+r} \\
&= 25,000 \times \frac{-0.02 + 0.04r - 0.12\pi - 0.04\pi r}{2+r} \\
&= \frac{-500 + 1,000r - 3,000\pi - 1,000\pi r}{2+r}
\end{aligned}
$$

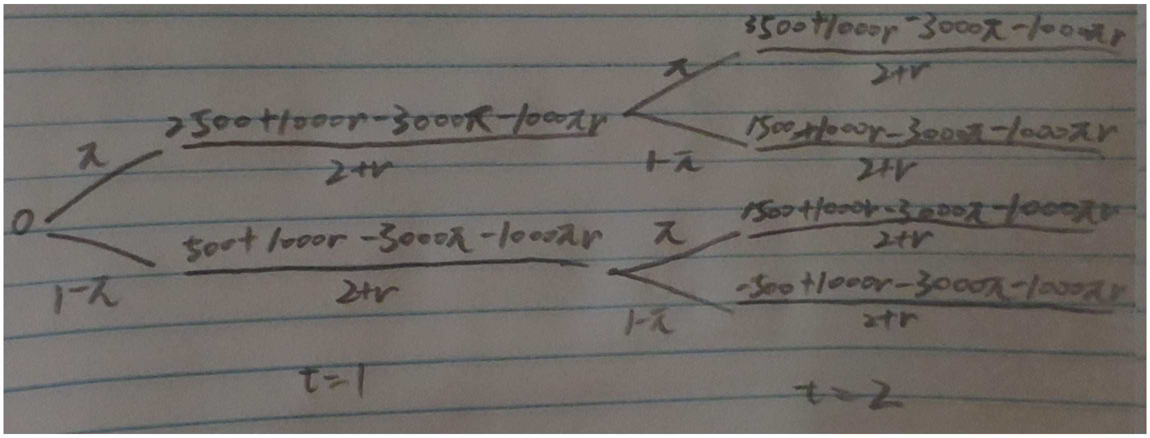

Figure 2: Time-decision tree of (c)

3

(a)

Let the risk-neutral probability measure $\pi = (\pi_1, \pi_2)$.

We have

$\pi_1 + \pi_2 = 1 \implies \pi_2 = 1 - \pi_1$

Look at the stock at first.

If this stock is arbitrage free, by the pricing formula (1.2) in the lecture note, we get:

$$\boxed{q_1 = \frac{\pi_1 g q_1 + (1 - \pi_1)\, b q_1}{1 + r}}$$

$\implies (1 + r)\, q_1 = \pi_1 g q_1 + b q_1 - \pi_1 b q_1$

$\implies (1 + r)\, q_1 - b q_1 = \pi_1 (g q_1 - b q_1)$

$\implies \pi_1 = \dfrac{(1 + r)\, q_1 - b q_1}{g q_1 - b q_1}$

$$
\begin{aligned}
\pi_2 &= 1 - \pi_1 \\
&= \frac{gq_1 - bq_1}{gq_1 - bq_1} - \frac{(1+r)\,q_1 - bq_1}{gq_1 - bq_1} \\
&= \frac{gq_1 - bq_1 - ((1+r)\,q_1 - bq_1)}{gq_1 - bq_1} \\
&= \frac{gq_1 - bq_1 - (1+r)\,q_1 + bq_1}{gq_1 - bq_1} \\
&= \frac{gq_1 - (1+r)\,q_1}{gq_1 - bq_1}
\end{aligned}
$$

$\because \pi_1 \geqslant 0, \pi_2 \geqslant 0$

$\therefore \dfrac{(1+r)\,q_1 - bq_1}{gq_1 - bq_1} \geqslant 0, \dfrac{gq_1 - (1+r)\,q_1}{gq_1 - bq_1} \geqslant 0$

$\because g > b$

$\therefore gq_1 - bq_1 > 0$

$\implies (1+r)\,q_1 - bq_1 \geqslant 0, gq_1 - (1+r)\,q_1 \geqslant 0$

$\implies (1+r)\,q_1 \geqslant bq_1, gq_1 \geqslant (1+r)\,q_1$

$\implies q_1 \geqslant \dfrac{bq_1}{1+r}, q_1 \leqslant \dfrac{gq_1}{1+r}$

$\implies \dfrac{bq_1}{1+r} \leqslant q_1 \leqslant \dfrac{gq_1}{1+r}$

$\implies \dfrac{b}{1+r} \leqslant 1 \leqslant \dfrac{g}{1+r}$

$1 - \dfrac{g}{1+r} \leqslant 0$, and $1 - \dfrac{b}{1+r} \geqslant 0$

$(1+r) - g \leqslant 0$, and $(1+r) - b \geqslant 0$

$\therefore \boxed{\dfrac{(1+r) - g}{(1+r) - b} \leqslant 0}$

In order to make sure that $\pi_1 \leqslant 1$ and $\pi_2 \leqslant 1$, $g > b \geqslant 1 + r$.

(b)

If $K \geqslant \max\{r_{11}, r_{21}\} = gq_1$, the buyer always choose to put.

If $K \leqslant \min\{r_{11}, r_{21}\} = bq_1$, the buyer always choose not to put.

If $bq_1 < K < gq_1$,

Let α_p be the shares of the stock, β_p be the shares of the bond.

We have the system of equation:

$0 = gq_1\alpha_p + Rq_2\beta_p$

$K - bq_1 = bq_1\alpha_p + Rq_2\beta_p$

$\implies$

$bq_1 - K = (g-b)\,q_1\alpha_p$

$$\alpha_p = \frac{bq_1 - K}{(g-b)\,q_1}$$

$$Rq_2\beta_p = -gq_1\alpha_p = -gq_1 \cdot \frac{bq_1 - K}{(g-b)\,q_1} = g \cdot \frac{K - bq_1}{g-b} = \frac{g\,(K - bq_1)}{g-b}$$

$$\beta_p = \frac{g\,(K - bq_1)}{Rq_2\,(g-b)}$$

$$\therefore q_3 = q_1\alpha_p + q_2\beta_p = q_1\frac{bq_1 - K}{(g-b)\,q_1} + q_2\frac{g\,(K-bq_1)}{Rq_2\,(g-b)} = \boxed{\frac{bq_1 - K}{g-b} + \frac{g\,(K-bq_1)}{R\,(g-b)}}$$

On the other hand,

$$q_1 = \frac{1}{R}\,(\pi_1 gq_1 + \pi_2 bq_1) = \frac{1}{R}\,(\pi_1 gq_1 + (1-\pi_1)\,bq_1) = \frac{1}{R}\,(\pi_1 gq_1 + bq_1 - \pi_1 bq_1) = \frac{q_1}{R}\,(\pi_1\,(g-b) + b)$$

$\implies R = \pi_1\,(g-b) + b \implies \pi_1\,(g-b) = R - b$

$$\implies \boxed{\pi_1 = \frac{R-b}{g-b}}$$

$$\implies \pi_2 = 1 - \pi_1 = 1 - \frac{R-b}{g-b} = \frac{g-b}{g-b} - \frac{R-b}{g-b} = \frac{g-b-(R-b)}{g-b} = \frac{g-b-R+b}{g-b}$$

$$\boxed{\pi_2 = \frac{g-R}{g-b}}$$

$$\mathbb{E}^{\pi}\,[r_3] = \pi_1 r_{13} + \pi_2 r_{23} = 0 + \frac{g-R}{g-b} \cdot (K - bq_1) = \frac{g\,(K-bq_1)}{g-b} - \frac{R\,(K-bq_1)}{g-b}$$

$$R^{-1}\mathbb{E}^{\pi}\,[r_3] = \frac{g\,(K-bq_1)}{R\,(g-b)} - \frac{K-bq_1}{g-b} = \frac{g\,(K-bq_1)}{R\,(g-b)} + \frac{bq_1-K}{g-b} = q_3$$