8 Network Control

8.1 Introduction

Since the pioneering work of Wiener [1], automatic control theory and technology have been rapidly developed, and are recently being further extended from a single (albeit higher-dimensional) dynamical system to a network of many (higher-dimensional) dynamical systems mutually connected in various topologies.

Chaos control, as a special subject in general control theory and engineering, aims at utilizing the extreme sensitivity of chaotic dynamics to system initial conditions to achieve significant effects by tiny control actions [2,3]. Synchronization of two coupled chaotic oscillators, or a network of coupled chaotic oscillators thereafter, have been investigated, as briefly discussed in Chapter 7. When a given network of dynamical systems is not synchronizable, control becomes a necessary means for guiding or forcing the network to synchronize. In the past, regarding the special structures of complex networks, a simple and effective control strategy named "pinning control" was commonly used [4,5]. The pinning control technique was at the beginning to add a constant control to a small fraction of nodes in the network, where the constant control input remains unchanged (just like a "pin") until the end of the control process when the control objective such as synchronization is achieved. Lately, this constant control input had been extended to a general controller such as state-feedback, time-delay feedback, nonlinear feedback, etc., which in a broader sense can be any kind of conventional controllers [6-10]. No matter what kind of controllers are used, pinning control differs from many other control strategies such as switching control and impulsive control in that pinning controllers will not be disconnected or turned off throughout the entire control process after being put at a fraction of nodes of the network.

In the following, some special pinning control examples are first reviewed.

8.2 Spatiotemporal Chaos Control on Regular CML

Consider a continuous-time nearest-coupled network of the form

$$\begin{cases} \dot{x}_i = f(x_i) + \dfrac{\varepsilon}{2}\left(x_{i+1} + x_{i-1} - 2x_i\right) + \dfrac{r}{2}\left(x_{i-1} - x_{i+1}\right), \qquad i = 1,2,\ldots,N-1 \\ \dot{x}_0 = f(x_0) + \dfrac{\varepsilon}{2}\left(x_1 + x_{N-2} - 2x_0\right) + \dfrac{r}{2}\left(x_{N-2} - x_1\right) - \lambda x_0 \end{cases} \tag{8-1}$$

where $f: R \to R$ is a chaotic map with state variable $x_i \in R$ of node i, satisfying $f(0) = 0$ for simplicity, $\varepsilon > 0$ is the coupling strength, and $r \geq 0$ and $\lambda \geq 0$ are the control gains of the state-feedback controllers $\frac{1}{2}(x_{i-1}-x_i)$ and $-x_0$, respectively, with periodic boundary conditions as described by the second equation in (8-1), $i=1,2,...,N$.

The objective is to stabilize the network to its zero equilibrium, in the sense that

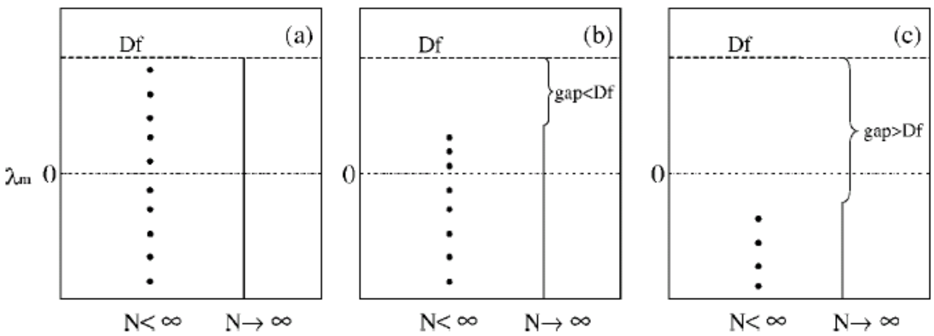
$$\lim_{t\to\infty} \| x_i \| = 0, \quad i=1,2,...,N$$

or, to achieve network synchronization, in the sense that

$$\lim_{t\to\infty} \| x_i - x_j \| = 0, \quad i,j=1,2,...,N$$

where $\|\cdot\|$ is the Euclidean norm of the state vector (for a scalar variable, it is just the absolute value).

Simulations show [11] that for $r=0$, it is impossible to achieve control, as shown by Fig. 8-1 (a) [11], where λ_m is the maximum Lyapunov exponent ($\lambda_m>0$ means the network is chaotic, $\lambda_m=0$ means the network is oscillatory, $\lambda_m<0$ means the networked can be stabilized to its equilibrium). As $r>0$ is increased to be larger than a threshold value $r_c>0$, the network is being controlled to its equilibrium, as shown by Fig. 8-1 (c) [11].



(a) $r=0$; (b) $0<r<r_c$; (c) $r>r_c$

Fig. 8-1 Control performance of pinning a nearest-coupled network [11]

The above investigation can be easily extended to the discrete-time setting. Consider a discrete-time Coupled Map Lattice (CML) of the form [4]

$$x_{n+1}(i) = (1-\varepsilon) f[x_n(i)] + \frac{\varepsilon}{2}\left\{ f[x_n(i-1)] + f[x_n(i+1)]\right\} \tag{8-2}$$

or, when pinning control is added,

$$x_{n+1}(i) = (1-\varepsilon) f[x_n(i)] + \frac{\varepsilon}{2}\{f[x_n(i-1)] + f[x_n(i+1)]\} + \sum_{k=0}^{L/I} \delta(i - Ik - 1) g_n \qquad (8\text{-}3)$$

where same notations are defined as above; L is the total number of nodes; I is the distance between two controlled nodes; g_n is a controller; δ determines if a controller is to be added: $\delta(j) = 1$ for $j = 0$, and $\delta(j) = 0$ otherwise. Clearly, in this controlled CML, there are a total of L/I controllers used.

Research shows that in order to effectively control a CML, to its equilibrium for example, one must use a large enough number of pinning controllers [4]. Also, this pinning control strategy can be applied to achieve network synchronization [12], where numerical simulations show that for this kind of discrete-time regular CML, the fraction of controllers used depends on the control gains of the controllers: the smaller the control gains are, the more the pinning controllers will be needed in general.

The above pinning controller is of closed-loop state-feedback type. One may also consider using open-loop pinning controllers such as the following [13]:

$$x_{n+1}(i) = (1-\varepsilon) f(x_n(i)) + \frac{\varepsilon}{2}[f(x_n(i-1)) + f(x_n(i+1))] + p_n(i) \qquad (8\text{-}4)$$

where $p_n(i)$ represents the pinning input strength of node i at time n, with $p_n(i) = \delta(i - i_p) p$, in which $p > 0$ is a constant, and $\delta(j) = 1$ for $j = 0$ and $\delta(j) = 0$ otherwise, therefore only nodes i_p are pinned by $p_n(i_p) = p$ and the other nodes receive $p_n(i) = 0$.

Simulations show [13] that both uniformly and randomly distributed pinning control input strengths p can stabilize the spatiotemporal network to its equilibrium, and that the number of controllers and the control input strength both have significant effects on the control performance.

As seen above, when controlling a network (or a system) from a chaotic state to a non-chaotic or even an equilibrium state, it is referred to as "control of chaos", while the opposite task, i.e., controlling a network (or system) from a regular state to a chaotic state, is called "anti-control of chaos" [3,14].

Some study [15] shows that pinning can also achieve anti-control of chaos in the above CML network (8-4), with $f(\cdot)$ being a logistic map. Figure 8-2 (a) [15] shows that the spatiotemporal chaotic CML (left) is controlled to a periodic state (middle) and non-chaotic state (right); while Fig. 8-2 (b) and (c) [15] show that the CML is controlled to chaotic and strongly chaotic states, respectively, all by means of pinning.

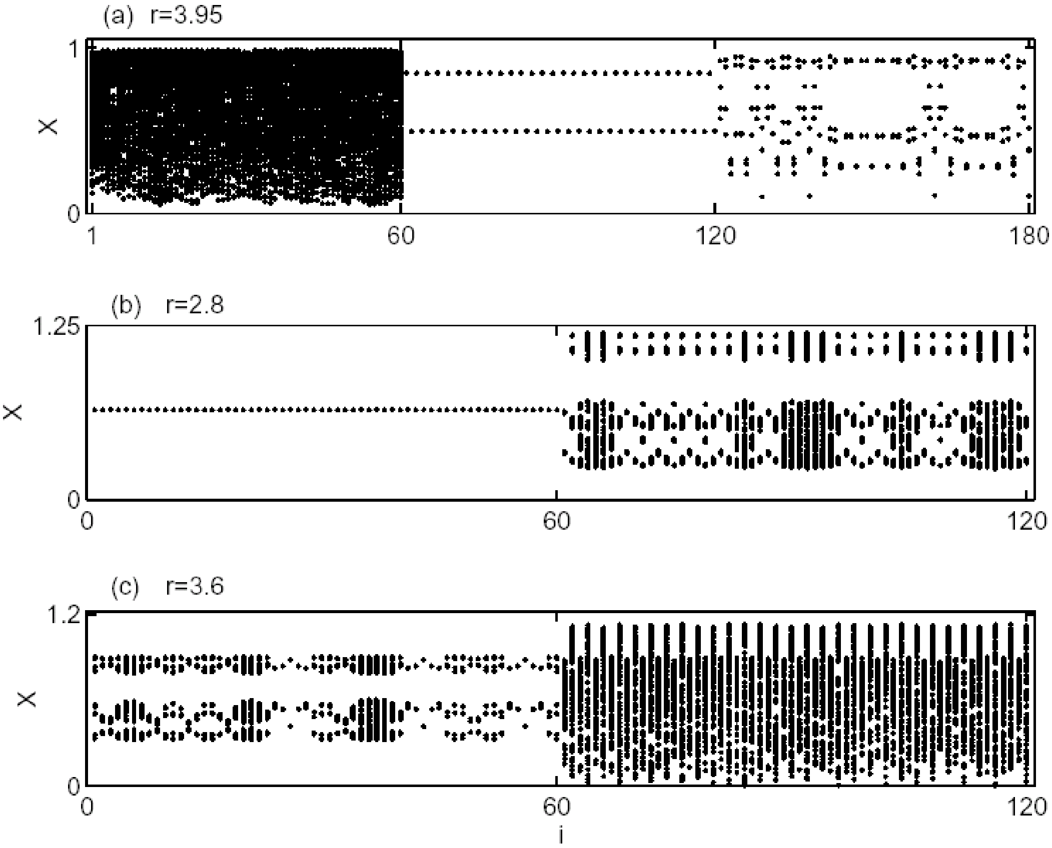


Fig. 8-2 Chaos control and anti-control of CML network by pinning [15]

8.3 Pinning Control of Scale-Fee Networks: Robustness and Fragility

Consider the following linearly and diffusively coupled scale-free network [6]

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij}\Gamma x_j, \qquad i = 1,2,\cdots,N \tag{8-5}$$

where, as studied in Chapter 7, $x_i = \left[x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(n)}\right]^T \in R^n$ is the state vector, constant $c > 0$ is the coupling strength, $\Gamma = diag\{r_1, r_2, ..., r_n\}$ is the inner coupling matrix describing the connections among different components of a state vector, and $A = \left[a_{ij}\right] \in R^{N\times N}$ is the outer coupling matrix satisfying the following conditions: if there is a connection between node i and node j, for $i \neq j$, then $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij}a_{ji} = 0$, for $i \neq j$; and the diagonal elements are defined by

$$a_{ii} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ji} = -k_i, \qquad i = 1, 2, \cdots, N \tag{8-6}$$

in which k_i is the degree of node i, $i = 1,2,...,N$.

Recall from Chapter 7 that for a connected network, matrix A is a symmetric and irreducible matrix, with a zero eigenvalue of multiplicity 1 and all other eigenvalues being strictly negative. It should be noted, however, that unlike the study of network self-synchronization, which was investigated in Chapter 7, since external control inputs are involved now, the eigenvalues of the outer coupling matrix A (hence, those of the Laplacian matrix $L = -A$) are not as important any more. Instead, the eigenvalues of the linearized controlled network become most significant, in which the controller gains will be involved as further discussed below. For this reason, the minus sign in front of the couplings in the network model (8-5) needs not be negative when control inputs are added to the network, therefore for notational convenience it will be changed to plus, as shown in the controlled network (8-8) below.

Now, return to network (8-5). The objective here is, once again, to control the network to its equilibrium $\bar{x}$, which satisfies $f(\bar{x}) = 0$:

$$x_1 = x_2 = \cdots = x_N = \bar{x} \tag{8-7}$$

Assume that only a fraction δ of nodes are pinned, $0 < \delta << 1$, and let these nodes be labeled as $i_1, i_2, \ldots, i_l$, where $l = \lfloor \delta N \rfloor$ is the integer part of the real number δN. Thus, the controlled network can be written as

$$\begin{cases} \dot{x}_{i_k} = f(x_{i_k}) - c\sum_{j=1_k}^{N} a_{i_k j}\Gamma x_j - cd\Gamma\left(x_{i_k} - \bar{x}\right), & k = 1,2,\cdots,l \\ \dot{x}_{i_k} = f(x_{i_k}) - c\sum_{j=1_k}^{N} a_{i_k j}\Gamma x_j, & k = l+1, l+2,\cdots,N \end{cases} \tag{8-8}$$

where $d > 0$ is the control gain of the linear state-feedback controller $\left(x_{k_k} - \bar{x}\right)$, $k = 1,2,...,l$.

Linearizing network (8-8) around its equilibrium $\bar{X} = \left[\bar{x}^T \quad \bar{x}^T \quad \cdots \quad \bar{x}^T\right]^T$ gives the following linear matrix equation:

$$\dot{\eta} = \eta\left[Df(\bar{x})\right] - cB\eta\Gamma \tag{8-9}$$

where $Df(\bar{x})$ is the Jacobi matrix of $f(x)$ evaluated at $\bar{x}$, $\eta = \left(\eta_1, \eta_2, \cdots, \eta_N\right)^T$, $\eta_i(t) = x_i(t) - \bar{x}$, matrix $B = A - D$ and matrix $D = diag(d_1, d_2, \cdots, d_N)$, in which for $1 \le k \le l$, $d_{i_k} = d$, and for $l+1 \le k \le N$, $d_i = 0$.

Thus, the problem of controlling the network (8-8) to its equilibrium $\bar{x}$ is studied locally around $\bar{x}$, which is transformed to be a stabilization problem for system (8-9). Consequently, the classical linear stability theory can be applied, leading to the conclusion that if there exists a constant $\rho < 0$ such that $[Df(\bar{x}) + \rho\Gamma]$ is Hurwitz stable and that

$$c \ge \left| \frac{\rho}{\lambda_1(B)} \right| \tag{8-10}$$

then the dynamical system (8-8) will be controlled to its equilibrium $\bar{x}$, where $\lambda_1(B)$ is the largest eigenvalue of matrix B.

Notice that as $d \to \infty$, one has $\lim_{d\to\infty} \lambda_1(B) = \lambda_1(\tilde{A})$, where the matrix $\tilde{A}$ is obtained from the outer coupling matrix A by removing those rows and columns that correspond to the controlled nodes $i_1, i_2, \ldots, i_l$. When $d \to \infty$, the stability of the equilibrium of network (8-8) is equivalent to the stability of the following dynamical systems:

$$\begin{cases} \dot{x}_{i_k} = \bar{x}, \qquad k = 1,2,\cdots,l \\ \dot{x}_{i_k} = f(x_{i_k}) - c\sum_{j=1_k}^{N} a_{i_k j}\Gamma x_j, \qquad k = l+1, l+2, \cdots, N \end{cases} \tag{8-11}$$

In this case, the stability condition (8-10) becomes

$$c \ge \left| \frac{\rho}{\lambda_1(\tilde{A})} \right| \tag{8-12}$$

For scale-free networks, the pinning control strategy can be classified into two different types, *random pinning* and *selective pinning*, where the former means to randomly pin l nodes among the N nodes of the network while the latter means to successively pin the l big nodes with the largest degree, second largest degree, and so on.

To evaluate the control performance, according to condition (8-12), one may compare different control schemes by examining their corresponding largest eigenvalues $\lambda_{1r}(\tilde{A})$ and $\lambda_{1s}(\tilde{A})$ versus δ (i.e., versus l). In a BA scale-free network (8-8) with $N = 3{,}000$, simulation shows that it suffices to control only a few large-degree nodes to stabilize such a network, as shown in Fig. 8-3 [6], where solid curve corresponds to random pinning and dash curve to selective pinning. It can be seen that selective pinning is much more effective than random pinning for scale-free networks, consistent with the experience about the robustness and fragility of this kind of networks.

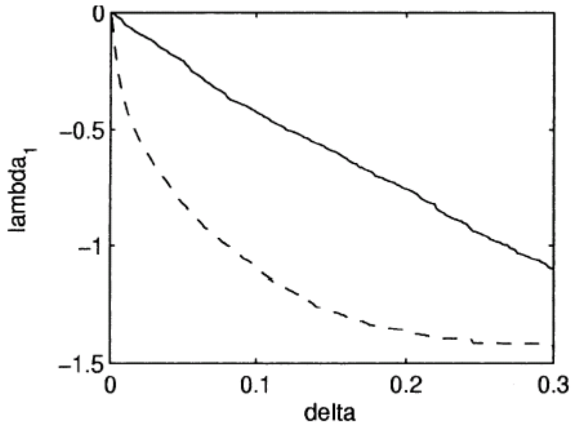


Fig. 8-3 Comparison of two pinning schemes on scale-free networks [6]

Example 8-1 [6]

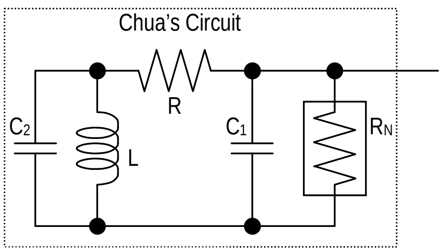
Consider a scale-free network of Chua's circuits shown in Fig. 8-4 (a), with dynamical equations

$$\begin{aligned} C_1\dot{v}_{C_1} &= R^{-1}(v_{C_2} - v_{C_1}) - f(v_{C_1}) \\ C_2\dot{v}_{C_2} &= R^{-1}(v_{C1} - v_{C2}) + i_L \\ L\,\dot{i}_L &= -v_{C_2} \end{aligned} \tag{8-13}$$

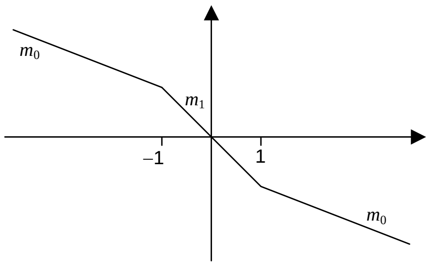
where L is a conductor, i_L is the current through L, C_1 and C_2 are two capacitors, v_{C_1} and v_{C_2} are the voltages across C_1 and C_2, respectively, R is a resistor, and R_N is a nonlinear resistor described by $f(\cdot)$ in (8-13) with expression

$$f(v_{C_1}) = m_0 v_{C_1} + \frac{1}{2}(m_1 - m_0)\ (\ |v_{C_1}+1| - |v_{C_1}-1|\ )$$

in which the two constants $m_0 < 0$ and $m_1 > 0$, as shown in Fig. 8-4 (b).



(a) Chua's circuit



(b) piecewise linear function

Fig. 8-4 Chua's circuit and its nonlinear resistor

This circuit can produce complex chaotic behaviors, as shown in Fig. 8-5.

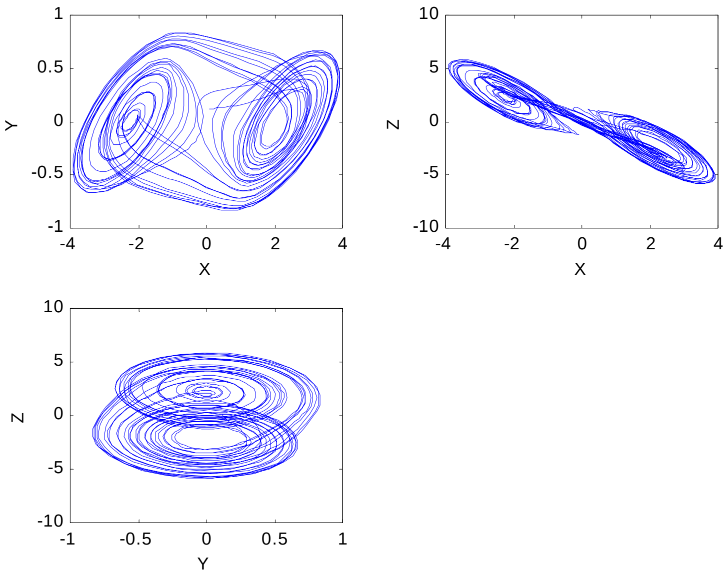


Fig. 8-5 Chaotic attractor of Chua's circuit

The circuit equation (8-13) can be rewritten in a dimensionless form via a simple linear nonsingular transformation, as follows:

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} \alpha(x_{i1} - x_{i2} + f(x_{i1})) - c\sum_{j=1}^{N} a_{ij} x_{i1} \\ x_{i1} - x_{i2} + x_{i3} \\ -\beta x_{i2} + \gamma x_{i3} \end{pmatrix}, \qquad i = 1,2,\cdots,N$$

where

$$f(x_1) = \begin{cases} -bx_1 - a + b & x_1 > 1 \\ -ax_1 & |x_1| < 1 \\ -bx_1 + a - b & x_1 < -1 \end{cases}$$

and, with parameters

$$\alpha = 10, \quad \beta = 15, \quad \gamma = 0.0385, \quad a = -1.27, \quad b = -0.68$$

the circuit has three unstable equilibrium points:

$$x^{\pm} = \begin{bmatrix} \pm 1.8586 & \pm 0.0048 & \mp 1.8539 \end{bmatrix}^T, \qquad x^0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

The objective here is to control the circuit to its equilibrium x^+ by pinning:

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} \alpha(x_{i1} - x_{i2} + f(x_{i1})) - c\sum_{j=1}^{N} a_{ij} x_{i1} + u_i \\ x_{i1} - x_{i2} + x_{i3} \\ -\beta x_{i2} + \gamma x_{i3} \end{pmatrix}, \qquad i = 1,2,\cdots,N$$

where the controllers are

$$u_i = \begin{cases} cd\left(x_1^+ - x_{i1}\right) & i = i_1,\dots,i_l \\ 0 & otherwise \end{cases}$$

It follows from condition (8-12) that $\rho = -4.71$. Under the two pinning control schemes, the results are compared in Fig. 8-6, where the dash curve corresponds to random pinning while the solid curve to selective pinning.

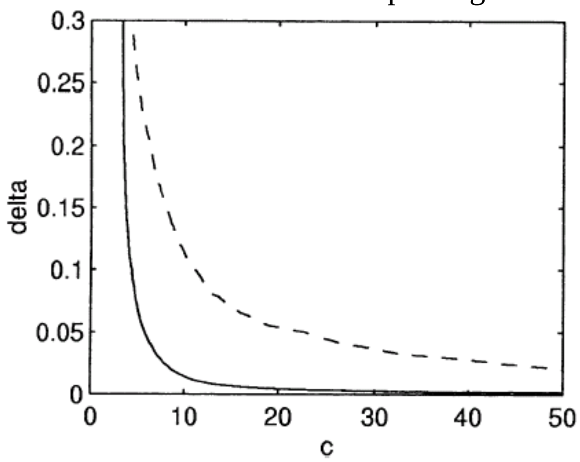


Fig. 8-6 Comparison of two pinning schemes on a network of Chua's circuit [6]

8.4 Pinning Control of General Complex Networks

8.4.1 Stability Analysis of General Networks under Pinning Control

Consider a general network

$$\dot{x}_i = f(x_i) - \sum_{\substack{j=1 \\ j \neq i}}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i), \qquad i = 1,2,\cdots,N \tag{8-14}$$

where all notations are defined as above, and $c_{ij} > 0$ is the coupling strength of the connection between node i and node j, $i, j = 1,2,...,N$.

In order to pin this network to its equilibrium $\bar{x}$, consider the following pinning controlled model:

$$\begin{cases} \dot{x}_{i_k} = f(x_{i_k}) - \sum c_{i_k j} a_{i_k j} \Gamma(x_j - x_{i_k}) - c_{i_k i_k} d_{i_k} \Gamma(x_{i_k} - \bar{x}), & k = 1,2,...,l \\ \dot{x}_{i_k} = f(x_{i_k}) - \sum c_{i_k j} a_{i_k j} \Gamma(x_j - x_{i_k}), & k = l+1, l+2,...,N \end{cases} \tag{8-15}$$

where the control gains $d_{i_k} > 0$, assuming the diffusive condition

$$c_{i_k i_k} a_{i_k i_k} + \sum_{\substack{j=1 \\ j \neq i}}^{N} c_{i_k j} a_{i_k j} = 0, \quad k = 1,2,...,l, \quad i, j = 1,2,...,N$$

Without loss of generality, re-label the nodes such that the controlled nodes are arranged to be listed first: $i_k = k$, $k = 1,2,...,l$, and define

$$D = diag\{d_1,...,d_l,0,...,0\} \in R^{N \times N}$$

$$D' = diag\{c_{11}d_1,...,c_{ll}d_l,0,...,0\} \in R^{N \times N}$$

Moreover, using the Kronecker notation, one can write

$$\dot{x} = f(x) - [(G+D) \otimes \Gamma]x + (D' \otimes \Gamma)\bar{X} = I_N \otimes f(x_i) - [(G+D) \otimes \Gamma]x + (D' \otimes \Gamma)\bar{X} \tag{8-16}$$

where $G = [g_{ij}] \in R^{N \times N}$ is a symmetric and semi-positive definite matrix, with $g_{ij} = -c_{ij} a_{ij}$, $i, j = 1,2,...,N$, and $G + D$ is positive definite, with the smallest eigenvalue $\lambda_{\min}(G+D) > 0$.

Theorem 8-1 [6] *Suppose that* $f(x)$ *is Lipschitz continuous with Lipschitz constant* $L_c^f > 0$, *matrix* Γ *is symmetric and positive definite. If*

$$\lambda_{\min}(G+D) > \alpha \equiv \frac{L_c^f}{\lambda_{\min}(\Gamma)} \tag{8-17}$$

where $\lambda_{\min}(\Gamma)$ *and* $\lambda_{\min}(G+D)$ *are the smallest eigenvalues of matrices* Γ *and*

$G+D$, respectively, then the equilibrium $\bar{x}$ of the pinning controlled network (8-15) *is globally and asymptotically stable.*

Condition (8-17) reveals a sufficient condition that the coupling strength matrix $C_{couple} = [c_{ij}]^{N\times N}$ needs to satisfy in order to guarantee the equilibrium of network (8-15) to be globally and asymptotically stable.

In the special case where $c_{ij} = c$ and $d_i = cd$, condition (8-17) reduces to

$$c > \frac{L_c^f}{\lambda_{\min}(-A + diag(d,\ldots,d,0,\ldots,0)) \cdot \lambda_{\min}(\Gamma)} \tag{8-18}$$

and, if $\Gamma = [\tau_{ij}] \in R^{n\times n}$ is a 0-1matrix, then the above condition furthermore reduces to

$$c > \frac{L_c^f}{\lambda_{\min}(-A + diag(d,\ldots,d,0,\ldots,0))} \tag{8-19}$$

For a pinning controlled chaotic network (8-15), the Lyapunov exponent of each isolated node $\dot{x}_i = f(x_i)$ can characterize the local stability of the network, as follows.

Theorem 8-2 [6] *Consider the pinning control network* (8-15) *of chaotic nodes described by $\dot{x}_i = f(x_i)$, $i = 1,2,\cdots,N$, with the maximum Lyapunov exponent $h_{\max} > 0$. If $c_{ij} = c$, $d_i = cd$, $\Gamma = I_m$, and*

$$c > \frac{h_{\max}}{\lambda_{\min}(-A + diag(d,\ldots,d,0,\ldots,0))} \tag{8-20}$$

then the network equilibrium is locally asymptotically stable.

It can be seen that in many cases the constant ρ in conditions (8-10) and (8-12) can be taken to be the Lipschitz constant $L_c^f > 0$ or the maximum Lyapunov exponent $h_{\max} > 0$, but in general the condition (8-20) has to be satisfied by choosing some appropriate coupling strengths $C_{couple} = [c_{ij}]^{N\times N}$.

8.4.2 Pinning Control and Virtual Control of General Networks

Consider the simple case where $c_{ij} = c$ and $d_i = cd$ in the network (8-15), and rewrite it as

$$\begin{aligned} \dot{x}_i &= f(x_i) - c\sum_{j=1}^{N} a_{ij}\Gamma x_j - cd\Gamma(x_i - \bar{x}), \qquad i = 1,2,\cdots,l \\ \dot{x}_i &= f(x_i) - c\sum_{j=1}^{N} a_{ij}\Gamma x_j, \qquad i = l+1, l+2,\cdots,N \end{aligned} \tag{8-21}$$

According to the above analysis, if

$$\begin{cases} \Gamma = I_m \\ c > \dfrac{C}{\lambda_{\min}(-A + diag(d,\ldots,d,0,\ldots 0))} \end{cases} \tag{8-22}$$

then the equilibrium of network (8-21) is globally asymptotically stable when $C = L_c^f$ and is locally asymptotically stable when $C = h_{\max}$.

Recall also from the last subsection that as the control gain $d \to +\infty$, the network can be stabilized to its equilibrium by choosing appropriate nodes to pin. Notice that

$$a_{ii} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij} = -\left(\sum_{\substack{j=l+1 \\ j\neq i}}^{N} a_{ij} + \sum_{j=1}^{l} a_{ij} \right), \quad i = l+1, l+2, \ldots, N$$

therefore, as $d \to +\infty$, network (8-21) becomes [9]

$$\begin{cases} x_i = \bar{x}, \qquad i = 1,2,\ldots,l \\ \dot{x}_i = f(x_i) - c\sum_{j=l+1}^{N} \tilde{b}_{ij}\Gamma x_j + \tilde{u}_i, \qquad i = l+1, l+2,\ldots,N \end{cases} \tag{8-23}$$

where $\tilde{B} = \left[\tilde{b}_{ij}\right] \in R^{(N-l)\times(N-l)}$ with

$$\begin{aligned} \tilde{b}_{ij} &= a_{ij}, \qquad j \neq i, j = l+1,\ldots,N, i = l+1, l+2,\ldots,N \\ \tilde{b}_{ii} &= -\sum_{\substack{j=l+1 \\ j\neq i}}^{N} a_{ij} \end{aligned}$$

Note that $\tilde{u}_i$ here are not real controllers being added to the nodes, therefore they are referred to as *virtual control*, which are give by

$$\tilde{u}_i = -c\tilde{d}_i(x_i - \bar{x}), \qquad i = l+1, l+2,\ldots,N \tag{8-24}$$

where $\tilde{d}_i = \sum_{j=1}^{l} a_{ij}$. It can be verified that $\tilde{A} = \tilde{B} + diag\left(\tilde{d}_{l+1}, \tilde{d}_{l+2}, \cdots, \tilde{d}_N\right)$.

Theorem 8-3 [9] *If* $\tilde{A}$ *is irreducible,* $\Gamma = I_m$ *, and*

$$c > \frac{C}{\lambda_{\min}(-\tilde{A})} \tag{8-25}$$

then the equilibrium of the virtually controlled network (8-23) *is globally stable when* $C = L_c^f$ *and is locally asymptotically stable when* $C = h_{\max}$.

If $\tilde{A}$ is reducible, then $\tilde{A}$ can be decomposed into a sequence of irreducible sub-matrices, $\tilde{A}_1, \cdots \tilde{A}_m$, so condition (8-25) becomes

$$c > \frac{C}{\max\left\{\lambda_{\min}(-\tilde{A}_j)\middle| j = 1,2,\cdots,m\right\}} \tag{8-26}$$

In this case,

$$\begin{aligned} 0 = \lambda_{\min}(-A) &< \lambda_{\min}(-A + diag\{d,\cdots,d,0,\cdots,0\}) \\ &< \lim_{d \to +\infty} \lambda_{\min}(-A + diag\{d,\cdots,d,0,\cdots,0\}) \\ &= \lambda_{\min}(-\tilde{A}) \le \max\left\{ \lambda_{\min}(-\tilde{A}_j) \;\middle|\; j = 1,2,\cdots,m \right\} \end{aligned} \tag{8-27}$$

It should be pointed out that condition (8-27) actually is a sufficient condition for network (8-23) to synchronize. Therefore, during the process of virtual control, those uncontrolled nodes in the network are being “controlled” through synchronization to those controlled nodes. This reflects the close relationship between control and synchronization from another point of view.

8.4.3 Pinning Control and Virtual Control of Scale-Free Networks

As discussed above, selective pinning is more effective than random pinning for scale-free networks, and if the number of pinned nodes is relatively small comparing to the network size, all the uncontrolled nodes are being virtually controlled to the network equilibrium via synchronizing to the controlled nodes.

To better describe the virtual control principle under selective pinning control scheme, consider a small network of 10 nodes, as shown in Fig. 8-7 [9]. First, 4 nodes are pinned to the network equilibrium $\bar{x}$. Then, through coupling (dash curves) they indirectly affect those uncontrolled nodes. In this case, the network is divided into two parts: node 1 and the rest nodes on the right are separated (Fig. 8-7 (b)). As the selective pinning scheme controls more nodes, the network is further divided into more and smaller sub-nets. Finally, the entire controlled network as shown in Fig. 8-5 (d), in which there are 5 nodes being pinned and the other 5 are being virtually controlled.

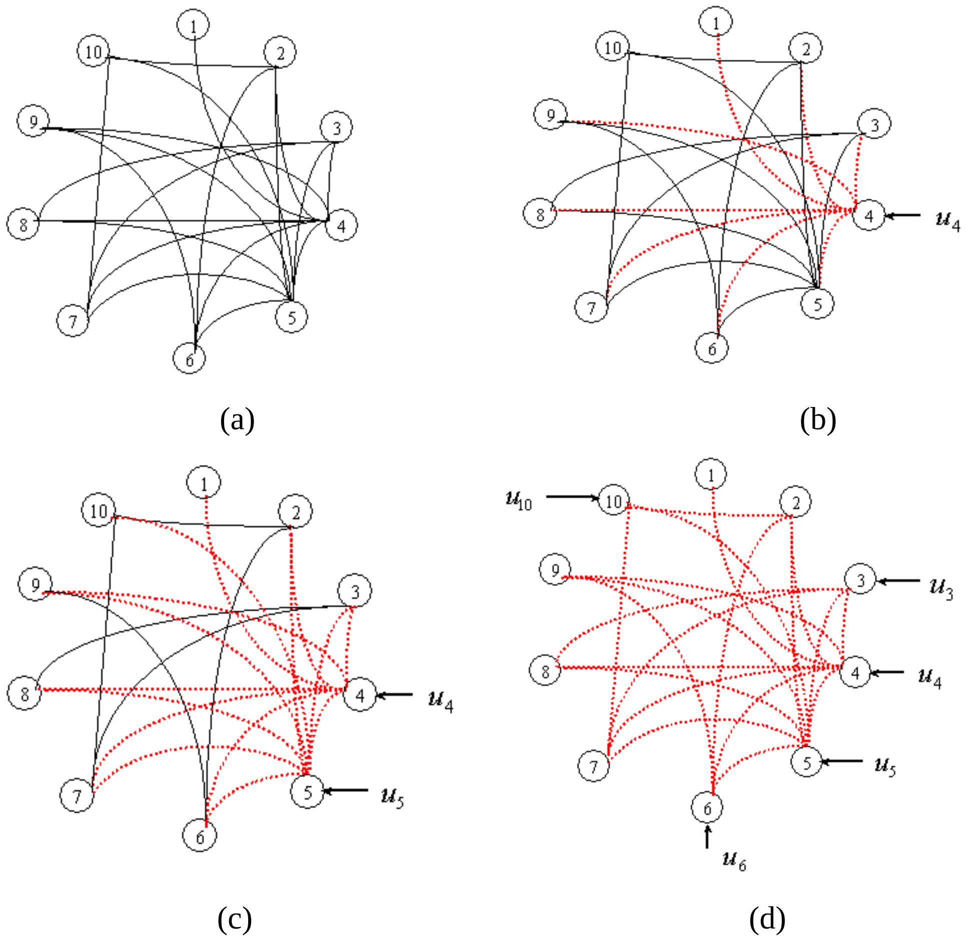
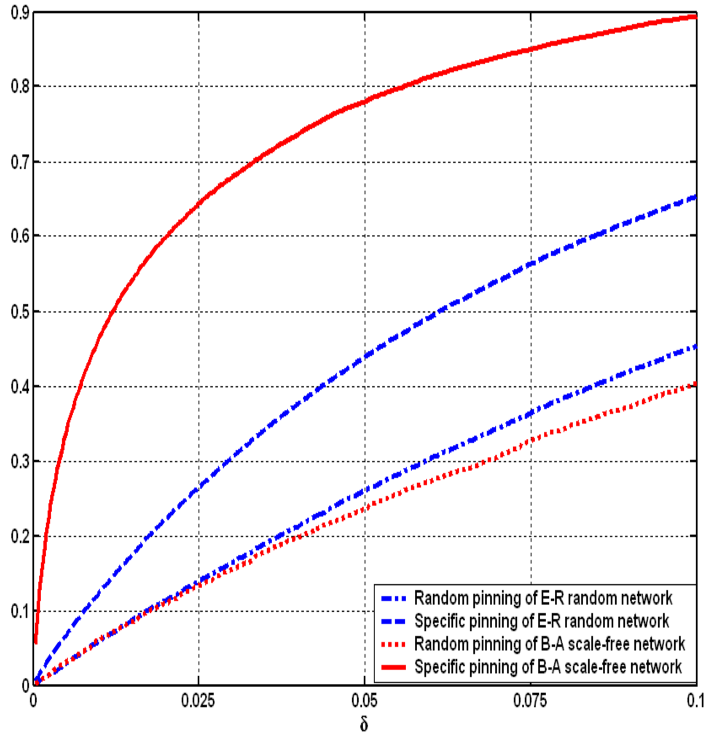


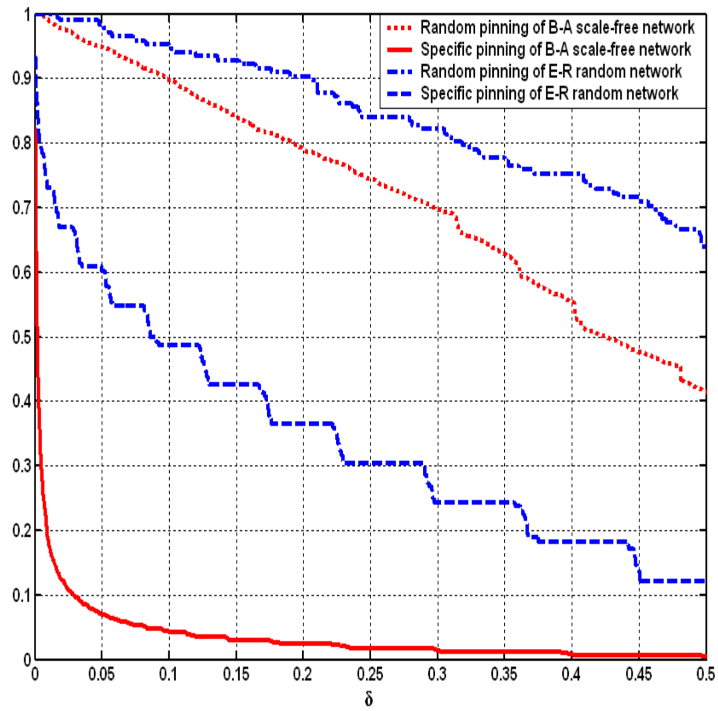
Fig. 8-7 Illustration of virtual control by selective pinning scheme [9]

The virtual control performances of different pinning schemes are generally not the same, which depend on the percentage of virtually controlled nodes in the network and the maximum sub-nets divided by pinning control.

Figure 8-8 (a) [9] shows the percentages of controlled nodes on a BA scale-free network and an ER random-graph network under selective and random pinning schemes, respectively. Both network models have 3,000 nodes and 9,000 edges. It can be seen from the simulation results depicted in this figure that for the BA model, selectively pinning 5% big nodes can virtually control 80% of other nodes over the network, while randomly pinning 5% nodes can only virtually affect less than 30% other nodes; while for the ER model, the two pinning schemes do not have significant differences, as expected.



(a) percentage of controlled nodes



(b) percentage of sub-nets

Fig. 8-8 Comparison of two pinning schemes on two network models [9]

Figure 8-8 (b) [9] compares the percentage of the resulting maximum sub-nets on both BA and ER models under the two pinning control schemes. For the BA model, selectively pinning 5% big nodes rapidly reduces this percentage to be below 10%. Since, according to conditions (8-25) and (8-26), the coupling strengths required by this scheme are much smaller than that required by the random pinning scheme, it once again shows the advantage of utilizing the heterogeneity of scale-free networks for control. Similarly, for the ER mode, the two pinning schemes have about the same performance.

8.4.4 Pinning Control of Scale-Free Chaotic Networks

The above analysis is now illustrated and visualized by an example of a scale-free chaotic network.

Example 8-2 [9]

Consider the chaotic Chen system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} p_1(x_2 - x_1) \\ (p_3 - p_2)x_1 - x_1x_3 + p_3x_2 \\ x_1x_2 - p_2x_3 \end{pmatrix}$$

When the constant parameters $p_1 = 35, p_2 = 3, p_3 = 28$, this system has some very complex dynamical behaviors: for example, it produces a chaotic attractor, as shown in Fig. 8-9. In this case, the system has three an unstable equilibria, among which one is $x^+ = [7.9373 \quad 7.9373 \quad 21]^T$.

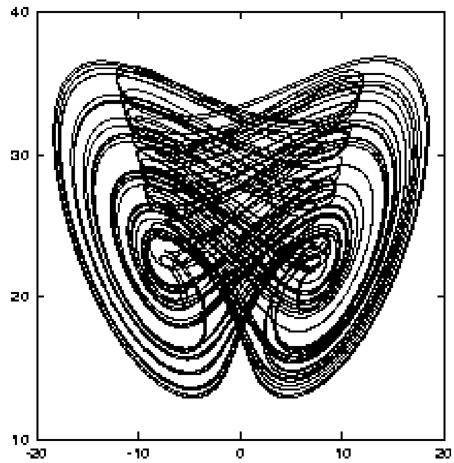


Fig. 8-9 Chen's chaotic attractor (projection on x_1 - x_3 plane)

Now, consider the following scale-free network of N Chen systems:

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} p_1(x_{i2} - x_{i1}) - c\sum_{j=1}^{N} a_{ij} x_{j1} + u_{i1} \\ (p_3 - p_2)x_{i1} - x_{i1}x_{i3} + p_3 x_{i2} + c\sum_{j=1}^{N} a_{ij} x_{j2} + u_{i2} \\ x_{i1}x_{i2} - p_2 x_{i3} + c\sum_{j=1}^{N} a_{ij} x_{j3} + u_{i3} \end{pmatrix}, \qquad i = 1,2,\cdots,N$$

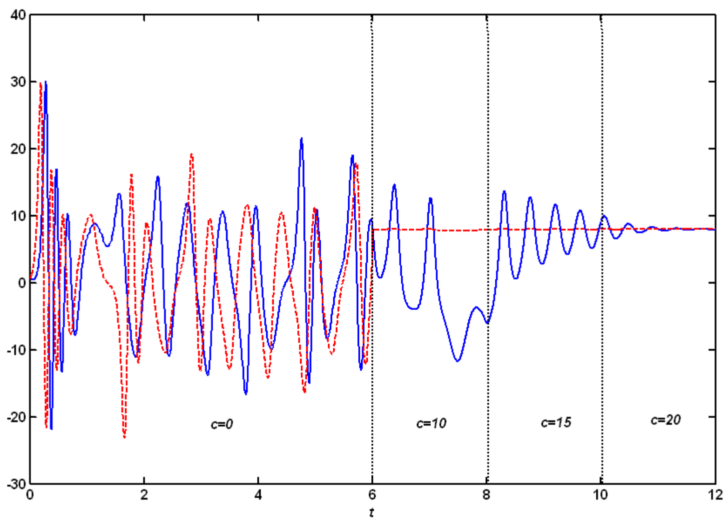
The objective is to stabilize it to the unstable equilibrium $\bar{x}$, shown above, by the pinning controllers

$$u_{ij} = \begin{cases} -cd\left(x_{ij} - x_j^+\right) & i = i_1,\ldots,i_l,\ j = 1,2,3 \\ 0 & otherwise \end{cases}$$

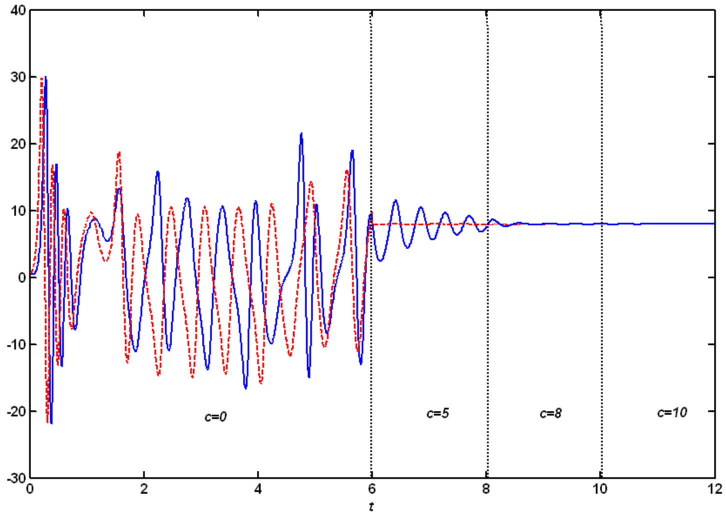
It follows from condition (8-25) that $C = 2.01745$, and in this case $d = 1{,}000$ (high gain controllers).

Simulations on a BA network of $N = 50$ nodes, each node is a Chen system, show that pining only the biggest node needs a much larger coupling strength than pinning two biggest nodes, and the virtual control process is much slower, as shown in Fig. 8-10, where solid curves correspond to the biggest node and the dash curves to the smallest node.

Simulations also show that selective pinning control scheme requires a much smaller coupling strength and a small number of nodes to pin, as compared to random pinning control. Figure 8-11 shows the performance by selectively controlling 2 biggest nodes (solid curves) and randomly pinning 2 or 5 nodes, under different coupling strengths (other curves).



(a) pinning the biggest node



(b) pinning two biggest nodes

Fig. 8-10 Selective pinning control of a BA network of chaotic Chen systems [9]

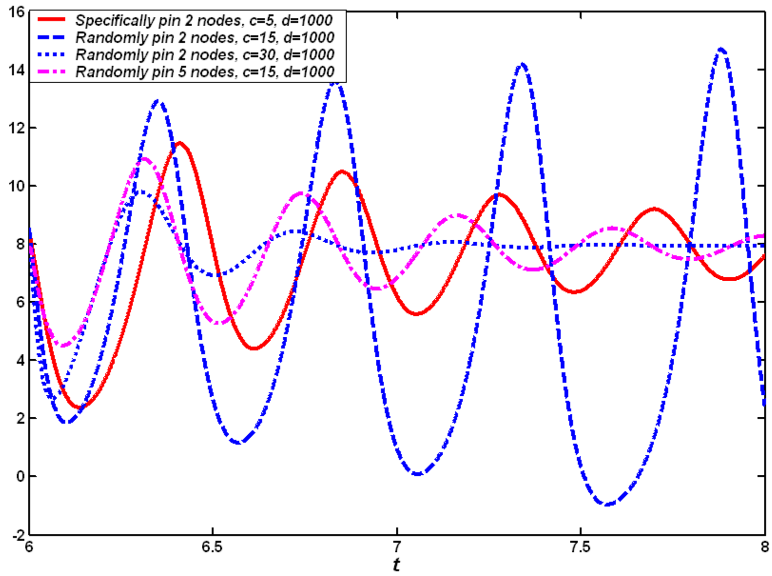


Fig. 8-11 Orbits of the smallest node under selective and random pinning schemes [9]

8.5 Time-Delay Pinning Control of Complex Networks

Consider once again a linearly and diffusively coupled network

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij}\Gamma x_j, \qquad i = 1,2,\cdots,N \tag{8-28}$$

where all notations are defined as before. The objective is to stabilize the network onto the equilibrium $\bar{x} \in R^n$ of the network, in the sense that

$$x_1(t) \to x_2(t) \to \cdots \to x_N(t) \to \bar{x}, \quad as \quad t \to \infty$$

where for simplicity it is assumed that $f(0)=0$, so $\bar{x}=0$. Without loss of generality, suppose that nodes $1,2,\cdots,l,$ are selected for pinning control. Thus, the controlled network can be written as

$$\begin{cases} \dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij}\Gamma x_j + u_i,\ i=1,2,\cdots,l \\ \dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij}\Gamma x_j, i = l+1, l+2,\cdots,N \end{cases} \tag{8-29}$$

in which u_i is the controller which pins node i, $i=1,2,...,l$.

By using the identity matrix $I \in R^{l\times l}$ and zero matrix $\Omega \in R^{(N-l)\times(N-l)}$, which has all elements $\Omega_{i,j}=0$, and defining matrix $B=\left[b_{ij}\right]=diag\{I,\Omega\}\in R^{N\times N}$, network (8-29) can be rewritten as

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij}\Gamma x_j + b_{ii}u_i,\ \ i=1,2,\cdots,N \tag{8-30}$$

where b_{ii} are the diagonal elements of matrix B, satisfying $b_{ii}=1$ if node i is pinned, and $b_{ii}=0$ otherwise.

Now, the following time-delay feedback controller is applied for pinning:

$$u_i = -k_i\Gamma(x_i(t) - x_i(t-\tau)) \tag{8-31}$$

where k_i is a constant control gain and τ is the constant delay time. Then, network (8-30) becomes

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij}\Gamma x_j - b_{ii}k_i\Gamma(x_i(t) - x_i(t-\tau)), \qquad i=1,2,\cdots,N \tag{8-32}$$

Let $e_i(t) = x_i(t) - \bar{x}$ be the control errors, $i=1,2,...,N$. Then, one has

$$\dot{e}_i = f(\bar{x}+e_i(t)) - f(\bar{x}) - c\sum_{j=1}^{N} a_{ij}\Gamma e_j - b_{ii}k_i\Gamma(e_i(t) - e_i(t-\tau)),\ i=1,2,\cdots,N \tag{8-33}$$

Assume that f is continuously differentiable. Then, linearizing the above equation at its zero equilibrium gives

$$\dot{e}_i = (J(t) + b_{ii}k_i\Gamma)e_i(t) - c\sum_{j=1}^{N} a_{ij}\Gamma e_j - b_{ii}k_i\Gamma e_i(t-\tau),\ i = 1,2,\cdots,N \qquad (8\text{-}34)$$

Theorem 8-4 [16] *The equilibrium point of the controlled network* (8-32) *is locally asymptotically stable if there are symmetrical and positive-definite matrices* $W, X, Z \in R^{n\times n}$ *such that the following Linear Matrix Inequality (LMI) holds:*

$$M = \begin{bmatrix} \widehat{A} & ca_{i1}\Gamma W & \cdots & ca_{iN}\Gamma W & 0 & \cdots & -b_{ii}\Gamma X & \cdots & 0 \\ ca_{i1}W\Gamma & Z & & & & & & & \\ \vdots & & \ddots & & & & & & \\ ca_{iN}W\Gamma & & & Z & & & & & \\ 0 & & & & -Z & & & & \\ \vdots & & & & & \ddots & & & \\ -b_{ii}X\Gamma & & & & & & & & \\ \vdots & & & & & & & & \\ 0 & & & & & & & & -Z \end{bmatrix} < 0$$

where $\widehat{A} = WJ^T + JW + b_{ii}X\Gamma + b_{ii}\Gamma X$.

Proof. Construct a Lyapunov functional of the form

$$V = \sum_{i=1}^{N}\left\{e_i^T(t)Pe_i(t) + \sum_{j=1}^{N}\int_{t-\tau}^{t} e_j^T(\sigma)Re_j(\sigma)d\sigma\right\}$$

where P and R are symmetric and positive-definite matrices. The derivative of $V(e_1, e_2, \cdots e_N)$ along the solution of system (8-32) is

$$\begin{aligned}\dot{V}(e_1, e_2, \cdots e_N) &= \sum_{i=1}^{N}\left[\dot{e}_i^T(t)Pe_i(t) + e_i^T(t)P\dot{e}_i(t)\right] \\ &\quad + \sum_{i=1}^{N}\left[\sum_{j=1}^{N} e_j^T(t)\,\mathrm{Re}_j(t) - \sum_{j=1}^{N} e_j^T(t-\tau)\,\mathrm{Re}_j(t-\tau)\right] \\ &= \sum_{i=1}^{N}\left\{e_i^T(t)\Big(\big(J^T(t) + b_{ii}k_{ii}\Gamma\big)P + P\big(J(t) + b_{ii}k_{ii}\Gamma\big)\Big)e_i(t)\right. \\ &\quad + 2c\left[\sum_{j=1}^{N} a_{ij}\Gamma e_j(t)\right]^T Pe_i(t) - 2b_{ii}k_i e_i^T(t-\tau)\Gamma Pe_i(t) \\ &\quad \left. + \sum_{j=1}^{N} e_j^T(t)\,\mathrm{Re}_j(t) - \sum_{j=1}^{N} e_j^T(t-\tau)\,\mathrm{Re}_j(t-\tau)\right\}.\end{aligned}$$

Obviously, the derivative of $V(e_1, e_2, \cdots e_N)$ is negative for all $e_i(t)$ if $\Xi(t) < 0$, where

$$\Xi(t) = w^T(t)\left(J^T(t) + b_{ii}k_i\Gamma\right)Pw(t) + w^T(t)P\left(J(t) + b_{ii}k_i\Gamma\right)w(t)$$
$$+ 2c\left[\sum_{j=1}^{N} a_{ij}\Gamma e_j(t)\right]^T Pw(t) - 2b_{ii}k_i e_i^T(t-\tau)\Gamma Pw(t)$$
$$+ \sum_{j=1}^{N} e_j^T(t)\,\mathrm{Re}_j(t) - \sum_{j=1}^{N} e_j^T(t-\tau)\,\mathrm{Re}_j(t-\tau).$$

This is equivalent to $Y^T(t)MY(t) < 0$, where

$$M = \begin{bmatrix} A & ca_{i1}P\Gamma & \cdots & ca_{iN}P\Gamma & 0 & \cdots & -b_{ii}k_i\Gamma P & \cdots & 0 \\ ca_{i1}\Gamma P & R & & & & & & & \\ \vdots & & \ddots & & & & & & \\ ca_{iN}\Gamma P & & & R & & & & & \\ 0 & & & & -R & & & & \\ \vdots & & & & & \ddots & & & \\ -b_{ii}k_i\Gamma P & & & & & & -R & & \\ \vdots & & & & & & & \ddots & \\ 0 & & & & & & & & -R \end{bmatrix}$$

$$Y(t) = \left[w(t) \quad e_1(t) \quad \cdots \quad e_N(t) \quad e_1(t-\tau) \quad \cdots \quad e_i(t-\tau) \quad \cdots \quad e_N(t-\tau)\right]^T$$

and

$$A = (J^T(t) + b_{ii}k_i\Gamma)P + P(J(t) + b_{ii}k_i\Gamma)$$

Therefore, if $M < 0$, then system (8-32) is asymptotically stable about zero. By multiplying $diag\{P^{-1},...,P^{-1}\}$ to both sides of the matrix M, one can easily verify that the LMI in the theorem statement is equivalent to $M < 0$. □

Example 8-3 [16]

Consider a BA scale-free network consisting of N cellular neural nodes, with $\Gamma = diag(1,1,1,1)$ and each isolated node described by

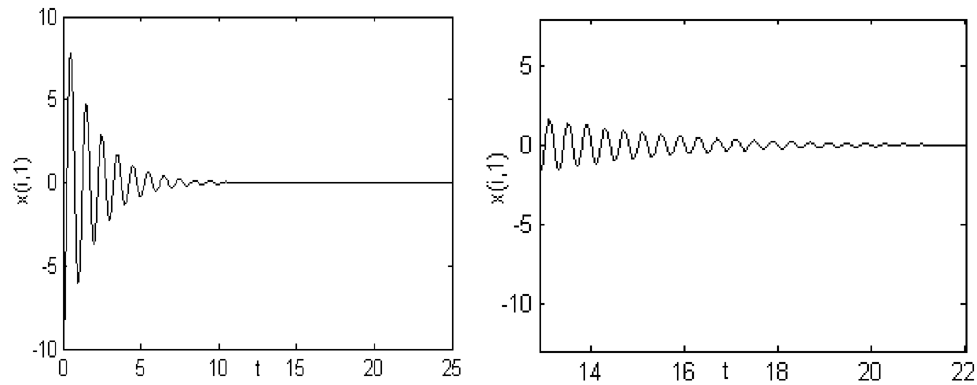
$$\begin{cases} \dot{x}_1 = -x_3 - x_4 \\ \dot{x}_2 = 2x_2 + x_3 \\ \dot{x}_3 = 14x_1 - 14x_2 \\ \dot{x}_4 = 100x_1 - 100x_4 + 100(|x_4 + 1| - |x_4 - 1|) \end{cases}$$

The entire network is described by

$$\dot{x}_i = \begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \\ \dot{x}_{i4} \end{pmatrix} = \begin{pmatrix} -x_{i3} - x_{i4} + c\sum_{j=1}^{N} a_{ij} x_{j1} \\ 2x_{i2} + x_{i3} + c\sum_{j=1}^{N} a_{ij} x_{j2} \\ 14x_{i1} - 14x_{i2} + c\sum_{j=1}^{N} a_{ij} x_{j3} \\ 100x_{i1} - 100x_{i4} \\ \quad + 100(|x_{i4}+1| - |x_{i4}-1|) \\ \quad + c\sum_{j=1}^{N} a_{ij} x_{j4} \end{pmatrix}, \quad i = 1,2,\cdots N$$

To stabilize this network with $N = 60$ to its zero equibrium, choose the coupling strength $c = 8.2$ and the number of controlled nodes $l = 15$. In the selective pinning control scheme, set $b_{ii} = 1$ for the first 15 nodes of largest degrees. The control gains are found to be $k_i = 29.7603$ according to the condition given in Theorem 8-4. For comparison, the control gains $k_i = 513.3709$ are needed by the random pinning scheme, which pins also $l = 15$ randomly selected nodes.

Figure 8-12 shows plots of the first state components of the largest node in the network, using the selective pinning control scheme versus the random pinning control scheme. It can be seen that the random pinning control scheme not only uses higher gains, about 17 times larger than the selective pinning scheme, but also takes doubly longer time, to achieve the same control performance.



(a) selective pinning (b) random pinning

Fig. 8-12 Comparison of two pinning control schemes [16]

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