Online Primal-Dual Algorithms for Maximizing Ad-Auctions Revenue

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Model

Algorithm

Analyze

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- provider's goal: Maximize revenue

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- Result: Revenue $= \$20$

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- "Clever" strategy: \$14 (User 1 wins item 1 & 3 & 5)

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- "Adjusted Bid": $(1 - x(i))b(i, j)$

Online Algorithm

Allocation Algorithm: Initially $\forall i \;\; x(i) \leftarrow 0$.
Upon arrival of a new product j allocate the product to the buyer i that maximizes $b(i,j)(1-x(i))$.
If $x(i) \geq 1$ then do nothing. Otherwise:

1. Charge the buyer the minimum between $b(i,j)$ and its remaining budget and set $y(i,j) \leftarrow 1$
2. $z(j) \leftarrow b(i,j)(1-x(i))$
3. $x(i) \leftarrow x(i)\left(1+\frac{b(i,j)}{B(i)}\right)+\frac{b(i,j)}{(c-1)\cdot B(i)}$ (c is determined later). $(x(i) \leftarrow x(i) + \Delta)$

Fractional Dual and Primal

Primal (Covering)		Dual (Packing)	
Maximize:	$\sum_{j=1}^{m} \sum_{i=1}^{n} b(i,j)y(i,j)$	Minimize :	$\sum_{i=1}^{n} B(i)x(i) + \sum_{j=1}^{m} z(j)$
Subject to:		Subject to:	
For each $1 \leq j \leq m$:	$\sum_{i=1}^{n} y(i,j) \leq 1$	For each (i,j):	$b(i,j)x(i) + z(j) \geq b(i,j)$
For each $1 \leq i \leq n$:	$\sum_{j=1}^{m} b(i,j)y(i,j) \leq B(i)$	For each i, j:	$x(i), z(j) \geq 0$
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- 3. The primal is satisfied?

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- Greedily choose: $\max_i \{b(i,j)(1 - x(i))\}$
- And sets $z(j)$ to this value
- Also notice: $x(i)$ only increases

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- $\Delta D = B(i)\Delta + z(j)$

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- So the updating equation for $x(i)$ is

$$x(i) \leftarrow x(i)\left(1 + \frac{b(i,j)}{B(i)}\right) + \frac{b(i,j)}{(c-1)B(i)}$$

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- $B(i) + b(i,j) \leq B(i)(1+R)$
- Actual competitive ratio is

$$\frac{1}{R+1} \cdot (1 + \frac{1}{c-1})$$

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- Conclusion is:

$$c \leq (1+R)^{1/R}$$

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- The ratio is close to $e/(e-1)$ (or 1.58-competitive)