The Multi-shop Ski Rental Problem

Ski Rental Problem

- Buy: $b
- Rent: $r
- Consumer strategy: rent x days and then buy
- Nature decides the ending time: y

- Randomized strategy: prob. of buying at day x is p(x)
- Optimal strategy

Multi-shop Ski Rental Problem (MSR)

$$0 < r_1 < r_2 < \cdots < r_n$$

$$b_1 > b_2 > \cdots > b_n > 0$$

- Consumer strategy: choose shop j, rent until day x, and then buy
- Randomized strategy: $p_j(x)$
- Nature strategy: ending time y

Extensions

- Multi-shop Ski Rental Problem (MSR)
- With switching cost (MSR-S)

- With entry fee (MSR-E)
- With entry fee and switching (MSR-ES)

Applications

- Scheduling in distributed computing
- Cost management in IaaS cloud

Vendor	Option	Upfront($)	Hourly($)
Amazon	On-Demand	0	0.145
	1 yr Term	161	0.09
	3 yr Term	243	0.079
ElasticHosts	1 mo Term	97.60	0
	1 yr Term	976.04	0

Strategy Space

$$\mathcal{P} = \left\{ \mathbf{p} : \quad \begin{aligned} & \sum_{j=1}^{n} \int_0^{\infty} p_j(x) \mathrm{d}x = 1, \\ & p_j(x) \geq 0, \forall x \in [0, +\infty) \cup \{+\infty\}, \forall j \in [n] \end{aligned} \right\}$$

Expected Cost

$$c_j(x,y) \triangleq \begin{cases} r_j y, & y < x \\ r_j x + b_j, & y \geq x \end{cases}$$

$$C(\mathbf{p}, y) \triangleq \sum_{j=1}^{n} C_j(p_j, y)$$

$$\begin{aligned} C_j(p_j, y) &\triangleq \int_0^\infty c_j(x,y) p_j(x) \mathrm{d}x \\ &= \int_0^y (r_j x + b_j) p_j(x) \mathrm{d}x + \int_y^\infty y r_j p_j(x) \mathrm{d}x \end{aligned}$$

Competitive Ratio

$$\begin{aligned} \text{minimize} \quad & \max_{y>0} \left\{ \frac{C(\mathbf{p}, y)}{\mathrm{OPT}(y)} \right\} \\ \text{subject to} \quad & \mathbf{p} \in \mathcal{P} \end{aligned}$$

$$\mathrm{OPT}(y) = \begin{cases} r_1 y, & y \in (0, B] \\ b_n, & y > B \end{cases}$$

where B is defined as $B \triangleq \frac{b_n}{r_1}$.

Competitive Ratio

which is equivalent to the following:

$$
\begin{aligned}
\text{minimize} \quad & \lambda \\
\text{subject to} \quad & \frac{C(\mathbf{p}, y)}{r_1 y} \le \lambda \\
& \sum_{j=1}^{n} \int_0^{\infty} p_j(x) \mathrm{d}x = 1 \\
& p_j(x) \ge 0 \quad \forall x \in [0, +\infty) \cup \{+\infty\} \\
& \forall y \in (0, +\infty) \cup \{+\infty\}, \forall j \in [n]
\end{aligned}
$$

Simplifying the Zero-sum Game

- Lemma 1: We only need to consider

$$\forall x \in [0, B], \forall y \in (0, B]$$

Optimal Strategy

LEMMA 2. *$\forall y \in (0, B]$, $\exists$ a constant λ, such that $\mathbf{p}^*$ satisfies that*

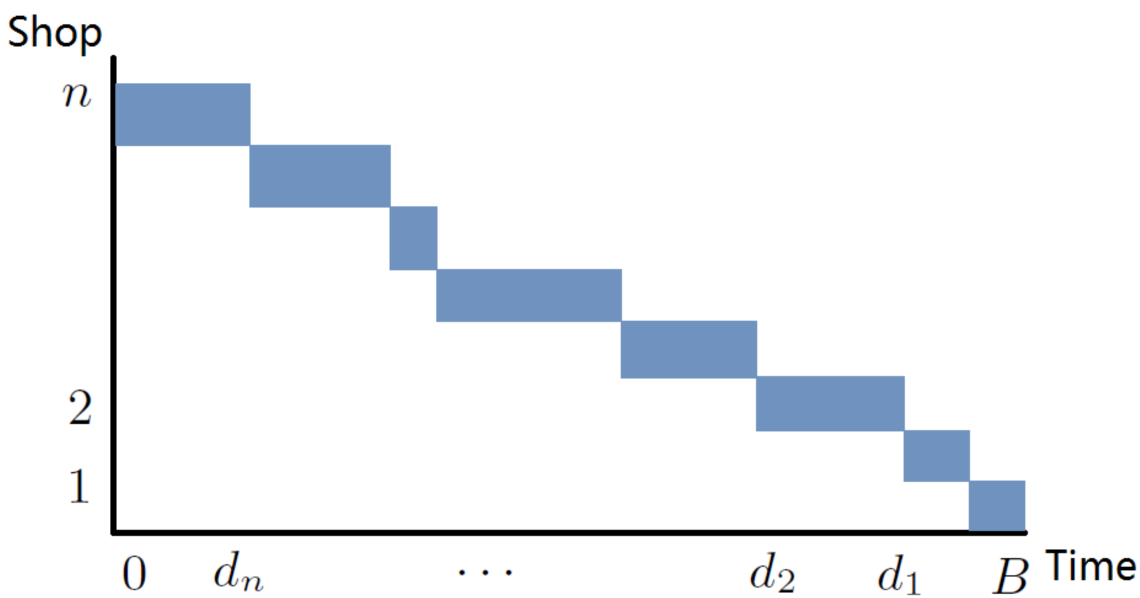
$$\frac{C(\mathbf{p}^*, y)}{r_1 y} = \lambda \tag{5}$$

Optimal Strategy

- Lemma 3

There exist $n+1$ *breakpoints:* $d_1, d_2, \cdots, d_{n+1}$*, such that* $B = d_1 \geq d_2 \geq \cdots \geq d_n \geq d_{n+1} = 0$*, and* $\forall j \in [n]$*, we have*

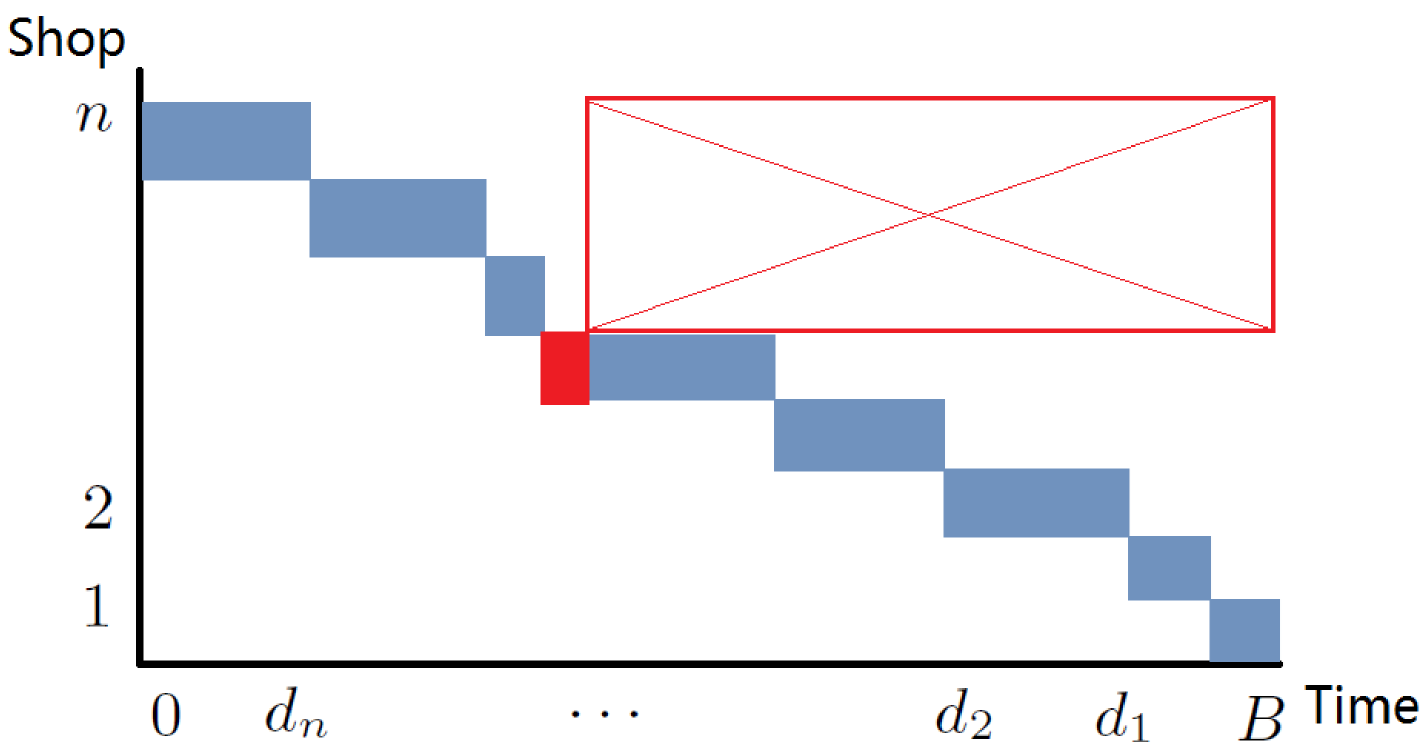
$$p_j^*(x) = \begin{cases} \alpha_j e^{r_j x/b_j}, & x \in (d_{j+1}, d_j) \\ 0, & otherwise \end{cases}$$



Why Only One Shop at Any Time?

- Claim: If $p^*{}_j(x) \geq 0$, for some j, x. Then:

$$\forall j' > j, x' \geq x, \text{ we must have } \int_{x'}^{B} p^*_{j'}(t)\mathrm{d}t = 0.$$



Final Steps

$$C(\mathbf{p}^*, y) = \lambda r_1 y$$

and taking twice derivatives

$$b_i \frac{\mathrm{d}p_j^*(x)}{dx} = r_j p_j^*(x) \quad \forall x \in (d_{j+1}, d_j)$$

$$p_j^*(x) = \begin{cases} \alpha_j e^{r_j x / b_j}, & x \in (d_{j+1}, d_j) \\ 0, & otherwise \end{cases}$$

- We can solve α_j in linear time.

With switching cost (MSR-S)

- Claim: At most one switch, right before buying

- Define virtual buying cost
- Reduce to MSR