Superconducting pairing in lattice systems

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Outline

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Pairing symmetries

Generally, the superconducting order parameter can be expressed as

$$\Delta(r,\theta) =\mid \Delta(\mathbf{r}) \mid e^{i\phi(\theta)}. \tag{1}$$

$\phi(\theta)$ is the phase of the order parameter.
s-wave symmetry: ϕ does not depend on θ
p-wave symmetry: $\phi(\theta+\pi)=\phi(\theta)+\pi$; $\Delta(r,\theta+\pi)=-\Delta(r,\theta)$
d-wave symmetry: $\phi(\theta+\pi/2)=\phi(\theta)+\pi$; $\Delta(r,\theta+\pi/2)=-\Delta(r,\theta)$
f-wave symmetry: $\phi(\theta+\pi/3)=\phi(\theta)+\pi$; $\Delta(r,\theta+\pi/3)=-\Delta(r,\theta)$
g-wave symmetry: $\phi(\theta+\pi/4)=\phi(\theta)+\pi$; $\Delta(r,\theta+\pi/4)=-\Delta(r,\theta)$
s, d, g-pairing symmetries: $\Delta(-\mathbf{r})=\Delta(\mathbf{r})$: even parity
p, f-pairing symmetries: $\Delta(-\mathbf{r})=-\Delta(\mathbf{r})$: odd parity
In a lattice system, the pairing order parameter may be real (imaginary parts do not necessarily exist)

Superconducting pairing in the square lattice

On-site pairing

$$H_\Delta = \sum_{\mathbf{i}} (\Delta_{\mathbf{ii}} c^\dagger_{\mathbf{i}\uparrow} c^\dagger_{\mathbf{i}\downarrow} + h.c.) \tag{2}$$

s-wave pairing: $\Delta_{\mathbf{ii}} = \Delta_0$
Fourier Transform: $c^\dagger_{\mathbf{i}\sigma} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} (e^{i\mathbf{k}\cdot\mathbf{i}} c^\dagger_{\mathbf{k}\sigma})$

$$H_\Delta = \frac{1}{N} \sum_{\mathbf{i}} \sum_{\mathbf{k_1 k_2}} \left[\Delta_0 e^{i(\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{i}} c^\dagger_{\mathbf{k_1}\uparrow} c^\dagger_{\mathbf{k_2}\downarrow} + h.c. \right]$$

$$H_\Delta = \sum_{\mathbf{k}} (\Delta_0 c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + h.c.) \tag{3}$$

Superconducting pairing in the square lattice

NN-site pairing

$$H_\Delta = \sum_{\mathbf{ij}} (\Delta_{\mathbf{ij}} c^\dagger_{\mathbf{i}\uparrow} c^\dagger_{\mathbf{j}\downarrow} + h.c.) \tag{4}$$

Each site $\mathbf{i}$ has four NN sites, with:
$\mathbf{j_1} = \mathbf{i} + \hat{x}$, $\mathbf{j_2} = \mathbf{i} + \hat{y}$, $\mathbf{j_3} = \mathbf{i} - \hat{x}$, $\mathbf{j_4} = \mathbf{i} - \hat{y}$
For the NN-site pairing, the following three pairing symmetries are possible:

1. s-wave pairing: $\Delta_{\mathbf{ij}} \equiv \Delta_0$
2. p-wave pairing: $\Delta_{\mathbf{ij}} = (\Delta_0, i\Delta_0, -\Delta_0, -i\Delta_0)$
3. d-wave pairing: $\Delta_{\mathbf{ij}} = (\Delta_0, -\Delta_0, \Delta_0, -\Delta_0)$

Superconducting pairing in the square lattice

Express the pairing Hamiltonian in the momentum space:

1. s-wave pairing symmetry

Fourier Transform: $c^{\dagger}_{\mathbf{i}\sigma} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} (e^{i\mathbf{k}\cdot\mathbf{i}} c^{\dagger}_{\mathbf{k}\sigma})$

$$H_{\Delta} = \frac{1}{N} \sum_{\mathbf{i}} \sum_{\mathbf{k_1 k_2}} [\Delta_0 e^{i(\mathbf{k_1}\cdot\mathbf{i}+\mathbf{k_2}\cdot\mathbf{i})} (e^{ik_{2x}} + e^{ik_{2y}} + e^{-ik_{2x}} + e^{-ik_{2y}}) c^{\dagger}_{\mathbf{k_1}\uparrow} c^{\dagger}_{\mathbf{k_2}\downarrow} + h.c.] \tag{5}$$

$$H_{\Delta} = \sum_{\mathbf{k}} (\Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + h.c.) \tag{6}$$

$$\Delta_{\mathbf{k}} = 2\Delta_0 (\cos k_x + \cos k_y) \tag{7}$$

Superconducting pairing in the square lattice

2. p-wave pairing symmetry

Fourier Transform: $c^{\dagger}_{\mathbf{i}\sigma} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} (e^{i\mathbf{k}\cdot\mathbf{i}} c^{\dagger}_{\mathbf{k}\sigma})$

$$H_{\Delta} = \frac{1}{N} \sum_{\mathbf{i}} \sum_{\mathbf{k_1 k_2}} [\Delta_0 e^{i(\mathbf{k_1}\cdot\mathbf{i}+\mathbf{k_2}\cdot\mathbf{i})} (e^{ik_{2x}} + ie^{ik_{2y}} - e^{-ik_{2x}} - ie^{-ik_{2y}}) c^{\dagger}_{\mathbf{k_1}\uparrow} c^{\dagger}_{\mathbf{k_2}\downarrow} + h.c.] = \sum_{\mathbf{k}} (\Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + h.c.) \quad (8)$$

$$\Delta_{\mathbf{k}} = -2\Delta_0 (\sin k_y - i \sin k_x) \quad (9)$$

3. d-wave pairing symmetry

Fourier Transform: $c^{\dagger}_{\mathbf{i}\sigma} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} (e^{i\mathbf{k}\cdot\mathbf{i}} c^{\dagger}_{\mathbf{k}\sigma})$

$$H_{\Delta} = \frac{1}{N} \sum_{\mathbf{i}} \sum_{\mathbf{k_1 k_2}} [\Delta_0 e^{i(\mathbf{k_1}\cdot\mathbf{i}+\mathbf{k_2}\cdot\mathbf{i})} (e^{ik_{2x}} - e^{ik_{2y}} + e^{-ik_{2x}} - e^{-ik_{2y}}) c^{\dagger}_{\mathbf{k_1}\uparrow} c^{\dagger}_{\mathbf{k_2}\downarrow} + h.c.] = \sum_{\mathbf{k}} (\Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + h.c.) \quad (10)$$

$$\Delta_{\mathbf{k}} = 2\Delta_0 (\cos k_x - \cos k_y) \quad (11)$$

Superconducting pairing in the square lattice

NNN-site pairing

Each site $\mathbf{i}$ has four NNN sites, with:

$\mathbf{j_1} = \mathbf{i} + \hat{x} + \hat{y}$, $\mathbf{j_2} = \mathbf{i} - \hat{x} + \hat{y}$, $\mathbf{j_3} = \mathbf{i} - \hat{x} - \hat{y}$, $\mathbf{j_4} = \mathbf{i} + \hat{x} - \hat{y}$

Three pairing symmetries

1. s-wave pairing: $\Delta_{\mathbf{ij}} \equiv \Delta_0$
2. p-wave pairing: $\Delta_{\mathbf{ij}} = (\Delta_0, i\Delta_0, -\Delta_0, -i\Delta_0)$
3. d-wave pairing: $\Delta_{\mathbf{ij}} = (\Delta_0, -\Delta_0, \Delta_0, -\Delta_0)$

Superconducting pairing in the square lattice

The pairing function in the momentum space:

1. s-wave pairing symmetry

$$\Delta_{\mathbf{k}} = 2\Delta_0 \left[e^{i(k_x+k_y)} + e^{i(-k_x+k_y)} + e^{i(-k_x-k_y)} + e^{i(k_x-k_y)} \right] \tag{12}$$

$$\Delta_{\mathbf{k}} = 4\Delta_0 \cos k_x \cos k_y \tag{13}$$

2. p-wave pairing symmetry: $\Delta_{\mathbf{k}} = 2\Delta_0 \left[\sin(k_x - k_y) + i \sin(k_x + k_y) \right]$
3. d-wave pairing symmetry: $\Delta_{\mathbf{k}} = -4\Delta_0 \sin k_x \sin k_y$

The g-wave pairing symmetry can be obtained with 5-th neighbor pairing, with:

$$\Delta_{\mathbf{k}} = 2\Delta_0 (\sin 2k_x \sin k_y - \sin 2k_y \sin k_x) \tag{14}$$

Superconducting pairing in the triangle lattice

Each site $\mathbf{i}$ has six NN sites,

For the NN-site pairing, the following pairing symmetries are possible:

1. s-wave pairing: $\Delta_{\mathbf{ij}} \equiv \Delta_0$
2. p-wave pairing:

$\Delta_{\mathbf{ij}} = (\Delta_0, \Delta_0 e^{i\pi/3}, \Delta_0 e^{i2\pi/3}, -\Delta_0, \Delta_0 e^{i4\pi/3}, \Delta_0 e^{i5\pi/3})$

3. d-wave pairing:

$\Delta_{\mathbf{ij}} = (\Delta_0, \Delta_0 e^{i2\pi/3}, \Delta_0 e^{i4\pi/3}, \Delta_0, \Delta_0 e^{i8\pi/3}, \Delta_0 e^{i10\pi/3})$

4. f-wave pairing: $\Delta_{\mathbf{ij}} = (\Delta_0, -\Delta_0, \Delta_0, -\Delta_0, \Delta_0, -\Delta_0)$

For the p-wave and d-wave pairing symmetry, the order parameters are complex. Also written as $p + ip$-wave and $d + id$-wave

Superconducting pairing in the graphene lattice

Each site $\mathbf{i}$ has three NN sites,

For the NN-site pairing, the following pairing symmetries are possible:

1. s-wave pairing: $\Delta_{\mathbf{ij}} \equiv \Delta_0$ and $\Delta_{\mathbf{ji}} \equiv \Delta_{\mathbf{ij}}$
2. p-wave pairing: $\Delta_{\mathbf{ij}} = (\Delta_0, \Delta_0 e^{i2\pi/3}, \Delta_0 e^{i4\pi/3})$ and $\Delta_{\mathbf{ji}} \equiv -\Delta_{\mathbf{ij}}$
3. d-wave pairing: $\Delta_{\mathbf{ij}} = (\Delta_0, \Delta_0 e^{i4\pi/3}, \Delta_0 e^{i8\pi/3})$ and $\Delta_{\mathbf{ji}} \equiv \Delta_{\mathbf{ij}}$
4. f-wave pairing: $\Delta_{\mathbf{ij}} = (\Delta_0, \Delta_0, \Delta_0)$ and $\Delta_{\mathbf{ji}} \equiv -\Delta_{\mathbf{ij}}$

FF state and LO state

Considersing the On-site pairing, with

$$H_\Delta = \sum_{\mathbf{i}} (\Delta_{\mathbf{ii}} c^\dagger_{\mathbf{i}\uparrow} c^\dagger_{\mathbf{i}\downarrow} + h.c.) \tag{15}$$

$\Delta_{\mathbf{ii}}$ depends on the site $\mathbf{i}$, with
1. $\Delta_{\mathbf{ii}} = \Delta_0 e^{i\mathbf{i}\cdot\mathbf{Q}}$ (FF state) 2. $\Delta_{\mathbf{ii}} = \Delta_0 \cos(\mathbf{i}\cdot\mathbf{Q})$ (LO state)
1. FF state: Fourier Transform: $c^\dagger_{\mathbf{i}\sigma} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} (e^{-i\mathbf{k}\cdot\mathbf{i}} c^\dagger_{\mathbf{k}\sigma})$

$$\begin{aligned} H_\Delta &= \tfrac{1}{N} \textstyle\sum_{\mathbf{i}} \sum_{\mathbf{k_1 k_2}} \left[\Delta_0 e^{i(-\mathbf{k_1}-\mathbf{k_2}+\mathbf{Q})\cdot\mathbf{i}} c^\dagger_{\mathbf{k_1}\uparrow} c^\dagger_{\mathbf{k_2}\downarrow} + h.c. \right] \\ &= \textstyle\sum_{\mathbf{k}} (\Delta_0 c^\dagger_{\mathbf{k+Q}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + h.c.) \end{aligned} \tag{16}$$

2: LO state:

$$H_\Delta = \sum_{\mathbf{k}} \left(\frac{\Delta_0}{2} c^\dagger_{\mathbf{k+Q}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + \frac{\Delta_0}{2} c^\dagger_{\mathbf{k-Q}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + h.c. \right) \tag{17}$$

Outline

Mean-field approach

The superconducting pairing terms generally come from an attractive interaction.
The pairing order parameter Δ_0 is a mean-field order parameter.
We consider two operators: A and B
Considering $A - \langle A \rangle$ and $B - \langle B \rangle$ are small.
At the mean-field level, we have,

$$(A - \langle A \rangle)(B - \langle B \rangle) \approx 0 \tag{18}$$

As a result, we have

$$AB \approx \langle A \rangle B + \langle B \rangle A - \langle A \rangle \langle B \rangle \tag{19}$$

Mean-field approach on on-site attractive interaction

We start from an on-site attractive interaction in square lattice, with

$$H_I = -V \sum_{\mathbf{i}} n_{\mathbf{i}\uparrow} n_{\mathbf{i}\downarrow} = -V \sum_{\mathbf{i}} c^{\dagger}_{\mathbf{i}\uparrow} c_{\mathbf{i}\uparrow} c^{\dagger}_{\mathbf{i}\downarrow} c_{\mathbf{i}\downarrow} \tag{20}$$

$$H_I = -V \sum_{\mathbf{i}} c^{\dagger}_{\mathbf{i}\uparrow} c^{\dagger}_{\mathbf{i}\downarrow} c_{\mathbf{i}\downarrow} c_{\mathbf{i}\uparrow} \tag{21}$$

Defining the mean-field order parameter

$$\Delta_{\mathbf{ii}} = V \langle c_{\mathbf{i}\uparrow} c_{\mathbf{i}\downarrow} \rangle ; \Delta^{*}_{\mathbf{ii}} = V \langle c^{\dagger}_{\mathbf{i}\downarrow} c^{\dagger}_{\mathbf{i}\uparrow} \rangle \tag{22}$$

Then the interaction Hamiltonian can be written as the superconducting pairing Hamiltonian

$$H_{\Delta} = \sum_{\mathbf{i}} (\Delta_{\mathbf{ii}} c^{\dagger}_{\mathbf{i}\uparrow} c^{\dagger}_{\mathbf{i}\downarrow} + h.c.) + \sum_{\mathbf{i}} \frac{|\Delta_{\mathbf{ii}}|^2}{V} \tag{23}$$

Mean-field approach on off-site attractive interaction

Considering the $S_z = 0$ pairing channel, the off-site interaction is written as,

$$H_I = -V \sum_{\mathbf{ij}} n_{\mathbf{i}\uparrow} n_{\mathbf{j}\downarrow} = -V \sum_{\mathbf{i}} c^{\dagger}_{\mathbf{i}\uparrow} c_{\mathbf{i}\uparrow} c^{\dagger}_{\mathbf{j}\downarrow} c_{\mathbf{j}\downarrow} \tag{24}$$

To avoid double counting, the term $n_{\mathbf{i}\downarrow} n_{\mathbf{j}\uparrow}$ is not considered.

$$\Delta_{\mathbf{ij}} = V \langle c_{\mathbf{i}\uparrow} c_{\mathbf{j}\downarrow} \rangle ; \Delta^{*}_{\mathbf{ij}} = V \langle c^{\dagger}_{\mathbf{j}\downarrow} c^{\dagger}_{\mathbf{i}\uparrow} \rangle \tag{25}$$

$$H_{\Delta} = \sum_{\mathbf{ij}} (\Delta_{\mathbf{ij}} c^{\dagger}_{\mathbf{i}\uparrow} c^{\dagger}_{\mathbf{j}\downarrow} + h.c.) + \sum_{\mathbf{ij}} \frac{| \Delta_{\mathbf{ij}} |^2}{V} \tag{26}$$

$\Delta_{\mathbf{ij}} = \pm \Delta_{\mathbf{ji}}$, $+$ and $-$ are for the spin single/triplet pairing symmetries, respectively.

Outline

Self-consistent calculation for the order parameters

Considering the interaction V as the input parameter, how to obtain the pairing order parameter?
Generally, there are two methods to calculate the order parameter self-consistently
1. Based on the definition of the order parameters: $\Delta_{\mathbf{ij}} = V\langle c_{\mathbf{j}\downarrow} c_{\mathbf{i}\uparrow}\rangle$
2. Express the free energy as functions of the order parameter, then the oder parameters can be determined through minimizing the free energy

$$\frac{\partial F}{\partial \Delta_{\mathbf{ij}}} = 0; \qquad \frac{\partial F}{\partial \mu} = -\mathcal{N}, \tag{27}$$

where $\mathcal{N}$ is the whole particle number of the system.
The two methods are equivalent. At the first step, one needs to diagonalize the Hamiltonian. In the following we introduce the first method briefly.

Self-consistent calculation for the order parameters

Generally, for a diagonalized Hamiltonian, with

$$H = \sum_i E_i \alpha_i^\dagger \alpha_i \tag{28}$$

The mean-value of the diagonalized term can be obtained as

$$\langle \alpha_i^\dagger \alpha_i \rangle = f(E_i); \qquad \langle \alpha_i \alpha_i^\dagger \rangle = 1 - f(E_i) \tag{29}$$

The mean-value of non-diagonalized term is zero.
The order parameter $\Delta_{\mathbf{ij}} = V\langle c_{\mathbf{i}\uparrow} c_{\mathbf{j}\downarrow} \rangle$ cannot be calculated directly. Firstly, one needs to diagonalize the Hamiltonian. The operators c and the quasiparticle operators α are connected through the eigen-vectors.

Self-consistent calculation for the order parameters

Let us consider a superconducting Hamiltonian, with

$$H = -t \sum_{\langle ij \rangle \sigma} (c^{\dagger}_{\mathbf{i}\sigma} c_{\mathbf{j}\sigma} + h.c.) - \mu \sum_{\mathbf{i}\sigma} c^{\dagger}_{\mathbf{i}\sigma} c_{\mathbf{i}\sigma} + \sum_{\mathbf{ij}} (\Delta_{\mathbf{ij}} c^{\dagger}_{\mathbf{i}\uparrow} c^{\dagger}_{\mathbf{j}\downarrow} + h.c.), \quad (30)$$

with $\Delta_{\mathbf{ij}} = V \langle c_{\mathbf{i}\uparrow} c_{\mathbf{j}\downarrow} \rangle$

The self-consistent calculation can be performed either in the real space or in the momentum space. In the real-space, a rather large matrix needs to be considered. In the momentum space, the order parameter should be uniform, and the pairing symmetry should be assumed at first.

Self-consistent calculation in the real space

Considering a system with N lattice sites, we define a 2N order column wave vector Ψ, with

$$\Psi = (c_{1\uparrow}, c_{2\uparrow}, \cdots, c_{N\uparrow}, c^{\dagger}_{1\downarrow}, c^{\dagger}_{2\downarrow}, \cdots, c^{\dagger}_{N\downarrow})^{\mathrm{T}} \tag{31}$$

The Hamiltonian can be written as the matrix form, with

$$H = \Psi^{\dagger} M \Psi \tag{32}$$

M is a $2N \times 2N$ matrix. Diagonalizing the matrix, with $Q = U^{\dagger} M U$. The Hamiltonian can be rewritten as

$$H = C^{\dagger} Q C = C^{\dagger} U^{\dagger} M U C \tag{33}$$

Q is the diagonalized matrix with the elements E_1, E_2, $\cdots$, E_{2N}

Self-consistent calculation in the real space

We have: $\Psi = UC$.

$$c_{i\uparrow} = \sum_n U_{in} C_n; \qquad c_{j\downarrow} = \sum_n U^*_{j+N,n} C^\dagger_n \tag{34}$$

$$\Delta_{\mathbf{ij}} = V \sum_n U_{in} U^*_{j+N,n} [1 - f(E_n)] \tag{35}$$

The chemical potential is determined by the particle number of the system (N), with

$$N = \sum_{i,n} [\mid U_{in} \mid^2 f(E_n) + \mid U_{i+N,n} \mid^2 (1 - f(E_n))] \tag{36}$$

Self-consistent calculation in the momentum space

For the on-site s-wave pairing symmetry in the square lattice,

$$H = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} (\Delta_0 c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + h.c.), \tag{37}$$

with $\varepsilon_{\mathbf{k}} = -2t(\cos k_x + \cos k_y) - 4t' \cos k_x \cos k_y - \mu$.
The order parameter Δ_0 and the chemical potential μ is determined as,

$$\Delta_0 = \frac{V}{N} \sum_{\mathbf{i}} \langle c_{\mathbf{i}\uparrow} c_{\mathbf{i}\downarrow} \rangle \tag{38}$$

$$\Delta_0 = \frac{V}{N} \sum_{\mathbf{k}} \langle c_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow} \rangle \tag{39}$$

$$\mathcal{N} = \sum_{\mathbf{k}\sigma} \langle c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \rangle \tag{40}$$

Self-consistent calculation in the momentum space

For the extended s-wave symmetry in the square lattice, we have

$$H=\sum_{\mathbf{k}\sigma}\varepsilon_{\mathbf{k}}c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+\sum_{\mathbf{k}}(\Delta_{\mathbf{k}}c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}+h.c.), \tag{41}$$

with $\varepsilon_{\mathbf{k}}=-2t(\cos k_x+\cos k_y)-4t'\cos k_x\cos k_y-\mu$ and
$\Delta_{\mathbf{k}}=2\Delta_0(\cos k_x+\cos k_y)$.
The order parameter Δ_0 and the chemical potential μ is determined as,

$$\Delta_0=\frac{V}{4N}\sum_{\mathbf{i}}(\langle c_{\mathbf{i}\uparrow}c_{\mathbf{i}+\hat{x}\downarrow}+c_{\mathbf{i}\uparrow}c_{\mathbf{i}-\hat{x}\downarrow}+c_{\mathbf{i}\uparrow}c_{\mathbf{i}+\hat{y}\downarrow}+c_{\mathbf{i}\uparrow}c_{\mathbf{i}-\hat{y}\downarrow}\rangle) \tag{42}$$

$$\Delta_0=\frac{V}{2N}\sum_{\mathbf{k}}(\cos k_x+\cos k_y)\langle c_{\mathbf{k}\uparrow}c_{-\mathbf{k}\downarrow}\rangle \tag{43}$$

$$\mathcal{N}=\sum_{\mathbf{k}\sigma}\langle c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}\rangle \tag{44}$$

Self-consistent calculation in the momentum space

For the $d_{x^2-y^2}$-wave symmetry in the square lattice, we have

$$H=\sum_{\mathbf{k}\sigma}\varepsilon_{\mathbf{k}}c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+\sum_{\mathbf{k}}(\Delta_{\mathbf{k}}c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}+h.c.), \tag{45}$$

with $\varepsilon_{\mathbf{k}}=-2t(\cos k_x+\cos k_y)-4t'\cos k_x\cos k_y-\mu$ and
$\Delta_{\mathbf{k}}=2\Delta_0(\cos k_x-\cos k_y)$.
The order parameter Δ_0 and the chemical potential μ is determined as,

$$\Delta_0=\frac{V}{4N}\sum_{\mathbf{i}}(\langle c_{\mathbf{i}\uparrow}c_{\mathbf{i}+\hat{x}\downarrow}+c_{\mathbf{i}\uparrow}c_{\mathbf{i}-\hat{x}\downarrow}-c_{\mathbf{i}\uparrow}c_{\mathbf{i}+\hat{y}\downarrow}-c_{\mathbf{i}\uparrow}c_{\mathbf{i}-\hat{y}\downarrow}\rangle) \tag{46}$$

$$\Delta_0=\frac{V}{2N}\sum_{\mathbf{k}}(\cos k_x-\cos k_y)\langle c_{\mathbf{k}\uparrow}c_{-\mathbf{k}\downarrow}\rangle \tag{47}$$

$$\mathcal{N}=\sum_{\mathbf{k}\sigma}\langle c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}\rangle \tag{48}$$

Self-consistent calculation in the momentum space

For the $p+ip$-wave symmetry in the square lattice, we have

$$H=\sum_{\mathbf{k}\sigma}\varepsilon_{\mathbf{k}}c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+\sum_{\mathbf{k}}(\Delta_{\mathbf{k}}c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}+h.c.), \tag{49}$$

with $\varepsilon_{\mathbf{k}}=-2t(\cos k_x+\cos k_y)-4t'\cos k_x\cos k_y-\mu$ and
$\Delta_{\mathbf{k}}=-2\Delta_0(\sin k_y-i\sin k_x)$.
The order parameter Δ_0 and the chemical potential μ is determined as,

$$\Delta_0=\frac{V}{4N}\sum_{\mathbf{i}}(\langle c_{\mathbf{i}\uparrow}c_{\mathbf{i}+\hat{x}\downarrow}-c_{\mathbf{i}\uparrow}c_{\mathbf{i}-\hat{x}\downarrow}-ic_{\mathbf{i}\uparrow}c_{\mathbf{i}+\hat{y}\downarrow}+ic_{\mathbf{i}\uparrow}c_{\mathbf{i}-\hat{y}\downarrow}\rangle) \tag{50}$$

$$\Delta_0=\frac{V}{2N}\sum_{\mathbf{k}}(\sin k_y+i\sin k_x)\langle c_{\mathbf{k}\uparrow}c_{-\mathbf{k}\downarrow}\rangle \tag{51}$$

$$\mathcal{N}=\sum_{\mathbf{k}\sigma}\langle c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}\rangle \tag{52}$$