Yu Xia's Problem Set 4

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1

$r_{15}^m = \frac{r_{15}^a}{12} = 0.003$

We have:

$$M_{15} = \frac{P \cdot r_{15}^m}{1 - \frac{1}{\left(1 + r_{15}^m\right)^{180}}}$$

If $P = \$160\text{K}$,

$M_{15} = \boxed{1,151.69}$

$TP_0 = 180 M_{15} = \boxed{207,303}$

If $P = \$150\text{K}$,

$M_{15} = \boxed{1,079.71}$

$TP_1 = 180 M_{15} = \boxed{194,346}$

In 5.1, $\rho^a = 0.02$ is given.

$\rho^m = \frac{\rho^a}{12} = \frac{0.02}{12} = \frac{0.01}{6} = \frac{1}{600} \approx 0.017$

The present value of the total annual rent is

$$C\left(c\right) = c\left(1 + \frac{1}{1+\rho^m} + \cdots + \frac{1}{\left(1+\rho^m\right)^{11}}\right) = c\frac{1+\rho^m}{\rho^m}\left(1 - \frac{1}{\left(1+\rho^m\right)^{12}}\right)$$

If $C = \$1,500$, $\boxed{C\left(1500\right) = 17,836}$

$$TP_0 - C(c) - \frac{TP_1}{1+\rho^a} = 207,303 - 17,836 - \frac{194,346}{1.02} \approx 189,467 - 190,535.29 \approx \boxed{-1,068.29}$$

If $C = \$2,000$, $\boxed{C(2000) = 23,782}$

$$TP_0 - C(c) - \frac{TP_1}{1+\rho^a} = 207,303 - 23,782 - \frac{194,346}{1.02} \approx 183,521 - 190,535.29 \approx \boxed{-7,014.29}$$

$$\begin{aligned}
c^* &= \frac{\rho^m}{1+\rho^m} \frac{TP_0 - \dfrac{TP_1}{1+\rho^a}}{1 - \dfrac{1}{(1+\rho_m)^{12}}} \\
&= \frac{\dfrac{1}{600}}{1+\dfrac{1}{600}} \frac{207,303 - \dfrac{194,346}{1+0.02}}{1 - \dfrac{1}{\left(1+\dfrac{1}{600}\right)^{12}}} \\
&= \frac{\dfrac{1}{600}}{\dfrac{601}{600}} \frac{207,303 - \dfrac{194,346}{1.02}}{1 - \dfrac{1}{\left(\dfrac{601}{600}\right)^{12}}} \\
&= \frac{600}{601} \frac{207,303 - \dfrac{194,346}{1.02}}{1 - \left(\dfrac{600}{601}\right)^{12}} \\
&\approx \boxed{1410.1423}
\end{aligned}$$

2

If $r_u^a = 0.04$

$$r_u^m = \frac{r_u^a}{12} = \frac{0.04}{12} = \frac{0.01}{3} = \frac{1}{300} \approx 0.00333$$

When $P = \$150\text{K}$,

$$M_u = \frac{P \cdot r_u^m}{1 - \dfrac{1}{(1+r_u^m)^{180}}} = \frac{150,000 \cdot \dfrac{1}{300}}{1 - \dfrac{1}{\left(\dfrac{301}{300}\right)^{180}}} = \frac{1500 \cdot \dfrac{1}{3}}{1 - \left(\dfrac{300}{301}\right)^{180}} \approx \frac{500}{1 - 0.5493595} \approx \frac{500}{0.45} \approx \boxed{1,109.53}$$

$$TP_u = 180 M_u \approx 180 \times 1,109.53 \approx 199,715.7$$

$$\mathbb{E}[TP] = \pi TP_u + (1-\pi) TP_1 \approx 0.5 \times 199,715.7 + 0.5 \times 194,346 \approx \boxed{197,030.9}$$

3

If the prices go up by 10 percent, $P_u = 170\text{K}$.

$$M_u = \frac{P_u \cdot r_{15}^m}{1 - \dfrac{1}{(1+r_{15}^m)^{180}}} = \frac{170,000 \cdot 0.003}{1 - \dfrac{1}{\left(1+\dfrac{3}{1000}\right)^{180}}} = \frac{170 \cdot 3}{1 - \dfrac{1}{\left(\dfrac{1003}{1000}\right)^{180}}} = \frac{510}{1 - \left(\dfrac{1000}{1003}\right)^{180}} \approx \boxed{1,223.6658}$$

$TP_u = 180 M_u \approx \boxed{220,259.84}$

If the prices go down by 10 percent, $P_d = 130\text{K}$.

$$M_d = \frac{P_d \cdot r_{15}^m}{1 - \dfrac{1}{(1+r_{15}^m)^{180}}} = \frac{130,000 \cdot 0.003}{1 - \dfrac{1}{\left(1+\dfrac{3}{1000}\right)^{180}}} = \frac{130 \cdot 3}{1 - \dfrac{1}{\left(\dfrac{1003}{1000}\right)^{180}}} = \frac{390}{1 - \left(\dfrac{1000}{1003}\right)^{180}} \approx \boxed{935.7442}$$

$TP_d = 180 M_d \approx \boxed{168,433.99}$

$\mathbb{E}[TP] = 0.45 TP_u + 0.55 TP_d \approx 0.45 \times 220,259.8 + 0.55 \times 168,434 \approx \boxed{191755.62}$

4

By **1**, we have the critical value $c^* \approx 1410.1423$, comparing buying now and buying a year afterward.

Assume you are in the end of year 1. You are considering whether to buy in the end of $t = 1$ or the end of $t = 2$.

$P^{t=2} = \$200,000 - 60,000 = \$140\text{K}$.

On the other hand,

$$M_{15}^{t=2} = \frac{P^{t=2} \cdot r_{15}^m}{1 - \dfrac{1}{(1+r_{15}^m)^{180}}} = \frac{140,000 \cdot 0.003}{1 - \dfrac{1}{(1+0.003)^{180}}} = \frac{140 \cdot 3}{1 - \dfrac{1}{(1.003)^{180}}} = \frac{420}{1 - \dfrac{1}{(1.003)^{180}}} \approx 1007.7248$$

$TP_2 = 180 M_{15}^{t=2} \approx 181,390.46$

The critical value, comparing $t = 1$ and $t = 2$, is:

$$c^{*\prime} = \frac{\rho^m}{1+\rho^m} \frac{TP_1 - \dfrac{TP_2}{1+\rho^a}}{1 - \dfrac{1}{(1+\rho_m)^{12}}} = \frac{600}{601} \frac{194,346 - \dfrac{181,390.5}{1.02}}{1 - \left(\dfrac{600}{601}\right)^{12}} \approx \boxed{1388.6526}$$

If the rent at the end of $t = 1$ is higher than \$1388.6526, buy at the end of $t = 1$ (if you have already been in $t = 1$). If the rent at the end of $t = 1$ is less than \$1388.6526, wait another year and buy at the end of $t = 2$ (if you have already been in $t = 1$).

(i)

If $TP_0 - C(c_0) - 0.9c_0 \left(\frac{1}{(1+\rho^m)^{12}} + \frac{1}{(1+\rho^m)^{13}} + \cdots + \frac{1}{(1+\rho^m)^{23}} \right) - \frac{TP_2}{(1+\rho^a)^2} < 0$,

better to buy at $t = 0$ than $t = 2$ regardless of future change of rent.

That is,

$$\begin{aligned} &207,303 - c_0 \frac{1+\rho^m}{\rho^m} \left(1 - \frac{1}{(1+\rho^m)^{12}} \right) \\ &\qquad - 0.9c_0 \frac{1}{(1+\rho^m)^{12}} \left(1 + \frac{1}{1+\rho^m} + \cdots + \frac{1}{(1+\rho^m)^{11}} \right) - \frac{181,390.46}{(1+\rho^a)^2} < 0 \end{aligned}$$

$$\begin{aligned} \Longrightarrow\ &207,303 - c_0 \frac{1+\rho^m}{\rho^m} \left(1 - \frac{1}{(1+\rho^m)^{12}} \right) \\ &\qquad - \frac{0.9c_0}{(1+\rho^m)^{12}} \frac{1+\rho^m}{\rho^m} \left(1 - \frac{1}{(1+\rho^m)^{12}} \right) - \frac{181,390.46}{(1+\rho^a)^2} < 0 \end{aligned}$$

$$\Longrightarrow 207,303 - c_0 \frac{1+\rho^m}{\rho^m} \left(1 - \frac{1}{(1+\rho^m)^{12}} \right) \left(1 + \frac{0.9}{(1+\rho^m)^{12}} \right) - \frac{181,390.46}{(1+\rho^a)^2} < 0$$

$$\Longrightarrow 207,303 - c_0 \frac{1+\frac{1}{600}}{\frac{1}{600}} \left(1 - \frac{1}{\left(1+\frac{1}{600}\right)^{12}} \right) \left(1 + \frac{0.9}{\left(1+\frac{1}{600}\right)^{12}} \right) - \frac{181,390.46}{(1+0.02)^2} < 0$$

$$\Longrightarrow 207,303 - c_0 \frac{\frac{601}{600}}{\frac{1}{600}} \left(1 - \frac{1}{\left(\frac{601}{600}\right)^{12}} \right) \left(1 + \frac{0.9}{\left(\frac{601}{600}\right)^{12}} \right) - \frac{181,390.46}{1.0404} < 0$$

$$\Longrightarrow 207,303 - 601c_0 \left(1 - \left(\frac{600}{601}\right)^{12} \right) \left(1 + 0.9 \left(\frac{600}{601}\right)^{12} \right) - \frac{181,390.46}{1.0404} < 0$$

$$\Longrightarrow 207,303 - \frac{181,390.46}{1.0404} < 601c_0 \left(1 - \left(\frac{600}{601}\right)^{12} \right) \left(1 + 0.9 \left(\frac{600}{601}\right)^{12} \right)$$

$$\Longrightarrow 601c_0 \left(1 - \left(\frac{600}{601}\right)^{12} \right) \left(1 + 0.9 \left(\frac{600}{601}\right)^{12} \right) > 207,303 - \frac{181,390.46}{1.0404}$$

$$\implies c_0 > \frac{207,303 - \dfrac{181,390.46}{1.0404}}{601\left(1-\left(\dfrac{600}{601}\right)^{12}\right)\left(1+0.9\left(\dfrac{600}{601}\right)^{12}\right)} \approx \boxed{1472.52} > 1410.1423$$

Buy at $t = 0$ if $c_0 > 1472.52$ regardless of future change.

If $TP_0 - C(c_0) - 1.1c_0\left(\frac{1}{(1+\rho^m)^{12}} + \frac{1}{(1+\rho^m)^{13}} + \cdots + \frac{1}{(1+\rho^m)^{23}}\right) - \frac{TP_2}{(1+\rho^a)^2} > 0$,

better to buy at $t = 2$ than $t = 0$ regardless of future change of rent.

That is,

$$\implies c_0 < \frac{207,303 - \dfrac{181,390.46}{1.0404}}{601\left(1-\left(\dfrac{600}{601}\right)^{12}\right)\left(1+1.1\left(\dfrac{600}{601}\right)^{12}\right)} \approx \boxed{1333.62} < 1410.1423$$

Buying at $t = 0$ is the wrost.

Now let's consider:

$1.1 \times 1333.62 \approx 1466.9779 > 1388.6526$

$0.9 \times 1333.62 < 1333.62 < 1388.6526$

Still cannot tell $t = 1$ and $t = 2$.

If $c_1 = 1.1c_0 < 1388.6526 \iff c_0 < \boxed{1262.4115} < 1333.62 < 1410.1423$, $t = 2$ is better than $t = 1$, $t = 1$ is better than $t = 0$,even if future increase in rate. Thus buying at $t = 2$ is the best regardless of future rent change.

If $c_1 = 0.9c_0 > 1388.6526 \iff c_0 > \boxed{1542.9474} > 1410.1423 > 1333.62$, buying at $t = 2$ is wrose than buying at $t = 1$ regardless of future change. But at the same time, $t = 0$ becomes the best choice.

If $t = 1$ is the best time,

$c_0 < 1410.1423$ and $c_1 > 1388.6526$.

$\because c_1 \geqslant 0.9c_0$

$\therefore 0.9c_0 > 1388.6526$ regardless of rent change. Follows the conclusion above.

It follows that:

Buy at $t = 0$ if $c_0 > 1472.52$ regardless of future change.

Buy at $t = 2$ if $c_0 < 1262.4115$ regardless of future change.

But it's still hard to draw a conslusion if $1262.4115 < c_0 < 1472.52$, unless you know exactly how rate will change.

(ii)

Your expectation of the rent the next year is

$\mathbb{E}[c_1] = 0.5 \cdot 1.1c_0 + 0.5 \cdot 0.9c_0 = c_0$

If $c_0 > 1410.1423 > 1388.6526$, buy the house at $t = 0$.

If $c_0 < 1388.6526 < 1410.1423$, buy the house at $t = 2$.

If $1388.6526 < c_0 < 1410.1423$,

Buying at $t = 1$ is better than buying at $t = 0$, at the same time, $t = 1$ is better than $t = 2$.

So buying at $t = 1$ is the optimal choice if $1388.6526 < c_0 < 1410.1423$.

<u>To sum up,</u>

If $c_0 > 1410.1423$, buy the house at $t = 0$.

If $c_0 = 1410.1423$, indifferent between $t = 0$ and $t = 1$.

If $1388.6526 < c_0 < 1410.1423$, buying at $t = 1$ is the optimal choice.

If $c_0 = 1388.6526$, indifferent between $t = 1$ and $t = 2$.

If $c_0 < 1388.6526$, buy the house at $t = 2$.

(iii)

If $1262.4115 < c_0 < 1472.52$:

Case 1: If rent goes up at the end of $t = 1$, $c_1 = 1.1c_0 > 1.1 \times 1262.4115 \approx 1388.6526$, buying a house at the end of $t = 1$ is better than $t = 2$.

If $c_0 < 1410.1423$, buying at $t = 1$ is the best choice.

If $c_0 > 1410.1423$, buying at $t = 0$ is the best choice.

Case 2: If rent goes down at the end of $t = 1$, $c_1 = 0.9c_0 < 0.9 \times 1472.52 \approx 1325.27 < 1388.6526$, buying a house at the end of $t = 2$ is better than buying a house at the end of $t = 1$.

Since at $t = 0$, you don't know the rent at $t = 1$,

if $TP_0 - C(c_0) - \mathbb{E}[c_1]\left(\frac{1}{(1+\rho^m)^{12}} + \frac{1}{(1+\rho^m)^{13}} + \cdots + \frac{1}{(1+\rho^m)^{23}}\right) - \frac{TP_2}{(1+\rho^a)^2} < 0$,

buy at $t = 0$ according to your expectation, $\mathbb{E}[c_1] = c_0$.

That implies,

$$c_0 > \frac{207,303 - \dfrac{181,390.46}{1.0404}}{601\left(1 - \left(\dfrac{600}{601}\right)^{12}\right)\left(1 + \left(\dfrac{600}{601}\right)^{12}\right)} \approx \boxed{1399.6309}$$

If $c_0 > 1399.6309$, buying at $t = 0$ is better than $t = 2$.

If $c_0 < 1399.6309$, buying at $t = 2$ is better than $t = 0$.

If $1410.1423 < c_0 < 1472.52$:

Buying at $t = 0$ is better than buying at $t = 1$, at the same time, $t = 0$ is better than $t = 2$.

If $1399.6309 < c_0 < 1410.1423$:

Buying at $t = 1$ is better than buying at $t = 0$, at the same time, $t = 0$ is better than $t = 2$.

At the same time, $c_1 = 0.9c_0 < 0.9 \times 1410.1423 \approx 1269.1281 < 1388.6526$. $t = 2$ is actually better than $t = 1$.

The case is that, you know $t = 1$ is better than buying at $t = 0$ at the beginning. You wait to $t = 1$. You find that rent goes down. $t = 2$ becomes the best choice. But you cannot know the rent change in advance. At $t = 0$, your expectation is that $t = 1$ is the best.

If $1262.4115 < c_0 < 1399.6309$:

Buying at $t = 2$ is better than buying at $t = 0$, at the same time,

$c_1 = 0.9c_0 < 0.9 \times 1399.6309 \approx 1259.6662 < 1388.6526$

$t = 2$ is better than $t = 1$,if rent goes down.

<u>In conclusion,</u>

If $c_0 > 1472.52$, buy at $t = 0$.

If $c_0 < 1262.4115$, buy at $t = 2$.

If $1262.4115 \leqslant c_0 \leqslant 1472.52$:

Case 1: rent increases.

If $c_0 = 1262.4115$, indifferent with $t = 1$ and $t = 2$.

If $1262.4115 < c_0 < 1410.1423$, buy at $t = 1$.

If $c_0 = 1410.1423$, indifferent with $t = 0$ and $t = 1$.

If $1410.1423 < c_0 \leqslant 1472.52$, buy at $t = 0$.

Case 2: rent decreases,

If $1410.1423 < c_0 \leqslant 1472.52$, best choice is to buy at $t = 0$.

If $1262.4115 \leqslant c_0 < 1410.1423$, best choice is to buy at $t = 2$.

(iv)

Follows from (iii), but plug in $0.9c_0$ in case 2.

It implies $c_0 > \dfrac{207,303 - \dfrac{181,390.46}{1.0404}}{601\left(1 - \left(\dfrac{600}{601}\right)^{12}\right)\left(1 + 0.9\left(\dfrac{600}{601}\right)^{12}\right)} \approx 1472.52$

If $c_0 > 1472.52$, buying at $t = 0$ is the best.

If $c_0 < 1472.52$, buying at $t = 2$ is better than $t = 0$ if price goes down.

Case 2 is under the condition $1262.4115 < c_0 < 1472.52$, where buying a house at the end of $t = 2$ is better than buying a house at the end of $t = 1$ if rent goes down.

So in Case 2, rent goes down, buy at $t = 2$ is optimal.

<u>To summarize:</u>

If $c_0 > 1472.52$, buy at $t = 0$.

If $c_0 < 1262.4115$, buy at $t = 2$.

If $1262.4115 \leqslant c_0 \leqslant 1472.52$:

Case 1, rent goes up,

If $c_0 = 1262.4115$, indifferent with $t = 1$ and $t = 2$

if $1262.4115 < c_0 < 1410.1423$, buy at $t = 1$,

if $c_0 = 1410.1423$, indifferent with $t = 0$ and $t = 1$,

if $1410.1423 < c_0 \leqslant 1472.52$, buy at $t = 0$,

Case 2, rent goes down, buy at $t = 2$.

(v)

If $1262.4115 < c_0 < 1472.52$:

When $1410.1423 < c_0 < 1472.52$, $t = 0$ is better than $t = 1$. You are comparing $t = 0$ and $t = 2$.

Plug in $\mathbb{E}[c_1]$ like (iii), we have $1399.6309 < 1410.1423 < c_0 < 1472.52$. $t = 0$ is better than $t = 2$.

When $1262.4115 < c_0 < 1410.1423$, $t = 1$ is better than $t = 0$. You wait for another year.

If price goes up, $c_1 = 1.1c_0 > 1.1 \times 1262.4115 \approx 1388.6526$, you buy at $t = 1$.

If price goes down, $c_1 = 0.9c_0 < 0.9 \times 1410.1423 \approx 1269.1281 < 1388.6526$, you buy at $t = 2$.

In short,

If $c_0 > 1410.1423$, buy at $t = 0$.

If $c_0 < 1262.4115$, buy at $t = 2$.

If $1262.4115 \leqslant c_0 \leqslant 1410.1423$:

Wait for one year.

When rent goes up, buy at $t = 1$.

When rent goes down, buy at $t = 2$.

(vi)

If $1262.4115 < c_0 < 1472.52$:

When $1410.1423 < c_0 < 1472.52$, $t = 0$ is better than $t = 1$. You are comparing $t = 0$ and $t = 2$.

If rent goes up, $c_0 > 1333.62$, buy at $t = 0$.

If rent goes down, $c_0 < 1472.52$, buy at $t = 2$.

When $1262.4115 < c_0 < 1410.1423$:

Wait for one year.

If rent goes up, $c_1 = 1.1c_0 > 1.1 \times 1262.4115 \approx 1388.6526$, you buy at $t = 1$.

If price goes down, $c_1 = 1.1c_0 < 0.9 \times 1410.1423 \approx 1269.1281 < 1388.6526$, you buy at $t = 2$.

<u>Thus,</u>

If $c_0 > 1472.52$, buy at $t = 0$.

If $c_0 < 1262.4115$, buy at $t = 2$.

If $1410.1423 < c_0 < 1472.52$:

If rent goes up, buy at $t = 0$.

If rent goes down, buy at $t = 2$.

If $1262.4115 \leqslant c_0 \leqslant 1410.1423$:

Wait for one year.

When rent goes up, buy at $t = 1$.

When rent goes down, buy at $t = 2$.

I think (v) is closest to professor's meaning.