Yu Xia's Answer for Problem Set 9

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1

(a)

Denote the risk-neutral probability measure as $\{\pi, 1-\pi\}$.

By the non-arbitrage pricing formula:

$$100 = \frac{120\pi + 80\left(1-\pi\right)}{1.04}$$

$\Longleftrightarrow$

$$104 = 120\pi + 80 - 80\pi$$

$\Longleftrightarrow$

$$24 = 40\pi$$

$\Longleftrightarrow$

$$\pi = \frac{24}{40} = \frac{8\times 3}{8\times 5} = 0.6$$

Check other cases:

$$120 = \frac{144\pi + 96\left(1-\pi\right)}{1.04} \quad \Longleftrightarrow \quad \pi = 0.6$$

$$80 = \frac{96\pi + 64\left(1-\pi\right)}{1.04} \quad \Longleftrightarrow \quad \pi = 0.6$$

which is reasonable because the price change factor is $u = 1.2$ when price goes up and $d = 0.8$ when price goes down. It holds for all 3 periods.

(b)

$\because K \in (80, 96)\,,$

$\therefore$ The option will be exercised only if price never goes up.

The value of exercise at $t = 2$ is:

$$\frac{(1-\pi)^2}{(1+r)^2}(K-64)=\frac{0.4^2}{1.04^2}(K-64)$$

The value of exercise at $t=1$ is:

$$\frac{1-\pi}{1+r}(K-80)=\frac{0.4}{1.04}(K-80)$$

The value of exercise at $t=1$ is: $K-100$

American put option will be exercised prematurely iff:

$$\frac{0.4^2}{1.04^2}(K-64)<\min\left\{K-100,\frac{0.4}{1.04}(K-80)\right\}$$

When $\frac{0.4^2}{1.04^2}(K-64)<K-100$

$$\Longleftrightarrow$$

$$\frac{0.4^2}{1.04^2}K-\frac{0.4^2}{1.04^2}\times 64<K-100$$

$$\Longleftrightarrow$$

$$100-\frac{0.4^2}{1.04^2}\times 64<K-\frac{0.4^2}{1.04^2}K$$

$$\Longleftrightarrow$$

$$\left(1-\frac{0.4^2}{1.04^2}\right)K>100-\frac{0.4^2}{1.04^2}\times 64$$

$$\Longleftrightarrow$$

$$K>\frac{100-\frac{0.4^2}{1.04^2}\times 64}{1-\frac{0.4^2}{1.04^2}}=106.25$$

When $\frac{0.4^2}{1.04^2}(K-64)<\frac{0.4}{1.04}(K-80)$

$$\Longleftrightarrow$$

$$\frac{0.4}{1.04}(K-64)<K-80$$

$$\Longleftrightarrow$$

$$\frac{0.4}{1.04}K-\frac{0.4}{1.04}\times 64<K-80$$

$$80-\frac{0.4}{1.04}\times 64<K-\frac{0.4}{1.04}K$$

$$\frac{1}{1.04}K>80-\frac{0.4}{1.04}\times 64$$

$$K>1.04\times 80-0.4\times 64=90$$

So American put option will be exercise prematurely iff $K\in(90,96)$

(c)

We have:

$$P_0(K) = \begin{cases} \dfrac{0.4^2}{1.04^2}(K-64) & 80 < K < 90, \\ \dfrac{0.4}{1.04}(K-80) & 90 \leqslant K < 96. \end{cases}$$

2

In high price level case ω_1, the operational profit each year T is:

$$\Pi_T(\omega_1) = 5 \cdot 50 - 150 = 100$$

In low price level case ω_2:

$$\Pi_T(\omega_2) = 5 \cdot 30 - 150 = 0$$

Thus

$$V(\omega_i) = \sum_{t=1}^{\infty} \beta^t \Pi_t(\omega_i) = \frac{\Pi_T(\omega_i)}{r} = 10\Pi_T(\omega_i)$$

We have

$$V(\omega_1) = 10\Pi_T(\omega_1) = 1000$$

$$V(\omega_2) = 10\Pi_T(\omega_2) = 0$$

The PV of future profit is:

$$V_0 = 0.6 \cdot 1000 + 0.4 \cdot 0 = 600$$

$$NPV_{0;0} = \max\{V_0 - I, 0\} = (600 - I)_+$$

Instead, assume that the manager waits one period. Then, if price goes up, the PV of profit (without accounting for fixed investment cost I) is:

$$\sum_{t=1}^{\infty} \beta^t \Pi_t(\omega_1) = 1000$$

Therefore,

$$NPV_{1;1}(\omega_1) = (1000 - I)_+$$

Similarly,

$$NPV_{1;1}(\omega_2) = (-I)_+$$

Hence,

$$NPV_{0;1} = \mathbb{E}^{\mathbb{Q}}[\beta NPV_{1;1}] = \beta \sum_{\omega \in \Omega} p(\omega) NPV_{1;\geqslant 1}(\omega) = \frac{1}{1.1}\left(0.6(1000 - I)_+ + 0.4(-I)_+\right)$$

$$V_{\text{opt}} = NPV_{0;1} - NPV_{0;0} = \frac{1}{1.1}\left(0.6\,(1000-I)_+ + 0.4\,(-I)_+\right) - (600-I)_+$$

(a)

Never invest if: $(600-I)_+ = (1000-I)_+ = (-I)_+ = 0$

$I > 1000$

If $I = 1000$, it would be indifferent to whether or not to invest in the highest price case.

(b)

Invest at $t = 1$ is optimal if: $\max\left\{(1000-I)_+, (-I)_+\right\} > 0$ and $V_{\text{opt}} > 0$.

That is:

$$I < 1000 \text{ and } \frac{1}{1.1}\left(0.6\,(1000-I)_+ + 0.4\,(-I)_+\right) - (600-I)_+ > 0$$

$\Longleftrightarrow$

$$(600-I)_+ < \frac{1}{1.1}\left(600 - 0.6I + 0.4\,(-I)_+\right)$$

If $1000 > I \geqslant 600$, $(600-I)_+ = 0$, $(-I)_+ = 0$, $(1000-I)_+ > 0$ holds.

The firm will invest at $t = 1$ rather than $t = 0$. It is reasonable because the firm will wait for a period, and invest when the price goes up.

If $0 < I \leqslant 600$, $(600-I)_+ = 600 - I$, $(1000-I)_+ = 1000 - I$, $(-I)_+ = 0$

To satisfy $(600-I)_+ < \frac{1}{1.1}\left(600 - 0.6I + 0.4\,(-I)_+\right)$,

$$600 - I < \frac{1}{1.1}(600 - 0.6I)$$

$$1.1\,(600 - I) < 600 - 0.6I$$

$$660 - 1.1I < 600 - 0.6I$$

$$600 - 0.6I > 660 - 1.1I$$

$$1.1I + 600 - 0.6I > 660$$

$$1.1I - 0.6I > 60$$

$$0.5I > 60$$

$$I > 120$$

If $I = 120$, the investor will be indifferent with investing at $t = 0$ and wait for the price to go up for a period to invest.

In conclusion, if $120 < I < 1000$, invest at time $t = 1$.

(c)

Invest at $t = 0$ if: $(600 - I)_+ > 0$ and $V_{\text{opt}} < 0$

That is,

$$I < 600, \text{ and } \quad \frac{1}{1.1}\left(0.6\,(1000 - I)_+ + 0.4\,(-I)_+\right) - (600 - I)_+ < 0$$

$$600 - I > \frac{1}{1.1}\left(0.6\,(1000 - I) + 0.4\,(-I)_+\right)$$

$$1.1\,(600 - I) > (600 - 0.6I) + 0.4\,(-I)_+$$

$$660 - 1.1I > 600 - 0.6I + 0.4\,(-I)_+$$

$$60 - 0.5I > 0.4\,(-I)_+$$

$(-I)_+ = 0$ if $I > 0$, which is usually the case.

If a negative I is allowed,

$$60 - 0.5I > -0.4I$$

$$60 - 0.1I > 0$$

$$0.1I < 60$$

$$I < 600$$

So invest at time $t = 0$ if $I < 120$. (If a negative I is not allowed, $0 < I < 120$.)

3

Time 2 PV of the future profit:

If price goes up:

$$\sum_{s=1}^{\infty} \beta^s \left(R_2\,(40) - C\right) = \frac{40 \cdot 5 - 150}{r} = \frac{200 - 150}{0.1} = 500$$

$$V_2\,(40) = 500 \text{ if } 50 + \frac{Sc^h}{1.1} < 500$$

If $50 + \dfrac{Sc^h}{1.1} > 500$:

$$\frac{Sc^h}{1.1} > 450$$

$\iff$

$$Sc^h > 450 \cdot 1.1 = 495$$

$$V_2\,(40) = 50 + \frac{Sc^h}{1.1} \text{ if } Sc^h > 495$$

The firm would be indifferent if $Sc^h = 495$.

If price falls:

$$\Pi_2(20) = 20 \cdot 5 - 150 = -50 < 0$$

$$V_2(20) = -50 + \frac{Sc^l}{1.1}$$

Usually $Sc^l > 0$. It is optimal to exit as well if:

$$-50 + \frac{Sc^l}{1.1} > -500$$

$$\frac{Sc^l}{1.1} > -450$$

$$Sc^l > -450 \cdot 1.1 = -495$$

Value of staying active at $t = 1$:

$$V_1(S_1) = V_1(30) = \frac{1}{1.1}\left(0.6V_2(40) + 0.4V_2(20)\right)$$

Value of exit at $t = 1$:

$$NPV_1(S_1) = NPV_1(30) = \beta Sc = \frac{121}{1.1} = 110$$

(a)

Equation holds when the firm is indifferent.

If exit at $t = 1$ is better than staying in $t = 1$ in terms of expectation:

$$V_1(S_1) < NPV_1(S_1)$$

$$\iff$$

$$\frac{1}{1.1}\left(0.6V_2(40) + 0.4V_2(20)\right) < \frac{Sc}{1.1}$$

$$\iff$$

$$0.6V_2(40) + 0.4V_2(20) < Sc = 121$$

If $Sc^h > 495$ and $Sc^l > -495$:

$$0.6\left(50 + \frac{Sc^h}{1.1}\right) + 0.4\left(-50 + \frac{Sc^l}{1.1}\right) < 121$$

$$\iff$$

$$30 + \frac{6}{11}Sc^h - 20 + \frac{4}{11}Sc^l < 121$$

$$\iff$$

$$10 + \frac{6}{11}Sc^h + \frac{4}{11}Sc^l < 121$$

$$\iff$$

$$\frac{6}{11}Sc^h + \frac{4}{11}Sc^l < 111$$

If $Sc^h < 495$ and $Sc^l > -495$:

$$0.6 \cdot 500 + 0.4 \left(-50 + \frac{Sc^l}{1.1} \right) < Sc = 121$$

$$\Longleftrightarrow$$

$$6 \cdot 50 + \left(-0.4 \cdot 50 + \frac{0.4}{1.1} Sc^l \right) < 121$$

$$\Longleftrightarrow$$

$$300 + \left(-4 \cdot 5 + \frac{4}{11} Sc^l \right) < 121$$

$$\Longleftrightarrow$$

$$-20 + \frac{4}{11} Sc^l < -179$$

$$\Longleftrightarrow$$

$$\frac{4}{11} Sc^l < -159$$

$$\Longleftrightarrow$$

$$Sc^l < -159 \cdot \frac{11}{4} = -437.25$$

If $Sc^h > 495$ and $Sc^l < -495$:

$$0.6 \left(50 + \frac{Sc^h}{1.1} \right) + 0.4 \left(-500 \right) < Sc = 121$$

$$30 + \frac{6}{11} Sc^h - 0.4 \cdot 500 < 121$$

$$\frac{6}{11} Sc^h - 4 \cdot 50 < 91$$

$$\frac{6}{11} Sc^h < 291$$

$$Sc^h < 291 \cdot \frac{11}{6} = 533.5$$

If $Sc^h < 495$ and $Sc^l < -495$:

$$0.6 V_2 \left(40 \right) + 0.4 V_2 \left(20 \right) = 0.6 \cdot 500 + 0.4 \cdot \left(-500 \right) = 0.2 \cdot 500 = 2 \cdot 50 = 100 < 121 = Sc$$

In conclusion,

case 1:

$$Sc^h > Sc^l,\ Sc^h > 495,\ Sc^l > -495 \text{ and } \frac{6}{11} Sc^h + \frac{4}{11} Sc^l < 111,$$

case 2:

$$Sc^h > Sc^l,\ Sc^h < 495,\ -495 < Sc^l < -437.25,$$

if a negative Sc^l is alowed,

case 3:

$Sc^h > Sc^l$, $495 < Sc^h < 533.5$ and $Sc^l < -495$,

case 4:

$Sc^h > Sc^l$, $Sc^h < 495$ and $Sc^l < -495$.

(b)

If not exit in $t = 1$:

$Sc^h > Sc^l$, $Sc^h > 495$, $Sc^l > -495$ and $\frac{6}{11}Sc^h + \frac{4}{11}Sc^l > 111$

or

$Sc^h > Sc^l$, $Sc^h < 495$ and $Sc^l > -437.25$

or if negative values are allowed,

$Sc^h > Sc^l$, $Sc^h > 533.5$ and $Sc^l < -495$

Quit if price goes down:

$Sc^l > -495$

Quit if price goes up:

$Sc^h > 495$

Quit if no matter what happens:

$Sc^h > 495$ and $Sc^l > -495$ as well.

4

Assume the firm is active at $t = 2$.

Let $S_2 = 50$.

The PV of the profit is:

$$\frac{50 \cdot 5 - 150}{0.1} = \frac{250 - 150}{0.1} = \frac{100}{0.1} = 1000 > 400 = Sc$$

Not optimal to quit.

$V_2^{\text{ac}}(50) = 1000$

Let $S_2 = 40$.

The PV of the profit is:

$$\frac{40 \cdot 5 - 150}{0.1} = \frac{200 - 150}{0.1} = \frac{50}{0.1} = 500 > 400 = Sc$$

Not optimal to quit.

$V_2^{\text{ac}}(40) = 500$

Let $S_2 = 20$.

The PV of the profit at this period is:

$20 \cdot 5 - 150 = 100 - 150 = -50 = -50 < 0$

The firm will exit if the price goes down to 20.

$$V_2^{\text{ac}}(20) = -50 + \frac{400}{1.1} \approx 313.64$$

Now look at the firm's decision at $t = 1$:

Let $S_1 = 50$.

The PV of the profit is:

$$\frac{50 \cdot 5 - 150}{0.1} = \frac{250 - 150}{0.1} = \frac{100}{0.1} = 1000 > 400 = Sc$$

Let $S_1 = 30$, the EPV of the profit is:

$$\mathbb{E}_1^{\mathbb{Q}}\left[\beta V_2^{\text{ac}}\right] = \mathbb{E}^{\mathbb{Q}}\left[\beta V_2^{\text{ac}} | S_1 = 40\right] \approx 0.6 \cdot 500 + 0.4 \cdot \frac{313.64}{1.1} \approx 414.05$$

If the firm, instead, decides to exit at $t = 1$, when $S_1 = 50$, it will gain $\frac{Sc}{1.1} = \frac{400}{1.1} \approx 363.64 < 414.05$.

It's not optimal to exit at $t = 1$ if $S_1 = 30$.

To summarize:

$V_1^{\text{ac}}(50) = 1000$

$V_1^{\text{ac}}(30) = 414.05$

At $t = 0$, the EPV of future gains, is:

$$V_0^{\text{ac}} = \mathbb{E}^{\mathbb{Q}}\left[\beta V_1^{\text{ac}}\right] = 0.6 \cdot V_1^{\text{ac}}(50) + 0.4 \cdot V_1^{\text{ac}}(30) \approx 0.6 \cdot 1000 + 0.4 \cdot \frac{414.05}{1.1} \approx 750.56$$

Backward induction:

At $t = 2$:

$G_2(50) = V_2^{\text{ac}}(50) - I = 1000 - 500 = 500$

$G_2(40) = V_2^{\text{ac}}(40) - I = 500 - 500 = 0$

$G_2(20) = \left(V_2^{\text{ac}}(20) - I, 0\right)_+ = (313.64 - 500, 0)_+ = 0$

Hence the firm invests at $t = 2$ only if $S_2 = 50$ or $S_2 = 40$.

At $t = 1$:

$$G_1(50) = V_1^{\text{ac}}(50) - I = 1000 - 500 = 500 > \frac{500}{1.1} = \frac{G_2}{1.1}$$

$G_1(30) = \left(V_1^{\text{ac}}(30) - I, 0\right)_+ = (414.05 - 500, 0)_+ = 0$

Hence the firm invests at $t = 1$ immediately only if $S_1 = 50$.

At $t = 0$:

$$\beta \mathbb{E}^{\mathbb{Q}} [G_1] = \frac{1}{1.1} (0.6 G_1 (50) + 0.4 G_1 (30)) = \frac{1}{1.1} (0.6 \cdot 500 + 0) = \frac{6 \cdot 50}{1.1} = \frac{300}{1.1} \approx 272.73$$

$$V_0^{\mathrm{ac}} - I \approx 750.56 - 500 \approx 250.56 < 272.73$$

In conclusion:

(a)

Don't invest immediately at $t = 0$.

If $S_1 = 30$, wait until $t = 2$.

Do not invest at $t = 2$ if $S_2 = 20$ after waiting in the previous period.

(b)

If $S_1 = 50$, the firm invests at $t = 1$ immediately.

If $S_1 = 30$, wait until $t = 2$. If $S_2 = 40$, invest immediately.

5

Assume the firm is active at $t = 2$.

Let $S_2 = 60$.

The PV of the profit is:

$$\frac{60 \cdot 5 - 150}{0.1} = \frac{300 - 150}{0.1} = \frac{150}{0.1} = 1500 > 200 = Sc$$

Not optimal to quit.

$$V_2^{\mathrm{ac}} (60) = 1500$$

Let $S_2 = 40$.

The PV of the profit is:

$$\frac{40 \cdot 5 - 150}{0.1} = \frac{200 - 150}{0.1} = \frac{50}{0.1} = 500 > 200 = Sc$$

Not optimal to quit.

$$V_2^{\mathrm{ac}} (40) = 500$$

Let $S_2 = 20$.

The PV of the profit at this period is:

$$20 \cdot 5 - 150 = 100 - 150 = -50 = -50 < 0$$

The firm will exit if the price goes down to 20.

$V_2^{\text{ac}}(20) = -50 + \frac{200}{1.1} \approx 131.82$

Now look at the firm's decision at $t = 1$:

Let $S_1 = 50$, the EPV of the profit is:

$\mathbb{E}^{\mathbb{Q}}\left[\beta V_2^{\text{ac}} | S_1 = 50\right] \approx 0.6 \cdot 1500 + 0.4 \cdot 500 = 6 \cdot 150 + 4 \cdot 50 = 3 \cdot 300 + 200 = 1100$

Let $S_1 = 30$, the EPV of the profit is:

$\mathbb{E}^{\mathbb{Q}}\left[\beta V_2^{\text{ac}} | S_1 = 30\right] \approx 0.6 \cdot 500 + 0.4 \cdot \frac{131.82}{1.1} \approx 347.93$

If the firm, instead, decides to exit at $t = 1$, it will gain $\frac{Sc}{1.1} = \frac{200}{1.1} \approx 181.82 < 347.93 < 1100$.

The firm will not exit at $t = 1$ before observing the price at $t = 2$.

To summarize:

$V_1^{\text{ac}}(50) = 1100$

$V_1^{\text{ac}}(30) = 347.93$

At $t = 0$, the EPV of future gains, is:

$V_0^{\text{ac}} = \mathbb{E}^{\mathbb{Q}}\left[\beta V_1^{\text{ac}}\right] = 0.6 \cdot V_1^{\text{ac}}(50) + 0.4 \cdot V_1^{\text{ac}}(30) \approx 0.6 \cdot 1100 + 0.4 \cdot \frac{347.93}{1.1} \approx 786.52$

Backward induction:

At $t = 2$:

$G_2(60) = V_2^{\text{ac}}(60) - I = 1500 - 400 = 1100$

$G_2(40) = V_2^{\text{ac}}(40) - I = 500 - 400 = 100$

$G_2(20) = \left(V_2^{\text{ac}}(20) - I, 0\right)_+ = (131.82 - 400, 0)_+ = 0$

Hence the firm invests at $t = 2$ only if $S_2 = 60$ or $S_2 = 40$.

At $t = 1$:

$G_1(50) = V_1^{\text{ac}}(50) - I = 1100 - 400 = 700$

$G_1(30) = \left(V_1^{\text{ac}}(30) - I, 0\right)_+ = (347.93 - 400, 0)_+ = 0$

Hence the firm invests at $t = 1$ immediately only if $S_1 = 50$.

At $t = 0$:

$\beta \mathbb{E}^{\mathbb{Q}}\left[G_1\right] = \frac{1}{1.1}\left(0.6 G_1(50) + 0.4 G_1(30)\right) = \frac{1}{1.1}(0.6 \cdot 700 + 0) = \frac{420}{1.1} \approx 381.82$

$V_0^{\text{ac}} - I \approx 786.52 - 400 \approx 386.52 > 381.82$

In conclusion:

(a)

If $S_1 = 30$, wait until $t = 2$.

Do not invest at $t = 2$ if $S_2 = 20$ after waiting in the previous period.

(b)

Invest at $t = 0$.

If $S_1 = 50$, the firm invests at $t = 1$ immediately.

If $S_1 = 40$, wait until $t = 2$. If $S_2 = 40$, invest immediately.