1. (a)

By no-arbitrage pricing formula, we have:

$0 = \frac{1}{1+r}(\lambda(\Phi_{1u} - \Phi_0) + (1-\lambda)(\Phi_{1d} - \Phi_0))$

$+ \frac{\lambda}{(1+r)^2}(\lambda(0.94 - \Phi_{1u}) + (1-\lambda)(0.90 - \Phi_{1u}))$

$+ \frac{1-\lambda}{(1+r)^2}(\lambda(0.90 - \Phi_{1d}) + (1-\lambda)(0.86 - \Phi_{1d}))$ ①

$0 = \frac{1}{1+r}(\lambda(0.94 - \Phi_{1u}) + (1-\lambda)(0.90 - \Phi_{1u}))$ ②

$0 = \frac{1}{1+r}(\lambda(0.90 - \Phi_{1d}) + (1-\lambda)(0.86 - \Phi_{1d}))$ ③

From ② & ③, respectively,

$\Phi_{1u} = 0.94\lambda + 0.90(1-\lambda)$

$= 0.92$

$\Phi_{1d} = 0.90\lambda + .86(1-\lambda)$

$= 0.88$

Plug them into ①, we get:

$\lambda(0.92 - \Phi_0) + (1-\lambda)(0.88 - \Phi_0) = 0$

$\Rightarrow \Phi_0 = 0.92\lambda + 0.88(1-\lambda) = 0.90$

The notional amount is:

$1000 \cdot \Phi_0 = €900$

The amount of contract is:

$\frac{100,000}{1,000} = 100$

$100 \times (\Phi_0 - \Phi_{1u}) \times 1000$

$= 100 \times (0.90 - 0.92) \times 1000$

$= -100 \times 0.02 \times 1000$

$= -2000$

$100 \times (\Phi_0 - \Phi_{1d}) \times 1000$

$= 100 \times (0.90 - 0.88) \times 1000$

$= 2000$

$\cancel{100 \times (\Phi_0 - \Phi_{1d})}$

$100 \times (\Phi_{1u} - 0.94) \times 1000$

$= 100 \times (0.92 - 0.94) \times 1000$

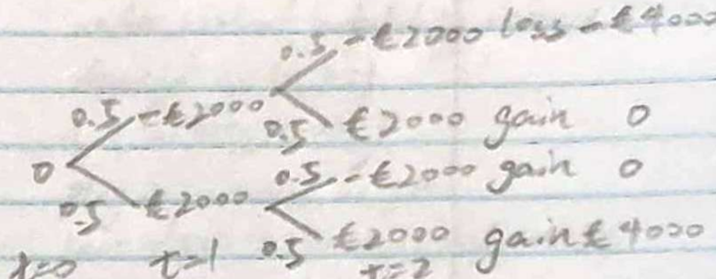
$= -2000$

$100 \times (\Phi_{1u} - 0.90) \times 1000 = 2000$

$100 \times (\Phi_{1d} - 0.90) \times 1000 = -2000$

$100 \times (\Phi_{1d} - 0.86) \times 1000 = 2000$

The evolution of the short position in 100 futures' cash stream:



$100,000 \times 0.94 = 94,000$

$100,000 \times 0.90 = 90,000$

$100,000 \times 0.86 = 86,000$

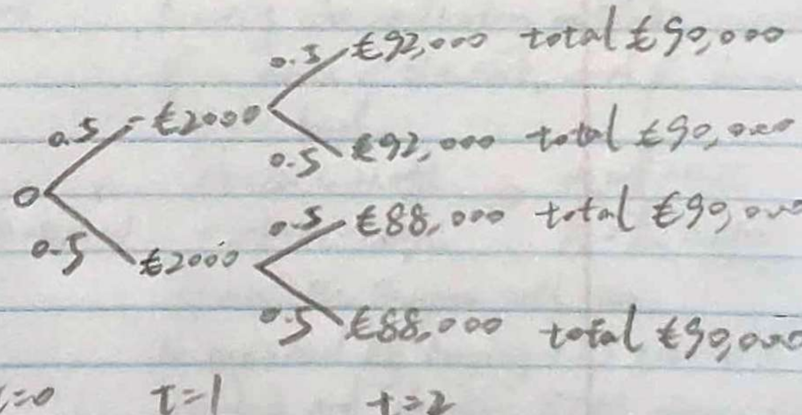
$94,000 - 2000 = 92,000$

$90,000 + 2000 = 92,000$

$90,000 - 2000 = 88,000$

$86,000 + 2000 = 88,000$

The producer's cash flow is:



(b) By the no-arbitrage pricing formula, we have:

$0=\frac{1}{1+r}(\lambda(0.92-X)+(1-\lambda)(0.88-X))$
$\quad+\frac{\lambda}{(1+r)^2}(\lambda(0.94-X)+(1-\lambda)(0.90-X))$
$\quad+\frac{1-\lambda}{(1+r)^2}(\lambda(0.90-X)+(1-\lambda)(0.86-X))$

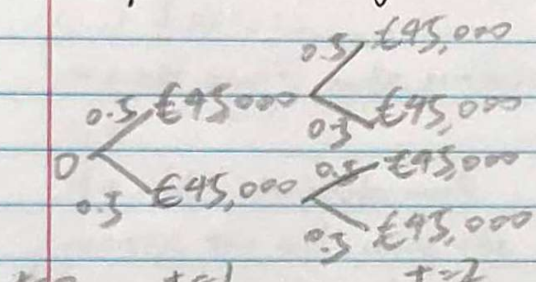
$\Rightarrow X(1+\frac{1}{1+r})=0.88+0.04\lambda+\frac{1}{1+r}(0.86+0.08\lambda)$

$\Rightarrow X=\frac{1.005}{2.005}\times(0.88+0.02+\frac{1}{1.005}(0.86+0.04))$

$\Rightarrow X=£0.90$

$5000\cdot X=£45{,}000$

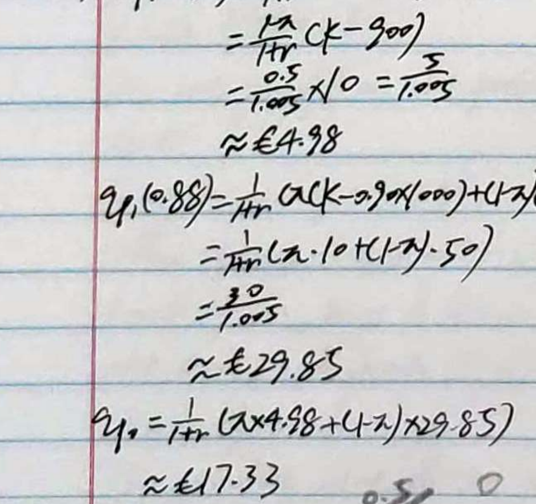
The producer's cash flow:



(c) $q_1(0.92)=\frac{1}{1+r}(\lambda\cdot 0+(1-\lambda)(K-0.90\times 1000))$
$\quad=\frac{1-\lambda}{1+r}(K-900)$
$\quad=\frac{0.5}{1.005}\times 10=\frac{5}{1.005}$
$\quad\approx £4.98$

$q_1(0.88)=\frac{1}{1+r}(\lambda(K-0.90\times1000)+(1-\lambda)(K-0.86\times1000))$
$\quad=\frac{1}{1+r}(\lambda\cdot 10+(1-\lambda)\cdot 50)$
$\quad=\frac{30}{1.005}$
$\quad\approx £29.85$

$q_0=\frac{1}{1+r}(\lambda\times 4.98+(1-\lambda)\times 29.85)$
$\quad\approx £17.33$



2.(a) Not correct.

The investor shouldn't use historic probabilites to estimate the price. There is no arbitrage only if pricing under neutral probability measure, denoted as $\{\lambda, 1-\lambda\}$.

$q_0=\frac{1}{1+r}(\lambda q_u+(1-\lambda)q_d)$

plug in numbers, we get:

$100=\frac{1}{1.04}(120\lambda+80(1-\lambda))=\frac{1}{1.04}(40\lambda+80)$

$\Rightarrow 104=40\lambda+80\Rightarrow 24=40\lambda$

$\Rightarrow\lambda=0.6$

$1-\lambda=0.4$

(b) If $q_1=\$120>K=\$90$, exercise the option, payoff is $\$120-\$90=\$30$.

If $q_1=\$80<K=\$90$, the investor won't exercise, the payoff is 0.

Denoting the replicating portfolio consisting of α_c shares of the stock, and β_c shares of the bond.

$30=120\alpha_c+1.04\beta_c$
$0=80\alpha_c+1.04\beta_c$
$\Rightarrow \alpha_c=\frac{3}{4}=0.75$, $\beta_c=\frac{-60}{1.04}\approx -57.69$

Hence, the European call can be replicated by buying 0.75 shares of the stock and shorting 57.69 shares of bond.

The option price at t=0 is:

$q_c=\frac{3}{4}\times 100-57.69=\$17.31$

(c) Suppose that $q_c<\$17.31$.

Consider a portfolio consisting of 1 shares of option, shorting $\alpha_c=\frac{3}{4}=0.75$ shares of stock, and buys 57.69 shares of bond.

The market value of the portfolio is:
$q_c - 100\alpha_c - \beta_c < 0$.
Suppose $q_h = \$120$, then investor exercise the call, gets \$30, and gets back $1.04 \times 57.69 = 60$ on the bond holding.
$\therefore$ The investor ends up \$90 to get.
The investor uses the payment to buy $\alpha_c = \frac{3}{4} = 0.75$ shares of stock at \$120 and pays back the debt.
Since $\frac{3}{4} \times 120 = 90$,
the net payoff on the portfolio is 0.
If $q_1 = \$80$, the investor won't exercise the call, using the payment of $1.04 \times 57.69 = 60$ to buy $\frac{3}{4}$ shares of stock at \$80, paying back the debt.
Since $\frac{3}{4} \times 80 = 60$,
the net payoff on the portfolio is 0.
Suppose now $q_c > \$17.31$.
Then at $t=0$, the investor shorts 1 call option, buys $\frac{3}{4}$ shares of the stock, and shorts 57.69 shares of bond.
The market value of the portfolio is: $-q_c + 100\alpha_c + \beta_c < 0$.
If $q_1 = \$120$, the value of the call is \$30, the investor sells $\frac{3}{4}$ shares of stock, get $\frac{3}{4} \times \$120 = \$90$.
The investor returns \$30 for the call option, and pays back the debt on bond: $1.04 \times 57.69 = 60$.
Since $\$30 + \$60 = \$90$, the net payoff is 0.
If $q_1 = \$80$, the value of the call is 0, the investor sells $\frac{3}{4}$ shares of the stock, gets $\frac{3}{4} \times \$80 = \$60$
The debt on the bond this investor needs to payback is $\$1.04 \times \$57.69 = \$60$.
The net payoff is 0.
Hence, there is arbitrage opportunity.

d)
The value of immediate exercise is
$$V_0(K) = \begin{cases} 0 & \text{if } K \le 100 \\ K - 100 & \text{if } K > 100. \end{cases}$$
If not exercised at $t=0$, it's equivalent to the European call with strike K and maturity data $t=1$.
The European put option will be exercised iff $q_1 = \$80$.
The value of this European put option:
$$q_{p_0}(K) = \frac{1-\pi}{1+r}(K - q_1) = \frac{0.4}{1.04}(K - 80)$$
$$V_0(K) \ge q_{p_0}(K)$$
$$K - q_0 \ge \frac{1-\pi}{1+r}(K - q_1)$$
$$K - 100 \ge \frac{0.4}{1.04}(K - 80)$$
$$K \ge 112.5$$
The American put option will be exercised prematurely iff
$120 > K \ge 112.5$.