The Mass Transport Model

I. Model Overview

Considering the mass transport model, we can build up a stochastic model of P2P network, capturing the budget transport and distribution.

- The model is a generalization of Zero-Range Process (ZRP) [1].
- The mass transport model is based on arbitrary graph and stochastic process.
- A stochastic portion of budgets transports from one peer to others at each time step.
- System is closed: Total amount of budgets does not change.
- The system dynamically comes to a steady state eventually.
- Sufficient conditions are given. [2]

II. Model Specifications

A. Signal Table for Reference

$\mathcal{G}$	Directed graph as a P2P network
i, j, k	Nodes in the graph, representing peers in P2P netwrok
L	Number of peers in the network
$[\mu]$	Transfer matrix of budget flow
μ_{ij}	Amount of budgets transferred from i to j
m_i	$m_i = \sum_{j=1}^{L} \mu_{ij}$, total amount of budgets transfer from i
$\varphi_i(\{\mu_{ij}\}\|m_i)$	"chipping kernel" that represents the joint distribution of the budgets transported from peer i in one step
$P(\underline{m}, t \to \infty)$	stready state joint probability distribution of budgets
$\delta()$	Dirac delta function
$f_i(m_i)$, ν_{ij}	will be explained in the sufficient condition.

B. Transport Matrix

Consider an arbitrary directed graph $\mathcal{G}$ as a P2P network which consists L peers in total, with each directed link denoting the direction of transport. The $L \times L$ transfer matrix is denoted as $[\mu]$ whose element μ_{ij} denotes the stochastic amount of budgets transferred from peer i to peer j.

We also define the sum of each row $m_i = \sum_{j=1}^{L} \mu_{ij}$.

Here gives an example:

$$[\mu] = \begin{pmatrix} \mu_{11} & 0 & \mu_{13} \\ \mu_{21} & 0 & \mu_{23} \\ 0 & \mu_{32} & \mu_{33} \\ \mu_{41} & \mu_{42} & 0 \end{pmatrix} \tag{1}$$

C. Chipping Kernel

The name of "chipping kernel" is adopted from the original paper. $\varphi_i(\{\mu_{ij}\}|m_i)$ represents the joint distribution of the budgets transported from peer i in a single time step, where $\{\mu_{ij}\}$ denotes non-zero elements in the ith row of matrix $[\mu]$. We normalizing the chipping kernels to unity at each peer i,

$$\int \varphi_i(\{\mu_{ij}\}|m_i)\delta\Big(\sum_j \mu_{ij} - m_i\Big)\Pi_j d\mu_{ij} = 1 \tag{2}$$

In this way, the model implies that the amount of budgets transported from a given peer i in one step depends only on itself, and how much amount decided according to the $\varphi_i(\{\mu_{ij}\}|m_i)$.

In the next subsections, we discuss the steady state joint probability distribution relating to the conditions of the chipping kernel.

D. Steady State Joint Probability Distribution

The steady state joint probability distribution of budgets is defined:

$$P(\underline{m}, t \to \infty), \underline{m} \equiv \{m_1, m_2, ..., m_L\}. \tag{3}$$

We can determine the properties of "chipping kernel" $\phi_i(\{\mu_{ij}\}|m_i)$ in order to guarantee that the steady state joint probability distribution is factorizable as follows:

$$P(\underline{m}, t \to \infty) = Z(M,L)^{-1} \Big[\prod_{i=1}^{L} f_i(m_i) \Big] \delta \Big(\sum_{i=1}^{L} m_i - M \Big) \tag{4}$$

where $Z(M, L)$ as a normalization constant is given, the $f_i(m_i)$ is a factor for peer i whose property to be discuss later. :

$$Z(M,L) = \prod_{i=1}^{L} \Big[\int_0^{\infty} dm_i f_i(m_i) \Big] \delta \Big(\sum_{i=1}^{L} m_i - M \Big) \tag{5}$$

The sufficient conditions is:

$$\varphi_i(\{\mu_{ij}\}|m_i) = \frac{\Pi_j \nu_{ij}(\mu_{ij})}{\Big[\Pi_{*k} \nu_{ki}\Big](m_i)} \tag{6}$$

where ν_{ij} is a non-negative function that satisfies:

$$\Big[\prod_{*j} \nu_{ji} \Big](m) = \Big[\prod_{*j} \nu_{ij} \Big](m) \tag{7}$$

and where $f_i(m)$ has the convolution form related to ν_{ij}:

$$f_i(m) = \Big[\Pi_{*j} \nu_{ji} \Big](m) \tag{8}$$

III. Distribution Analysis of ZRP as an Example

This is a section we still need to explore, the reference [3] gives us some ideas:

Since the above model we have discussed is a generalization of Zero-Range Process, we can analyze the distribution of ZRP on some scale-free topology of networks.

The rate that n budgets transport from a peer with to its neighbors: $p(n) = n^{\sigma}$, and the γ is the degree distribution exponent of the underlying networks.

When $\sigma \leq \frac{1}{\gamma - 1}$, complete condensation occurs.

IV. Problem Remaining

How to explain the convoluting operations?

TABLE I

MAPPING BETWEEN THE QUEUING NETWORK MODEL AND THE P2P VOD SYSTEM

A queuing network	A overlay
A node Q_i	A peer p_i
A job	a unit of budget
num. of jobs in a node	a peer's budget
routing probability	probability of budget transfer
num. of routing arrows ending at Q_i	num. of p_i's upstream neighbors
num. of routing arrows heading at Q_i	num. of p_i's downstream neighbors
u_i	p_i's budget average spending in a unit of time
λ_i	p_i's avarage net income of budget in a unit of time

TABLE II

NOTATIONS USED IN THE MODEL

N	num. of peers

V. SPECIAL CASE: JACKSON NETWORK

The Jackson Network is a special case of the mass transport model, below is Xuanjia's research on that case:

λ_i can reflect peer i's upload capacity.

The paper "Condensation in Large Closed Jackson Networks" [4], gives sufficient condition of condenstion in its Theorem 2.2.

a BCMP network is a class of queueing network for which a product form equilibrium distribution exists. It is a significant extension to a Jackson network allowing virtually arbitrary customer routing and service time distributions, subject to particular service disciplines.

The term of “a product form equilibrium distribution” seems to be similar to “factorized steady states” quoted from the paper “Factorized steady states in mass transport models on an arbitrary graph” [5].

The paper [5] offers sufficient and necessary conditions for the “factorized stead states” and argues that “having a factorized steady state opens the door for the study of condensation” and “Thus one should be able to analyze condensation in various geometries or even on scale-free networks”. This tells us that even if we don’t need to use the result of this paper to get the sufficient and necessary conditions (but, instead, find out the “factorized steady state” based on the assumptions of our specific model), we have more confidence to analyze the condensation once we have the “factorized steady state”.

[6] also discusses that if agents based on a relationship topology of scale-free network exchange mass following a Zero Range Process, it is possible to achieve condensation under some conditions.

REFERENCES

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