Truthful and Competitive Double Auctions

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Double auction

- A market with multiple buyers and sellers. (Cloud providers and users)
- An auctioneer (broker) organizes the double auction.
- Profitable or not
- Another example of application: stock market.

Model

- Single-round sealed-bid double-auction mechanism:
- n buyers and n sellers, each one has one identical item to buy or sell.
- Buyers: $\mathbf{b} = (b_1, \ldots, b_n)$. Sellers: $\mathbf{s} = (s_1, \ldots, s_n)$.
- The mechanism calculate on input $\mathbf{b}, \mathbf{s}$, output $\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}$.
- Allocation vector: $\mathbf{x} \in \{0, 1\}^n$. If $x_i = 1$, then the i-th buyer get the item. The same for $\mathbf{y}$.
- Payment vector: $\mathbf{p} \in \mathbb{R}^n$. How much a buyer pays.
- Profit: $M(\mathbf{b}, \mathbf{s}) = \sum_i p_i - \sum_i q_i$
- Constraint: $\sum_i x_i = \sum_i y_i$
- Constraint: $0 \leq p_i \leq b_i$ for winning buyers and $p_i = 0$ for losing buyers.

Assumptions

- Each buyer/seller has a private utility value for the item. u_i, v_i.
- Each bidder try to maximize their profit: $u_i x_i - p_i$
- The algorithm used by auctioneer is public
- The private utility value of others are secret.
- No collude.

Truthfulness

- Bidding $b_i = u_i$ is a dominant strategy for each bidder.
- In the remainder part, we always assume that $b_i = u_i$. And we guarantee this property by proving its truthfulness.

Bid Independence

- $\mathbf{b_{-i}} = (b_1, \ldots, b_{i-1}, ?, b_{i+1}, \ldots, b_n)$
- Compute the threshold $t_i = f(\mathbf{b_{-i}, s})$
- For each buyer i, if $t_i < b_i$, set $x_i = 1$ and $p_i = t_i$.
- Otherwise, $x_i = p_i = 0$

Theorem

A double auction is truthful if and only if it is bid-independent.

Monotonicity

Definition

For any pair of buyers i and j with $b_i \leq b_j$, $\forall x \leq b_i$,
$\mathbf{Pr}[\text{buyer i wins at price} \leq x] \leq \mathbf{Pr}[\text{buyer j wins at price} \leq x]$

- If you bid higher, then your chance to win becomes higher.
- If your competitor bids higher, then your chance to lose becomes higher.
- In another word, the function $f(\mathbf{b_{-i}}, \mathbf{s})$ in 'Bid Independence' is monotonous.

Basic Auction

- n buyers and one auctioneer. The auctioneer is the only seller.
- A special case of the double auction with all sell bids at value zero.

Vickrey Basic Auction

Definition

The ith highest bidding buyer is $b_{(i)}$. The ith smallest seller bid is $s_{(i)}$.

Definition

The k-item Vickery basic auction on bids $\mathbf{b}$, is $V_k(\mathbf{b})$. It sells to the highest k bidders at the k+1st bid value, i.e., $b_{(k+1)}$.

Vickrey Double Auction

Definition

The k-item Vickery double auction on bids $\mathbf{b}$ and $\mathbf{s}$, is $V_k(\mathbf{b}, \mathbf{s})$. It sells to the highest k bidders at $b_{(k+1)}$.
And buys from the lowest k sellers at price $s_{(k+1)}$. Its revenue is

$$V_k(\mathbf{b}, \mathbf{s}) = k(b_{(k+1)} - s_{(k+1)})$$

Vickrey Double Auction

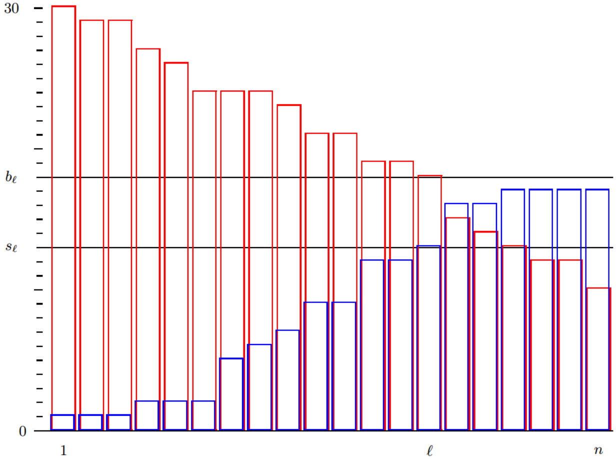

Figure 2. In this and the remaining figures, the red bars represent the buyer bids sorted in order of decreasing value, the blue bars represent the seller bids sorted in

Multiple-price Omniscient Mechanism, T

- Just require each buyer to pay their utility value.
- And pays to each seller their utility value (as long as positive profit can be made.)
- If $\forall i, b_i \geq s_i$,
- $T(\mathbf{b}, \mathbf{s}) = \sum_i b_i - \sum_i s_i$
- n items are traded.
- Problem with this mechanism? Too powerful!

Single-price Omniscient Mechanism, F

- Find an optimal i,
- $F(\mathbf{b}, \mathbf{s}) = \max\{i(b_{(i)} - s_{(i)})\}$
- For some technical reason, we restrict that F must trade at least 2 items. $(i \geq 2)$

Single-price Omniscient Mechanism

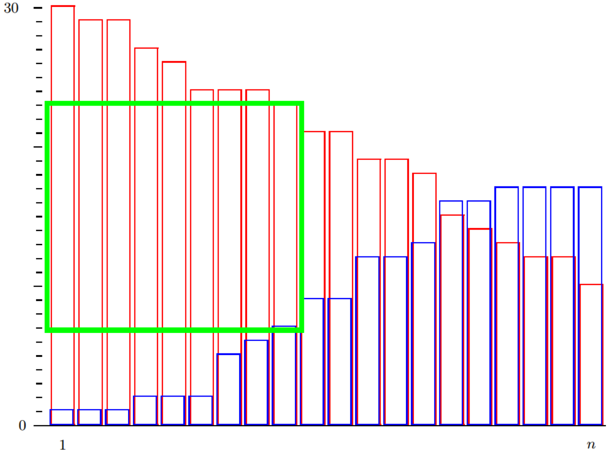

Figure 1. In this figure, the red bars represent the buyer bids sorted in order of decreasing value, the blue bars represent the seller bids sorted in order of increasing value and the area of the green rectangle represents the value of $\mathcal{F}(\mathbf{b}, \mathbf{s})$.

β-competitive

Definition

A truthful randomized double auction M is β-competitive against F if, for all bid vectors $\mathbf{b}, \mathbf{s}$, the expected profit of M satisfies;

$$E[M(\mathbf{b}, \mathbf{s})] \geq F(\mathbf{b}, \mathbf{s})/\beta$$

Upper Bound

- No monotone basic auction has expected profit more than F.
- (For basic auction, at least 1-competitive)
- $F(\mathbf{b}, \mathbf{s})$ can be $\Theta(T(\mathbf{b}, \mathbf{s})/\log n)$
- So comparing with T is meaningless, (at least $\log n$-competitive).

Upper Bound

Theorem

Any truthful monotone double auction, M, has expected profit at most $2F(\mathbf{b}, \mathbf{s})$.

Goal

- We have a basic auction mechanism A. It is β-competitive (comparing with omniscient basic auction).
- We want to: convert A to a double auction M_A, with 2β-competitive (comparing with omniscient double auction).

Reduction Algorithm

- Let l be the largest value such that $b_{(l)} \leq s_{(l)}$
- For special case that $l \leq 2$, we will: (Omitted here)
- For $l \geq 3$:
- For $\forall i$, let $b'_{(i)} = b_{(i)} - s_{(l)}$, $s'_{(i)} = b_{(l)} - s_{(i)}$. And let $\mathbf{b}'$ be the $(l-1)$ dimensional vector consisting $b'_{(1)} \ldots b'_{(l-1)}$.
- With probability $1/2$, simulate $A(\mathbf{b}')$.
- If a buyer wins in A at price p'_i, then he wins in M_A at price $p_i = \max\{(b_{(l)}, p'_i + s_{(l)})\}$.
- Let k be the number of winners in $A(\mathbf{b}')$. Run $V_k(\mathbf{s})$ to decide the outcome for the sellers.
- With probability $1/2$, simulate $A(\mathbf{s}')$. And do the opposite things.

Reduction Algorithm

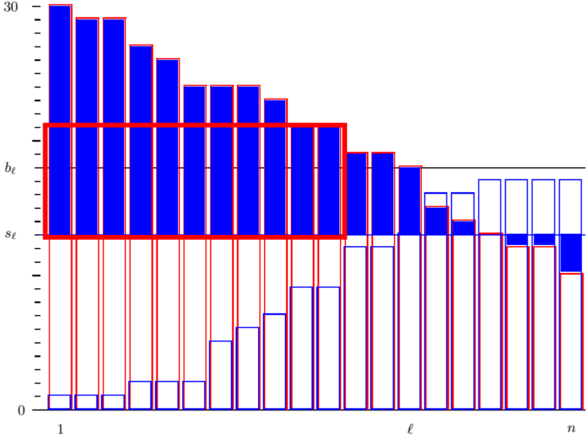

Figure 4. The solid blue bars represent the bids $b'_i = b_i - s_{(\ell)}$ sorted in order of decreasing value. In step 1 of case 3, we run the auction $\mathcal{A}$ on the first $\ell - 1$ of these (all those with $b'_i > 0$). The area of the red rectangle represents the optimal fixed price revenue for this set of bids. The auction $\mathcal{A}$ is guaranteed to return a profit from these

Truthful

- Prove it directly (not using bid-independent).
- For winning buyers, they pay at least $b_{(l)}$. (Either in A or in V_k)
- For lth highest buyer and all lower buyers, if they want to win, they need to bid higher than $b_{(l-1)}$. Then, all winners would pay at least $b_{(l-1)}$. Negative profit.
- For the $l-1$ high buyers, they can not control value $b_{(l)}$ and l. For V_k part, the truthful is trivial because of the truthfulness of Vickrey, and the value of k is independent.
- For A part, suppose A use an independent function $f'()$, then the function for M_A is $\max\{(b_{(l)}, f'() + s_{(l)})\}$. This function is bid-independent.

2β-competitive

- Let k be the number of items traded by omniscient F. $k \leq l$ Thus,

$$F(\mathbf{b}, \mathbf{s}) = k(b_{(k)} - s_{(k)}) \leq F(\mathbf{b}') + F(\mathbf{s}')$$

Reduction Algorithm

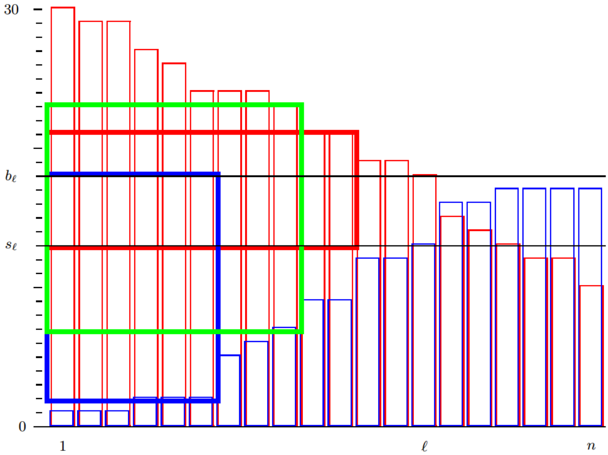

Figure 7. In this figure, the area of the red rectangle is the optimal fixed price revenue from the buyer bids $\mathbf{b}'$ and the area of the blue rectangle is the optimal fixed price revenue from the seller bids $\mathbf{s}'$. The green rectangle is the optimal fixed price revenue

2β-competitive

- $A \geq F(\mathbf{b'})/\beta$ and $A \geq F(\mathbf{s'})/\beta$ respectively.
- And both cases happens with probability $1/2$. The expectation is the sum divided by 2. Thus,

$$E[M_A(\mathbf{b}, \mathbf{s})] \geq \frac{1}{2\beta}(F(\mathbf{b'}) + F(\mathbf{s'})) \geq \frac{1}{2\beta}F(\mathbf{b}, \mathbf{s})$$

Usage

- We have a 4-competitive CORE basic auction. Now we get a double auction with competitive ratio of 8.
- Using the 3.39-competitive CRE basic auction, we get a ratio of 6.78.
- We can do better if we customize a CORE mechanism for double auction directly.

Bid Independent Consensus Estimate

- Independent function, which means with only $2n - 1$ input.
- Try to guess the value of $F(\mathbf{b}, \mathbf{s})$, which is easy to calculate if we have total $2n$ inputs.
- This guess should be close and smaller: $R \leq F(\mathbf{b}, \mathbf{s})$.
- This guess should be consensus. For all i, R is the same.

Profit Extractor

- An α-profit extractor is pe_R tries to gain a profit at least R/α.
- We present a profit extractor here:
- With input R, computes the largest k such that $k(b_{(k)} - s_{(k)}) \geq R$, buys from lowest to top highest.

Theorem

This profit extractor is a $\frac{k}{k-1}$ extractor. (k is the amount of item it exchanges.)

Consensus Estimate

- Let U be a uniform random variable from $[0, 1]$.
- $r()$ calculates $F(\mathbf{b}, \mathbf{s})$, supposing the missing value to be 0, and round down the result to the nearest c^{i+U} for some integer i.

Putting together

- Two parameters: c, p
- With probability $1 - p$, run consensus estimator, and the profit extractor. With probability p, run V_1.
- Choose $c = 2.62, p = 0.54$, we can have a 3.75-competitive mechanism against F.