Yu Xia's Answer for Problem Set 5

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(a)

$$\boxed{\begin{aligned} &\max_{(x_0,x_1)} \sqrt{x_0}+\sqrt{x_1} \\ \text{subject to}& \\ &x_0 = e_0 = 121, \\ &x_1 = e_1 = 49, \\ &x_0 \geqslant 0, x_1 \geqslant 0. \end{aligned}}$$

Plug in we have:

$$\boxed{V(121,49) = \sqrt{121}+\sqrt{49} = 11+7 = 18}$$

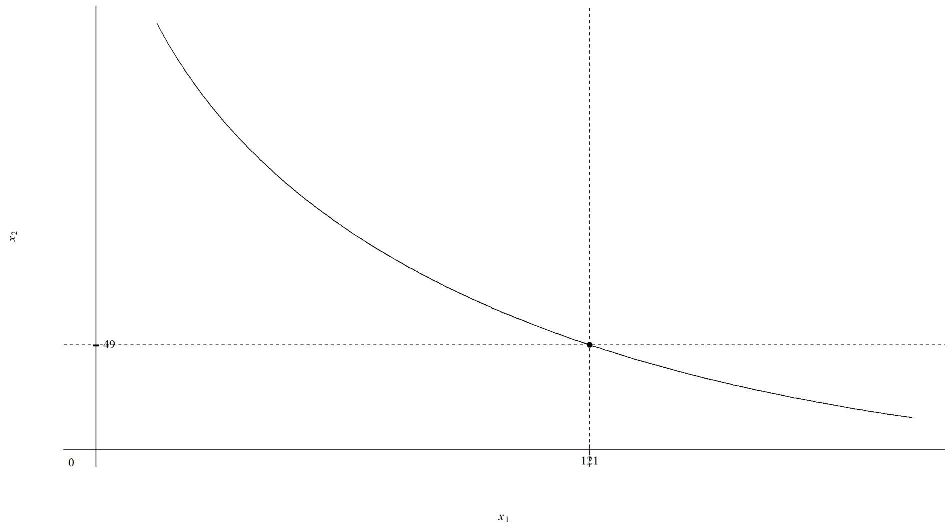

Figure 1: Consumer's indifference curve through the optimal consumption bundle of 1(a)

(b)

The payoff matrix is:

$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} 1+r & 1 \\ 1+r & 1 \end{pmatrix}$

The asset prices are $(1, 1)$.

The consumer's problem is:

$$\boxed{\begin{aligned} &\max_{(x_0, x_1)} \sqrt{x_0} + \sqrt{x_1} \\ &\text{subject to} \\ &\quad x_0 + b = 121, \\ &\quad x_1 = 49 + (1+r)\, b, \\ &\quad x_0 \geqslant 0, x_1 \geqslant 0. \end{aligned}}$$

$b = 121 - x_0$

$x_1 = 49 + (1+r)(121 - x_0) = 49 + 121(1+r) - x_0(1+r)$

$x_0(1+r) + x_1 = 49 + 121(1+r)$, the budget constraint.

$MU_0 = \frac{1}{2}(x_0)^{-\dfrac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x_0}}$

$MU_1 = \frac{1}{2}\left(x_1\right)^{-\dfrac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x_1}}$

$$\text{MRS}= \frac{MU_0}{MU_1} = \frac{\frac{1}{2} \cdot \frac{1}{\sqrt{x_0}}}{\frac{1}{2} \cdot \frac{1}{\sqrt{x_1}}} = \frac{\frac{1}{\sqrt{x_0}}}{\frac{1}{\sqrt{x_1}}} = \frac{1}{\frac{\sqrt{x_0}}{\sqrt{x_1}}} = \frac{\sqrt{x_1}}{\sqrt{x_0}} = \sqrt{\frac{x_1}{x_0}}$$

On the other hand,

MRS$= 1 + r$

$\therefore \sqrt{\frac{x_1}{x_0}} = 1 + r$

$\implies \frac{x_1}{x_0} = (1 + r)^2$

$\implies x_1 = (1 + r)^2 x_0$

Plug into constraint:

$x_0 (1 + r) + (1 + r)^2 x_0 = 49 + 121 (1 + r)$

$\implies x_0 + x_0 (1 + r) = \frac{49}{1 + r} + 121$

$\implies x_0 (2 + r) = \frac{49}{1 + r} + 121$

$x_0 = \frac{49}{(1 + r)(2 + r)} + \frac{121}{2 + r}$

$$x_1 = (1 + r)^2 \left(\frac{49}{(1 + r)(2 + r)} + \frac{121}{2 + r}\right) = \frac{49 (1 + r)}{2 + r} + \frac{121 (1 + r)^2}{2 + r} = \frac{49 (1 + r) + 121 (1 + r)^2}{2 + r}$$

$\because r = 0.03$

$\therefore$ The consumer's problem is:

$$\boxed{\begin{aligned} &\max_{(x_0, x_1)} \sqrt{x_0} + \sqrt{x_1} \\ &\text{subject to} \\ &\quad x_0 + b = 121, \\ &\quad x_1 = 49 + 1.03b, \\ &\quad x_0 \geqslant 0, x_1 \geqslant 0. \end{aligned}}$$

The budget constraint: $1.03x_0 + x_1 = 49 + 121 \times 1.03 = 173.63$

$x_0 = \frac{49}{1.03 \times 2.03} + \frac{121}{2.03} \approx \boxed{83.0408}$

$x_1 = \frac{49 \times 1.03 + 121 (1.03)^2}{2.03} \approx \boxed{88.09798}$

$b = 121 - x_0 \approx \boxed{37.9592}$

The maximal utility level is:

$$\begin{aligned}
V(x_0, x_1) &= V\left(\frac{49}{(1+r)(2+r)} + \frac{121}{2+r}, \frac{49(1+r) + 121(1+r)^2}{2+r}\right) \\
&= \sqrt{\frac{49}{(1+r)(2+r)} + \frac{121}{2+r}} + \sqrt{\frac{49(1+r) + 121(1+r)^2}{2+r}} \\
&= \sqrt{\frac{49}{1.03 \times 2.03} + \frac{121}{2.03}} + \sqrt{\frac{49 \times 1.03 + 121(1.03)^2}{2.03}} \\
&\approx \boxed{18.49872}
\end{aligned}$$

Check the corner solution:

If $x_0 = 0$, $0 + x_1 = 49 + 121(1+r) \implies x_1 = 49 + 121 \times 1.03 = 173.63$

$V(0, 173.63) = \sqrt{173.63} \approx 13.17687 < V\left(\frac{49}{1.03 \times 2.03} + \frac{121}{2.03}, \frac{49 \times 1.03 + 121 \times 1.03^2}{2.03}\right)$

If $x_1 = 0$, $(1+r)x_0 + 0 = 49 + 121(1+r) \implies x_0 = \frac{49}{1+r} + 121 = \frac{49}{1.03} + 121 \approx 168.5728$

$V\left(\frac{49}{1+r} + 121, 0\right) = \sqrt{\frac{49}{1+r} + 121} = \sqrt{\frac{49}{1.03} + 121} \approx 12.98356$

$V\left(\frac{49}{1.03 \times 2.03} + \frac{121}{2.03}, \frac{49 \times 1.03 + 121 \times 1.03^2}{2.03}\right)$

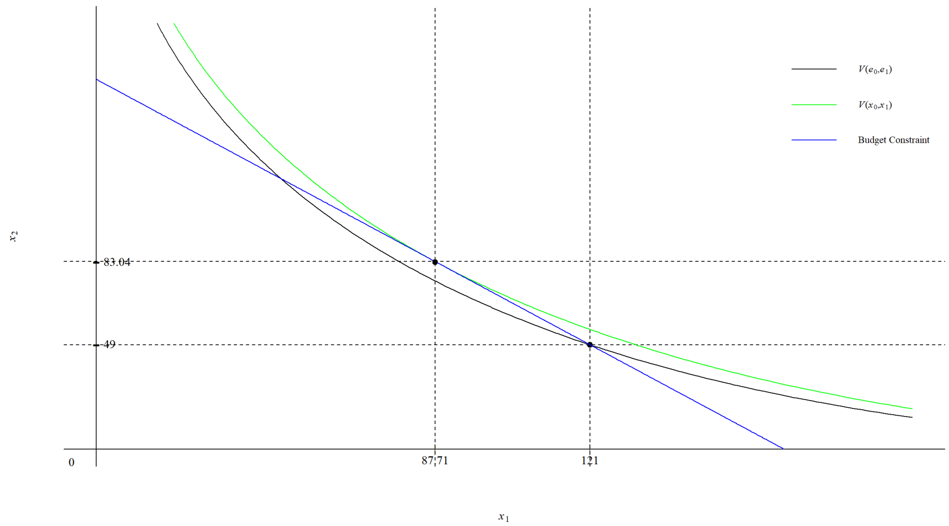

Figure 2: Consumer's indifference curve through the optimal consumption bundle of 1(b)

(c)

The consumer is better off. We can see this from the diagram.

We can also prove it by math:

$$V\left(121,49\right)=11+7=18<V\left(\frac{49}{\left(1+r\right)\left(2+r\right)}+\frac{121}{2+r},\frac{49\left(1+r\right)+121\left(1+r\right)^{2}}{2+r}\right)$$

The consumer saves part of his/her money, and invests in the risk-free bond at the same time, so that it pays back in the future. The consumer do so in order to maximize their utility.

2

(a)

$$\max_{(x_1,x_2)} \frac{1}{4}\sqrt{x_1}+\frac{3}{4}\sqrt{x_2}$$

subject to

$$x_1=e_1=121,$$
$$x_2=e_2=49,$$
$$x_1\geqslant 0, x_2\geqslant 0.$$

Plug in we have:

$$\boxed{V(121,49)=\frac{1}{4}\sqrt{121}+\frac{3}{4}\sqrt{49}=\frac{1}{4}\times 11+\frac{3}{4}\times 7=\frac{11+21}{4}=\frac{32}{4}=\boxed{8}}$$

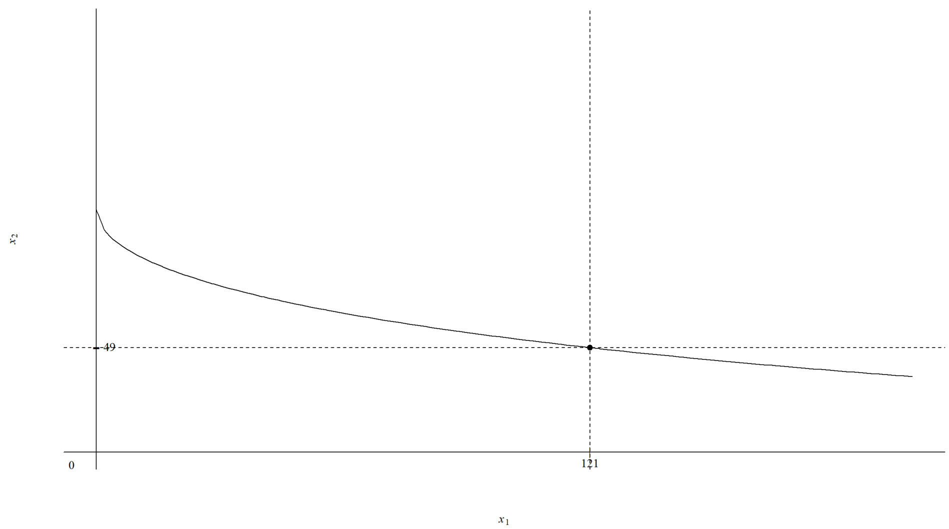

Figure 3: consumer's indifference curve through the optimal consumption bundle of 2(a)

(b)

The payoff matrix is:

$$R=\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}=\begin{pmatrix} 1+r & 1 \\ 1+r & 0 \end{pmatrix}$$

The asset prices are $(1,q)$.

The consumer's problem is:

$$\boxed{\begin{aligned} &\max_{(x_1,x_2)} \frac{1}{4}\sqrt{x_1}+\frac{3}{4}\sqrt{x_2} \\ \text{subject to} \quad & \\ & b+qk=0, \\ & x_1=121+(1+r)\,b+k, \\ & x_2=49+(1+r)\,b, \\ & x_1\geqslant 0, x_2\geqslant 0. \end{aligned}}$$

$$\implies b=-qk \implies x_1=121+(1+r)(-qk)+k=121-qk-rqk+k=121+(1-q-rq)\,k$$

It also implies:

$x_2 = 49 + (1+r)(-qk) = 49 - qk - rqk \implies x_2 + qk + rqk = 49 \implies (1+r)\,qk = 49 - x_2$
$\implies k = \dfrac{49 - x_2}{(1+r)\,q}$

Thus,

$\implies x_1 = 121 + (1 - q - rq) \cdot \dfrac{49 - x_2}{(1+r)\,q}$

$\implies (1+r)\,qx_1 = 121\,(1+r)\,q + (1 - q - rq)\,(49 - x_2)$

$= 121\,(1+r)\,q + (1 - q - rq) \cdot 49 + (1 - q - rq)\,(-x_2)$

$\implies (1+r)\,qx_1 + (1 - q - rq)\,x_2 = 121\,(1+r)\,q + 49\,(1 - q - rq)$

$\implies (1+r)\,qx_1 + (1 - q - rq)\,x_2 = 121q + 121rq + 49 - 49q - 49rq$

$\implies (1+r)\,qx_1 + (1 - q - rq)\,x_2 = 72\,(1+r)\,q + 49$, the budget constraint.

$MU_1 = \dfrac{1}{4} \cdot \dfrac{1}{2}\,(x_1)^{-\frac{1}{2}} = \dfrac{1}{8} \cdot \dfrac{1}{\sqrt{x_1}}$

$MU_2 = \dfrac{3}{4} \cdot \dfrac{1}{2}\,(x_2)^{-\frac{1}{2}} = \dfrac{3}{8} \cdot \dfrac{1}{\sqrt{x_2}}$

MRS$= \dfrac{MU_1}{MU_2} = \dfrac{\dfrac{1}{8} \cdot \dfrac{1}{\sqrt{x_1}}}{\dfrac{3}{8} \cdot \dfrac{1}{\sqrt{x_2}}} = \dfrac{\dfrac{1}{\sqrt{x_1}}}{\dfrac{3}{\sqrt{x_2}}} = \dfrac{1}{\dfrac{3\sqrt{x_1}}{\sqrt{x_2}}} = \dfrac{\sqrt{x_2}}{3\sqrt{x_1}}$

On the other hand,

MRS$= \dfrac{(1+r)\,q}{1 - q - rq}$

$\therefore \dfrac{\sqrt{x_2}}{3\sqrt{x_1}} = \dfrac{(1+r)\,q}{1 - q - rq}$

In order to generate a positive supply and demand for the risky asset, the expected value of the consumer's portfolio should be 0, that is,

$pk - qk = 0$, where $p = 0.25 \implies \boxed{q = \dfrac{1}{4}}$

Rewriting budget constraint:

$\dfrac{(1+r)\,x_1}{4} + \left(1 - \dfrac{1}{4} - \dfrac{r}{4}\right) x_2 = 18\,(1+r) + 49$

$\implies (1+r)\,x_1 + (3 - r)\,x_2 = 72\,(1+r) + 196$

$\implies (1+r)\,x_1 + (3 - r)\,x_2 = 72 + 72r + 196$

$(1+r)\,x_1 + (3-r)\,x_2 = 268 + 72r$

Also,

$$\frac{\sqrt{x_2}}{3\sqrt{x_1}} = \frac{\frac{1}{4}\,(1+r)}{1-\frac{1}{4}-\frac{1}{4}r} = \frac{1+r}{3-r} \implies \frac{x_2}{9x_1} = \frac{(1+r)^2}{(3-r)^2} \implies x_2 = \frac{(1+r)^2}{(3-r)^2}\cdot 9x_1$$

The budget constraint becomes:

$$\frac{1}{4}\,(1+r)\,x_1 + \left(1-\frac{1}{4}-\frac{1}{4}r\right)\cdot\frac{(1+r)^2}{(3-r)^2}\cdot 9x_1 = 72\,(1+r)\cdot\frac{1}{4} + 49$$

$$\implies \frac{1}{4}\,(1+r)\,x_1 + \frac{3-r}{4}\cdot\frac{(1+r)^2}{(3-r)^2}\cdot 9x_1 = 72\,(1+r)\cdot\frac{1}{4} + \frac{4\times 49}{4}$$

$$\implies (1+r)\,x_1 + \frac{(1+r)^2}{3-r}\cdot 9x_1 = 72\,(1+r) + 4\times 49$$

$$\implies (1+r)\,x_1\left(1+\frac{9\,(1+r)}{3-r}\right) = 72\,(1+r) + 196$$

$$\implies x_1\left(\frac{3-r}{3-r}+\frac{9+9r}{3-r}\right) = 72 + \frac{196}{1+r}$$

$$\implies x_1\cdot\frac{12+8r}{3-r} = 72 + \frac{196}{1+r}$$

$$\implies x_1\cdot\frac{3+2r}{3-r} = 18 + \frac{49}{1+r} = \frac{18+18r+49}{1+r} = \frac{67+18r}{1+r}$$

$$\implies x_1 = \frac{67+18r}{1+r}\cdot\frac{3-r}{3+2r}$$

$$\implies x_1 = \frac{(3-r)\,(67+18r)}{(1+r)\,(3+2r)}$$

$$x_2 = \frac{(1+r)^2}{(3-r)^2}\cdot 9x_1 = \frac{(1+r)^2}{(3-r)^2}\cdot 9\cdot\frac{(67+18r)\,(3-r)}{(1+r)\,(3+2r)} = \frac{1+r}{3-r}\cdot 9\cdot\frac{67+18r}{3+2r}$$

$$x_2 = \frac{9\,(1+r)\,(67+18r)}{(3-r)\,(3+2r)}$$

$$(1+r)\,b = x_2 - 49 = \frac{9\,(1+r)\,(67+18r)}{(3-r)\,(3+2r)} - 49$$

$$b = \frac{9\,(67+18r)}{(3-r)\,(3+2r)} - \frac{49}{1+r}$$

$$k = \frac{49 - \frac{9\,(1+r)\,(67+18r)}{(3-r)\,(3+2r)}}{(1+r)\cdot\frac{1}{4}} = \frac{196}{1+r} - \frac{36\,(67+18r)}{(3-r)\,(3+2r)}$$

The maximal utility level is:

$$
\begin{aligned}
V\left(x_1,x_2\right) &= V\left(\frac{(3-r)\,(67+18r)}{(1+r)\,(3+2r)},\frac{9\,(1+r)\,(67+18r)}{(3-r)\,(3+2r)}\right)\\
&= \frac{1}{4}\sqrt{\frac{(3-r)\,(67+18r)}{(1+r)\,(3+2r)}}+\frac{3}{4}\sqrt{\frac{9\,(1+r)\,(67+18r)}{(3-r)\,(3+2r)}}\\
&= \frac{1}{4}\sqrt{\frac{(3-r)\,(67+18r)}{(1+r)\,(3+2r)}}+\frac{9}{4}\sqrt{\frac{(1+r)\,(67+18r)}{(3-r)\,(3+2r)}}\\
&= \frac{1}{4}\sqrt{\frac{67+18r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}}+9\sqrt{\frac{1+r}{3-r}}\right)
\end{aligned}
$$

Check the corner solution:

If $x_1=0$, $0+(3-r)\,x_2=268+72r \implies x_2=\dfrac{268+72r}{3-r}$

$$V\left(0,\frac{268+72r}{3-r}\right)=\frac{3}{4}\sqrt{\frac{268+72r}{3-r}}=\frac{3}{4}\sqrt{\frac{4\cdot(67+18r)}{3-r}}=\frac{3}{2}\sqrt{\frac{67+18r}{3-r}}$$

If $x_2=0$, $(1+r)\,x_1+0=268+72r \implies x_1=\dfrac{268+72r}{1+r}$

$$V\left(\frac{268+72r}{1+r},0\right)=\frac{1}{4}\sqrt{\frac{268+72r}{1+r}}=\frac{1}{2}\sqrt{\frac{67+18r}{1+r}}$$

$$
\begin{aligned}
\frac{V\left(\frac{(3-r)\,(67+18r)}{(1+r)\,(3+2r)},\frac{9\,(1+r)\,(67+18r)}{(3-r)\,(3+2r)}\right)}{V\left(0,\frac{268+72r}{3-r}\right)} &= \frac{\frac{1}{4}\sqrt{\frac{67+18r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}}+9\sqrt{\frac{1+r}{3-r}}\right)}{\frac{3}{2}\sqrt{\frac{67+18r}{3-r}}}\\
&= \frac{1}{4}\sqrt{\frac{67+18r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}}+9\sqrt{\frac{1+r}{3-r}}\right)\cdot\frac{2}{3}\sqrt{\frac{3-r}{67+18r}}\\
&= \frac{1}{4}\cdot\frac{2}{3}\sqrt{\frac{3-r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}}+9\sqrt{\frac{1+r}{3-r}}\right)\\
&= \frac{1}{6}\left(\frac{3-r}{\sqrt{(3+2r)\,(1+r)}}+9\sqrt{\frac{1+r}{3+2r}}\right)>1
\end{aligned}
$$

In the interval $r\in(0,1]$, the fraction above minimizes at $r=1$ with value about 1.054093.

$$\frac{V\left(\frac{(3-r)(67+18r)}{(1+r)(3+2r)}, \frac{9(1+r)(67+18r)}{(3-r)(3+2r)}\right)}{V\left(\frac{268+72r}{1+r}, 0\right)} = \frac{\frac{1}{4}\sqrt{\frac{67+18r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}}+9\sqrt{\frac{1+r}{3-r}}\right)}{\frac{1}{2}\sqrt{\frac{67+18r}{1+r}}}$$

$$= \frac{1}{4}\sqrt{\frac{67+18r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}}+9\sqrt{\frac{1+r}{3-r}}\right)\cdot 2\sqrt{\frac{1+r}{67+18r}}$$

$$= \frac{1}{2}\sqrt{\frac{1+r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}}+9\sqrt{\frac{1+r}{3-r}}\right)$$

$$= \frac{1}{2}\left(\sqrt{\frac{3-r}{3+2r}}+9\frac{1+r}{\sqrt{(3+2r)(3-r)}}\right) \geqslant 2$$

In the interval $r \in (0,1)$, the fraction above minimizes at about $r = 0$ with value approximately 2.

Hence,

$$\boxed{\begin{cases} x_1 = \frac{(3-r)(67+18r)}{(1+r)(3+2r)} \\ x_2 = \frac{9(1+r)(67+18r)}{(3-r)(3+2r)} \\ b = \frac{9(67+18r)}{(3-r)(3+2r)} - \frac{49}{1+r} \\ k = \frac{196}{1+r} - \frac{36(67+18r)}{(3-r)(3+2r)} \\ \max V = \frac{1}{4}\sqrt{\frac{67+18r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}}+9\sqrt{\frac{1+r}{3-r}}\right) \end{cases}}$$

If $r = 0.03$,

$$x_1 = \frac{67+18\times 0.03}{1.03}\cdot\frac{3-0.03}{3+2\times 0.03} \approx 63.6442$$

$$x_2 = \frac{9\times 1.03\,(67+18\times 0.03)}{(3-0.03)(3+2\times 0.03)} \approx 68.89107$$

$$b = \frac{9(67+18\times 0.03)}{(3-0.03)(3+2\times 0.03)} - \frac{49}{1.03} \approx 19.31172$$

$$k = \frac{196}{1.03} - \frac{36(67+18\times 1.03)}{(3-0.03)(3+2\times 0.03)} \approx -77.24686$$

$$V(x_1, x_2) = \frac{1}{4}\sqrt{\frac{67+18\times 0.03}{3+2\times 0.03}}\left(\sqrt{\frac{3-0.03}{1.03}}+9\sqrt{\frac{1.03}{3-0.03}}\right) \approx 8.219481$$

(c)

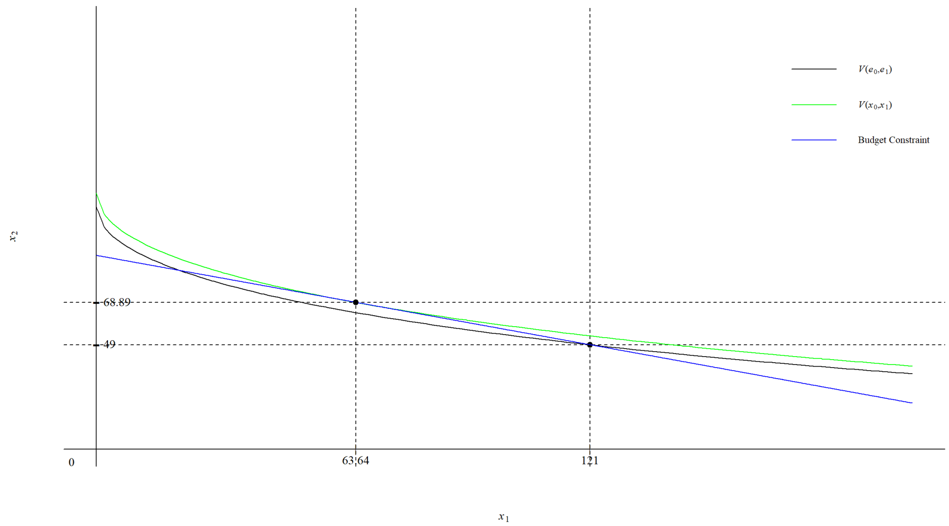

Figure 4: consumer's indifference curve through the optimal consumption bundle of 2(b)

(d)

The consumer is better off. His/her indifference curve is higher, as shown in the figure above. We can also prove it by math that

$\frac{1}{4} \times 11 + \frac{3}{4} \times 7 = \frac{11}{4} + \frac{21}{4} = \frac{32}{4} = 8$

While $\max V = \frac{1}{4}\sqrt{\frac{67+18r}{3+2r}}\left(\sqrt{\frac{3-r}{1+r}} + 9\sqrt{\frac{1+r}{3-r}}\right) > 8$.

$b = \frac{9\left(67+18r\right)}{\left(3-r\right)\left(3+2r\right)} - \frac{49}{1+r} > 0$, which means the consumer has long position of bond, and short position on risky asset, in order to maximize his/her expected utility.

Also, $x_1 = x_2$ doesn't hold generally. It implies that the consumer doesn't hedge all his/her risk.

3

(a)

$$\boxed{\begin{aligned} &\max_{(x_1^A, x_2^A)} \frac{1}{2}\ln x_1^A + \frac{1}{2}\ln x_2^A \\ &\text{subject to} \\ &\hat{p}_1 x_1^A + \hat{p}_2 x_2^A = \hat{p}_1 + 2\hat{p}_2. \end{aligned}}$$

For Alice:

$MU_1^A = \frac{1}{2} \cdot \frac{1}{x_1^A}$

$MU_2^A = \frac{1}{2} \cdot \frac{1}{x_2^A}$

$\text{MRS}^A = \frac{MU_1^A}{MU_2^A} = MU_1^A \cdot \frac{1}{MU_2^A} = \frac{1}{2} \cdot \frac{1}{x_1^A} \cdot 2x_2^A = \frac{x_2^A}{x_1^A}$

On the other hand,

$\text{MRS}^A = \frac{\hat{p}_1}{\hat{p}_2}$

$\therefore \frac{x_2^A}{x_1^A} = \frac{\hat{p}_1}{\hat{p}_2}$

Substituting $x_2^A = \frac{\hat{p}_1}{\hat{p}_2} x_1^A$ into budget constraint for Alice:

$\hat{p}_1 x_1^A + \hat{p}_2 \frac{\hat{p}_1}{\hat{p}_2} x_1^A = \hat{p}_1 + 2\hat{p}_2$

$\implies \hat{p}_1 x_1^A + \hat{p}_1 x_1^A = \hat{p}_1 + 2\hat{p}_2$

$\implies 2\hat{p}_1 x_1^A = \hat{p}_1 + 2\hat{p}_2$

$\implies x_1^A = \frac{\hat{p}_1 + 2\hat{p}_2}{2\hat{p}_1} = \boxed{\frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1}}$

$x_2^A = \frac{\hat{p}_1}{\hat{p}_2} x_1^A = \frac{\hat{p}_1}{\hat{p}_2}\left(\frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1}\right) = \boxed{\frac{\hat{p}_1}{2\hat{p}_2} + 1}$

(b)

$$\boxed{\begin{aligned} &\max_{(x_1^B, x_2^B)} \frac{1}{2}\ln x_1^B + \frac{1}{2}\ln x_2^B \\ \text{subject to}& \\ &\hat{p}_1 x_1^B + \hat{p}_2 x_2^B = 3\hat{p}_1 + \hat{p}_2. \end{aligned}}$$

For Bob:

$MU_1^B = \frac{1}{2} \cdot \frac{1}{x_1^B}$

$MU_2^B = \frac{1}{2} \cdot \frac{1}{x_2^B}$

$\text{MRS}^B = \frac{MU_1^B}{MU_2^B} = MU_1^B \cdot \frac{1}{MU_2^B} = \frac{1}{2} \cdot \frac{1}{x_1^B} \cdot 2x_2^B = \frac{x_2^B}{x_1^B}$

On the other hand,

$\text{MRS}^B = \frac{\hat{p}_1}{\hat{p}_2}$

$\therefore \frac{x_2^B}{x_1^B} = \frac{\hat{p}_1}{\hat{p}_2}$

Substituting $x_2^B = \frac{\hat{p}_1}{\hat{p}_2} x_1^B$ into budget constraint for Bob:

$\hat{p}_1 x_1^B + \hat{p}_2 \frac{\hat{p}_1}{\hat{p}_2} x_1^B = 3\hat{p}_1 + \hat{p}_2$

$\implies \hat{p}_1 x_1^B + \hat{p}_1 x_1^B = 3\hat{p}_1 + \hat{p}_2$

$\implies 2\hat{p}_1 x_1^B = 3\hat{p}_1 + \hat{p}_2$

$\implies x_1^B = \frac{3\hat{p}_1 + \hat{p}_2}{2\hat{p}_1} = \boxed{\frac{3}{2} + \frac{\hat{p}_2}{2\hat{p}_1}}$

$x_2^B = \frac{\hat{p}_1}{\hat{p}_2}\left(\frac{3}{2} + \frac{\hat{p}_2}{2\hat{p}_1}\right) = \boxed{\frac{3\hat{p}_1}{2\hat{p}_2} + \frac{1}{2}}$

(c)

The market for consumption in state 1:

$x_1^A + x_1^B = e_1^A + e_1^B = 1 + 3 = 4$

$\iff \frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1} + \frac{3}{2} + \frac{\hat{p}_2}{2\hat{p}_1} = 4$

$$\iff 2 + \frac{2\hat{p}_2}{2\hat{p}_1} + \frac{\hat{p}_2}{2\hat{p}_1} = 4$$

$$\iff \frac{3\hat{p}_2}{2\hat{p}_1} = 2$$

$$\iff \frac{\hat{p}_2}{\hat{p}_1} = 2 \times \frac{2}{3}$$

$$\iff \boxed{\frac{\hat{p}_1}{\hat{p}_2} = \frac{3}{4}}$$

$$x_1^A = \frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1} = \frac{1}{2} + \frac{4}{3} = \frac{3}{6} + \frac{8}{6} = \frac{11}{6}$$

$$x_1^B = \frac{3}{2} + \frac{\hat{p}_2}{2\hat{p}_1} = \frac{3}{2} + \frac{1}{2} \times \frac{4}{3} = \frac{3}{2} + \frac{2}{3} = \frac{9}{6} + \frac{4}{6} = \frac{13}{6}$$

$$x_2^A = \frac{\hat{p}_1}{\hat{p}_2} x_1^A = \frac{3}{4} \times \frac{11}{6} = \frac{1}{4} \times \frac{11}{2} = \frac{11}{8}$$

$$x_2^B = \frac{\hat{p}_1}{\hat{p}_2} x_1^B = \frac{3}{4} \times \frac{13}{6} = \frac{1}{4} \times \frac{13}{2} = \frac{13}{8}$$

Alice and Bob bear some risk:

$x_1^A \neq x_2^A$, $x_1^B \neq x_2^B$

They hedge some risks, but not completely, to maximize their expected utility.

(d)

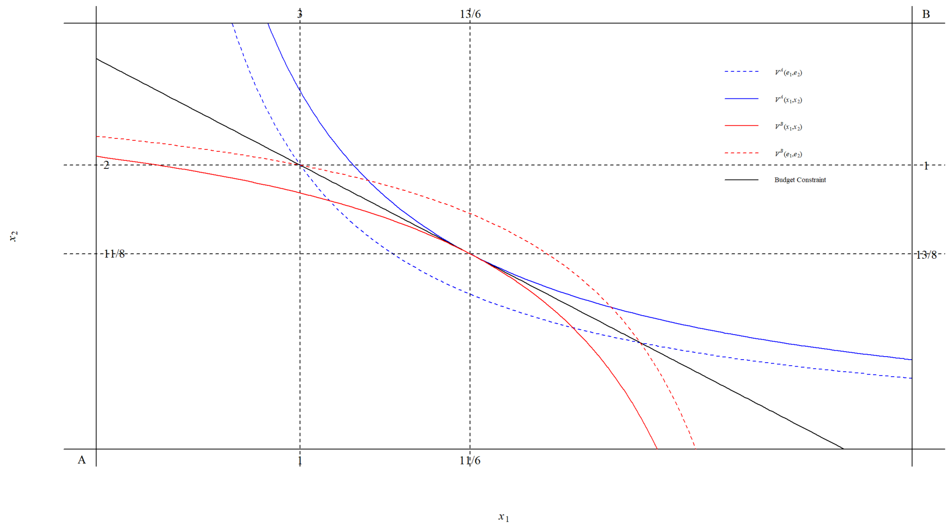

Figure 5: Edgeworth box diagram for Problem 3 equilibrium

It's impossible to increase one's utility without hurting the other one. Thus it is Pareto optimal.

4

(a)

$$\max_{(x_1^A, x_2^A)} \frac{1}{3} \ln x_1^A + \frac{2}{3} \ln x_2^A$$

subject to

$$\hat{p}_1 x_1^A + \hat{p}_2 x_2^A = 3\hat{p}_1 + 2\hat{p}_2.$$

For Alice:

$MU_1^A = \dfrac{1}{3} \cdot \dfrac{1}{x_1^A}$

$MU_2^A = \dfrac{2}{3} \cdot \dfrac{1}{x_2^A}$

$\text{MRS}^A = \dfrac{MU_1^A}{MU_2^A} = MU_1^A \cdot \dfrac{1}{MU_2^A} = \dfrac{1}{3} \cdot \dfrac{1}{x_1^A} \cdot \dfrac{3}{2} x_2^A = \dfrac{x_2^A}{2x_1^A}$

On the other hand,

$\text{MRS}^A = \dfrac{\hat{p}_1}{\hat{p}_2}$

$\therefore \dfrac{x_2^A}{2x_1^A} = \dfrac{\hat{p}_1}{\hat{p}_2}$

Substituting $x_2^A = \dfrac{2\hat{p}_1}{\hat{p}_2} x_1^A$ into budget constraint for Alice:

$\hat{p}_1 x_1^A + \hat{p}_2 \dfrac{2\hat{p}_1}{\hat{p}_2} x_1^A = 3\hat{p}_1 + 2\hat{p}_2$

$\implies \hat{p}_1 x_1^A + 2\hat{p}_1 x_1^A = 3\hat{p}_1 + 2\hat{p}_2$

$\implies 3\hat{p}_1 x_1^A = 3\hat{p}_1 + 2\hat{p}_2$

$\implies x_1^A = \dfrac{3\hat{p}_1 + 2\hat{p}_2}{3\hat{p}_1} = \boxed{1 + \dfrac{2}{3} \cdot \dfrac{\hat{p}_2}{\hat{p}_1}}$

$x_2^A = \dfrac{2\hat{p}_1}{\hat{p}_2} x_1^A = \dfrac{2\hat{p}_1}{\hat{p}_2}\left(1 + \dfrac{2}{3} \cdot \dfrac{\hat{p}_2}{\hat{p}_1}\right) = \boxed{\dfrac{2\hat{p}_1}{\hat{p}_2} + \dfrac{4}{3}}$

(b)

$$\boxed{\begin{aligned} &\max_{(x_1^B, x_2^B)} \frac{1}{2}\ln x_1^B + \frac{1}{2}\ln x_2^B \\ \text{subject to} \\ &\hat{p}_1 x_1^B + \hat{p}_2 x_2^B = \hat{p}_1 + 2\hat{p}_2. \end{aligned}}$$

For Bob:

$MU_1^B = \dfrac{1}{2} \cdot \dfrac{1}{x_1^B}$

$MU_2^B = \dfrac{1}{2} \cdot \dfrac{1}{x_2^B}$

$\text{MRS}^B = \dfrac{MU_1^B}{MU_2^B} = MU_1^B \cdot \dfrac{1}{MU_2^B} = \dfrac{1}{2} \cdot \dfrac{1}{x_1^B} \cdot 2x_2^B = \dfrac{x_2^B}{x_1^B}$

On the other hand,

$\text{MRS}^B = \dfrac{\hat{p}_1}{\hat{p}_2}$

$\therefore \dfrac{x_2^B}{x_1^B} = \dfrac{\hat{p}_1}{\hat{p}_2}$

Substituting $x_2^B = \dfrac{\hat{p}_1}{\hat{p}_2} x_1^B$ into budget constraint for Bob:

$$\hat{p}_1 x_1^B + \hat{p}_2 \frac{\hat{p}_1}{\hat{p}_2} x_1^B = \hat{p}_1 + 2\hat{p}_2$$

$$\implies \hat{p}_1 x_1^B + \hat{p}_1 x_1^B = \hat{p}_1 + 2\hat{p}_2$$

$$\implies 2\hat{p}_1 x_1^B = \hat{p}_1 + 2\hat{p}_2$$

$$\implies x_1^B = \frac{\hat{p}_1 + 2\hat{p}_2}{2\hat{p}_1} = \boxed{\frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1}}$$

$$x_2^B = \frac{\hat{p}_1}{\hat{p}_2} x_1^B = \frac{\hat{p}_1}{\hat{p}_2}\left(\frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1}\right) = \boxed{\frac{\hat{p}_1}{2\hat{p}_2} + 1}$$

(c)

The market for consumption in state 1:

$$x_1^A + x_1^B = e_1^A + e_1^B = 3 + 1 = 4$$

$$\iff 1 + \frac{2}{3} \cdot \frac{\hat{p}_2}{\hat{p}_1} + \frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1} = 4$$

$$\iff \left(\frac{2}{3} + 1\right) \cdot \frac{\hat{p}_2}{\hat{p}_1} + \frac{1}{2} = 3$$

$$\iff \frac{5}{3} \cdot \frac{\hat{p}_2}{\hat{p}_1} = \frac{5}{2}$$

$$\iff \frac{\hat{p}_2}{\hat{p}_1} = \frac{5}{2} \times \frac{3}{5}$$

$$\iff \boxed{\frac{\hat{p}_1}{\hat{p}_2} = \frac{2}{3}}$$

$$x_1^A = 1 + \frac{2}{3} \cdot \frac{\hat{p}_2}{\hat{p}_1} = 1 + \frac{2}{3} \cdot \frac{3}{2} = 2$$

$$x_1^B = \frac{1}{2} + \frac{\hat{p}_2}{\hat{p}_1} = \frac{1}{2} + \frac{3}{2} = 2$$

$$x_2^A = \frac{2\hat{p}_1}{\hat{p}_2} x_1^A = 2 \times \frac{2}{3} \times 2 = \frac{8}{3}$$

$$x_2^B = \frac{\hat{p}_1}{\hat{p}_2} x_1^B = \frac{2}{3} \times 2 = \frac{4}{3}$$

Alice and Bob bear some risk:

$x_1^A \neq x_2^A$, $x_1^B \neq x_2^B$

They hedge some risks, but not completely, to maximize their expected utility.

(d)

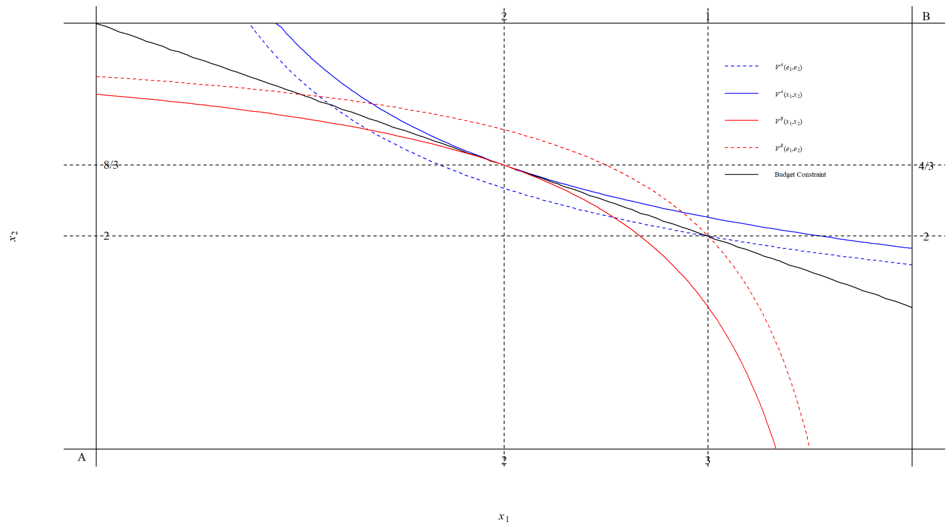

Figure 6: Edgeworth box diagram for Problem 4 equilibrium

It's impossible to increase one's utility without hurting the other one. Thus it is Pareto optimal.