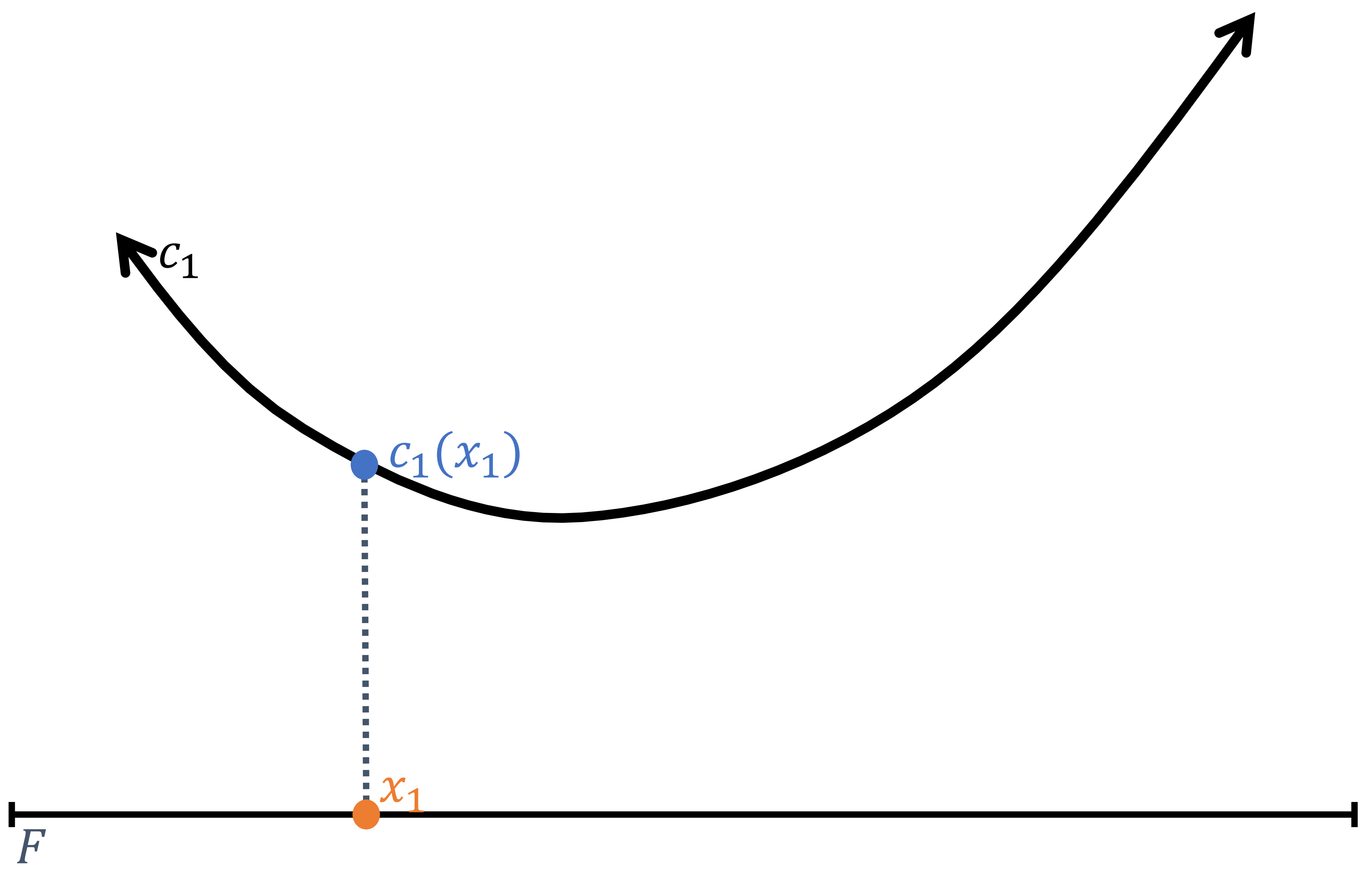
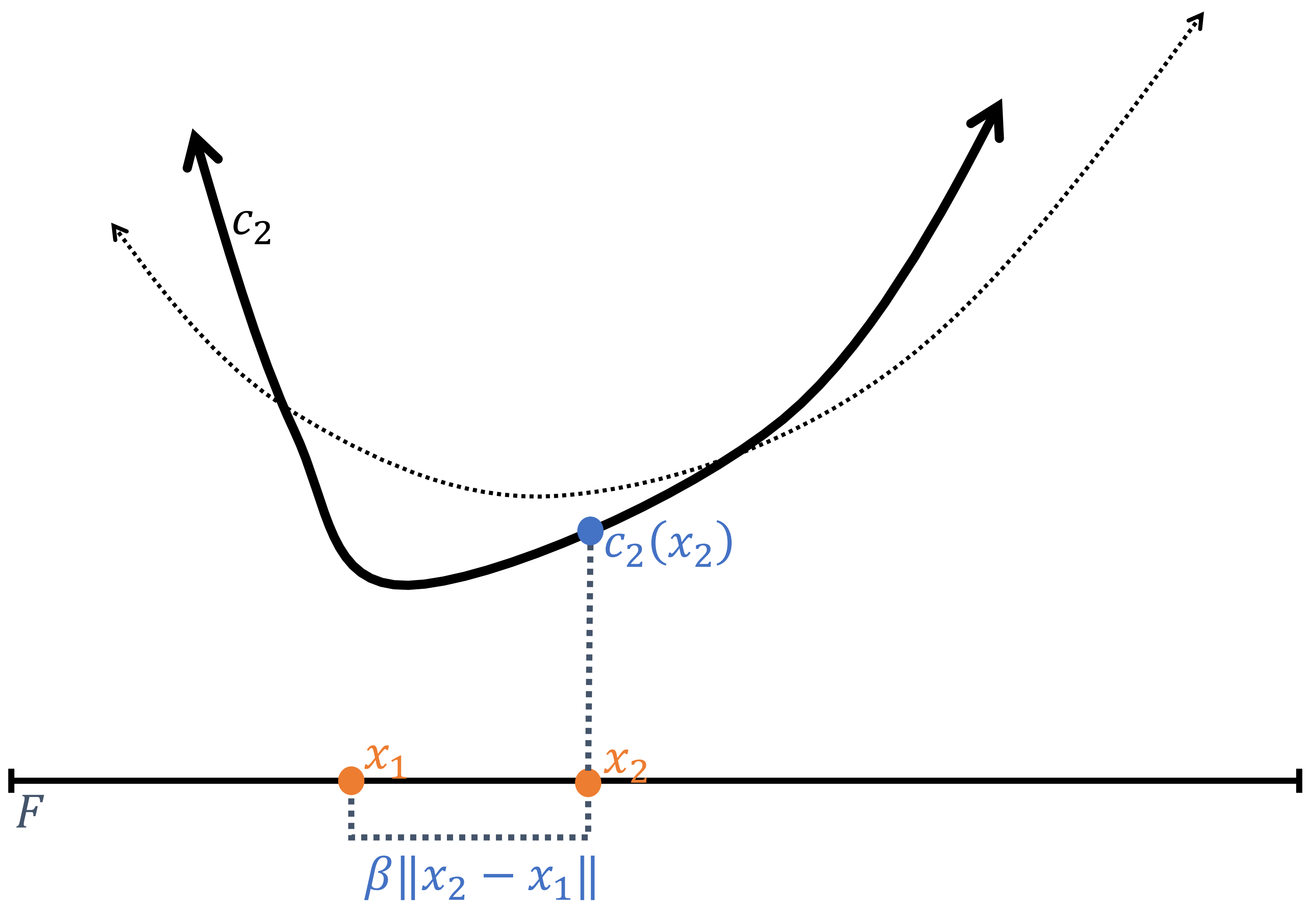
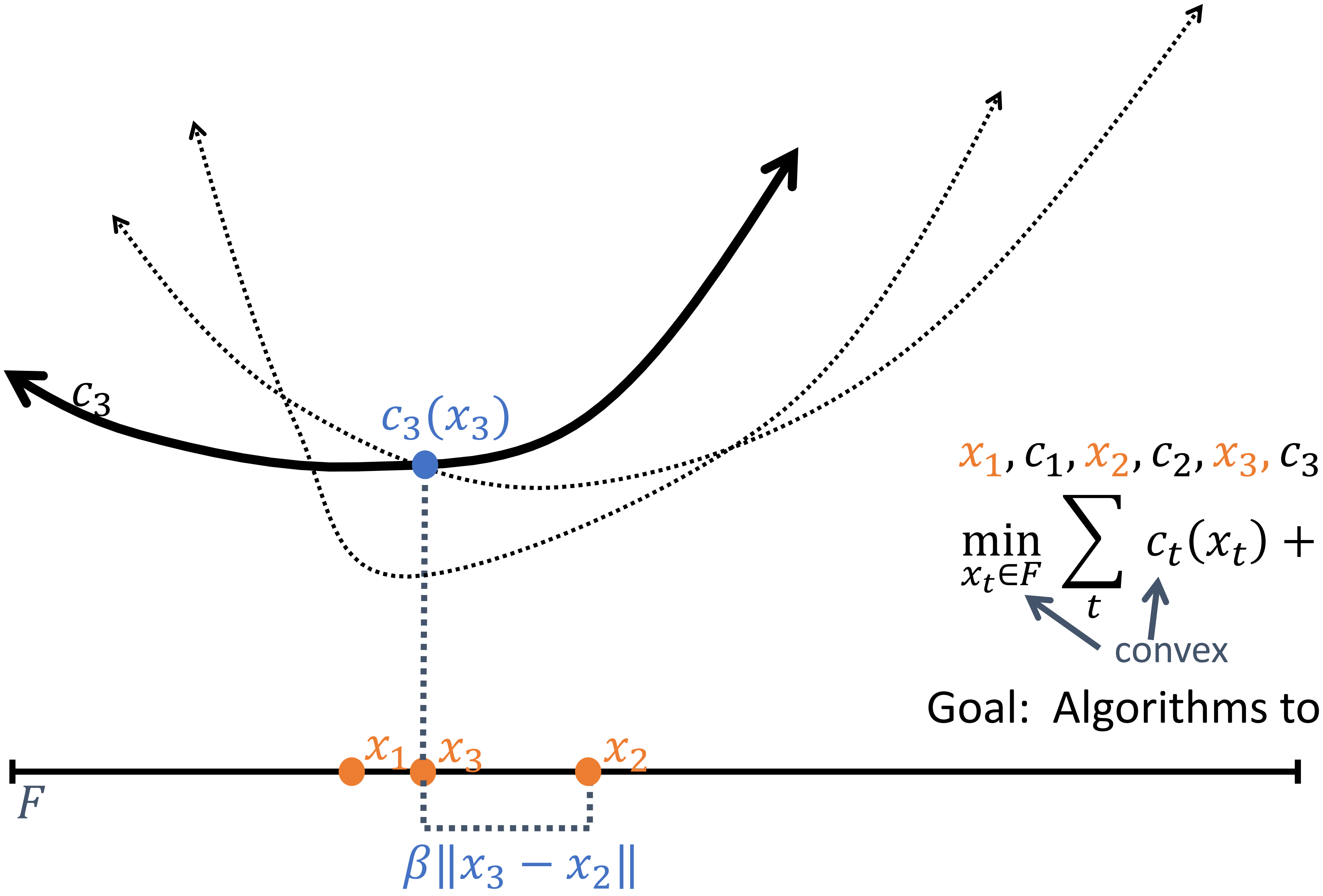
Online Convex Optimization Using Predictions

SIGMETRICS 2015


c_1
$c_1(x_1)$
x_1
F



c_2
$c_2(x_2)$
x_1
x_2
F
$\beta \|x_2 - x_1\|$



$x_1, c_1, x_2, c_2, x_3, c_3 \ldots \leftarrow$ online

$$\min_{x_t \in F} \sum_t c_t(x_t) + \underbrace{\beta \|x_t - x_{t-1}\|}_{\text{switching cost}}$$

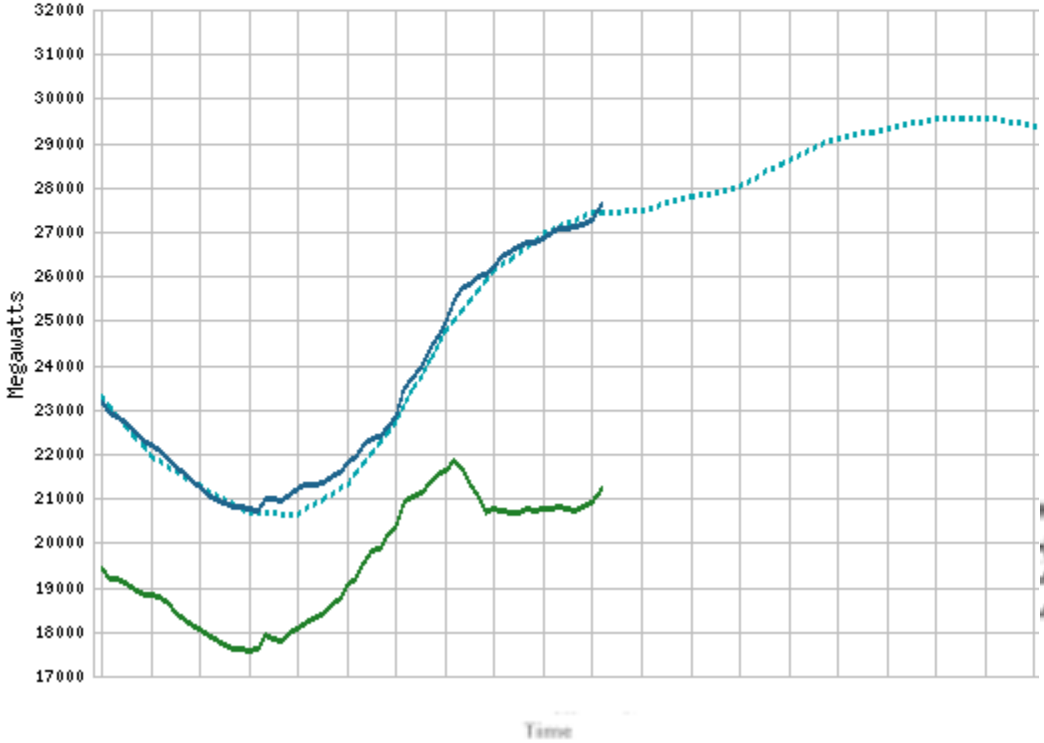
convex

Goal: Algorithms to minimize cost

Lots of applications …

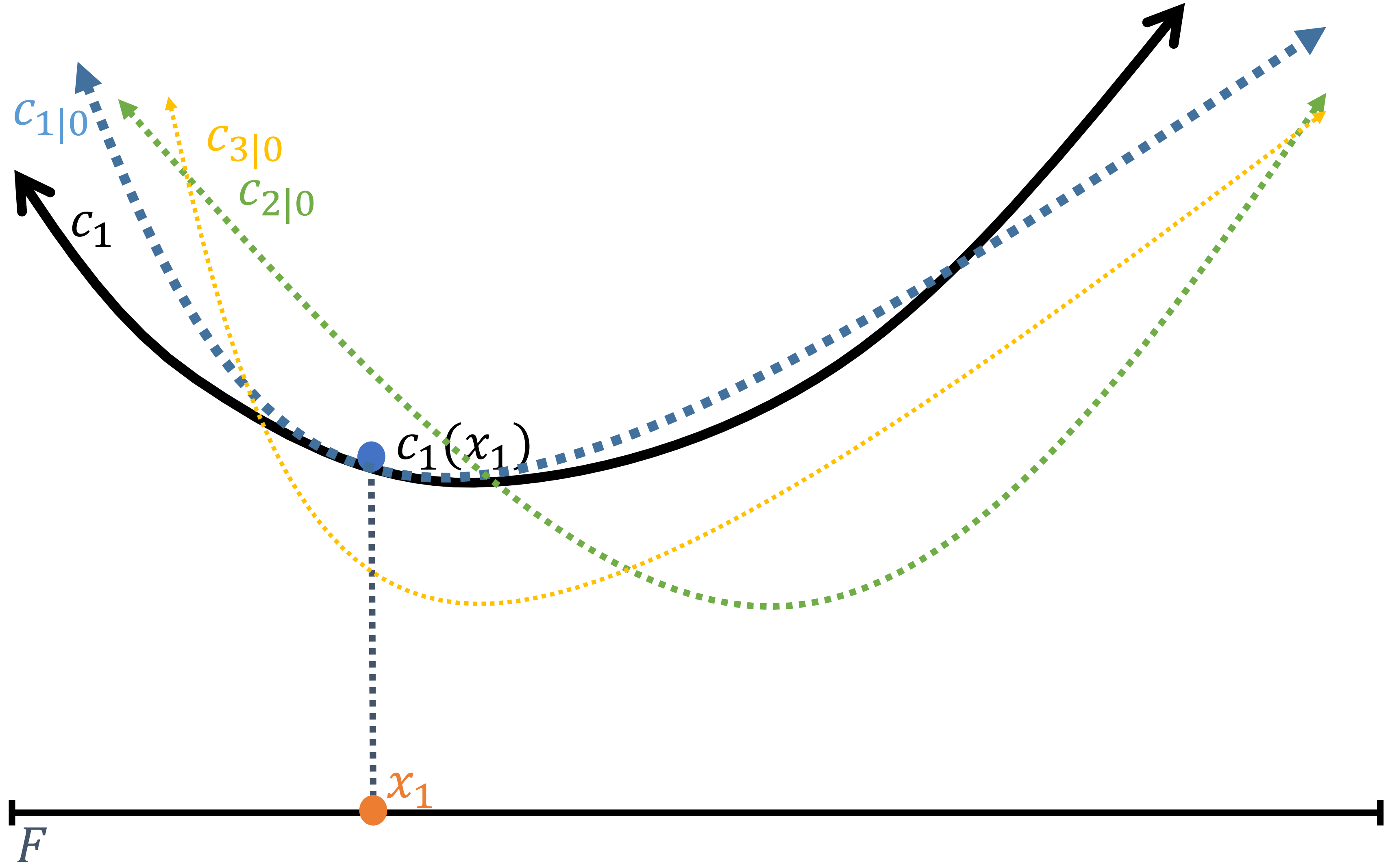
Dynamic capacity management in data centers [Tu et al. 2013]
Power system generation/load scheduling[Lu et al. 2013]
Portfolio management [Cover 1991][Boyd et al. 2012]
Video streaming [Sen et al. 2000][Liu et al. 2008]
Network routing [Bansal et al. 2003][Kodialam et al. 2003]
Geographical load balancing [Hindman et al. 2011] [Lin et al. 2012]
…

In most applications, predictions are crucial

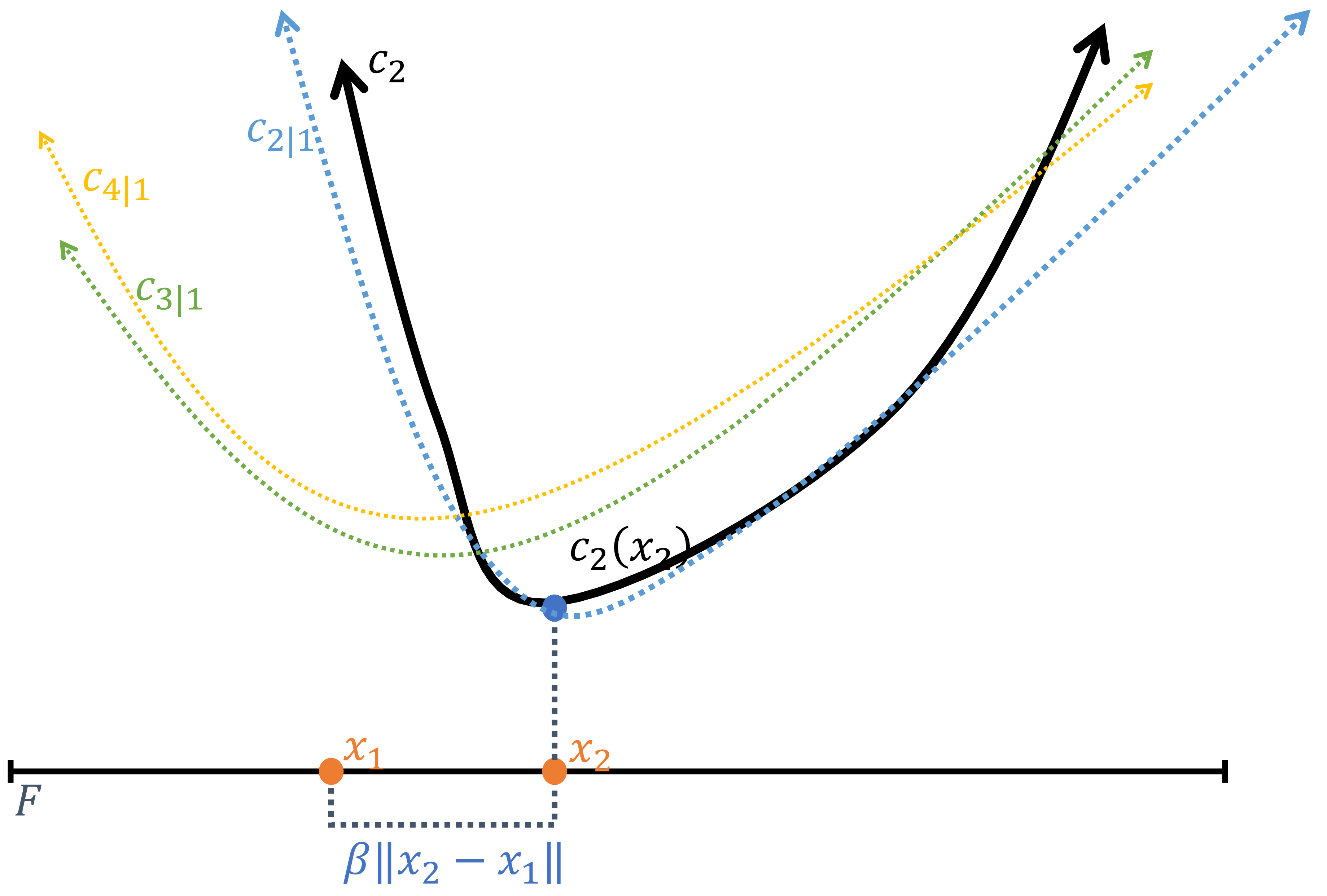


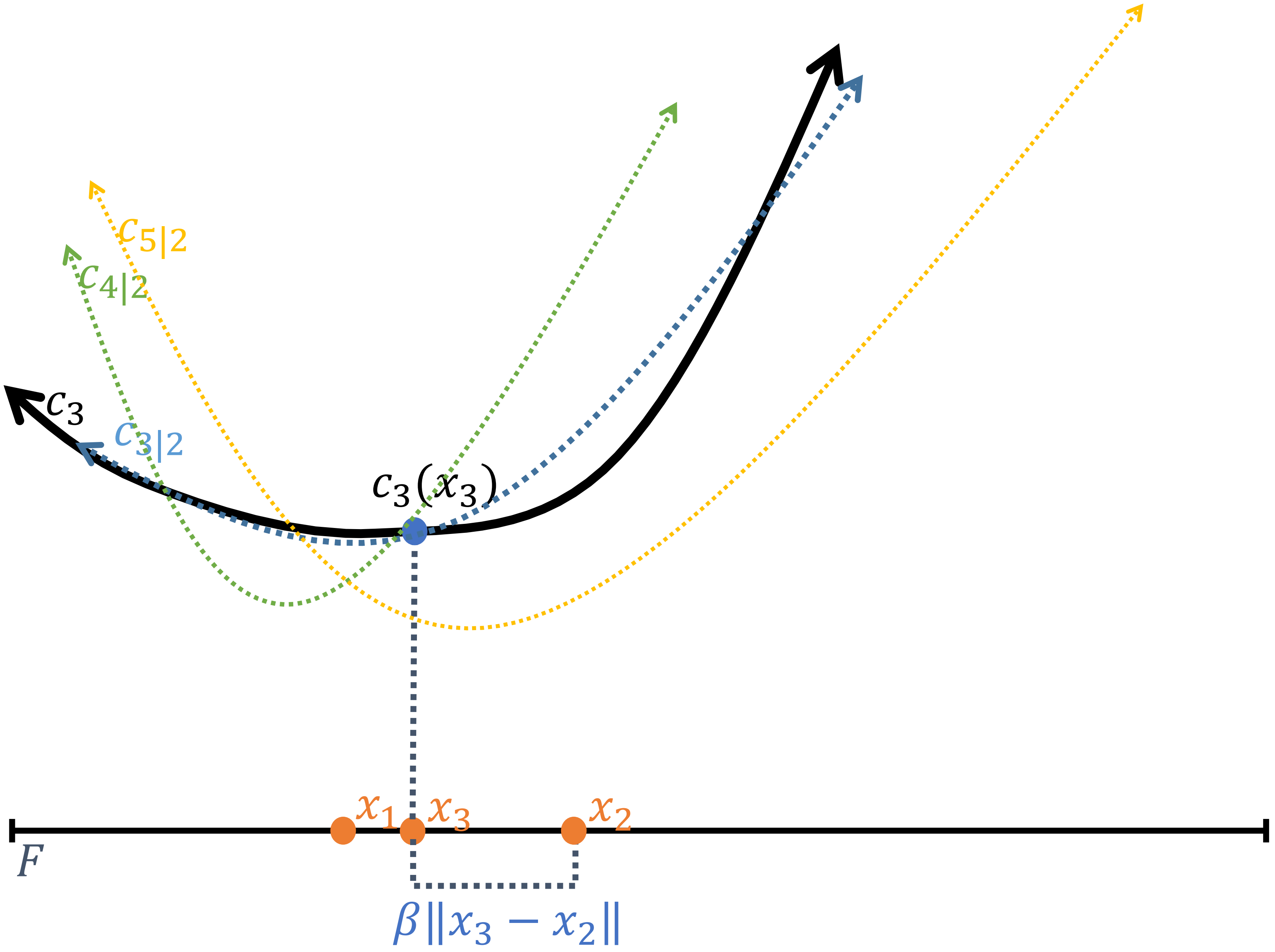
But we do not have a good understanding about how (imperfect) predictions impact online algorithm design

This talk: Online Convex Optimization Using Predictions



$c_{1|0}$
$c_{3|0}$
$c_{2|0}$
c_1
$c_1(x_1)$
x_1
F


c_2
$c_{2|1}$
$c_{4|1}$
$c_{3|1}$
$c_2(x_2)$
x_1
x_2
F
$\beta\|x_2 - x_1\|$



$c_{5|2}$
$c_{4|2}$
c_3
$c_{3|2}$
$c_3(x_3)$
x_1
x_3
x_2
F
$\beta\|x_3 - x_2\|$

Online convex optimization using predictions

$x_1, y_1, x_2, y_2, x_3, y_3 \ldots$ ← online

$$\min_{x_t \in F} \sum_t \boxed{c(x_t, y_t)} + \beta \underbrace{\|x_t - x_{t-1}\|}_{\text{switching cost}}$$

convex

e.g. online tracking cost

$$c(x_t, y_t) = \frac{1}{2}\left\|y_t - Kx_t\right\|^2$$

Given prediction of y_t at time τ, $y_{t|\tau}$

Time	Information Available				Decision
1	$y_{1\|0}$	$y_{2\|0}$	$y_{3\|0}$	…	x_1

Outline

1. **Background : regret and competitive ratio**
   OCO without prediction
   OCO with worst case prediction

2. Our prediction noise model

3. Algorithm design

4. OCO with stochastic prediction noise

$$\min_{x \in \mathbb{R}^n_+} \quad C(x) := f(x) \quad \text{subject to} \quad Ax \geq \mathbf{1}.$$

Online Convex Optimization (OCO)

input: A convex set S
for $t = 1, 2, \ldots$
  predict a vector $\mathbf{w}_t \in S$
  receive a convex loss function $f_t : S \to \mathbb{R}$
  suffer loss $f_t(\mathbf{w}_t)$

Two communities, two metrics

Online Learning

Regret(Alg) = $\sup_y[\mathrm{Cost}(\mathrm{Alg}) - \mathrm{Cost}(\mathrm{STA})]$

Goal: sublinear regret

Online Algorithm

Competitive ratio(Alg) = $\sup_y\left[\frac{\mathrm{Cost}(\mathrm{Alg})}{\mathrm{Cost}(\mathrm{OPT})}\right]$

Goal: constant competitive ratio

Real applications want both

Guarantees without prediction

- ➢Sublinear regret?

  Yes, [Kivinen & Vempala 2002] [Bansal et al. 2003] [Zinkevich 2003] [Hazan et al. 2007] [Lin et al. 2012] …

- ➢Constant CR?

  Yes, but only for scalar case

  [Blum et al. 1992] [Borodin et al. 1992][Blum & Burch 2000] [Lin et al. 2011][Lin et al. 2012] …

- ➢Sublinear regret *and* constant CR?

  Not even in scalar case! [Andrew et al. 2013]

Guarantees with prediction

1st cut, perfect lookahead:

$$y_{t|\tau} = y_t \text{ for any time } t \leq \tau + w$$

- Sublinear regret?
  - Yes, [Kivinen & Vempala 2002] [Bansal et al. 2003] [Zinkevich 2003] [Hazan et al. 2007] [Lin et al. 2012] …
- Constant CR?
  - Yes <u>in general</u> [Lin et al. 2013]

- Sublinear regret *and* constant CR?
  - Not without a lot of prediction [Chen et al. 2015]

Outline

1. Background : regret and competitive ratio
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2. **Prediction noise model**

3. Algorithm design

4. OCO with stochastic prediction noise

What do we want in a prediction noise model?

- Predictions are “refined” as time goes forward
- Predictions are more noisy as you look further ahead
- Prediction errors can be correlated
- Should be general enough to incorporate detailed models

A more realistic prediction noise model

$$y_t = y_{t|\tau} + \underbrace{\sum_{s=\tau+1}^{t} f(t-s)e(s)}$$

Realization that algorithm is trying to track

prediction error

Prediction for time t given to algorithm at time τ

A more realistic prediction noise model

Per-step noise

$$y_t = y_{t|\tau} + \sum_{s=\tau+1}^{t} f(t-s)\boxed{e(s)}$$

How much uncertainty is there one step ahead?

$$y_t - y_{t|t-1} = f(0)e(t)$$

where $e(t)$ are white, mean zero (unbiased)
and $f(0)$= I, $\mathbb{E}e(t)e(t)^T = R_e$

A more realistic prediction noise model

$$y_t = y_{t|\tau} + \sum_{s=\tau+1}^{t} \overset{\text{Weighting factor}}{\boxed{f(t-s)}}\, e(s)$$

How important is the noise at time $t - s$ for the prediction of t?

A more realistic prediction noise model

$$y_t = y_{t|\tau} + \underbrace{\sum_{s=\tau+1}^{t} f(t-s)e(s)}_{\text{prediction error}}$$

- Predictions are “refined” as time goes forward
- Predictions are more noisy as you look further ahead

$$\mathbb{E}\left|\left|y_t - y_{t|\tau}\right|\right|^2 = \sigma^2 \sum_{s=0}^{t-\tau-1} \left|\left|f(s)\right|\right|^2$$

- Prediction errors can be correlated
- **Form of errors matches many classic models**

A more realistic prediction noise model

$$y_t = y_{t|\tau} + \underbrace{\sum_{s=\tau+1}^{t} f(t-s)e(s)}_{\text{prediction error}}$$

This form of prediction error matches what occurs in

- Prediction of a wide-sense stationary process using a Weiner filter
- Prediction of a linear dynamical system using a Kalman filter

A more realistic prediction noise model

$$y_t = y_{t|\tau} + \sum_{s=\tau+1}^{t} f(t-s)e(s)$$

Key observation: No assumption about y_t or how predictions are made

Allows adversarial analysis using stochastic prediction noise

$$\mathbf{Regret}(\mathbf{Alg}) = \sup_y \mathbb{E}_e \operatorname{cost}(\mathrm{Alg}) - \operatorname{cost}(\mathrm{STA})$$

$$\mathbf{Competitive\ Ratio}(\mathbf{Alg}) = \sup_{\mathrm{y}} \mathbb{E}_e \frac{\operatorname{cost}(\mathrm{Alg})}{\operatorname{cost}(\mathrm{Opt})}$$

Outline

1. Background : regret and competitive ratio
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4. OCO with stochastic prediction noise

A natural suggestion:
Model Predictive Control (MPC)

$$\boxed{y_{t+1|t}, y_{t+2|t}, \dots, y_{t+w|t}}, y_{t+w+1|t}, y_{t+w+2|t}, \cdots$$

$$x_{t+1}, x_{t+2}, \dots, x_{t+w} = \operatorname{argmin}\left\{ \sum_{s=t+1}^{t+w} \frac{1}{2} \left|\left| y_{s|t} - K x_t \right|\right|^2 + \beta \left|\left| x_t - x_{t-1} \right|\right|_1 \right\}$$

A natural suggestion: Model Predictive Control (MPC)

$$y_{t+1|t}, y_{t+2|t}, \dots, y_{t+w|t}, y_{t+w+1|t}, y_{t+w+2|t}, \dots$$

$$y_{t+2|t+1}, y_{t+3|t+1}, \dots, y_{t+w+1|t+1}, y_{t+w+2|t+1}, y_{t+w+3|t+1}, \dots$$

$$x_{t+2}, x_{t+3}, \dots x_{t+w+1}$$

A natural suggestion:
Model Predictive Control (MPC)

$$y_{t+1|t}, y_{t+2|t}, \dots, y_{t+w|t}, y_{t+w+1|t}, y_{t+w+2|t}, \cdots$$

$$y_{t+2|t+1}, y_{t+3|t+1}, \dots, y_{t+w+1|t+1}, y_{t+w+2|t+1}, y_{t+w+3|t+1}, \cdots$$

$$y_{t+3|t+2}, y_{t+4|t+2}, \dots, y_{t+w+2|t+2}, y_{t+w+3|t+2}, y_{t+w+4|t+2}, \cdots$$

$$x_{t+3}, x_{t+4}, \dots x_{t+w+2}$$

But MPC doesn't work well in this setting …

A more stable alternative: Averaging Fixed Horizon Control (AFHC)

Fixed Horizon Control (FHC)

$$\boxed{y_{t+1|t}, y_{t+2|t}, \dots, y_{t+w|t},}$$

$$x_{t+1}, x_{t+2}, \dots, x_{t+w} = \operatorname{argmin}\left\{\sum_{s=t+1}^{t+w} \frac{1}{2}\left|\left|y_{s|t} - Kx_t\right|\right|^2 + \beta\left|\left|x_t - x_{t-1}\right|\right|_1\right\}$$

A more stable alternative: Averaging Fixed Horizon Control (AFHC)

Fixed Horizon Control (FHC)

$$y_{t+1|t}, y_{t+2|t}, \dots, y_{t+w|t}, y_{t+w+1|t+w}, y_{t+w+2|t+w}, \dots$$

$$x_{t+1}, x_{t+2}, \dots, x_{t+w} \qquad x_{t+w+1}, x_{t+w+2}, \dots, x_{t+2w}$$

A more stable alternative: Averaging Fixed Horizon Control (AFHC)

Average choices of FHC algorithms

$x_{AFHC} = \frac{1}{w} \sum_{k=1}^{w} x_{FHC}^{(k)}$

w FHC algorithms

$y_{t+1|t}, y_{t+2|t}, \cdots, y_{t+w|t}, y_{t+w+1|t+w}, y_{t+w+2|t+w}, \cdots$

$y_{t+2|t+1}, y_{t+3|t+1}, \cdots, y_{t+w+1|t+1}, y_{t+w+2|t+w+1}, y_{t+w+3|t+w+1}, \cdots$

$y_{t+3|t+2}, y_{t+4|t+2}, \cdots, y_{t+w+2|t+2}, y_{t+w+3|t+w+2}, y_{t+w+4|t+w+2}, \cdots$

$y_{t+4|t+3}, y_{t+5|t+3}, \cdots, y_{t+w+3|t+3}, y_{t+w+4|t+w+3}, y_{t+w+5|t+w+3}, \cdots$

$\vdots$

Outline

1. Background : regret and competitive ratio
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   OCO with worst case prediction

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3. Algorithm design

4. **OCO with stochastic prediction noise**

Theorem: AFHC(w) with $w = O(1)$ has sublinear regret and is constant competitive (in expectation) when $cost(OPT) = \Omega(T)$, and $cost(STA) \geq \alpha_1 T - o(T)$.

Theorem: AFHC(w) with $w = O(1)$ has sublinear regret and is constant competitive (in expectation) when $\mathbf{cost}(\boldsymbol{OPT}) = \boldsymbol{\Omega}(\boldsymbol{T})$, **and** $\mathbf{cost}(\boldsymbol{STA}) \geq \boldsymbol{\alpha_1 T} - \boldsymbol{o}(\boldsymbol{T})$.

How tight is this condition?

Theorem: Any online algorithm that chooses action independent of $e(t)$ has cost at least $\left\|R_e^{1/2}\right\|^2 T + o(T)$

No online algorithm can do well if cost$(OPT) \in o(T)$ or cost$(STA) \leq \left(\left\|R_e^{1/2}\right\|^2 - \gamma\right) T$ for some $\gamma > 0$.

Theorem: **AFHC(w)** with $w = O(1)$ has sublinear regret and is constant competitive (in expectation) when $cost(OPT) = \Omega(T), and\ cost(STA) \geq \alpha_1 T - o(T)$.

How to choose w?

Lemma: $\sup_y \mathbb{E}[cost(AFHC) - cost(OPT)] \leq \frac{T}{w}[\boxed{F(w)} + \boxed{g(\beta, K, w)}]$

Cumulative prediction error over w timesteps

loss due to switching

We can compute the optimal lookahead w

Theorem: AFHC(w) with $w = O(1)$ has sublinear regret and is constant competitive (in expectation) when $\text{cost}(OPT) = \Omega(T)$, and $\text{cost}(\text{STA}) \geq \alpha_1 T - o(T)$.

How likely is large deviation from expected performance for AFHC?

Theorem: When $e(t)$ is independent, sub-Gaussian for all t, for sufficiently large u, $\exists \text{a}, \text{b}, \text{c} > 0$ such that

$$\mathbb{P}(\text{cost}(\text{AFHC}) - \text{cost}(\text{Opt}) > t + \mu) \leq c \cdot \exp\left(-\frac{t^2}{a + bt}\right)$$

Intuition: the competitive difference of AFHC is a "smooth" function of $e(t)$

Contributions: a general and tractable model for prediction

Key message: prediction allows

1. Overcoming "impossibility" results for OCO with minimal structural assumption

    AFHC can achieve sublinear regret and constant CR

2. Balance between average case and worst case analysis

    Concentration of AFHC around its mean performance

Take-aways

- Two different types of online optimization problems
- Competitive ratio and Regret
- Analysis in a specific problem

Thanks!

Online Convex Optimization Using Predictions

Niangjun Chen

Joint work with Anish Agarwal, Lachlan Andrew, Sid Barman, and Adam Wierman

Backup Slides

Predicting stationary process with Wiener Filter

- $\{y_t\}_{t=0}^{T}$ wide sense stationary, with $\mathbb{E}y_t = \hat{y}_t$

and $\mathbb{E}(y_i - \hat{y}_i)(y_j - \hat{y}_j)^T = R_y(i-j)$

- Optimal prediction

$$y_{t|\tau} = \hat{y}_t + \sum_{s=1}^{\tau} \langle y_t, e(s) \rangle \, ||e(s)||^{-2} \boxed{e(s)}$$

Innovation process, white, mean 0 Variance R_e

$$y_t = y_{t|\tau} + \sum_{s=\tau+1}^{t} \underbrace{\boxed{R_y(t-s) R_e^{-1}}}_{f(t-s)} e(s)$$

Predicting Linear Dynamical System Using Kalman Filter

- $x_{t+1} = Ax_t + Bu_t, \;\; y = Cx_t + v_t$, stationary

$$\left\langle \begin{bmatrix} u_i \\ v_i \\ x_0 \end{bmatrix}, \begin{bmatrix} u_j \\ v_j \\ x_0 \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} Q\delta_{ij} & S\delta_{ij} & 0 & 0 \\ S^*\delta_{ij} & R\delta_{ij} & 0 & 0 \\ 0 & 0 & \Pi_0 & 0 \end{bmatrix}$$

- $y_{t|\tau} = \sum_{s=1}^{\tau} \underbrace{\boxed{CA^{t-s-1}(APC^* + BS)R_e^{-1}}}_{f(t-s)} \; \boxed{e(s)}$

Innovation process, white, mean 0
Variance R_e

Proof Sketch

1. Within a lookahead window

$$\underbrace{y_{t+1|t}, y_{t+2|t}, \ldots, y_{t+w|t}}_{\downarrow} \qquad \underbrace{y_{t+1}, y_{t+2}, \ldots, y_{t+w}}_{\downarrow}$$

$$x_{t+1}, x_{t+2}, \ldots, x_{t+w} \qquad x^*_{t+1}, x^*_{t+2}, \ldots, x^*_{t+w}$$

$$\mathrm{p(x,y)} \qquad \min. \sum_t \frac{1}{2}\left|\left|y_t - Kx_t\right|\right|^2 + \beta\left|\left|x_t - x_{t-1}\right|\right|_1$$

$$x = argmin_x\, p\left(x, y_{\bullet|t}\right) \qquad x^* = argmin_x p(x, y)$$

By perturbation analysis using Fenchel-Rockafellar duality

$$p(x, y) - p(x^*, y) \le \sum_t \frac{1}{2}\left|\left|KK^\dagger \underbrace{\left(y_{\bullet|t} - y_t\right)}_{\text{Prediction error}}\right|\right|^2$$

Proof Sketch

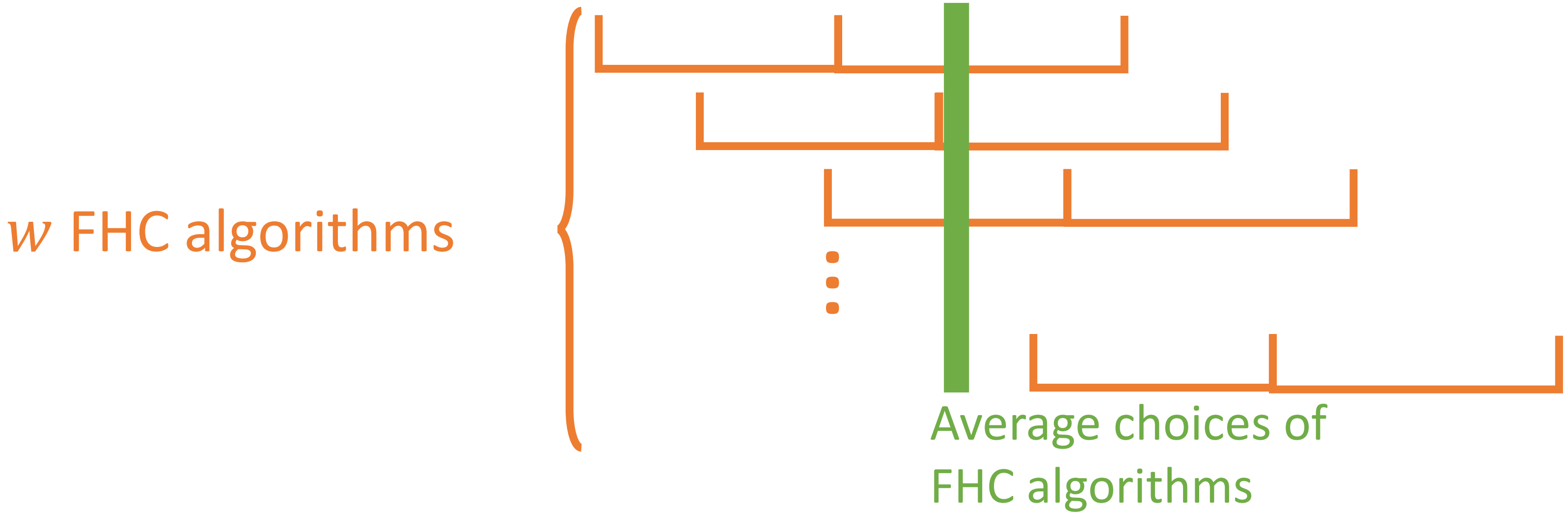
2. Between lookahead windows

$$y_{t+1|t}, y_{t+2|t}, \dots, y_{t+w|t}, y_{t+w+1|t+w}, y_{t+w+2|t+w}, \dots$$

$$x_{t+1}, x_{t+2}, \dots, \underbrace{x_{t+w} \quad x_{t+w+1}}_{\text{cost between lookahead window}}, x_{t+w+2}, \dots, x_{t+2w}$$

$$cost(FHC) - cost(OPT) \le \sum_{\tau=\Omega_k} \left(\overbrace{\beta \left|\left|x^*_{\tau-1} - x_{\tau-1}\right|\right|_1} + \sum_{t=\tau}^{\tau+w} \frac{1}{2} \left|\left|KK^{\dagger}(y_t - y_{t|\tau-1})\right|\right|^2\right)$$

Proof Sketch



3. By Jensen's inequality and taking expectation

$\sup_{y} \ \mathbb{E}_e \mathrm{cost}(AFHC) - \mathrm{cost}(OPT) \leq VT$, where

$$V = \frac{\underbrace{\beta \left|\left|K^{\dagger}\right|\right|_1 ||f_w|| + 3\beta^2 ||(K^T K)^{-1}\mathbf{1}||}_{\text{Switching cost}} + \underbrace{F(w)/2}_{\text{Prediction error}}}{w+1}$$

Lookahead window w $\downarrow$ $\uparrow$ $\downarrow$

Proof Sketch

$\mathbb{E}\mathrm{cost}(AFHC) - \mathrm{cost}(OPT) \leq VT$ implies

Constant competitive ratio if $cost(OPT) \in \Omega(T)$

4. Similarly for regret

> <u>Theorem:</u> AFHC(w) with $w = O(1)$ has sublinear regret and is constant competitive (in expectation) when $\mathrm{cost}(Opt) = \Omega(T)$, and $\mathrm{cost}(\mathrm{STA}) \geq \alpha_1 T - o(T)$.

Theorem: AFHC(w) with $w = O(1)$ has sublinear regret and is constant competitive (in expectation) when $\text{cost}(Opt) = \Omega(T)$, and $\text{cost}(\text{STA}) \geq \alpha_1 T - o(T)$.

Averaging Fixed Horizon Control

Fixed Horizon Control

$$\boxed{y_{t+1|t}, y_{t+2|t}, \dots, y_{t+w|t}}$$

$$x_{t+1}, x_{t+2}, \dots, x_{t+w} = \operatorname{argmin}\left\{\sum_{s=t+1}^{t+w} \frac{1}{2}\left|\left|y_{s|t} - Kx_t\right|\right|^2 + \beta\left|\left|x_t - x_{t-1}\right|\right|_1\right\}$$

Online Convex Optimization

Using Predictions

$x_1, y_1, x_2, y_2, x_3, y_3, \ldots$ ← online

$$\min_{x_t \in F} \sum_t \boxed{c(x_t, y_t)} + \beta \underbrace{\|x_t - x_{t-1}\|}_{\text{switching cost}}$$

convex

e.g. $c(x_t, y_t) = \frac{1}{2}||y_t - Kx_t||^2$

Goal: Algorithms to minimize cost

Time	Information Available				Decision
1	$y_{1\|0}$	$y_{2\|0}$	$y_{3\|0}$	…	x_1
2	y_1	$y_{2\|1}$	$y_{3\|1}$	…	x_2
3	y_1	y_2	$y_{3\|2}$	…	x_3
4	y_1	y_2	y_3	…	x_4

Theorem: AFHC(w) with $w = O(1)$ has sublinear regret and is constant competitive (in expectation) when $\text{cost}(Opt) = \Omega(T)$, and $\text{cost}(\text{STA}) \geq \alpha_1 T - o(T)$.

Averaging Fixed Horizon Control

Fixed Horizon Control

$y_{t+1|t}, y_{t+2|t}, \dots, y_{t+w|t}, y_{t+w+1|t+w}, y_{t+w+2|t+w}, \cdots$

$x_{t+1}, x_{t+2}, \dots, x_{t+w}$ $\quad$ $x_{t+w+1}, x_{t+w+2}, \dots, x_{t+2w}$

Online Convex Optimization

Using Predictions

$x_1, y_1, x_2, y_2, x_3, y_3, \ldots \longleftarrow$ online

$$\min_x \sum_t \frac{1}{2} ||y_t - Kx_t||^2 + \beta \|x_t - x_{t-1}\|_1$$

Using given prediction $y_{\tau|t}$ that satisfies

$$y_t = y_{t|\tau} + \sum_{s=\tau+1}^{t} f(t-s)e(s)$$

Time	Information Available				Decision
1	$y_{1\|0}$	$y_{2\|0}$	$y_{3\|0}$	…	x_1
2	y_1	$y_{2\|1}$	$y_{3\|1}$	…	x_2
3	y_1	y_2	$y_{3\|2}$	…	x_3
4	y_1	y_2	y_3	…	x_4

Is there online algorithm that achieve sublinear regret and constant CR?

Yes!

Theorem: AFHC(w) with $w = O(1)$ has sublinear regret and is constant competitive (in expectation) when $\text{cost}(Opt) = \Omega(T)$, and $\text{cost}(\text{STA}) \geq \alpha_1 T - o(T)$.

Averaging Fixed Horizon Control

w FHC algorithms

$y_{t+1|t}, y_{t+2|t}, \cdots, y_{t+w|t}, y_{t+w+1|t+w}, y_{t+w+2|t+w}, \cdots$

$y_{t+2|t+1}, y_{t+3|t+1}, \cdots, y_{t+w+1|t+1}, y_{t+w+2|t+w+1}, y_{t+w+3|t+w+1}, \cdots$

$y_{t+3|t+2}, y_{t+4|t+2}, \cdots, y_{t+w+2|t+2}, y_{t+w+3|t+w+2}, y_{t+w+4|t+w+2}, \cdots$

$y_{t+4|t+3}, y_{t+5|t+3}, \cdots, y_{t+w+3|t+3}, y_{t+w+4|t+w+3}, y_{t+w+5|t+w+3}, \cdots$

$\vdots$

Average choices of FHC algorithms