Competitive Analysis via Regularization (SODA'14)

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Motivation

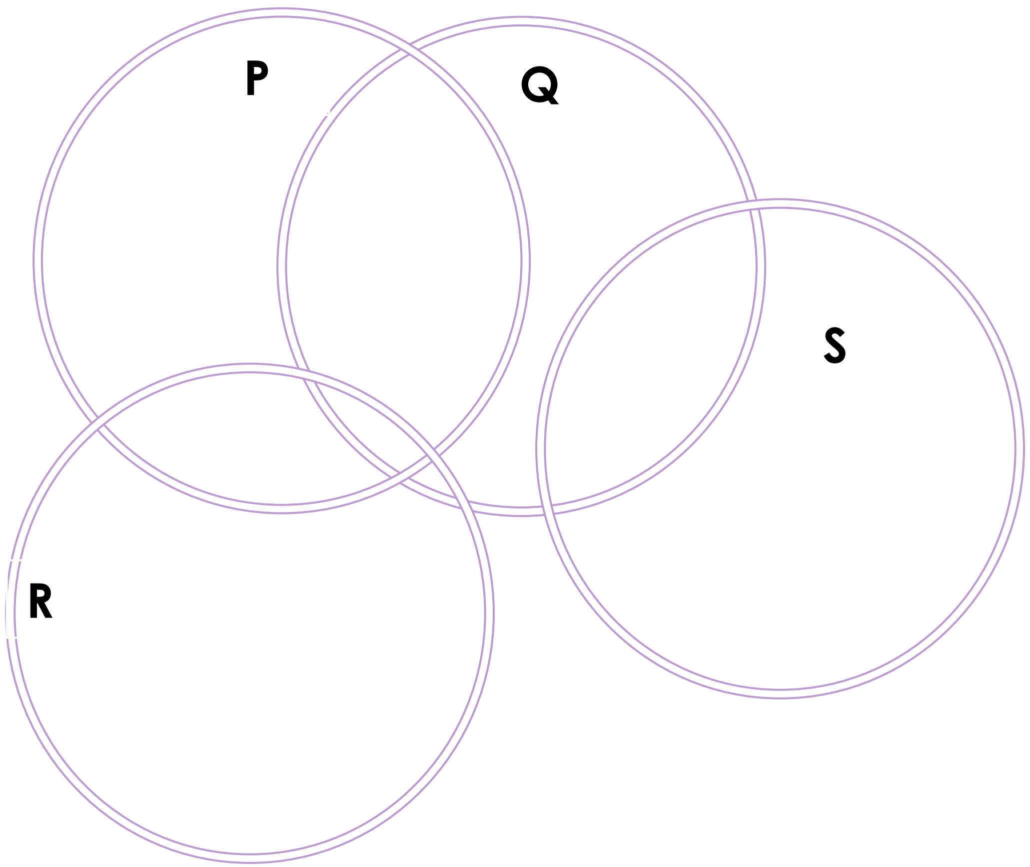
- Understand the model
- Understand primal-dual combined with KKT optimality conditions

Outline

- Model
- Regularization Algorithm (fractional version)
- Competitive Analysis (fractional version)

- Extensions:
  - More general constraints
  - Rounding a fractional solution to the online set cover with service cost problem

Online Set Covering (simple and integer version)



- What are known in advance ?
  The sets(circles) on the plane are fixed in advance.

- Goal:
  to minimize the # of sets (circles) covering all the nodes

- What comes online?
  The nodes on this plane will appear gradually, reporting the set of circles(Set) which can cover them.

Online Set Covering
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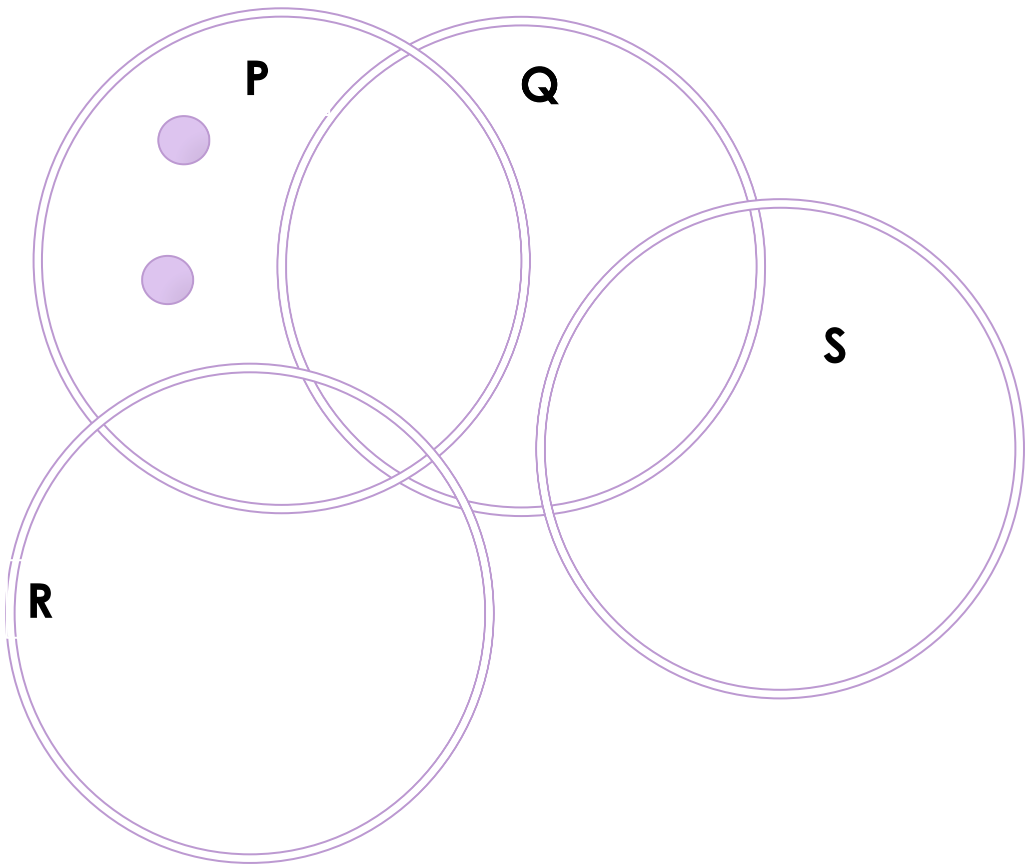

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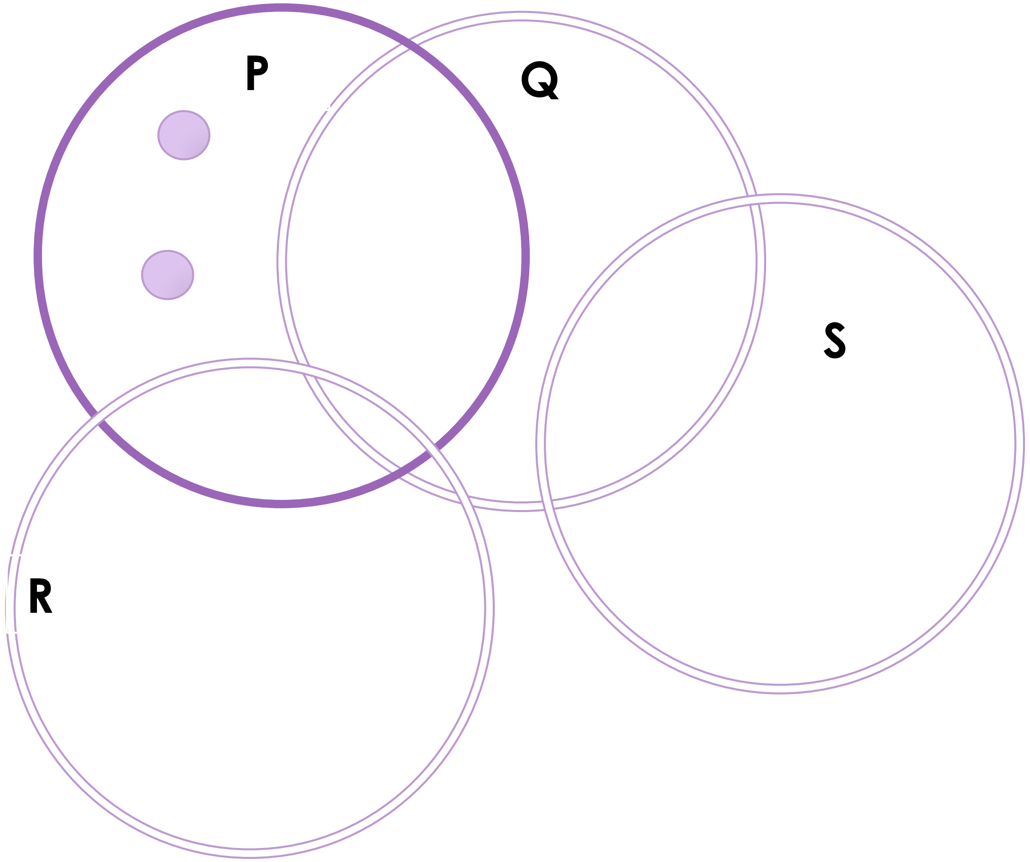

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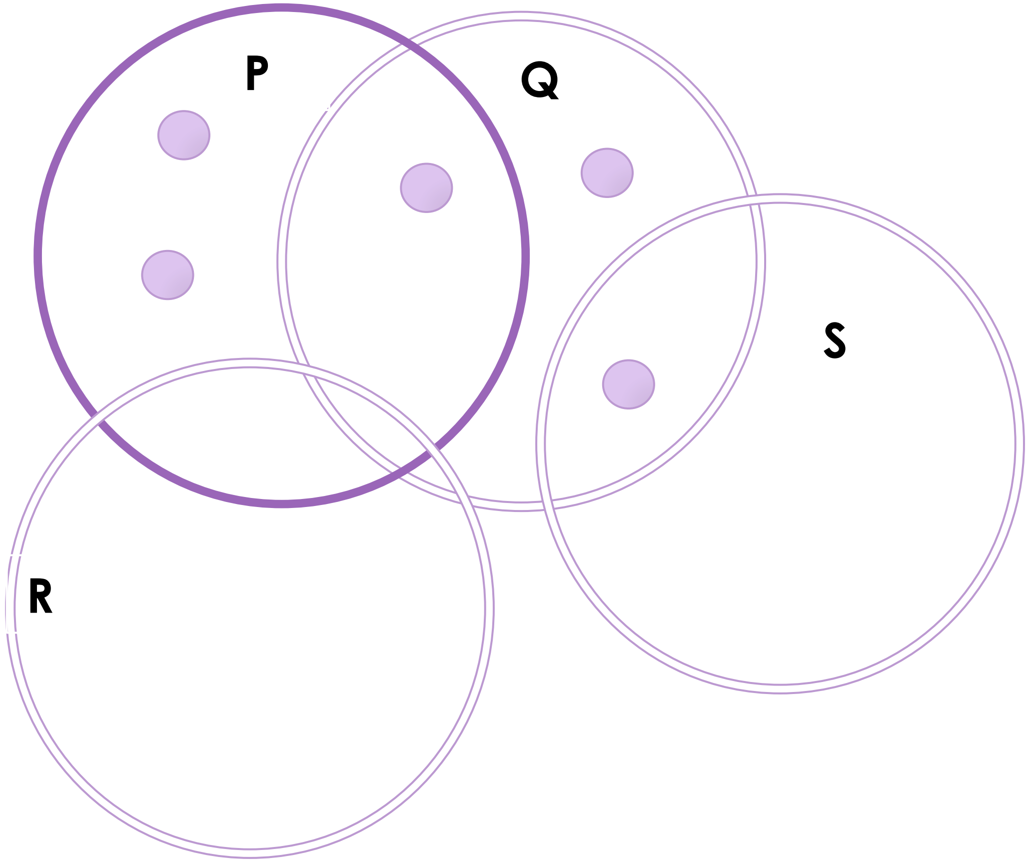

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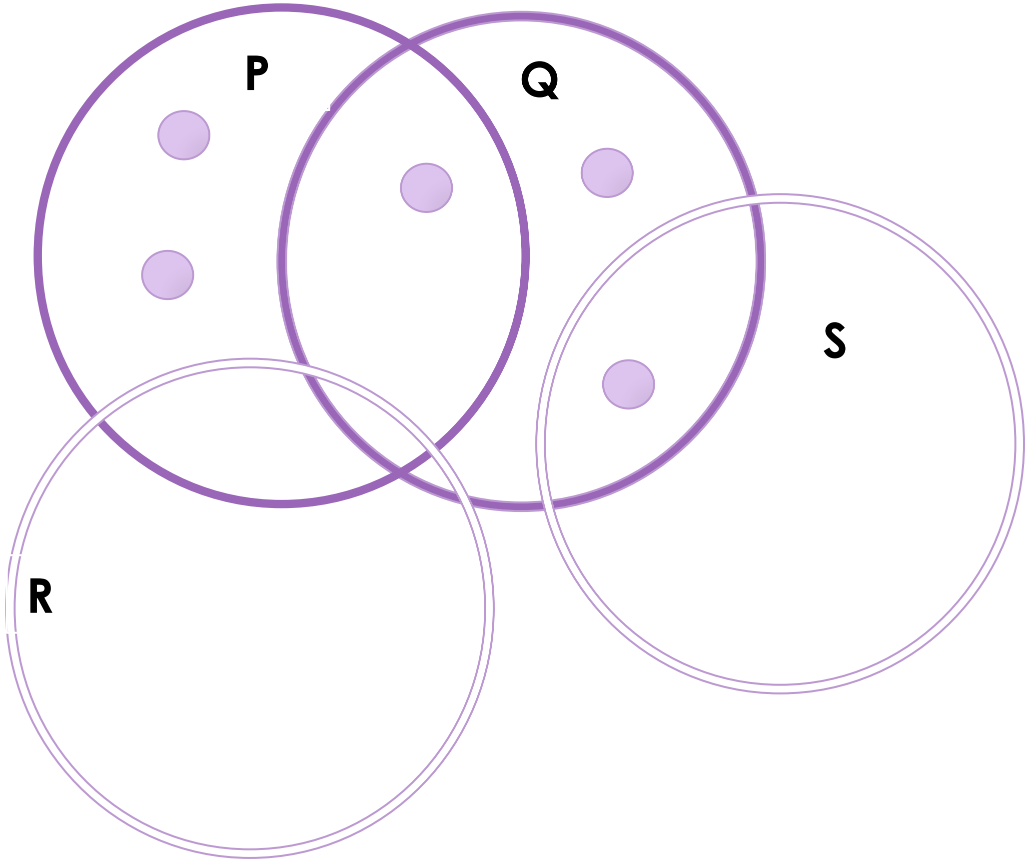

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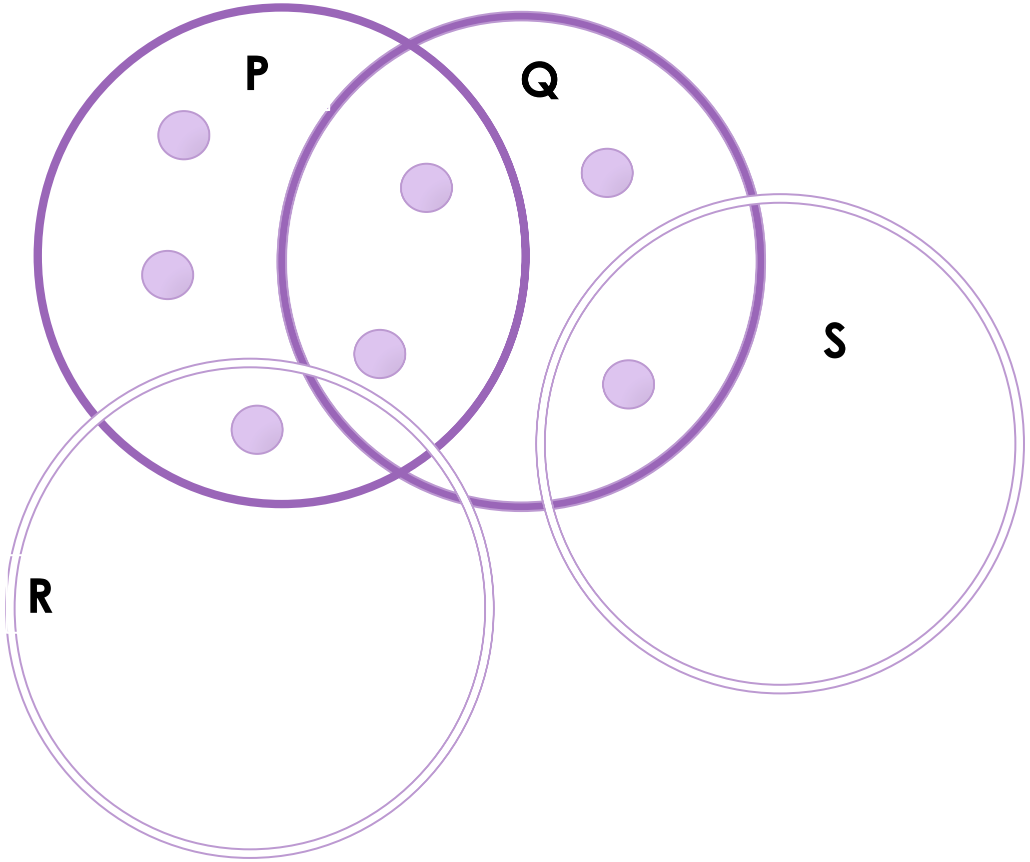

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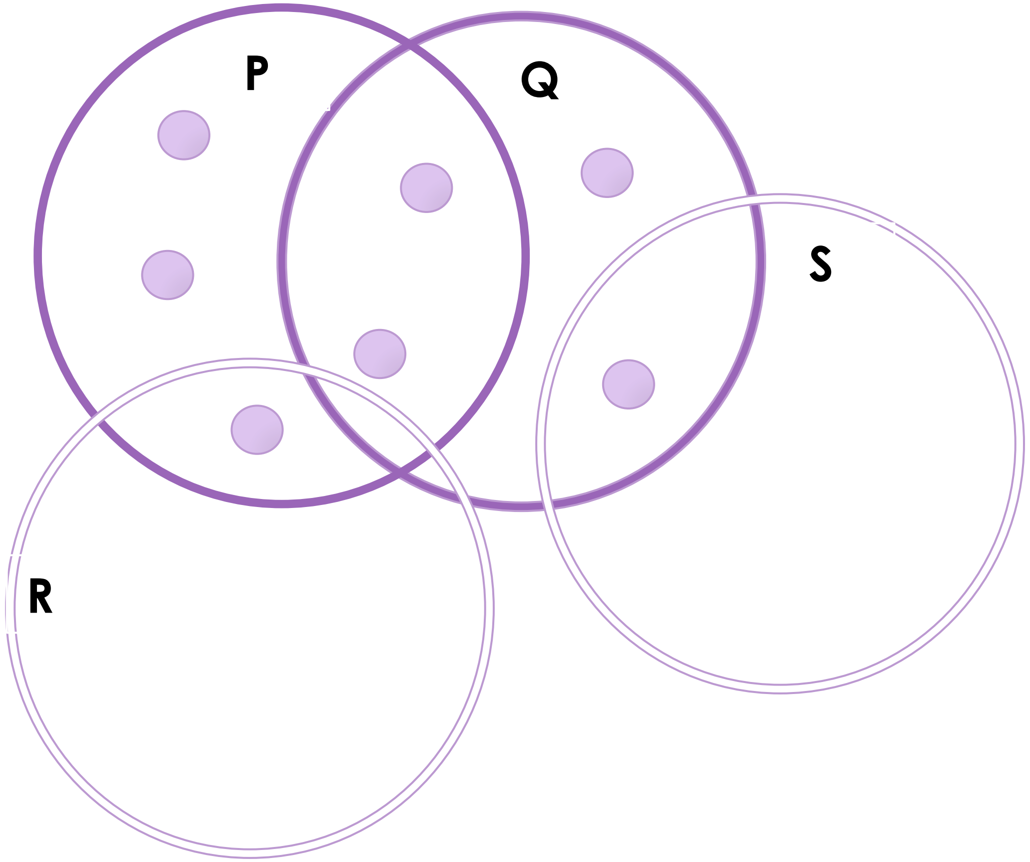

- What if nodes can disappear some time in the future？
- Or even worse, the disappearing time is revealed only when the node disappear？
- What if the cost of turning on the sets are different？
- What if there is cost for keeping a set active after the set is turning on？

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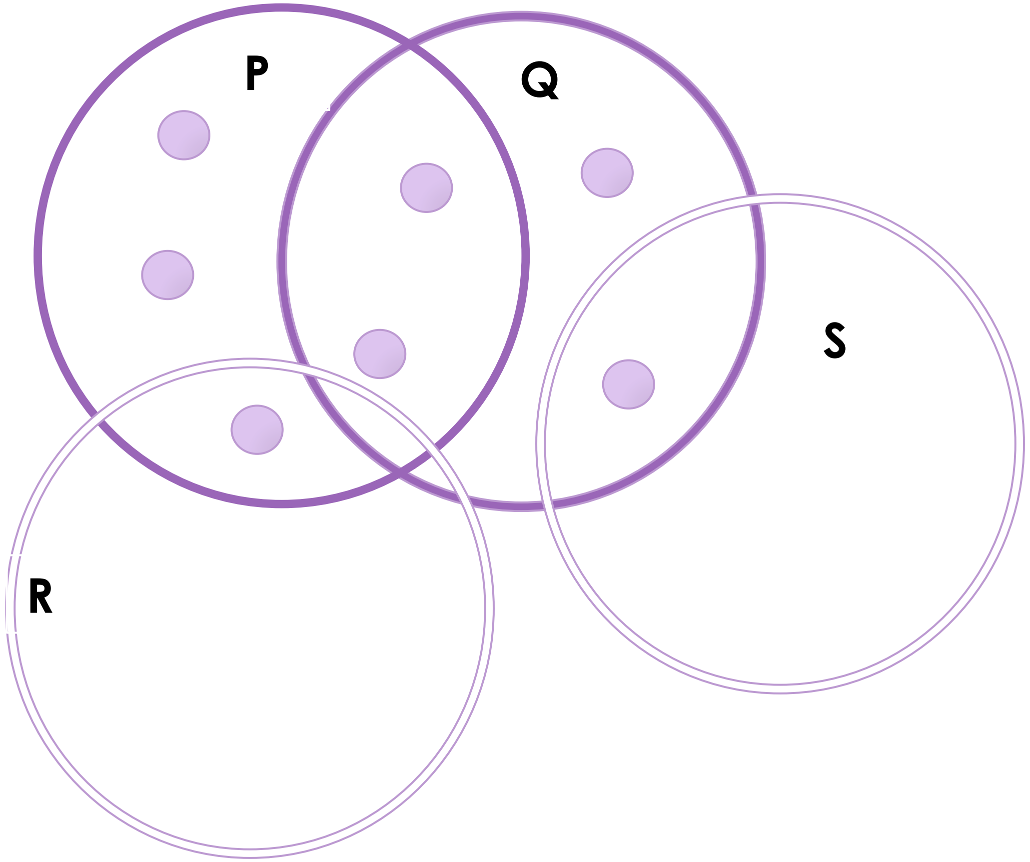

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- **How the situation will change you can think of ?**

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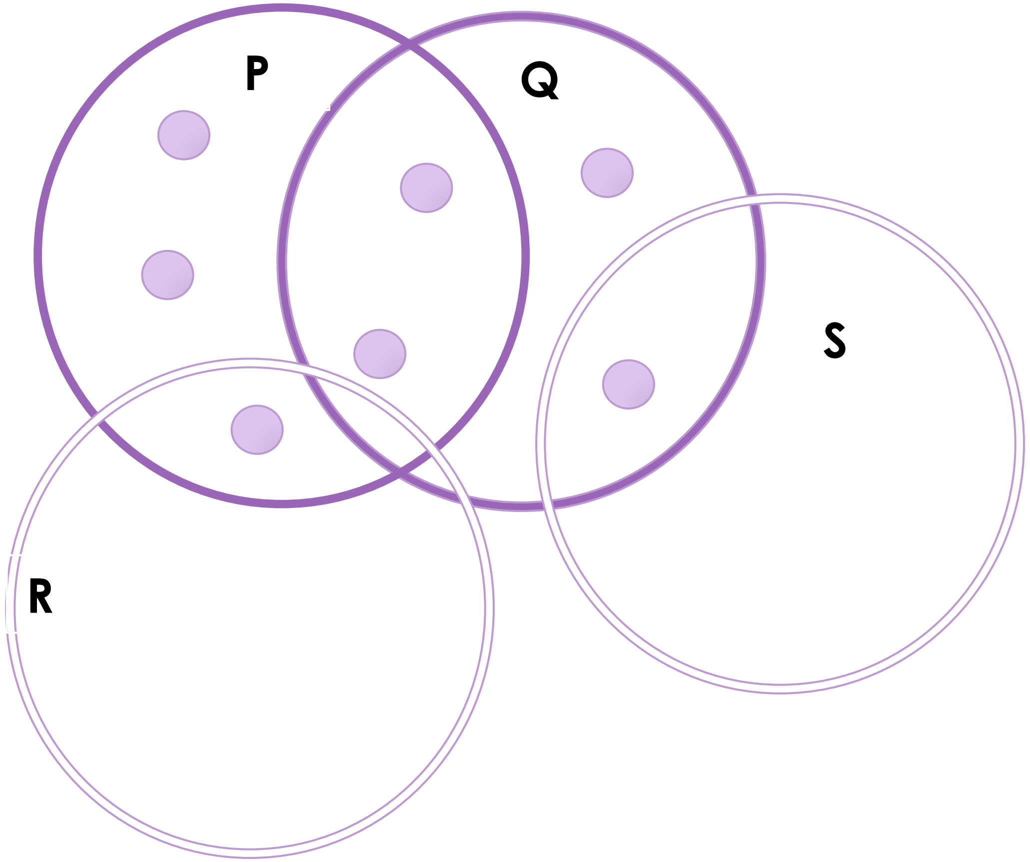

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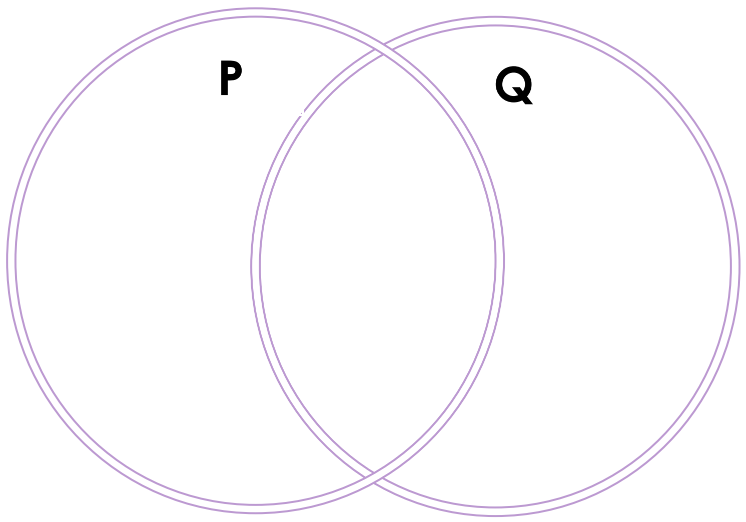
- What if the cost of turning on the sets are different ?
- What if there is cost for keeping a set active after the set is turning on ?
- Some sets may be turned off

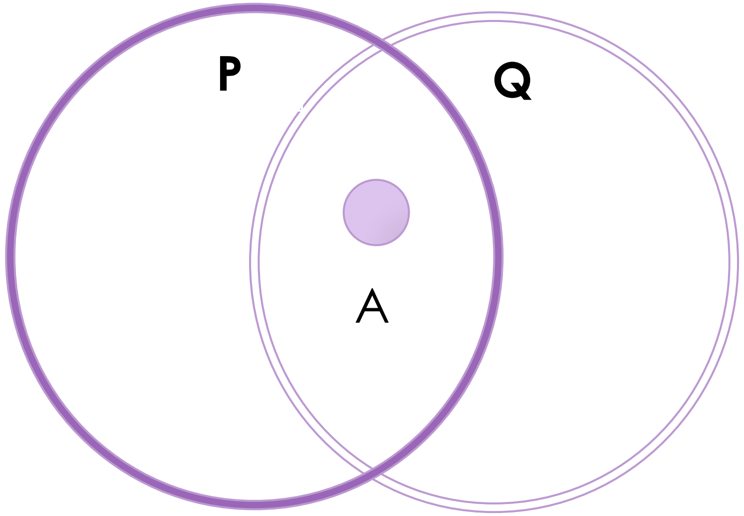
Online Set Covering with Service Cost (simple and integer version)



Set	service cost/unit time slot	turn-on cost
P	3	1
Q	1	4

- Online setting: the nodes on this plane will appear and disappear gradually, reporting the set of circles(Set) which can cover them.
- The appearing time and disappearing time will not be revealed in advance
- Our goal: to minimize the sum of total serving cost and turn-on cost.

Online Set Covering with Service Cost
(simple and integer version)



Greedy

Set	service cost/unit time slot	turn-on cost
P	3	1
Q	1	4

Time	active nodes	associated circles	active circles	updated total cost
t=1	A	P,Q	P	3+1=4

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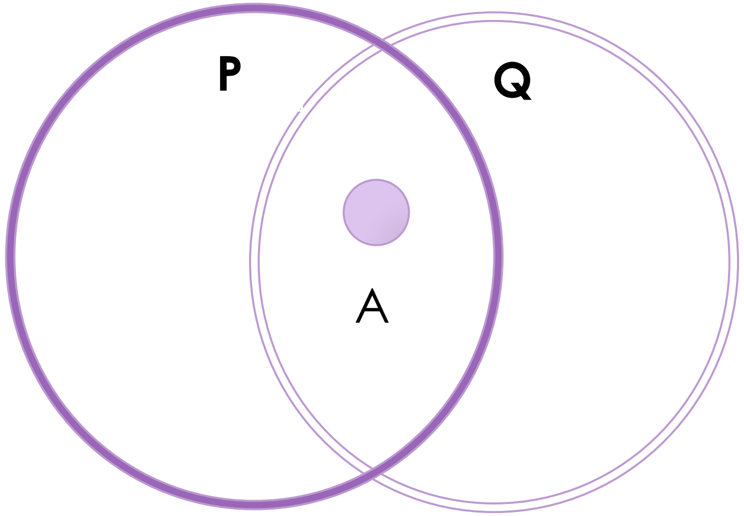

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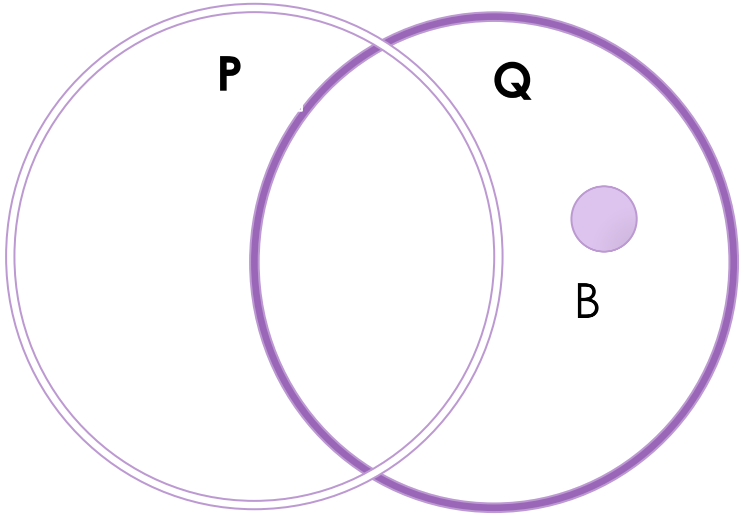
t=2	A	P,Q	P	4+3=7

Online Set Covering with Service Cost
(simple and integer version)



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Set	service cost/unit time slot	turn-on cost
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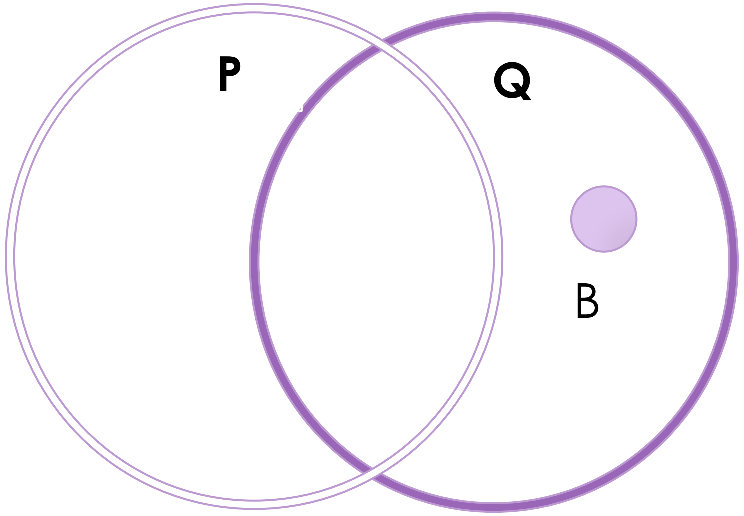
Q	1	4

Time	active nodes	associated circles	active circles	updated total cost
t=1	A	P,Q	P	3+1=4
t=2	A	P,Q	P	4+3=7
t=3	B	Q	Q	7+1+4=12

Online Set Covering with Service Cost
(simple and integer version)



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t=1	A	P,Q	P	3+1=4
t=2	A	P,Q	P	4+3=7
t=3	B	Q	Q	7+1+4=12

OPT: Turn on Q. Optimal Cost= 4+1+1+1=7

Model

$$\text{(P)} \qquad \min \sum_{t=1}^{T} \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \sum_{t=1}^{T} \sum_{i=1}^{n} w_i \cdot z_{i,t}$$

$$\begin{array}{ll} \forall t \geq 1 \text{ and } 1 \leq j \leq m_t & \sum_{i \in S_{j,t}} y_{i,t} \geq 1 \\ \forall t \geq 1 \text{ and } 1 \leq i \leq n & z_{i,t} \geq y_{i,t} - y_{i,t-1} \\ \forall t \text{ and } 1 \leq i \leq n & z_{i,t}, y_{i,t} \geq 0 \end{array}$$

What arrives online?

- m_t the # of items at each time slot t arrives at t
- S_{j,t} the associated set of item j at time slot arrives at t,
  containing the elements I
  1<=|S_{j,t}|<=n

Model

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Difficult to understand !!!

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Online Set Covering with Service Cost (fractional version)

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turn-on cost of circle i

Service cost / unit time slot

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turn-on cost of circle i

the # of the nodes appearing at t

i: circle(set);
j: node at each time slot

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The fraction at which circle i is active at time t

turn-on cost of circle i

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Associated set of circles which can cover the node j at t

Break Ties !!!

the # of the nodes appearing at t

i: circle(set);
j: node at each time slot

Service cost / unit time slot

Multiple slots in the duration of each node may have different indexes

Outline

- Model
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Online Regularization Algorithm

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Offline

Online Goal

$$\min \textstyle\sum_{t=1}^{T} \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \displaystyle\sum_{t=1}^{T} \sum_{i=1}^{n} w_i \cdot |y_{i,t} - y_{i,t-1}|,$$

Actual cost

$$\min \textstyle\sum_{t=1}^{T} \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \sum_{t=1}^{T} \displaystyle\sum_{i=1}^{n} w_i \cdot \max\{0, y_{i,t} - y_{i,t-1}\}.$$

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Online Regularization Algorithm

Online Optimization

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Regularization Function for l_1-norm

$$\Delta(\mathbf{w} \| \mathbf{u}) = \sum_i \left(\mathbf{w}_i \ln \frac{\mathbf{w}_i}{\mathbf{u}_i} + \mathbf{u}_i - \mathbf{w}_i \right)$$

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Online Optimization

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Modified Online optimization

$$\min \sum_{t=1}^{T} \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \sum_{t=1}^{T} \frac{1}{\eta} \sum_{i=1}^{n} w_i \left(\left(x_i + \frac{\epsilon}{n} \right) \ln \left(\frac{x_i + \epsilon/n}{y_{i,t-1} + \epsilon/n} \right) - x_i \right)$$

$$\forall t \geq 1 \text{ and } 1 \leq j \leq m_t \qquad \sum_{i \in S_{j,t}} y_{i,t} \geq 1$$

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Algorithm 1 Regularization Algorithm

parameters: $\epsilon > 0, \eta = \ln(1 + n/\epsilon)$.
initialize $y_{i,0} = 0$ for all $i = 1, \ldots, n$.
for $t = 1, 2, \ldots, T$ **do**
  let $c_t \in \mathbb{R}^n_+$ be the cost vector and let P_t be the feasible set of solutions at time t.
  solve the following convex program to obtain y_t,

$$(P') \qquad y_t = \arg\min_{x \in P_t} \left\{ \langle c_t, x \rangle + \frac{1}{\eta} \sum_{i=1}^{n} w_i \left(\left(x_i + \frac{\epsilon}{n} \right) \ln \left(\frac{x_i + \epsilon/n}{y_{i,t-1} + \epsilon/n} \right) - x_i \right) \right\}.$$

end for

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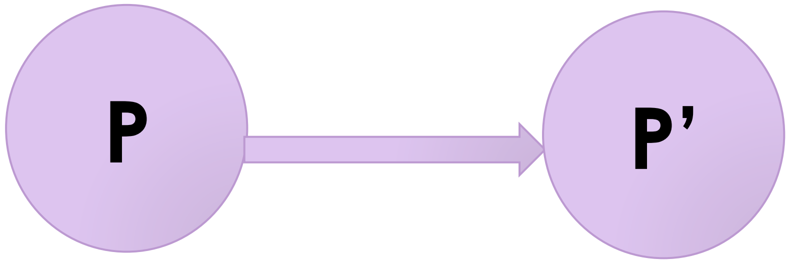
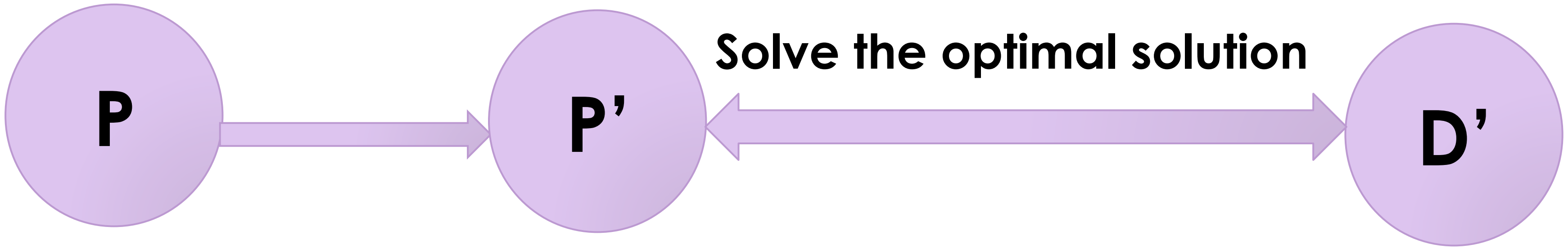
Competitive Analysis —— Primal-dual Framework

- Primal $\xrightarrow{\text{KKT Condition}}$ Dual
- Objective consists of two parts:
  - Service Cost $\sum_{t=1}^{T}\sum_{i=1}^{n} c_{i,t} \cdot y_{i,t}$
  - Movement Cost $\sum_{t=1}^{T}\sum_{i=1}^{n} w_i \cdot \max\{0, y_{i,t} - y_{i,t-1}\}.$
- Two inequalities
  - a-b <=a ln (a/b)
  - $\sum_i a_i \log(a_i/b_i) \geq (\sum_i a_i) \log(\frac{\sum_i a_i}{\sum_i b_i}).$

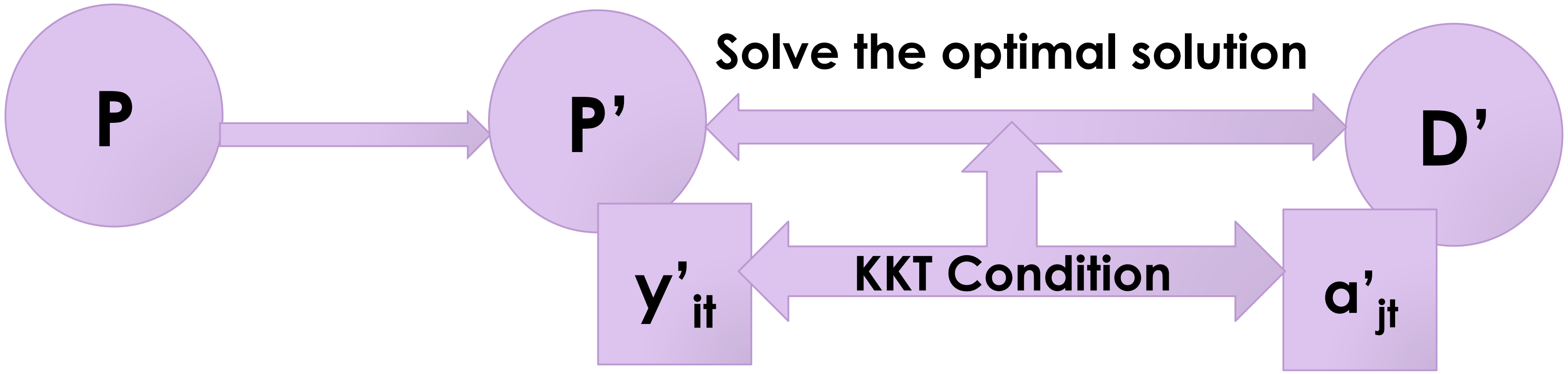
Competitive Analysis ——big picture



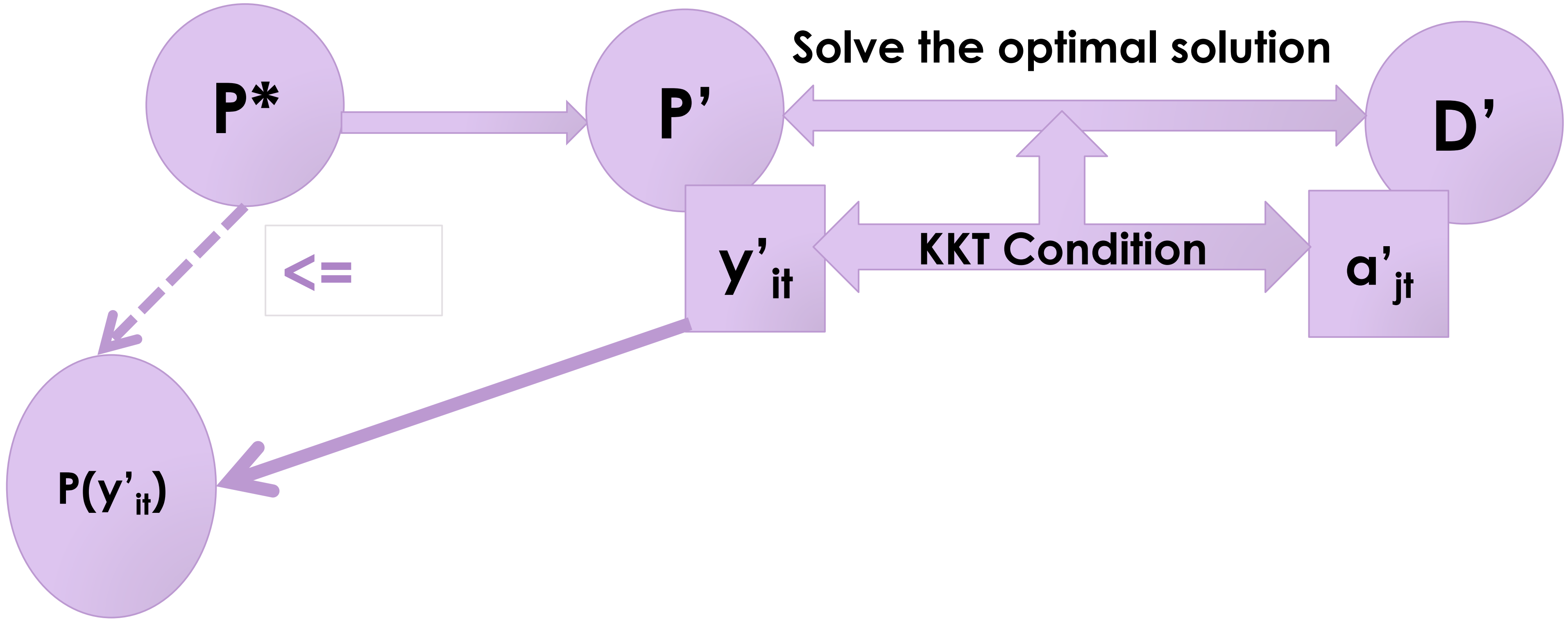
Competitive Analysis ——big picture



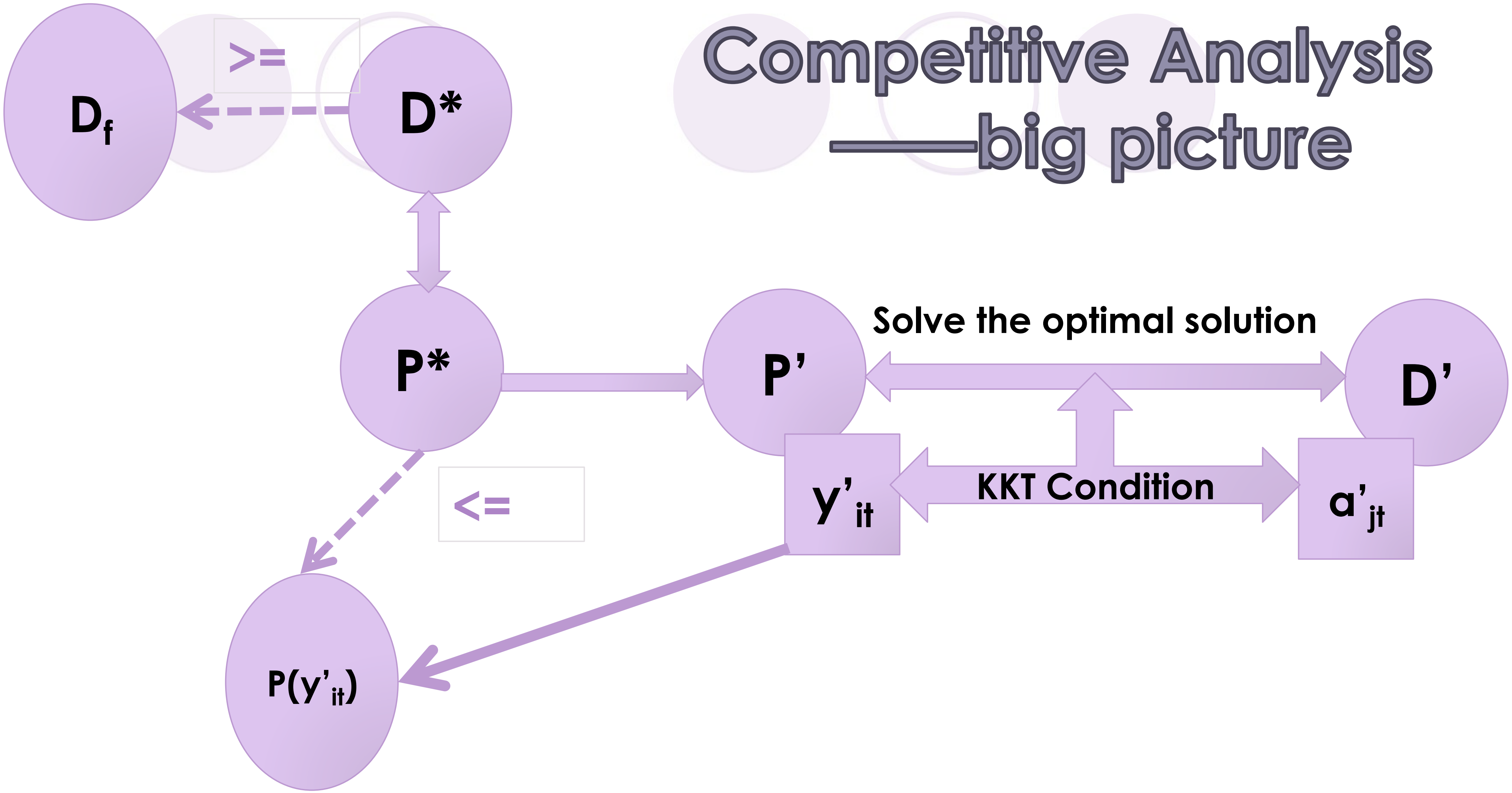
Competitive Analysis ——big picture



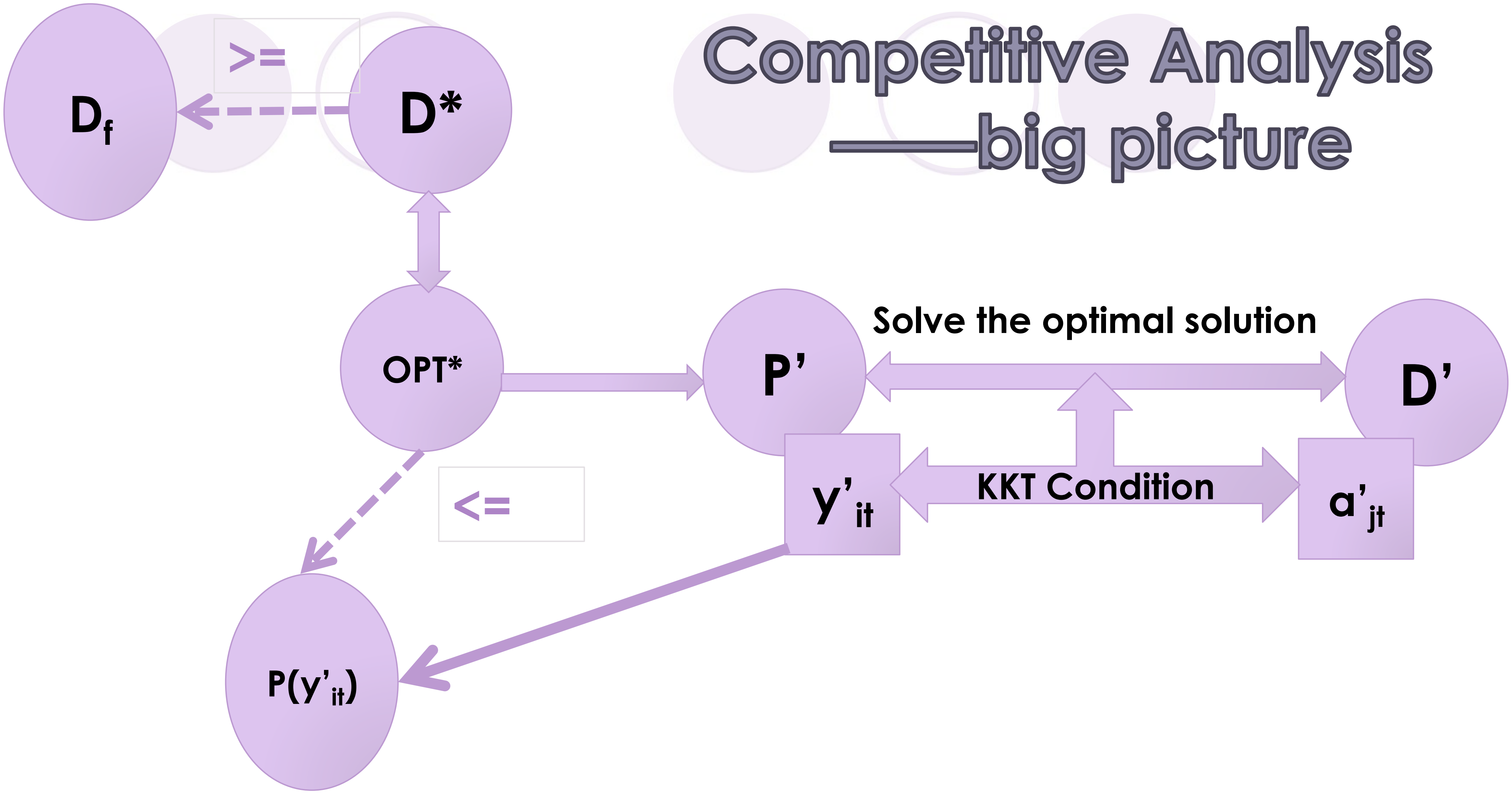
Competitive Analysis ——big picture



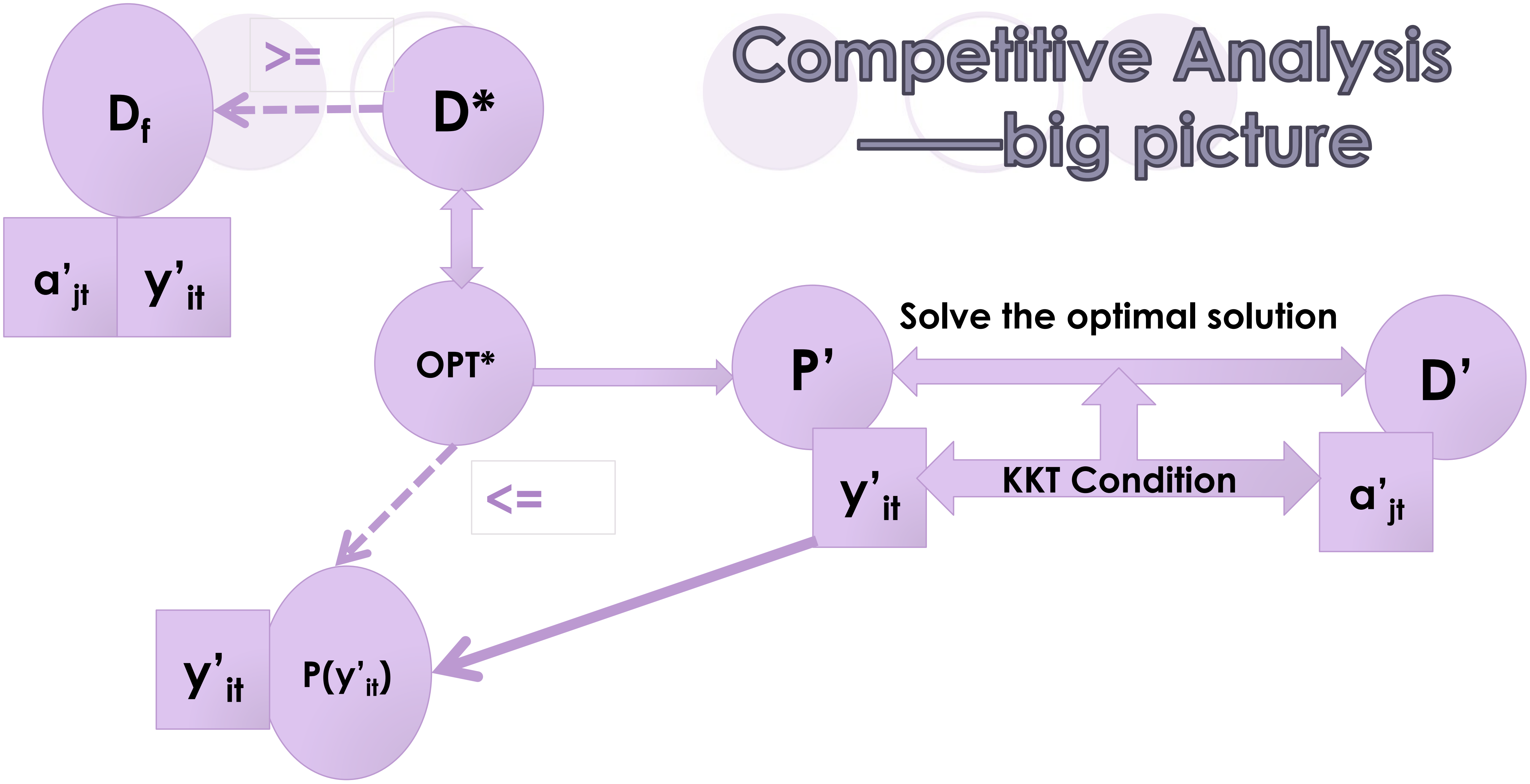
Competitive Analysis ——big picture



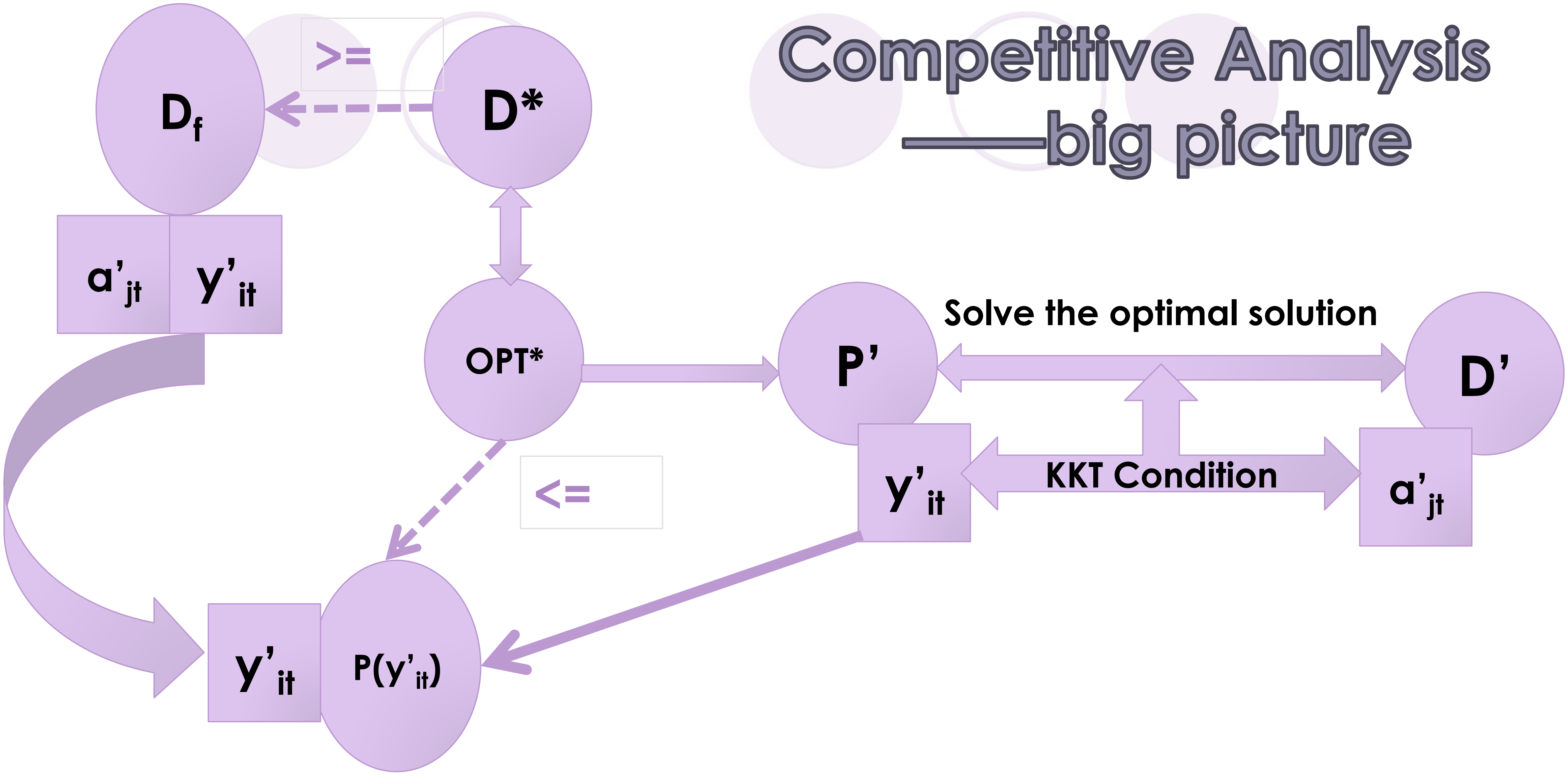
Competitive Analysis ——big picture



Competitive Analysis ——big picture



Competitive Analysis ——big picture



Competitive Analysis
—— Primal-dual Framework

- Since we solve P'^* and use the solution $\vec{y}'$ (primal variable vector) as that of P, we need to guarantee :

1. $\vec{y}'$ is feasible for P. ($P(\vec{y}') >= P^* = OPT$)
2. $P(\vec{y}') <= c\ P^* = c\ OPT$ (now we only have $P(\vec{y}') >= P^* = OPT$ due to 1.)

- How to achieve $P(\vec{y}') <= c\ OPT$?

$P(y') <= c\ D <= c\ D^* <= c\ OPT <= c\ P^*$

Competitive Analysis —— Primal-dual Framework

- Since we solve P'* and use the solution $\vec{y'}$ (primal variable vector) as that of P, we need to guarantee ：

1. $\vec{y'}$ is feasible for P. (P($\vec{y'}$)>=P*=OPT)
2. P($\vec{y'}$)<= c P*=c OPT (now we only have P($\vec{y'}$)>=P*=OPT due to 1.)

- How to achieve P($\vec{y'}$)<= c OPT ？

P(y')<=c D<=c D* <=c OPT<=c P*

D is exactly a lower bound of OPT we need !

Competitive Analysis
—— Primal-dual Framework

- Since we solve P’* and use the solution $\vec{y}'$ (primal variable vector) as that of P, we need to guarantee ：

1. $\vec{y}'$ is feasible for P. ($P(\vec{y}')>=P^*=OPT$)
2. $P(\vec{y}')<= c\ P^*=c\ OPT$ (now we only have $P(\vec{y}')>=P^*=OPT$ due to 1.)

- How to achieve $P(\vec{y}')<= c\ OPT$?

$P(y')<=c\ D<=c\ D^* <=c\ OPT<=c\ P^*$

How to obtain such a feasible dual solution ？？？

Competitive Analysis
—— Primal-dual Framework

- Since we solve P'* and use the solution $\vec{y'}$ (primal variable vector) as that of P, we need to guarantee ：

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- How to achieve P($\vec{y'}$)<= c OPT ？

P(y')<=c D<=c D* <=c OPT<=c P*

Design appropriate values for dual variables

How to obtain such a feasible dual solution ？ ？ ？

Competitive Analysis
—— Primal-dual Framework

- Primal $\xrightarrow{\text{KKT Condition}}$ Dual
  - Optimality
  - $\Longleftrightarrow$ primal and dual variables satisfy KKT Condition
  - Which one we list the KKT Condition for ?

   Ans: P'* and D'*.

  Let $\vec{y}'$ denote the solution of P'*.

KKT Optimality Condition

$$\min \quad \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \frac{1}{\eta}\sum_{i=1}^{n} w_i \left(\left(x_i + \frac{\epsilon}{n} \right) \ln \left(\frac{x_i + \epsilon/n}{y_{i,t-1} + \epsilon/n} \right) - x_i \right) \qquad \forall t$$

$1 \le j \le m_t \quad \sum_{i \in S_{j,t}} y_{i,t} \ge 1$ → Associated dual variable : a_{jt}

$1 \le i \le n \qquad z_{i,t}, y_{i,t} \ge 0$

In fact, we do not need to derive the specific dual program from P'. We only need to know the dual variables and list the kkt condition for P' and D'.

KKT Optimality Condition

2. KKT

We now assume additionally that f_i and h_i are differentiable (but general otherwise). By (1.2), x^* minimizes $L(x, \lambda^*, \nu^*)$ over x. Thus, its gradient must vanish at x^*:

$$\nabla f_0(x^*) + \sum \lambda_i^* \nabla f_i(x^*) + \sum \nu_i^* \nabla h_i(x^*) = 0.$$

Thus, we have:

$$
\begin{aligned}
f_i(x^*) &\leq 0 \\
h_i(x^*) &= 0 \\
a_{jt} = \lambda_i^* &\geq 0 \\
\lambda_i^* f_i(x^*) &= 0
\end{aligned}
\tag{2.1}
$$

$$\nabla f_0(x^*) + \sum \lambda_i^* \nabla f_i(x^*) + \sum \nu_i^* \nabla h_i(x^*) = 0.$$

This system is called the *Karush-Kuhn-Tucker (KKT)* conditions.

References

[1] S. Boyd, L. Vandenberghe, *Convex Optimization*.

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$$\nabla f_0(x^*) + \sum \lambda_i^* \nabla f_i(x^*) + \sum \nu_i^* \nabla h_i(x^*) = 0.$$

= Obj

Thus, we have:

$$f_i(x^*) \leq 0$$

$$h_i(x^*) = 0$$

(2.1) $a_{jt} = \lambda_i^* \geq 0$

$$\lambda_i^* f_i(x^*) = 0$$

$$\nabla f_0(x^*) + \sum \lambda_i^* \nabla f_i(x^*) + \sum \nu_i^* \nabla h_i(x^*) = 0.$$

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KKT Optimality Condition

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$$\nabla \underbrace{f_0(x^*)}_{=\ \text{Obj}} + \sum \lambda_i^* \nabla f_i(x^*) + \sum \nu_i^* \nabla h_i(x^*) = 0.$$

Thus, we have:

$$\begin{aligned} f_i(x^*) &\le 0 \\ \cancel{h_i(x^*)} &\cancel{= 0} \\ a_{jt} = \lambda_i^* &\ge 0 \\ \lambda_i^* f_i(x^*) &= 0 \end{aligned} \tag{2.1}$$

$$\nabla f_0(x^*) + \sum \lambda_i^* \nabla f_i(x^*) + \cancel{\sum \nu_i^* \nabla h_i(x^*)} = 0.$$

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$$\nabla f_0(x^*) + \sum \lambda_i^* \nabla f_i(x^*) + \sum \nu_i^* \nabla h_i(x^*) = 0.$$

$=$ Obj

Thus, we have:

$$1 \leq j \leq m_t \quad \sum_{i \in S_{j,t}} y_{i,t} \geq 1$$

(2.1)

$$\begin{aligned} f_i(x^*) &\leq 0 \\ \cancel{h_i(x^*)} &\cancel{= 0} \\ a_{jt} = \lambda_i^* &\geq 0 \\ \lambda_i^* f_i(x^*) &= 0 \end{aligned}$$

$$\nabla f_0(x^*) + \sum \lambda_i^* \nabla f_i(x^*) + \cancel{\sum \nu_i^* \nabla h_i(x^*)} = 0.$$

This system is called the *Karush-Kuhn-Tucker (KKT)* conditions.

REFERENCES

[1] S. Boyd, L. Vandenberghe, *Convex Optimization*.

KKT Optimality Condition

$\forall t$

$$\min \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \frac{1}{\eta} \sum_{i=1}^{n} w_i \left(\left(x_i + \frac{\epsilon}{n} \right) \ln \left(\frac{x_i + \epsilon/n}{y_{i,t-1} + \epsilon/n} \right) - x_i \right)$$

$1 \le j \le m_t \quad \sum_{i \in S_{j,t}} y_{i,t} \ge 1$ → Associated dual variable: a_{jt}

$1 \le i \le n \quad z_{i,t}, y_{i,t} \ge 0$

$$\nabla f_0(x^*) + \sum \lambda_i^* \nabla f_i(x^*) + \cancel{\sum \nu_i^* \nabla h_i(x^*)} = 0.$$

$$c_{i,t} + \frac{w_i}{\eta} \ln \left(\frac{y_{i,t} + \frac{\epsilon}{n}}{y_{i,t-1} + \frac{\epsilon}{n}} \right) - \sum_{j: i \in S_{j,t}} a_{j,t} = 0.$$

For all $1 \le j \le m_t$,

$$\sum_{i \in S_{j,t}} y_{i,t} - 1 \ge 0, \tag{2.9}$$

$$a_{j,t} \left(\sum_{i \in S_{j,t}} y_{i,t} - 1 \right) = 0, \tag{2.10}$$

For all $1 \le i \le n$,

$$c_{i,t} + \frac{w_i}{\eta} \ln \left(\frac{y_{i,t} + \frac{\epsilon}{n}}{y_{i,t-1} + \frac{\epsilon}{n}} \right) - \sum_{j: i \in S_{j,t}} a_{j,t} \ge 0, \tag{2.11}$$

$$y_{i,t} \left(c_{i,t} + \frac{w_i}{\eta} \ln \left(\frac{y_{i,t} + \frac{\epsilon}{n}}{y_{i,t-1} + \frac{\epsilon}{n}} \right) - \sum_{j: i \in S_{j,t}} a_{j,t} \right) = 0. \tag{2.12}$$

KKT Optimality Condition

$\forall t$

$$\min \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \frac{1}{\eta} \sum_{i=1}^{n} w_i \left(\left(x_i + \frac{\epsilon}{n} \right) \ln \left(\frac{x_i + \epsilon/n}{y_{i,t-1} + \epsilon/n} \right) - x_i \right)$$

$1 \leq j \leq m_t \quad \sum_{i \in S_{j,t}} y_{i,t} \geq 1$ → Associated dual variable: a_{jt}

$1 \leq i \leq n \qquad z_{i,t}, y_{i,t} \geq 0$

$\vec{y'}$ association $\vec{a'}$

For all $1 \leq j \leq m_t$,

(2.9) $$\sum_{i \in S_{j,t}} y_{i,t} - 1 \geq 0,$$

(2.10) $$a_{j,t} \left(\sum_{i \in S_{j,t}} y_{i,t} - 1 \right) = 0,$$

For all $1 \leq i \leq n$,

(2.11) $$c_{i,t} + \frac{w_i}{\eta} \ln \left(\frac{y_{i,t} + \frac{\epsilon}{n}}{y_{i,t-1} + \frac{\epsilon}{n}} \right) - \sum_{j: i \in S_{j,t}} a_{j,t} \geq 0,$$

(2.12) $$y_{i,t} \left(c_{i,t} + \frac{w_i}{\eta} \ln \left(\frac{y_{i,t} + \frac{\epsilon}{n}}{y_{i,t-1} + \frac{\epsilon}{n}} \right) - \sum_{j: i \in S_{j,t}} a_{j,t} \right) = 0.$$

KKT Optimality Condition

$\vec{y}' \xleftrightarrow{\text{association}} \vec{a}'$

How KKT benefits Competitive Analysis

(P) $\min \sum_{t=1}^{T} \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \sum_{t=1}^{T} \sum_{i=1}^{n} w_i \cdot z_{i,t}$

$\forall t \geq 1$ and $1 \leq j \leq m_t$ $\quad \sum_{i \in S_{j,t}} y_{i,t} \geq 1$

$\forall t \geq 1$ and $1 \leq i \leq n$ $\quad z_{i,t} \geq y_{i,t} - y_{i,t-1}$

$\forall t$ and $1 \leq i \leq n$ $\quad z_{i,t}, y_{i,t} \geq 0$

(D) $\max \sum_{t=1}^{T} \sum_{j=1}^{m_t} \underline{a_{j,t}}$

?

$\forall t \geq 1$ and i $\quad \underline{b_{i,t}} \leq w_i$

$\forall t \geq 1$ and $1 \leq i \leq n$ $\quad \underline{b_{i,t+1} - b_{i,t}} \leq c_{i,t} - \underline{\sum_{j | i \in S_{j,t}} a_{j,t}}$

$\forall t \geq 1$ and i, j $\quad \underline{a_{j,t}, b_{i,t}} \geq 0$

$\vec{y}'$ $\xleftrightarrow{\text{association}}$ $\vec{a}'$

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$\forall t \geq 1$ and $1 \leq j \leq m_t$	$\sum_{i \in S_{j,t}} y_{i,t} \geq 1$
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(D) $\max \sum_{t=1}^{T} \sum_{j=1}^{m_t} a_{j,t}$

$\forall t \geq 1$ and i	$b_{i,t} \leq w_i$
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$\forall t \geq 1$ and i, j	$a_{j,t}, b_{i,t} \geq 0$

=

$\vec{y'}$ association $\vec{a'}$

$b_{it} = func(y_{it})$

How KKT benefits Competitive Analysis

(P) $\min \sum_{t=1}^{T} \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \sum_{t=1}^{T} \sum_{i=1}^{n} w_i \cdot z_{i,t}$

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$\forall t \geq 1$ and i, j	$a_{j,t}, b_{i,t} \geq 0$

=

$\vec{y}'$ association $\vec{a}'$

$$b_{i,t+1} = \frac{w_i}{\eta} \ln\left(\frac{1+\epsilon/n}{y_{i,t}+\epsilon/n}\right)$$

How KKT benefits Competitive Analysis

(P)	$\min \sum_{t=1}^{T} \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \sum_{t=1}^{T} \sum_{i=1}^{n} w_i \cdot z_{i,t}$	(D)	$\max \sum_{t=1}^{T} \sum_{j=1}^{m_t} a_{j,t}$
$\forall t \geq 1$ and $1 \leq j \leq m_t$	$\sum_{i \in S_{j,t}} y_{i,t} \geq 1$	$\forall t \geq 1$ and i	$b_{i,t} \leq w_i$
$\forall t \geq 1$ and $1 \leq i \leq n$	$z_{i,t} \geq y_{i,t} - y_{i,t-1}$	$\forall t \geq 1$ and $1 \leq i \leq n$	$b_{i,t+1} - b_{i,t} \leq c_{i,t} - \sum_{j \mid i \in S_{j,t}} a_{j,t}$
$\forall t$ and $1 \leq i \leq n$	$z_{i,t}, y_{i,t} \geq 0$	$\forall t \geq 1$ and i, j	$a_{j,t}, b_{i,t} \geq 0$

=

$\overrightarrow{y'}$ association $\overrightarrow{a'}$

$$c_{i,t} + \frac{w_i}{\eta} \ln\left(\frac{y_{i,t} + \frac{\epsilon}{n}}{y_{i,t-1} + \frac{\epsilon}{n}}\right) - \sum_{j:i \in S_{j,t}} a_{j,t} \geq 0, \tag{2.11}$$

$$b_{i,t+1} = \frac{w_i}{\eta} \ln\left(\frac{1+\epsilon/n}{y_{i,t}+\epsilon/n}\right)$$

How KKT benefits Competitive Analysis

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$\forall t \geq 1$ and i, j $\quad a_{j,t}, b_{i,t} \geq 0$

feasible!

=

$\overrightarrow{y'}$ association $\overrightarrow{a'}$

$$(2.11) \qquad c_{i,t} + \frac{w_i}{\eta} \ln\left(\frac{y_{i,t} + \frac{\epsilon}{n}}{y_{i,t-1} + \frac{\epsilon}{n}}\right) - \sum_{j: i \in S_{j,t}} a_{j,t} \geq 0,$$

$$b_{i,t+1} = \frac{w_i}{\eta} \ln\left(\frac{1+\epsilon/n}{y_{i,t}+\epsilon/n}\right)$$

How KKT benefits Competitive Analysis

(P)	$\min \sum_{t=1}^{T} \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \sum_{t=1}^{T} \sum_{i=1}^{n} w_i \cdot z_{i,t}$	(D)	$\max \sum_{t=1}^{T} \sum_{j=1}^{m_t} a_{j,t}$
$\forall t \geq 1$ and $1 \leq j \leq m_t$	$\sum_{i \in S_{j,t}} y_{i,t} \geq 1$	$\forall t \geq 1$ and i	$b_{i,t} \leq w_i$
$\forall t \geq 1$ and $1 \leq i \leq n$	$z_{i,t} \geq y_{i,t} - y_{i,t-1}$	$\forall t \geq 1$ and $1 \leq i \leq n$	$b_{i,t+1} - b_{i,t} \leq c_{i,t} - \sum_{j \mid i \in S_{j,t}} a_{j,t}$
$\forall t$ and $1 \leq i \leq n$	$z_{i,t}, y_{i,t} \geq 0$	$\forall t \geq 1$ and i, j	$a_{j,t}, b_{i,t} \geq 0$

feasible!

=

$\vec{y'}$ association $\vec{a'}$

$$(2.11) \qquad c_{i,t} + \frac{w_i}{\eta} \ln\left(\frac{y_{i,t} + \frac{\epsilon}{n}}{y_{i,t-1} + \frac{\epsilon}{n}}\right) - \sum_{j: i \in S_{j,t}} a_{j,t} \geq 0,$$

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How KKT benefits Competitive Analysis

$$\text{(P)} \quad \min \sum_{t=1}^{T} \sum_{i=1}^{n} c_{i,t} \cdot y_{i,t} + \sum_{t=1}^{T} \sum_{i=1}^{n} w_i \cdot z_{i,t}$$

$\forall t \geq 1$ and $1 \leq j \leq m_t$	$\sum_{i \in S_{j,t}} y_{i,t} \geq 1$
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$\forall t \geq 1$ and i	$b_{i,t} \leq w_i$
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$\forall t \geq 1$ and i, j	$a_{j,t}, b_{i,t} \geq 0$

feasible!

=

$$\eta = \ln(1+n/\epsilon)$$

$\vec{y'}$ association $\vec{a'}$

$$b_{i,t+1} = \frac{w_i}{\eta} \ln\left(\frac{1+\epsilon/n}{y_{i,t}+\epsilon/n}\right)$$

Feasible is not enough, we need P(y')<=c D !

How KKT benefits Competitive Analysis

$$b_{i,t+1} = \frac{w_i}{\eta} \ln\left(\frac{1+\epsilon/n}{y_{i,t}+\epsilon/n}\right)$$

$a_{jt}=a'_{jt}$

$$M_t = \eta \sum_{y_{i,t}>y_{i,t-1}} \frac{w_i}{\eta}(y_{i,t}-y_{i,t-1})$$

a-b<=a ln(a/b)

$$(2.1) \quad M_t = \eta \sum_{y_{i,t}>y_{i,t-1}} \frac{w_i}{\eta}(y_{i,t}-y_{i,t-1}) \le \eta \sum_{y_{i,t}>y_{i,t-1}} \left(y_{i,t}+\frac{\epsilon}{n}\right)\cdot\left(\frac{w_i}{\eta}\ln\left(\frac{y_{i,t}+\frac{\epsilon}{n}}{y_{i,t-1}+\frac{\epsilon}{n}}\right)\right)$$

$$(2.2) \quad = \eta \sum_{y_{i,t}>y_{i,t-1}} \left(y_{i,t}+\frac{\epsilon}{n}\right)\cdot\left(\sum_{j|i\in S_{j,t}} a_{j,t} - c_{i,t}\right)$$

$$(2.3) \quad \le \eta \sum_{i=1}^{n}\left(y_{i,t}+\frac{\epsilon}{n}\right)\sum_{j|i\in S_{j,t}} a_{j,t} = \eta\sum_{j=1}^{m_t} a_{j,t}\left(\sum_{i\in S_{j,t}} y_{i,t} + \frac{\epsilon|S_{j,t}|}{n}\right)$$

$$(2.4) \quad \le \underline{\eta\left(1+\frac{\epsilon k}{n}\right)\sum_{j=1}^{m_t} a_{j,t}}.$$

For all $1 \le j \le m_t$,

$$(2.9) \quad \sum_{i\in S_{j,t}} y_{i,t} - 1 \ge 0,$$

$$(2.10) \quad a_{j,t}\left(\sum_{i\in S_{j,t}} y_{i,t} - 1\right) = 0,$$

For all $1 \le i \le n$,

$$(2.11) \quad c_{i,t} + \frac{w_i}{\eta}\ln\left(\frac{y_{i,t}+\frac{\epsilon}{n}}{y_{i,t-1}+\frac{\epsilon}{n}}\right) - \sum_{j:i\in S_{j,t}} a_{j,t} \ge 0,$$

$$(2.12) \quad y_{i,t}\left(c_{i,t} + \frac{w_i}{\eta}\ln\left(\frac{y_{i,t}+\frac{\epsilon}{n}}{y_{i,t-1}+\frac{\epsilon}{n}}\right) - \sum_{j:i\in S_{j,t}} a_{j,t}\right) = 0.$$

How KKT benefits Competitive Analysis

$$b_{i,t+1} = \frac{w_i}{\eta} \ln\left(\frac{1+\epsilon/n}{y_{i,t}+\epsilon/n}\right)$$

$$S = \sum_{t=1}^{T}\sum_{i=1}^{n} c_{i,t} y_{i,t}$$

$a_{jt}=a'_{jt}$

$$S = \sum_{t=1}^{T}\sum_{i=1}^{n} c_{i,t}y_{i,t} = \sum_{t=1}^{T}\sum_{j=1}^{m_t} a_{j,t} \sum_{i\in S_{j,t}} y_{i,t} - \frac{1}{\eta}\sum_{t=1}^{T}\sum_{i=1}^{n} w_i \cdot y_{i,t} \ln\left(\frac{y_{i,t}+\frac{\epsilon}{n}}{y_{i,t-1}+\frac{\epsilon}{n}}\right) \tag{2.5}$$

$$= \sum_{t=1}^{T}\sum_{j=1}^{m_t} a_{j,t} - \frac{1}{\eta}\sum_{i=1}^{n} w_i \left\{\sum_{t=1}^{T}\left(y_{i,t}+\frac{\epsilon}{n}\right)\ln\left(\frac{y_{i,t}+\frac{\epsilon}{n}}{y_{i,t-1}+\frac{\epsilon}{n}}\right) - \frac{\epsilon}{n}\sum_{t=1}^{T}\ln\left(\frac{y_{i,t}+\frac{\epsilon}{n}}{y_{i,t-1}+\frac{\epsilon}{n}}\right)\right\} \tag{2.6}$$

$$\leq \sum_{t=1}^{T}\sum_{j=1}^{m_t} a_{j,t} - \frac{1}{\eta}\sum_{i=1}^{n} w_i \left\{\left(\sum_{t=1}^{T}\left(y_{i,t}+\frac{\epsilon}{n}\right)\right)\ln\left(\frac{\sum_{t=1}^{T}(y_{i,t}+\frac{\epsilon}{n})}{\sum_{t=1}^{T}(y_{i,t-1}+\frac{\epsilon}{n})}\right) - \frac{\epsilon}{n}\ln\left(\frac{y_{i,T}+\frac{\epsilon}{n}}{y_{i,0}+\frac{\epsilon}{n}}\right)\right\} \tag{2.7}$$

$$\leq \sum_{t=1}^{T}\sum_{j=1}^{m_t} a_{j,t} = \text{value of } (D). \tag{2.8}$$

$$\boxed{\sum_i a_i \log(a_i/b_i) \geq \left(\sum_i a_i\right)\log\left(\frac{\sum_i a_i}{\sum_i b_i}\right).}$$

For all $1 \leq j \leq m_t$,

$$\sum_{i\in S_{j,t}} y_{i,t} - 1 \geq 0, \tag{2.9}$$

$$a_{j,t}\left(\sum_{i\in S_{j,t}} y_{i,t} - 1\right) = 0, \tag{2.10}$$

For all $1 \leq i \leq n$,

$$c_{i,t} + \frac{w_i}{\eta}\ln\left(\frac{y_{i,t}+\frac{\epsilon}{n}}{y_{i,t-1}+\frac{\epsilon}{n}}\right) - \sum_{j:i\in S_{j,t}} a_{j,t} \geq 0, \tag{2.11}$$

$$y_{i,t}\left(c_{i,t} + \frac{w_i}{\eta}\ln\left(\frac{y_{i,t}+\frac{\epsilon}{n}}{y_{i,t-1}+\frac{\epsilon}{n}}\right) - \sum_{j:i\in S_{j,t}} a_{j,t}\right) = 0. \tag{2.12}$$

Outline

- Model
- Regularization Algorithm (fractional version)
- Competitive Analysis (fractional version)

- Extensions:
  - More general constraints
  - Rounding a fractional solution to the online set cover with service cost problem

General Covering Constraints with Variable Upper Bounds

- Each node needs multiple circles to cover him $\sum_{i \in S_{j,t}} y_{i,t} \geq r_{j,t}$ (where $r_{j,t} \in \mathbb{N}$)
- Upper Bound for each variable

  $0 \leq y_{i,t} \leq 1$

$0 \leq y_{i,t} \leq 1$ is naturally guaranteed in P'

General Covering Constraints with Variable Upper Bounds

- ✓ $\sum_{i \in S_{j,t}} y_{i,t} \geq r_{j,t}$
- ✓ $0 \leq y_{i,t} \leq 1$

Knapsack Constraints:

If we need to guarantee $\sum_{i \in S} x_i \geq r$

Then for every $S' \subset S : |S'| < r,$

we should have:

$$\sum_{i \in S \setminus S'} x_i \geq r - |S'|.$$

By adding the KC constraints, the original box constraints become redundant; consider the first round t in which a variable $y_{i,t}$ is strictly larger than 1. The KC constraints imply that for every S containing i,

$$\sum_{\ell \in S \setminus \{i\}} y_{\ell,t} \geq r_{j,t} - 1,$$

Outline

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Rounding a fractional solution to integer solution (minor weakness: w_i=1)

Algorithm 2 Rounding Algorithm

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parameter: $\alpha \geq 0$
for each $S \in \mathcal{S}$, choose i.i.d random variable $Z_S \sim \exp(1)$.
for each $e \in \mathcal{E}$, choose i.i.d random variable $Z_e \sim \exp(1)$.
at any time t, let $y_{S,t}$ denote the current fractional value of S.
for $t = 1, 2, \ldots, T$ **do**
    let $A_t = \left\{ S \in \mathcal{S} \middle| \frac{Z_S}{y_{S,t}} < \alpha \right\}$.
    let $B_t = \cup_{e \in \mathcal{E}} \left\{ S \middle| \, S = \arg\min_{S' | e \in S'} \left\{ \frac{Z_{S'}}{y_{S',t}} \right\}, \text{ and } \frac{Z_S}{y_S} < \frac{Z_e}{\max\{0, 1 - \sum_{S | e \in S} y_{S,t}\}} \right\}$.
    output $A_t \cup B_t$.
end for

Thanks!
Q&A

The MTS Work Fuction Algorithm

- Let (S,d) be any MTS and let s_0 be an initial state
- Fix any task sequence $\sigma = r_1 r_2, \ldots, r_n$
- Let $\sigma_i = r_1 r_2, \ldots, r_i$ be the prefix of σ
- For each state $s \in S$, define $w_i(s)$ to be the minimum (offline) cost to process σ_i starting from s_0 and ending in state s

The MTS Work Function Algorithm

- For each state $s \in S$, define $w_i(s)$ to be the minimum (offline) cost to process σ_i starting from s_0 and ending in state s
- **Optimal offline cost** $\mathrm{OPT}(\sigma) = \min_{x \in S} w_n(x)$
- To compute $w_n(s)$, we have:

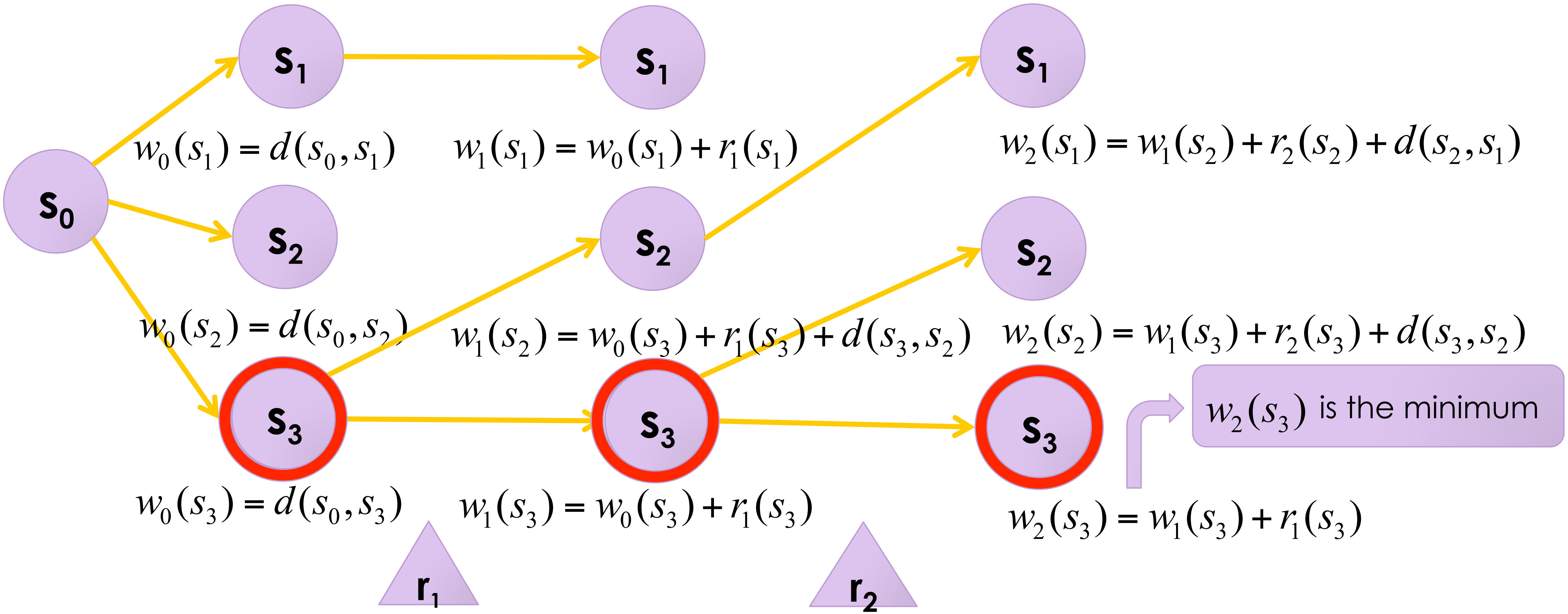
$$w_{i+1}(s) = \min_{x \in S} \{w_i(x) + r_{i+1}(x) + d(x, s)\},$$

$$w_0(s) = d(s_0, s)$$

Dynamic programming

The MTS Work Function Algorithm

$$w_{i+1}(s) = \min_{x\in S}\{w_i(x) + r_{i+1}(x) + d(x,s)\}, w_0(s) = d(s_0,s)$$



The MTS Work Function Algorithm

- Online:

  Algorithm WFA: Suppose that the algorithm is in state s_i after processing the i tasks in σ_i. Then to process r_i ,the algorithm moves to a state

$$\begin{cases} s_{i+1} = \arg\min_{x \in S} \{w_{i+1}(x) + d(s_i, x)\}, \\ \qquad w_{i+1}(s_{i+1}) = w_i(s_{i+1}) + r_{i+1}(s_{i+1}) \end{cases}$$

WFA Can Always Choose An Appropriate State s_{i+1}

Offline: $w_{i+1}(s) = \min_{x \in S}\{w_i(x) + r_{i+1}(x) + d(x,s)\}, w_0(s) = d(s_0,s)$

Algorithm WFA:
$$\begin{cases} s_{i+1} = \arg\min_{x \in S}\{w_{i+1}(x) + d(s_i, x)\}, & (1) \\ w_{i+1}(s_{i+1}) = w_i(s_{i+1}) + r_{i+1}(s_{i+1}) & (2) \end{cases}$$

Proof: **Let A be the set of the states satisfying (1)and (2).**
We firstly define a set A' satisfying (1). Clearly, A' isn't empty.
Then we prove that there is an element of A' that satisfies (2).