Theory of Distributions

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Def. Principal value. $P.V.\frac{1}{x}$ is a distribution $\left(P.V.\frac{1}{x}\right)(\varphi) = \lim_{\varepsilon\to 0^+}\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}dx \quad \forall\, \varphi\in C_c^\infty(\mathbb{R})$.

Prop. $\left(PV\frac{1}{x}\right)(\varphi) = \int \log|x|\,\varphi'(x)\,dx, \quad \forall\,\varphi\in C_c^\infty(\mathbb{R})$.

Hint: $\int_{|x|\geq\varepsilon}\frac{\varphi}{x}dx = \int_\varepsilon^\infty\frac{\varphi}{x}dx + \int_{-\infty}^{-\varepsilon}\frac{\varphi}{x}dx$

$y=-x,\ dx=-dy$

$$= \int_\varepsilon^\infty\frac{\varphi}{x}dx + \int_\infty^\varepsilon\frac{\varphi(-y)}{-y}(-dy)$$

$$= \int_\varepsilon^\infty\frac{\varphi(x)-\varphi(-x)}{x}dx$$

MVT

$$= \int_\varepsilon^\infty\frac{x\cdot\psi(x)}{x}dx \quad \text{where } \psi(x)\in C_c^\infty$$

procedure to define a singular distribution called ~~*a principal value distribution,*~~ denoted by p.v.$(1/x)$. We define its action on a test function $\varphi\in\mathcal{S}(\mathbb{R})$ by

$$\text{p.v.}\frac{1}{x}(\varphi) = \lim_{\epsilon\to 0^+}\int_{|x|>\epsilon}\frac{\varphi(x)}{x}\,dx.$$

The limit is finite because of a cancellation between the nonintegrable contributions of $1/x$ for $x<0$ and $x>0$:

$$\text{p.v.}\frac{1}{x}(\varphi) = \lim_{\epsilon\to 0^+}\int_\epsilon^\infty\frac{\varphi(x)-\varphi(-x)}{x}\,dx = \int_0^\infty\frac{\varphi(x)-\varphi(-x)}{x}\,dx.$$

The integrand is bounded at $x=0$ since φ is smooth. For $x>0$, we have

$$\left|\frac{\varphi(x)-\varphi(-x)}{x}\right| \leq \frac{1}{x}\int_{-x}^{x}|\varphi'(t)|\,dt \leq 2\|\varphi'\|_\infty,$$

so the continuity of p.v.$(1/x)$ on $\mathcal{S}$ follows from the estimate

$$\begin{aligned}\left|\text{p.v.}\frac{1}{x}(\varphi)\right| &\leq \int_0^1\left|\frac{\varphi(x)-\varphi(-x)}{x}\right|dx + \int_1^\infty\left|\frac{x[\varphi(x)-\varphi(-x)]}{x^2}\right|dx\\ &\leq 2\|\varphi'\|_\infty + 2\|x\varphi\|_\infty\\ &= 2\left(\|\varphi\|_{0,1}+\|\varphi\|_{1,0}\right).\end{aligned}$$

Def. Derivative of a distribution. if $u\in\mathcal{D}'(\mathbb{R}^n)$, define $\frac{\partial}{\partial x_j}u$ to be:

$$\left(\frac{\partial}{\partial x_j}u\right)(\varphi) = -u\left(\frac{\partial\varphi}{\partial x_j}\right) \qquad \forall\,\varphi\in C_c^\infty(\mathbb{R}^n)$$

Prop. $\frac{\partial}{\partial x_j}u\in\mathcal{D}'(\mathbb{R}^n)$ if $u\in\mathcal{D}'(\mathbb{R}^n)$.

Similarly, we can define $D^\alpha u\in\mathcal{D}'(\mathbb{R}^n)$ as: $(D^\alpha u)(\varphi) := (-1)^{|\alpha|}u(D^\alpha\varphi)$

One can differentiate as much as he wants.

Example. Heaviside function $H(x) = \begin{cases}1 & x>0\\ 0 & x<0\end{cases} \in L^1_{loc}(\mathbb{R})$

$$T_H(\varphi) := \int_{-\infty}^{+\infty}H(x)\varphi(x)\,dx$$

$$\left(\frac{d}{dx}T_H\right)(\varphi) = -T_H(\varphi'(x)) = -\int_{-\infty}^{+\infty}H(x)\varphi'(x)\,dx = -\int_0^{+\infty}H\varphi'\,dx = \varphi(0)$$

$\Rightarrow \delta_0 = \frac{d}{dx}T_H$ in $\mathcal{D}'(\mathbb{R})$

Example: $f=\log|x|\in L^1_{loc}$ but $f'=\frac{1}{x}\notin L^1_{loc}$

$$T_f(\varphi) = \int_{-\infty}^{+\infty}\log|x|\,\varphi(x)\,dx, \quad \varphi\in C_c^\infty(\mathbb{R})$$

$$\left(\frac{d}{dx}T_f\right)(\varphi) = -T_f(\varphi') = -\int\log|x|\,\varphi'(x)\,dx = \left(P.V.\frac{1}{x}\right)(\varphi) = \lim_{\varepsilon\to 0^+}\int_{|x|\geq\varepsilon}\frac{\varphi}{x}dx$$

Now look at 1D wave equation $\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u(x,t) = 0$. $u \in C^2(\mathbb{R}\times\mathbb{R})$

General sol: $u(x,t) = f(x+ct) + g(x-ct)$, $f, g \in C^2(\mathbb{R})$

Now what if $f, g \in \mathcal{D}'(\mathbb{R})$? How to define $f(x+ct)$, $g(x-ct)$?

Assume $f, g \in L^1_{loc}(\mathbb{R})$, take $\varphi \in C_c^\infty(\mathbb{R})$ and test it against $T_{f(x+ct)}$:

$$\int_{\mathbb{R}^2} f(x+ct)\, \varphi(x,t)\, dx\, dt = \int_{\mathbb{R}^2} f(y)\, \varphi(y-ct, t)\, dy\, dt \qquad y := x+ct$$

Hence define $f(x+ct)$ as a distribution: $f(x+ct)(\varphi) = \int_{\mathbb{R}^2} f(y)\, \varphi(y-ct, t)\, dy\, dt$

Similarly, $g(x-ct)(\varphi) = \int_{\mathbb{R}^2} g(y)\, \varphi(y+ct, t)\, dy\, dt$ as a distribution.

$u_0 \in \mathcal{D}'(\mathbb{R})$, $u_0(x \pm ct) \in \mathcal{D}'(\mathbb{R}^2)$ acting on functions of 2 variable. defined as follows: take $\varphi(x,t) \in C_c^\infty(\mathbb{R}^2)$

$$u_0(x \pm ct)(\varphi) = \int \underbrace{\boxed{u_0(\varphi(\cdot \mp ct, t))}}_{\substack{u_0 \text{ acting on a function} \\ \text{of } x \text{, one variable}}}^{\text{output a function of } t} dt$$

$\varphi(\cdot \mp ct, t) \in C_c^\infty(\mathbb{R}_x)$ a function of x

$\varphi(\cdot \mp ct, t)(x) = \varphi(x \mp ct, t)$, $x \in \mathbb{R}$

the "dot" variable

Claim: $h(t) := u_0(\varphi(\cdot \mp ct, t)) \in C_c(\mathbb{R}_t) \Rightarrow$ the integral above makes sense.

Proof: For large t, $\varphi(\cdot \mp ct, t) = 0$ since $\varphi \in C_c^\infty(\mathbb{R}_x, \mathbb{R}_t)$

Let $t_j \to t_0$. WANT: $u_0(\varphi(\cdot \mp ct_j, t_j)) \to u_0(\varphi(\cdot \mp ct_0, t_0))$ as $j \to \infty$.

$\varphi(\cdot \mp ct_j, t_j) \to \varphi(\cdot \mp ct_0, t_0)$ in $C_c^\infty(\mathbb{R})$ since φ is smooth.

Then by continuity of u_0, we are done.

Prop. $u \in \mathcal{D}'(\mathbb{R}^2)$ $\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$. Factorization holds for distributions.

Proof: $\left(\frac{\partial}{\partial x}\frac{\partial}{\partial t} u\right)(\varphi) = -\left(\frac{\partial}{\partial t} u\right)\left(\frac{\partial \varphi}{\partial x}\right)$

$= u\left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\varphi\right)$ $\varphi \in C_c^\infty$ Hence switch the order of differentiation.

$= \left(\frac{\partial}{\partial t}\frac{\partial}{\partial x} u\right)(\varphi)$

$\Rightarrow$ Mixed partial derivative thm holds for distributions.

$\Rightarrow$ It implies the prop.

Prop. $u_0 \in \mathcal{D}'(\mathbb{R})$ $\quad u_0(x \pm ct)(\varphi) := \int u_0(\varphi(\cdot \mp ct, t))\,dt$ solves the equation

$$\left(\frac{\partial}{\partial t} \mp c\frac{\partial}{\partial x}\right) u_0(x \pm ct)(\varphi) = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^2)$$

RMK. $u_0(x \pm ct)$ then solves the wave equation $u_{tt} = c^2 u_{xx}$

Proof. Just prove "−" case:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u_0(x-ct)(\varphi) = -u_0(x-ct)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)(\varphi)$$

$$= -\int u_0\left[\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)(\varphi(\cdot + ct, t))\right] dt$$

$$= -\int u_0\left(\frac{d}{dt}\varphi(\cdot + ct, t)\right) dt$$

$$\overset{?}{=} -\int \frac{d}{dt}\left[u_0(\varphi(\cdot + ct, t))\right] dt$$

$$= u_0(\varphi(\cdot + ct, t))\Big|_{-\infty}^{+\infty} = 0 \quad \text{since compactly supported.}$$

$$\frac{d}{dt}\underbrace{u_0(\varphi(\cdot + ct, t))}_{h(t)} \overset{\text{linearity}}{=} \lim_{h\to 0} \overset{\text{continuous}}{u_0}\left(\frac{\varphi(\cdot - c(t+h), t+h) - \varphi(\cdot - ct, t)}{h}\right)$$

$$= u_0\left(\lim_{h\to 0}\frac{\varphi(\cdot, t+h) - \varphi(t)}{h}\right)$$

$$= u_0\left(\frac{d}{dt}\varphi(\cdot)\right) \quad \text{since } \varphi \in C_c^\infty$$

Example. $\delta_0(x \pm ct)(\varphi) = \int \delta_0(\varphi(\cdot \mp t, t))\,dt$

$$= \int \varphi(\mp ct, t)\,dt, \quad \varphi \in C_c^\infty(\mathbb{R}^2)$$

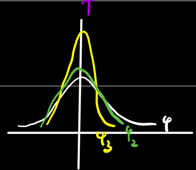
Notation. $T \in \mathcal{D}'$, $\varphi \in C_c^\infty$. $\quad \langle T, \varphi \rangle := T(\varphi)$

Convergence of distribution.

Let $u_j \in \mathcal{D}'$, $u \in \mathcal{D}'$. We say $u_j \to u$ in $\mathcal{D}'$ if $\underbrace{u_j(\varphi) \to u(\varphi)}_{\text{conv. in } \mathbb{C}}$ $\forall \varphi$ in C_c^∞. Weak convergence.

Example (Approximation of the identity) take $\varphi \in C_c^\infty(\mathbb{R}^n)$ $\int_{\mathbb{R}^n} \varphi\,dx = 1$

$\varphi_j(x) = j^n \varphi(jx)$



Then $\int \varphi_j(x)\,dx = 1$. And $T_{\varphi_j}(\psi) = \int \varphi_j \psi \, dx \to \delta_0(\psi) = \psi(0)$ $\forall \psi \in C_c^\infty(\mathbb{R}^n)$.

Hence $\varphi_j \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$.

Multiplication by C_c^∞: $u \in \mathcal{D}'$. $f \in C_c^\infty$. define fu as:

$$(fu)(\varphi) = u(f\varphi).$$

Example: $f(x) = x$. $u = \delta_0$. $(f\delta_0)(\varphi) \overset{def}{=} \delta_0(f\varphi) = f(0)\varphi(0) = 0$. Here f, δ_0 both not zero but $f\delta_0 \equiv 0$.

Theorem. $u \in \mathcal{D}'(X) \iff$

① $u : C_c^\infty(X) \to \mathbb{C}$ is linear

(*) ② $\forall K \subset X$ compact, $\exists C_K$ $N_K \in \mathbb{N}$.

$$|u(\varphi)| \leq C_K \sum_{|\alpha| \leq N_K} \sup_{x \in K} |\varphi^{(\alpha)}(x)|$$

$$\forall \varphi \in C_c^\infty(K)$$

Proof: $\Leftarrow$: Suppose ① and ② holds.

Then for any converging sequence $\{\varphi_n\} \subseteq C_c^\infty(X)$, $\varphi_n \to 0$.

$\iff \exists K \subseteq X$ compact. $\|\partial^\alpha \varphi_j(x)\|_\infty \to 0$ $\forall \alpha$. on K.

$$|u(\varphi_j) - 0| = |u(\varphi_j)| \le C_K \sum_{|\alpha| \le N_K} \|\partial^\alpha \varphi_j\|_\infty$$

$$\to 0 \quad \text{as } j \to \infty.$$

Hence u is continuous; $u \in \mathcal{D}'(X)$

$\Rightarrow$: Suppose now $u \in \mathcal{D}'$: u is linear.

proof by contradiction, Assume ② is not true.

$\exists K \subseteq X$ compact. $\forall N \in \mathbb{N}$:

$\exists \varphi_N \in C_c^\infty(K)$.

$$|u(\varphi_N)| > N \sum_{|\alpha| \le N} \|\partial^\alpha \varphi_N\|_\infty.$$

Take $\psi_N = \dfrac{\varphi_N}{\sum_{|\alpha| \le N} \|\partial^\alpha \varphi_N\|_\infty} \in C_c^\infty(K).$

$$|u(\psi_N)| > N.$$

$$\Leftrightarrow \left| u\left(\frac{\psi_N}{N}\right) \right| > 1.$$

Claim: $\left(\frac{\psi_N}{N}\right)_{N=1}^\infty \to 0$ in $C_c^\infty(K)$.

proof: $$\frac{\psi_N}{N} = \frac{\varphi_N}{N \sum_{|\alpha| \le N} \|\partial^\alpha \varphi_N\|_\infty}.$$

$$\left|\partial^\beta\left(\frac{\psi_N}{N}\right)\right| = \frac{|\partial^\beta \varphi_N|}{N \sum_{|\alpha| \le N} \|\partial^\alpha \varphi_N\|_\infty} \le \frac{1}{N}$$

as long as $|\beta| \le N$.

$$\Rightarrow \quad \sup_K \left|\partial^\beta\left(\frac{\psi_N}{N}\right)\right| \le \frac{1}{N}. \text{ if } |\beta| \le N$$

$$\Rightarrow \quad \frac{\psi_N}{N} \longrightarrow 0 \text{ in } C_c^\infty(K).$$

However. $u\left(\frac{\psi_N}{N}\right) \ge 1.$

(Contradiction to the fact $u \in \mathcal{D}'(X)$.

$\square$

Definition. Order of distribution

min N s.t. $(*)$ is valid for any compact set $K \subseteq X$.

(Might be infinity)

Example. 1) Order of $\delta(x)$:

$$\delta(\varphi) = \varphi(0)$$

order $= 0$.

2) $p.v.\frac{1}{x}$: $\quad p.v.\frac{1}{x}(\varphi) = -\int \varphi' \log|x|\, dx$

order $= 1$.

Exercise: Find a distribution of order ∞.

Hint: $\sum_{N=0}^{\infty} \delta_N^{(N)}$.

Definition. Restriction of distribution.

$X \subseteq \mathbb{R}^n$ open. $V \subseteq X$ open.

$u \in \mathcal{D}'(X)$.

$u|_V \in \mathcal{D}'(V)$ is a distribution on V

$$u|_V(\varphi) := u(\varphi)$$

$$\forall \varphi \in C_c^\infty(V)$$

Definition $X \subseteq \mathbb{R}^n$ open
$V \subseteq X$ open.
$u = 0$ on V if $u(\varphi) = 0 \ \forall \varphi \in C_c^\infty(V)$

proposition $X_i \subseteq X$ open. $i = 1, 2, \dots, N$
$u = 0$ on $X_i \ \forall i$
Then $u = 0$ on $\bigcup_{i=1}^N X_i$

proof: Partition of Unity: $X := \bigcup_{i=1}^N X_i$
$K \subseteq X$ compact.
then $\exists \psi_i \in C_c^\infty(X_i)$. $0 \le \psi_i \le 1$

$\sum \psi_i \equiv 1$ on a neighbourhood of K.

Now $\varphi \in C_c^\infty(\bigcup_{i=1}^N X_i)$.
Suppose $K := \operatorname{supp} \varphi$.

$$\varphi = \sum_{i=1}^N \underbrace{(\psi_i \varphi)}_{\in C_c^\infty(X_i)}$$

$$u(\varphi) = \sum_{i=1}^N u(\varphi \psi_i) \underset{u=0 \text{ on } X_i}{=} \sum_{i=1}^N 0 = 0.$$

$\square$

Definition. Support of distribution

$\operatorname{supp} u :=$ complement of the largest open set where $u=0$.

existence: $V = \bigcup_{\alpha\in A} X_\alpha$ where $u=0$ on X_α

prove $u=0$ on V: $\forall \varphi \in C_c^\infty(V)$.

$K := \operatorname{supp} \varphi$.

$\exists$ finite number $X_{\alpha_1}, \dots, X_{\alpha_N}$ covers K.

Apply the previous proposition. Done.

Example.

1) $\delta(x)$: $\operatorname{supp}\delta = \{0\}$ since

$$\forall \varphi \in C_c^\infty(\mathbb{R}\setminus\{0\}), \quad \delta(\varphi)=0.$$

2) $\operatorname{supp}\left(p.v.\frac{1}{x}\right) = \mathbb{R}$

$$p.v.\frac{1}{x}(\varphi) = \lim_{\varepsilon\to 0^+}\int_{|x|\geq\varepsilon}\frac{\varphi}{x}\,dx$$

$$= -\int \log|x|\cdot\varphi'(x)\,dx.$$

On $\mathbb{R}\setminus\{0\}$: $p.v.\frac{1}{x} = \frac{1}{x}$.

$$\operatorname{supp}\frac{1}{x} = \overline{\mathbb{R}\setminus\{0\}} = \mathbb{R}$$

$\Rightarrow \operatorname{supp}(\mathrm{pv}\frac{1}{x}) = \mathbb{R}$.

Definition $\mathcal{E}'(X) := \{ u \in \mathcal{D}'(X) \mid \operatorname{supp} u \text{ is compact} \}$

is the space of compactly supported distributions.

Example 1) $\delta(x)$ supported at $\{0\}$.

Theorem Given $u \in \mathcal{D}'(X)$.

$u \in \mathcal{E}'(X)$ iff $\exists K \subseteq X$ compact.

$\exists C > 0$. $N \in \mathbb{N}$.

(**) $$|u(\varphi)| \leq C \sum_{|\alpha| \leq N} \sup_K \|\partial^\alpha \varphi\|.$$

$\forall \varphi \in C_c^\infty(X)$

Proof of Theorem:

$\Leftarrow$: take $\varphi \in C_c^\infty(X)$

$\operatorname{supp} \varphi \cap K = \emptyset$.

$u(\varphi) = 0$ by (**)

$\Rightarrow \operatorname{supp} u \subseteq K$.

$\rightarrow$ u is compactly supported.

$\Rightarrow$: $u \in \mathcal{D}'(X)$ has compact support.

$\operatorname{supp} u = K \subseteq X$ compact.

By (*), $|u(\varphi)| \le C \sum_{|\alpha| \le N} \sup_K |\partial^\alpha \varphi|$

$\forall \varphi \in C_c^\infty(K)$.

Take a bump function f s.t.

$f \equiv 1$ on a nbhd of K.

Let $\psi \in C_c^\infty(X)$.

$$\psi = f\psi + (1-f)\psi$$

$$u(\psi) = u(f\psi) + u((1-f)\psi)$$

$(1-f)\psi = 0$ on a nbhd of K

Hence $u((1-f)\psi) = 0$. since $\operatorname{supp} u = K$.

$\Rightarrow u(\psi) = u(f\psi)$

Now $\operatorname{supp} f\psi = \tilde{K}$ compact

Apply (*) on $\tilde{K}$:

$$|u(\psi)| = |u(f\psi)| \le C_{\tilde{K}} \sum_{|\alpha| \le N} \sup_{\tilde{K}} |\partial^\alpha (f\psi)|$$

$$\left|\partial^\alpha(f\psi)\right| = \left|\sum_\beta C_{\alpha,\beta}\, \partial^\beta f \cdot \partial^{\alpha-\beta}\psi\right|$$

$$\lesssim \sum_{|\alpha|\le N} \sup_X |\partial^\alpha \psi|.$$

$$\forall \psi \in C_c^\infty(X)$$

(**) is shown.

$\square$

(**) $u \in \mathcal{E}'(X)$ $\quad |u(\varphi)| \le C \sum_{|\alpha|\le N} \sup_K |\partial^\alpha \varphi|$

for some $C>0$, $N \in \mathbb{N}$, $K \subseteq X$ compact.

– any $\varphi \in C_c^\infty(X)$.

Corollary The order of $u \in \mathcal{E}'$ is finite

Definition. Convergence in $C^\infty(X)$:

$\{f_j\} \subseteq C^\infty(X)$ $\quad f_j \to 0$ in $C^\infty(X)$

iff $\forall K \subseteq X$ compact,

$$\sup_K |\partial^\alpha f_j| \longrightarrow 0$$

$\forall$ multi-index α.

Definition. $u: C^\infty(X) \to \mathbb{C}$ linear is continuous if $\forall \{f_j\} \subseteq C^\infty(X)$ $f_j \to 0$ in $C^\infty(X)$ then $u(f_j) \to 0$ in $\mathbb{C}$.

Theorem $u \in \mathcal{E}'(X)$ then u has an extension $\tilde{u}: C^\infty(X) \to \mathbb{C}$ linear, continuous

proof. Take $f \in C^\infty(X)$. $\operatorname{supp} u = K$ compact.

Let $\psi \in C_c^\infty(X)$ $\psi \equiv 1$ on a nbhd of K.

Then $f\psi \in C_c^\infty(X)$, $f\psi \equiv f$ on K

$$f = \psi f + (1-\psi) f$$

Define $\tilde{u}(f) := u(\psi f)$.

Check def is independent of ψ:

Let $\tilde{\psi}$ be another bump function s.t. $\tilde{\psi} \equiv 1$ on a nbhd of K

$u((\psi - \tilde{\psi}) f) = 0$ since $\tilde{\psi} f = \psi f$ on a nbhd of K.

Check continuity of extended $\tilde{u}$:

As $\{f_j\} \in C^\infty$ $f_j \to 0$: $\psi f_j \to 0$ in $C_c^\infty(X)$

Since $\operatorname{supp} \psi f_j \subseteq \operatorname{supp} \psi = K$.

$\partial^\alpha(\psi f_j) := \cdots$ Libniz rule.

$\to 0$ uniformly

$\tilde{u}(f_j) = u(\psi f_j) \longrightarrow 0$

continuity of u

$\square$

Summary $u \in \mathcal{D}' \iff u : C_c^\infty \to \mathbb{C}$ satisfies $(*)$

$u \in \mathcal{E}' \iff u : C^\infty \to \mathbb{C}$ satisfies $(**)$

However, multiplication of distribution could be a problem.

proposition. δ^2 can not be defined in $\mathcal{D}'$ in the sense that if $\varphi_j \to \delta$,

then $\varphi_j^2 \to \delta^2$.

proof. Take $\varphi \in C_c^\infty$ with $\int_{\mathbb{R}^n} \varphi = 1$

$$\varphi_j(x) := j^n \varphi(jx).$$

$\int \varphi_j = 1 \ \forall j.$ $\quad \varphi_j \to \delta$ in $\mathcal{D}'$

What about φ_j^2 ?

$$\varphi_j^2 = j^{2n} \varphi(jx)^2$$

Take $\psi \in C_c^\infty(\mathbb{R}^n)$

$$\int \psi \varphi_j^2 \, dx = \int j^{2n} \psi \varphi^2(jx) \, dx$$

$y = jx$

$dy = j^n dx$

$$= \underbrace{j^n}_{\to \infty} \underbrace{\int \psi\left(\frac{y}{j}\right) \varphi^2(y) \, dy}_{\to \psi(0) \int \varphi^2} \to \infty$$

as $j \to \infty$.

$\square$

Theorem Suppose $u \in \mathcal{D}'(X)$. $x_0 \in X$

$\operatorname{supp} u = \{x_0\}$, then $\exists N \in \mathbb{N}$.

$$u = \sum_{|\alpha| \le N} a_\alpha \partial^\alpha \delta_{x_0}, \quad a_\alpha \in \mathbb{C}.$$

proof: W.L.O.G. Suppose $x_0 = 0$. $X = \mathbb{R}$.

Since u is compactly supported,

order $u = N < +\infty$.

take $\varphi \in C_c^\infty(\mathbb{R})$

$$\varphi = \varphi(0) + \varphi'(0)x + \frac{x^2}{2}\varphi''(0) + \cdots + \frac{x^N}{N!}\varphi^{N}(0) + x^{N+1}\underbrace{R_{N+1}(x)}_{\in C^\infty(\mathbb{R})}$$

Take $\psi \in C_c^\infty(\mathbb{R})$ s.t. $\psi \equiv 1$ on $|x| \le \frac{1}{2}$
$\psi \equiv 0$ on $|x| \ge 1$

$$f_\varepsilon(x) := \varphi(x)\,\psi\left(\frac{x}{\varepsilon}\right)$$

$f_\varepsilon \equiv 0$ on $|x| \ge \varepsilon$.

Write $f_\varepsilon(x) = f_\varepsilon(0) + f_\varepsilon'(0)x + \cdots + \frac{f_\varepsilon^{(N)}}{N!}x^N + R_{n+1}(x)x^{N+1}$

$$= \varphi(0) + \varphi'(0)x + \cdots + \frac{\varphi^{(N)}(0)}{N!}x^N + x^{N+1}\tilde{R}_{N+1}(x)$$

where $\tilde{R}_{N+1}(x,\varepsilon) = \left(\varphi(\eta)\psi\left(\frac{1}{\varepsilon}\right)\right)^{(N+1)}$

$0 < \eta < x$. (MVT). Langrange remainder.

$u(f_\varepsilon) = u(\psi(\frac{x}{\varepsilon})\varphi) = u(\varphi) \quad \forall \varepsilon > 0$

since $\operatorname{supp} u = \{0\}$

$u(\varphi) = \varphi(0)\underbrace{u(1)} + \varphi'(0)u(x) + \dots + \frac{\varphi^{(N)}(0)}{N!}u(x^{N+1})$

u applied to const function 1

$+ u(x^{N+1}\tilde{R}_{N+1}(x))$

$= c_0\delta_0(\varphi) + c_1\delta_0'(\varphi) + \dots + c_N\delta_0^{(N)}(\varphi)$

$+ \underbrace{u(x^{N+1}\tilde{R}_{N+1}(x,\varepsilon))}$ want it to be zero.

$\varphi = \varphi\psi(\frac{x}{\varepsilon}) + \varphi(\underbrace{1-\psi(\tfrac{x}{\varepsilon})}_{=0 \text{ near zero}})$

$u(\varphi) = u(\varphi\psi(\frac{x}{\varepsilon})) + \underbrace{\cancel{u(\varphi(1-\psi(\tfrac{x}{\varepsilon})))}}_{=0 \text{ since } \operatorname{supp} u = \{0\}}$

$\forall \varepsilon > 0$.

Now use supp u compact again:

$$u(f) \leq C \sum_{|\alpha| \leq N} \sup_{K} |\partial^\alpha f| \quad f \in C_c^\infty(\mathbb{R})$$

$K = [-1, 1]$. $\quad$ ($\forall\, K \supseteq \{0\}$ would be fine)

$$u(x^{N+1} \tilde{R}_{N+1}(x, \varepsilon)) = u\left[x^{N+1} \quad (f_\varepsilon)^{(N+1)}\Big|_{x=\eta}\right].$$

Claim: $|\partial^\alpha (x^{N+1} \underbrace{f_\varepsilon}_{\text{supp} \subseteq \{|x| \leq \varepsilon\}})| \leq C\varepsilon \quad |\alpha| \leq N.$

proof of the claim: $\alpha = 0$: $\partial^0 (x^{N+1} f_\varepsilon)$

$$= x^{N+1} \psi(\tfrac{x}{\varepsilon}) \varphi(x)$$

$$\sup_{[-1,1]} |x^{N+1} \psi(\tfrac{x}{\varepsilon}) \varphi(x)| \leq C \varepsilon^{N+1}$$

$\alpha = 1$: $(x^{N+1} f_\varepsilon)' = f_\varepsilon' x^{N+1} + (N+1) f_\varepsilon x^N.$

$$\sup |(x^{N+1} f_\varepsilon)'| \leq \sup |f_\varepsilon'| \cdot \varepsilon^{N+1} + (N+1) \sup |f_\varepsilon| \varepsilon^N.$$

$$\lesssim C \varepsilon^N$$

The rest can be shown by induction. $(x^{N+1} f_\varepsilon)^{(\alpha)} \lesssim \varepsilon^{N+1-\alpha}$

Now $u(x^{N+1} f_\varepsilon) \leq C \varepsilon$

Take $\varepsilon \to 0^+$, we are done. $\square$

Convolutions. $f \in C_c^\infty(\mathbb{R}^n)$. $g \in L^1_{loc}(\mathbb{R}^n)$

$$(f*g)(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

Extendible to $g \in \mathcal{D}'(\mathbb{R}^n)$.

Proposition: 1) $(f*g)(x) = (g*f)(x)$.

2) $\operatorname{supp} f*g \subseteq \operatorname{supp} f + \operatorname{supp} g$.

Proofs are omitted.

Theorem(*) (Mollifier) $f \in C_c^\infty(\mathbb{R}^n)$, $g \in L^1_{loc}(\mathbb{R}^n)$

$\Rightarrow f*g \in C^\infty(\mathbb{R}^n)$. $\partial^\alpha(f*g) = \partial^\alpha f * g = f * \partial^\alpha g$.

proof of the theorem. 1) $f*g$ is continuous:

Suppose $x_j \to x_0$.

$$(f*g)(x_j) = \int f(x_j - y)\, g(y)\, dy \xrightarrow{\text{LDCT}} \int f(x-y) g(y)\, dy = (f*g)(x).$$

since $f(x_j - y)\, g(y)$ is integrable.

Fourier transform.

$f \in C_c^\infty$. $\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx$, $\xi \in \mathbb{R}^n$.

But $C_c^\infty(\mathbb{R}^n)$ is not very good space for F.T.

later show $f \in C_c^\infty$ then $\hat{f}$ cannot be compactly supp.

Definition. (Schwartz)

$$\mathcal{S}(\mathbb{R}^n) := \left\{ \begin{array}{l} f : \mathbb{R}^n \to \mathbb{C} \text{ smooth} \\ \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha f| \le C_{\alpha,\beta} \end{array} \right\}$$

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}. \quad \beta = (\beta_1, \cdots, \beta_n)$$

Example

1) $C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$

2) $f(x) = e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$

Theorem: 1) $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \partial^\alpha f \in \mathcal{S}(\mathbb{R}^n)$ $\forall \alpha$.

$$\widehat{\partial_j f}(\xi) = i\xi_j \hat{f}(\xi)$$

proof:

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = \int e^{-ix\cdot\xi} \frac{\partial f}{\partial x_j}(x)\,dx$$

$$\overset{\text{integrate by parts}}{=} \int i\xi_j e^{-ix\cdot\xi} f(x)\,dx = i\xi_j \hat{f}(\xi) \quad \square$$

no boundary term since f decays fast

In particular, $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$

$$\Rightarrow \widehat{\Delta f}(\xi) = \sum_{i=1}^{n} \widehat{\frac{\partial^2 f}{\partial x_i^2}}(\xi) = -\sum_j \xi_j^2 \hat{f}(\xi)$$
$$= -|\xi|^2 \hat{f}(\xi)$$

2) $\frac{\partial}{\partial \xi_j} \hat{f}(\xi) = -i\,\widehat{x_j f}(\xi)$

proof: $\partial_j \hat{f}(\xi) = \partial_j \int f(x) e^{-ix\cdot\xi} dx$

$$\overset{\text{LDCT}}{=} \int f(x)\left(-ix_j e^{-ix\cdot\xi}\right) dx$$
$$= -i\,\widehat{x_j f}(\xi) \quad \square$$

Example: F.T. of Gaussian $e^{-|x|^2/2}$:

1D: $f(x) = e^{-x^2/2}$. Note $\frac{d}{dx} f + xf = 0$.

Take F.T. on both sides:

$$i\xi \hat{f}(\xi) + i\frac{d}{d\xi}\hat{f}(\xi) = 0.$$

$\Rightarrow \hat{f}(\xi)$ satisfies the same ODE as f.

$\Rightarrow \hat{f} = cf$

$$\hat{f}(0) = c f(0) = c = \int e^{-x^2/2} dx = \sqrt{2\pi}$$

$$\Rightarrow \hat{f} = \sqrt{2\pi}\, e^{-\xi^2/2}$$

n D: $\hat{f}(\xi) = \int e^{-(x_1\xi_1 + \dots + x_n\xi_n)} e^{-x_1^2/2} e^{-x_2^2/2} \cdots e^{-x_n^2/2} dx$

$$= \prod_{i=1}^{n} \int e^{-\xi_i x_i} e^{-x_i^2/2} dx_i$$

$$= \prod_{i=1}^{n} \sqrt{2\pi}\, e^{-\xi_i^2/2}$$

$$= (2\pi)^{\frac{n}{2}} e^{-|\xi|^2/2}$$

3) $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \hat{f} \in \mathcal{S}(\mathbb{R}^n)$

proof: $\xi^\beta \partial^\alpha \hat{f}(\xi) = \xi^\beta (-i)^{|\alpha|} \widehat{x^\alpha f}(\xi)$

$$= (-i)^{|\alpha|} (i)^{|\beta|} \widehat{\partial^\beta (x^\alpha f)}(\xi)$$

$$\sup_{\xi \in \mathbb{R}^n} |\xi^\beta \partial^\alpha \hat{f}(\xi)| \le \sup_{\xi \in \mathbb{R}^n} |\widehat{\partial^\beta (x^\alpha f)}(\xi)|$$

$$f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \partial^\beta (x^\alpha f) \in \mathcal{S}(\mathbb{R}^n)$$

It suffices to show $g \in \mathcal{S}(\mathbb{R}^n)$

$$\sup_{\xi \in \mathbb{R}^n} |\hat{g}| \le +\infty$$

$$|\hat{g}| = \left|\int e^{-ix\cdot\xi} g(x)\,dx\right| \leq \|g\|_\infty < +\infty$$

since $g \in \mathcal{S}(\mathbb{R}^n)$. $\square$

Hence $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear operator over Schwartz space

proposition. F.T. under linear trans.

Take $A \in GL(n, \mathbb{R})$ $f \in \mathcal{S}(\mathbb{R}^n)$

$$f_A(x) := f(Ax) \in \mathcal{S}(\mathbb{R}^n)$$

Then
$$\hat{f_A}(\xi) = \frac{1}{|\det(A)|} \hat{f}\left((A^T)^{-1}\xi\right)$$

proof:
$$\hat{f_A}(\xi) = \int e^{-ix\cdot\xi} f_A(x)\,dx$$

$$= \int e^{-ix\cdot\xi} f(Ax)\,dx \qquad y = Ax, \quad dy = |\det A|\,dx$$

$$= \int \frac{1}{|\det A|} \cdot e^{-i(A^{-1}y)\cdot\xi} f(y)\,dy$$

$$= \frac{1}{|\det A|} \int e^{-iy\cdot((A^{-1})^T\xi)} f(y)\,dy$$

$$= \frac{1}{|\det A|} \hat{f}\left((A^{-1})^T\xi\right)$$

Examples 1) $Ax = ax$, $a \neq 0$. $x \in \mathbb{R}^n$

$$\hat{f}_A(\xi) = \frac{1}{|a|^n} \hat{f}\left(\frac{\xi}{a}\right)$$

2) $\widehat{e^{-\frac{\varepsilon|x|^2}{2}}} = ?$

$f(x) = e^{-\frac{|x|^2}{2}}$, $a = \sqrt{\varepsilon}$ as in (1)

$$\Rightarrow \widehat{e^{-\frac{\varepsilon|x|^2}{2}}} \underset{\uparrow}{=} \frac{1}{\varepsilon^{n/2}} \frac{1}{(2\pi)^{n/2}} e^{-\frac{|\xi|^2}{2\varepsilon}}$$

verified later

3) $O \in SO(n)$. $O^{-1} = O^T$. $\det O = 1$

$$\hat{f}_O(\xi) = \hat{f}((O^{-1})^T \xi) = \hat{f}(O\xi)$$

Hence F.T. is invariant (commute with) under rotations

proposition Translation $a \in \mathbb{R}^n$, $f \in \mathcal{S}(\mathbb{R}^n)$ $f_a(x) := f(x+a)$

$$\Rightarrow \hat{f}_a(\xi) = e^{ia\xi} \hat{f}(\xi)$$

proof is omitted.

* Proposition. $f, g \in \mathcal{S}(\mathbb{R}^n)$

$$\int \hat{f}(\xi) g(\xi) \, d\xi = \int f(\xi) \hat{g}(\xi) \, d\xi$$

proof: Fubini Theorem.

proposition. $f, g \in \mathcal{S}(\mathbb{R}^n)$. then

$$\widehat{f * g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$$

proof: Fubini Theorem.

Theorem. (Fourier Inversion on $\mathcal{S}(\mathbb{R}^n)$)

$$f(x) = \frac{1}{(2\pi)^n} \int e^{ix \cdot \xi} \hat{f}(\xi) \, d\xi.$$

$\forall f \in \mathcal{S}(\mathbb{R}^n)$

Corollary. $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear isomorphism.

proof of Corollary: $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \hat{f}(\mathbb{R}^n)$ ✓

Now if $\hat{f}(\xi) \in \mathcal{S}(\mathbb{R}^n)$.

Inversion formula $\Rightarrow$ $f(x) = \frac{1}{(2\pi)^n} \hat{\hat{f}}(-x)$

Hence $f \in \mathcal{S}(\mathbb{R}^n)$. $\square$

proof of the Theorem:

$$\text{RHS} = \frac{1}{(2\pi)^n} \int \underbrace{e^{ix\xi}}_{\notin \mathcal{S}(\mathbb{R}^n)} \hat{f}(\xi)\, d\xi$$

$\notin \mathcal{S}(\mathbb{R}^n)$. cannot apply $\int \hat{f}(\xi) g = \int \hat{g} \cdot f$

$$= \lim_{\varepsilon \to 0^+} \int e^{-\frac{\varepsilon|\xi|^2}{2}} e^{ix\xi} \hat{f}(\xi)\, d\xi.$$

$$= \lim_{\varepsilon \to 0^+} \int \underbrace{e^{-\frac{\varepsilon|\xi|^2}{2}}}_{\in \mathcal{S}(\mathbb{R}^n)} \underbrace{\hat{f_x}(\xi)}_{\in \mathcal{S}(\mathbb{R}^n)}\, d\xi$$

where $f_x(y) = f(x+y)$.

$$= \lim_{\varepsilon \to 0^+} \int \widehat{e^{-\frac{\varepsilon|\xi|^2}{2}}}(y)\, f_x(y)\, dy$$

$$= \lim_{\varepsilon \to 0^+} (2\pi)^{n/2} \int \boxed{e^{-\frac{|y|^2}{2\varepsilon}} \cdot \frac{1}{\varepsilon^{n/2}}} f(x+y)\, dy$$

δ-like sequence. $\to C\delta$ for some const. C

$$= \cdot f(x).$$

Then calculate the const. $C = (2\pi)^{-n/2}$ $\square$

Now suppose $f : \mathbb{R}^n \to \mathbb{C}$

$$f = \mathrm{Re} f + i \,\mathrm{Im}(f)$$

proposition. $\widehat{\bar{f}}(\xi) = \overline{\hat{f}}(-\xi) \quad f \in \mathcal{S}(\mathbb{R}^n)$.

proof:

$$\widehat{\bar{f}}(\xi) = \int e^{-ix\xi} \bar{f}(x)\,dx$$

$$= \overline{\int e^{ix\xi} f(x)\,dx}$$

$$= \overline{\hat{f}}(-\xi) \quad \square$$

Definition.

$$L^2(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{C} \;\middle|\; \int |f|^2 dx < +\infty \right\}.$$

$$\|f\|_{L^2} = \left(\int |f|^2 dx \right)^{\frac{1}{2}}$$

$$\langle f, g \rangle = \int f(x) \bar{g}(x)\,dx.$$

Note:

$$\langle \hat{f}, \hat{g} \rangle = \int \hat{f}(\xi)\, \overline{\hat{g}}(\xi)\,d\xi$$

Recall $\int \hat{f} g = \int \hat{g} f$, $f, g \in \mathcal{S}(\mathbb{R}^n)$

$$= \int \hat{f}\; \widehat{\bar{g}}(-\xi)\,d\xi$$

$$= \int \hat{\hat{f}}(x)\, \bar{g}(-x)\,dx$$

Inversion formula

$$\overset{=}{} (2\pi)^n \int f(-x)\, \bar{g}(-x)\, dx$$

$$= \int f(x)\bar{g}(x)\, dx = (2\pi)^n \langle f, g \rangle$$

$\Rightarrow$ Fourier Transform preserves L^2-inner product on $\mathcal{S}(\mathbb{R}^n)$.

(Plancherel)

Corollary. $\|f\|_{L^2} = (2\pi)^{-n} \|\hat{f}\|_{L^2}$.

Exercise: $\mathcal{S}(\mathbb{R}^n)$ is dense in L^2.

Definition, F.T. on L^2 space

Take $f \in L^2(\mathbb{R}^n)$ Let $\{\varphi_n\} \in \mathcal{S}(\mathbb{R}^n)$

$\varphi_n \to f$ in L^2 then $\hat{\varphi}_n$ is Cauchy in L^2 by Plancherel

Define

$$\hat{f} := \lim_{n\to\infty} \hat{\varphi}_n \text{ in } L^2\text{-sense},$$

Exercise: Check independence of $\{\varphi_n\}$.

Proposition. $\widehat{fg}(\xi) = (2\pi)^{-n}\, \hat{f} * \hat{g}(\xi)$

proof: Recall $\widehat{f * g}(\xi) = \hat{f}\, \hat{g}$

take F.T. on LHS:

$$\hat{\hat{fg}} = (2\pi)^{n} f(-x)\, g(-x)$$

take F.T. on RHS:

$$(2\pi)^{-n}\, \widehat{\hat{f} * \hat{g}} = \hat{\hat{f}}(x) \cdot \hat{\hat{g}}(x) \cdot (2\pi)^{-n}$$

$$= (2\pi)^{\cancel{2}n} f(-x)\, g(-x) \cdot \cancel{(2\pi)^{-n}}$$

Hence F.T. of LHS = F.T. of RHS.

$\Longleftrightarrow$ LHS = RHS, by bijectivity of F.T.

$\square$

Summary F.T. on $\mathcal{S}(\mathbb{R}^n)$

a) $\widehat{\partial_j f}(\xi) = i\xi_j \hat{f}(\xi)$

b) $\frac{\partial}{\partial \xi_j}\hat{f}(\xi) = -i\,\widehat{x_j f}(\xi)$

c) $A \in GL(n,\mathbb{R})$. $f_A(x) = f(Ax)$

$\hat{f_A}(\xi) = \frac{1}{|\det A|}\hat{f}((A^{-1})^T\xi)$

d) $a \in \mathbb{R}^n$. $f_a(x) = f(a+x)$

$\hat{f_a}(\xi) = e^{ia\xi}\hat{f}(\xi)$

e) $\int \hat{f} g = \int \hat{g} f$.

f) $f(x) = \int \hat{f}(\xi) e^{ix\cdot\xi} d\xi \cdot \frac{1}{(2\pi)^n}$

g) $\hat{\hat{f}}(x) = \frac{1}{(2\pi)^n} f(-x)$

h) $\overline{\hat{f}}(\xi) = \hat{\bar{f}}(-\xi)$

i) $\widehat{f * g} = \hat{f}\,\hat{g}$

j) $\widehat{fg}(\xi) = (2\pi)^{-n}\,\hat{f} * \hat{g}(\xi)$