2

Decomposition in Linear Programming: Complicating Constraints

2.1 Introduction

The size of a linear programming problem can be very large. One can encounter in practice problems with several hundred thousands of equations and/or unknowns. To solve these problems the use of some special techniques is either convenient or required. Alternatively, a distributed solution of large problems may be desirable for technical or practical reasons. Decomposition techniques allow certain type of problems to be solved in a decentralized or distributed fashion. Alternatively, they lead to a drastic simplification of the solution procedure of the problem under study.

For a decomposition technique to be useful, the problem at hand must have the appropriate structure. Two such cases arise in practice: the complicating constraint and the complicating variable structures. The first one is analyzed below, and the second is analyzed in Chap. 3.

In a linear programming problem, the complicating constraints involving variables from different blocks drastically complicate the solution of the problem and prevent its solution by blocks. The following example illustrates how complicating constraints impede a solution by blocks.

Illustrative Example 2.1 (Complicating constraints that prevent a distributed solution). Consider the problem

$$\underset{x_1, x_2, x_3, y_1, y_2, v_1, v_2, v_3}{\text{minimize}} \quad a_1x_1 + a_2x_2 + a_3x_3 + b_1y_1 + b_2y_2 + c_1v_1 + c_2v_2 + c_3v_3$$

subject to

$$\begin{array}{lllllllll}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & & & & & & = e_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & & & & & & = e_2 \\
 & b_{11}y_1 & + b_{12}y_2 & & & & = f_1 \\
 & & & c_{11}v_1 & + c_{12}v_2 & + c_{13}v_3 & = g_1 \\
 & & & c_{21}v_1 & + c_{22}v_2 & + c_{23}v_3 & = g_2 \\
d_{11}x_1 + d_{12}x_2 + d_{13}x_3 & + d_{14}y_1 & + d_{15}y_2 & + d_{16}v_1 & + d_{17}v_2 & + d_{18}v_3 & = h_1 \\
 & & & & & x_1, x_2, x_3, y_1, y_2, v_1, v_2, v_3 & \geq 0 \,.
\end{array}$$

If the last equality constraint is not enforced, i.e., it is relaxed, the above problem decomposes into the following three problems:

Subproblem 1:

$$\underset{x_1, x_2, x_3}{\text{minimize}} \quad a_1x_1 + a_2x_2 + a_3x_3$$

subject to

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= e_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= e_2 \\
x_1, x_2, x_3 &\geq 0 \,.
\end{aligned}$$

Subproblem 2:

$$\underset{y_1, y_2}{\text{minimize}} \quad b_1y_1 + b_2y_2$$

subject to

$$\begin{aligned}
b_{11}y_1 + b_{12}y_2 &= f_1 \\
y_1, y_2 &\geq 0 \,.
\end{aligned}$$

Subproblem 3:

$$\underset{v_1, v_2, v_3}{\text{minimize}} \quad c_1v_1 + c_2v_2 + c_3v_3$$

subject to

$$\begin{aligned}
c_{11}v_1 + c_{12}v_2 + c_{13}v_3 &= g_1 \\
c_{21}v_1 + c_{22}v_2 + c_{23}v_3 &= g_2 \\
v_1, v_2, v_3 &\geq 0 \,.
\end{aligned}$$

Since the last equality constraint of the original problem

$$d_{11}x_1 + d_{12}x_2 + d_{13}x_3 + d_{14}y_1 + d_{15}y_2 + d_{16}v_1 + d_{17}v_2 + d_{18}v_3 = h_1$$

involves all variables, preventing a solution by blocks, it is a complicating constraint. □

Complicating constraints may prevent a straightforward solution of the linear programming problem being considered. The next example illustrates this situation.

Illustrative Example 2.2 (Complicating constraints that prevent an efficient solution). Consider the problem

$$\underset{x_1,\dots,x_7}{\text{minimize}} \quad c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6 + c_7x_7$$

subject to

$$\begin{array}{rcl}
a_{11}x_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + a_{17}x_7 &=& b_1 \\
a_{22}x_2 \qquad\qquad\qquad\qquad\qquad + a_{27}x_7 &=& b_2 \\
a_{37}x_7 &=& b_3 \\
a_{41}x_4 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + a_{45}x_5 + a_{46}x_6 + a_{47}x_7 &=& b_4 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq& 0 \; ,
\end{array}$$

where

$$c_1, c_2, c_3, c_4, c_5, c_6, c_7 \geq 0 \; .$$

If the last equality constraint is dropped, the optimal solution is trivially obtained solving the resulting system of equations, i.e.,

$$\begin{aligned}
x_7 &= \frac{b_3}{a_{37}} \\
x_1 &= \frac{b_1}{a_{11}} - \frac{a_{17}}{a_{37}a_{11}}b_3 \\
x_2 &= \frac{b_2}{a_{22}} - \frac{a_{27}}{a_{37}a_{22}}b_3 \\
x_3, x_4, x_5, x_5 &= 0 \; .
\end{aligned}$$

Since the last constraint of the original problem

$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + a_{45}x_5 + a_{46}x_6 + a_{47}x_7 = b_4$$

prevents a straightforward solution of the problem, it is a complicating constraint. □

Decomposition procedures are computational techniques that indirectly consider the complicating constraints. The price that has to be paid for such a simplification is repetition. That is, instead of solving the original problem with complicating constraints, two problems are solved iteratively (i.e., repetitively): a simple so-called master problem and a problem similar to the original one but without complicating constraints. In this manner, complicating constraints are progressively taken into account. The decomposition techniques analyzed in this chapter attain optimality within a finite number of iterations. This robust behavior is of particular interest in many practical applications. Decomposition techniques for problems with complicating constraints are developed in the following sections.

2.2 Complicating Constraints: Problem Structure

Consider the linear programming problem

$$\underset{x_1, x_2, \ldots, x_n}{\text{minimize}} \quad \sum_{j=1}^{n} c_j x_j \tag{2.1}$$

subject to

$$\sum_{j=1}^{n} e_{ij} x_j = f_i; \qquad i = 1, \ldots, q \tag{2.2}$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i; \qquad i = 1, \ldots, m \tag{2.3}$$

$$0 \le x_j \le x_j^{\text{up}}; \quad j = 1, \ldots, n \ , \tag{2.4}$$

where constraints (2.2) have a decomposable structure in r blocks, each of size n_k $(k = 1, 2, \ldots, r)$, i.e., they can be written as

$$\sum_{j=n_{k-1}+1}^{n_k} e_{ij} x_j = f_i; \qquad i = q_{k-1}+1, \ldots, q_k; \qquad k = 1, 2, \ldots, r \ . \tag{2.5}$$

Note that $n_0 = q_0 = 0$, $q_r = q$ and $n_r = n$.

On the other hand, since constraints (2.3) have no decomposable structure, they are the complicating constraints.

Note that upper bounds x_j^{up} $(j = 1, \ldots, n)$ are considered for all optimization variables x_j $(j = 1, \ldots, n)$. This assumption allows dealing with a compact (finite) feasible region, leading to a simpler theoretical analysis of the problem (2.1)–(2.4). This assumption is justified by the bounded nature of most engineering variables.

Figure 2.1 shows the structure of the above problem for the case $r = 3$. This particular problem can be written as

$$\underset{\boldsymbol{x}^{[1]}, \boldsymbol{x}^{[2]}, \boldsymbol{x}^{[3]}}{\text{minimize}} \quad \left((\boldsymbol{c}^{[1]})^T \mid (\boldsymbol{c}^{[2]})^T \mid (\boldsymbol{c}^{[3]})^T \right) \begin{pmatrix} \boldsymbol{x}^{[1]} \\ - \\ \boldsymbol{x}^{[2]} \\ - \\ \boldsymbol{x}^{[3]} \end{pmatrix} ,$$

where the superindices in brackets refer to partitions, subject to

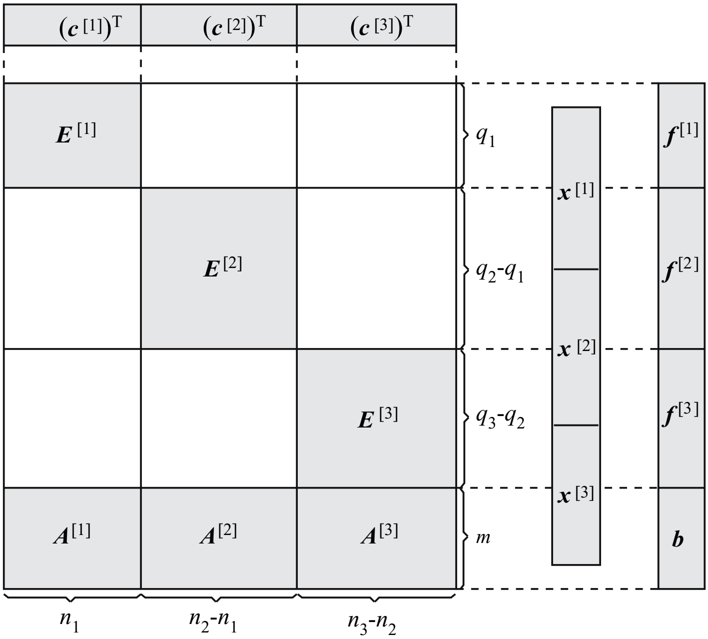

Fig. 2.1. Decomposable matrix with complicating constraints

$$\begin{pmatrix} \boldsymbol{E}^{[1]} & | & & | & \\ -- & - & -- & - & -- \\ & | & \boldsymbol{E}^{[2]} & | & \\ -- & - & -- & - & -- \\ & | & & | & \boldsymbol{E}^{[3]} \\ -- & - & -- & - & -- \\ \boldsymbol{A}^{[1]} & | & \boldsymbol{A}^{[2]} & | & \boldsymbol{A}^{[3]} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^{[1]} \\ -- \\ \boldsymbol{x}^{[2]} \\ -- \\ \boldsymbol{x}^{[3]} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}^{[1]} \\ -- \\ \boldsymbol{f}^{[2]} \\ -- \\ \boldsymbol{f}^{[3]} \\ -- \\ \boldsymbol{b} \end{pmatrix}$$

$$\begin{array}{ccccc} 0 & \leq & \boldsymbol{x}^{[1]} & \leq & \boldsymbol{x}^{[1]\mathrm{up}} \\ 0 & \leq & \boldsymbol{x}^{[2]} & \leq & \boldsymbol{x}^{[2]\mathrm{up}} \\ 0 & \leq & \boldsymbol{x}^{[3]} & \leq & \boldsymbol{x}^{[3]\mathrm{up}} \end{array} .$$

In general, the initial problem can be written as

$$\underset{\boldsymbol{x}^{[1]}, \mathbf{x}^{[2]}, \dots, \mathbf{x}^{[r]}}{\text{minimize}} \quad \sum_{k=1}^{r} \left(\boldsymbol{c}^{[k]}\right)^T \boldsymbol{x}^{[k]} \tag{2.6}$$

subject to

$$\boldsymbol{E}^{[k]} \boldsymbol{x}^{[k]} = \boldsymbol{f}^{[k]};\ k = 1, \dots, r \tag{2.7}$$

$$\sum_{k=1}^{r} \boldsymbol{A}^{[k]} \boldsymbol{x}^{[k]} = \boldsymbol{b} \tag{2.8}$$

$$0 \leq \boldsymbol{x}^{[k]} \leq \boldsymbol{x}^{[k]\mathrm{up}};\ k = 1, \dots, r\ , \tag{2.9}$$

where for each $k = 1, 2, \ldots, r$, we have

$$\left(\boldsymbol{c}^{[k]}\right)^T = \left(c_{n_{k-1}+1} \ldots c_{n_k} \right) \tag{2.10}$$

$$\left(\boldsymbol{x}^{[k]}\right)^T = \left(x_{n_{k-1}+1} \ldots x_{n_k} \right) \tag{2.11}$$

$$\left(\boldsymbol{x}^{[k]\text{up}}\right)^T = \left(x^{\text{up}}_{n_{k-1}+1} \ldots x^{\text{up}}_{n_k} \right) \tag{2.12}$$

$$\boldsymbol{E}^{[k]} = (e_{ij})\,; \quad i = q_{k-1}+1, \ldots, q_k; \quad j = n_{k-1}+1, \ldots, n_k \tag{2.13}$$

$$\left(\boldsymbol{f}^{[k]}\right)^T = \left(f_{q_{k-1}+1} \ldots f_{q_k} \right) \tag{2.14}$$

$$\boldsymbol{A}^{[k]} = (a_{ij})\,; \qquad i = 1, \ldots, m; \quad j = n_{k-1}+1, \ldots, n_k \tag{2.15}$$

$$\boldsymbol{b}^T = \left(b_1 \ldots b_m \right) . \tag{2.16}$$

If complicating constraints are ignored, i.e., they are relaxed, the original problem becomes

$$\underset{\boldsymbol{x}^{[k]}, \boldsymbol{x}^{[2]}, \ldots, \boldsymbol{x}^{[r]}}{\text{minimize}} \quad \sum_{k=1}^{r} \left(\boldsymbol{c}^{[k]}\right)^T \boldsymbol{x}^{[k]} \tag{2.17}$$

subject to

$$\boldsymbol{E}^{[k]}\boldsymbol{x}^{[k]} = \boldsymbol{f}^{[k]}; \quad k = 1, \ldots, r \tag{2.18}$$

$$0; \leq \ \boldsymbol{x}^{[k]} \leq \boldsymbol{x}^{[k]\text{up}}; \ k = 1, \ldots, r \ . \tag{2.19}$$

This problem (2.17)–(2.19) is called the relaxed version of the problem. The decomposed kth-subproblem is therefore

$$\underset{\boldsymbol{x}^{[k]}}{\text{minimize}} \quad \left(\boldsymbol{c}^{[k]}\right)^T \boldsymbol{x}^{[k]} \tag{2.20}$$

subject to

$$\boldsymbol{E}^{[k]}\boldsymbol{x}^{[k]} = \boldsymbol{f}^{[k]} \tag{2.21}$$

$$0 \leq \ \boldsymbol{x}^{[k]} \leq \boldsymbol{x}^{[k]\text{up}} \tag{2.22}$$

or

$$\underset{x_{n_{k-1}+1}, x_{n_{k-1}+2}, \ldots, x_{n_k}}{\text{minimize}} \quad \sum_{j=n_{k-1}+1}^{n_k} c_j x_j \tag{2.23}$$

subject to

$$\sum_{j=n_{k-1}+1}^{n_k} e_{ij} x_j = f_i; \qquad i = q_{k-1}+1, \ldots, q_k \tag{2.24}$$

$$0 \leq \ x_j \leq x_j^{\text{up}}; \quad j = n_{k-1}+1, \ldots, n_k \ . \tag{2.25}$$

The following example shows a decomposable linear programming problem with complicating constraints.

Illustrative Example 2.3 (Problem with decomposable structure). The problem

$$\underset{x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3,w_1}{\text{minimize}} \quad -4x_1 - y_1 - 6z_1$$

subject to

$$\begin{array}{lllllllllll}
x_1 & -x_2 & & & & & & & & & = 1\\
x_1 & & +x_3 & & & & & & & & = 2\\
 & & & y_1 & -y_2 & & & & & & = 1\\
 & & & y_1 & & +y_3 & & & & & = 2\\
 & & & & & & z_1 & -z_2 & & & = 1\\
 & & & & & & z_1 & & +z_3 & & = 2\\
3x_1 & & & +2y_1 & & & +4z_1 & & & +w_1 & = 17\\
 & & & & & & & & & x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3,w_1 & \geq 0
\end{array}$$

has a decomposable structure in three blocks, where the last equality

$$3x_1 + 2y_1 + 4z_1 + w_1 = 17$$

is the complicating constraint. □

2.3 Decomposition

This section motivates the decomposition algorithm developed in the following sections.

Suppose that each of the subproblems (relaxed problem) is solved p times with different and arbitrary objective functions, and assume that the p basic feasible solutions of the relaxed problems are

$$\begin{array}{cccc}
x_1^{(1)}, & x_2^{(1)} & \dots & x_n^{(1)}\\
\vdots & \vdots & \vdots & \vdots\\
x_1^{(p)}, & x_2^{(p)} & \dots & x_n^{(p)}\ ,
\end{array} \tag{2.26}$$

where $x_j^{(s)}$ is the jth component of solution s, where all variables from all the subproblems are considered, and the corresponding p optimal objective function values are

$$\begin{array}{c}
z^{(1)}\\
z^{(2)}\\
\vdots\\
z^{(p)}\ ,
\end{array} \tag{2.27}$$

where $z^{(s)}$ is the objective function value of solution s. Note that we use superindices of the form (s) to refer to the sth solution.

The values of the m complicating constraints for the above p solutions are

$$\begin{array}{cccc} r_1^{(1)}, & r_2^{(1)} & \dots & r_m^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ r_1^{(p)}, & r_2^{(p)} & \dots & r_m^{(p)} \end{array} , \tag{2.28}$$

where $r_i^{(s)}$ is the value of the ith complicating constraint for the sth solution, i.e.,

$$r_i^{(s)} = \sum_{j=1}^{n} a_{ij} x_j^{(s)} .$$

For the derivations below, it should be emphasized that a linear convex combination of basic feasible solutions of a linear programming problem is a feasible solution of that problem.

The above p basic feasible solutions of the relaxed problem can be used to produce a feasible solution of the original (nonrelaxed) problem. The case in which a feasible solution cannot be generated is treated below. This is done by solving the weighting problem below, which is the so-called master problem

$$\underset{u_1,\dots,u_p}{\text{minimize}} \quad \sum_{s=1}^{p} z^{(s)} u_s \tag{2.29}$$

subject to

$$\sum_{s=1}^{p} r_i^{(s)} u_s = b_i : \ \lambda_i; \qquad i = 1,\dots,m \tag{2.30}$$

$$\sum_{s=1}^{p} u_s = 1 : \ \sigma \tag{2.31}$$

$$u_s \geq 0; \ s = 1,\dots,p , \tag{2.32}$$

where the corresponding dual variables λ_i and σ are indicated.

The following observations are in order:

1. Any solution of the above problem is a convex combination of basic feasible solutions of the relaxed problem; therefore, it is itself a basic feasible solution of the relaxed problem.
2. Complicating constraints are enforced; therefore, the solution of the above problem is a basic feasible solution for the original (nonrelaxed) problem.

Consider that a prospective new basic feasible solution is added to the problem above. The objective function value of this solution is z and its complicating constraints values are $r_1,\dots,r_m$.

The new weighting problem becomes

$$\underset{u_1,\dots,u_p,u}{\text{minimize}} \quad \sum_{s=1}^{p} z^{(s)} u_s + zu \tag{2.33}$$

subject to

$$\sum_{s=1}^{p} \left(r_i^{(s)} u_s + r_i u \right) = b_i : \ \lambda_i; \qquad i = 1, \dots, m \tag{2.34}$$

$$\sum_{s=1}^{p} u_s + u = 1 : \ \sigma \tag{2.35}$$

$$u, u_s \geq 0; \ s = 1, \dots, p \ . \tag{2.36}$$

The reduced cost (see Bazaraa et al. [20], Castillo et al. [21]) of the new weighting variable u, associated with the tentative new basic feasible solution, can be computed as

$$d = z - \sum_{i=1}^{m} \lambda_i r_i - \sigma \ . \tag{2.37}$$

Taking into account that

$$z = \sum_{j=1}^{n} c_j x_j \ , \tag{2.38}$$

where $x_j (j = 1, \dots, n)$ is the new prospective basic feasible solution, and that

$$r_i = \sum_{j=1}^{n} a_{ij} x_j \ , \tag{2.39}$$

the reduced cost of weighting variable u becomes

$$d = \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) - \sigma \ , \tag{2.40}$$

which reduces to

$$d = \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} \lambda_i a_{ij} \right) x_j - \sigma \ . \tag{2.41}$$

If the tentative basic feasible solution is to be added to the set of previous ones, the reduced cost associated with its weighting variable should be negative and preferably a minimum. To find the minimum reduced cost associated with the weighting variable u and corresponding to a basic feasible solution of the relaxed problem, the following problem is solved.

$$\underset{x_1, x_2, \dots, x_n}{\text{minimize}} \quad \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} \lambda_i a_{ij} \right) x_j - \sigma$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} e_{ij}x_j = f_i \; ; & \qquad i = 1, \ldots, q \\ 0 \le \; x_j \; \le x_j^{\text{up}} \; ; & \qquad j = 1, \ldots, n \; . \end{aligned}$$

Note that the constraints of the relaxed problem must be added.

If constant σ is dropped from the objective function, the problem above becomes

$$\underset{x_1, x_2, \ldots, x_n}{\text{minimize}} \quad \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} \lambda_i a_{ij} \right) x_j \tag{2.42}$$

subject to

$$\sum_{j=1}^{n} e_{ij}x_j = f_i \; ; \qquad i = 1, \ldots, q \tag{2.43}$$

$$0 \le \; x_j \; \le x_j^{\text{up}} \; ; \qquad j = 1, \ldots, n \; . \tag{2.44}$$

It should be noted that the above problem has the same structure as the relaxed problem but with different objective functions. Therefore, it can be solved by blocks. The subproblem associated with block k is

$$\underset{x_{n_{k-1}+1}, x_{n_{k-1}+2}, \ldots, x_{n_k}}{\text{minimize}} \quad \sum_{j=n_{k-1}+1}^{n_k} \left(c_j - \sum_{i=1}^{m} \lambda_i a_{ij} \right) x_j \tag{2.45}$$

subject to

$$\sum_{j=n_{k-1}+1}^{n_k} e_{ij}x_j = f_i \; ; \qquad i = q_{k-1}+1, \ldots, q_k \tag{2.46}$$

$$0 \le \; x_j \; \le x_j^{\text{up}} \; ; \qquad j = n_{k-1}+1, \ldots, n_k \; . \tag{2.47}$$

From the analysis carried out, the following conclusions can be drawn:

1. To determine whether or not a given basic feasible solution of the relaxed problem should be added to the weighting problem, the subproblems associated with the relaxed problem should be solved with modified objective functions.
2. The modified objective function of subproblem k is

$$\sum_{j=n_{k-1}+1}^{n_k} \left(c_j - \sum_{i=1}^{m} \lambda_i a_{ij} \right) x_j \; . \tag{2.48}$$

In the above objective function, each cost coefficient has the form

$$\bar{c}_j = c_j - \sum_{i=1}^{m} \lambda_i a_{ij} \; , \tag{2.49}$$

that is, the cost coefficient $\bar{c}_j$ depends on the dual variable of every complicating constraint and on the column j of the corresponding complicating constraint matrix $\boldsymbol{A}$.

3. Once the subproblems are solved, the optimal value v^o of the objective function of the relaxed problem is computed as

$$v^o = \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} \lambda_i^o a_{ij} \right) x_j^o \ , \tag{2.50}$$

   where x_j^o $(j = 1, \ldots, n)$ is the solution of the relaxed problem, and λ_i^o $(i = 1, \ldots, m)$ are the optimal values of the dual variables associated with complicating constraints.
   The minimum reduced cost is then

$$d^o = v^o - \sigma^o \ , \tag{2.51}$$

   where σ^o is the optimal value of the dual variable associated with the convex combination constraint (2.35).
4. Based on the minimum reduced cost value, it can be decided whether or not to include the tentative basic feasible solution associated with the weighting variable u. This is done as follows:
   a) If $v^o \geq \sigma^o$, the tentative basic feasible solution cannot improve the current solution of the weighting problem because its reduced cost is nonnegative.
   b) If, on the other hand, $v^o < \sigma^o$, the tentative basic feasible solution should be included in the weighting problem because its reduced cost is negative and this can be used to attain a basic feasible solution with smaller objective function value than the current one.

The above remarks allow us to propose the solution algorithm described in the next section.

Because of its importance, we dedicate a section to the Dantzig-Wolfe decomposition method.

2.4 The Dantzig-Wolfe Decomposition Algorithm

In this section the Dantzig-Wolfe decomposition algorithm is described in detail.

2.4.1 Description

The Dantzig-Wolfe decomposition algorithm works as follows.

Algorithm 2.1 (The Dantzig-Wolfe decomposition algorithm).

Input. A linear programming problem with complicating constraints.
Output. The solution of the linear programming problem obtained after using the Dantzig-Wolfe decomposition algorithm.

Step 0: Initialization. Initialize the iteration counter, $\nu = 1$. Obtain $p^{(\nu)}$ distinct solutions of the relaxed problem by solving $p^{(\nu)}$ times ($\ell = 1, \dots, p^{(\nu)}$) each of the r subproblems ($k = 1, \dots, r$) below

$$\underset{x_{n_{k-1}+1}, \dots, x_{n_k}}{\text{minimize}} \quad \sum_{j=n_{k-1}+1}^{n_k} \hat{c}_j^{(\ell)} x_j \tag{2.52}$$

subject to

$$\sum_{j=n_{k-1}+1}^{n_k} e_{ij} x_j = f_i \; ; \qquad i = q_{k-1}+1, \dots, q_k \tag{2.53}$$

$$0 \leq x_j \leq x_j^{\text{up}} \; ; \qquad j = n_{k-1}+1, \dots, n_k \; , \tag{2.54}$$

where $\hat{c}_j^{(\ell)}$ ($j = n_{k-1}+1, \dots, n_k; k = 1, \dots, r; \ell = 1, \dots, p^{(\nu)}$) are arbitrary cost coefficients to attain the $p^{(\nu)}$ initial solutions of the r subproblems.

Step 1: Master problem solution. Solve the master problem

$$\underset{u_1, \dots, u_{p^{(\nu)}}}{\text{minimize}} \quad \sum_{s=1}^{p^{(\nu)}} z^{(s)} u_s \tag{2.55}$$

subject to

$$\sum_{s=1}^{p^{(\nu)}} r_i^{(s)} u_s = b_i : \; \lambda_i; \qquad i = 1, \dots, m \tag{2.56}$$

$$\sum_{s=1}^{p^{(\nu)}} u_s = 1 : \; \sigma \tag{2.57}$$

$$u_s \geq 0; \; s = 1, \dots, p^{(\nu)} \tag{2.58}$$

to obtain the solution $u_1^{(\nu)}, \dots, u_{p^{(\nu)}}^{(\nu)}$, and the dual variable values $\lambda_1^{(\nu)}, \dots, \lambda_m^{(\nu)}$ and $\sigma^{(\nu)}$.

Step 2: Relaxed problem solution. Generate a solution of the relaxed problem by solving the r subproblems ($k = 1, \dots, r$) below.

$$\underset{x_{n_{k-1}+1}, x_{n_{k-1}+2}, \dots, x_{n_k}}{\text{minimize}} \quad \sum_{j=n_{k-1}+1}^{n_k} \left(c_j - \sum_{i=1}^{m} \lambda_i^{(\nu)} a_{ij} \right) x_j \tag{2.59}$$

subject to

$$\sum_{j=n_{k-1}+1}^{n_k} e_{ij}x_j = f_i \ ; i = q_{k-1}+1,\dots,q_k \tag{2.60}$$

$$0 \ \leq \ x_j \leq x_j^{\text{up}} \ ; j = n_{k-1}+1,\dots,n_k \tag{2.61}$$

to obtain a solution of the relaxed problem, i.e., $x_1^{\left(p^{(\nu)}+1\right)},\dots,x_n^{\left(p^{(\nu)}+1\right)}$, and its objective function value $v^{(\nu)}$, i.e.,

$$v^{(\nu)} = \sum_{j=1}^{n}\left(c_j - \sum_{i=1}^{m}\lambda_i^{(\nu)}a_{ij}\right)x_j^{\left(p^{(\nu)}+1\right)} \ . \tag{2.62}$$

The objective function value of the original problem is

$$z^{\left(p^{(\nu)}+1\right)} = \sum_{j=1}^{n} c_j x_j^{\left(p^{(\nu)}+1\right)} \tag{2.63}$$

and the value of every complicating constraint is

$$r_i^{\left(p^{(\nu)}+1\right)} = \sum_{j=1}^{n} a_{ij}x_j^{\left(p^{(\nu)}+1\right)}; \qquad i = 1,\dots,m \ . \tag{2.64}$$

Step 3: Convergence checking. If $v^{(\nu)} \geq \sigma^{(\nu)}$, the optimal solution of the original problem has been achieved. It is computed as

$$x_j^* = \sum_{s=1}^{p^{(\nu)}} u_s^{(\nu)}x_j^{(s)}; \qquad j = 1,\dots,n \tag{2.65}$$

and the algorithm concludes.

Else if $v^{(\nu)} < \sigma^{(\nu)}$, the relaxed problem current solution can be used to improve the solution of the master problem. Update the iteration counter, $\nu \leftarrow \nu + 1$, and the number of available solutions of the relaxed problem, $p^{(\nu+1)} = p^{(\nu)} + 1$. Go to Step 1. □

A GAMS implementation of the Dantzig-Wolfe decomposition algorithm is given in the Appendix A, p. 397.

Computational Example 2.1 (The Dantzig-Wolfe decomposition). Consider the problem below

$$\underset{x_1,x_2,x_3}{\text{minimize}} \quad z = -4x_1 - x_2 - 6x_3$$

subject to

$$\begin{array}{rcl}
-x_1 \qquad\qquad\qquad & \leq & -1 \\
x_1 \qquad\qquad\qquad & \leq & 2 \\
-x_2 \qquad\quad & \leq & -1 \\
x_2 \qquad\quad & \leq & 2 \\
-x_3 & \leq & -1 \\
x_3 & \leq & 2 \\
3x_1 + 2x_2 + 4x_3 & \leq & 17 \\
x_1, x_2, x_3 & \geq & 0 \; .
\end{array}$$

Note that this problem has a decomposable structure and one complicating constraint. Its optimal solution is

$$x_1^* = 2, \;\; x_2^* = 3/2, \;\; x_3^* = 2 \; .$$

This problem is solved in the following steps using the Dantzig-Wolfe decomposition algorithm as previously stated in Subsect. 2.4.1.

Step 0: Initialization. The iteration counter is initialized, $\nu = 1$. Two ($p^{(1)} = 2$) solutions for the relaxed problem are obtained solving the three subproblems twice.

First, cost coefficients $\hat{c}_1^{(1)} = -1$, $\hat{c}_2^{(1)} = -1$, and $\hat{c}_3^{(1)} = -1$ are used. The subproblems for the first solution are

$$\underset{x_1}{\text{minimize}} \quad -x_1$$

subject to

$$1 \leq x_1 \leq 2 \; ,$$

whose solution is $x_1^{(1)} = 2$, and

$$\underset{x_2}{\text{minimize}} \quad -x_2$$

subject to

$$1 \leq x_2 \leq 2 \; ,$$

whose solution is $x_2^{(1)} = 2$, and

$$\underset{x_3}{\text{minimize}} \quad -x_3$$

subject to

$$1 \leq x_3 \leq 2$$

whose solution is $x_3^{(1)} = 2$.

The objective function of the relaxed problem is $z^{(1)} = -22$ and the complicating constraint value is $r_1^{(1)} = 18$.

Using cost coefficients $\hat{c}_1^{(2)} = 1$, $\hat{c}_2^{(2)} = 1$, and $\hat{c}_3^{(2)} = -1$, the subproblems are solved again to derive the second solution for the relaxed problem. This solution is $x_1^{(2)} = 1$, $x_2^{(2)} = 1$, and $x_3^{(2)} = 2$, leading to an objective function value $z^{(2)} = -17$ and a complicating constraint value $r_1^{(2)} = 13$.

Step 1: Master problem solution. The master problem below is solved.

$$\underset{u_1, u_2}{\text{minimize}} \quad -22u_1 - 17u_2$$

subject to

$$\begin{aligned} 18u_1 + 13u_2 &\leq 17 : \ \lambda_1 \\ u_1 \quad + u_2 &= 1 : \ \sigma \\ u_1, u_2 &\geq 0 \ . \end{aligned}$$

Its solution is $u_1^{(1)} = \frac{4}{5}$ and $u_2^{(1)} = \frac{1}{5}$ with dual variable values $\lambda_1^{(1)} = -1$ and $\sigma^{(1)} = -4$.

Step 2: Relaxed problem solution. The subproblems are solved below to obtain a solution for the current relaxed problem.

The objective function of the first subproblem is

$$\left(c_1 - \lambda_1^{(1)} a_{11}\right) x_1 = (-4 + 3)x_1 = -x_1$$

and its solution, obtained by inspection, is $x_1^{(3)} = 2$.

The objective function of the second subproblem is

$$\left(c_2 - \lambda_1^{(1)} a_{12}\right) x_2 = (-1 + 2)x_2 = x_2$$

and its solution is $x_2^{(3)} = 1$.

Finally, the objective function of the third subproblem is

$$\left(c_3 - \lambda_1^{(1)} a_{13}\right) x_3 = (-6 + 4)x_3 = -2x_3$$

and its solution is $x_3^{(3)} = 2$.

For this relaxed problem solution ($x_1^{(3)} = 2$, $x_2^{(3)} = 1$, $x_3^{(3)} = 2$), the objective function value of the original problem is $z^{(3)} = -21$ and the value of the complicating constraint $r_1^{(3)} = 16$.

Step 3: Convergence checking. The objective function value of the current relaxed problem is

$$v^{(1)} = -x_1^{(3)} + x_2^{(3)} - 2x_3^{(3)} = -5 \ .$$

Note that $v^{(1)} < \sigma^{(1)}$ $(-5 < -4)$ and therefore the current solution of the relaxed problem can be used to improve the solution of the master problem.

The iteration counter is updated, $\nu = 1 + 1 = 2$, and the number of available solutions of the relaxed problem is also updated, $p^{(2)} = 2 + 1 = 3$. The algorithm continues in Step 1.

Step 1: Master problem solution. The master problem below is solved.

$$\underset{u_1, u_2, u_3}{\text{minimize}} \quad -22u_1 - 17u_2 - 21u_3$$

subject to

$$\begin{aligned} 18u_1 + 13u_2 + 16u_3 &\leq 17 : \ \lambda_1 \\ u_1 \quad + u_2 \quad + u_3 &= 1 : \ \sigma \\ u_1, u_2, u_3 &\geq 0 \,. \end{aligned}$$

Its solution is $u_1^{(2)} = \frac{1}{2}$, $u_2^{(2)} = 0$, and $u_3^{(2)} = \frac{1}{2}$ with dual variable values $\lambda_1^{(2)} = -\frac{1}{2}$ and $\sigma^{(2)} = -13$.

Step 2: Relaxed problem solution. The subproblems are solved below to obtain a solution for the current relaxed problem.

The objective function of the first subproblem is

$$\left(c_1 - \lambda_1^{(2)} a_{11}\right) x_1 = (-4 + \frac{3}{2})x_1 = -\frac{5}{2}x_1$$

that renders $x_1^{(4)} = 2$.

The objective function of the second subproblem is

$$\left(c_2 - \lambda_1^{(2)} a_{12}\right) x_2 = (-1 + 1)x_2 = 0$$

that renders $x_2^{(4)} = 1$.

Finally, the objective function of the third subproblem is

$$\left(c_3 - \lambda_1^{(2)} a_{13}\right) x_3 = (-6 + 2)x_3 = -4x_3$$

and its solution is $x_3^{(4)} = 2$.

For this relaxed problem solution ($x_1^{(4)} = 2$, $x_2^{(4)} = 1$, $x_3^{(4)} = 2$), the objective function of the original problem is $z^{(4)} = -21$ and the value of the complicating constraint $r_1^{(4)} = 16$.

Step 3: Convergence checking. The objective function value of the current relaxed problem is

$$v^{(2)} = -\frac{5}{2}x_1^{(4)} - 4x_3^{(4)} = -13 \,.$$

Note that $v^{(2)} \geq \sigma^{(2)}$ ($-13 \geq -13$) and therefore the optimal solution of the original problem has been attained, i.e.,

$$\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = u_1^{(2)} \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} + u_2^{(2)} \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{pmatrix} + u_3^{(2)} \begin{pmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{pmatrix}$$

and

$$\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{3}{2} \\ 2 \end{pmatrix} .$$

□

The following example has been designed to illustrate a geometric interpretation of the Dantzig-Wolfe decomposition technique.

Computational Example 2.2 (The Dantzig-Wolfe algorithm). Consider the problem

$$\underset{x_1, x_2}{\text{minimize}} \quad z = 2x_1 + x_2 \tag{2.66}$$

subject to

$$\begin{array}{rcl} x_1 & \leq & 5 \\ x_2 & \leq & 5 \\ x_1 + x_2 & \leq & 9 \\ x_1 \ -x_2 & \leq & 4 \\ -x_1 \ -x_2 & \leq & -2 \\ -3x_1 \ -x_2 & \leq & -3 \\ x_1, x_2 & \geq & 0 , \end{array} \tag{2.67}$$

where the last four constraints in (2.67) are the complicating constraints.

As shown in Fig. 2.2a, where the original feasible region has been shaded, and the objective function contours drawn, it is clear that the global solution of this problem is

$$z^* = 2.5, \quad x_1^* = 0.5, \quad x_2^* = 1.5 .$$

Next, we use the Dantzig-Wolfe algorithm, which is illustrated in Table 2.1, where the solutions of the master problems and subproblems are shown for each iteration, until convergence.

Step 0: Initialization. As it has been explained, the initialization part of the algorithm requires a minimum of $p^{(\nu)} \geq 1$ solutions of the relaxed problem with arbitrary objective functions, to obtain a set of extreme points of the relaxed problem. We consider the following two arbitrary optimization problems:

$$\underset{x_1, x_2}{\text{minimize}} \quad z = -x_1 - x_2$$

and

$$\underset{x_1, x_2}{\text{minimize}} \quad z = -2x_1 + x_2$$

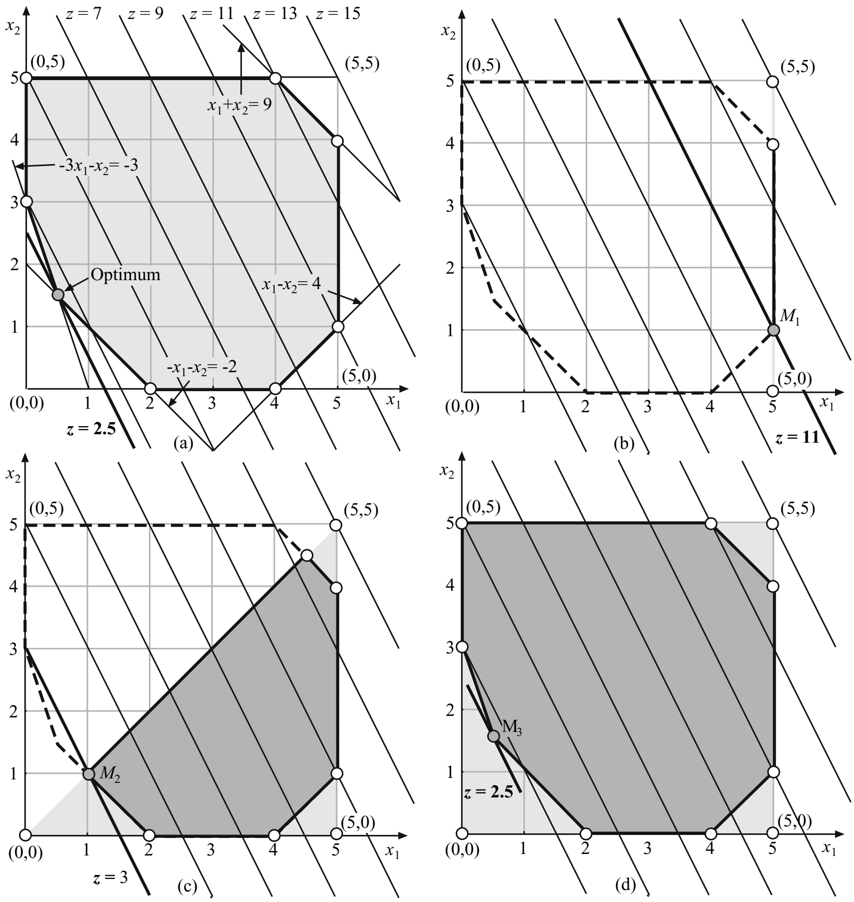

Fig. 2.2. Graphical illustration of the Dantzig-Wolfe decomposition algorithm

subject to

$$\begin{aligned} x_1 \quad &\leq 5 \\ x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \,, \end{aligned}$$

whose solutions are the extreme points $(5,5)$ and $(5,0)$, respectively. Next, we evaluate the complicating constraints [the last four constraints in (2.67)], not including the non-negativity constraints, and the target objective function (2.66), and obtain the values of r_1, r_2, r_3, r_4, and z in Table 2.1 (iteration 0). This ends the initialization step.

Step 1: Master problem solution. The master problem finds the point that minimizes the original objective function in the intersection of the set of linear convex combinations of the extreme points $(5,5)$ and $(5,0)$ [this is the

Table 2.1. Solutions of the master problem and the subproblems in Example (option 1)

Iteration	Initial solutions for the subproblems							
ν	$x_1^{(\nu)}$	$x_2^{(\nu)}$	$r_1^{(\nu)}$	$r_2^{(\nu)}$	$r_3^{(\nu)}$	$r_4^{(\nu)}$	$z^{(\nu)}$	$v^{(\nu)}$
0–1	5.0	5.0	10.0	0.0	–10.0	–20.0	15.0	–
0–2	5.0	0.0	5.0	5.0	-5.0	–15.0	10.0	–
	Solutions for the subproblem							
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	5.0	5.0	–5.0	–5.0	–5.0	5.0	–2.5
3	–	–	–	–	–	–	–	0.0

Iteration	Master problem solutions								
ν	$u_1^{(\nu)}$	$u_2^{(\nu)}$	$u_3^{(\nu)}$	$u_4^{(\nu)}$	$\lambda_1^{(\nu)}$	$\lambda_2^{(\nu)}$	$\lambda_3^{(\nu)}$	$\lambda_4^{(\nu)}$	$\sigma^{(\nu)}$	Feasible
1	0.2	0.8	–	–	0.0	–1.0	0.0	0.0	15.0	Yes
2	0.2	0.0	0.8	–	0.0	0.0	–1.5	0.0	0.0	Yes
3	0.1	0.0	0.7	0.2	0.0	0.0	–0.5	–0.5	0.0	Yes

segment $(5,5) - (5,0)$], and the original feasible region, i.e., in the segment $(5,4) - (5,1)$ [see Fig. 2.2b]. Since the master problem looks for the optimal solution in this intersection region, the point denoted by M_1 in Fig. 2.2b is obtained, with associated values of the primal variables, u_1 and u_2, and the dual variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and σ, as shown in Table 2.1 (master problem iteration 1). Note that only the second complicating constraint is active; this implies that the only λ-value different from zero is λ_2. Similarly, since the solution of the master problem is on the boundary of the feasible region of the relaxed problem, the value of σ is different from zero.

Step 2: Relaxed problem solution. The subproblems look for the extreme point in the relaxed feasible region to be added to the master problem so that the largest improvement in the objective function value is obtained. We have two options for selecting the extreme point to be incorporated (see Fig. 2.2c): $(0,0)$ and $(0,5)$. The extreme point $(0,5)$ would allow us to move to point $P:(0,5)$, and the extreme point $(0,0)$, to the point $M_2:(1,1)$.

Clearly, the optimum is obtained by adding the point $(0,0)$, and this is the point obtained after solving the relaxed problem. Next, the complicating constraints and the target objective function are evaluated at this point, and the values of r_1, r_2, r_3, r_4, and z are obtained (see subproblem iteration 1 in Table 2.1)

Step 3: Convergence checking. Since $v^{(1)} = 0 < \sigma^{(1)} = 15$, we go to Step 1.

Step 1: Master problem solution. Now the intersection of the set of linear convex combinations of the points in the set $\{(5,5),(5,0),(0,0)\}$ with the original feasible region, that is also indicated in Fig. 2.2c, is obtained. From this, the solution of the master problem can be easily obtained (point M_2 in the figure). The associated values of the primal, u_1, u_2, and u_3, and the dual variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and σ, are shown in Table 2.1 (master problem iteration 2). Note that only the third complicating constraint is active; this implies that the only λ-value different from zero is λ_3. Similarly, since the solution of the master problem is not in the boundary of the feasible region of relaxed problem, the value of σ is zero.

Step 2: Relaxed problem solution. Then, the subproblem step looks for the extreme point in the relaxed feasible region to be added, which is the point $(0,5)$ because it is the only one remaining. Next, the complicating constraints and the original objective function are evaluated at this point, and the values of r_1, r_2, r_3, r_4, and z are obtained (see subproblem iteration 2 in Table 2.1).

Step 3: Convergence checking. Since $v^{(2)} = -2.5 < \sigma^{(2)} = 0$, we go to Step 1.

Step 1: Master problem solution. The new relaxed feasible region (gray and shaded region), becomes the linear convex combination of the points in the set $\{(5,5),(5,0),(0,0),(0,5)\}$. The intersection of this region with the initial feasible region is also indicated in the Fig. 2.2d, from which the solution of the master problem can be easily obtained (point M_3 in the figure). The associated values for the primal u_1, u_2, u_3, and u_4 and dual variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and σ, are shown in Table 2.1 (master problem iteration 3). Note that only the third and the fourth complicating constraints are active; this implies that the only λ-value different from zero are λ_3 and λ_4. Similarly, since the solution of the master problem is not at the boundary of the feasible region of the relaxed problem, the value of σ is zero.

Step 2: Relaxed problem solution. Since no extreme point can be added, the algorithm continues in Step 3.

Step 3: Convergence checking. Since $\nu^{(3)} = 0 \geq \sigma^{(3)} = 0$, the optimal solution has been obtained and the algorithm returns the optimal solution using the following expressions:

$$\begin{aligned} z^* &= u_1^{(3)} z^{(0-1)} + u_2^{(3)} z^{(0-2)} + u_3^{(3)} z^{(1)} + u_4^{(3)} z^{(2)} \\ &= 0.1 \times 15 + 0.0 \times 10 + 0.7 \times 0 + 0.2 \times 5 = 2.5 \end{aligned}$$

and

$$\begin{aligned}(x_1^*, x_2^*) &= u_1^{(3)}(x_1^{(0-1)}, x_2^{(0-1)}) + u_2^{(3)}(x_1^{(0-2)}, x_2^{(0-2)}) + u_3^{(3)}(x_1^{(1)}, x_2^{(1)}) \\ &\quad + u_4^{(3)}(x_1^{(2)}, x_2^{(2)}) \\ &= 0.1 \times (5,5) + 0.0 \times (5,0) + 0.7 \times (0,0) + 0.2 \times (0,5) \\ &= (0.5, 1.5)\ ,\end{aligned}$$

where superscript $(0-\nu)$ indicates initial solution ν. □

2.4.2 Bounds

An upper and a lower bound of the objective function value that are obtained as the Dantzig-Wolfe algorithm progresses, are derived below. These bounds are of interest to stop the solution procedure once a prespecified error tolerance is satisfied.

The upper bound is readily available once the master problem objective function value is available. The master problem is, in fact, a restricted version of the original problem and, therefore, its objective function value is an upper bound of the optimal objective function value of the original problem. At iteration ν, the objective function of the master problem is

$$\sum_{i=1}^{p^{(\nu)}} z^{(i)} u_i^{(\nu)}\ .$$

An upper bound of the optimal objective function value of the original problem is therefore

$$z_{\text{up}}^{(\nu)} = \sum_{i=1}^{p^{(\nu)}} z^{(i)} u_i^{(\nu)}\ . \tag{2.68}$$

A lower bound is easily obtained from the solutions of the subproblems. The relaxed problem at iteration ν is

$$\underset{x_j;\, j=1,\ldots,n}{\text{minimize}} \quad \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} \lambda_i^{(\nu)} a_{ij} \right) x_j \tag{2.69}$$

subject to

$$\sum_{j=1}^{n} e_{ij} x_j = f_i\ ; \qquad i = 1, \ldots, q \tag{2.70}$$

$$0 \ \le\ x_j \le x_j^{\text{up}}\ ; \qquad j = 1, \ldots, n \tag{2.71}$$

and its optimal objective function value is $v^{(\nu)}$.

Consider a feasible solution x_j $(j = 1, \ldots, n)$ of the original problem. Therefore, this solution meets the complicating constraints, i.e.,

$$\sum_{j=1}^{n} a_{ij} x_j = b_i; \qquad i = 1, \ldots, m. \tag{2.72}$$

Since $v^{(\nu)}$ is the optimal objective function value of problem (2.69)–(2.71) above, substituting the arbitrary feasible solution into the objective function of that problem renders

$$\sum_{j=1}^{n}\left(c_j - \sum_{i=1}^{m}\lambda_i^{(\nu)}a_{ij}\right)x_j \geq v^{(\nu)} \tag{2.73}$$

and

$$\sum_{j=1}^{n} c_j x_j \geq v^{(\nu)} + \sum_{i=1}^{m}\lambda_i^{(\nu)}\sum_{j=1}^{n} a_{ij}x_j \tag{2.74}$$

so that using (2.72) yields

$$\sum_{j=1}^{n} c_j x_j \geq v^{(\nu)} + \sum_{i=1}^{m}\lambda_i^{(\nu)} b_i \ . \tag{2.75}$$

Due to the fact that x_j $(j = 1, \ldots, n)$ is an arbitrary feasible solution of the original problem, the inequality above allows writing

$$z_{\text{down}}^{(\nu)} = v^{(\nu)} + \sum_{i=1}^{m}\lambda_i^{(\nu)} b_i \ , \tag{2.76}$$

where $z_{\text{down}}^{(\nu)}$ is a lower bound of the optimal objective function value of the original problem.

2.4.3 Issues Related to the Master Problem

The possible infeasibility of the master problem is considered in this subsection and an alternative always-feasible master problem is formulated. The price paid is a larger number of variables and dissimilar cost coefficients in the objective function.

The selection of p solutions of the relaxed problem are usually motivated by engineering considerations and it allows us the formulation of a feasible master problem. However, in some instances, the formulation of a feasible master problem might not be simple. In such a situation, an always feasible master problem that includes artificial variables can be formulated. This problem has the form

$$\underset{u_1,\ldots,u_p;v_1,\ldots,v_m,w}{\text{minimize}} \quad \sum_{i=1}^{p} z^{(i)}u_i + M\left(\sum_{j=1}^{m} v_j + w\right) \tag{2.77}$$

subject to

$$\sum_{i=1}^{p} r_j^{(i)} u_i + v_j - w = b_j \, : \; \lambda_j; \; j = 1, \ldots, m \tag{2.78}$$

$$\sum_{i=1}^{p} u_i = 1 \, : \; \sigma \tag{2.79}$$

$$u_i \geq 0; \; i = 1, \ldots, p \tag{2.80}$$

$$0 \leq v_j \leq v_j^{\mathrm{up}}; \; j = 1, \ldots, m \tag{2.81}$$

$$0 \leq w \leq w^{\mathrm{up}} \, , \tag{2.82}$$

where v_j $(j = 1, \ldots, m)$ and w are the artificial variables, and v_j^{up} $(j = 1, \ldots, m)$ and w^{up} are their respective upper bounds.

Note that the artificial variable makes problem (2.77)–(2.82) always feasible.

To illustrate how this master problem behaves, and its geometrical interpretation, we include the following example.

Computational Example 2.3 (The Dantzig-Wolfe example revisited). Consider the same problem as in Example 2.2, but assume that only the first arbitrary objective function is used in the initialization Step 0.

Step 0: Initialization. Then, we consider the following arbitrary optimization problems

$$\underset{x_1, x_2}{\text{minimize}} \quad z = -x_1 - x_2$$

subject to

$$\begin{array}{rcl} x_1 & \leq & 5 \\ x_2 & \leq & 5 \, , \end{array}$$

whose solution is the extreme point $(5, 5)$. Then, we evaluate the complicating constraints [the last four constraints in (2.67)] and the original objective function (2.66), and obtain the values of r_1, r_2, r_3, r_4, and z in Table 2.2 (iteration 0).

Step 1: Master problem solution. The master problem is infeasible because the intersection of the region generated by all linear convex combinations of the point $(5, 5)$ reduces to this point, which is not in the original feasible region.

In the modified master problem, we obtain feasibility by modifying the hyperplane boundaries, translating them by the minimum amount required to attain feasibility. This means that those constraints that lead to feasibility are kept ($v_j = w$), and those that lead to infeasibility are modified the minimum amount to get feasibility.

In our example, only the constraint $x_1 + x_2 = 9$ is not satisfied by the point $(5, 5)$, so it is replaced by $x_1 + x_2 = 10$, as illustrated in Fig. 2.3a.

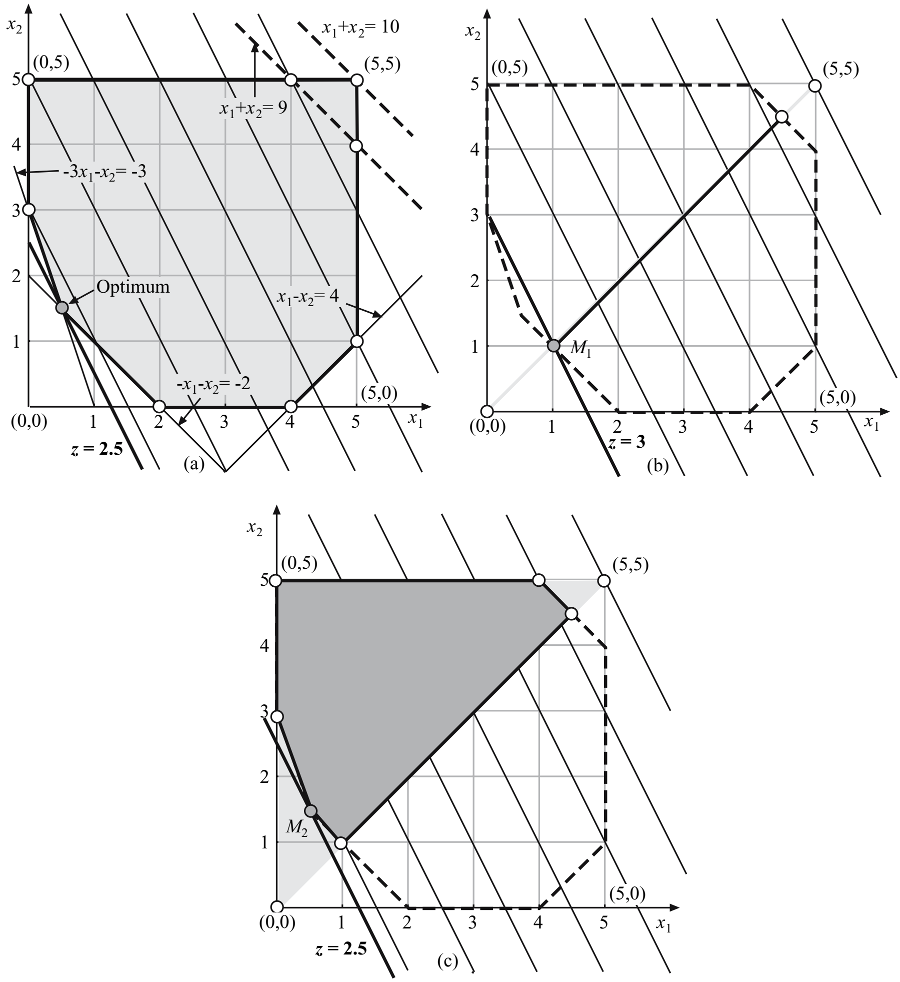

Fig. 2.3. Graphical illustration of how the feasible region is modified to make the master problem feasible

Since the master problem looks for the optimal solution in this modified intersection region, the point $(5,5)$ is obtained, whose associated values of the primal, u_1, and the dual variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and σ, are shown in Table 2.2 (master problem iteration 1).

Step 2: Relaxed problem solution. The subproblems look for the extreme point in the relaxed feasible region to be added to the master problem so that the largest improvement in the objective function value is obtained. We have three options for selecting the extreme point to be incorporated (see Fig. 2.3b): $(0,0), (0,5)$, and $(5,0)$. The extreme point $(0,5)$ would allow us to move to point $P : (0,5)$, the extreme point $(0,0)$, to the point $M_1 : (1,1)$, and the extreme point $(5,0)$, to the point $(5,1)$.

Clearly, the optimum is obtained by adding the point $(0,0)$, and this is the point obtained after solving the relaxed problem. Next, the complicating constraints and the original objective function are evaluated at this point, and the values of r_1, r_2, r_3, r_4, and z are obtained (see subproblem iteration 1 in Table 2.2).

Step 3: Convergence checking. Since $v^{(1)} = 0 < \sigma^{(1)} = 215$, the procedure continues in Step 1.

Step 1: Master problem solution. Now the intersection of the set of linear convex combinations of the points in the set $\{(5,5),(0,0)\}$ with the target feasible region that is the segment $(1,1)$–$(4.5,4.5)$, is obtained. From it, the solution of the master problem can be easily obtained (point M_1 in Fig. 2.3b). The associated values of the primal, u_1 and u_2, and the dual variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and σ, are shown in Table 2.2 (master problem iteration 2).

Step 2: Relaxed problem solution. Next, the subproblem step looks for the extreme point in the relaxed feasible region to be added. We have two options for selecting the extreme point to be incorporated (see Fig. 2.3c): $(0,5)$ and $(5,0)$. The extreme point $(0,5)$ would allow us to move to point $M_2 : (0.5, 1.5)$, and the extreme point $(5,0)$ would not allow us further improvement. So, we add the point $(0,5)$. Next, the complicating constraints and the original objective function are evaluated at this point, and the values of r_1, r_2, r_3, r_4, and z are obtained (see subproblem iteration 2 in Table 2.2).

Step 3: Convergence checking. Since $v^{(2)} = -2.5 < \sigma^{(2)} = 0$, the algorithm continues in Step 1.

Step 1: Master problem solution. The new relaxed feasible region (gray and shaded region), becomes the linear convex combination of the points in the set $\{(5,5),(0,0),(0,5)\}$. The intersection of this region with the initial feasible region is also indicated in the Fig. 2.3c, from which the solution of the master problem can be easily obtained (point M_2 in Fig. 2.3c). The associated values for the primal u_1, u_2, and u_3 and dual variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and σ, are shown in Table 2.2 (master problem iteration 3).

Step 2: Relaxed problem solution. Since no extreme point can be added, the algorithm continues in Step 3.

Step 3: Convergence checking. Since $v^{(3)} = 0 \geq \sigma^{(3)} = 0$, the optimal solution has been obtained and the algorithm returns the optimal solution using the following expressions:

Table 2.2. Solutions of the master problems and subproblems in Example 2.3

Iteration	Bounds		Initial solutions for the subproblems							
l	Lower	Upper	$x_1^{(\nu)}$	$x_2^{(\nu)}$	$r_1^{(\nu)}$	$r_2^{(\nu)}$	$r_3^{(\nu)}$	$r_4^{(\nu)}$	$z^{(\nu)}$	$v^{(\nu)}$
0	$-\infty$	∞	5.0	5.0	10.0	0.0	-10.0	-20.0	15.0	–
			Solutions for the subproblem							
1	-180.0	35.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.50	3.00	0.0	5.0	5.0	-5.0	-5.0	-5.0	5.0	-2.5
3	2.50	2.50	–	–	–	–	–	–	–	0.0

Iteration	Bounds		Master solutions								
ν	Lower	Upper	$u_1^{(\nu)}$	$u_2^{(\nu)}$	$u_3^{(\nu)}$	$\lambda_1^{(\nu)}$	$\lambda_2^{(\nu)}$	$\lambda_3^{(\nu)}$	$\lambda_4^{(\nu)}$	$\sigma^{(\nu)}$	Feasible
1	$-\infty$	35.0	1.0	–	–	-20.0	0.0	0.0	0.0	215	No
2	-180.0	3.00	0.2	0.8	–	0.0	0.0	-1.5	0.0	0.0	Yes
3	0.50	2.50	0.1	0.7	0.2	0.0	0.0	-0.5	-0.5	0.0	Yes

$$z^* = u_1^{(3)} z^{(0)} + u_2^{(3)} z^{(1)} + u_3^{(3)} z^{(2)} = 0.1 \times 15 + 0.7 \times 0 + 0.2 \times 5 = 2.5$$

and

$$\begin{aligned}(x_1^*, x_2^*) &= u_1^{(3)}(x_1^{(0)}, x_2^{(0)}) + u_2^{(3)}(x_1^{(1)}, x_2^{(1)}) + u_3^{(3)}(x_1^{(2)}, x_2^{(2)}) \\ &= 0.1 \times (5,5) + 0.7 \times (0,0) + 0.2 \times (0,5) = (0.5, 1.5)\ .\end{aligned}$$

Using expressions (2.68)–(2.76) lower and upper bounds on the solution of Example 2.3 are computed and plotted in Fig. 2.4. Observe how bounds approach each other until the optimal solution is attained. □

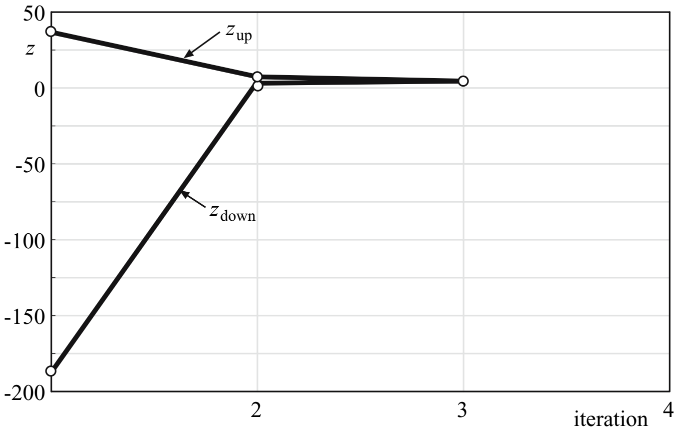

Fig. 2.4. Evolution of the upper and lower bounds of the objective function in Example 2.3

To comprehend the Dantzig-Wolfe decomposition, we encourage the reader to determine the solutions of the master and subproblems corresponding to the following cases:

1. The initialization step leads to the extreme point: $(5, 0)$.
2. The initialization step leads to the extreme point: $(0, 5)$.
3. The initialization step leads to the extreme point: $(0, 0)$.
4. The initialization step leads to the extreme points: $(5, 0)$ and $(0, 5)$.

2.4.4 Alternative Formulation of the Master Problem

In some practical applications, it is convenient to formulate the master problem (2.29)–(2.32) in the alternative format stated below.

$$\underset{u_{ij};\, i=1,\dots,p;\, j=1,\dots,r}{\text{minimize}} \quad \sum_{i=1}^{p}\sum_{j=1}^{r} z^{(ij)} u_{ij} \tag{2.83}$$

subject to

$$\sum_{i=1}^{p}\sum_{j=1}^{r} r_{ij}^{(\ell)} u_{ij} = b^{(\ell)} \,:\; \lambda^{(\ell)} \;;\; \ell = 1, \dots, m \tag{2.84}$$

$$\sum_{i=1}^{p} u_{ij} = 1 \,:\; \sigma_j \;;\; j = 1, \dots, r \tag{2.85}$$

$$u_{ij} \geq 0 \;;\; i = 1, \dots, p; j = 1, \dots, r \;, \tag{2.86}$$

where $z^{(ij)}$ is the objective function value for the solution i of the subproblem j and $r_{ij}^{(\ell)}$ is the contribution to the right-hand-side values of the complicating constraint ℓ of the solution i of the subproblem j. In Fig. 2.5 a graphical comparison between the master problem formulation presented in this section and the one presented in Sect. 2.4.3 is shown.

Note that the above formulation relies on the fact that a convex combination of any number of basic feasible solutions of any subproblem is a feasible solution of this subproblem. Note also that $u_{1j}, \dots, u_{pj}$ are the convex combination variables corresponding to subproblem j ($j = 1, \dots, r$). The r constraints (2.85) in the problem above are equivalent to the single constraint (2.31) in the original formulation of the master problem.

The following observations are in order:

1. Any solution of problem (2.83)–(2.86) is a convex linear combination of basic feasible solutions of the subproblems and therefore of the relaxed problem; thus, it is itself a basic feasible solution of the relaxed problem.
2. Complicating constraints are enforced; therefore, the solution of the above problem is a basic feasible solution for the original (nonrelaxed) problem.

A valuable geometrical interpretation associated with this alternative approach is illustrated in the example below.

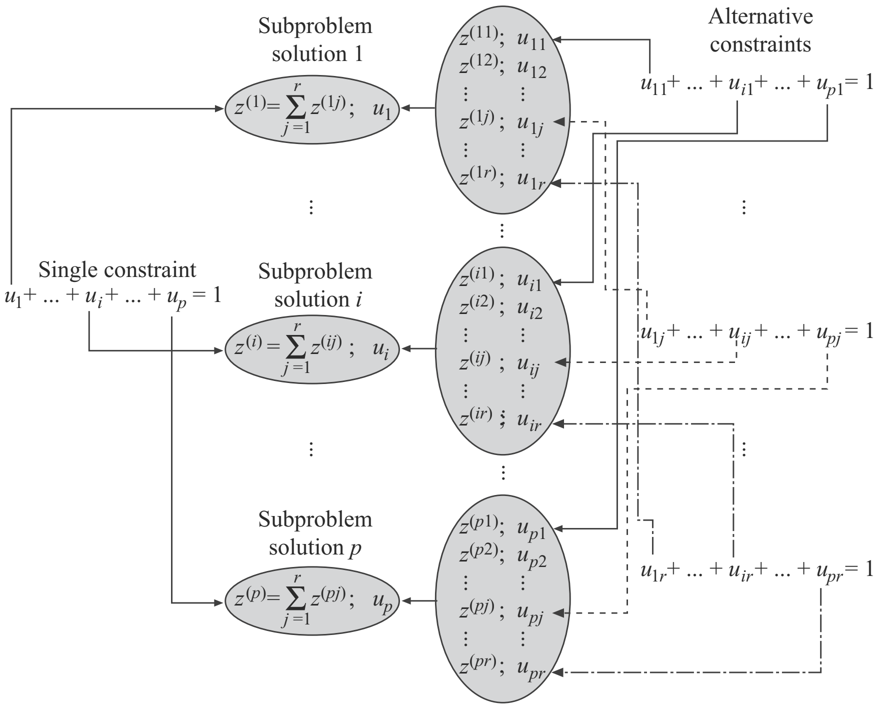

Fig. 2.5. Relationship of the two alternative master problem formulations

Computational Example 2.4 (Alternative master problem). Consider the problem

$$\underset{x_1, x_2, x_3}{\text{minimize}} \quad z = x_1 + x_2 + x_3 \tag{2.87}$$

subject to

$$\begin{array}{rcr} x_1 \qquad\qquad\quad & \leq & 5 \\ x_2 \qquad\quad & \leq & 5 \\ 1/2x_1 \;\; -x_2 \qquad\quad & \leq & -1/2 \\ x_3 & \leq & 4 \\ -x_1 + x_2 + x_3 & \leq & 0 \\ x_1, x_2, x_3 & \geq & 0\;. \end{array} \tag{2.88}$$

Note that this problem has a decomposable structure in two blocks and one complicating constraint. Its optimal solution is

$$z^* = 2,\ x_1^* = 1,\ x_2^* = 1,\ x_3^* = 0\ .$$

Figure 2.6 illustrates the feasibility region of problem (2.87)–(2.88), where Fig. 2.6a represents the feasible region of that problem not considering the single complicating constraint (relaxed problem), Fig. 2.6b shows the feasible

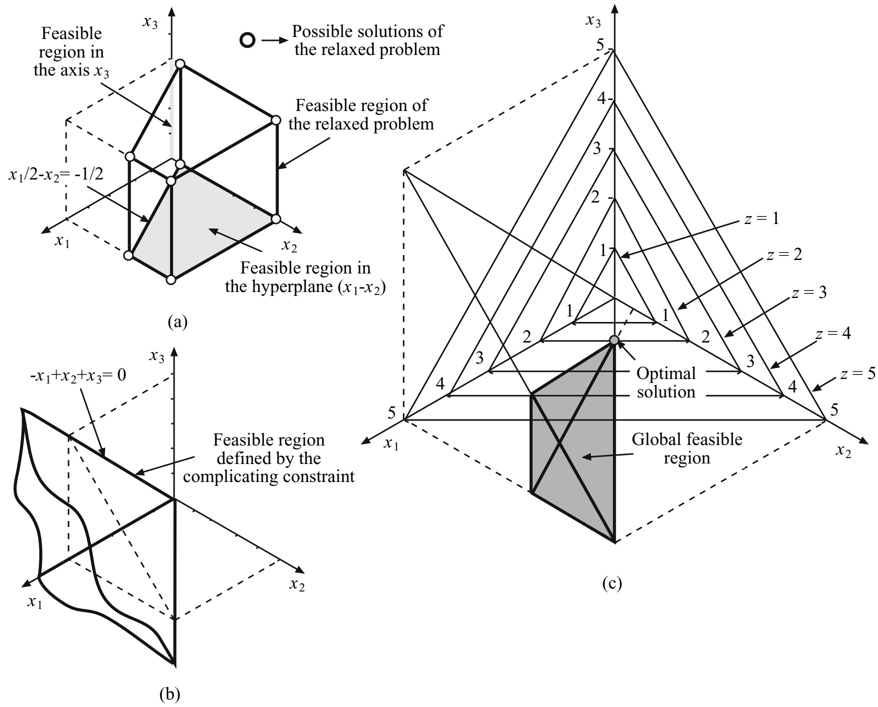

Fig. 2.6. Graphical illustration of the feasibility region of the Computational Example 2.4

region defined by the complicating constraint, and Fig. 2.6c shows the feasible region of the original problem, and the objective function contours.

The problem above is solved using the Dantzig-Wolfe algorithm and the two alternative master problem definitions.

Step 0: Initialization. Consider the objective function

$$z = 5x_1 + 2x_2 - x_3,$$

to obtain an initial feasible solution of the relaxed problem for both decomposition algorithms. This solution, represented in Fig. 2.7, is $S_1 = (0, 0.5, 4)$. The complicating constraint [last constraint in (2.88)] and the objective function (2.87), are then evaluated to obtain $r^{(0)}$ and $z^{(0)}$ that are shown in Table 2.3 and $r_1^{(0)}, r_2^{(0)}, z_1^{(0)}$, and $z_2^{(0)}$ that are shown in Table 2.4.

Step 1: Master problem solutions. The master problems are solved obtaining the solution $M_1 = (0, 0.5, 4)$ shown in Fig. 2.7. Values for the primal variables $u_1^{(1)}, u_{11}^{(1)}, u_{12}^{(1)}$ and the dual variables $\lambda^{(1)}, \sigma^{(1)}, \sigma_1^{(1)}, \sigma_2^{(1)}$ are shown in Tables 2.3 and 2.4, respectively.

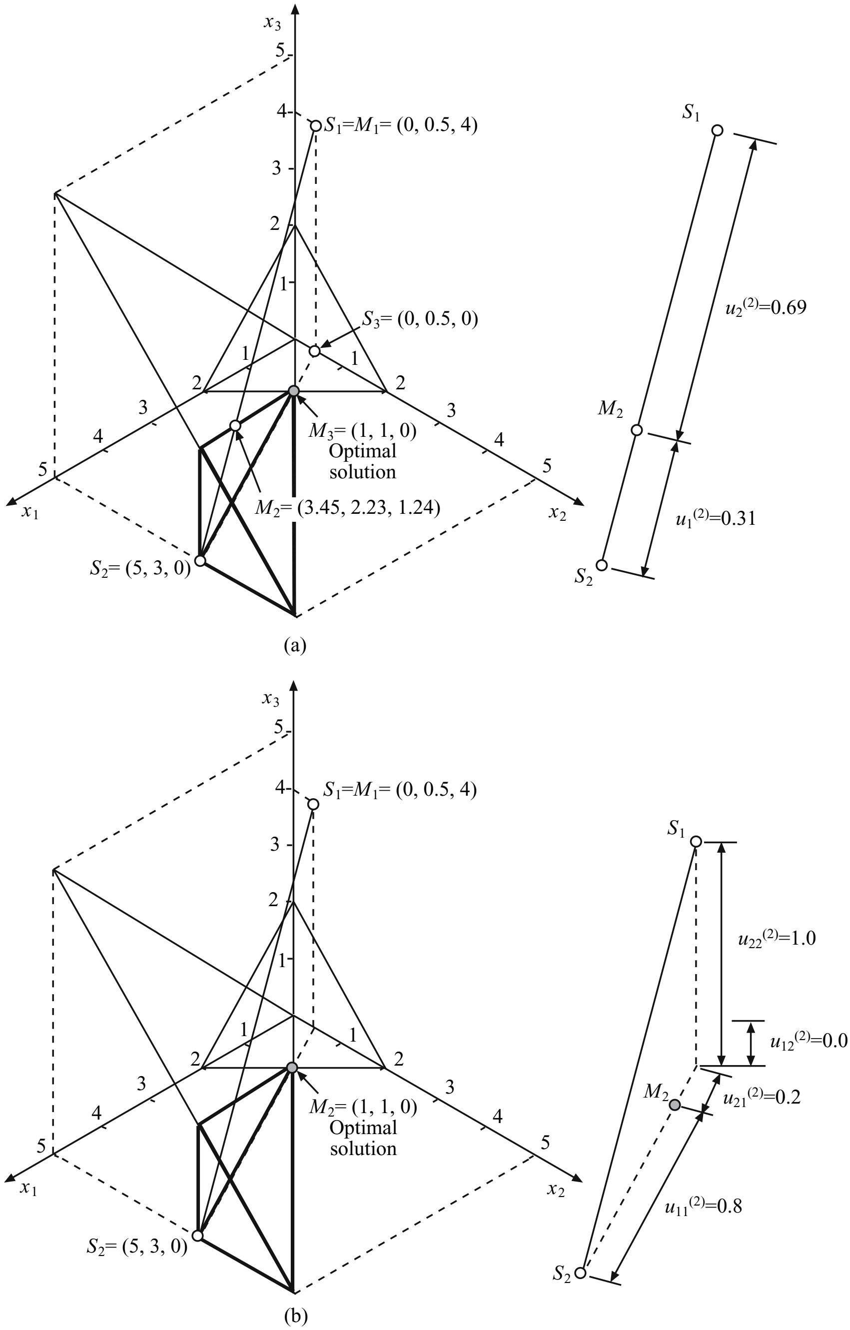

Fig. 2.7. Graphical illustration of the functioning of the two alternative algorithms used in Computational Example 2.4

Table 2.3. Solutions of the master problem and the subproblems in Computational Example 2.4 using the first master problem formulation

Iteration	Bounds		Initial solutions for the subproblem				
ν	Lower	Upper	$x_1^{(\nu)}$	$x_2^{(\nu)}$	$x_3^{(\nu)}$	$r^{(\nu)}$	$z^{(\nu)}$
0	−1.0E+8	1.0E+8	0.0	0.5	4.0	4.5	4.5
			Solutions for the subproblem				
1	−32.00	94.50	5.0	3.0	0.0	−2.0	8.0
2	0.77	6.92	0.0	0.5	0.0	0.5	0.5
3	2.00	2.00	–	–	–	–	–

Iteration	Master solutions					
ν	$u_1^{(\nu)}$	$u_2^{(\nu)}$	$u_3^{(\nu)}$	$\lambda^{(\nu)}$	$\sigma^{(\nu)}$	Feasible
1	1.0	0.0	0.0	−20.0	94.5	No
2	0.31	0.69	0.0	−0.54	6.92	Yes
3	0.0	0.2	0.8	−3.0	2.0	Yes

Table 2.4. Solutions of the master problem and the subproblems in Computational Example 2.4 using the alternative master problem formulation

Iteration	Bounds		Initial solutions for the subproblem								
ν	Lower	Upper	$x_1^{(\nu)}$	$x_2^{(\nu)}$	$x_3^{(\nu)}$	$r_1^{(\nu)}$	$r_2^{(\nu)}$	$r^{(\nu)}$	$z_1^{(\nu)}$	$z_2^{(\nu)}$	$z^{(\nu)}$
0	−1.0E+8	1.0E+8	0.0	0.5	4.0	0.5	4.0	4.5	0.5	4.0	4.5
	Solutions for the subproblem										
1	−32.00	94.50	5.0	3.0	0.0	−2.0	0.0	−2.0	8.0	0.0	8.0
2	2.00	2.00	–	–	–	–	–	–	–	–	

Iteration	Master solutions							
ν	$u_{11}^{(\nu)}$	$u_{12}^{(\nu)}$	$u_{21}^{(\nu)}$	$u_{22}^{(\nu)}$	$\lambda^{(\nu)}$	$\sigma_1^{(\nu)}$	$\sigma_2^{(\nu)}$	$\sigma^{(\nu)}$	Feasible
1	1.0	1.0	0.0	0.0	–20.0	10.5	84.0	94.5	No
2	0.8	0.0	0.2	1.0	–3.0	2.0	0.0	2.0	Yes

Step 2: Relaxed problem solution. The objective of the subproblems is to attain the most convenient solution inside the relaxed feasible region to be added to the master problem. This results in solution $S_2 = (5, 3, 0)$ shown in Fig. 2.7. The complicating constraint and the objective function are then evaluated to obtain $r^{(1)}$ and $z^{(1)}$ that are shown in Table 2.3 and $r_1^{(1)}, r_2^{(1)}, z_1^{(1)}$, and $z_2^{(1)}$ that are shown in Table 2.4.

Step 3: Convergence checking. Since

$$v^{(1)} = z^{(1)} = 8 < \sigma^{(1)} = 94.5$$

and

$$v^{(1)} = z_1^{(1)} + z_2^{(1)} = 8 < \sigma_1^{(1)} + \sigma_2^{(1)} = 94.5 \,,$$

the current solution of the relaxed problem can be used to improve the solution of the master problem.

Step 1: Master problem solutions. Up to this point, both decomposition algorithms work identically, as shown in Fig. 2.7. In this step, they follow different paths.

- The master problem (2.77)–(2.82) is solved finding the solution M_2, shown in Fig. 2.7a. This solution minimizes the original objective function in the intersection of (i) the set of linear convex combinations of solutions S_1 and S_2, and (ii) the original feasible region. The values of the primal variables $u_1^{(2)}, u_2^{(2)}$ and dual variables $\lambda^{(2)}$, and $\sigma^{(2)}$ are shown in Table 2.3.

$$M_2 = u_1^{(2)} S_1 + u_2^{(2)} S_2 = 0.31 \begin{pmatrix} 0 \\ 0.5 \\ 4 \end{pmatrix} + 0.69 \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.45 \\ 2.23 \\ 1.24 \end{pmatrix} .$$

- The alternative master problem (2.83)–(2.86) is solved finding the solution M_2 shown in Fig. 2.7b. Note that this solution minimizes the original objective function in the intersection of (i) the set of linear convex combinations decomposed by blocks of the solutions S_1 and S_2, and (ii) the global feasible region. The associated values of the primal variables $u_{11}^{(2)}, u_{12}^{(2)}, u_{21}^{(2)}, u_{22}^{(2)}$ and dual variables $\lambda^{(2)}, \sigma_1^{(2)}$, and $\sigma_2^{(2)}$ are shown in Table 2.4.

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= u_{11}^{(2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{S_1} + u_{21}^{(2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{S_2} \\ &= 0.8 \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} + 0.2 \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} (x_3) &= u_{12}^{(2)} \, (x_3)_{S_1} + u_{22}^{(2)} \, (x_3)_{S_2} \\ &= 0.0 \, (4) + 1.0 \, (0) = (0) \end{aligned}$$

$$M_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} .$$

Step 2: Relaxed problem solution. The subproblem to be solved is the same for both decomposition approaches. Its target is to find the most convenient solution inside the relaxed feasible region to be added to the master problem. This solution is $S_3 = (0, 0.5, 0)$ shown in Fig. 2.7. The complicating constraint and the target objective function are then evaluated to obtain $r^{(1)}$ and $z^{(1)}$, which are shown in Table 2.3.

Step 3: Convergence checking. Note that this step is different for both decompositions algorithms.

- As
$$v^{(2)} = z^{(2)} = 0.5 < \sigma^{(2)} = 6.92 \; ,$$
the current solution of the relaxed problem can be used to improve the solution of the master problem in the first approach.
- Since
$$v^{(2)} = z_1^{(1)} + z_2^{(1)} = \sigma_1^{(2)} + \sigma_2^{(2)} = 2 \; ,$$
the optimal solution has been obtained using the second alternative. This optimal solution is M_2, shown in Fig. 2.7b. Therefore, this algorithm concludes.

Step 1: Master problem solution. The master problem solution using the first approach is $M_3 = (1, 1, 0)$, shown in Fig. 2.7a. The values of the primal variables $u_1^{(3)}, u_2^{(3)}, u_3^{(3)}$ and the dual variables $\lambda^{(3)}$, and $\sigma^{(3)}$ are shown in Table 2.3. Note that these values are optimal. Therefore the algorithm concludes.

□

Concerning the alternative formulation (2.83)–(2.86) of the master problem, the following observations are in order:

1. Convex combination of basic feasible solutions of any subproblem are treated independently from other subproblems. This may provide the master problem with more flexibility to attain the optimal solution of the original problem.
2. Despite of a larger number of variables in the master problem, it has been observed in practical applications (see [9]) that this alternative master problem performs usually better than the initial one.

2.5 Concluding Remarks

This chapter analyzes linear problems that include complicating constraints. If these constraints are relaxed, the original problem decomposes by blocks or attains such a structure that its solution is straightforward. This circumstance occurs often in engineering and science problems. The Dantzig-Wolfe decomposition algorithm is motivated, derived, and illustrated in this chapter. Alternative formulations of the master problem are considered, and bounds on the optimal value of the objective function are provided. Diverse geometrical interpretations enrich the algebra-oriented algorithms and quite a few illustrative examples are analyzed in detail.

The decomposition technique analyzed in this chapter to solve the original linear problem is the so-called Dantzig-Wolfe decomposition procedure. This decomposition is also analyzed in the excellent references by Bazaraa et al.[5], Chvatal [22], and Luenberger [23], in the application-oriented manual by Bradley et al. [1], and in the historical book by Dantzig [2].

2.6 Exercises

Exercise 2.1. The problem faced by a multinational company that manufactures one product in different countries is analyzed in Sect. 1.3.1, p. 8. This problem is formulated as a linear programming problem that includes complicating constraints.

Solve the numerical example presented in that section using Dantzig-Wolfe decomposition. Analyze the numerical behavior of the decomposition algorithm and show that the result obtained are identical to those provided in Sect. 1.3.1.

Exercise 2.2. Given the problem

$$\underset{x_1,x_2,x_3,x_4}{\text{minimize}} \quad z = -2x_1 - x_2 - x_3 + x_4$$

subject to

$$\begin{aligned}
x_1 + 2x_2 \quad\quad\quad\quad\quad &\le 5\\
-x_3 + x_4 &\le 2\\
2x_3 + x_4 &\le 6\\
x_1 \quad\quad + x_3 \quad\quad &\le 2\\
x_1 + x_2 \quad\quad + 2x_4 &\le 3\\
x_1, x_2, x_3, x_4 &\ge 0\ .
\end{aligned}$$

1. Check that the following vector (x_1, x_2, x_3, x_4) is a solution:

$$x_1 = 1,\quad x_2 = 2,\quad x_3 = 1,\quad x_4 = 0,\quad z = -5.$$

2. Using the Dantzig-Wolfe decomposition algorithm and by minimizing the objective functions,

$$\begin{aligned}
z_1 &= -x_1 - x_2 + x_4\\
z_2 &= x_1 + x_2 - x_3\\
z_3 &= x_1 - x_3 + x_4\\
z_4 &= 2x_1 + x_2 + 3x_4\ ,
\end{aligned}$$

obtain the two different feasible solutions (x_1, x_2, x_3, x_4) of the relaxed problem and the associated values of r_i and z, shown in Table 2.5.

Table 2.5. Initial solutions for the subproblems and new added solutions using the Dantzig-Wolfe decomposition algorithm for Exercise 2.2

Iteration	Bounds		Initial solutions for the subproblems						
ν	Lower	Upper	$x_1^{(\nu)}$	$x_2^{(\nu)}$	$x_3^{(\nu)}$	$x_4^{(\nu)}$	$r_1^{(\nu)}$	$r_2^{(\nu)}$	$z^{(\nu)}$
0–1	$-\infty$	∞	5.00	0.00	0.00	0.00	5.00	5.00	−10.00
0–2	$-\infty$	∞	0.00	0.00	3.00	0.00	3.00	0.00	−3.00
			Subproblem solutions						
1	−42.50	17.00	0.00	2.50	0.00	0.00	0.00	2.50	−2.50
2	−5.00	−5.00	–	–	–	–	–	–	–

Iteration	Bounds		Master solutions						
ν	Lower	Upper	$u_1^{(\nu)}$	$u_2^{(\nu)}$	$u_3^{(\nu)}$	$\lambda_1^{(\nu)}$	$\lambda_2^{(\nu)}$	$\sigma^{(\nu)}$	Feasible
1	$-\infty$	17.00	0.00	1.00	0.00	−20.00	0.00	57.00	No
2	−42.50	−5.00	0.30	0.10	0.50	−1.00	−1.00	0.00	Yes

3. Show that using the Dantzig-Wolfe decomposition algorithm the following solution is obtained

$$x_1 = 1.6, \quad x_2 = 1.4, \quad x_3 = 0.4, \quad x_4 = 0, \quad z = -5 \; .$$

4. Compare and discuss the resulting solution and that given in item 1 above.

Exercise 2.3. David builds electrical cable using 2 type of alloys, A and B. Alloy A contains 80% of copper and 20% of aluminum, whereas alloy B contains 68% of copper and 32% of aluminum. Costs of alloys A and B are \$80 and \$60, respectively. In order to produce 1 unit of cable and ensuring that the cable manufactured does not contain more that 25% of aluminum, which are the quantities of alloys A and B that David should use to minimize his manufacturing cost?

Consider the constraint limiting the amount of aluminum a complicating constraint and solve the problem using Dantzig-Wolfe decomposition.

Exercise 2.4. Given the problem

$$\underset{x_1, x_2, \ldots, x_{10}}{\text{minimize}} \qquad z = -4x_1 - x_4 - 6x_7$$

subject to

$$
\begin{array}{rrrrrrrrrrl}
x_1 & -x_2 & & & & & & & & & = \ 1 \\
x_1 & & +\,x_3 & & & & & & & & = \ 2 \\
 & & & x_4 & -x_5 & & & & & & = \ 1 \\
 & & & x_4 & & +\,x_6 & & & & & = \ 2 \\
 & & & & & & x_7 & -x_8 & & & = \ 1 \\
 & & & & & & x_7 & & +\,x_9 & & = \ 2 \\
3x_1 & & & +\,2x_4 & & & +\,4x_7 & & & +\,x_{10} & = 17 \\
 & & & & & & & & & x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} & \geq \ 0\,.
\end{array}
$$

Show, using the Dantzig-Wolfe decomposition algorithm, that its solution is

$$x_1 = 2,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1.5,\ x_5 = 0.5,$$

$$x_6 = 0.5,\ x_7 = 2,\ x_8 = 1,\ x_9 = 0,\ x_{10} = 0,\ \ z = -21.5\,.$$

Exercise 2.5. The multireservoir hydroelectric operating planning problem stated in Sect. 1.3.3, p. 19, of Chap. 1 is linear and includes complicating constraints.

Solve the numerical example described in Sect. 1.3.3 using Dantzig-Wolfe decomposition and compare the results obtained with those provided in that section.

Exercise 2.6. Given the problem

$$\underset{x_1, x_2, x_3, x_4}{\text{minimize}} \quad z = -2x_1 - x_2 - x_3 + x_4$$

subject to

$$
\begin{array}{rrrrl}
x_1 & -x_2 & & & \leq 0 \\
x_1 & +\,2x_2 & & & \leq 3 \\
 & & -x_3 & +\,x_4 & \leq 0 \\
 & & 3x_3 & +\,x_4 & \leq 4 \\
x_1 & & +\,x_3 & & \leq 2 \\
x_1 & +\,4x_2 & & +\,2x_4 & \leq 7 \\
 & & & x_1, x_2, x_3, x_4 & \geq 0\,,
\end{array}
$$

show, using the Dantzig-Wolfe decomposition algorithm, that its solution is

$$x_1 = 1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 0,\ \ z = -4\,.$$

Exercise 2.7. Consider the hydroelectric river system depicted in Fig. 2.8. The system should be operated so that the demand for electricity is served in every time period of the planning horizon in such a way that total cost is minimum. Data is provided in the Tables 2.6, 2.7, and 2.8. The conversion factor is used to convert water discharge volume to energy.

Solve this multiperiod operation planning problem using the Dantzig-Wolfe decomposition so that the problem decomposes by reservoir.

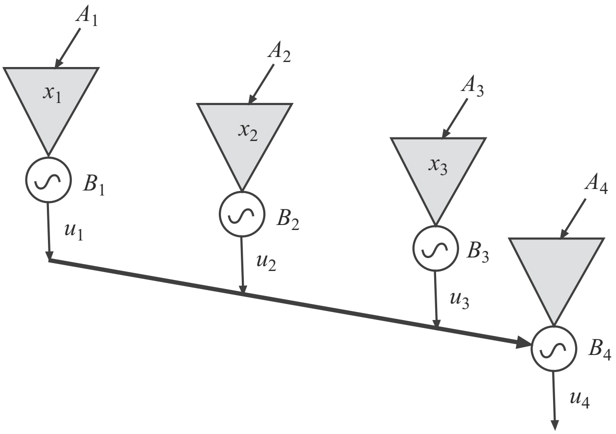

Fig. 2.8. Hydroelectric river system for Exercise 2.7

Table 2.6. Hydroelectric plant data for Exercise 2.7

Hydro plant data				
Unit	1	2	3	4
Initial Volume (hm^3)	104	205	55	0
Maximum Volume (hm^3)	1,000	1,000	1,000	0
Minimum Volume (hm^3)	0	0	0	0
Maximum Discharge (hm^3/h)	30	30	30	80
Minimum Discharge (hm^3/h)	0	0	0	0
Cost ($)	20	10	5	0
Conversion Factor (MWh/hm^3)	10	10	10	10

Table 2.7. Inflow to reservoirs for Exercise 2.7

Inflow to reservoirs (hm^3)				
Reservoir	1	2	3	4
Period 1	35	25	20	10
Period 2	36	26	21	9
Period 3	37	27	22	8
Period 4	36	26	21	7
Period 5	35	25	20	6

Table 2.8. Demand data for Exercise 2.7

Demand data					
Hour	1	2	3	4	5
Demand (MWh)	1,000	1,200	1,300	1,400	1,200

Exercise 2.8. The stochastic programming linear problem formulated in Sect. 1.3.2, p. 12, consists in determining the production policy of a hydroelectric system under water inflow uncertainty. The target is to achieve maximum expected benefit from selling energy. This problem includes complicating constraints and therefore can be conveniently solved using the Dantzig-Wolfe decomposition.

Use the Dantzig-Wolfe decomposition to solve the hydro scheduling numerical example stated in Sect. 1.3.2, and compare the results obtained with those stated in that section.

Exercise 2.9. Electric Power Alpha serves a system including four nodes and four lines. The generating nodes are 1 and 2 and the consumption nodes 3 and 4. Similarly, electric Power Beta serves other system that also includes four nodes and four lines. Nodes 5 and 6 are generating nodes and nodes 7 and 8 are demand nodes. Both companies have agreed in interconnecting their system to minimize cost and to improve security. The interconnection line connects nodes 4 and 8.

1. Compute the saving resulting from using the interconnection. To do this, solve the operation problem of the interconnected system, and compare its optimal solution with the optimal solutions obtained if the two systems are operated independently and without taking into account the interconnection.
2. Solve the operation problem of the interconnected system using the Dantzig-Wolfe decomposition.
3. Give an economic interpretation of the coordinating parameters of the decomposition.

Electric Power Alpha data are provided in the Tables 2.9 and 2.10. Electric Power Beta data are given in the Tables 2.11 and 2.12. Interconnection line data are provided in the Table 2.13.

Note that equations describing how electricity is transmitted through a transmission line are explained in Sect. 1.5.2, p. 42.

Table 2.9. Line data for system served by Electric Power Alpha

From/to	Conductance	Susceptance	Maximum capacity (MW)
1–2	−0.0064	0.4000	0.3
1–3	−0.0016	0.2857	0.5
2–4	−0.0033	0.3333	0.4
3–4	−0.0016	0.2500	0.6

Table 2.10. Node data system served by Electric Power Alpha

Node	Generating cost ($/MWh)	Demand (MW)
1	6	0.00
2	7	0.00
3	0	0.35
4	0	0.45

Table 2.11. Line data system served by Electric Power Beta

From/to	Conductance	Susceptance	Maximum capacity (MW)
5–6	−0.0056	0.3845	0.3
5–7	−0.0021	0.2878	0.5
6–8	−0.0033	0.3225	0.8
7–8	−0.0014	0.2439	0.6

Table 2.12. Node data system served by Electric Power Beta

Node	Generating cost ($/MWh)	Demand (MW)
5	8	0.00
6	9	0.00
7	0	0.35
8	0	0.45

Table 2.13. Interconnection line data

From/to	Conductance	Susceptance	Maximum Capacity (MW)
4–8	−0.0033	0.3333	0.6

Exercise 2.10. To supply the energy demand depicted in Fig. 2.9, five production devices are available. Their respective production costs ($/MWh) and powers (MW) are 1, 2, 3, 4, 5 and 1, 2, 3, 3, 5. Knowing that the joint production of devices 1 and 3 should be below 3 and that the joint production devices 4 and 5 should be above 4, find the optimal schedule of the production devices. In order to do so, analyze first the structure of the problem and then solve it using the Dantzig-Wolfe decomposition procedure. Provide an economical interpretation of the equivalent costs of the subproblems.

Exercise 2.11. The multi-year energy model studied in Sect. 1.3.4, p. 23, is a large-scale linear programming problem that includes complicating constraints. These complicating constraints are few while the noncomplicating ones are many. If the complicating constraints are ignored, the resulting problem attains polymatroid structure and its solution is straightforwardly obtained using a greedy algorithm.

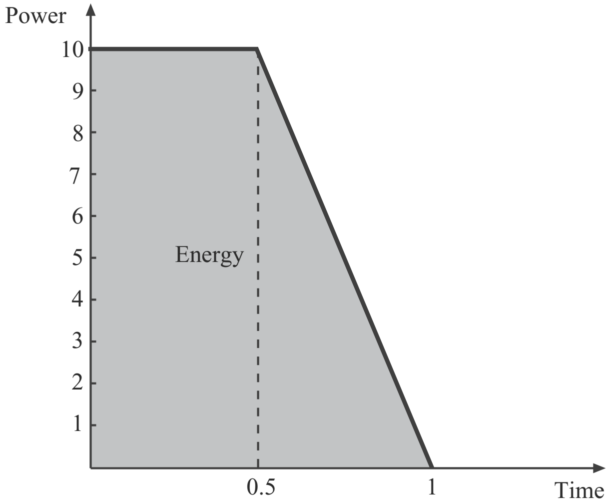

Fig. 2.9. Energy demand for Exercise 2.10

Use the Dantzig-Wolfe decomposition technique to solve the numerical example stated in Sect. 1.3.4, solving subproblems through a greedy algorithm. Analyze the numerical behavior of the Dantzig-Wolfe algorithm for this particular problem.

Exercise 2.12. A transnational plane maker manufactures engines centrally but fuselages locally in three different locations where planes are built and sold. In the first location available labor time and fuselage material are respectively 100 and 55. In the second location 120 and 40, and in the third are 60 and 60.

The manufacture of a plane requires 10 labor units and 15 fuselage material unit plus one engine.

1. Write an optimization problem to determine the maximum number of planes that can be manufactured.
2. Consider the number of planes a real variable and solve the problem using the Dantzig-Wolfe decomposition algorithm.
3. How to solve this problem if the number of planes is considered an integer variable?