CHAPTER 2

The Poisson Process

2.1 The Poisson Process

A stochastic process $\{N(t),\, t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of 'events' that have occurred up to time t. Hence, a counting process $N(t)$ must satisfy:

(i) $N(t) \geq 0$.

(ii) $N(t)$ is integer valued.

(iii) If $s < t$, then $N(s) \leq N(t)$.

(iv) For $s < t$, $N(t) - N(s)$ equals the number of events that have occurred in the interval $(s, t]$.

A counting process is said to possess *independent increments* if the numbers of events that occur in disjoint time intervals are independent. For example, this means that the number of events that have occurred by time t (that is, $N(t)$) must be independent of the number of events occurring between times t and $t + s$ (that is, $N(t + s) - N(t)$).

A counting process is said to possess *stationary increments* if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval. In other words, the process has stationary increments if the number of events in the interval $(t_1 + s, t_2 + s]$ (that is, $N(t_2 + s) - N(t_1 + s)$) has the same distribution as the number of events in the interval $(t_1, t_2]$ (that is, $N(t_2) - N(t_1)$) for all $t_1 < t_2$, and $s > 0$.

One of the most important types of counting processes is the Poisson process, which is defined as follows.

Definition 2.1.1

The counting process $\{N(t), t \geq 0\}$ is said to be a *Poisson process having rate* λ, $\lambda > 0$, if:

(i) $N(0) = 0$.

(ii) The process has independent increments.

(iii) The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geq 0$,

$$P\{N(t+s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \qquad n = 0, 1, \ldots.$$

Note that it follows from condition (iii) that a Poisson process has stationary increments and also that

$$E[N(t)] = \lambda t,$$

which explains why λ is called the rate of the process.

In order to determine if an arbitrary counting process is actually a Poisson process, we must show that conditions (i), (ii), and (iii) are satisfied. Condition (i), which simply states that the counting of events begins at time $t = 0$, and condition (ii) can usually be directly verified from our knowledge of the process. However, it is not at all clear how we would determine that condition (iii) is satisfied, and for this reason an equivalent definition of a Poisson process would be useful.

As a prelude to giving a second definition of a Poisson process, we shall define the concept of a function f being $o(h)$.

Definition

The function f is said to be $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0.$$

We are now in a position to give an alternative definition of a Poisson process.

Definition 2.1.2

The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate λ, $\lambda > 0$, if:

(i) $N(0) = 0$.
(ii) The process has stationary and independent increments.
(iii) $P\{N(h) = 1\} = \lambda h + o(h)$.
(iv) $P\{N(h) \geq 2\} = o(h)$.

THEOREM 2.1.1

Definitions 2.1.1 and 2.1.2 are equivalent.

Proof We first show that Definition 2.1.2 implies Definition 2.1.1. To do this let

$$P_n(t) = P\{N(t) = n\}.$$

We derive a differential equation for $P_0(t)$ in the following manner:

$$\begin{aligned}P_0(t+h) &= P\{N(t+h) = 0\}\\ &= P\{N(t) = 0, N(t+h) - N(t) = 0\}\\ &= P\{N(t) = 0\}P\{N(t+h) - N(t) = 0\}\\ &= P_0(t)[1 - \lambda h + o(h)],\end{aligned}$$

where the final two equations follow from Assumption (ii) and the fact that (iii) and (iv) imply that $P\{N(h) = 0\} = 1 - \lambda h + o(h)$. Hence,

$$\frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \frac{o(h)}{h}.$$

Letting $h \to 0$ yields

$$P_0'(t) = -\lambda P_0(t)$$

or

$$\frac{P_0'(t)}{P_0(t)} = -\lambda,$$

which implies, by integration,

$$\log P_0(t) = -\lambda t + c$$

or

$$P_0(t) = Ke^{-\lambda t}.$$

Since $P_0(0) = P\{N(0) = 0\} = 1$, we arrive at

(2.1.1) $$P_0(t) = e^{-\lambda t}.$$

Similarly, for $n \geq 1$,

$$\begin{aligned}P_n(t+h) &= P\{N(t+h) = n\}\\ &= P\{N(t) = n, N(t+h) - N(t) = 0\}\\ &\quad + P\{N(t) = n-1, N(t+h) - N(t) = 1\}\\ &\quad + P\{N(t+h) = n, N(t+h) - N(t) \geq 2\}.\end{aligned}$$

However, by (iv), the last term in the above is $o(h)$; hence, by using (ii), we obtain

$$\begin{aligned} P_n(t+h) &= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h) \\ &= (1-\lambda h)P_n(t) + \lambda h P_{n-1}(t) + o(h). \end{aligned}$$

Thus,

$$\frac{P_n(t+h)-P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}.$$

Letting $h \to 0$,

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t),$$

or, equivalently,

$$e^{\lambda t}[P_n'(t) + \lambda P_n(t)] = \lambda e^{\lambda t} P_{n-1}(t).$$

Hence,

$$\frac{d}{dt}(e^{\lambda t}P_n(t)) = \lambda e^{\lambda t}P_{n-1}(t). \tag{2.1.2}$$

Now by (2.1.1) we have when $n = 1$

$$\frac{d}{dt}(e^{\lambda t}P_1(t)) = \lambda$$

or

$$P_1(t) = (\lambda t + c)e^{-\lambda t},$$

which, since $P_1(0) = 0$, yields

$$P_1(t) = \lambda t e^{-\lambda t}.$$

To show that $P_n(t) = e^{-\lambda t}(\lambda t)^n/n!$, we use mathematical induction and hence first assume it for $n - 1$. Then by (2.1.2),

$$\frac{d}{dt}(e^{\lambda t}P_n(t)) = \frac{\lambda(\lambda t)^{n-1}}{(n-1)!}$$

implying that

$$e^{\lambda t}P_n(t) = \frac{(\lambda t)^n}{n!} + c,$$

or, since $P_n(0) = P\{N(0) = n\} = 0$,

$$P_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

Thus Definition 2.1.2 implies Definition 2.1.1. We will leave it for the reader to prove the reverse.

Remark The result that $N(t)$ has a Poisson distribution is a consequence of the Poisson approximation to the binomial distribution. To see this subdivide the interval $[0, t]$ into k equal parts where k is very large (Figure 2.1.1). First we note that the probability of having 2 or more events in any subinterval goes to 0 as $k \to \infty$. This follows from

$$\begin{aligned} &P\{2 \text{ or more events in any subinterval}\} \\ &\quad \le \sum_{i=1}^{k} P\{2 \text{ or more events in the } i\text{th subinterval}\} \\ &\quad = ko(t/k) \\ &\quad = t\frac{o(t/k)}{t/k} \\ &\quad \to 0 \qquad \text{as } k \to \infty. \end{aligned}$$

Hence, $N(t)$ will (with a probability going to 1) just equal the number of subintervals in which an event occurs. However, by stationary and independent increments this number will have a binomial distribution with parameters k and $p = \lambda t/k + o(t/k)$. Hence by the Poisson approximation to the binomial we see by letting k approach ∞ that $N(t)$ will have a Poisson distribution with mean equal to

$$\lim_{k\to\infty} k\left[\lambda\frac{t}{k} + o\left(\frac{t}{k}\right)\right] = \lambda t + \lim_{k\to\infty}\left[t\frac{o(t/k)}{t/k}\right] = \lambda t.$$

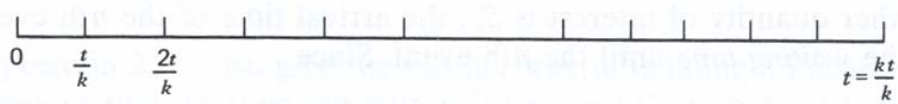

Figure 2.1.1

2.2 Interarrival and Waiting Time Distributions

Consider a Poisson process, and let X_1 denote the time of the first event. Further, for $n \geq 1$, let X_n denote the time between the $(n-1)$st and the nth event. The sequence $\{X_n, n \geq 1\}$ is called the *sequence of interarrival times.*

We shall now determine the distribution of the X_n. To do so we first note that the event $\{X_1 > t\}$ takes place if, and only if, no events of the Poisson process occur in the interval $[0, t]$, and thus

$$P\{X_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}.$$

Hence, X_1 has an exponential distribution with mean $1/\lambda$. To obtain the distribution of X_2 condition on X_1. This gives

$$\begin{aligned} P\{X_2 > t \mid X_1 = s\} &= P\{0 \text{ events in } (s, s+t] \mid X_1 = s\} \\ &= P\{0 \text{ events in } (s, s+t]\} \qquad \text{(by independent increments)} \\ &= e^{-\lambda t} \qquad \text{(by stationary increments).} \end{aligned}$$

Therefore, from the above we conclude that X_2 is also an exponential random variable with mean $1/\lambda$, and furthermore, that X_2 is independent of X_1. Repeating the same argument yields the following.

PROPOSITION 2.2.1

X_n, $n = 1, 2, \ldots$ are independent identically distributed exponential random variables having mean $1/\lambda$.

Remark The proposition should not surprise us. The assumption of stationary and independent increments is equivalent to asserting that, at any point in time, the process *probabilistically* restarts itself. That is, the process from any point on is independent of all that has previously occurred (by independent increments), and also has the same distribution as the original process (by stationary increments). In other words, the process has no *memory,* and hence exponential interarrival times are to be expected.

Another quantity of interest is S_n, the arrival time of the nth event, also called the *waiting time* until the nth event. Since

$$S_n = \sum_{i=1}^{n} X_i, \qquad n \geq 1,$$

it is easy to show, using moment generating functions, that Proposition 2.2.1 implies that S_n has a gamma distribution with parameters n and λ. That is, its probability density is

$$f(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \qquad t \geq 0.$$

The above could also have been derived by noting that the nth event occurs prior or at time t if, and only if, the number of events occurring by time t is at least n. That is,

$$N(t) \geq n \Leftrightarrow S_n \leq t.$$

Hence,

$$\begin{aligned} P\{S_n \leq t\} &= P\{N(t) \geq n\} \\ &= \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}, \end{aligned}$$

which upon differentiation yields that the density function of S_n is

$$\begin{aligned} f(t) &= -\sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} + \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!} \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}. \end{aligned}$$

Remark Another way of obtaining the density of S_n is to use the independent increment assumption as follows:

$$\begin{aligned} P\{t < S_n < t + dt\} &= P\{N(t) = n-1, 1 \text{ event in } (t, t+dt)\} + o(dt) \\ &= P\{N(t) = n-1\} P\{1 \text{ event in } (t, t+dt)\} + o(dt) \\ &= \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \lambda dt + o(dt) \end{aligned}$$

which yields, upon dividing by $d(t)$ and then letting it approach 0, that

$$f_{S_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$$

Proposition 2.2.1 also gives us another way of defining a Poisson process. For suppose that we start out with a sequence $\{X_n, n \geq 1\}$ of independent identically distributed exponential random variables each having mean $1/\lambda$. Now let us define a counting process by saying that the nth event of this process occurs at time S_n, where

$$S_n \equiv X_1 + X_2 + \cdots + X_n.$$

The resultant counting process $\{N(t), t \geq 0\}$ will be Poisson with rate λ.

2.3 Conditional Distribution of the Arrival Times

Suppose we are told that exactly one event of a Poisson process has taken place by time t, and we are asked to determine the distribution of the time at which the event occurred. Since a Poisson process possesses stationary and independent increments, it seems reasonable that each interval in $[0, t]$ of equal length should have the same probability of containing the event. In other words, the time of the event should be uniformly distributed over $[0, t]$. This is easily checked since, for $s \leq t$,

$$\begin{aligned}
P\{X_1 < s | N(t) = 1\} &= \frac{P\{X_1 < s, N(t) = 1\}}{P\{N(t) = 1\}} \\
&= \frac{P\{1 \text{ event in } [0, s), 0 \text{ events in } [s, t)\}}{P\{N(t) = 1\}} \\
&= \frac{P\{1 \text{ event in } [0, s)\}P\{0 \text{ events in } [s, t)\}}{P\{N(t) = 1\}} \\
&= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} \\
&= \frac{s}{t}.
\end{aligned}$$

This result may be generalized, but before doing so we need to introduce the concept of order statistics.

Let $Y_1, Y_2, \ldots, Y_n$ be n random variables. We say that $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ are the order statistics corresponding to $Y_1, Y_2, \ldots, Y_n$ if $Y_{(k)}$ is the kth smallest value among $Y_1, \ldots, Y_n$, $k = 1, 2, \ldots, n$. If the Y_i's are independent identically distributed continuous random variables with probability density f, then the joint density of the order statistics $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ is given by

$$f(y_1, y_2, \ldots, y_n) = n! \prod_{i=1}^{n} f(y_i), \qquad y_1 < y_2 < \cdots < y_n.$$

The above follows since (i) $(Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)})$ will equal $(y_1, y_2, \ldots, y_n)$ if $(Y_1, Y_2, \ldots, Y_n)$ is equal to any of the $n!$ permutations of $(y_1, y_2, \ldots, y_n)$,