A Tale of Two Metrics: Simultaneous Bounds on Competitiveness and Regret

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- Realistic scenario is rare
- Idea is inspiring

Two communities

- Online convex optimization (OCO) $\sum_{t=1}^{T} c^t(x^t)$
- Metric task system (MTS)

$$\sum_{t=1}^{T}(c^t(s^t) + d(s^t, s^{t-1}))$$

A constant number of states, a cost c^t for each given state, a fixed switching cost d for given s^t, s^{t-1}. At each t, c^t is revealed. Then we choose a state s^t for this time t.

Example:dynamic datacenter right sizing

$$\text{minimize } \sum_{t=1}^{T}\sum_{i=1}^{x^t} f(\lambda_i^t) + \beta \sum_{t=1}^{T}(x^t - x^{t-1})^+$$

$$\text{subject to: } 0 \leq \lambda_i^t \leq 1 \text{ and } \sum_{i=1}^{x_t} \lambda_i^t = \lambda_t$$

- $f(\cdot)$ is convex, x^t, λ_i^t are fractional variables.
- β is fixed. At each t, λ^t arrives first. Algorithm makes decisions then.

Example:dynamic datacenter right sizing

$$\text{minimize} \ \sum_{t=1}^{T} \underline{x^t f(\lambda^t / x^t)} + \beta \sum_{t=1}^{T} (x^t - x^{t-1})^+$$

$$\text{subject to: } x^t \geq \lambda^t$$

- $f(\cdot)$ is convex, x^t, λ_i^t are fractional variables.
- β is fixed. At each t, λ^t arrives first. Algorithm makes decisions then.

Smoothed online convex optimization (SOCO)

- OCO + switching cost

At each t, algorithm chooses x^t, then experiences loss function $c^t(x^t) + ||x^t - x^{t-1}||$. $c^t(x^t)$ is convex.

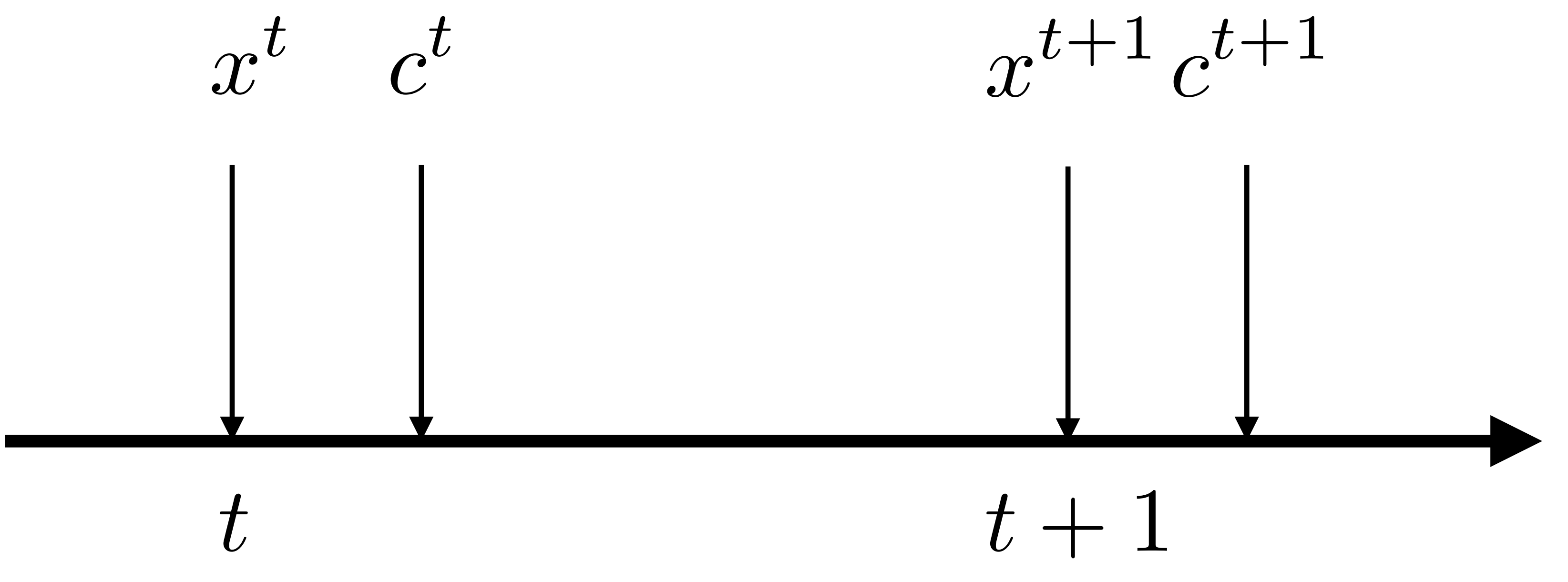
How to connect the two problems

Problem formulation

$$C_i^{\alpha}(A,T) = \sum_{t=1}^{T} c^t(x^{t+i}) + \alpha||x^{t+i} - x^{t+i-1}||$$

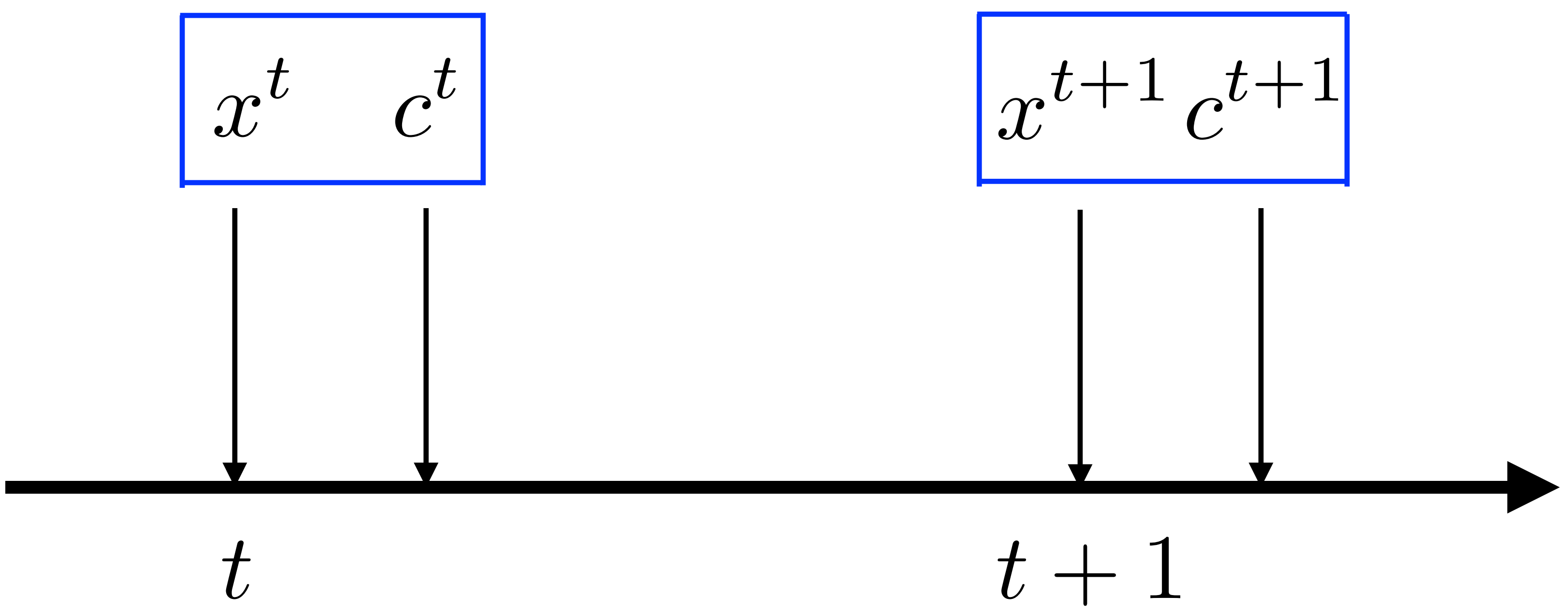
- α -penalized cost with lookahead i
- Euclidean norms of subgradients of c^t are bounded

$x^i = 0$


x^t
c^t
x^{t+1}
c^{t+1}
t
$t+1$

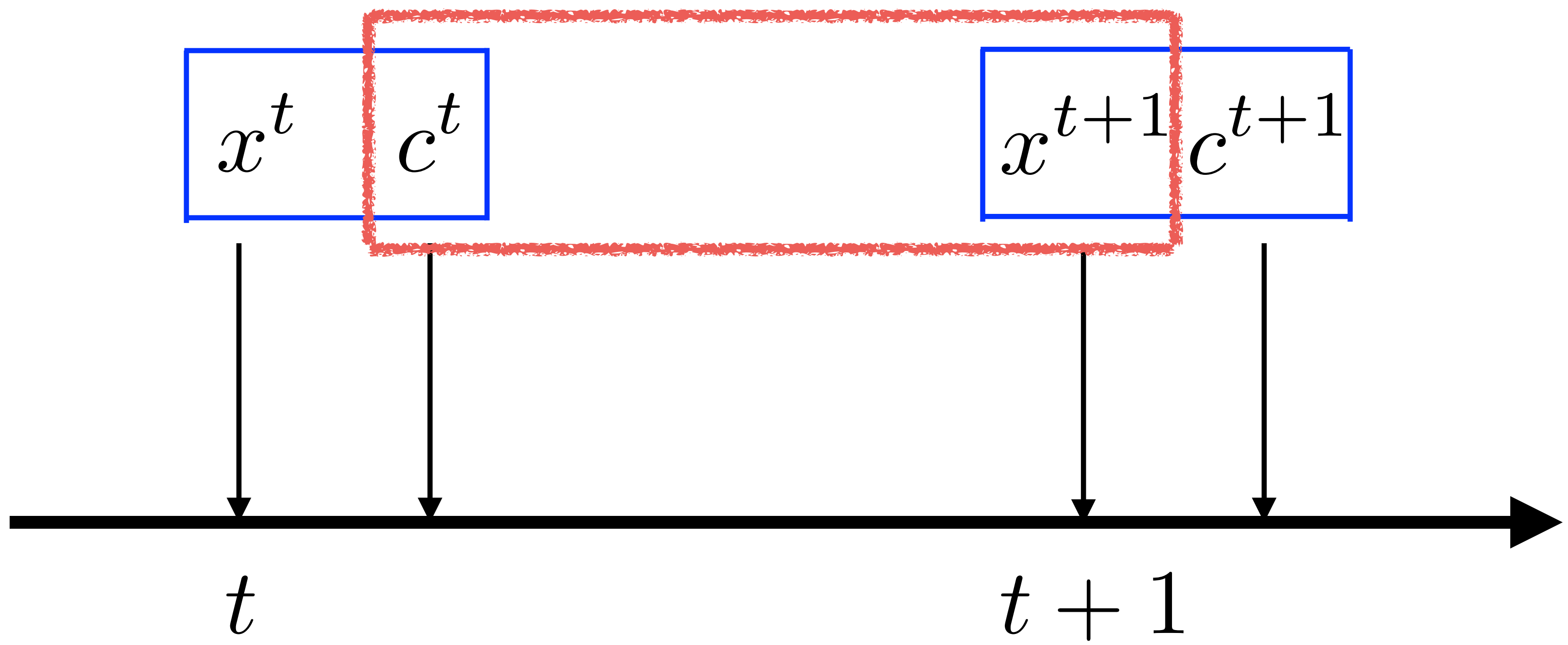
$x^0 = 0$

i=0: $$C_0^\alpha(A,T) = \sum_{t=1}^{T} c^t(x^t) + \alpha||x^t - x^{t-1}||$$



$x^0 = 0$

i=0: $C_0^\alpha(A,T) = \sum_{t=1}^{T} c^t(x^t) + \alpha||x^t - x^{t-1}||$



i=1: $C_1^\alpha(A,T) = \sum_{t=1}^{T} c^t(x^{t+1}) + \alpha||x^{t+1} - x^t||$

$x^0 = 0 \quad x^1 = 0$

Traditional performance metrics

$$C_i^{\alpha}(A,T) = \sum_{t=1}^{T} c^t(x^{t+i}) + \alpha||x^{t+i} - x^{t+i-1}||$$

- OCO: R_0^0
- MTS: CR_1^1

Traditional performance metrics

- In OCO literature: $OPT_s = \min_{x \in \mathcal{F}} \sum_{t=1}^{T} c^t(x)$

$$R_0^0(A,T) = \max_{\vec{c}}(C_0^0(A) - OPT_s)$$

- In MTS literature:

$$OPT_d = \min_{x \in \mathcal{F}^T} \sum_{t=1}^{T} c^t(x^{t+1}) + ||x^{t+1} - x^t||$$

$$CR_1^1(A,T) = \max_{\vec{c}}(C_1^1(A)/OPT_s)$$

Traditional algorithms

- Online context optimization (OCO) $$\sum_{t=1}^{T} c^t(x^t)$$

  Regret sub-linear with T

- Metric task system (MTS) $$\sum_{t=1}^{T} (c^t(s^t) + d(s^t, s^{t-1}))$$

  Competitive ratio is independent with T

In the unified problem

$$C_i^{\alpha}(A,T) = \sum_{t=1}^{T} c^t(x^{t+i}) + \alpha||x^{t+i} - x^{t+i-1}||$$

- i connects online algorithm and online learning
- α connects dynamic offline optimal and static offline optimal

Contributions

- Connecting OCO and (convex)MTS into one general problem
- Finding the incompatibility of regret and competitive ratio
- Designing an algorithm trading off the performance between the two metrics

Traditional algorithm

Online Gradient Decent (OGD)

- works for OCO
- works for SOCO (contribution of this paper)
- has unbounded competitive ratio for (convex)MTS

- Is there fundamental incompatibility between these two problems?

With window i

- In OCO literature: $OPT_s = \min_{x \in \mathcal{F}} \sum_{t=1}^{T} c^t(x)$

$$R_i^0(A) = \max_{\vec{c}}(C_i^0(A) - OPT_s)$$

- In MTS literature:

$$OPT_d = \min_{x \in \mathcal{F}^T} \sum_{t=1}^{T} c^t(x^{t+i}) + \alpha||x^{t+i} - x^{t+i-1}||$$

$$CR_{i+1}^{\alpha}(A) = \max_{\vec{c}}(C_{i+1}^{\alpha}(A)/OPT_s)$$

Incompatibility of regret and competitive ratio

$$R_i^0(A) = \max_{\vec{c}}(C_i^0(A) - OPT_s)$$

$$CR_{i+1}^{\alpha}(A) = \max_{\vec{c}}(C_{i+1}^{\alpha}(A)/OPT_s)$$

- For any algorithm A , $\alpha \geq 1$, $\gamma \geq 0$, either $R_i^0(A) = O(T)$ or $CR_{i+1}^{\alpha}(A) \geq \gamma$

Both with switching cost, Both without lookahead window

$$R_0^1(A) = \max_{\vec{c}}(C_0^1(A) - OPT_s)$$

$$CR_0^\alpha(A) = \max_{\vec{c}}(C_0^\alpha(A)/OPT_s)$$

- For any algorithm A, $\alpha \geq 1$, $\gamma \geq 0$, either $R_0^1(A,T) = O(T)$ or $CR_0^\alpha(A,T) \geq \lambda$

- Regret = $O(T^{1-\epsilon})$, Competitive ratio= $O(T^{\epsilon})$

$$\epsilon \to 0$$

A unified algorithm

Algorithm 2 (Randomly Biased Greedy, RBG(N))
Given a norm N, define $w^0(x) = N(x)$ for all x and $w^t(x) = \min_y\{w^{t-1}(y) + c^t(y) + N(x - y)\}$. Generate a random number r uniformly in $(-1, 1)$. For each time step t, go to the state x^t which minimizes $Y^t(x^t) = w^{t-1}(x^t) + rN(x^t)$.

Theorem 7 *For a SOCO problem in a one-dimensional normed space $\|\cdot\|$, running RBG(N) with a one-dimensional norm having $N(1) = \theta\|1\|$ as input (where $\theta \geq 1$) attains an α-unfair competitive ratio CR_1^α of $(1+\theta)/\min\{\theta, \alpha\}$ and a regret R_0' of $O(\max\{T/\theta, \theta\})$.*

A unified algorithm

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MTS: $s^t = \text{argmin}_x w^t(x) + ||x - s^{t-1}||$

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RBG(||•||) is 2-competitive, has linear regret

RBG(N) encourages its actions to change less

Drawbacks of RBG(N)

- Metrics are still unfair
- Can't guarantee two kinds of performance in one problem setting $c^t(x^t),\ c^t(x^{t+1})$
- How to simultaneously guarantee good $CR_0^\alpha(A)$ and $R_0^\alpha(A)$, or competitive difference?

Lessons learned

- Negative results are wonderful
- New metrics matter if meaningful. (α-unfair competitive ratio, competitive difference)
- Classical algorithms (or variants) may have good performance in other metrics (WFA)

Open questions

- What if input is i.i.d.?
- Does their algorithm work well in the combinatorial version of MTS?