Weekly Report (2009-01-31)

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I. Centralized Algorithm

We present a solution to the optimization problem with *Dantzig-Wolfe Decomposition* and *branch-and-price*.

The *Dantzig-Wolfe Decomposition* method is to decompose the original optimization problem into several subproblems, based on which we can have a master problem. By solving the subproblems and master problems iteratively, we can update the variable values and converge to the optimal solution. The whole method can be divided into four steps: *Initialization*, *Master problem solution*, *Relaxed problem solution* and *Convergence checking*.

A. Step 0: Initialization

$$\begin{aligned}
minimize \quad & c_t^{(y)} \sum_t y_t + \sum_{i,j,t} c_{ijt}^{(P)} P_{ijt} \\
s.t. \quad & \sum_{i,j} s_{ijt} \le n^2 y_t,\ \forall t \in [1,n] \\
& 0 \le P_{ijt} \le P_{max} s_{ijt}(P_{max}) \ \ \forall t,i,j \in [1,n] \\
& G_{ij}P_{ijt} - \beta \sum_{u \ne i} \sum_{w \ne j} G_{uj} P_{uwt} - \beta \sum_{u > i} G_{uj} P_{ujt} - \beta N_0 \ge \Phi(s_{ijt} - 1) \ \ \forall t,i,j \in [1,n] \\
& s_{ijt}, y_t \in \{0,1\},\ \forall i,j,t \in [1,n]
\end{aligned}$$

First set of subproblems: We set $c_t^{(y)} = 1$ and $c_{ijt}^{(P)} = 1$.

So the interpretation for this case is that the objective function is to find a minimum-latency and minimum-power solution. Since $\sum_{i,j} s_{ijt} \le n^2 y_t$, we know that $y_t = 0$ if and only if $\forall i,j \in [1,n], s_{ijt} = 0$. Meanwhile, $P_{ijt} = 0$ if and only if $\forall i,j \in [1,n], s_{ijt} = 0$. So the optimal solution for this subproblem is

$$\begin{aligned}
y_t &= 0,\ \forall t \in [1,n] \\
s_{ijt} &= 0,\ \forall t \in [1,n] \\
P_{ijt} &= 0,\ \forall t \in [1,n]
\end{aligned}$$

and the optimal value for the objective function of the subproblem is 0.

Second set of subproblems: We set $c_t^{(y)} = -\Phi$, where Φ is a large enough positive value, and $c_{ijt}^{(P)} = 1$.

So the interpretation for this case is that the objective function is to find a maximum-latency and minimum-power solution. The value of Φ is set to be a large enough value such that y_t should be maximized first. Since $\sum_{i,j} s_{ijt} \le n^2 y_t$, we know that $y_t = 1$ if and only if $\exists i,j \in [1,n], s_{ijt} = 1$. On the other hand, $\sum_{i,j,t} P_{ijt}$ should be minimized. As at least one link should be scheduled in time slot t, $\sum_{i,j,t} P_{ijt}$ is minimized when the shortest link is scheduled alone. So the optimal solution for this subproblem is

$$\begin{aligned}
y_t &= 1,\ \forall t \in [1,n] \\
s_{uvt} &= 1,\ \forall t \in [1,n] \& \forall i,j \in [1,n], d_{uv} \le d_{ij} \\
s_{ijt} &= 0,\ \forall t \in [1,n] \& d_{uv} \le d_{ij} \\
P_{uvt} &= \beta N_0 d_{uv}^{\alpha} \\
P_{ijt} &= 0,\ \forall t \in [1,n]
\end{aligned}$$

and the optimal value for the objective function of the subproblem is $-\Phi + \beta N_0 d_{uv}^{\alpha}$.

B. Step 1: Master problem solution

$$\begin{aligned}
minimize_u \quad & \sum_{\nu} u_\nu z_\nu \\
s.t. \quad & \sum_{\nu} u_\nu \sum_{j,t} s_{ijt}^{(\nu)} = 1,\ \forall i \in [1, n-1] \\
& \sum_{\nu} u_\nu \sum_{j,t} s_{njt}^{(\nu)} = 0 \\
& \sum_{\nu} u_\nu (\sum_{j} s_{ijt}^{(\nu)} + \frac{1}{n^2} \sum_{l \ge t, j} s_{jil}^{\nu}) \le 1,\ \forall i,t \in [1,n]
\end{aligned}$$

Since u_ν is real number, the master problem is a standard linear optimization problem. We solve this problem with *Simplex* method.

The lagrangian function for the master problem is

$$\begin{aligned}
L(u,\lambda,\sigma) &= \sum_{\nu} u_\nu z_\nu - \sum_{it} \lambda_{it} (\sum_{\nu} u_\nu (\sum_{j} s_{ijt}^{(\nu)} + \frac{1}{n^2} \sum_{l \ge t, j} s_{jil}^{(\nu)}) - 1) - \sum_{i} \sigma_{i=1}^{n-1} (\sum_{\nu} u_\nu \sum_{j,t} s_{ijt}^{(\nu)} - 1) - \sigma_n (\sum_{\nu} u_\nu \sum_{j,t} s_{njt}^{(\nu)}) \\
&= \sum_{\nu} u_\nu (z_\nu - \sum_{it} \lambda_{it} (\sum_{j} s_{ijt}^{(\nu)} + \frac{1}{n^2} \sum_{l \ge t, j} s_{jil}^{(\nu)}) - \sum_{i} \sigma_i \sum_{j,t} s_{ijt}^{(\nu)}) - \sum_{it} \lambda_{it} + \sum_{i=1}^{n-1} \sigma_i
\end{aligned}$$

The Dual problem of the master problem is

$$\begin{aligned}
minimize_u \quad & -\sum_{it} \lambda_{it} + \sum_{i=1}^{n-1} \sigma_i \\
s.t. \quad & z_\nu - \sum_{it} \lambda_{it} (\sum_{j} s_{ijt}^{(\nu)} + \frac{1}{n^2} \sum_{l \ge t, j} s_{jil}^{(\nu)}) - \sum_{i} \sigma_i \sum_{j,t} s_{ijt}^{(\nu)} = 0 \\
& \lambda_{it} \ge 0,\ \forall i,t \in [1,n]
\end{aligned}$$

Then we can get the value for dual variables for this round.

C. Step 2: Relaxed problem solution

On the basis of the solution for variables in relaxed subproblems and dual variables in master problem, we update the relaxed the problem as follows,

$$\begin{aligned}
minimize \quad & ay_t + b\sum_{i,j} P_{ijt} - \sum_{i,j} s_{ijt}(\lambda_{it} + \frac{1}{n^2}\sum_{l\le t}\lambda_{jl}) \\
s.t. \quad & \sum_{i,j} s_{ijt} \le n^2 y_t \\
& 0 \le P_{ijt} \le P_{max} s_{ijt} \quad \forall i,j \in [1,n] \\
& G_{ij}P_{ijt} - \beta\sum_{u\neq i}\sum_{w\neq j} G_{uj}P_{uwt} - \beta\sum_{u>i} G_{uj}P_{ujt} - \beta N_0 \ge \Phi(s_{ijt}-1) \quad \forall i,j \in [1,n] \\
& s_{ijt}, y_t \in \{0,1\}, \ \forall i,j \in [1,n]
\end{aligned}$$

Since each dual variable $\lambda_{it} \ge 0$, which is one constraint in dual problem, we know that $\lambda_{it} + \frac{1}{n^2}\sum_{l\le t}\lambda_{jl} \ge 0$. As a result, the relaxed subproblems can be interpreted as, for each time slot t, minimizing the aggregation latency and power consumption while maximizing the number of links scheduled in that time slot.

The optimal value may require non-polynomial time complexity. I came up with a heuristic algorithm for solving this Binary Mixed Integer Programming (BMIP) problem, which is a branch and price method as follows.

Initial Point: Check the value of $a - \sum_{i,j}(\lambda_{it} + \frac{1}{n^2}\sum_{l\le t}\lambda_{jl})$. If it is no less than zero, we set the initial point as

$$\begin{aligned}
y_t &= 0 \\
s_{ijt} &= 0, \ \forall i,j \in [1,n] \\
P_{ijt} &= 0, \ \forall i,j \in [1,n]
\end{aligned}$$

Otherwise, we set the initial point as

$$\begin{aligned}
y_t &= 1, \ \forall t \in [1,n] \\
s_{ijt} &= 1, \ \forall t \in [1,n]
\end{aligned}$$

and solve the value for P_{ijt} as a linear programming problem without considering the power upper bound.

Branch and bound: If the initial point is the later one, we calculate the price for the scheduling of each link. The price for link e_{ij} is defined as

$$bP_{ijt}(1 + \beta\sum_{u\neq i}\sum_{v\neq j} G_{iv}/G_{uv} + \beta\sum_{u<i} G_{ij}/G_{uj}) - \sum_j(\lambda_{it} + \frac{1}{n^2}\sum_{l\le t}\lambda_{jl})$$

Then we sort the prices of each link in descending order. If the price of the first link in the ordered sequence is no less than zero, we set $P_{ijt} = 0$ and $s_{ijt} = 0$. Then the sorting and removing procedure is repeated until the price of the first link in the ordered sequence is less than zero.

Up to now, we get all values for each variable and update the summed value for objective functions of each subproblem as

$$X = \sum_t (ay_t + b\sum_{i,j} P_{ijt} - \sum_{i,j} s_{ijt}(\lambda_{it} + \frac{1}{n^2}\sum_{l\le t}\lambda_{jl}))$$

And the value for objective function of original problem is updated as

$$z = a\sum_{t} + b\sum_{ijt} P_{ijt}$$

and the value for each complicating constraint is

$$\sum_{\nu} u_{\nu} \sum_{j,t} s_{ijt}^{(\nu)},\ \forall i \in [1, n-1]$$

$$\sum_{\nu} u_{\nu} \sum_{j,t} s_{njt}^{(\nu)}$$

$$\sum_{\nu} u_{\nu} (\sum_{j} s_{ijt}^{(\nu)} + \frac{1}{n^2} \sum_{l \geq t, j} s_{jil}^{\nu}),\ \forall i, t \in [1, n]$$

D. Step 4: Convergence checking

If $X \geq \sigma$, the optimal solution of original problem is achieved as

$$y_t^* = \sum_{\nu} u_{\nu} y_t^{(\nu)},\ \forall t \in [1, n]$$

$$P_{ijt}^* = \sum_{\nu} u_{\nu} P_{ijt}^{\nu},\ \forall i, j, t \in [1, n]$$

$$s_{ijt}^* = \sum_{\nu} u_{\nu} s_{ijt}^{\nu},\ \forall i, j, t \in [1, n]$$

Otherwise, $\nu \leftarrow \nu + 1$ and repeat step 1 - 3.