Online Mixed Packing and Covering

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Outline

- Definition of OMPC
- Online algorithm for OMPC
- Competitive ratio analysis
- Another online mixed problem CCFL problem
  - Capacity Constrained Facility Location

Online Mixed Packing and Covering (OMPC)

- Packing constraints are given offline
  - $\mathbf{Px} \leq \lambda \mathbf{p}$
- Covering constraints arrive online one at a time
  - $\mathbf{Cx} \geq \mathbf{c}$
- Goal: minimize λ

$$
\begin{array}{lrcl}
\min & & \lambda & \\
\text{s.t.} & \mathbf{Cx} & \geq & \mathbf{1}\,, \\
& \mathbf{Px} & \leq & \lambda\,, \\
& \mathbf{x}, \lambda & \geq & \mathbf{0}\,.
\end{array}
$$

$$\kappa = \max_{i,j} c_{ij} / \min_{i,j:c_{ij}>0} c_{ij}$$

$$\rho = \max_{k,j} p_{kj} / \min_{k,j:p_{kj}>0} p_{kj}$$

$$\mu = 1 + \frac{1}{3\ln(em)}$$

$$\sigma = e^2 \ln(\mu d^2 \rho\kappa)$$

(d denotes the maximum number of variables in any constraint)

$$\lambda(\mathbf{x}) = \max_{k\in[m]} (\mathbf{P}\mathbf{x})_k .$$

$$\Phi(\mathbf{x}) := \ln\left(\sum_{k\in[m]} \exp(\mathbf{P}\mathbf{x})_k\right)$$

$$\begin{aligned} \mathrm{rate}_j(\mathbf{x}) &= \frac{\partial\Phi(\mathbf{x})}{\partial x_j} \\ &= \frac{\sum_{k\in[m]} p_{kj} \exp(\mathbf{P}\mathbf{x})_k}{\sum_{k\in[m]} \exp(\mathbf{P}\mathbf{x})_k} . \end{aligned}$$

$$\epsilon_i(\mathbf{x}) = (\mu - 1) \min_{j:c_{ij}>0} \mathrm{rate}_j(\mathbf{x})/c_{ij} ,$$

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & \mathbf{C}\mathbf{x} \geq \mathbf{1} , \\ & \mathbf{P}\mathbf{x} \leq \lambda , \\ & \mathbf{x}, \lambda \geq \mathbf{0} . \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_i y_i \\ \text{s.t.} \quad & \mathbf{C}^{\mathbf{T}}\mathbf{y} \leq \mathbf{P}^{\mathbf{T}}\mathbf{z} , \\ & \sum_{k=1}^{m} z_k \leq 1 , \\ & \mathbf{y}, \mathbf{z} \geq \mathbf{0} . \end{aligned}$$

Algorithm

$$\Phi(\mathbf{x}) := \ln\left(\sum_{k\in[m]} \exp(\mathbf{P}\mathbf{x})_k\right)$$

$$\begin{aligned}\text{rate}_j(\mathbf{x}) &= \frac{\partial\Phi(\mathbf{x})}{\partial x_j}\\ &= \frac{\sum_{k\in[m]} p_{kj}\exp(\mathbf{P}\mathbf{x})_k}{\sum_{k\in[m]}\exp(\mathbf{P}\mathbf{x})_k}.\end{aligned}$$

- Multiplicative weight update

- Initial: $x_j \leftarrow 1/(d_1^2\rho\kappa_1)$
- For covering constraint i, while $(Cx)_i < 1$, do:

Let $\mathbf{x}^l$ be the current value of $\mathbf{x}$. Increase each x_j to $x_j\left(1+\epsilon_i(\mathbf{x}^l)\frac{c_{ij}}{\text{rate}_j(\mathbf{x}^l)}\right)$.

Increment dual variable y_i by $e\epsilon_i(\mathbf{x}^l)$.

$$\sigma = e^2 \ln(\mu d^2 \rho \kappa)$$

Result and Proof Sketch

- Result: The algorithm is $8\sigma \ln(em)$ competitive.

- Proof:

$$\max_k(\mathbf{Px})_k \;\leq\; \Phi(\mathbf{x}) \;\leq\; \max_k(\mathbf{Px})_k + \ln m.$$

LEMMA 2.1. *For the variables as initialized,* $\lambda(\mathbf{x}^0) \leq \text{OPT}$*, and hence* $\Phi(\mathbf{x}^0) \leq \text{OPT} + \ln m$.

LEMMA 2.2. *The increase in* $\sum_i y_i$ *is an upper bound on the increase in* $\Phi(\mathbf{x})$ *in every phase.*

- Bound the result by dual variables

$$\lambda(\mathbf{x}) \leq \sum_i y_i + \ln m + \text{OPT}$$

Cont'd

- Dual variables **y** might violate the constraints in dual problem

LEMMA 2.4. *For any* $j \in [n]$,

$$(\mathbf{C}^{\mathbf{T}}\mathbf{y})_j \;\leq\; \sigma \max_{l \in L} \mathrm{rate}_j(\mathbf{x}^l)\,.$$

- Then we choose

$$\max_{l \in L} \mathrm{rate}_j(\mathbf{x}^l) \;=\; \max_{l \in L} \frac{\sum_{k \in [m]} p_{kj} \exp(\mathbf{P}\mathbf{x}^l)_k}{\sum_{k \in [m]} \exp(\mathbf{P}\mathbf{x}^l)_k}$$

$$\leq\; \sum_{k \in [m]} p_{kj} \max_{l \in L} \frac{\exp(\mathbf{P}\mathbf{x}^l)_k}{\sum_{k \in [m]} \exp(\mathbf{P}\mathbf{x}^l)_k}\,.$$

$$z_k \;:=\; \max_{l \in L} \frac{\exp(\mathbf{P}\mathbf{x}^l)_k}{\sum_{k \in [m]} \exp(\mathbf{P}\mathbf{x}^l)_k}\,.$$

LEMMA 2.5. $\displaystyle\sum_{k \in [m]} z_k \;\leq\; \ln(em) + \max_{l \in L} \lambda(\mathbf{x}^l).$

$$\sigma = e^2 \ln(\mu d^2 \rho \kappa)$$

Cont'd

- Define $\nu := \ln(em) + \max_l \lambda(x^l)$
- $\boldsymbol{z}/\nu$ and $\boldsymbol{y}/(\sigma\nu)$ are feasible for the dual problem
- $\sum_i y_i \leq OPT \times (\sigma\nu)$

$$\begin{aligned}\lambda(\mathbf{x}) &\leq 4\sigma \ln(em)\mathrm{OPT} + \mathrm{OPT} + 4\sigma \ln m \mathrm{OPT} \\ &\leq 8\sigma \ln(em) OPT .\end{aligned}$$

Capacity Constrained Facility Location (CCFL)

- A set of facilities F, each has opening cost and capacity
- Clients arrive online, they should be assigned to facilities.
  - Assignment cost, facility capacity
- Determine whether to open new facilities (paying opening cost), and where to assign the client (paying assignment cost)
- Goal: minimize the sum of opening costs and assignment costs
  - Subject to the capacity constraints

Integer Scheduling LP (ISLP) for the CCFL problem

$$\text{Minimize} \sum_{i \in \mathcal{F}} c_i x_i \; + \; \sum_{j \in \mathcal{C}} \sum_{i \in \mathcal{F}} a_{ij} y_{ij} \text{ subject to:}$$

$$
\begin{array}{llcll}
(3.11) & \displaystyle\sum_{j \in \mathcal{C}} p_{ij} y_{ij} & \leq & x_i & \forall i \in \mathcal{F} \\
(3.12) & y_{ij} & \leq & x_i & \forall\, i \in \mathcal{F},\; j \in \mathcal{C} \\
(3.13) & \displaystyle\sum_{i \in \mathcal{F}} y_{ij} & \geq & 1 & \forall\, j \in \mathcal{C} \\
(3.14) & x_i, y_{ij} & \in & \{0, 1\} & \forall i \in \mathcal{F},\; j \in \mathcal{C}
\end{array}
$$

Fractional Algorithm

- Relax integrality constraints x, y
- Relax "packing constraints"

$$\begin{aligned} \sum_{j \in \mathcal{C}} p_{ij} y_{ij} &\leq 9x_i \quad \forall\, i \in \mathcal{F} \\ y_{ij} &\leq 2x_i \quad \forall\, i \in \mathcal{F},\ j \in \mathcal{C} \end{aligned}$$

- Facility i is said to be closed, partially open or fully open when $x_i = 0, x_i \in (0,1), x_i = 1$

Cont'd

- Define a virtual cost for client j on facility i

$$\eta_i(j) = \begin{cases} c_i A^{\ell_i - 1} p_{ij} + a_{ij}, & \text{if facility } i \text{ is fully open,} \\ & \text{i.e., } x_i = 1 \\ c_i p_{ij} + a_{ij}, & \text{otherwise} \end{cases}$$

- Order all facilities in non-decreasing order of virtual cost M(j)
- Let P(j) denote the maximal prefix of M(j) such that $\sum_{i \in P(j)} x\_i < 1$
- Let k(j) denote the first facility in M(j) and not in P(j)

Cont'd

- When client j arrives, increase x_i and y_{ij} until $\sum_{i \in \mathrm{F}} y_{ij} \geq 1$ (Covering constraint satisfied)
  - Increase x_i for $i \in P(j)$ by $x_i / c_i n$
  - And

• $x_{k(j)} < 1$ **(i.e. facility** $k(j)$ **is partially open).** We increase $x_{k(j)}$ (and correspondingly $\Delta x_{k(j)}$) by $\delta x_{k(j)}$ for facility $k(j)$; further, we set the value of $y_{k(j)j}$ to the effective capacity created on facility $k(j)$ for client j. We call this an algorithmic step of **type A**.

• $x_{k(j)} = 1$ **(i.e. facility** $k(j)$ **is fully open).** We keep the value of $x_{k(j)}$ unchanged at 1 but *increase* $y_{k(j)j}$ by $9/\eta_{k(j)}(j)n$. We call this an algorithmic step of **type B**.

Analysis

LEMMA 3.1. *The fractional assignment produced by the online algorithm satisfies*

$$\sum_{j \in \mathcal{C}} y_{ij} p_{ij} = O(\log m)$$

for each facility i, *and*

$$\sum_{i \in \mathcal{F}} c_i x_i + \sum_{j \in \mathcal{C}} \sum_{i \in \mathcal{F}} a_{ij} y_{ij} = O(m \log m).$$

- Lemma is proved by using this potential function

$$\phi_i = \begin{cases} c_i A^{\ell_i - 1} + \frac{1}{9} \sum_{j \in \mathcal{C}} a_{ij} y_{ij}, & \text{if facility } i \text{ is fully open, i.e., } x_i = 1 \\ c_i x_i + \frac{1}{9} \sum_{j \in \mathcal{C}} a_{ij} y_{ij}, & \text{otherwise.} \end{cases}$$

Cont'd

- Bound increase of potential function for different operation type

  Lemma 3.4. *The increase in potential in a single algorithmic step of type A is at most $4/n$.*

  Lemma 3.5. *For any constant $1 < A < 19/18$, the increase in potential in a single algorithmic step of type B is at most $3/n$.*

  Lemma 3.3. *For any constant $1 < A < 19/18$, the increase in potential in a single algorithmic step of either type A or type B is at most $4/n$.*

- Then bound the number of algorithmic step

  Lemma 3.7. *The total number of algorithmic steps (of either type A or type B) in the second and third categories for a client j is at most $2\eta_{\mathrm{OPT}(j)}(j)n/9$.*

Online Randomized Rounding

- **Fractional step.** The fractional solution is updated (via multiple algorithmic steps) as described in the fractional algorithm. Let $x_i(j)$ be the value of x_i after this update.

- **Activation step.** Each facility i that satisfies $5x_i(j)\ln(mn) \geq r_i$ (and is not already open) is opened. Let $M_{(j)}$ denote the set of open facilities after this step.
- **Assignment step.** Let

$$z_{ij} = \begin{cases} \frac{y_{ij}}{2x_i(j)} & \text{if } x_i(j) < \frac{1}{5\ln(mn)} \\ y_{ij} & \text{otherwise} \end{cases}$$

and

$$q_{ij} = \frac{z_{ij}}{\sum_{i \in \mathcal{F}_A(j)} z_{ij}}$$

for any facility $i \in \mathcal{F}_A(j)$. We assign client j to a facility $i \in \mathcal{F}_A(j)$ with probability q_{ij}.

LEMMA 3.11. *The total opening cost of all facilities opened in the integer schedule is* $O(m \log m \log(mn))$ *in expectation.*

LEMMA 3.13. *The total assignment cost of all clients in the integer assignment is* $O(m \log m \log(mn))$ *in expectation.*

LEMMA 3.14. *The maximum congestion on any facility is* $O(\log m)$ *with probability* $1 - 2/\sqrt{m}$.

Lessons Learned

- Potential function is useful for analysis.
- Multiplicative weight update.
- In these two problems, packing constraints are not satisfied strictly.
- Several techniques for analyzing inequalities.