Watch and Learn: Optimizing from Revealed Preferences Feedback

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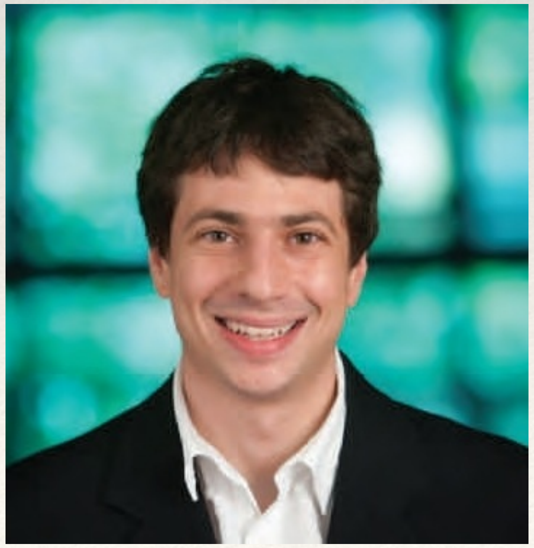
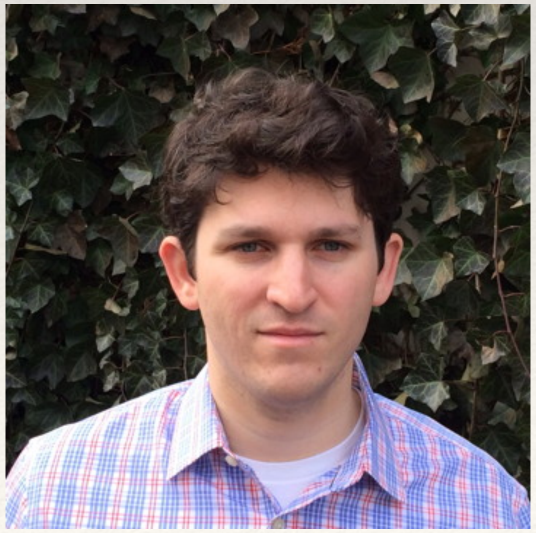
Joint work with Aaron Roth and Jonathan Ullman





What kind of Data?

Prices

Purchase Behavior (Revealed Preferences)

Tolls

Traffic (Equilibrium Flow)

Learning from Revealed Preferences

- d divisible goods
- Producer posts prices:

$$p = (p_1, \ldots, p_d) \in \mathbb{R}^d_+$$

- Buyer purchases utility-maximizing bundle:

$$x^*(p) = \arg \max_{x \in C \subseteq \mathbb{R}^d_+} v(x) - \langle p, x \rangle$$

- v : valuation function ***unknown*** to producer;
- C is the set of feasible bundles
- v : Strongly concave & Lipschitz over C, Non-decreasing

Producer's Goal: Profit Maximization

Learning: The producer can adaptively set prices over rounds, and observe the purchased bundle by the buyer and the profit

Objective: Find the (approximately) optimal price vector under a small number rounds

$$\mathrm{Profit}(p) = \langle p, x^*(p) \rangle - c(x^*(p))$$

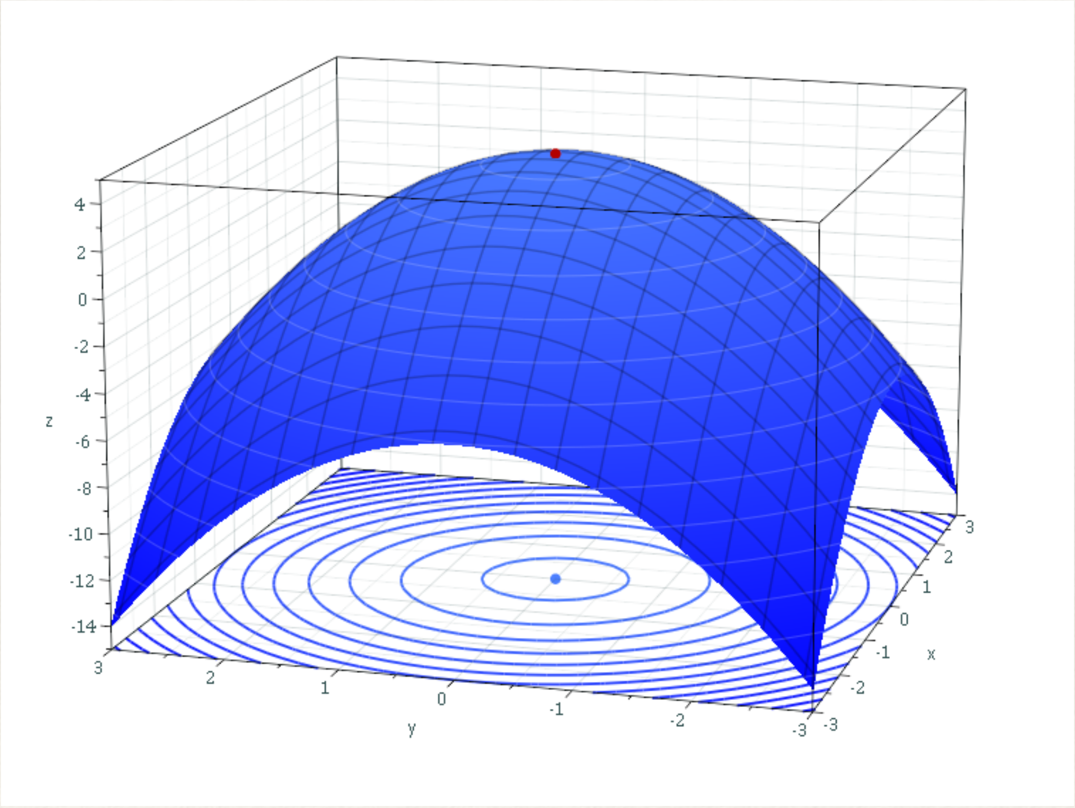
Revenue

Convex Production Cost

Unknown Objective

Zeroth Order Optimization?

Zeroth Order Optimization: given query access to an unknown **concave** function *f*, can find an approximately optimal solution with *poly*(*d*) queries



Unfortunately, the Profit function is not **concave** in the decision variables p

For example, if $v(x) = \sqrt{x}$

then $\text{Profit}(p) = \frac{1}{4p} - \frac{1}{4p^2}$

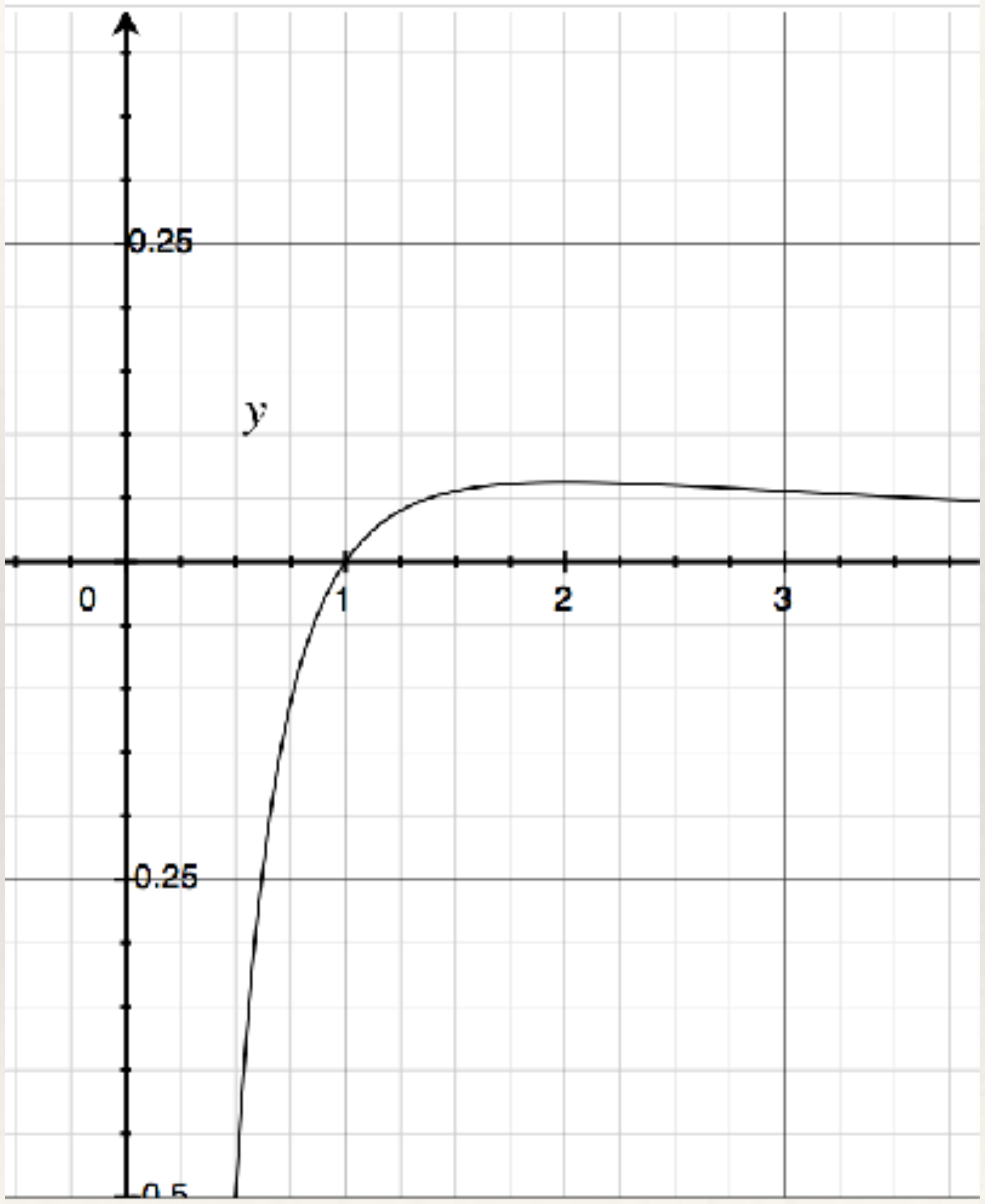
Illustration

$$v(x) = \sqrt{x}$$

$$x^*(p) = \arg\max_{x \in R^+} \sqrt{x} - p \cdot x$$

$$= \frac{1}{4p^2}$$

$$Profit(p) = \frac{1}{4p} - \frac{1}{4p^2}$$



Switching Decision Variables

- What if the producer could magically control what buyer buys (the variable *x*)?

Switching Decision Variables to Bundles

$$\mathrm{Profit}(x) = \max_{p: x^*(p) = x} \langle x, p \rangle - c(x)$$

$p^*(x)$ is the best price vector to induce bundle x

What is the best price vector?

- Lemma: if the buyer is allowed to buy nothing $(\mathbf{0} \in C)$, then

$$p^*(x) = \nabla v(x)$$

Now the profit is simpler!

$$\text{Profit}(x) = \langle x, \nabla v(x) \rangle - c(x)$$

simple is beautiful.

Is Profit(x) concave?

$$\text{Profit}(x) = \langle x, \nabla v(x) \rangle - c(x)$$

- The answer is yes for a large class of economically meaningful valuation functions — *homogeneous functions*

$$\exists k \geq 0, \quad v(a\, x) = a^k\, v(x)$$

- Scale Invariance: $x^*(p)$ is unchanged even switched to different units
- Example: CES & Cobb-Douglas

CES and Cobb-Douglas

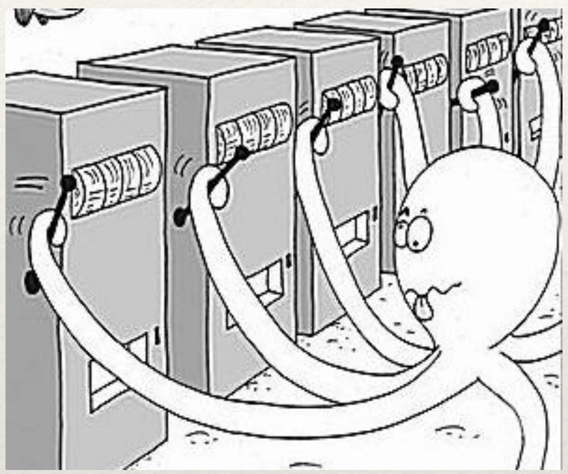
CES:

$$v(x) = \left(\sum_{i=1}^{d} \alpha_i x_i^{\rho} \right)^{\beta}$$

Cobb-Douglas:

$$v(x) = \prod_{i=1}^{d} x_i^{\alpha_i}$$

New Plan



At step t, query $\mathrm{Profit}(x^t)$

Return (approximate) evaluation for

$\mathrm{Profit}(x^t)$

Bandit Algorithm
Optimize over function
Profit(x)

Oracle

Optimal bundle x^*

Price vector to induce bundle x^*

Need to simulate query access to Profit(x)

Technical Problem

- Given a target bundle, find a price vector to (approximately) induce it.

$$\text{for any } \hat{x}, \text{ find } \hat{p} \text{ s.t. } \|\hat{x} - x^*(\hat{p})\| \leq \varepsilon$$

- Due to Lipschitzness,

$$\text{Profit}(\hat{x}) \approx \text{Profit}(x^*(\hat{p}))$$

Tâtonnement

For each good j, $$\hat{p}_j^{t+1} = \left[p_j - \eta(\hat{x}_j - x^*(p^t)_j)\right]$$
$$p^{t+1} = \Pi_P\left[\hat{p}^{t+1}\right]$$

Projected Gradient Descent

- If the buyer buys too much good *j*, raise the price
- If the buyer buys too little good *j*, lower the price

Why does it work?

- Consider the following convex program

$$\max_{x \in C} v(x)$$

$$\text{for each good } j \qquad x_j \leq \hat{x}_j$$

$$\hat{x} \text{ is the optimal solution}$$

- For each good, introduce a price (dual) variable

$$\text{Lagrangian: } \mathcal{L}(x, p) = v(x) - \langle p, x - \hat{x} \rangle$$

Lagrangian Zero-sum Game

- Strong duality

$$\max_{x \in C} \min_{p \in \mathbb{R}^d_+} \mathcal{L}(x,p) = \min_{p \in \mathbb{R}^d_+} \max_{x \in C} \mathcal{L}(x,p) = v(\hat{x})$$

- Minimax theorem continues to hold for

$$P = \{p \in \mathbb{R}^d_+ \mid \|p\| \leq \sqrt{d}\}$$

$$\max_{x \in C} \min_{p \in P} \mathcal{L}(x,p) = \min_{p \in P} \max_{x \in C} \mathcal{L}(x,p) = v(\hat{x})$$

Payoff $\mathcal{L}(x,p)$

Price Player p

Bundle Player x

No Regret vs. Best Response

For $t = 1, \ldots, T$

Price Player plays

online gradient descent

$$p^{t+1} = \Pi_P \left[p^t - \eta(\hat{x} - x^t) \right]$$

Bundle Player plays

best response

$$x^t = \arg\max_{x \in C} \mathcal{L}(x, p^t)$$

Rewrite the Best Response

$$x^t = \arg\max_{x \in C} \left[v(x) - \langle p^t, x - \hat{x} \rangle \right]$$

$$= \arg\max_{x \in C} \left[v(x) - \langle p^t, x \rangle + \langle p^t, \hat{x} \rangle \right]$$

We could always remove the constants in argmax

$$x^t = \arg\max_{x \in C} \left[v(x) - \langle p^t, x \rangle \right] = x^*(p^t)$$

Best response is just
the observed purchased bundle!

No Regret vs. Best Response

Price Player plays
online gradient descent

$$p^{t+1} = \Pi_P \left[p^t - \eta(\hat{x} - x^*(p^t)) \right]$$

Bundle Player plays
best response

$$x^*(p^t)$$

The average plays $p' = 1/T \sum_t p^t$ and $x' = 1/T \sum_t x^t$
forms approximate minimax equilibrium [FreudSchapire'96]

Approximate Equilibrium

Lemma: Let (x', p') be any approximate minimax equilibrium of the Lagrangian zero-sum game, then

$$\|\hat{x} - x^*(p')\| \leq \varepsilon$$

Proof idea: equilibrium condition and strong concavity

$$\mathcal{L}(\hat{x}, p') \approx \mathcal{L}(x^*(p), p')$$

Our Roadmap

Profit(p) non-concave objective

switch decision variables to *bundles*

Profit(x) concave (Homogeneous v)

optimize over x

Bandit Algorithm [BLNR'15]

requires query access Profit(x)

Learning dynamics (OGD by [Zin'03])

Optimizing Traffic Routing from Revealed Behavior

- The same framework can find tolls to induce an approximately *optimal* traffic flow

- In the process, we also solve the problem of finding tolls to induce target flow introduced by [BLSS'14]

Stackelberg Games

- More general settings: a class of ***Stackelberg Games***:
  - Optimize *leader's* utility given observations on the *follower's actions*
  - e.g. Contract design in principal-agent Problems

- Extension to noisy observations

Open Problem

- Stochastic Revealed Preferences
  - producer sets prices p, and a buyer with valuation v drawn from some unknown prior
  - Goal: find the prices that maximize expected profit

Time for Coffee and more Problems!

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