3 Network Topologies: Basic Models and Properties

3.1 Introduction

In order to understand the relationships between the topology and dynamics of a complex network, it is necessary to study and comprehend the structural characteristics of real-world complex networks thereby establishing appropriate mathematical network models. Since the discoveries of Watts and Strogatz on small-world networks [1] and of Barabási and Albert on scale-free networks [2], many case studies on various real-world networks have been carried out and reported from different points of view. Based on such insights and experiences, a number of new models of complex networks have been proposed, as summarized in [3,4]. This chapter introduces several basic network models, such as regular networks, random-graph networks, small-world networks, scale-free networks, hierarchical networks, and local-world evolving networks, and so on. In addition to the notions of average path length, clustering coefficient, and degree distribution introduced in Chapter 1, some more new concepts such as motif and self-similarity will be further introduced and discussed in this chapter.

3.2 Regular Networks

The name regular network here does not necessarily refer to the mathematically defined concept of regular graph (Chapter 2). One of the most typical regular networks is a ***fully connected network***. In such a network, between every pair of nodes there exists an edge connecting them together (Fig. 3-1 (a)).

Theorem 3-1 *A fully connected network has the average path length*

$$\overline{L}_{full} = 1 \tag{3-1}$$

and the largest clustering coefficient

$$C_{ful} = 1 \tag{3-2}$$

Moreover, a fully connected network with N nodes has a total of $N(N-1)/2$ edges.

Proof. They can be calculated directly based on the definitions. □

Note that among all networks with the same number of nodes, the fully connected network has the shortest average path length 1 and the largest clustering coefficient 1.

Note also that although fully connected networks can describe the fact that many real-world networks possess high-clustering and small-world features, they are nevertheless too regular to be realistic. For example, a fully or almost fully connected network with N nodes has the number of edges in the order of $O(N^2)$, but most real-world networks are known to be relatively sparse, with the number of edges only in the order of $O(N)$. A typical sparse regular network is a ***nearest-neighbor coupled network***, where every node is connected to its nearest neighbors. In particular, such a network with a periodic boundary connectivity condition is a ***ring-shaped network*** (Fig. 3-1 (b)).

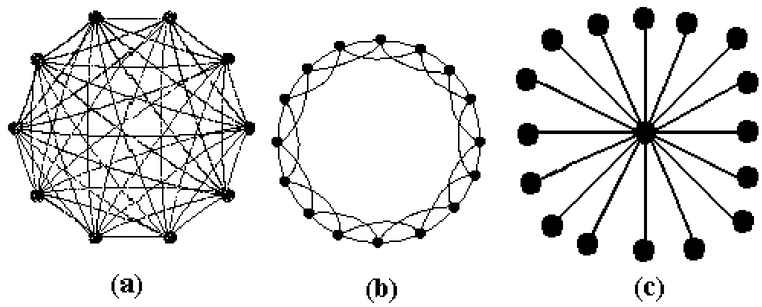


Fig. 3-1 Some simple networks: (a) fully connected network
(b) ring-shaped network (c) star-shaped network

Theorem 3-2 Assume that in a large-scale ring-shaped network, with a large enough number N of nodes, if each node is connected to $2K$ nearest neighbors where $K > 0$ is an integer relatively small as compared to N, then the clustering coefficient can be well approximated by

$$C_{ring} \approx \frac{3(K-1)}{2(2K-1)} \to \frac{3}{4} \qquad (K \to \infty) \tag{3-3}$$

Moreover, let M be the total number of edges of such a ring-shaped network. Then, the average path length can be well approximated by

$$\bar{L}_{ring} \approx \frac{M(M+1) - 2(K-1)(M-K+1)}{2M} \to \infty \qquad (M \to \infty) \tag{3-4}$$

Proof. When $K = 1$, the resulting network is a ring, for which, according to formula (1-4) in Chapter 1, $C_{ring} = 0$. When $K = 2$, the resulting ring-shaped network is as shown in Fig. 3-1 (b), from which it is also clear that there are 6 triangular graphs and 3 complete triangles at each node, so $C_{ring} = \frac{3}{6} = \frac{1}{2}$, which verifies formula (3-3). Formula (3-3) can then be proved by mathematical induction. When K becomes large enough so that $2K$ is approaching N, however, the formula becomes less accurate due to some multiple counting of common edges when such edges are connecting back towards the starting point along the circular path; therefore, the formula is only a good approximation for relatively small N and relatively large K.

To verify formula (3-4), first notice that when $K = 1$, namely, for a perfect ring, every node has the same average path length therefore one may only calculate a single node, denoted V_0, for its average path length in the ring, which is equal to the overall average path length: $\bar{L}_{ring} = (1 + 2 + \cdots + M)/M$. When $K > 1$, all distances to node V_0 are reduced by $K-1$ each, except the $(K-1)$ nearest-neighboring nodes of V_0; namely, the total reduction is $(K-1)(M-(K-1))$. Therefore,

$$\bar{L}_{ring} = \frac{(1+2+\cdots+M)-(K-1)(M-(K-1))}{M}$$

which is formula (3-4). Similarly, when K (hence, M) becomes large enough so that $2K$ (hence, M) is approaching N, the formula becomes less accurate due to some multiple counting of common edges in the circular path; so the formula is only a good approximation for relatively large K (hence, relatively large M). □

Note that the asymptotic result of (3-4) explains, in general, though not always, why locally connected networks are difficult to achieve some global cooperative dynamical behaviors (such as synchronization, a topic to be further studied later).

Another typical regular network is the ***star-shaped network***, which has a center, and all the other nodes are connected to, and only connected to the center (Fig. 3-1 (c)).

Theorem 3-3 *For a star-shaped network of size N, counting the central node, the average path length is*

$$\bar{L}_{star} = 2 - \frac{2}{N} \to 2 \qquad (N \to \infty) \tag{3-5}$$

and the clustering coefficient is

$$C_{star} = 0 \tag{3-6}$$

Proof. On one hand, regarding the total path length, first observe that the central node has a total path length $(N-1)$. Since the central node has $(N-1)$ neighboring nodes, the total path length of all the neighboring nodes of the central node is equal to $(N-1)(1+2(N-2)) = (N-1)(2N-3)$. Then, each neighboring node has a path length 1 to the central node and a path length 2 to any other neighboring node which yields a total path length $(1+2(N-2))$. Thus, the total path length of the whole star-shaped network is given by $(N-1)+(N-1)(2N-3)$.

On the other hand, regarding the total number of paths, notice that the central node has $(N-1)$ paths, and each neighboring node has $(1+(N-2))$ paths, therefore all neighboring nodes of the central node have a total of $(1+(N-2))(N-1) = (N-1)^2$ paths. Consequently, there are $(N-1)+(N-1)^2 = N(N-1)$ paths in total. As a result, one has

$$\bar{L}_{star} = \frac{(N-1)+(N-1)(2N-3)}{N(N-1)} = 2 - \frac{2}{N}$$

Next, formula (1-4) in Section 3.1 of Chapter 1 is applied to calculate the clustering coefficient. Since there are no complete triangles in a star-shape network, one has $C_{star} = 0$. □

3.3 Random-Graph Networks

The other extreme to the regular networks are completely random networks, where a typical model is the *random graph* introduced by Erdös and Rényi [5,6].

To obtain an ER random graph of size N, one picks up all possible pairs of nodes, once and once only, from the pool of N given nodes, and then connect each pair of nodes by an edge with probability p. Statistically, namely, after averaging over a large enough number of trials, this will yield a graph of N nodes and $pN(N-1)/2$ edges. Generally, the larger the p is, the denser the resultant network will be, as illustrated by Fig. 3-2. This type of network is referred to as an ***ER random-graph network***.

Note that the above process of generating a random-graph network will not introduce multiple edges, nor self-loop at any single node, therefore the results are all simple graphs.

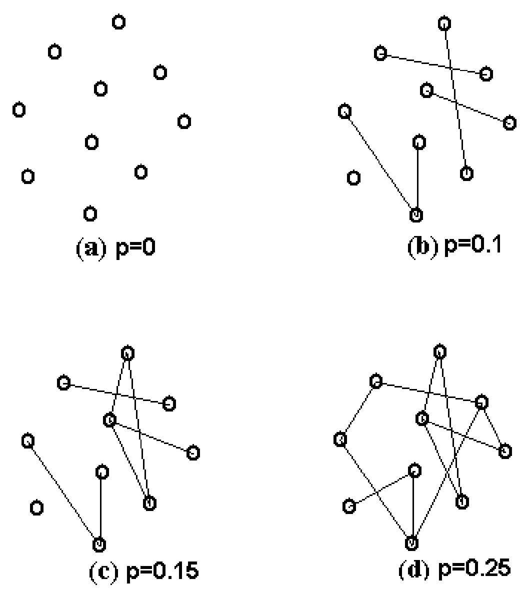


Fig. 3-2 Some random graphs generated with different values of p

The main concern in studying ER random-graph networks has been the following: for what value(s) of p, the resultant graph will have certain specified properties? In the endeavor of searching for answers, the most significant discovery of Erdös and Rényi is that many important properties of such random graphs *emerge* suddenly, in the sense that for a set of graphs generated by a given probability p, either almost all these graphs possess a certain property P or almost all of them do not possess this property P. For instance, when p is larger than a certain threshold $p_c \sim \ln N / N$ (where $a \sim b$ means "a is proportional to b" or "a is in the same order of b"), almost all random graphs generated by using this probability p in the above-described procedure will suddenly become connected, before which almost all such graphs were unconnected networks (i.e., had isolated clusters).

ER random-graph networks have the following basic structural properties.

Theorem 3-4 *In an ER random-graph network, the average degree of nodes is*

$$\langle k \rangle_{ER} = p(N-1) \tag{3-7}$$

the average path length is

$$\overline{L}_{ER} \sim \ln N / \ln \langle k \rangle_{ER} \tag{3-8}$$

and the clustering coefficient is

$$C \approx \langle k \rangle_{ER} / N = p \tag{3-9}$$

Moreover, the node-degree distribution follows a Poisson distribution

$$P(k) = \frac{\mu^k}{k!} e^{-\mu} \tag{3-10}$$

in which μ is the expectation value, $\mu = pN \approx \langle k \rangle_{ER}$.

Proof. For any node, there are $N-1$ other nodes for possible connections and each connecting edge has a probability p to appear; therefore, this node has a total of $p(N-1)$ possible edges. Since in average each node has degree $\langle k \rangle_{ER}$, which is equal to the number of its edges, $p(N-1)$.

For any node of degree $\langle k \rangle_{ER}$, the number of other nodes that it can connect to is $N_1 = \langle k \rangle_{ER}$. Then, moving forward from this node to the next one, the number of other nodes it can connect to in two steps will be $N_2 = \langle k \rangle_{ER} N_1 = \langle k \rangle_{ER}^2$, and so on. In general, $N_n = \langle k \rangle_{ER} N_{n-1} = \langle k \rangle_{ER}^n$, and this holds for every node in the network. On the other hand, the average distance between any pair of nodes is $\overline{L}_{ER}$, therefore one can only go $n \sim \overline{L}_{ER}$ steps forward from any node. Consequently, the total number of nodes equals $N \sim \langle k \rangle_{ER}^{\overline{L}_{ER}}$, which yields $\overline{L}_{ER} \sim \ln N / \ln \langle k \rangle_{ER}$.

The clustering coefficient of node i is given via formula (1-3) in Section 1.3 of Chapter 1 by $C_i = 2\dfrac{E_i}{k_i(k_i-1)}$, where k_i is the degree of node i and E_i is the number of actual edges among the neighbors of node i. Over the whole ER random-graph network, in average, one has $E \approx p \langle k \rangle_{ER} (\langle k \rangle_{ER} - 1) / 2$, so

$$C \approx 2 \frac{p \langle k \rangle_{ER} (\langle k \rangle_{ER} - 1) / 2}{\langle k \rangle_{ER} (\langle k \rangle_{ER} - 1)} = p \approx \frac{\langle k \rangle_{ER}}{N}$$

Finally, the Poisson node-degree distribution formula (3-10) was proved in Theorem 1-1, Chapter 1. □

Note that the characteristic of average path length being equal to a logarithmic function of the network growth rate, formula (3-8), is a typical small-world feature. Since the growth of $\ln N$ is much slower than that of N, it means that a very large-scale network of this kind usually has a relatively small average path length.

Yet, the ER random-graph networks are not small-world networks. They just have some small-world features, for example, with a small average path length, but they generally do not have another important small-world network feature—a large clustering coefficient. As a matter of fact, as seen from formula (3-9), the clustering coefficient of an ER random-graph network is fairly small: $C = p \approx \langle k \rangle_{ER} \; / \; N \; << 1$. Therefore, large-scale random-graph networks generally do not have prominent clustering features.

One illustrative example about the Poisson distribution of an ER random-graph network is shown in Fig. 3-3 [7].

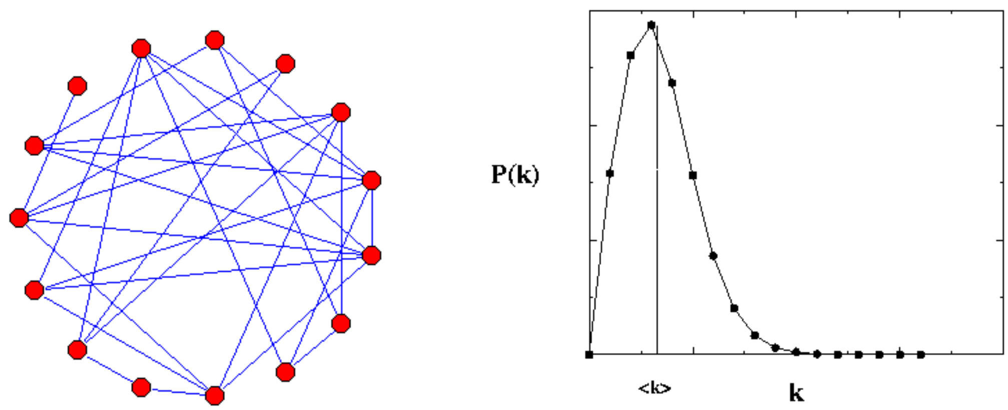


Fig. 3-3 Illustration of ER random-graph networks [7]

Many real-world examples of complex networks, however, do have considerably large clustering coefficients, showing that they are not typical ER random-graph networks. Nevertheless, the classical ER random-graph network theory has already been applied, generalized and extended for better describing many real-world complex networks. For instance, for a given desirable node-degree distribution $P(k)$, which represents the proportion of nodes with degree k in the network, one can generate many degree sequences $\{k_i\}_{i=1}^{n}$, so as to obtain many networks with N nodes, in such a way that they all have the same node-degree distribution $P(k)$. The collection of all these networks is called a ***configuration model***, useful for describing many real-world networks [4].

3.4 Small-World Network Models

As mentioned above, a regular nearest-neighboring network has a large clustering coefficient and an ER random-graph network has a short average path length. Are there some kinds of networks that have both features of large clustering coefficients along with short average path lengths? The answer is *yes*, which are the so-called small-world networks discovered by Watts and Strogatz in 1998 [1], named the WS small-world network model hereafter.

3.4.1 The WS Small-World Network Model

A *WS small-world network* can be generated by the following algorithm (WS algorithm):

1. Start from a regular nearest-neighboring network with N nodes, in which each node is connected to its $2K$ neighbors, where $K > 0$ is a (usually small) integer.

2. At every step, randomly pick a node and operate on its connections to the K nearest neighbors one by one in a clockwise (or counterclockwise) manner: an edge connecting this node to its neighbor will be kept in place (i.e., this edge is unchanged); another end of the edge will be re-connected to a node randomly chosen from anywhere in the network (i.e., this picked edge is being rewired) with probability p.

Here, at Step 2, all random operations follow a uniform distribution in the sense that every node has equal probability to be picked at random.

As always, it is required that between any pair of nodes there can have at most one edge connecting them and no node can have self-loops.

Clearly, Step 2 above will introduce some long-range connections. More importantly, in the above algorithm, the case of $p = 0$ corresponds to a regular network (the original nearest-neighboring network) while the case of $p = 1$ corresponds to a kind of ER random-graph networks, except that it starts with a ring-shaped network as the initial condition and the total possible number of edges has been determined beforehand which may not equal $N(N-1)/2$. By tuning the value of p, one can obtain a transition from a completely regular network to a random-like network, as illustrated by Fig. 3-4 (a).

3.4.2 The NW Small-World Network Model

Notice that the above WS algorithm may destroy the network connectivity during the rewiring process, yielding possibly some unconnected clusters (sub-networks). As a remedy, Newman and Watts [8] slightly modified the above algorithm, by replacing "random rewiring edges" with "random adding edges," as illustrated by Fig. 3-4 (b).

The NW algorithm is as follows:

1. Start from a regular nearest-neighboring network with N nodes, in which each node is connected to its $2K$ neighbors, where $K > 0$ is a (usually small) integer.

2. At every step, with probability p, add an edge to each possible pair of originally unconnected nodes.

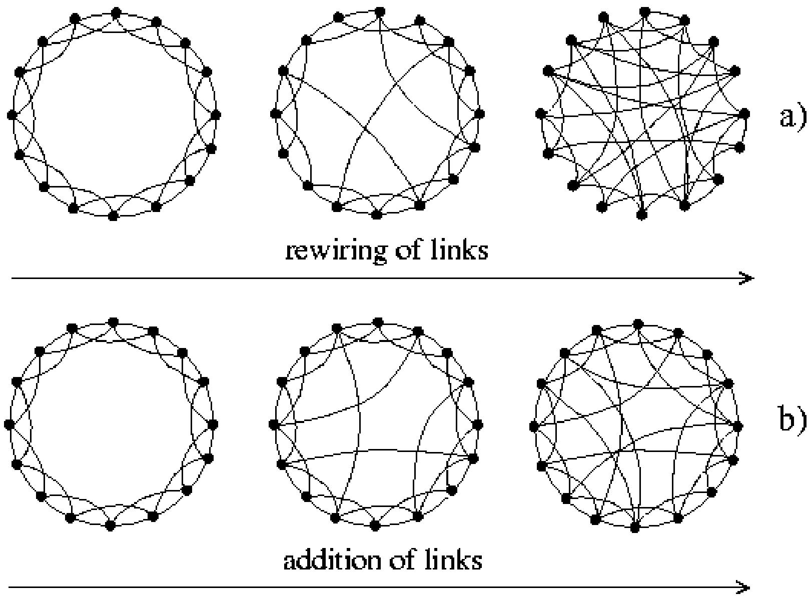


Fig. 3-4 Illustration of two small-world network models:
(a) WS small-world network model (b) NW small-world network model

In this process, between any pair of nodes there will not be multiple edges and no node will have self-loops.

In the *NW small-world model*, the case of $p=0$ corresponds to the original regular nearest-neighboring network while $p=1$ eventually yields a fully connected network. By tuning the value of p ($0<p<<1$), one can obtain a transition from a regular sparse network to a regular dense network, as illustrated by Fig. 3-4 (b). For small enough values of p, though, the WS and NW models are about the same.

Afterwards, there appeared some other variants of the WS and NW small-world network models [4], but they are not to be further discussed here.

3.4.3 Statistical Properties of Small-World Network Models

For the WS network model, the clustering coefficient is re-defined to be the ratio of the mean number of edges among the neighbors of a node and the number of all possible edges among the neighbors of the node:

$$C_{WS} = \frac{\text{average number of neighboring edges}}{\text{total possible number of neighboring edges}} \tag{3-11}$$

Note that this definition differs from the original one only by a small amount of order $O(1/N)$, as further explained in the proof given below.

Theorem 3-5 [4,9] *For large enough size N, the clustering coefficient of the WS small-world network model is given by* [9]

$$C(p) = \frac{3(K-1)}{2(2K-1)}(1-p)^3 \tag{3-12}$$

and the clustering coefficient of the NW small-world network model is given by [4]

$$C(p) = \frac{3(K-2)}{4(K-1)+4Kp(p+2)} \tag{3-13}$$

Proof. An outline of the proof of (3-12) can be found in [8], as follows.

For $p=0$, each node has $2K$ neighbors, so the number of edges among these neighbors is $N_0 = 3K(K-1)/2$, while the number of all possible edges among these nodes is $2K(2K-1)/2$; therefore, $C(0) = 3(K-1)/(2(2K-1))$. For $p>0$, two neighbors of node i, which were connected at $p=0$ are still neighbors of i and remain being connected with probability $(1-p)^3$, up to some terms of order $O(1/N)$. Thus, the mean number of edges among the neighbors of a node is $N_0(1-p)^3 + O(1/N)$. Consequently, the clustering coefficient is given by $N_0(1-p)^3/K(2K-1)$, which is formula (3-12), with a possible difference of order $O(1/N)$.

An outline of the proof of (3-13) can be found in [4]. □

Theorem 3-6 [8,11] The average path length of the WS small-world network model is given by [8]

$$\overline{L}(p) = \frac{2N}{K} f(2Np/K) \tag{3-14}$$

with

$$f(x) = \begin{cases} c & x << 1 \\ \ln x / x & x >> 1 \end{cases} \qquad \text{(typically, } c = 1/4\text{)} \tag{3-15}$$

The average path length of the NW small-world network model is also given by (3-14), with [11]

$$f(x) \approx \frac{1}{2\sqrt{x^2+2x}} \tanh^{-1}\sqrt{\frac{x}{x^2+2x}} \tag{3-16}$$

Proof. The proof of the first part can be found in [8], as follows.

For the WS network model, with p fixed, first perform the renormalization process (Fig. 3-5) and let the number of sites of the resultant renormalized network be S. The average path length $\overline{L}(p)$ is less than 1 and is increasing linearly as S gradually increases. But at some threshold value of S, denoted S^*, $\overline{L}(p)$ will become bigger than 1. This leads to a phase transition, after which $\overline{L}(p)$ will increase only logarithmically. To be more precise, consider only the case of $K=1$, namely, a perfect ring, and assume that $0 < p << 1$ and

$S^* = 1/p$, thus $S^* >> 1$. In this case, $\bar{L}(p)$ should obey a finite-size scaling law of the form $\bar{L}(p) = Sf(S/S^*) = Sf(pS)$, where $f(x)$ is given by (3-15). From the renormalized network, it can be seen that $S = 2N/K$; so, formula (3-14) follows.

For the NW network model, a proof of formula (3-16) is quite tedious, which can be found in [11]. □

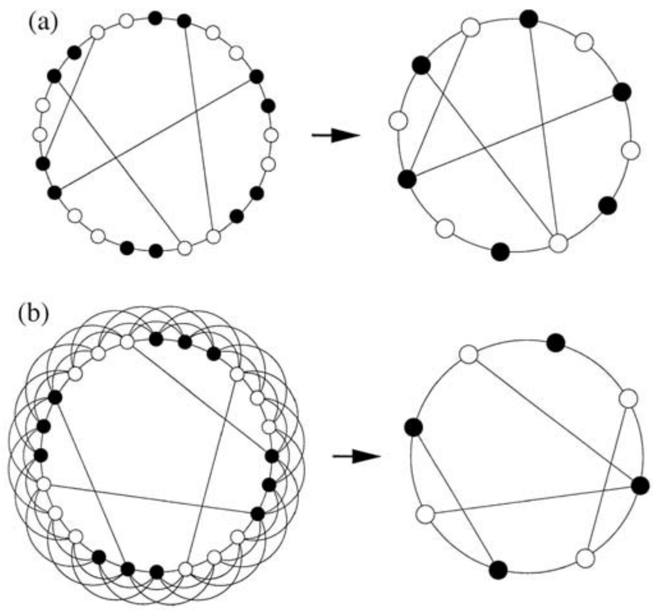


Fig. 3-5 Renormalization of a small-world network model: (a) $K = 1$ (b) $K = 3$ [8]

Figure 3-5 depicts a typical evolutionary result on a small-sized network generated by the WS algorithm, where both values of the clustering coefficient and the average path length have been normalized for a better comparison. Clearly, the regular nearest-neighboring network (corresponding to $p = 0$) has very high clustering ($C(0) \approx 3/4$ in this example) with a relatively long average path length (here, $\bar{L}(0) \approx 2N/K >> 1$). For a small p ($0 < p << 1$), the local properties of both the original network and the rewired network do not change by too much, therefore their clustering coefficients remain about the same ($C(p) \approx C(0)$); however, the average path length of the rewired network becomes much smaller than the original one ($\bar{L}(p) << \bar{L}(0)$). This implies that, after rewiring, the resultant network has become a small-world network with a large clustering coefficient and a small average path length.

Theorem 3-7 [4,9] The node-degree distribution of the WS small-world network model is given by [9]

$$P_p(k) = \sum_{i=0}^{\min(k-K,K)} \binom{K}{i} (1-p)^i p^{K-i} \frac{(Kp)^{k-K-i}}{(k-K-i)!} \exp(-Kp) \quad (k \geq K) \qquad (3\text{-}17)$$

with $P_p(k)=0$ for $k<K$; and the node-degree distribution of the NW small-world network model is given by [4]

$$P(k)=\binom{N}{k-K}\left(\frac{Kp}{n}\right)^{k-K}\left(1-\frac{Kp}{N}\right)^{n-k+K} \qquad (k\geq K) \qquad \text{(3-18)}$$

with $P(k)=0$ for $k<K$.

Proof. The proof of formula (3-17) can be found in [9], as follows.

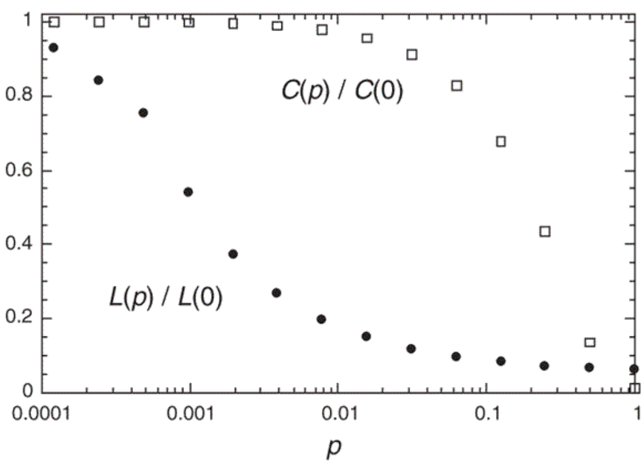


Fig. 3-6 Clustering coefficient and average path length versus rewiring probability p in the WS small-world network model [1]

For the WS network model, with $p=0$, each node has the same connectivity $2K$. For the nodes being rewired with probability $p>0$, they introduce non-uniformity to the network while maintaining a fixed average path length $\bar{k}=2K$. Let this non-uniform probability distribution of network connectivity (degree) be denoted by $P_p(k)$. Since K of the initial $2K$ connections of each node are left unchanged by the construction, the degree of node i is $k_i=K+n_i$ with $n_i\geq 0$, where $n_i=n_i^1+n_i^2$ in which $n_i^1\leq K$ is the number of edges left unchanged (each one with probability $1-p$) and n_i^2 is the number of edges that have been reconnected to some another node from node i (each one with probability p/N). Thus, one has

$$P_1(n_i^1)=\binom{K}{n_i^1}(1-p)^{n_i^1}p^{K-n_i^1}$$

$$P_2(n_i^2)=\frac{(Kp)^{n_i^2}}{n_i^2!}\exp(-Kp) \qquad \text{for large } N$$

and, generally,

$$P_p(k)=\sum_{i=0}^{\min(k-K,K)}\binom{K}{i}(1-p)^i p^{K-i}\frac{(Kp)^{k-K-i}}{(k-K-i)!}\exp(-Kp) \quad \text{for } k\geq K$$

For the NW network model, formula (3-17) can be found in [4]. □

3.5 Scale-Free Network Models

A common feature of the ER random-graph networks and the WS/NW small-world networks is that their node-degree distributions are (approximately) described by the Poisson distribution, which peaks at the average degree $\langle k \rangle$ and then decays exponentially fast on both sides as $k \to 0, \infty$. In particular, nodes with very high degrees $k >> \langle k \rangle$ almost do not exist. For this reason, this kind of networks is called *homogeneous networks;* sometimes, due to its two-sided decaying tails, they are also called *exponential networks*.

It has been found, however, that many real-world networks, including such typical ones as the Internet, WWW, metabolic networks, etc., are not homogeneous networks; instead, their connectivity is heterogeneous. More importantly, their node-degree distributions have a power-law form and are independent of the connectivity scale, therefore are referred to as scale-free networks (see Section 1.3.4, Chapter 1).

3.5.1 The BA Scale-Free Network Model

As mentioned before, both the ER random-graph network model and the WS/NW small-world network models do not grow in size, but most real-world networks grow and indeed grow very fast; for instance, journals have new papers being published every month and WWW has new web-sites being added every day. Furthermore, in the growth of such a network, a forthcoming node has the tendency to connect itself to some "big" nodes (with large degrees), referred to as *preferential attachment*. This is the so-called "rich gets richer" phenomenon, or Matthew effect; for example, a new paper is more likely to cite an already well-cited reference and a new web-site tends to link to famous sites like Yahoo and Google. Motivated by some good observations like these, Barabási and Albert [2] proposed a new network modeling algorithm, known as the *BA scale-free network* model today.

The BA modeling algorithm is as follows:

1. *Growth*: Starting from a fully-connected network of small size $m_0 \geq 1$, introduce one new node to the existing network each time, and this new node is connected to m existing nodes in the network, where $1 \leq m \leq m_0$.

2. (*Linear*) *Preferential Attachment*: The above incoming new node is connected to an existing node i of degree k_i according to the following probability:

$$\Pi_i = \frac{k_i}{\sum_{j=1}^{N} k_j} \qquad (3\text{-}19)$$

   where $N = m_0 + t - 1$ is the total number of existing nodes at the $(t-1)$ st step of the node-adding process.

Here, the preferential attachment probability (3-19) is proportional to $k_i + 1$, so that even isolated node (with $k_i = 0$) has a nonzero probability to acquire new edges; otherwise it will be an isolated node forever. Of course, if the initial network is connected, then this $+1$ in (3-19) is no needed; nevertheless, comparing to a large number of N, this $+1$ does not change much in the formula anyway.

Clearly, after t steps, the network will have a total of $N = t + m_0$ nodes and $m\,t$ edges. Figure 3-6 illustrates the evolving process of an BA scale-free network, with $m = m_0 = 2$, where the green nodes represent the new comers and the red nodes are the existing ones at each step.

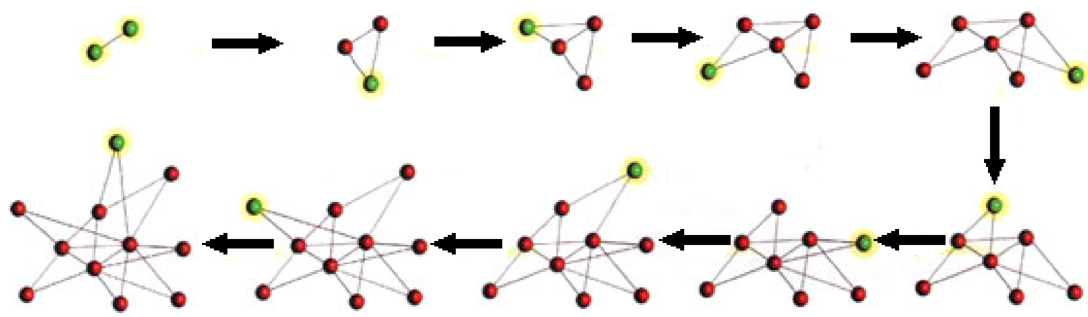


Fig. 3-7 Illustration of the growth of a BA scale-free network [17]

Theorem 3-8 [12,13] The average path length of the BA scale-free network is given by

$$\bar{L} \sim \frac{\ln N}{\ln \ln N} \tag{3-20}$$

Proof. A proof of this formula is rather mathematically involved, and is therefore referred to [12,13] for details. □

Note that this $\bar{L}$ actually is not very large, implying that the BA scale-free network model also has a certain small-world network feature—two randomly chosen nodes are connected by a rather short path.

Theorem 3-9 [14] The clustering coefficient of the BA scale-free network model is given by

$$C = \frac{m^2(m+1)^2}{4(m-1)} \left(\ln\left(\frac{m+1}{m}\right) - \frac{1}{m+1} \right) \frac{(\ln t)^2}{t} \tag{3-21}$$

Proof. A proof of this formula is rather mathematically involved, and is therefore referred to [14] for details. □

Note that this result shows that the BA scale-free network model also has a certain random-graph network feature—the clustering property is not prominent.

Theorem 3-10 [15] The node-degree distribution of the BA scale-free network is given by a power-law of the form

$$P(k) \sim 2m^2 k^{-3} \tag{3-22}$$

Proof. Let i be a node, which was added to the network at instant t_i, and let $p(k,t_i,t)$ be the probability that this node i has degree k when it is being picked up at time t ($t \geq t_i$).

Imaging that k is a continuous variable, so that probability $\Pi(k_i) = k_i / \sum_j k_j$ may be viewed as a continuous rate of change of k_i; therefore,

$$\frac{\partial k_i}{\partial t} = a\,\Pi(k_i) = a\,\frac{k_i}{\sum_j k_j}$$

for some constant a. Since the new node brings in m new edges, which has degree m at time t_i, so the change of connectivity at t_i is m, implying that $a = m$. Also, at every step m new edges have been added, so the total node degree of the network at time t is $\sum_j k_j \approx 2\,m\,t$. Thus, the above equation reduces to

$$\frac{\partial k_i}{\partial t} = \frac{k_i}{2t}$$

Solving this equation, with the initial condition that node i was added to the network at time t_i with connectivity $k_i(t_i) = m$ yields

$$k_i(t) = m\sqrt{\frac{t}{t_i}}$$

which gives

$$t_i = \frac{m^2 t}{k_i^2}$$

On the other hand,

$$P(k_i(t) < k) = P\left(t_i > \frac{m^2 t}{k^2}\right)$$

Assuming that the new nodes are being added at equal time intervals, the time variables $\{t_i\}$ have a uniform distribution with $P(t_i) = \dfrac{1}{t+m_0}$, so that

$$P\left(t_i > \frac{m^2 t}{k^2}\right) = 1 - P\left(t_i \leq \frac{m^2 t}{k^2}\right) = 1 - \frac{m^2 t}{k^2} \cdot \frac{1}{t+m_0}$$

Consequently, the degree distribution $P(k_i(t) = k)$ can be obtained as

$$P(k) = \frac{\partial\, P(k_i(t) < k)}{\partial\, k} = 2\frac{m^2 t}{t + m_0} \cdot k^{-3} \approx 2m^2 k^{-3}$$

This is the power law (3-21), in the form of $c \cdot k^{-\gamma}$ with $\gamma = 3$. □

Note that the above proof certainly is not very rigorous, but it is quite insightful. An elementary and yet rigorous proof can be found in [16].

Note also that as proved in Theorem 1-2, Chapter 1, the power-law distribution (3-22) is scale-free.

A numerical result is shown in Fig. 3-8, where $N = 1{,}000$. For some different views on the BA scale-free network model, see [15].

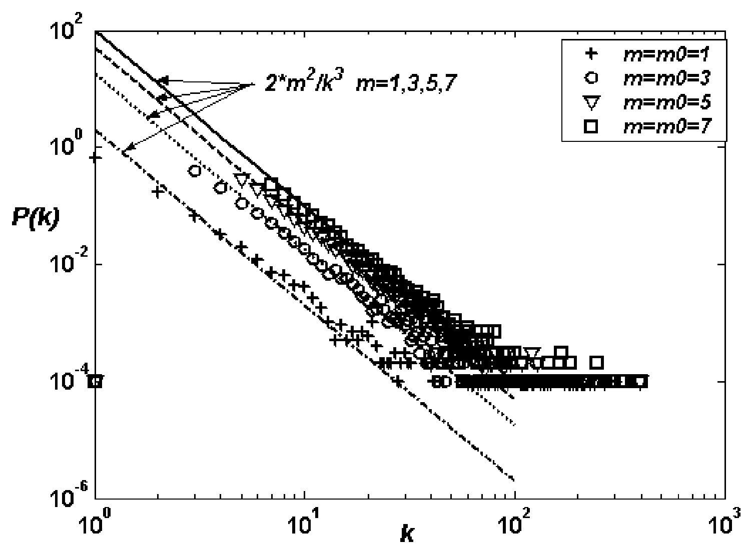


Fig. 3-8 The node-degree distribution of a BA scale-free network model

3.5.2 Robustness versus Fragility

In the Greek mythology, Achilles was the son of Thetis and Peleus, the bravest hero in the Trojan war. Stories told that when Achilles was born, his mother Thetis tried to make him immortal by dipping him in the river Styx. As she immersed him, she held him by one heel but forgot to dip him a second time to wet his other heel she held (Fig. 3-9 (a)). Thus, as a result, the place where she held him remained untouched by the magic water of the Styx and that part stayed vulnerable. Achilles was killed in a battle by an enemy's arrow hit exactly at his that very heel. Today, any weak point of a strong entity is called an "Achilles' heel". For instance, on July 27 of year 2000, the *Nature* magazine has a story about Achilles' heel of the Internet, as shown in Fig. 3-9 (b) [17].

(a) The Greek myth about Achilles' heel (b) Achilles' heel of the Internet

Fig. 3-9 Achilles' heel and the Internet [*Nature*, 27 July 2000]

For a given network of any kind, if one node is being removed at a time (consequently all the edges connecting it will also be removed), then at some point the network will be broken to become disconnected (Fig. 3-10). If the network remains being connected after some nodes have been removed, then the network is said to be *robust* against node-removal. Comparing two networks, when a certain number of nodes are removed, one network stays connected while another lost the connectivity, then the former is said to be more robust than the latter.

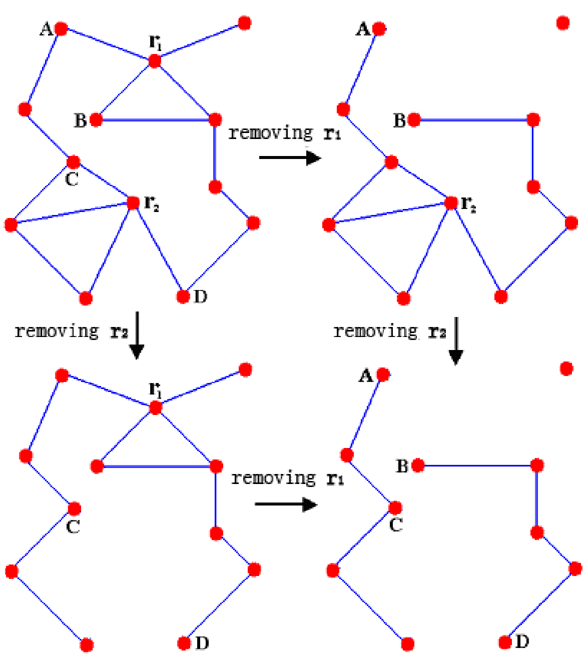


Fig. 3-10 A connected network becomes unconnected due to removal of nodes

Now, compare the robustness of the ER random-graph networks and the BA scale-free networks in the sense described above: their maintenance of connectivity against node-removal.

Apply two strategies for node removal: random removal (uniformly randomly remove a node from the network) and intentional removal (selectively remove the highest-degree node from the network).

Let $\overline{L}$ be the average path length of a network, f be the fraction of nodes being removed, and S represent the degree of the largest connected sub-network at the first time a network becomes unconnected due to node removal. Clearly, if there are several edges connecting node i and node j, then removing one edge between them will increase the distance d_{ij} between them, so that the average path length $\overline{L} = l$ of the whole network will be increased. Of course, if all edges connecting them are removed, then these two nodes will be disconnected, which can be considered as having average distance $\overline{L} = \infty$.

The simulation results shown in Fig. 3-11 [17] show that the ER random-graph networks and the BA scale-free networks are very different in this respect: the BA scale-free networks are very robust against random removal of nodes: comparing to the ER random-graph network, the size S of the largest sub-network in a scale-free network decreases to zero slowly and for a much larger fraction f of the removed nodes; yet its average path length grows also much slower.

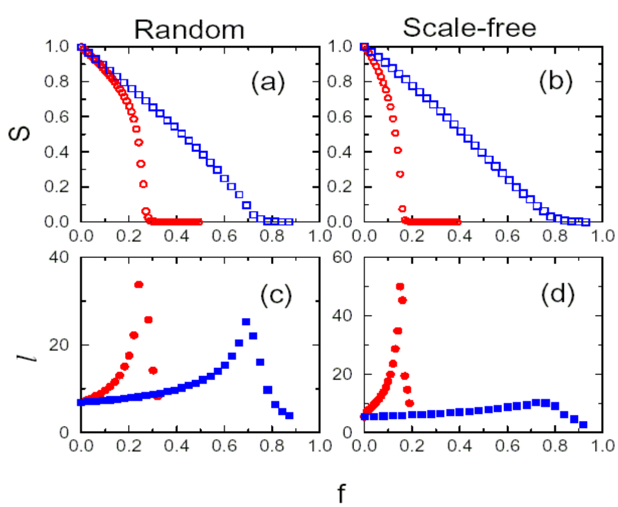
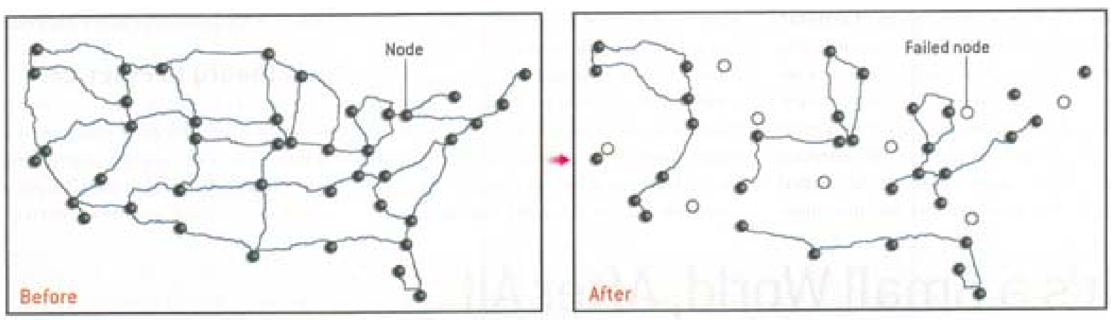


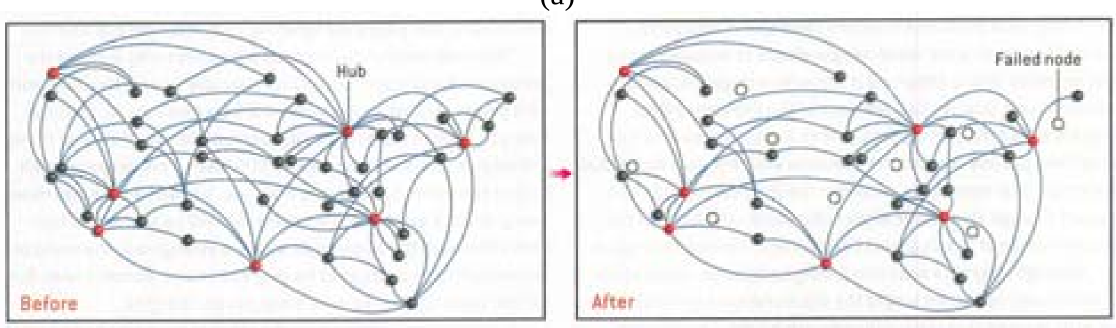
Fig. 3-11 Robustness and fragility of ER random-graph and BA scale-free networks [17]: (a) and (c): ER random-graph networks; (b) and (d): BA scale-free networks; squares—random removal of nodes; circles—intentional removal of nodes

The above-described robustness of scale-free networks against random removal of nodes is due to the heterogeneous distribution of nodes in the network: most nodes have very small degrees and only a few nodes have large degrees; thus, randomly removing a fraction of nodes will very likely remove some small nodes, which does not affect the network connectivity by too much. However, for exactly the same reason, any intentional removal of even a very small fraction of high-degree nodes will significantly affect the topology of the network, leading to drastic change of the network connectivity. Figure 3-12 [18] illustrates the robustness and fragility of the BA scale-free networks.

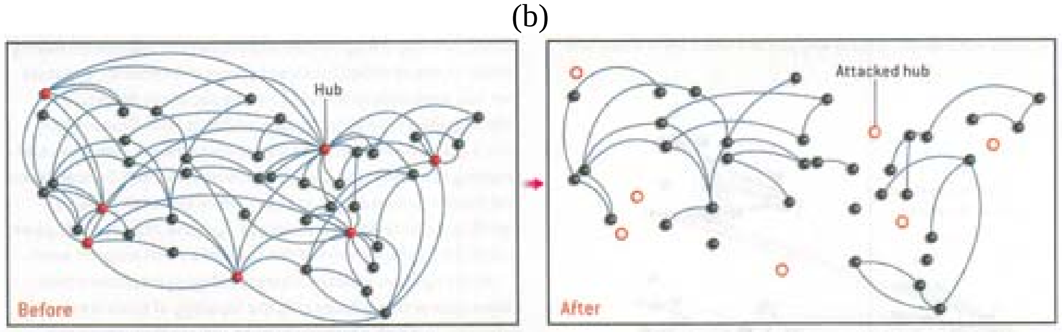
The Internet provides a typical example of the "robust yet fragile" phenomenon. The Internet has become an indispensable part of human life today. However, the Internet also faces with various failures and attacks everyday. Therefore, it is an extremely important issue to guarantee the robustness of the Internet against both random failures and intentional attacks, leaving a challenging technical issue for good solutions.



(a)



(b)



(c)

Fig. 3-12 Comparison of robustness and fragility of ER random-graph and BA scale-free networks [18]:
(a) effects on connectivity of ER networks due to random removal of nodes

(b) effects on connectivity of BA networks due to random removal of nodes
(c) effects on connectivity of BA networks due to intentional removal of nodes

In a study of the Internet by Albert, Jeong and Barabási [17], a model of the Internet at the AS (Autonomous Systems) level with 6,000 nodes, and a model of a subnet of the WWW with 326,000 nodes were investigated, showing that they behave in a way similar to the BA scale-free network (Fig. 3-13). This implies that the Internet and the WWW have the "robust yet fragile" property, discussed above, due to their heterogeneous topologies. It is believed that this "robust yet fragile" property is an essential feature of various systems complexity [19,20].

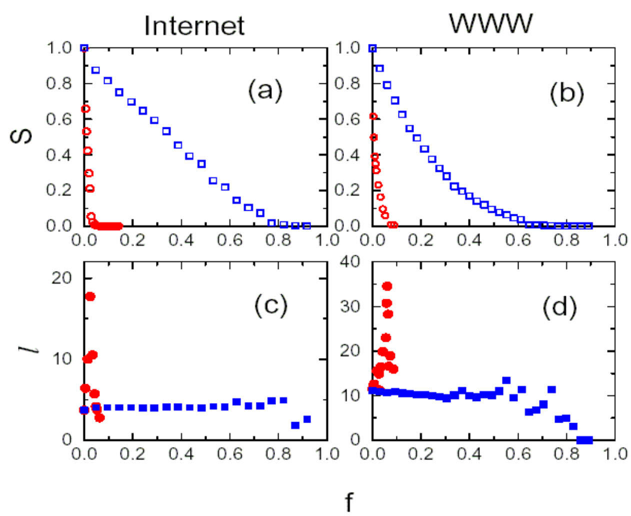


Fig. 3-13 Robustness and fragility of the Internet and WWW against intentional attacks [17]:
(a) and (c): Internet; (b) and (d): WWW;
squares—random failures, circles—intentional attacks

As a side note, Broder *et al.* [21] studied the WWW and found that only when all nodes with degree larger than 5 are removed the WWW can be disconnected, meaning that the WWW is quite robust against intentional attacks. Since the WWW is so large in size, the total number of degree-5 nodes is still a relatively small fraction of the entire huge network. This finding nevertheless is not conflicting with the observation reported in [17]. As to the Internet, Doyle *et al.* [22] argued that the Internet is a scale-free network but also is a highly optimized tolerance network, therefore its "robust yet fragile" property has to be measured from a somewhat different (i.e., technological) point of view. For more discussions on related issues, see [23-28].

3.5.3 Modified BA Models

The main contribution of the BA scale-free model is that it precisely captures the two essential features of many complex networks: growth and preferential attachment. Since complex networks are extremely complex, with many important parameters that can affect the network behaviors significantly, the BA model oversimplifies them thereby leading to some inconsistence with real-world complex networks. For example, in the BA model, the power-law form of degree distributions $P(k) \sim k^{\gamma}$ has exponent $\gamma = 3$ (see Theorem 3-10 above), but real data show that most real-world networks have their exponents satisfying $2 < \gamma < 3$. Knowing this, some generalized and improved versions of the BA model were developed lately, among which the most popular is the following extended BA model, referred to as the *EBA Model*.

The EBA Model

To allow for $2 < \gamma < 3$ in the power law $P(k) \sim k^{\gamma}$ of the BA model, Albert and Barabási [29] extended the original BA model by incorporating the idea of "rewiring".

The following is the EBA modeling algorithm:

1. *Growth* and *Preferential Attachment*: Start from a fully-connected network of small size $m_0 \geq 1$. At each step, with probability $1 - p - q$ a new node is added to the network and the new node bring m $(1 \leq m \leq m_0)$ new edges with probability p to the existing network: randomly select a node as the starting point of the new edge; the other end of the new edge is selected from the existing node i of degree k_i according to the following probability:
$$\Pi_i = \frac{k_i}{\sum_{j=1}^{n} k_j} \tag{3-23}$$
where n is the total number of existing nodes at this step; this process is repeated for m times.

2. *Rewiring*: With probability q, randomly rewire m existing edges: pick each connected pair of nodes i and j, and then reconnect it to another node with a certain probability; this process is repeated for m times.

Theorem 3-11 [29] *The node-degree distribution of the EBA model is given by*
$$P(k) \sim \left[k + c(p,q,m)\right]^{-\gamma(p,q,m)} \tag{3-24}$$
where $c(p,q,m) = 1 + A(p,q,m)$ *and* $\gamma(p,q,m) = 1 + B(p,q,m)$ *with*
$$A(p,q,m) = (p-q)\left[\frac{2m(1-q)}{1-p-q} + 1\right] \quad \textit{and} \quad B(p,q,m) = \frac{2m(1-q) + 1 - p - q}{m}$$
in which $0 \leq p < 1$ and $0 \leq q < 1 - p$.

Proof. [29] Here, for an insightful (though not very rigorous) proof, the degree k_i is consider as a variable, which is changing continuously with respect to time t, so that the probability $\Pi_i = \Pi(k_i)$ can be interpreted as the rate of change of k_i. Consequently, differentiation can be applied to it with respect to t.

Thus, adding m new edges with probability p to the network yields

$$\left(\frac{\partial k_i}{\partial t}\right)_{(1)} = p\ C_1 \frac{1}{n} + p\ C_1 \frac{k_i}{\sum_{j=1}^{n} k_j}$$

where n is the current network size. The first term on the right-hand side corresponds to the random selection of one end of the new edge, while the second term reflects the preferential attachment probability (3-23) used to select the other end of the edge. Since the total change in connectivity after one step is $\Delta k = 2m$, one has $C_1 = m$ at this stage.

Next, rewiring m edges with probability q gives

$$\left(\frac{\partial k_i}{\partial t}\right)_{(2)} = -q\ C_2 \frac{1}{n} + q\ C_2 \frac{k_i}{\sum_{j=1}^{n} k_j}$$

where the first term on the right-hand side incorporates the decreasing connectivity of the node from which the edge was removed, and the second term represents the increasing connectivity of the node to which the edge is reconnected. The total connectivity does not change during the rewiring process, however, so C_2 can be calculated by separating the two processes, resulting in $C_2 = m$ at this stage.

Moreover, adding a new node to the network with probability $1 - p - q$ leads to

$$\left(\frac{\partial k_i}{\partial t}\right)_{(3)} = (1 - p - q)\ C_3 \frac{k_i}{\sum_{j=1}^{n} k_j}$$

where the number of edges connecting the new node to the existing nodes is m, giving $C_3 = m$ at this stage.

Now, by combining the contributions from the above three processes, one obtains

$$\frac{\partial k_i}{\partial t} = (p - q)m \frac{1}{n} + m \frac{k_i}{\sum_{j=1}^{n} k_j}$$

at this stage. To this end, notice that both the network size n and the total number of edges $\sum_{j=1}^{n} k_j$ are actually varying with time in the process, given by

$$n = m_0 + (1 - p - q)t \quad \text{and} \quad \sum_{j=1}^{n} k_j = (1-q)2mt - m$$

Clearly, for large t, one may neglect the (relatively small) constants m_0 and m in the above two formulas. Consequently, a solution of the above differential equation, with initial condition that the connectivity of a node being added at time t_i is $k_i(t_i) = m$, is obtained as

$$k_i(t) = \left[A(p,q,m) + m + 1\right] \left(\frac{t}{t_i}\right)^{1/B(p,q,m)} - A(p,q,m) - 1$$

where $0 \le t_i \le t$, and $A(p,q,m)$ and $B(p,q,m)$ are given as in the theorem.

Furthermore, notice that the probability that a node i has degree (number of connections) $k_i(t) < k$, denoted $P(k_i(t) < k)$, can be written as

$$P(k_i(t) < k) = P(t_t > C(p,q,m)\ t) \quad \text{with} \quad C(p,q,m) = \left(\frac{m + A(p,q,m) + 1}{k + A(p,q,m) + 1}\right)^{B(p,q,m)}$$

There are three cases to consider: (i) if $C(p,q,m) > 1$, then $P(k_i(t) < k) = 0$, thus for $P(k)$ to be nonzero the condition is $k > m$; (ii) if $C(p,q,m) > 1$ is not real, then $P(k_i(t) < k)$ is not well-defined, so in order to calculate $P(k)$ the condition must be $[m + A(p,q,m) + 1]/[k + A(p,q,m) + 1] > 0$ for all $k > m$, which can be satisfied when $m + A(p,q,m) + 1 > 0$; (iii) if $0 < C(p,q,m) < 1$, then $P(k)$ can be arbitrary, and in this case, one may define the time unit as one attempt of growth/rewire/edge-adding, where the probability density of t_i is $P_i(t_i) = 1/(m_0 + t)$, leading to

$$P(k_i(t) < k) = 1 - C(p,q,m)\frac{t}{m_0 + t}$$

which, together with $P(k) = \partial P(k_i(t) < k)/\partial k$, yields

$$P(k) = \frac{t}{m_0 + t} D(p,q,m)\ \left[k + A(p,q,m) + 1\right]^{\ 1/(1+B(p,q,m))}$$

with $D(p,q,m) = [m + A(p,q,m) + 1]^{B(p,q,m)} B(p,q,m)$.

In summary, the degree (connectivity) distribution has the form of (3-24). □

It is easy to see that the power law (3-24) can generate scale-free networks with $2 < \gamma < 3$, which is a more practical situation than that with $\gamma = 3$, as has been verified by many simulated and real-world examples.

Fitness Model

Notice that during the growth of the BA model, the degree of a node is also changing according to the following law:

$$k_i(t) = \sqrt{\frac{t}{t_i}} \tag{3-25}$$

where $k_i(t)$ is the degree of node i at time t and t_i is the instant at which node i is being added into the network. This growth rate of the node degrees in the BA model implies that the older a node, the higher its degree. In real life, however, this is not always true. For instance, a young teenager can have more social connections than an elder man; a new website in WWW can have more links pointing to it than some old ones; a new scientific paper can receive more citations than many old articles, etc. All these depend on the importance or significance of the node in the network, which is referred to as the *fitness* by Bianconi and Barabási [30] in their modified version of the BA model, called the *fitness model*:

1. *Growth*: Start from a small network of size $m_0 \geq 1$ and introduce one new node to the existing network each time, and with probability $\rho(\eta)$ this node is given a fitness value, η_i **(**$0 < \eta_i < 1$**).**

2. *Preferential Attachment*: The new node is connected to m $(1 \leq m \leq m_0)$ existing nodes, each is according to the probability of connecting to node i of degree k_i with fitness value η_i given by

$$\Pi_i = \frac{\eta_i k_i}{\sum_{j=1}^{n} \eta_i k_j} \qquad (3\text{-}26)$$

   where $n = m_0 + t - 1$ is the total number of existing nodes at the $(t-1)$ st step of the process.

Clearly, after t steps, the network will have a total of $n = t + m_0$ nodes and mt edges. This model is the same as the BA model (see (3-18)), except that in this fitness model the preferential attachment is not only proportional to the degree but also proportional to the fitness of an existing node. Consequently, a young node with a higher fitness value can obtain more new edges than an old node with a lower fitness value.

Depending on the form of the fitness probability $\rho(\eta)$, there are two behaviors of the fitness model [30]: if this probability distribution has a finite domain, then similar to the BA model this fitness model also has a power-law form of node-degree distribution; if this probability distribution has an infinite domain, then the node with the highest fitness value will attract a large portion in the total number of newly coming edges—known as the "winner takes all" phenomenon.

3.5.4 Multi-Local-World Evolving Network Model

In most network models that generate scale-free features by means of preferential attachment, the mechanism responsible for the emergence of the scale-free topology has a global feature; namely, the probability that an existing node receives a new edge is with respect to the total number of existing edges in the whole network. However, this is not always the case in real life. For example, in the World Trade Web (WTW), it is reported [31] that the global preferential attachment mechanism does not work for those countries that have less than 20 trade connections with other countries. On the other hand, many countries are accelerating their economic cooperation in various regional economic-cooperative organizations, such as EU, ASEAN, and NAFTA. This indicates that very often preferential attachment mechanisms only exist within local economy regions of the WTW. Another typical example, with local preferential attachment but not global preferential attachment, is the Internet. In the Internet, due to the technical and economical limitations, a router in one autonomous system (AS) usually favors a shortest-path connection within the same AS when placing new edges. Therefore, the Internet can be divided into many sub-networks, and nodes in the same sub-network are relatively densely connected while nodes in different sub-networks have very few connections. This is a prominent localization effect, as illustrated by Fig. 3-14 for the Internet and Fig. 3-15 in general.

Obviously, models with the mechanism of global preferential attachment, such as the BA, EBA and Fitness network models, cannot well fit the topological structures of networks with the localization feature. Motivated from this observation, Li and Chen [33] developed a local-world evolving network model, followed by a somewhat more general model, the so-called Multi-Local-World network model, developed by Fan and Chen [34,35,36].

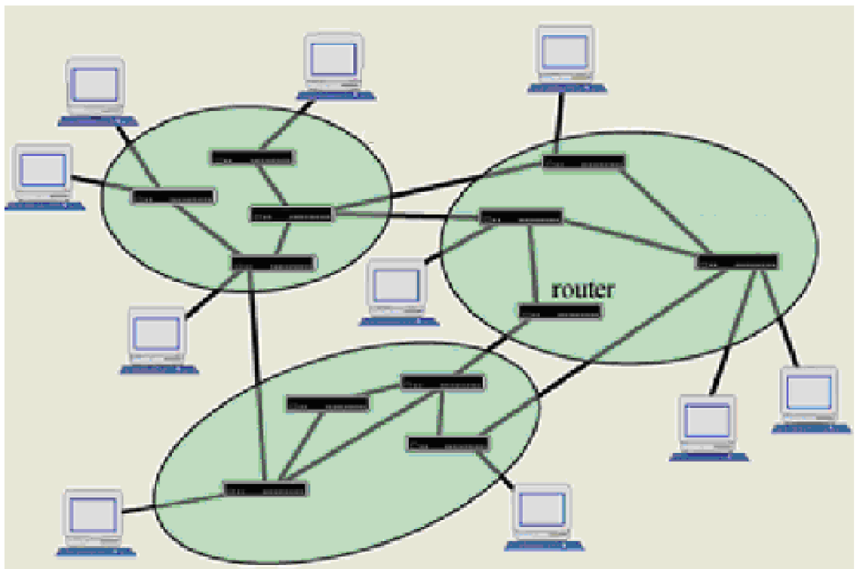


Fig. 3-14 Illustration of the Internet topology

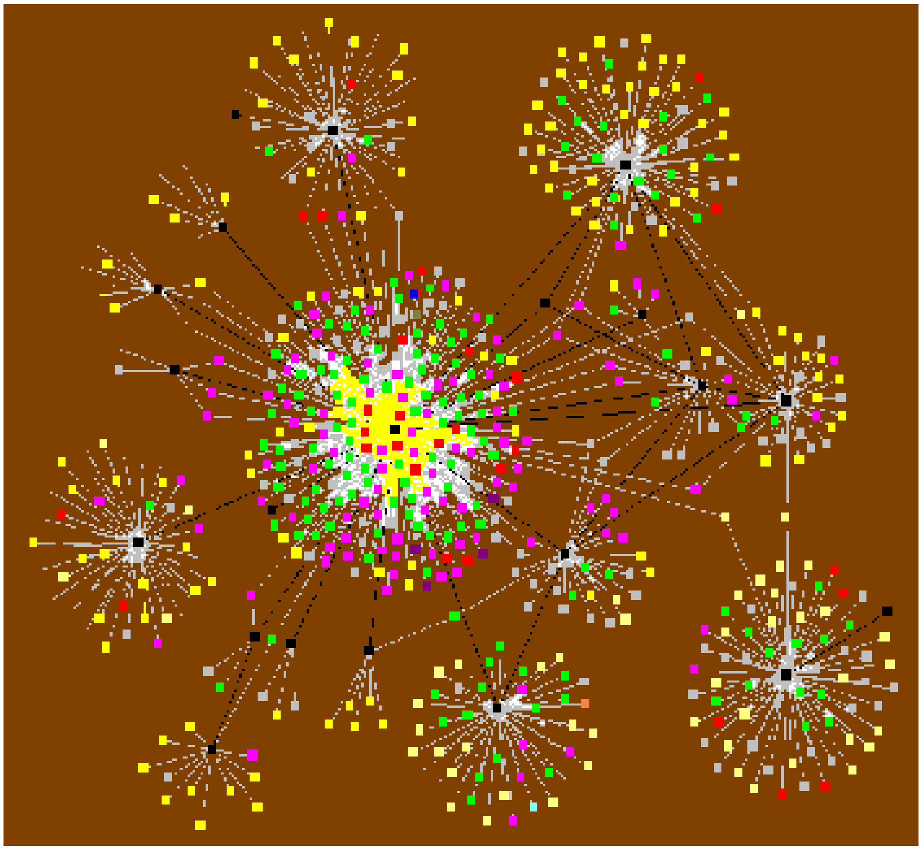


Fig. 3-15 Illustration of topology of networks with localization feature [32]

The algorithm of the MLW model is outlined as follows:

Start with $m \geq 1$ isolated (small) sub-networks (called *local-worlds*), with m_0 nodes and e_0 edges in each local-world, where $1 \leq m_0 \leq m$ and $0 \leq e_0 \leq m_0(m_0 - 1)/2$.

At each step, perform one of the following five operations:

(i) With probability p, a new local-world is created, which contains m_0 nodes and e_0 edges. Meanwhile, for convenience a unique name (e.g., A, B, C etc.) or number (e.g., I, II, III etc.) is generated to identify this new local-world.

(ii) With probability q, a new node is added to a randomly selected existing local-world, and the new node brings in m_1 edges connecting to some nodes within the same local-world. In doing so, first a local-world Ω is selected at random, and then a node to which the new node connects inside Ω is chosen with probability

$$\prod(k_i) = \frac{k_i + \alpha}{\sum_{j \in \Omega}(k_j + \alpha)} \tag{3-27}$$

where Ω is the selected local-world, in which node i locates, and the parameter $\alpha > 0$ represents "attractiveness" of node $i,$ which is used to guarantee a certain probability for "small" or "young" nodes to have a chance to receive new edges. This process is repeated m_1 times.

(iii) With probability $r,$ m_2 edges are added to a randomly chosen local-world. To do this, first a local-world Ω is selected at random, and then one end of an edge is chosen randomly while the other end of the edge is selected from Ω with probability (3-27). This process is repeated m_2 times.

(iv) With probability $s,$ m_3 edges are deleted within a randomly chosen local-world, describing the death of some old edges during the growth of the network. In doing so, first a local-world Ω is selected at random, and then one end of an edge is chosen randomly while the other end of the edge is selected with probability

$$\Pi'(k_i) = \frac{1}{N_\Omega(t) - 1}(1 - \Pi(k_i)), \qquad (3\text{-}28)$$

where $N_\Omega(t)$ represents the number of nodes within Ω, and $\Pi(k_i)$ is given by (3-26). (The negative terms are used to exclude selecting the other end of the edge from the same node i itself.) This process is repeated m_3 times.

(v) With probability $u,$ m_4 edges are added between two randomly selected existing local-worlds. To do this, randomly select a local-world and a node in the local-world with probability given by (3-27). The selected node is used as one end of an edge, and then the another node of the edge, which is in another local-world chosen at random, is selected with probability (3-27). This process is repeated m_4 times.

In this MLW model, the probability parameters satisfy $0 < q \le 1,$ $0 \le p, r, s, u \le 1,$ and $p + q + r + s + u = 1.$

The whole procedure of the MLW model is illustrated by schematic diagram Fig. 3-16 [34,36]. In Fig. 3-16 (a), the original network has $m = 3$ local-worlds (identified by the triangles A, B, and C), with $m_0 = 3$ nodes (represented by the black circles) and $e_0 = 2$ edges in each local-world. In Fig. 3-16 (b), a new local-world D is created, depending on the probability p. This new local-world has $m_0 = 3$ nodes and $e_0 = 2$ edges. In Fig. 3-16 (c-d), a new node j joins the network: first, it selects the local-world C in which it will locate, and then it connects to an existing node ($m_1 = 1$) inside this local-world with probability (3-27). In Fig. 3-16 (e), $m_2 = 2$ edges are added to a randomly selected local-world. Here, for illustration, local-world C is selected. In so doing, one end of an edge is selected at random, and the other end of the edge is chosen with probability (3-27). In Fig. 3-16 (f), a local-world is chosen at random, and then $m_3 = 1$ edge is deleted within this chosen local-world: an end of the edge is selected at random, and then the other end of the edge is chosen with probability (3-28). In Fig. 3-16 (g), depending on the probability u, $m_4 = 1$ edge is added between two nodes located in two different local-worlds, respectively. Both ends of the edge are chosen with probability (3-27). In Fig. 3-16 (h), at the next time step, one of the five possible operations listed above is performed, depending on the corresponding probability of occurrence. Here, for illustration, an edge is added between local-worlds B and D.

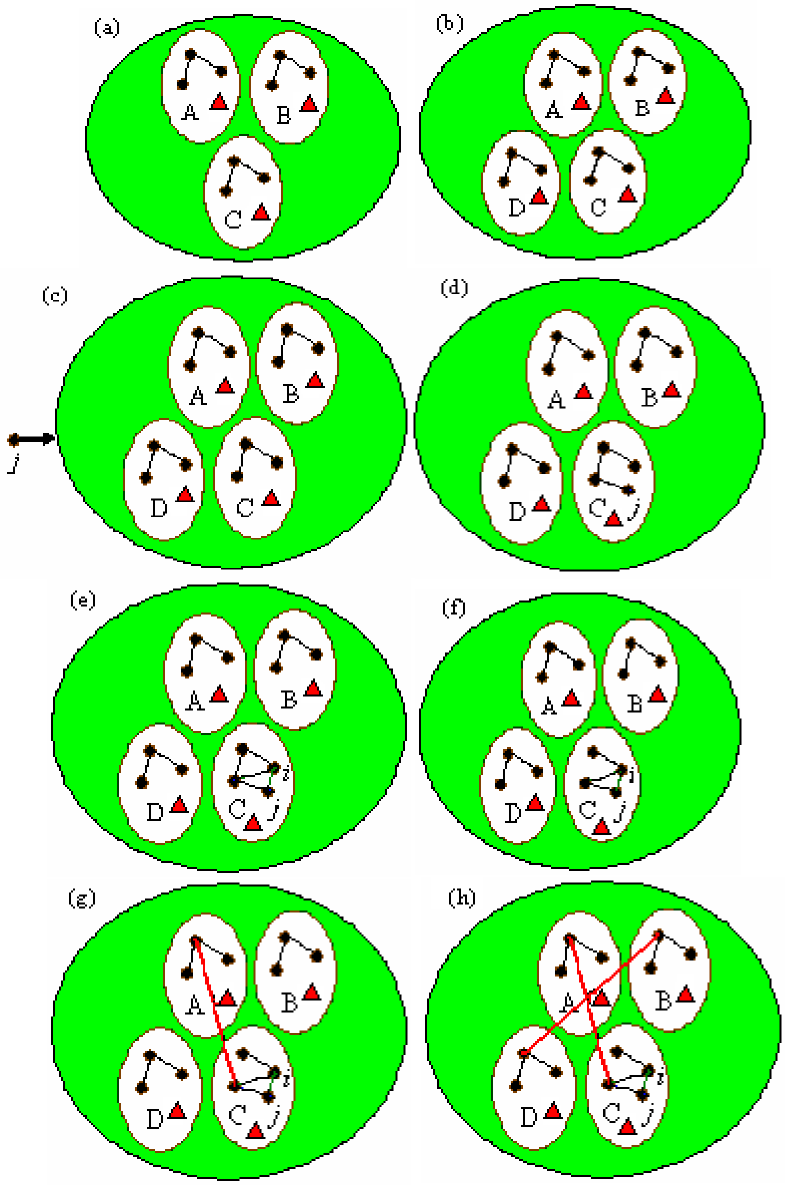


Fig. 3-16 Schematic illustration of the MLW model [34]

Theorem 3-12 [34] *The node-degree distribution of the MLW network model is given by*

$$P(k) = \frac{t}{a(3m + t(1+2p))}\left(m_1 + b/a\right)^{1/a}\left(k + b/a\right)^{-\gamma}$$

where $\gamma = 1 + 1/a$, with

$$a = \frac{qm_1}{c} + \frac{rm_2(q + m_0 p - p)}{(q + m_0 p)c} + \frac{sm_3 p}{(q + m_0 p)c} + \frac{2um_4}{c}$$

$$b = \frac{q\alpha m_1}{c} + \frac{rm_2}{(q + m_0 p)} + \frac{rm_2(q + m_0 p - p)\alpha}{(q + m_0 p)c}$$

$$+ \frac{sm_3 p\alpha}{(q + m_0 p)c} - \frac{2sm_3}{(q + m_0 p)} + \frac{2um_4\alpha}{c}$$

$$c = 2(pe_0 + qm_1 + rm_2 - sm_3 + um_4) + q\alpha$$

and all other probability parameters are defined as in the above-described algorithm.

Proof. [34] The following proof is similar to the proof of Theorem 3-11, in which the degree distribution of a node i in the local-world Ω is first derived, as outlined below.

Step (i) With probability p, a new local-world is created:

In this case, the degree of node i in the existing local-world Ω does not change over time, since the original nodes in the newly created local-world have no edges with any other nodes in the existing local-worlds. Thus,

$$\left(\frac{\partial k_i}{\partial t}\right)_{(1)} = 0$$

Step (ii) With probability q, a new node is being added to the local-world Ω:

$$\left(\frac{\partial k_i}{\partial t}\right)_{(2)} = \frac{m_1 q}{m + tp} \frac{k_i + \alpha}{\sum_{j \in \Omega}(k_j + \alpha)}$$

The term on the right-hand side corresponds to the random selection of a local-world, where a node is selected with probability (3-27). Since there are m_1 edges between the new node and the existing nodes, the coefficient is equal to m_1.

Step (iii) With probability r, m_2 edges are being added to the local-world Ω:

$$\left(\frac{\partial k_i}{\partial t}\right)_{(3)} = \frac{rm_2}{m + tp}\left[\frac{1}{N_\Omega(t)} + \left(1 - \frac{1}{N_\Omega(t)}\right)\frac{k_i + \alpha}{\sum_{j \in \Omega}(k_j + \alpha)}\right]$$

The first term on the right-hand side means the random selection of node i within the local-world Ω, which is also chosen at random; the second term represents the preferential selection of node i inside Ω.

Step (iv) With probability s, m_3 edges are being deleted from within a randomly chosen local-world Ω:

$$\left(\frac{\partial k_i}{\partial t}\right)_{(4)} = -\frac{sm_3}{m+tp}\left[\frac{1}{N_\Omega(t)} + \left(1-\frac{1}{N_\Omega(t)}\right)\frac{1}{N_\Omega(t)-1}\left(1-\frac{k_i+\alpha}{\sum_{j\in\Omega}(k_j+\alpha)}\right)\right]$$

The term on the right-hand side implies that the decrease of the degree of node i in the local-world Ω comes from two sources: one is that it acts as a randomly chosen end of a deleted edge, and the other is that it is the end of a deleted edge selected with probability (3-28).

Step (v) With probability u, m_4 edges are being added between two local-worlds in the network:

$$\left(\frac{\partial k_i}{\partial t}\right)_{(5)} = um_4\left[\frac{2}{m+tp}\frac{k_i+\alpha}{\sum_{j\in\Omega}(k_j+\alpha)} - \frac{1}{m+tp}\frac{1}{m+tp}\frac{k_i+\alpha}{\sum_{j\in\Omega}(k_j+\alpha)}\right]$$

Note that at time step t the total degree of any local-world Ω in the network, on average, is

$$\sum_{j\in\Omega} k_j = 2t(pe_0 + qm_1 + rm_2 - sm_3 + um_4)/(m+tp)$$

And the number of nodes in the local-world Ω, on average, is

$$N_\Omega(t) = m_0 + qt/(m+tp)$$

For convenience, let

$$c = 2(pe_0 + qm_1 + rm_2 - sm_3 + um_4) + q\alpha$$

Now, by combining all related equations together, one obtains

$$\begin{aligned}
\frac{\partial k_i}{\partial t} &= \frac{qm_1}{c}\frac{k_i}{t} + \frac{qm_1\alpha}{c}\frac{1}{t} + \frac{rm_2(q+m_0p-p)}{(q+m_0p)c}\frac{k_i}{t} \\
&\quad + \left(\frac{rm_2}{(q+m_0p)} + \frac{rm_2(q++m_0p-p)\alpha}{(q+m_0p)c}\right)\frac{1}{t} \\
&\quad - \frac{rm_2m}{(q+m_0p)c}\frac{(k_i+\alpha)}{t^2} + \frac{sm_3p}{(q+m_0p)c}\frac{k_i}{t} + \left(\frac{sm_3p\alpha}{(q+m_0p)c} - \frac{2sm_3}{(q+m_0p)}\right)\frac{1}{t} \\
&\quad + \frac{sm_3m}{(q+m_0p)c}\frac{(k_i+\alpha)}{t^2} + \frac{2um_4}{c}\frac{k_i}{t} + \frac{2um_4\alpha}{c}\frac{1}{t} - \frac{um_4}{c}\frac{(k_i+\alpha)}{t(m+tp)} \\
&= \left(\frac{qm_1}{c} + \frac{rm_2(q+m_0p-p)}{(q+m_0p)c} + \frac{sm_3p}{(q+m_0p)c} + \frac{2um_4}{c}\right)\frac{k_i}{t} \\
&\quad + \left(\frac{q\alpha m_1}{c} + \frac{rm_2}{(q+m_0p)} + \frac{rm_2(q+m_0p-p)\alpha}{(q+m_0p)c}\right)\frac{1}{t} \\
&\quad + \left(\frac{sm_3p\alpha}{(q+m_0p)c} - \frac{2sm_3}{(q+m_0p)} + \frac{2um_4\alpha}{c}\right)\frac{1}{t} \quad \text{(for large } t\text{)}
\end{aligned}$$

Next, for convenience, define

$$a = \frac{qm_1}{c} + \frac{rm_2(q+m_0p-p)}{(q+m_0p)c} + \frac{sm_3p}{(q+m_0p)c} + \frac{2um_4}{c}$$

$$b = \frac{q\alpha m_1}{c} + \frac{rm_2}{(q+m_0p)} + \frac{rm_2(q+m_0p-p)\alpha}{(q+m_0p)c} + \frac{sm_3p\alpha}{(q+m_0p)c} - \frac{2sm_3}{(q+m_0p)} + \frac{2um_4\alpha}{c}$$

Then, one has

$$\frac{\partial k_i}{\partial t} = a\frac{k_i}{t} + b\frac{1}{t}$$

Since $a \neq 0$, using the initial condition $k_i(t_i) = m_1$, one can solve the above differential equation to obtain

$$k_i(t) = -\frac{b}{a} + \left(m_1 + \frac{b}{a}\right)\left(\frac{t}{t_i}\right)^a$$

Furthermore, define the time unit in the above MLW model as (one local-world creation)/(one node increment)/(one edge deletion)/(one new edge within a local-world)/(one new edge between local-worlds). Then, the probability density of t_i is

$$P_i(t_i) = 1/(3m + t(1+2p))$$

Consequently, one obtains

$$P(k_i(t) < k) = P\left(t_i > \left(\frac{m_1 + b/a}{k + b/a}\right)^{1/a} t\right)$$

$$= 1 - \frac{1}{(3m + t(1+2p))}\left(\frac{m_1 + b/a}{k + b/a}\right)^{1/a} t$$

Since $P(k) = \dfrac{\partial(P(k_i(t) < k)}{\partial k}$, one has

$$P(k) = \frac{t}{a(3m + t(1+2p))}\left(m_1 + b/a\right)^{1/a}\left(k + b/a\right)^{-\gamma}$$

where $\gamma = 1 + 1/a$. ☐

To predict real-world networks, whose power-law exponents typically satisfy $2 < \gamma < 3$, the above MLW model can be applied, subject to the following conditions:

$$\begin{cases} (m_1 + b/a) > 0 \\ a < 1 \end{cases}$$

Obviously, if one takes $rm_2 \geq 2sm_3$, then the above conditions is satisfied.

Note that the power-law exponent increases with the increase of the attractiveness α of nodes in the above MLW model, which indicates that the attractiveness of nodes in a network may play an important role although the underlying mechanism is somewhat complicated and ambiguous.

Now, consider the following special cases of the MLW model.

Case A: When $m=1, q=1,$ and $p=r=s=u=0,$ the network has only one local-world, and the power-law exponent $\gamma=3+\alpha/m_1.$ The MLW model reduces to the BA model.

Case B: If a network consists of only one local-world, and the evolution of the network only includes the events of adding node and edges, namely, if $m=1, p=0, s=u=0$, then becomes the local-world network model developed in [33], with the exponent $\gamma=3+\alpha q/((m_1-m_2)q+m_2)$. This indicates that the event of adding edges between two existing nodes in the network also has a significant impact on the scale-free feature of this evolving network. However, the exponent keeps unchanged if one takes $\alpha=0$, which may indicate that the attractiveness of nodes plays a more important role than the addition of edges in the evolution of the network.

Case C: If a network consists of a fixed number of local-worlds, and the events of addition and deletion of edges between two nodes in the same local-world do not occur, then one has $p=0,\ \ r=0,\ \ s=0.$ In this case, the degree exponent of the resulting network is $\gamma=2+(m_1+\alpha)q/((m_1-2m_4)+2m_4)$.

Case D: If $rm_2=2sm_3$, then $b=\alpha\ a$, so $P(k)\propto(k+\alpha)^{-\gamma}$, which is different from $P(k)\propto k^{-\gamma}$ in the BA model. This clearly indicates that the attractiveness of nodes is very important to the evolution of the network.

To visualize the graph generated by the MLW model, a simple simulation is carried out as follows [34]: randomly assign a sub-region to every local-world in the plane which the entire network occupies. Within the assigned sub-region, nodes belonging to the same local-world are distributed randomly. Figure 3-17 shows such an example of a network generated by the MLW model, where the network consists of 500 nodes and includes 9 local-worlds. In this figure, nodes in the same local-world are relatively densely connected, while nodes between different local-worlds are connected sparsely.

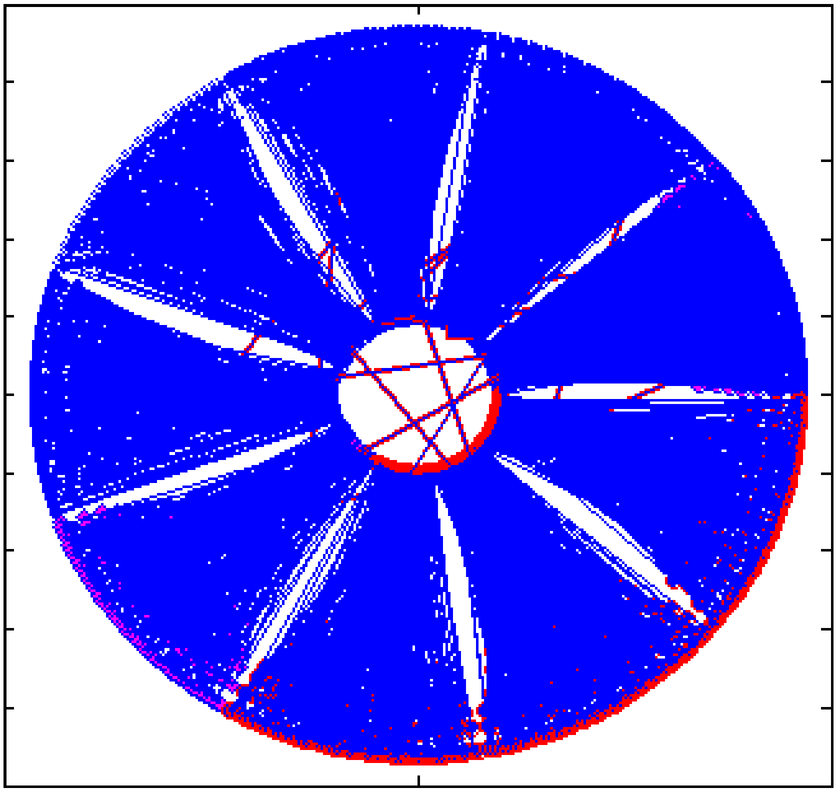


Fig. 3-17 An example of a network generated by the MLW model

3.5.5 A Simple Model with Power-Law Degree Distribution

It should be noted that the property of a power-law degree distribution is not only pertaining to scale-free networks but to some other types of networks as well, even some special sparse random-graph networks [37]. A simple example is the following evolving random-graph type of network model [38]:

1. Start with no nodes and no links.

2. At each time, a new node is added with probability p.

3. With probability q ($p+q=1$), a random edge is added to the existing nodes.

Theorem 3-13 [38] *The degree distribution of the above network model has a power law form,* $P(k) \sim k^{\gamma}$*, where* $\gamma = 1 + q^{-1}$. *Consequently, if* $1/2 < q < 1$ *then* $2 < \gamma < 3$.

Proof. See [38]. ☐

Problems

3-1 Verify Theorem 3-1.

3-2 Verify some simple cases of Theorem 3-2: for formula (3-3) with $K=3$, as illustrated by Fig. 3-18.

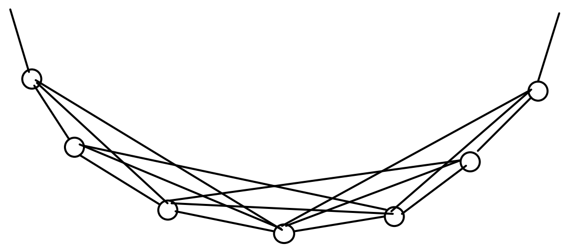


Fig. 3-18 Illustration of Problem 3-2

3-3 Verify formula (3-7) in Theorem 3-4 for simple cases of (a) $N=3,\ p=1$, (b) $N=4,\ p=1$, and (c) $N=5,\ p=1$.

3-4 Write a program to generate ER random-graph networks of size N with connection probability p. Plot some simulated results for networks of size $N=100$, with $p=0.1,\ 0.2,\ 0.5$, respectively.

3-5* Recall the clustering coefficient formula (1-3) from Section 1.3 of Chapter 1: $C_i = 2\dfrac{E_i}{k_i(k_i-1)}$, where k_i is the degree of node i and E_i is the number of actual edges among the neighbors of node i. Recall also that the clustering coefficient of the whole network is the average of all such C_i over all i: $\overline{C} = \sum_i C_i / N$. Verify that the number of edges among the neighbors of node i is equal to

$$E_i = \frac{1}{2}\sum_{jl} a_{ij}a_{jl}a_{li}$$

where $A = \lfloor a_{ij} \rfloor$ is the adjacent matrix of the network. Moreover, show that

$$\overline{C} = \frac{1}{n} Trace\{A^3 \Delta\}$$

where matrix $\Delta = \lfloor \Delta_{ij} \rfloor$ is defined by

$$\Delta_{ij} = \begin{cases} \dfrac{\delta_{ij}}{k_i(k_i-1)} & if \quad k_i > 1 \\ 0 & otherwise \end{cases}$$

in which $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i \neq j$.

3-6 Give some real-world network examples that can be well described by the ER random-graph network model, the WS small-world network model, and the NW small-world network model, respectively.

3-7 Verify that every power law of the form $\alpha\, k^{-\gamma}$, where k is an integer variable and α, γ are constants, satisfies the scale-free property $f(ax) = bf(x)$ shown in Theorem 1-2, Chapter 1.

3-8* a) Verify formula (3-13) in Theorem 3.5.

b) Verify formula (3-16) in Theorem 3.6.

c) Verify formula (3-18) in Theorem 3.7.

d) Verify formula (3-20) in Theorem 3.8.

e) Verify formula (3-21) in Theorem 3.9.

3-9* Verify formula (3-23).

3-10 Name some real-world examples of BA, EBA, Fitness and MLW networks and explain why you think they are such networks.

3-11 Write a program to generate some BA networks. Start with $m_0 = 1$ and proceed with $m = 1$, for steps $t = 1,\ldots,1000$. Plot two simulated results and then compare them.

3-12 Write a program to generate some EBA networks. Start with $m_0 = 1$ and proceed with $m = 1$, $p = 0.5$, and $q = 0.25$, for steps $t = 1,\ldots,1000$. Plot two simulated results and then compare them; also, compare them with the results of Problem 3-11.

3-13 It has been observed that many real-world scale-free networks have power-law node-distributions with exponents $2 < \gamma < 3$. Use Theorem 3-13 to provide a possible explanation.

3-14 Consider the following complex network models:

(a) Model A

- Step 1 (Initialization) Start with a large-sized fully-connected network.
- Step 2 (Process) For every possible pair of nodes, with probability p $(0 < p < 1)$ remove the edge between them. Remove all isolated nodes whenever they appear.
- Step 3 (End) After every possible pair of nodes has been operated once, and once only, stop.

What kind of network will you obtain?

(b) Model B

- Step 1 (Initialization) Start with a large-sized fully-connected network.
- Step 2 (Process) Pick up an edge: if removing it does not disconnect the whole network, then remove it; but if removing it will disconnect the network, then do nothing. Continue to pick up another edge from the resultant network and repeat the above possible edge-removal operation.
- Step 3 (End) After every possible edge has been operated once, and once only, stop.

What kind of network will you obtain?

(c) Model C

- Step 1 (Initialization) Start with a large-sized tree.
- Step 2 (Preferential Attachment) Pick up every possible pair of nodes from the tree. If this pair of nodes is directly connected already, do nothing; if this pair of nodes is not directly connected, then with a probability proportional to the degree of the larger node, add an edge between them.
- Step 3 (End) After every possible pair of nodes has been operated once, and once only, stop.

Is this resultant network a good model for an Internet-like network? If you think so, state three major advantages of this model; if you don't think so, state three major disadvantages of this model.

References

[1] Watts D J, Strogatz S H. Collective dynamics of 'small-world' networks. *Nature*, 1998, 393(6684): 440-442

[2] Barabási A-L, Albert R. Emergence of scaling in random networks. *Science*, 1999, 286(5439): 509-512

[3] Albert R, Barabási A-L. Statistical mechanics of complex networks. *Reviews of Modern Physics*, 2002，74: 47-97

[4] Newman M E J. The structure and function of complex networks. *SIAM Review*, 2003, 45: 167-256

[5] Erdös P, Rényi A. On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci.*, 1960, 5: 17-60

[6] Bollobás B. *Random Graphs*, New York: Academic Press, 2nd ed., 2001

[7] Crucitti P, Latora V, Marchiori M, Rapisarda, A. Complex systems: analysis and models of real-world networks, in *Energy and Information Transfer in Biological Systems*, M Francesco, L S Brizhik, M W Ho (eds.), World Scientific, 2003, pp 188-204

[8] Newman M E J, Watts D J. Renormalization group analysis of the small-world network model. *Phys. Lett. A*, 1999, 263: 341–346

[9] Barrat A, Weigt M. On the properties of small world networks. *Eur. Phys. J. B*, 2000, 13: 547–560

[10] Newman M E J. The structure and function of networks. *Computer Physics Communications*, 2002, 147, 40–45

[11] Newman M E J, Moore C, Watts D J. Mean field solution of the small-world network model. *Phys. Rev. Lett.*, 2000, 84: 3201-3204

[12] Bollobás B, Riordan O. Mathematical results on scale-free random graphs. In: Bornholdt, S, Schuster H G (ed.) *Handbook of Graphs and Networks: From the Genome to the Internet*, Berlin: Wiley-VCH, 2003, 1-34

[13] Cohen R, Havlin, S. Scale-free networks are ultrasmall. *Phys. Rev. Lett., 2003*, 86: 3682-3685

[14] Fronczak A, Fronczak P, and Holyst J A. Mean-field theory for clustering coefficients in Barabási-Albert networks, *Phys. Rev. E* 2003, 68: 046126

[15] Dorogovtsev S N, Mendes J F F, Samukhin A N. Structure of growing networks with preferential linking. *Phys. Rev. Lett.*, 2000, 85: 4633–4636

[16] Hou Z, Kong X, Shi D, and Chen G. Degree-distribution stability of scale-free networks, arXiv:0901.1418v1

[17] Albert R, Jeong H, Barabási A-L. Attack and error tolerance in complex networks. *Nature*, 2000, 406: 387-482

[18] Barabási A-L, Bonabeau E. Scale-free networks. *Scientific American*, May 2003, 50-59

[19] Carlson J, Doyle J. Highly optimized tolerance: Robustness and power laws in complex systems. *Phys. Rev. Lett.*, 2000, 84(11): 2529-2532

[20] Carlson J, Doyle J. Complexity and Robustness. *PNAS*, 2002, 99 (Suppl. 1): 2539-2545

[21] Broder A, Kumar R, Maghoul F *et al*. Graph structure in the web. *Comput. Networks*, 2000, 33: 309-320

[22] Doyle J, Alderson D, Li L *et al*. The “robust yet fragile” nature of the Internet. *Proceedings of the National Academy of Sciences USA*, 2005, 102(41): 14497-14502

[23] Callway D S, Newman M E J, Strogatz S H *et al.* Network robustness and fragility: Percolation on random graphs. *Phys. Rev. Lett.*, 2000, 85(25): 5468-5471

[24] Cohen R, Erez K, ben-Avraham D *et al.* Resilience of the internet to random breakdowns. *Phys. Rev. Lett.*, 2000, 85(21): 4626-4628

[25] Cohen R, Erez K, ben-Avraham D *et al*. Breakdown of the internet under intentional attack. *Phys. Rev. Lett.*, 2001, 86(16): 3682-3685

[26] Valente A X C N, Sarkar A, Stone H A. Two-peak and three-peak optimal complex networks. *Phys. Rev. Lett.*, 2004, 91: 118702

[27] Bollobás B, Riordan O. Robustness and vulnerability of scale-free random graphs. *Internet Math.*, 2003, 1: 1–35

[28] Holme P, Kim B J, Yoon C N *et al*. Attack vulnerability of complex networks. *Phys. Rev. E*, 2002, 65: 056109

[29] Albert R, Barabási A-L. Topology of evolving networks: local events and universality. *Phys. Rev. E*, 2000, 85(24): 5234-5237

[30] Bianconi G, Barabási A-L. Bose–Einstein condensation in complex networks. *Phys. Rev. Lett.*, 2001, 86: 5632–5635

[31] Li X, Jin Y Y, Chen G. Complexity and synchronization of the World Trade Web. *Physica A*, 2003, 328: 287-296

[32] Wang X F. Complex networks: Topology, dynamics and synchronization. *Int. J. Bifurcation and Chaos*, 2002, 12(5): 885-916

[33] Li X, Chen G. A local world evolving network model. *Physica A*, 2003, 328: 274-286

[34] Fan Z P. *Complex Networks: From Topology to Dynamics*. PhD Thesis, City University of Hong Kong, May 2006

[35] Chen G, Fan Z P, Li X. Modeling the complex Internet topology, in *Complex Dynamics in Communication Networks*, G Vattay and L Kocarev (eds.), Springer-Verlag, 2005, pp 213-234

[36] Fan, Z P, Chen G, Zhang Y. A comprehensive multi-local-world model for complex networks. *Physica A* 2009, 373: 1601-1605

[37] Aiello W, Chung F, Lu L Y. A random graph model for power law graphs. Experimental Math. 2001, 10: 53-66

[38] Aiello W, Chung F, Lu L Y. Random evolution of massive graphs, in *Handbook on Massive Data Sets*, J Abello, P M Pardalos and M G C Resende (eds.), Kluwer Academic, 2002, pp 97-122

