Online Primal-Dual Algorithms for Maximizing Ad-Auctions Revenue

Shi Weijie

September 24, 2013

Model

Algorithm

Analyze

Online Allocation

- n buyers (users), m items provided by the provider

Online Allocation

- n buyers (users), m items provided by the provider
- items arrive one-by-one

Online Allocation

- n buyers (users), m items provided by the provider
- items arrive one-by-one
- buyer i bid for item j at price $b(i,j)$

Online Allocation

- n buyers (users), m items provided by the provider
- items arrive one-by-one
- buyer i bid for item j at price $b(i, j)$
- provider decides which buyer wins item j

Online Allocation

- n buyers (users), m items provided by the provider
- items arrive one-by-one
- buyer i bid for item j at price $b(i,j)$
- provider decides which buyer wins item j
- provider's goal: Maximize revenue

Example 1

- 2 users, 5 items

Example 1

- 2 users, 5 items
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$

Example 1

- 2 users, 5 items
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$

Example 1

- 2 users, 5 items
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$
- Round 3: $b(1,3) = \$4, \quad b(2,3) = \$1$

Example 1

- 2 users, 5 items
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$
- Round 3: $b(1,3) = \$4, \quad b(2,3) = \$1$
- Round 4: $b(1,4) = \$4, \quad b(2,4) = \$1$

Example 1

- 2 users, 5 items
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$
- Round 3: $b(1,3) = \$4, \quad b(2,3) = \$1$
- Round 4: $b(1,4) = \$4, \quad b(2,4) = \$1$
- Round 5: $b(1,5) = \$3, \quad b(2,5) = \$1$

Example 1

- 2 users, 5 items
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$
- Round 3: $b(1,3) = \$4, \quad b(2,3) = \$1$
- Round 4: $b(1,4) = \$4, \quad b(2,4) = \$1$
- Round 5: $b(1,5) = \$3, \quad b(2,5) = \$1$
- Result: Revenue $= \$20$

User Budget

- User i has a total budget $B(i)$

User Budget

- User i has a total budget $B(i)$
- provider cannot charge more than his budget

Example 2

- 2 users, 5 items, $B(1) = \$10, B(2) = \$10$

Example 2

- 2 users, 5 items, $B(1) = \$10, B(2) = \$10$
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$

Example 2

- 2 users, 5 items, $B(1) = \$10, B(2) = \$10$
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$

Example 2

- 2 users, 5 items, $B(1) = \$10, B(2) = \$10$
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$
- Round 3: $b(1,3) = \$4, \quad b(2,3) = \$1$

Example 2

- 2 users, 5 items, $B(1) = \$10, B(2) = \$10$
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$
- Round 3: $b(1,3) = \$4, \quad b(2,3) = \$1$
- Round 4: $b(1,4) = \$4, \quad b(2,4) = \$1$

Example 2

- 2 users, 5 items, $B(1) = \$10, B(2) = \$10$
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$
- Round 3: $b(1,3) = \$4, \quad b(2,3) = \$1$
- Round 4: $b(1,4) = \$4, \quad b(2,4) = \$1$
- Round 5: $b(1,5) = \$3, \quad b(2,5) = \$1$

Example 2

- 2 users, 5 items, $B(1) = \$10, B(2) = \$10$
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$
- Round 3: $b(1,3) = \$4, \quad b(2,3) = \$1$
- Round 4: $b(1,4) = \$4, \quad b(2,4) = \$1$
- Round 5: $b(1,5) = \$3, \quad b(2,5) = \$1$
- "Greedy" strategy: \$12

Example 2

- 2 users, 5 items, $B(1) = \$10, B(2) = \$10$
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$
- Round 3: $b(1,3) = \$4, \quad b(2,3) = \$1$
- Round 4: $b(1,4) = \$4, \quad b(2,4) = \$1$
- Round 5: $b(1,5) = \$3, \quad b(2,5) = \$1$
- "Greedy" strategy: \$12
- Best offline strategy: \$16 (User 1 wins item 3 & 4 & 5)

Example 2

- 2 users, 5 items, $B(1) = \$10, B(2) = \$10$
- Round 1: $b(1,1) = \$5, \quad b(2,1) = \$3$
- Round 2: $b(1,2) = \$4, \quad b(2,2) = \$3$
- Round 3: $b(1,3) = \$4, \quad b(2,3) = \$1$
- Round 4: $b(1,4) = \$4, \quad b(2,4) = \$1$
- Round 5: $b(1,5) = \$3, \quad b(2,5) = \$1$
- "Greedy" strategy: \$12
- Best offline strategy: \$16 (User 1 wins item 3 & 4 & 5)
- "Clever" strategy: \$14 (User 1 wins item 1 & 3 & 5)

Difficulties

- No online algorithm can guarantee optimal result

Difficulties

- No online algorithm can guarantee optimal result
- Objective: a good competitive ratio

Difficulties

- No online algorithm can guarantee optimal result
- Objective: a good competitive ratio
- NP-hard, even with future information (offline problem)

Intuition 1

- We hope each user has a large budget $B(i)$, compared with his bid $b(i,j)$

Intuition 1

- We hope each user has a large budget $B(i)$, compared with his bid $b(i,j)$
- Then each decision only has a small impact on the final revenue

Intuition 1

- We hope each user has a large budget $B(i)$, compared with his bid $b(i,j)$
- Then each decision only has a small impact on the final revenue
- Define: $R = \max_{i,j}\{b(i,j)/B(i)\}$

Intuition 2

- The user with more remaining budget should have higher priority

Intuition 2

- The user with more remaining budget should have higher priority
- For the user with less remaining budget, maybe we should wait

Intuition 2

- The user with more remaining budget should have higher priority
- For the user with less remaining budget, maybe we should wait
- Reduce the original bid by some amount

Intuition 2

- The user with more remaining budget should have higher priority
- For the user with less remaining budget, maybe we should wait
- Reduce the original bid by some amount
- This “amount” depends on the remaining budget

Intuition 2

- The user with more remaining budget should have higher priority
- For the user with less remaining budget, maybe we should wait
- Reduce the original bid by some amount
- This "amount" depends on the remaining budget
- Define "Budget Index" $x(i)$

Intuition 2

- The user with more remaining budget should have higher priority
- For the user with less remaining budget, maybe we should wait
- Reduce the original bid by some amount
- This “amount” depends on the remaining budget
- Define “Budget Index” $x(i)$
- At the beginning $x(i) = 0$

Intuition 2

- The user with more remaining budget should have higher priority
- For the user with less remaining budget, maybe we should wait
- Reduce the original bid by some amount
- This "amount" depends on the remaining budget
- Define "Budget Index" $x(i)$
- At the beginning $x(i) = 0$
- $x(i)$ increases as remaining budget decreases

Intuition 2

- The user with more remaining budget should have higher priority
- For the user with less remaining budget, maybe we should wait
- Reduce the original bid by some amount
- This “amount” depends on the remaining budget
- Define “Budget Index” $x(i)$
- At the beginning $x(i) = 0$
- $x(i)$ increases as remaining budget decreases
- Budget used up, then $x(i) = 1$

Intuition 2

- The user with more remaining budget should have higher priority
- For the user with less remaining budget, maybe we should wait
- Reduce the original bid by some amount
- This "amount" depends on the remaining budget
- Define "Budget Index" $x(i)$
- At the beginning $x(i) = 0$
- $x(i)$ increases as remaining budget decreases
- Budget used up, then $x(i) = 1$
- "Adjusted Bid": $(1 - x(i))b(i, j)$

Online Algorithm

Allocation Algorithm: Initially $\forall i \;\; x(i) \leftarrow 0$.
Upon arrival of a new product j allocate the product to the buyer i that maximizes $b(i,j)(1-x(i))$.
If $x(i) \geq 1$ then do nothing. Otherwise:

1. Charge the buyer the minimum between $b(i,j)$ and its remaining budget and set $y(i,j) \leftarrow 1$
2. $z(j) \leftarrow b(i,j)(1-x(i))$
3. $x(i) \leftarrow x(i)\left(1+\frac{b(i,j)}{B(i)}\right)+\frac{b(i,j)}{(c-1)\cdot B(i)}$ (c is determined later). $(x(i) \leftarrow x(i) + \Delta)$

Fractional Dual and Primal

Dual (Packing)		Primal (Covering)	
Maximize:	$\sum_{j=1}^{m}\sum_{i=1}^{n} b(i,j)y(i,j)$	Minimize :	$\sum_{i=1}^{n} B(i)x(i) + \sum_{j=1}^{m} z(j)$
Subject to:		Subject to:	
For each $1 \le j \le m$:	$\sum_{i=1}^{n} y(i,j) \le 1$	For each (i,j):	$b(i,j)x(i) + z(j) \ge b(i,j)$
For each $1 \le i \le n$:	$\sum_{j=1}^{m} b(i,j)y(i,j) \le B(i)$	For each i,j:	$x(i), z(j) \ge 0$
For each i,j:	$y(i,j) \ge 0$		

Proof and calculate

- We hope this algorithm can be $(1 - 1/c)$-competitive

Proof and calculate

- We hope this algorithm can be $(1 - 1/c)$-competitive
- Note: competitive ratio less than 1

Proof and calculate

- We hope this algorithm can be $(1 - 1/c)$-competitive
- Note: competitive ratio less than 1
- Find the maximal c

Proof and calculate

- We hope this algorithm can be $(1 - 1/c)$-competitive
- Note: competitive ratio less than 1
- Find the maximal c
- 1. The primal is satisfied

Proof and calculate

- We hope this algorithm can be $(1 - 1/c)$-competitive
- Note: competitive ratio less than 1
- Find the maximal c
- 1. The primal is satisfied
- 2. The ratio between the primal and the dual

Proof and calculate

- We hope this algorithm can be $(1 - 1/c)$-competitive
- Note: competitive ratio less than 1
- Find the maximal c
- 1. The primal is satisfied
- 2. The ratio between the primal and the dual
- 3. The dual is satisfied?

The primal satisfied

- The Primal is always satisfied

The primal satisfied

- The Primal is always satisfied
- The constraint is: $b(i,j)x(i) + z(j) \geq b(i,j)$

The primal satisfied

- The Primal is always satisfied
- The constraint is: $b(i,j)x(i) + z(j) \geq b(i,j)$
- Equivalent: $z(j) \geq b(i,j)(1 - x(i))$

The primal satisfied

- The Primal is always satisfied
- The constraint is: $b(i,j)x(i) + z(j) \geq b(i,j)$
- Equivalent: $z(j) \geq b(i,j)(1 - x(i))$
- Greedy choice: $\max_i\{b(i,j)(1 - x(i))\}$

The ratio between 2 objectives

- Denote the dual's objective function by D, the primal's by P

The ratio between 2 objectives

- Denote the dual's objective function by D, the primal's by P
- $D/P \geq 1 - 1/c \Rightarrow$ competitive ratio

The ratio between 2 objectives

- Denote the dual's objective function by D, the primal's by P
- $D/P \geq 1 - 1/c \Rightarrow$ competitive ratio
- In each round

$$\frac{\Delta D}{\Delta P} \geq 1 - 1/c \quad \Rightarrow \quad \frac{D}{P} \geq 1 - 1/c$$

The ratio between 2 objectives

- Denote the dual's objective function by D, the primal's by P
- $D/P \geq 1 - 1/c \Rightarrow$ competitive ratio
- In each round

$$\frac{\Delta D}{\Delta P} \geq 1 - 1/c \quad \Rightarrow \quad \frac{D}{P} \geq 1 - 1/c$$

- If we can prove

$$B(i)\Delta + z(j) \leq (1 + \frac{1}{c-1})b(i,j)$$

The ratio between 2 objectives

- Denote the dual's objective function by D, the primal's by P
- $D/P \geq 1 - 1/c \Rightarrow$ competitive ratio
- In each round

$$\frac{\Delta D}{\Delta P} \geq 1 - 1/c \quad \Rightarrow \quad \frac{D}{P} \geq 1 - 1/c$$

- If we can prove

$$B(i)\Delta + z(j) \leq (1 + \frac{1}{c-1})b(i,j)$$

- So

$$\Delta \leq (x(i) + \frac{1}{c-1}) \cdot \frac{b(i,j)}{B(i)}$$

The dual

- Suppose we have this property: when user i's budget is used up, $\Rightarrow x(i) \geq 1$

The dual

- Suppose we have this property: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- We always stop selling him items when $x(i) \geq 1$

The dual

- Suppose we have this property: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- We always stop selling him items when $x(i) \geq 1$
- But this cannot guarantee that the dual is not violated! (Constraint 2)

The dual

- Suppose we have this property: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- We always stop selling him items when $x(i) \geq 1$
- But this cannot guarantee that the dual is not violated! (Constraint 2)
- Consider the final item

The dual

- Suppose we have this property: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- We always stop selling him items when $x(i) \geq 1$
- But this cannot guarantee that the dual is not violated! (Constraint 2)
- Consider the final item
- Before the final item, the budget is not used up

The dual

- Suppose we have this property: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- We always stop selling him items when $x(i) \geq 1$
- But this cannot guarantee that the dual is not violated! (Constraint 2)
- Consider the final item
- Before the final item, the budget is not used up
- Add this item, the total price may exceed the budget

The dual

- Suppose we have this property: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- We always stop selling him items when $x(i) \geq 1$
- But this cannot guarantee that the dual is not violated! (Constraint 2)
- Consider the final item
- Before the final item, the budget is not used up
- Add this item, the total price may exceed the budget
- Exceed by no more than $b(i, j)$

The dual

- Suppose we have this property: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- We always stop selling him items when $x(i) \geq 1$
- But this cannot guarantee that the dual is not violated! (Constraint 2)
- Consider the final item
- Before the final item, the budget is not used up
- Add this item, the total price may exceed the budget
- Exceed by no more than $b(i,j)$
- Total price is no more than $B(i) + b(i,j)$

The dual

- Suppose we have this property: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- We always stop selling him items when $x(i) \geq 1$
- But this cannot guarantee that the dual is not violated! (Constraint 2)
- Consider the final item
- Before the final item, the budget is not used up
- Add this item, the total price may exceed the budget
- Exceed by no more than $b(i, j)$
- Total price is no more than $B(i) + b(i, j)$
- $B(i) + b(i, j) \leq (1 + R)B(i)$

The dual

- Suppose we have this property: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- We always stop selling him items when $x(i) \geq 1$
- But this cannot guarantee that the dual is not violated! (Constraint 2)
- Consider the final item
- Before the final item, the budget is not used up
- Add this item, the total price may exceed the budget
- Exceed by no more than $b(i,j)$
- Total price is no more than $B(i) + b(i,j)$
- $B(i) + b(i,j) \leq (1 + R)B(i)$
- Actual competitive ratio is $(1 - 1/c)(1 - R)$

The property of $x(i)$

- Prove: when user i's budget is used up, $\Rightarrow x(i) \geq 1$

The property of $x(i)$

- Prove: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- If we can prove:

$$x(i) \geq \frac{1}{c-1}\left(c^{\sum_j (b(i,j)y(i,j))/B(i)} - 1\right)$$

The property of $x(i)$

- Prove: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- If we can prove:

$$x(i) \geq \frac{1}{c-1}\left(c^{\sum_j (b(i,j)y(i,j))/B(i)} - 1\right)$$

- Prove by induction ....

The property of $x(i)$

- Prove: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- If we can prove:

$$x(i) \geq \frac{1}{c-1}\left(c^{\sum_j (b(i,j)y(i,j))/B(i)} - 1\right)$$

- Prove by induction ....
- We need to prove:

$$1 + \frac{b(i,j)}{B(i)} \geq c^{\frac{b(i,j)}{B(i)}}$$

The property of $x(i)$

- Prove: when user i's budget is used up, $\Rightarrow x(i) \geq 1$
- If we can prove:

$$x(i) \geq \frac{1}{c-1}\left(c^{\sum_j (b(i,j)y(i,j))/B(i)} - 1\right)$$

- Prove by induction ....
- We need to prove:

$$1 + \frac{b(i,j)}{B(i)} \geq c^{\frac{b(i,j)}{B(i)}}$$

- Conclusion is:

$$c \leq (1+R)^{1/R}$$

Conclusion

- The competitive ratio is $(1 - R)(1 - (1 + R)^{-1/R})$

Conclusion

- The competitive ratio is $(1 - R)(1 - (1 + R)^{-1/R})$
- When R is a small value, the ratio is close to $1 - 1/e$ (0.632, or 1.58-competitive)