Optimal Control of Epidemic Evolution

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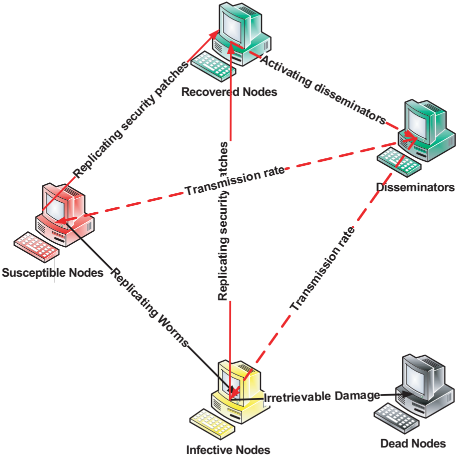
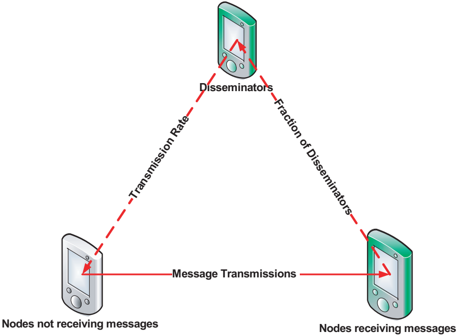
Motivation: Network Security Patch Distribution

- **The System**
  - System state: (susceptible nodes, infective nodes, recovered nodes, dead nodes)
  - System control parameters:
    - fraction of disseminators
    - dissemination rate
  - Incurred system cost:
    - number of infective and dead nodes
    - energy and bandwidth consumption in distribution
- **Problem**
  - What is the optimal control that minimizes the aggregate cost?



Motivation: Message Dissemination in Delay Tolerant Networks(DTNs)

- **The System**
  - System state: (nodes receiving messages, nodes not receiving messages)
  - System control parameters:
    - fraction of disseminators
    - dissemination rate
  - System costs:
    - message transmission delay
    - power consumption in activation and transmission of disseminators
- **Problem**
  - What is the optimal control that minimizes the aggregate cost due to delay and power consumption?



Methodology

Optimal control problem: find a set of differential equations of control variables that minimize the cost function.

- A control is to be chosen so as to minimize the **objective function**:

$$J(u) = \Phi\left(x(T)\right) + \int_0^T L(x,u,t)dt$$

- The **system state** $x(t)$ evolve according to the state equations:

$$\dot{x} = f(x,u,t)$$
$$x(0) = x_0, t \in [0,T]$$

- The **control** satisfies constraints:

$$a \leq u(t) \leq b, t \in [0,T]$$

Methodology

Pontryagin's Maximum Principle: The optimal control $u^\star \in \mathcal{U}$, the optimal state trajectory $x^\star$, and corresponding Lagrange multiplier vector $\lambda^\star$ must minimize the Hamiltonian defined as:

$$H(\lambda, x, u) = \lambda' f(x, u) + L(x, u)$$

Four necessary conditions for an optimal control:

- $$H(x^\star, u^\star, \lambda^\star) \leq H(x^\star, u, \lambda^\star)$$

- $$\Phi_T(x(T)) + H(T) = 0$$

- co-state equations:

  $$-\dot{\lambda}' = \lambda' f_x(x^\star, u^\star) + L_x(x^\star, u^\star)$$

- $$\lambda'(T) = \Phi_x(x(T))$$

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- **Controls:** the fraction of activated disseminators $\epsilon(t)$:

$$0 \leq \epsilon(t) \leq 1$$

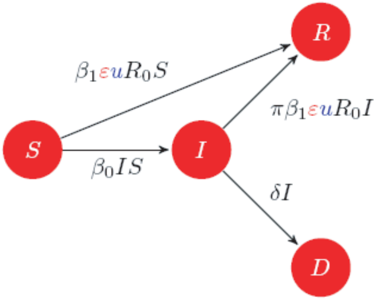
the transmission rate $\mu(t)$:

$$0 \leq \mu(t) \leq 1$$

.

- **System state equations:**

$$\dot{S} = -\beta_0 IS - \beta_1 R_0 \epsilon \mu S$$
$$\dot{I} = \beta_0 IS - \pi \beta_1 R_0 \epsilon \mu I - \delta I$$
$$\dot{D} = \delta I$$



[1] The paper discusses two modes of epidemic distribution: Non-replicative Dispatch and Replicative Dispatch, I take the non-replicative dispatch as an presentation example.

The general cost function:

$$J = \int_0^T \left(f(I(t)) + g(D(t)) - L(R(t)) + \epsilon(t) R_0 h(\mu(t)) \right) dt \\ + \kappa_I I(T) + \kappa_D D(T) - \kappa_R R(T).$$

Assume the system incurs costs at the rates of $f(I(t))$, $g(D(t))$ and benefits at the rate of $L(R(t))$. Each activated disseminator consumes resources at the rate $h(\mu(t))$.

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Apply Pontryagin's Maximum Principle:

- Hamiltonian:

$$\begin{aligned} H = f(I) + g(D) - L(R) + \epsilon R_0 h(\mu) + (\lambda_I - \lambda_S)\beta_0 IS \\ -\beta_1 R_0 \epsilon\mu\lambda_S S - \pi\beta_1 R_0\epsilon\mu\lambda_I I + (\lambda_D - \lambda_I)\delta I \end{aligned}$$

- optimal controls (ϵ, μ) satisfies:

$$(\epsilon, \mu) \in \arg \min_{\underline{\epsilon}, \underline{\mu} admissible} H(\vec{\lambda}, (S, I, D), (\underline{\epsilon}, \underline{\mu})).$$

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- We define $\varphi := \beta_1(\lambda_S S + \pi\lambda_I I)$, Hamiltonian can be written as:

$$H = f(I) + g(D) - L(R) + (\lambda_I - \lambda_S)\beta_0 IS$$
$$+(\lambda_D - \lambda_I)\delta I + \epsilon R_0(h(\mu) - \varphi\mu).$$

Apply Pontryagin's Maximum Principle:

- Hamiltonian:

$$H = f(I) + g(D) - L(R) + \epsilon R_0 h(\mu) + (\lambda_I - \lambda_S)\beta_0 IS \\ -\beta_1 R_0 \epsilon\mu\lambda_S S - \pi\beta_1 R_0 \epsilon\mu\lambda_I I + (\lambda_D - \lambda_I)\delta I$$

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- The Hamiltonian minimum problem converts to:

$$(\epsilon, \mu) \in \arg \min_{\underline{\epsilon}, \underline{\mu} admissible} \underline{\epsilon}(h(\underline{\mu}) - \varphi\underline{\mu}).$$

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$\varphi(t)$: $\varphi(t)$ is a strictly decreasing function of t for $t \in [0, T)$.

Some Properties of optimal controls:

- take $(\underline{\epsilon}, \underline{\mu}) = (0, 0)$, we get

$$\epsilon(h(\mu) - \varphi\mu) \leq 0$$

.

- When optimal $\mu = 0$, optimal ϵ can be chosen to be 0; When optimal $\mu > 0$, optimal $\epsilon = 1$.

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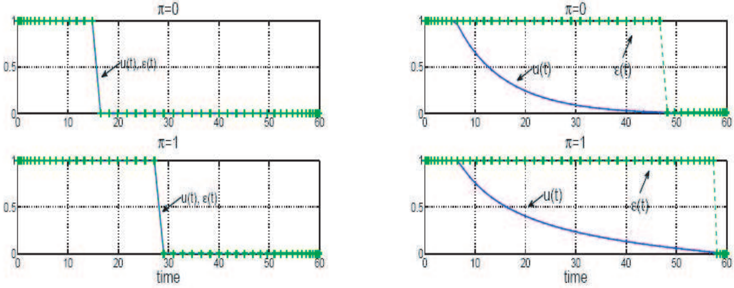
Theorem

For either one of the following two cases: (i) $L \equiv 0$ and $f(\cdot)$ is convex, (ii) $\delta = 0$, an optimal control $(\epsilon(\cdot), \mu(\cdot))$ has the following simple structure:
1) When $h(\cdot)$ is concave, $\exists t_1 \in [0 \dots T]$ such that (a) $\mu(t) = 1$ for $0 < t < t_1$, and (b) $\mu(t) = 0$ for $t_1 < t < T$.
2) When $h(\cdot)$ is strictly convex, $\exists t_0, t_1, 0 \leq t_0 \leq t_1 \leq T$ such that (a) $\mu(t) = 1$ for $0 < t \leq t_0$, (b) $\mu(t)$ strictly and continually decreases on $t_0 < t < t_1$, and (c) $\mu(t) = 0$ on $t_1 \leq t \leq T$.

Illustration of the theorems



(a) Concave $h(u)$, Non-replicative

(b) Convex $h(u)$, Replicative

Fig. 2: Illustration of the theorems. The common parameters are $\delta = 0.01$, $\beta = 0.15$, $I_0 = 0.2$, $R_0 = 0.25$, $D_0 = 0$, $T = 60$, $f(I) = 5I$, $g(D) = 10D$, $L(R) = 5R$. For concave $h(u)$ (fig.2(a)) we have used $h(u) = 10u$, and for convex $h(u)$ (fig.2(b)) we have used $h(u) = 10u^2$.

Conclusions

- The paper reveals the property of the optimal control of epidemic evolution
- The paper does not implement the optimal control of epidemic evolution