Online Submodular Welfare Maximization: Greedy Beats 1/2 in Random Order

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Submodular welfare maximization (SWM)

- Assign a set N of n items to a set M of m agents.
- Valuation set function $v_l: 2^N \rightarrow \mathbb{R}^+$
- Maximize $\sum v_l(S_l)$
- Submodularity:
  - $S \subseteq T \;\; v_l(S) \leq v_l(T)$

Online SWM

- Items in N arrive one by one

- Adversarial setting
  - Adversarially constructs a worst-case instance with order
  - Items in arbitrary order
- Stochastic setting
  - Adversarially constructs a worst-case instance
  - Measure average over random choice of arrival order

  - This paper's focus

Greedy Algorithm

- Assign each item to the agent whose marginal valuation increases the most
  - $v_l(e \cup A_l) - v_l(A_l)$
- Achieves a competitive ratio of 1/2 for the adversarial model (tight if NP=RP)
- The greedy algorithm has the competitive ratio at least 0.5052 for online SWM in the stochastic / random order model.

Outline

- Define a Gain function that capture the outcome when an item is assigned to the optimal agent under existing assignment.
- Divide Items into 3 groups:
  - First half, third quarter, last quarter
- Derive a lower bound for outcome of last quarter
- Solve the Factor-Revealing LP

Gain function

DEFINITION 5. *Fix an arbitrary permutation σ (such as the identity permutation). For any item j and any allocation A let $\ell = opt_j$, and $i+1$ denote the index of item j in σ. We define $Gain(j, A)$ as:*

$$v_\ell\left(\{j\} \cup A_\ell \cup (A_\ell^* \cap \sigma^i)\right) - v_\ell\left(A_\ell \cup (A_\ell^* \cap \sigma^i)\right).$$

- Marginal gain of assigning item j to agent $i = opt_j$ based on current allocation and a partial optimum allocation

Basic thought for ½ competitive

- Increase in welfare is at least Gain

  Lemma 6. *If item j arrives at position $i+1$ under permutation π, the increase in welfare that Greedy achieves by allocating this item is at least $Gain(j, A^i)$.*

  - Welfare at least $\sum_{i=1}^{n} Gain(\pi_i, A^{i-1}) \geq Gain(N, A^n)$

- Increase in welfare is at least Gain Reduction

  Lemma 7. *Greedy's increase in welfare from item π_i is at least $\sum_{j\in[n]} Gain(j, A^{i-1}) - Gain(j, A^i)$.*

  - Welfare at least $OPT - Gain(N, A^n)$

- Welfare at least ½ OPT

Go Beyond 1/2

- When bounding increase of welfare, we use

$$Gain(j, A^n) \leq Gain(j, A^i)$$

- $b_j = Gain(j, A^i) - Gain(j, A^n)$
- $\beta = \sum b_j$

- Competitive ratio at least $\frac{1}{2} + \frac{\beta}{2}$

Cont'd

- Expected welfare from greedily assigning the first t fraction of items is at least $w(t) = t - \frac{t^2}{2}$ that of the optimal
- Welfare increase is small for last few items
- Bound the welfare increases of items arrive at last
  - Bound the Gain when it arrives
  - Bound how it reduces Gain of other items

Formulating the Factor-Revealing LP

DEFINITION 9. *For each index $1 \leq i \leq n$, we define w_i to be the expected increase in welfare Greedy achieves by assigning the item in position i (that is, the ith item in the arrival order).*

Assigning item i possibly reduces the Gain values of items. We partition this effect into two parts; for any item i, we have two variables b_i and a_i defined as follows to capture the reduction in Gain values of other items:

- *b_i (we use b to denote* before*) is the expected reduction in Gain of items j that have already arrived. From our definition of β above, $\beta = \sum_{i=1}^{n} b_i$.*
- *a_i (a denotes* after*) is the expected reduction in Gain of items j that are going to arrive later.*

Clearly, allocating the item in position i reduces the total gain values of other items by $b_i + a_i$ in expectation.

Cont'd

- Greedy Algorithm's outcome $\sum_{i=1}^{n} w_i.$
- Two general constraint:
  - Increase of welfare of item is at least the reduction

$$w_i \geq b_i + a_i$$

  - LEMMA 10. *For any* $1 \leq i \leq n$, *if* $OPT = 1$, *we have* $w_i \geq \frac{1}{n} - \sum_{j=1}^{i-1} \frac{a_j}{n-j}$.
    - Original gain minus reduction of gain caused by item i

SECOND-ORDER SUPERMODULAR FUNCTIONS

Definition 2. *For a submodular function f, let $MG(A, e) = f(A \cup \{e\}) - f(A)$ denote the marginal gain from adding element e to set A. For sets A, S, we define $GR(A, S, e) = MG(A, e) - MG(A \cup S, e)$ as the amount by which S reduces the marginal gain from adding e to A. (Here, GR stands for Gain Reduction.) Note that by definition of submodularity, $GR(A, S, e)$ is always non-negative.*

- *The function f is said to be* second-order modular *if, for all sets A, B, S such that $A \subseteq B$, and $S \cap B = \emptyset$, and all elements e, we have: $GR(A, S, e) = GR(B, S, e)$.*
- *The function f is* second-order supermodular *if, for all sets A, B, S such that $A \subseteq B$, and $S \cap B = \emptyset$, and all elements e, we have: $GR(A, S, e) \geq GR(B, S, e)$. Equivalently, $MG(A, e) - MG(B, e) \geq MG(A \cup S, e) - MG(B \cup S, e)$.*
- *The function f is said to be* second-order submodular *if, for all sets A, B, S such that $A \subseteq B$, and $S \cap B = \emptyset$, and all elements e, we have: $GR(A, S, e) \leq GR(B, S, e)$.*

- Divide sequences into 3 parts
  - S1 S2 S3
  - S1 first half
  - S2 third quarter
  - S3 last quarter
- Reduction of Gains in S1 by allocating

For arbitrary submodular functions, we can no longer use the technique of the previous section to argue that we obtain sufficient welfare from items in $\langle S_2, S_3\rangle$. Instead, here we use a different constraint that provides a lower bound on $\sum_{i>3n/4} w_i$, which is Greedy's expected increase in welfare when assigning S_3, the last quarter of items. We obtain this constraint by considering the simulated sequence of items $S' = \langle S_1, S_2, S_3, S_2, S_3, S_2\rangle$. In other words, we will analyze how one could assign items if after the items in S_1 arrive, items in S_2 and S_3 arrive *multiple* times. We assign items of S' using the following allocation $A' = A^G(\langle S_1, S_2, S_3\rangle) \cup A^G(\langle S_2, S_3\rangle) \cup A^*(S_2)$. That is, we first use Greedy to assign the items of $\langle S_1, S_2, S_3\rangle$; then, we use the different allocation given by Greedy on $\langle S_2, S_3\rangle$ assuming nothing has been assigned so far (that is, ignoring the previous allocation of Greedy on $\langle S_1, S_2, S_3\rangle$). Finally we use the optimum allocation for S_2. A' is defined as the union of these three allocations.

CLAIM 1. $\mathbb{E}[Gain(S_1, A^G(S_1)) - Gain(S_1, A')] \geq \sum_{i=1}^{n/2} \frac{n/2}{n-i} a_i - \sum_{i=1}^{n/2} (\frac{n/2-i}{n-i} a_i + b_i)$.

CLAIM 2. $\mathbb{E}[Gain(S_2, A^G(S_1)) - Gain(S_2, A')] = \frac{1}{4} - \sum_{i=1}^{n/2} \frac{n/4}{n-i} a_i$.

CLAIM 3. $\mathbb{E}[Gain(S_3, A^G(S_1)) - Gain(S_3, A')] \geq \sum_{i=n/2+1}^{3n/4} \frac{n/4}{n-i} a_i$

From the three preceding claims, we conclude that $\mathbb{E}[V(A') - V(A^G(S1))]$ is at least:

$$\sum_{i=1}^{n/2} \frac{n/2}{n-i} a_i - \sum_{i=1}^{n/2} \left(\frac{n/2-i}{n-i} a_i + b_i \right) + \frac{1}{4} - \sum_{i=1}^{n/2} \frac{n/4}{n-i} a_i + \sum_{i=n/2+1}^{3n/4} \frac{n/4}{n-i} a_i$$

Finding lower bound of W3

LEMMA 12. *The expected increase in welfare that Greedy achieves for the last quarter of items* $\sum_{i>3n/4} w_i$ *is at least* $\frac{1}{24}+\sum_{i=1}^{n/2}(\frac{i-n/4}{6(n-i)}a_i-\frac{1}{6}b_i)+\sum_{i=n/2+1}^{3n/4}\frac{n/4}{6(n-i)}a_i-\frac{1}{6}\sum_{i=n/2+1}^{3n/4}w_i.$

- X: how much<S2, S3, S2, S3, S2> increases A’
- W2: outcome of S2
- W3: outcome of S3
- In X, two S3 provide at most 2*W3
- In <S2, S2, S2> three S2 provide at most X-2*W3
- By symmetry, three S3 provide at most X-2*W3
- In <S1, S2, S3, S3, S3, S3> first S3 provide more than following S3. Then
- W3 ≥ (X-2W3-W1-W2)/3
- W3 ≥ (X-W2)/6

Analyze Lower Bound

$$LB = \frac{1}{24} + \sum_{i=1}^{n/2}\left((1+\frac{i-n/4}{6(n-i)})a_i + \frac{5}{6}b_i\right) +$$

$$\sum_{i=n/2+1}^{3n/4}\left((\frac{5}{6}+\frac{n/4}{6(n-i)})a_i + \frac{5}{6}b_i\right) + \sum_{i=1}^{3n/4}\frac{5}{6}(w_i - a_i - b_i)$$

- Together with other 2 constrains
- Analyze Factor-Revealing LP
  - can assume $w_i = a_i + b_i$
  - Proved there is an optimum solution satisfy $a_i + b_i = \frac{1}{n} - \sum_{j=1}^{i-1}\frac{a_j}{n-j}$
- Solve a，find LB
- LB greater than 0.5052

LEMMA 13. *The expected welfare obtained by Greedy is at least* $\frac{1}{24} + \sum_{i=1}^{n/2}\left((1+\frac{i-n/4}{6(n-i)})a_i + \frac{5}{6}b_i\right) + \sum_{i=n/2+1}^{3n/4}\left((\frac{5}{6}+\frac{n/4}{6(n-i)})a_i + \frac{5}{6}b_i\right) + \sum_{i=1}^{3n/4}\frac{5}{6}(w_i - a_i - b_i)$.

Lessons learned

- Factor Revealing LP
- Greedy can be powerful but hard to prove good result
- Slicing the online sequence into several parts
  - Model the influence between each part