Worst Case Competitive Analysis of Online Algorithms for Conic Optimization

Reza Eghbali

University of Washington (NIPS 2016)

Novelty

Introduce new methods to prove the competitive ratio

Connection between online optimization and online learning (in algorithm perspective)

Problem Model

$$\begin{aligned} \text{maximize} \quad & \psi\left(\sum_{t=1}^{m} A_t x_t\right) \\ \text{subject to} \quad & x_t \in F_t, \quad \forall t \in [m], \end{aligned}$$

a proper convex cone $K \subset \mathbb{R}^n$

$\psi : K \mapsto \mathbb{R}$

$t \in [m] := \{1, 2, \ldots, m\}$

$x_t \in \mathbb{R}^k$

$A_t \in \mathbb{R}^{n \times k}$ maps F_t to K

Example

$$
\begin{aligned}
\text{maximize} \quad & \sum_{t=1}^{m} c_t^T x_t \\
\text{subject to} \quad & \sum_{t=1}^{m} B_t x_t \leq b \\
& \mathbf{1}^T x_t \leq 1, \; x_t \in \mathbf{R}_+^k, \quad \forall t \in [m].
\end{aligned}
$$

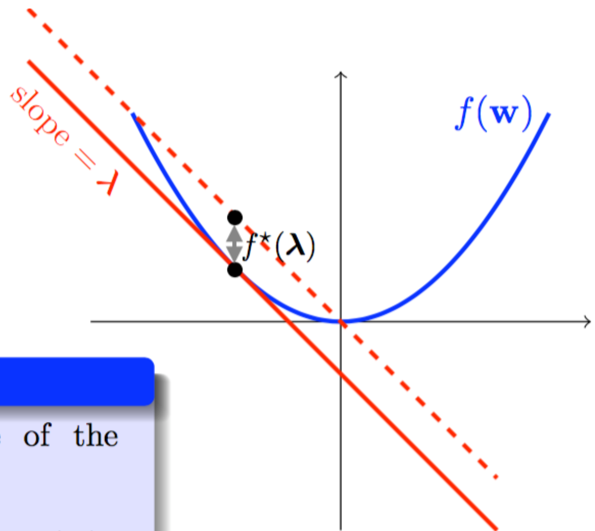
$$
\begin{aligned}
\text{maximize} \quad & \psi\left(\sum_{t=1}^{m} A_t x_t\right) \\
\text{subject to} \quad & x_t \in F_t, \quad \forall t \in [m],
\end{aligned}
$$

$$F_t = \{x \in \mathbf{R}_+^k \mid \mathbf{1}^T x \leq 1\}$$

$$A_t^T = [c_t, B_t^T]$$

$$\psi(u, v) = u + I_{\{\cdot \leq b\}}(v)$$

Fenchel Duality



Fenchel Conjugate

The Fenchel conjugate of the function $f : S \to \mathbb{R}$ is

$$f^\star(\boldsymbol{\lambda}) = \max_{\mathbf{w} \in S} \langle \mathbf{w}, \boldsymbol{\lambda} \rangle - f(\mathbf{w})$$

$$\text{minimize} \quad \sum_{t=1}^{m} \sigma_t(A_t^T y) - \psi^*(y),$$

$$\psi^*(y) = \inf_u \langle y, u \rangle - \psi(u),$$

$$\sigma_t(z) = \sup_{x \in F_t} \langle x, z \rangle$$

Comparison

Primal

$$\begin{aligned} \text{maximize} \quad & \psi\left(\sum_{t=1}^{m} A_t x_t\right) \\ \text{subject to} \quad & x_t \in F_t, \quad \forall t \in [m], \end{aligned}$$

dual

$$\text{minimize} \quad \sum_{t=1}^{m} \sigma_t(A_t^T y) - \psi^*(y),$$

Optimum

The optimal primal-dual pair if and only if

$$x_t^* \in \operatorname*{argmax}_{x \in F_t} \left\langle x, A_t^T y^* \right\rangle \quad \forall t \in [m],$$

$$y^* \in \partial\psi\Big(\sum_{t=1}^{m} A_t x_t^*\Big).$$

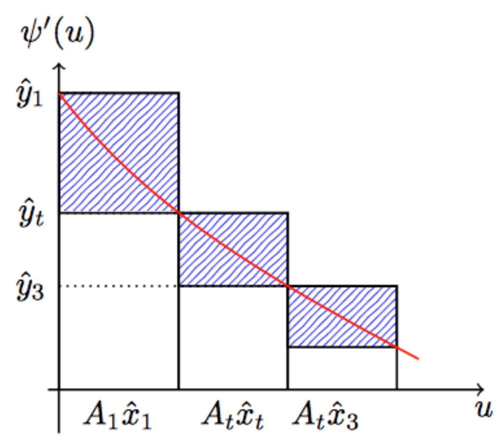
$$\partial\psi(u) = \operatorname*{argmin}_{y} \langle y, u \rangle - \psi^*(y).$$

Sequential algorithm

- Maintain dual variable y and use it to assign x.
- Related to the “Follow The Regularized Leader” update in online learning.
- Solve two optimization problems separately.

Algorithm 1 Sequential Update

Initialize $\hat{y}_1 \in \partial\psi(0)$
for $t \leftarrow 1$ to m **do**
  Receive A_t, F_t
  $\hat{x}_t \in \text{argmax}_{x \in F_t} \langle x, A_t^T \hat{y}_t \rangle$
  $\hat{y}_{t+1} \in \partial\psi(\sum_{s=1}^{t} A_s \hat{x}_s)$
end for



Simultaneously algorithm

- Solve a saddle-point problem

Algorithm 2 Simultaneous Update

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for $t \leftarrow 1$ to m **do**

  Receive A_t, F_t

  $(\tilde{y}_t, \tilde{x}_t) \in \arg\min_y \max_{x \in F_t} \left\langle y, A_t x + \sum_{s=1}^{t-1} A_s \tilde{x}_s \right\rangle - \psi^*(y)$

end for

Competitive ratio

Sequential

$$P_{\text{seq}} \geq \frac{1}{1-\bar{\alpha}_\psi}(D^\star - \sum_{t=1}^{m} \frac{1}{2\mu} \|A_t \hat{x}_t\|^2)$$

Simultaneous

$$P_{\text{sim}} \geq \frac{1}{1-\bar{\alpha}_\psi} D^\star$$

$$\alpha_\psi(u) = \inf_{y \in \partial\psi(u)} \frac{\psi^*(y)}{\psi(u)}$$

$$\bar{\alpha}_\psi = \inf\{\alpha_\psi(u) \mid u \in K\}$$

Example---online LP with non-separable budgets

Formalized problem

$$
\begin{aligned}
\text{maximize} \quad & \sum_{t=1}^{m} c_t^T x_t + I_{\{\cdot \leq 1\}} \left(\sum_{t=1}^{m} B_t x_t \right) \\
& x_t \in F_t, \quad \forall t \in [m].
\end{aligned}
$$

Exact penalty form

$$
\text{maximize}_{x_t \in F_t} \sum_{t=1}^{m} c_t^T x_t + G \left(\sum_{t=1}^{m} B_t x_t \right)
$$

$$
G(u) = -l \sum_{i=1}^{n} (u_i - 1)_+.
$$

$$
l > \max \left\{ \frac{c_{t,j}}{B_{t,ij}} \mid B_{t,ij} > 0,\ j \in [k],\ i \in [n] \right\}
$$

Conclusion

Concise algorithms

New updating methods/analysis for proving competitive ratio.

Exact penalty transformation