Cohesive 2-Topoi

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Contents

Chapter 1

Introduction

In these notes we briefly review cohesive topoi, the geometric structures existing on them and how they model the intrinsic cohomology and its differential refinement. As far as we know, part of the exposition in Chapter 3, specially concerning the internal/external incarnations of the delooping, is new and was conceived trying to make this project as clear as we could.

(...)

Chapter 2

(2,1)-Topoi

In this section we recall the basics of $(2,1)$-topos theory. We start talking about the $(2,1)$-categorical setting, and the $(2,1)$-topoi properly. In sequence we talk about internal groups and internal groupoids, discussing that in a $(2,1)$-topos every internal groupoid is effective. Finally, we consider other flavors of algebraic objects in $(2,1)$-topoi in the sense of algebras over $(2,1)$-operads.

2.1 (2,1)-Context

For a given $k \geq 0$, let $(n,r)\mathbf{Cat}_{k+1}$ be the $(k+1)$-category of (n,r)-categories, with $n,r \leq k$. If $n \leq r$, write simply $n\mathbf{Cat}_{k+1}$. Furthermore, if $r = 0$, $(n,0)\mathbf{Cat}_{k+1}$ agree with the $k+1$-category of n-groupoids. $n\mathbf{Grpd}_{k+1}$. If $n = 0$, then $(0,r)\mathbf{Cat}_{k+1}$ is equivalent to $\mathbf{Set}_{k+1}$, i.e., the category of sets trivially regarded as a $k+1$-category. For every k,n,r we have $(k+1)$-embeddings $\mathrm{disc}^k_{n,n+1} : (n,r)\mathbf{Cat}_{k+1} \hookrightarrow (n+1,r)\mathbf{Cat}_{k+1}$, that regard a (n,r)-category as a discrete $(n+1,r)$-category, and $(n,r)\mathbf{Cat}_{k+1} \hookrightarrow (n,r+1)\mathbf{Cat}_{k+1}$. The former has a left-adjoint $\tau^k_{\leq n-1} : (n,r)\mathbf{Cat}_{k+1} \hookrightarrow (n-1,r)\mathbf{Cat}_{k+1}$ given by truncation at $(n-1)$-morphisms, while the latter has a right-adjoint given by the core construction [??,??].

$$
\begin{tikzcd}
\mathbf{Set}_2 & \mathbf{Grpd}_2 & 2\mathbf{Grpd}_2 \\
(0,0)\mathbf{Cat}_2 \arrow[u, "\simeq"] \arrow[r, shift left, hook, "\mathrm{disc}_{0,1}"] \arrow[d, hook, "\simeq"'] & (1,0)\mathbf{Cat}_2 \arrow[u, "\simeq"'] \arrow[l, shift left, "\tau_{\leq 0}"] \arrow[r, shift left, hook, "\mathrm{disc}_{1,2}"] \arrow[d, hook] & (2,0)\mathbf{Cat}_2 \arrow[u, "\simeq"] \arrow[l, shift left, "\tau_{\leq 1}"] \arrow[d, hook] \\
(0,1)\mathbf{Cat}_2 \arrow[r, shift left, hook, "\mathrm{disc}_{0,1}"] \arrow[d, hook] & (1,1)\mathbf{Cat}_2 \arrow[l, shift left, "\tau_{\leq 0}"] \arrow[r, shift left, hook, "\mathrm{disc}_{1,2}"] \arrow[d, hook, "\simeq"'] & (2,1)\mathbf{Cat}_2 \arrow[l, shift left, "\tau_{\leq 1}"] \arrow[d, hook] \\
(0,2)\mathbf{Cat}_2 \arrow[r, shift left, hook, "\mathrm{disc}_{0,1}"] \arrow[d] & (1,2)\mathbf{Cat}_2 \arrow[l, shift left, "\tau_{\leq 0}"] \arrow[r, shift left, hook, "\mathrm{disc}_{1,2}"] \arrow[d, "\simeq"'] & (2,2)\mathbf{Cat}_2 \arrow[l, shift left, "\tau_{\leq 1}"] \arrow[d, "\simeq"'] \\
\mathbf{Set}_2 & \mathbf{Cat}_2 & 2\mathbf{Cat}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{Grpd}_2 & 2\mathbf{Grpd}_2 \arrow[d, hook] \arrow[dll, shift left, "\tau_{\leq 1}"] & \\
\mathbf{Set}_2 \arrow[urr, shift left, hook, "\mathrm{disc}_{0,1}"] \arrow[urr, shift right, hook, "\mathrm{disc}_{1,2}"'] \arrow[rrr] \arrow[drr, shift right, hook, "\mathrm{disc}_{0,1}"'] & & & (2,1)\mathbf{Cat}_2 \arrow[d, hook] \\
 & & \mathbf{Cat}_2 \arrow[ull, shift right, "\tau_{\leq 0}"'] \arrow[r, shift left, hook, "\mathrm{disc}_{1,2}"] & 2\mathbf{Cat}_2 \arrow[l, shift left, "\tau_{\leq 1}"]
\end{tikzcd}
\tag{2.1}
$$

Since we will focus on generalize gauge theories, in the following we will be interested in the $k = 1$ case, so that in the 2-categorical setting, with an special emphases to the $(2,1)$-setting. Higher values of k would be important in the development of generalizations for higher gauge theory, which are outside of the scope of this initial project. The diagram above summarize the previous adjunctions in the $k = 1$ case.

Remark 2.1.1. *In the following, if* $\mathbf{C}$ *is a 1-category or a 1-groupoid, we will denote its image under* $\mathrm{disc}_{1,2}$ *by* $\mathbf{C}_2$*. Notice this agree with the notation used above:* $\mathbf{Set}_2$*,* $\mathbf{Cat}_2$ *and* $\mathbf{Grpd}_2$ *are precisely the images of the 1-categories* $\mathbf{Set}$ *of sets,* $\mathbf{Cat}$ *of 1-categories and* $\mathbf{Grpd}$ *of 1-groupoids*

Remark 2.1.2. *Above we did not said what we consider as a "n-category". About this, some comments.*

1. *There is the "strict" setting, the "weak setting" and a lot of semi-strict intermediary settings [??]. There are also "strictfication functors" that connect those settings and commutes with truncation [??].*

2. *In the strict setting,* (n, r)*-category theory can be formalized via induction on* $\mathbf{Cat}$*-enriched category theory, followed by truncation [??].*

3. *For* $n = 2$*, there is a natural presentation of non-strict* 2*-categories by bicategories. Furthermore, there is a strictfication theory such that every bicategory is equivalent to a strict* 2*-category, so that non-strict* $(2, 1)$*-categories are always equivalent to strict* $(2, 1)$*-categories*[1] *[??].*

4. *For other* (n, r) *there is fully developed and globally accepted "*(n, r)*-category theory", specially for large* n, r*. Furthermore, for* $n > 2$ *there is no global strictification theorems (non-strict entities are only equivalent to certain semi-strict objects) [??].*

5. *The non-strict* $(\infty, 1)$*-case is closely related to homotopy theory and combinatorial presentations for* $(\infty, 1)$*-category theory have been extensively developed by Lurie and others [??,??]. Thus, presentations to non-strict* $(2, 1)$*-category theory could be obtained truncating the combinatorial presentations of* $(\infty, 1)$*-category theory.*

Remark 2.1.3. *In the strict setting, for every* k *and every* $n, m, r \leq k$ *there are also forgetful* $k+1$*-functors* $u^k_{n,n-m} : (n, r)\mathbf{Cat}_{k+1} \to (n - m, r)\mathbf{Cat}_{k+1}$ *that forgets* i*-morphisms for* $i \leq n \leq n - m$*. This* $k + 1$*-functor is not well-defined in the non-strict setting because compositions of* $n-1$*-morphisms are defined only up to* n*-morphisms, so that one cannot simply forget higher order data. An exception is for* $n = m$*, where* $u^k_{m,0} : (n, r)\mathbf{Cat}_{k+1} \to \mathbf{Set}_{k+1}$ *takes a* (n, r)*-category* $\mathbf{C}$ *and gives its collection of objects* $\mathrm{Ob}(\mathbf{C})$*. There is an obvious* $(k + 1)$*-morphism* $\pi : u^k_{m,0} \Rightarrow \tau_{\leq 0}$*: for every* $\mathbf{C}$*, it is just the projection* $\pi : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Iso}(\mathbf{C})$*.*

2.2 (2,1)-Topos

Recall that if $\mathbf{C}$ is a 1-category and $\mathbf{A}$ is a (2,1)-category we can form the $(2, 1)$-category $(2, 1)\,\mathrm{PShv}_{\mathbf{A}}(\mathbf{C})$ of 2-functors $F : \mathbf{C}^{op}_2 \to \mathbf{A}$. This actually produces a 2-functor $(2, 1)\,\mathrm{PShv}_{\mathbf{A}}(\cdot) : \mathbf{Cat}_2 \to (2, 1)\mathbf{Cat}_2$, which takes values into $(n, r)\mathbf{Cat}_2 \subset 2\mathbf{Cat}_2$ if $\mathbf{A}$ is a (n, r)-category for $0 \leq n, r < 2$. We say that $(2, 1)\,\mathrm{PShv}_{\mathbf{A}}(\mathbf{C})$ is the $(2, 1)$-category of $\mathbf{A}$-*valued* $(2, 1)$-*presheaves on* $\mathbf{C}$. In the following we will focus on the case $\mathbf{A} = \mathbf{Grpd}_2$.

We say that a $(2, 1)$-category $\mathbf{H}$ is a $(2, 1)$-*presheaf* $(2, 1)$-*category* if it is 2-equivalent to

[1] This, of course, does not means that one can forget non-strict 2-categories and work only with strict 2-categories. Indeed, if a construction/proof depends on a 2-category one can restrict the work to strict 2-categories only if it is invariant under 2-equivalences.

$(2,1)\,\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$ for some 1-category $\mathbf{C}$[2]. A *geometric morphism* $f : \mathbf{H} \to \mathbf{H}'$ between two $(2,1)$-presheaf (2,1)-categories is given by 2-functors $f_* : \mathbf{H} \to \mathbf{H}'$ and $f^* : \mathbf{H}' \to \mathbf{H}$ such that f^* preserve finite $(2,1)$-limits and is a left-adjoint to f_* (first diagram below). A *geometric transformation* $\xi : f \Rightarrow g$ between geometric morphisms $f, g : \mathbf{H} \to \mathbf{H}'$ is a pair (ξ_*, ξ^*) of 2-natural transformations $\xi_* : f_* \Rightarrow g_*$ and $\xi^* : f^* \Rightarrow g^*$ (second diagram below). We have the 2-category $(2,1)$**PShvTopos** of (2,1)-presheaf (2,1)-categories, geometric morphisms and geometric transformations [??].

$$\begin{tikzcd} \mathbf{H} \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathbf{H}' \arrow[l, bend right, "f^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{H} \arrow[r, bend right=50, "f_*"'] \arrow[r, bend right=10, "g_*"] & \mathbf{H}' \arrow[l, bend right=50, "g^*"'] \arrow[l, bend right=10, "g^*"] \end{tikzcd}$$

The 2-category $(2,1)$**PShvTopos** is a full sub-2-category of the 2-category $(2,1)$**PTopos** of (infinitary) $(2,1)$-*pretopoi*, i.e., $(2,1)$-categories which are 2-exact and (infinitary) 2-extensive [??]. It has, in turn, a full sub-2-category $(2,1)$**ShvTopos** of *Grothendieck* $(2,1)$-*topoi*, which are the $(2,1)$-categories 2-equivalent to sub-2-categories $\mathbf{H}_0 \subset \mathbf{H}$ of $(2,1)$-presheaf $(2,1)$-categories such that:

1. *are reflective*, so the inclusion 2-functor $\imath : \mathbf{H}_0 \hookrightarrow \mathbf{H}$ has a left-adjoint $L_0 : \mathbf{H} \to \mathbf{H}_0$;
2. *are exact*, so the left-adjoint L_0 preserves finite 2-limits.

Remark 2.2.1. *Of course, neither* $(2,1)$**PTopos** *nor its full sub-2-categories* $(2,1)$**PShvTopos** *and* $(2,1)$**ShvTopos** *are full sub-2-categories of* $(2,1)\mathbf{Cat}_2$. *There is, however, an identity on objects forgetful 2-functor* $U_{topos} : (2,1)\mathbf{PTopos} \to (2,1)\mathbf{Cat}_2$ *that takes a geometric morphism* (f_*, f^*) *and forgets* f^*.

Due 2-Giraud Theorem, the $(2,1)$**ShvTopos** is 2-equivalent to the 2-category $\mathbf{Stk}_2$ of *stack* $(2,1)$-*categories* [??]. These are $(2,1)$-presheaf $(2,1)$-categories $\mathbf{H} \simeq (2,1)\,\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$ such that $\mathbf{C}$ is endowed with a coverage J, such that $(\mathbf{C}, J)$ is a suitable site[3], being denoted by $\mathrm{Stk}(\mathbf{C}, J)$. The 2-equivalence is given by the assignment that takes $(2,1)\,\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C}$ and J, and gives the localization at the class W_J of local J-isomorphisms, i.e., identifying

$$\mathrm{Stk}(\mathbf{C}, J) \simeq (2,1)\,\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})[W_J^{-1}]. \tag{2.2}$$

Let $\mathbf{Site}_2$ be the discrete 2-category of sites and morphisms of sites. It is not a full sub-2-category of $\mathbf{Cat}_2$, but there is a forgetful functor $U_{site} : \mathbf{Site}_2 \to \mathbf{Cat}_2$ that takes a site $(\mathbf{C}, J)$ and forgets the coverage J. When working with general coverage, the 1-category $\mathbf{C}$ is not supposed to have pullback relatively to the coverings families. However, the sites appearing in (2.2) depends on pullbacks to build the localization at W_J. So, let $\mathbf{Cat}_{2,pb} \subset \mathbf{Cat}_2$ the 2-category of 1-categories with pullbacks and let $\mathbf{Site}_{2,pb} \subset \mathbf{Site}_2$ the 2-category of sites that factors through $\mathbf{Cat}_{2,pb}$, as below.

$$\begin{tikzcd} \mathbf{Cat}_{2,pb} \arrow[r, hook] & \mathbf{Cat}_2 \\ \mathbf{Site}_{2,pb} \arrow[r, hook] \arrow[u, dashed, "U_{site}"] & \mathbf{Site}_2 \arrow[u, "U_{site}"'] \end{tikzcd}$$

[2]One could considered here the more general context where $\mathbf{C}$ is a (2,1)-category, but for natural gauge theories the 1-categorical context is enough.

[3]More generally, $(\mathbf{C}, J)$ could be a $(2,1)$-site.

Restricting to it, the rule $(\mathbf{C}, J) \mapsto \mathrm{Stk}(\mathbf{C}, J)$ is not only well-defined but functorial, producing a 2-functor $\mathrm{Stk} : \mathbf{Cat}_{2,pb} \to \mathbf{Stk}_2$, and the localization at W_J extends to a natural transformation $L : \mathrm{PShv}_{\mathbf{Grpd}_2} \circ U_{site} \Rightarrow \mathrm{Stk}$, as below.

$$\begin{tikzcd}
(2,1)\mathbf{Cat}_2 & (2,1)\mathbf{PShvTopos}_2 \arrow[l, hook'] \arrow[dl, phantom, "\Leftarrow L"] & \\
\mathbf{Stk}_2 \arrow[u, hook] & \mathbf{Cat}_{2,pb} \arrow[r, hook] & \mathbf{Cat}_2 \arrow[ul, "2\,\mathrm{PShv}_{\mathbf{Grpd}_2}(-)"'] \\
& \mathbf{Site}_{2,pb} \arrow[r, hook] \arrow[u, "U_{site}"] \arrow[ul, bend left, "\mathrm{Stk}"] & \mathbf{Site}_2 \arrow[u, "U_{site}"']
\end{tikzcd} \tag{2.3}$$

Finally, we recall that there are many 2-functors $\mathrm{Site} : \mathbf{Cat}_{2,pb} \to \mathbf{Site}_{2,pb}$ that can be characterized by demanding additional properties on the corresponding localization transformation L_{Site}

$$\begin{tikzcd}
(2,1)\mathbf{Cat}_2 & (2,1)\mathbf{PShvTopos}_2 \arrow[l, hook'] \arrow[dl, phantom, "\Leftarrow L"] & \\
\mathbf{Stk}_2 \arrow[u, hook] & \mathbf{Cat}_{2,pb} \arrow[r, hook] \arrow[d, "\mathrm{Site}"'] & \mathbf{Cat}_2 \arrow[ul, "2\,\mathrm{PShv}_{\mathbf{Grpd}_2}(-)"'] \\
& \mathbf{Site}_{2,pb} \arrow[r, hook] \arrow[ul, bend left, "\mathrm{Stk}"] & \mathbf{Site}_2 \arrow[u, "U_{site}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Cat}_{2,pb} \arrow[r, bend left, "2\,\mathrm{PShv}_{\mathbf{Grpd}_2}(-)"] \arrow[r, bend right, "\mathrm{Stk}\circ\mathrm{Site}"'] \arrow[r, phantom, "\Downarrow L_{\mathrm{Site}}"] & (2,1)\mathbf{Cat}_2
\end{tikzcd}$$

For instance, we can put in every 1-category $\mathbf{C}$ with pullbacks a trivial coverage $\mathrm{Site}_0(\mathbf{C})$, characterized by being the universal coverage such that the localization becomes a natural isomorphisms, i.e., such that every (2,1)-presheaf (2,1)-topos defined by a trivial coverage is actually a Grothendieck (2,1)-topos in an universal way [??].

Another class of examples are the *subcanonical coverage*, $\mathrm{Site}_{sub}(\mathbf{C})$ such that the localization becomes an isomorphism in the class of representable 2-presheaves that are represented by elements of the coverage.

Explicitly, a trivial coverage is one formed only by identities. Similarly, if a category $\mathbf{C}$ has coproducts, a coverage J in $\mathbf{C}$ is subcanonical iff for every covering family $\{U_i \to X\}_i$ in J the following diagram (arising from universality of pullbacks) is a colimit.

$$\textstyle\bigsqcup_{i,j} U_i \cap U_j \rightrightarrows \bigsqcup_i U_i \longrightarrow X$$

Remark 2.2.2. *Recall that in a (2,1)-category $\mathbf{H}$ with (2,1)-pullbacks, a morphism $f : U \to X$ is an effective epimorphism if the corresponding diagram below, with arrows obtained by universality of (2,1)-pullbacks, is a (2,1)-colimit.*

$$U \times_X U \rightrightarrows U \longrightarrow X$$

If $\mathbf{C}$ is a cartesian closed 1-category with pullbacks and coproducts, then its pullbacks commutes with coproducts. Thus, if $\{U_i \to X\}_i$ is a covering family for a coverage J in $\mathbf{C}$, taking $U = \sqcup_i U_i$, we see that J is subcanonical iff $U \to X$ are effective epimorphisms.

2.3 Internal Groupoids

Let $\mathbf{Cat}_{2,c} \subset \mathbf{Cat}_2$ be the sub-2-category of cartesian 1-categories (i.e., that have binary products and terminal objects) with cartesian functors (i.e, that preserves binary products and terminal objects), and let $\mathbf{Cat}_{2,cp} = \mathbf{Cat}_{2,c} \cap \mathbf{Cat}_{2,pb}$ the full sub-2-category of those that also have pullbacks with functors that also preserve pullback.

Recall that to every $\mathbf{C} \in \mathbf{Cat}_{2,c}$ we have an internal notion of *group object*. These are objects $G \in \mathbf{C}$ endowed with morphisms $m : X \times X \to X$, $id : * \to X$ and $inv : X \to X$ satisfying commutative diagrams describing associativity, neutral element property and invertibility. There is an obvious notion of morphism between group objects defining a sub-1-category $\mathrm{Grp}(\mathbf{C}) \subset \mathbf{C}$ [??]. Let $\mathrm{Grp}_2(\mathbf{C})$ corresponding discrete $(2,1)$-category. Cartesian functors clearly factors through the groupoid object categories, so we have a 2-functor $\mathrm{Grp}_2 : \mathbf{Cat}_{2,c} \to (2,1)\mathbf{Cat}_2$.

Similarly, recall that in every $\mathbf{C} \in \mathbf{Cat}_{2,cp}$ we have a notion of *groupoid object* (or *internal groupoid*) as being given by a pair of objects G_0, G_1 and morphisms $s, t : G_1 \to G_0$, $\circ : \mathrm{pb}(s,t) \to G_1$, $id : * \to G_1$ and $inv : G_1 \to G_1$ satisfying similar commutativity and coherence conditions. We have corresponding notions of *internal functor* between internal groupoids and *internal natural transformation* between internal functors, defining a $(2,1)$-category $\mathrm{Grpd}_2(\mathbf{C})$. Each object $X \in \mathbf{C}$ defines an internal groupoid $\mathbf{E}X$ such that $\mathbf{E}X_0 = X$ and $\mathbf{E}_1 = *$. Thus, $\mathrm{disc}_{1,2}(\mathbf{C}) \subset \mathrm{Grpd}_2(\mathbf{C})$ meaning that for every $\mathbf{C}$ we have a sequence of inclusions

$$\mathrm{Grp}_2(\mathbf{C}) \hookrightarrow \mathrm{disc}_{1,2}(\mathbf{C}) \hookrightarrow \mathrm{Grpd}_2(\mathbf{C}).$$

Actually, taking internal groupoids also defines a 2-functor $\mathrm{Grpd}_2 : \mathbf{Cat}_{2,cp} \to (2,1)\mathbf{Cat}_2$, so we have a sequence of transformations

$$\mathrm{Grp}_2 \Longrightarrow \mathrm{disc}_{1,2} \Longrightarrow \mathrm{Grpd}_2 .$$

Let $\mathbf{Cat}_{2,ccp} \subset \mathbf{Cat}_{2,cp}$ be the sub-2-category of those 1-categories that, in addition, have coequalizers with functors preserving them. Fixed $\mathbf{C} \in \mathbf{Cat}_{2,ccp}$ to every groupoid object $\mathbf{G} \in \mathrm{Grpd}_2(\mathbf{C})$ one associate a morphism $\mathrm{coeq}_{\mathbf{C}}(G)$ obtained taking the coequalizer of the diagram defined by $\mathbf{G}$, as below.

The construction extends to internal functors (and trivially to internal natural transformations), defining a 2-functor $\mathrm{coeq}_{\mathbf{C}} : \mathrm{Grpd}_2(\mathbf{C}) \to \mathrm{Arr}_{2,ccp}(\mathbf{C})$, where $\mathrm{Arr}_{2,ccp}(\mathbf{C})$ is the arrow $(2,1)$-category. Varying $\mathbf{C} \in \mathbf{Cat}_{2,ccp}$ we find a natural transformation coeq as in the (noncommutative) diagram below, where the lower index $(\cdot)_{2,ccp}$ means restriction to $\mathbf{Cat}_{2,ccp}$. Furthermore, t is the target natural transformation.

$$\begin{tikzcd}
& & \mathrm{Arrow}_{2,ccp} \arrow[dl, Rightarrow, "t"'] \\
\mathrm{Grp}_{2,ccp} \arrow[r, Rightarrow] & \mathrm{disc}_{1,2,ccp} \arrow[r, Rightarrow] & \mathrm{Grpd}_{2,ccp} \arrow[u, Rightarrow, "\mathrm{coeq}"]
\end{tikzcd}$$

The universal morphism defined by a coequalizer is always an effective epimorphism. Thus, the natural transformation coeq factors objectwise to the sub-2-category $\mathrm{Eff}_{2,ccp}(\mathbf{C}) \subset$

$\mathrm{Arrow}_{2,ccp}(\mathbf{C})$ of effective morphisms.

$$\begin{tikzcd}
 & & \mathrm{Arrow}_{2,ccp} \arrow[dl, Rightarrow, "t"'] & \mathrm{Eff}_{2,ccp} \arrow[l, Rightarrow] \\
\mathrm{Grp}_{2,ccp} \arrow[r, Rightarrow] & \mathrm{disc}_{1,2,ccp} \arrow[r, Rightarrow] & \mathrm{Grpd}_{2,ccp} \arrow[u, Rightarrow, "\mathrm{coeq}"] \arrow[ur, Rightarrow, dashed, "\mathrm{coeq}"] &
\end{tikzcd}$$

For an internal groupoid $\mathbf{G} \in \mathrm{Grpd}_{2,ccp}(\mathbf{C})$ we say that $t(\mathrm{coeq}(\mathbf{G}))$ is its *quotient object* (or *representing object*). Of special interest are the internal groupoids which are "generated" by their representing object. These are the so-called *effective internal groupoids*, defined as follows.

Given a category $f : \mathbf{C} \in \mathbf{Cat}_{2,cp}$, to every morphism $U \to X$ in $\mathbf{C}$ we associate an internal groupoid $\check{C}_{\mathbf{C}}(f) \in \mathrm{Grpd}_2(\mathbf{C})$, its *Čech groupoid*, such that $\check{C}_{\mathbf{C}}(f)_0 = U$, $\check{C}(f)_1 = U \times_X U = \mathrm{pb}(f, f)$ and the structural morphisms obtained by universality. In other words, $\check{C}_{\mathbf{C}}(f)$ is the internal groupoid whose defining diagram is

$$\begin{tikzcd}
U \times_X U \arrow[r, "\hat{f}"] \arrow[d, "\hat{f}"'] & U \arrow[d, "f"] \\
U \arrow[r, "f"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
U \times_X U \arrow[r, shift left, "\hat{f}"] \arrow[r, shift right, "\hat{f}"'] & U.
\end{tikzcd}$$

The construction also extends to a functor $\check{C}_{\mathbf{C}} : \mathrm{Arrow}_{2,cp}(\mathbf{C}) \to \mathrm{Grpd}_2(\mathbf{C})$ and, varying $\mathbf{C} \in \mathbf{Cat}_{2,cp}$, to a natural transformation $\check{C} : \mathrm{Arrow}_{2,cp} \Rightarrow \mathrm{Grpd}_2$. Notice that, by definition, an effective morphism can be characterized as a morphism f such that $\mathrm{coeq}(\check{C}(f)) = f$. Thus, we have $\mathrm{coeq} \circ \check{C} = id$ when restricted to effective morphisms.

$$\begin{tikzcd}
 & & \mathrm{Arrow}_{2,ccp} & \mathrm{Eff}_{2,ccp} \arrow[l, Rightarrow] \\
\mathrm{Grp}_{2,ccp} \arrow[r, Rightarrow] & \mathrm{disc}_{1,2,ccp} \arrow[r, Rightarrow] & \mathrm{Grpd}_{2,ccp} \arrow[u, Rightarrow, "\mathrm{coeq}"] \arrow[ur, Rightarrow, "\mathrm{coeq}"] & \\
 & & \mathrm{Arrow}_{2,ccp} \arrow[u, Rightarrow, "\check{C}"] & \mathrm{Eff}_{2,ccp} \arrow[l, Rightarrow] \arrow[ul, Rightarrow, "\check{C}"] \arrow[uu, Rightarrow, "id"']
\end{tikzcd}$$

The opposite side composition $\check{C} \circ \mathrm{coeq}$ takes the quotient morphism of a groupoid and forms its Čech groupoid. The effective internal groupoids are precisely the internal groupoids for which such composition is the identity and, therefore, are equivalent to effective morphisms. For $\mathbf{C} \in \mathbf{Cat}_{2,ccp}$, let $\mathrm{Grpd}^{eff}_{2,ccp}(\mathbf{C}) \subset \mathrm{Grpd}_{2,ccp}(\mathbf{C})$ the sub-2-category of those effective internal groupoids.

The discussion naturally extends from $\mathbf{Cat}_{2,ccp}$ to $(2,1)\mathbf{Cat}_{2,ccp}$ with limits and colimits replaced by the corresponding $(2,1)$-limits and $(2,1)$-colimits. For presentations in the context of bicategories, see [??,??,??].

The first condition in the statement of 2-truncation of Giraud-Rezk-Lurie theorem says that every Grothendieck $(2,1)$-topos is locally presentable and, therefore, has all $(2,1)$-limits and $(2,1)$-colimits. Thus, $(2,1)\mathbf{ShvTopos}_2 \subset (2,1)\mathbf{Cat}_{2,ccp}$, so we can talk about internal groupoids in every Grothendieck $(2,1)$-topos $\mathbf{H}$. In turn, the fourth condition says that internal groupoids are always effective, so $\mathrm{Grpd}^{eff}_2(\mathbf{H}) \simeq \mathrm{Grpd}_2(\mathbf{H})$ for every Grothendieck $(2,1)$-topos $\mathbf{H}$.

2.4 Monoidal $(2,1)$-Categories

Recall that a monoidal $(2,1)$-category is a $(2,1)$-category $\mathbf{H}$ with a 2-bifunctor $\otimes : \mathbf{H}\times\mathbf{H} \to \mathbf{H}$ and a functor $1 : *_2 \to \mathbf{H}$ that satisfy associativity and neutral element properties up to 2-morphisms. Formally, this is the same thing as a monoidal 2-category which is also a 2-truncated $(\infty, 1)$-monoidal category in the sense of [??] and [??], respectively. A monoidal $(2,1)$-category is said to be *symmetric* if commutativity conditions up to 2-isomorphisms are satisfied too.

Let $(2,1)\mathbf{Mon}_2 \subset (2,1)\mathbf{Cat}_2$ the sub-2-category of monoidal $(2,1)$-categories with monoidal 2-functors and monoidal natural transformations, and let $(2,1)\mathbf{SymMon}_2 \subset (2,1)\mathbf{Mon}_2$ be the sub-2-category of the symmetric ones.

Example 2.4.1. *To us, the fundamental example of symmetric monoidal $(2,1)$-categories are the cartesian $(2,1)$-categories with the bifunctor $\otimes$ given by the categorical biproduct. We say that they are the cartesian monoidal $(2,1)$-categories. In particular, every Grothendieck $(2,1)$-topos has a canonical symmetric monoidal structure: the cartesian one. More precisely, we have an inclusion 2-functor $(2,1)\mathbf{Cat}_{c,2} \hookrightarrow (2,1)\mathbf{SymMon}_2$.*

In every monoidal $(2,1)$-category $(\mathbf{H}, \otimes, 1)$ we have a sub-2-category $\mathrm{Mon}_2(\mathbf{H}) \subset \mathbf{H}$ of the *monoid objects*. These are objects $X \in \mathbf{H}$ equipped with morphisms $* : X \otimes X \to X$ and $e : 1 \to X$ satisfying analogous associativity and neutral element properties up to 2-morphisms. The construction is 2-functorial in $\mathbf{H}$, defining a 2-functor $\mathrm{Mon} : (2,1)\mathbf{Mon}_2 \to (2,1)\mathbf{Cat}_2$.

If the monoidal $(2,1)$-category $(\mathbf{H}, \otimes, 1)$ happens to be symmetric, we also have additional sub-2-categories $\mathrm{BrMon}_2(\mathbf{H}), \mathrm{ComMon}_2(\mathbf{H}) \subset \mathrm{Mon}(\mathbf{H})$ of *braided monoid objects* and *commutative/abelian monoid objects*, respectively. Being braided means that certain "weaker" commutativity conditions are satisfied. In a commutative monoid objects the analogous commutativity conditions are satisfied. In a symmetric monoidal $(2,1)$-category we then have the following sequence of sub-2-categories.

$$\mathrm{ComMon}(\mathbf{H}) \hookrightarrow \mathrm{BrMon}(\mathbf{H}) \hookrightarrow \mathrm{Mon}(\mathbf{H}) \hookrightarrow \mathbf{H}$$

If $\mathbf{H}$ is a cartesian monoidal $(2,1)$-category, then there is a forgetful 2-functor $U : \mathrm{Grp}_2(\mathbf{H}) \to \mathrm{Mon}_2(\mathbf{H})$ that forgets the inverse maps. We then have the following diagram, where $\mathrm{BrGrp}_2(\mathbf{H})$ and $\mathrm{ComGrp}_2(\mathbf{H})$ are defined to be the universal sub-2-categories making the diagram commutative.

$$\begin{tikzcd}
\mathrm{ComGrp}_2(\mathbf{H}) \arrow[r, hook] \arrow[d, hook] & \mathrm{BrGrp}_2(\mathbf{H}) \arrow[r, hook] \arrow[d, hook] & \mathrm{Grp}_2(\mathbf{H}) \arrow[r, hook] \arrow[d, hook] & \mathbf{H} \arrow[r, hook] \arrow[d, equal] & \mathrm{Grpd}_2(\mathbf{H}) \\
\mathrm{ComMon}_2(\mathbf{H}) \arrow[r, hook] & \mathrm{BrMon}_2(\mathbf{H}) \arrow[r, hook] & \mathrm{Mon}_2(\mathbf{H}) \arrow[r, hook] & \mathbf{H}
\end{tikzcd}$$

The construction in functorial $\mathbf{H}$, so that we have 2-functors $\mathrm{Mon}_2, \mathrm{BrMon}_2, \mathrm{ComMon}_2 : (2,1)\mathbf{SymMon}_2 \to (2,1)\mathbf{Cat}_2$ and $\mathrm{BrGrp}_2, \mathrm{ComGrp}_2 : \mathbf{Cat}_{c,2} \to \mathbf{Cat}_2$ and the last diagrams extends to a diagram of natural transformations:

$$\begin{tikzcd}
\mathrm{ComGrp}_2 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{BrGrp}_2 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{Grp}_2 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & id \arrow[r, Rightarrow] \arrow[d, equal] & \mathrm{Grpd}_2 \\
\mathrm{ComMon}_2 \arrow[r, Rightarrow] & \mathrm{BrMon}_2 \arrow[r, Rightarrow] & \mathrm{Mon}_2 \arrow[r, Rightarrow] & id
\end{tikzcd}$$

2.5 Algebras Over $(2,1)$-Operads

In a symmetric monoidal $(2,1)$-category we can consider not only monoid objects, braided monoid objects and commutative monoid objects (and their groupal counterparts), but actually objects endowed with a lot of different kinds of algebraic structure. This is formalized through the notion of *algebra over a* $(2,1)$*-operad*, as we will briefly review.

The definition of $(2,1)$-operad properly does not matter to the following digression. Just recall that there is a 2-category $(2,1)\mathbf{Oper}_2$ of them. Furthermore, it has a sub-2-category $(2,1)\mathbf{SymOper}_2 \subset (2,1)\mathbf{Oper}_2$ of *symmetric* $(2,1)$*-operads* that comes equipped with a monoidal structure $\odot$ given by Boardman–Vogt tensor product [??,??].

The terminal object of $(2,1)\mathbf{SymOper}_2 \subset (2,1)\mathbf{Oper}_2$ is the $\mathbb{E}_\infty$-operad, whose truncation in k-ary operations is the $\mathbb{E}_k$-operad. For every $k \geq 0$ there is a canonical morphism $\mathbb{E}_k \to \mathbb{E}_{k+1}$, meaning that we have the following sequence of morphisms, and we can recover $\mathbb{E}_\infty$ as its direct limit (a $(2,1)$-colimit).

$$\mathbb{E}_0 \longrightarrow \mathbb{E}_1 \longrightarrow \mathbb{E}_2 \longrightarrow \cdots \longrightarrow \mathbb{E}_{k-1} \longrightarrow \mathbb{E}_k \longrightarrow \mathbb{E}_{k+1} \longrightarrow \cdots \tag{2.4}$$

One of the fundamental properties of $\mathbb{E}_k$ is the following additive theorem.

Theorem 2.5.1 (Dunn-Lurie). *For every $k, l \geq 0$ we have $E_k \odot E_l \simeq E_{k+l}$.*

Proof. This is the 2-truncation of Lurie's $(\infty,1)$-categorical extension of classical Dunn's theorem - see [HA, Dunn]. ☐

Recall, in addition, that for every $(2,1)$-operad $\mathcal{O}$ and every symmetric monoidal $(2,1)$-category $\mathbf{H}$ we have corresponding sub-2-category $\mathcal{O}\,\mathrm{Alg}_2(\mathbf{H}) \subset \mathbf{H}$ of $\mathcal{O}$*-algebra objects in* $\mathbf{H}$. One can assign to $\mathbf{H}$ its endomorphism operad $\mathcal{E}(\mathbf{H})$. As a 1-category, $\mathcal{O}\,\mathrm{Alg}_2(\mathbf{H})$ can be characterized as the category of morphism $\mathcal{O} \to \mathcal{E}(\mathbf{H})$ in the same way as R-modules are characterized by R-algebra representations.

Example 2.5.2. *The $\mathbb{E}_0$-algebras in $\mathbf{H}$ are just objects of $\mathbf{H}$. The $\mathbb{E}_0$-algebras are monoid objects, the $\mathbb{E}_2$-algebras are the braided monoid objects and $\mathbb{E}_3$-algebras are the commutative monoid objects. Thus, we have the following equivalences, which follows directly from the so-called Eckmann-Hilton argument.*

$$\begin{aligned}
\mathbb{E}_0(\mathrm{Alg})(\mathbf{H}) &\simeq \mathbf{H} \\
\mathbb{E}_1(\mathrm{Alg})(\mathbf{H}) &\simeq \mathrm{Mon}_2(\mathbf{H}) \\
\mathbb{E}_2(\mathrm{Alg})(\mathbf{H}) &\simeq \mathrm{BrMon}_2(\mathbf{H}) \\
\mathbb{E}_3(\mathrm{Alg})(\mathbf{H}) &\simeq \mathrm{ComMon}_2(\mathbf{H}).
\end{aligned}$$

Every morphism $\phi : \mathcal{O} \to \mathcal{O}'$ between $(2,1)$-operads induces, for every symmetric monoidal $(2,1)$-category $\mathbf{H}$, an adjunction between the corresponding $\mathcal{O}$-algebra objects, as below.

$$\begin{tikzcd}
\mathcal{O}\mathbf{Alg}_2(\mathbf{H}) \arrow[r, shift left, "\phi_*"] \arrow[r, phantom, "\perp"] & \mathcal{O}'\mathbf{Alg}_2(\mathbf{H}) \arrow[l, shift left, "\phi^*"]
\end{tikzcd}$$

It then follows that the equivalences of last example extends to the following commutative diagram.

$$\begin{tikzcd}
\mathbb{E}_3\operatorname{Alg}(\mathbf{H}) \arrow[r] \arrow[d, "\simeq"'] & \mathbb{E}_2\operatorname{Alg}(\mathbf{H}) \arrow[r] \arrow[d, "\simeq"] & \mathbb{E}_1\operatorname{Alg}(\mathbf{H}) \arrow[r] \arrow[d, "\simeq"] & \mathbb{E}_0\operatorname{Alg}(\mathbf{H}) \arrow[d, "\simeq"] \\
\operatorname{ComMon}_2(\mathbf{H}) \arrow[r, hook] & \operatorname{BrMon}_2(\mathbf{H}) \arrow[r, hook] & \operatorname{Mon}_2(\mathbf{H}) \arrow[r, hook] & \mathbf{H}
\end{tikzcd} \tag{2.5}$$

Suppose now that $\mathbf{H}$ is a cartesian monoidal $(2,1)$-category. Let $\mathcal{O}$ be a $(2,1)$-operad take a morphism $\theta : \mathbb{E}_1 \to \mathcal{O}$. Define the sub-2-category $\mathcal{O}\operatorname{Alg}^{grp}_{2,\theta}(\mathbf{H}) \subset \mathcal{O}\operatorname{Alg}_2(\mathbf{H})$ of *groupal $\mathcal{O}$-algebra objects relative to θ* as the universal sub-2-category for which the there is the factorization in diagram below.

$$\begin{tikzcd}
\mathcal{O}\operatorname{Alg}^{grp}_{2,\theta}(\mathbf{H}) \arrow[d, hook] \arrow[ddrr, dashed, bend right, "\theta^*"'] & & & \\
\mathcal{O}\operatorname{Alg}_2(\mathbf{H}) \arrow[r, "\theta^*"] & \mathbb{E}_1\operatorname{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, "\simeq"'] & \mathbb{E}_0\operatorname{Alg}_2(\mathbf{H}) \arrow[d, "\simeq"] \\
& \operatorname{Mon}_2(\mathbf{H}) \arrow[r, hook] & \mathbf{H} \\
& \operatorname{Grp}_2(\mathbf{H}) \arrow[u, hook] &
\end{tikzcd}$$

A direct inspection show that the groupal objects of $\mathbb{E}_k$-algebras with $k = 1,2,3$ (relative to the canonical morphism $\mathbb{E}_k \to \mathbb{E}_1$) are, respectively, group objects, braided group objects and commutative group objects. Therefore, diagram 2.5 extends to the following:

$$\begin{tikzcd}
\mathbb{E}_3\operatorname{Alg}^{grp}_2(\mathbf{H}) \arrow[r] \arrow[d, hook] \arrow[ddd, dashed, bend left, "\simeq"] & \mathbb{E}_2\operatorname{Alg}^{grp}_2(\mathbf{H}) \arrow[r] \arrow[d, hook] \arrow[ddd, dashed, bend left, "\simeq"] & \mathbb{E}_1\operatorname{Alg}^{grp}_2(\mathbf{H}) \arrow[d, hook] \arrow[ddd, dashed, bend left, "\simeq"] & \\
\mathbb{E}_3\operatorname{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, dashed, "\simeq"] & \mathbb{E}_2\operatorname{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, dashed, "\simeq"] & \mathbb{E}_1\operatorname{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, dashed, "\simeq"] & \mathbb{E}_0\operatorname{Alg}_2(\mathbf{H}) \arrow[d, dashed, "\simeq"] \\
\operatorname{ComMon}_2(\mathbf{H}) \arrow[r] & \operatorname{BrMon}_2(\mathbf{H}) \arrow[r] & \operatorname{Mon}_2(\mathbf{H}) \arrow[r] & \mathbf{H} \\
\operatorname{ComGrp}_2(\mathbf{H}) \arrow[r] \arrow[u, hook] & \operatorname{BrGrp}_2(\mathbf{H}) \arrow[r] \arrow[u, hook] & \operatorname{Grp}_2(\mathbf{H}) \arrow[u, hook] &
\end{tikzcd}$$

Finally, observe that the sequence (2.4) of $(2,1)$-operads does not ends in $\mathbb{E}_3$. Thus, one could consider $\mathbb{E}_k$-algebras in a $\mathbf{H}$ for $k > 3$ as ask what kind of monoid objects correspond to them. The following result shows, however, that the sequence stabilizes in $k = 3$.

Theorem 2.5.3. *For every symmetric monoidal* $(2,1)$*-category* $\mathbf{H}$ *and every* $k > 3$ *the canonical map in diagram below is an equivalence.*

$$\begin{tikzcd}
\mathbb{E}_k\operatorname{Alg}(\mathbf{H}) \arrow[r] \arrow[drr, dashed, "\simeq"'] & \cdots \arrow[r] & \mathbb{E}_3\operatorname{Alg}(\mathbf{H}) \arrow[r, hook] \arrow[d, "\simeq"'] & \mathbb{E}_2\operatorname{Alg}(\mathbf{H}) \arrow[r, hook] \arrow[d, "\simeq"] & \mathbb{E}_1\operatorname{Alg}(\mathbf{H}) \arrow[r, hook] \arrow[d, "\simeq"] & \mathbb{E}_0\operatorname{Alg}(\mathbf{H}) \arrow[d, "\simeq"] \\
& & \operatorname{ComMon}_2(\mathbf{H}) \arrow[r, hook] & \operatorname{BrMon}_2(\mathbf{H}) \arrow[r, hook] & \operatorname{Mon}_2(\mathbf{H}) \arrow[r, hook] & \mathbf{H}
\end{tikzcd}$$

Proof. This is the 2-truncation of Corollary 5.1.1.7, page 763 of [HA], or Corollary 1.1.10, page 8 of [DAGIV]. $\square$

Remark 2.5.4. *Of course, the theorem above does not implies that the only classes of algebraic objects in a symmetric monoidal $(2,1)$-category are monoid objects, braided monoid objects and commutative monoid objects. Indeed, it means that there is no other way to weak the commutativity condition. Generally, in a monoidal $(n,1)$-category, the sequence of $\mathbb{E}_k$-algebras stabilizes for $k > n$.*

Chapter 3

Delooping

Here we recall delooping groupoids in $(2,1)$-topoi. In the body of our project delooping are needed to give an internal definition of principal G-bundle. As we will see, the delooping construction can be realized internally or externally, producing equivalent internal groupoids. The later, however, has the advantage of making sense in every cartesian $(2,1)$-categories, even for that are not a $(2,1)$-topoi. We then talk about the delooping object of the delooping groupoid. Finally, we consider higher delooping, connecting them with algebra objects over the $\mathbb{E}_k$-operads.

3.1 Internal Delooping

Define the *cartesian cosmoi* as the full sub-2-category $(2,1)\mathbf{Cart}_2 \subset (2,1)\mathbf{Cat}_2$ of complete and cocomplete $(2,1)$-categories. Thus, in particular, $(2,1)\mathbf{Cat}_{c,2} \subset (2,1)\mathbf{Cart}_2$ and $(2,1)\mathbf{Cat}_{cp,2} \subset (2,1)\mathbf{Cart}_2$. Let $\mathrm{Grp}_2, \mathrm{Grpd}_2 : (2,1)Cart_2 \Rightarrow (2,1)\mathbf{Cat}_2$ denote the restrictions to $(2,1)Cart_2$ of the internal group and internal groupoid functors.

The *internal delooping* is a specific 2-natural transformation $\mathbf{B} : \mathrm{Grp}_2 \Rightarrow \mathrm{Grpd}_2$ such that for every $\mathbf{H} \in (2,1)\mathbf{Cart}_2$ and every group object $G \in \mathrm{Grp}_2(\mathbf{H})$ it assigns the internal object $\mathbf{B}_{\mathbf{H}}(G)$ such that $\mathbf{B}_{\mathbf{H}}(G)_0 = *_2$ and $\mathbf{B}_{\mathbf{H}}(G)_1 = G$, where $*_2 = \mathrm{disc}_{1,2}(*)$ is the terminal $(2,1)$-category, and such that $s, t : G \to *_2$ is the terminal morphism.

$$\begin{tikzcd}
(2,1)\mathbf{Cat}_2 \\
(2,1)\mathbf{Cart}_2 \arrow[u, bend left=60, "\mathrm{Grp}_2"] \arrow[u, bend right=60, "\mathrm{Grpd}_2"'] \arrow[u, phantom, "\xRightarrow{\mathbf{B}^{int}}"]
\end{tikzcd}$$

Notice that $(2,1)$-pretopoi need not be a cartesian cosmoi, because they typically lack of $(2,1)$colimits [??]. However, $(2,1)$-presheaves $(2,1)$-topoi and Grothendieck $(2,1)$-topoi are cartesian cosmoi (because they are locally presentable $(2,1)$-categories). Recall from Remark 2.2.1 that there is an identity on objects forgetful 2-functor $U_{topos} : (2,1)\mathbf{PTopos}_2 \to \mathbf{Cat}_2$.

What we are noticing is that this functor actually factors as below[1]

$$\begin{tikzcd}
(2,1)\mathbf{ShvTopos}_2 \arrow[r, hook] \arrow[rr, dotted, bend left, "U_{Shv}"] & (2,1)\mathbf{PShvTopos}_2 \arrow[r, dashed, "U_{PShv}"] \arrow[dr, hook] & (2,1)\mathbf{Cart}_2 \arrow[dr, hook] & \\
& & (2,1)\mathbf{PTopos}_2 \arrow[r, "U_{topos}"'] & (2,1)\mathbf{Cat}_2
\end{tikzcd}$$

This allows us to define the delooping for $(2,1)$-presheaf $(2,1)$-topoi and Grothendieck $(2,1)$-topoi by transportation along U_{PShv} and U_{Shv}, respectively.

$$\begin{tikzcd}
& & (2,1)\mathbf{Cat}_2 & \\
(2,1)\mathbf{ShvTopos}_2 \arrow[urr, dotted, bend left, "\mathrm{Grp}_2"] \arrow[urr, dotted, "\mathrm{Grpd}_2"'] \arrow[urr, phantom, bend left=15, "\Downarrow \mathbf{B}^{int}_{Shv}"] \arrow[r, hook] \arrow[rrr, dotted, bend right, "U_{Shv}"'] & (2,1)\mathbf{PShvTopos}_2 \arrow[ur, dashed, bend left, "\mathrm{Grp}_2"] \arrow[ur, dashed, bend right, "\mathrm{Grpd}_2"'] \arrow[ur, phantom, "\Rightarrow \mathbf{B}^{int}_{PShv}"] & & (2,1)\mathbf{Cart}_2 \arrow[l, dashed, "U_{PShv}"'] \arrow[ul, "\mathrm{Grp}_2"] \arrow[ul, bend right, "\mathrm{Grpd}_2"'] \arrow[ul, phantom, bend right=15, "\Uparrow \mathbf{B}^{int}"]
\end{tikzcd} \tag{3.1}$$

3.2 External Delooping

The above construction is “internal”, in the sense that it not depends on using that $(2,1)$-presheaf $(2,1)$-topoi and Grothendieck $(2,1)$-topoi are presented by $(2,1)$-presheaf $(2,1)$-categories and localization of them, respectively. An “external” presentation could also be obtained, as follows.

Consider the 2-functor

$$(2,1)\,\mathrm{PShv}_{(-)}(-) : \mathbf{Cat}_2 \times (2,1)\mathbf{Cat}_2 \to (2,1)\mathbf{Cat}_2$$

that to each $(\mathbf{C}, \mathbf{A})$ assigns $(2,1)\,\mathrm{PShv}_{\mathbf{A}}(\mathbf{C})$. Recall the Yoneda $(2,1)$-embedding such that to every $(2,1)$-category $\mathbf{C}$ it assigns the fully-faithful 2-functor $y_{\mathbf{C}} : \mathbf{C} \to 2\,\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$. Notice that $\mathbf{Grpd}_2 \simeq \mathrm{Grpd}_2(\mathbf{Set}_2)$, where $\mathbf{Set}_2 = \mathrm{disc}_{1,2}(\mathbf{Set})$. The rule $\mathbf{C} \mapsto y_{\mathbf{C}}$ is natural and, therefore, define 2-natural transformation y, as below. There we used that $\mathbf{Set}_2 \in (2,1)\mathbf{Cart}_2$ is a cartesian $(2,1)$category, and red color is to emphasize where the 2-functor $\mathrm{Grpd}_2 : (2,1)\mathbf{Cart}_2 \to (2,1)\mathbf{Cat}_2$ is used. In the second diagram, $2\,\mathrm{PShv}_{\mathrm{Grpd}_2}(-)$ denotes the corresponding composition.

$$\begin{tikzcd}
\mathbf{Cat}_2 \times *_2 \arrow[d, "id \times \mathbf{Set}_2"'] & & \mathbf{Cat}_2 \arrow[ll, "\simeq"'] \arrow[dd, "\mathrm{Grpd}_2"] \arrow[ddl, phantom, "\Swarrow y"] \\
\mathbf{Cat}_2 \times (2,1)\mathbf{Cart}_2 \arrow[d, "id \times \mathrm{Grpd}_2"'] & & \\
\mathbf{Cat}_2 \times (2,1)\mathbf{Cat}_2 \arrow[rr, "2\,\mathrm{PShv}_{(-)}(-)"'] & & (2,1)\mathbf{Cat}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Cat}_2 \arrow[dd, bend right=50, "2\,\mathrm{PShv}_{\mathrm{Grpd}_2}(-)"'] \arrow[dd, bend left=50, "\mathrm{Grpd}_2"] \arrow[dd, phantom, "\overset{y}{\Leftarrow}"] \\
\\
(2,1)\mathbf{Cat}_2
\end{tikzcd} \tag{3.2}$$

Now, recall that we have inclusion 2-natural transformations, as in diagram below. The cyan color is to remember that it used blue arrows and green arrows. Similarly, the orange

[1] This factorization follows precisely because the 2-functor $\mathbf{A} \mapsto (2,1)\,\mathrm{PShv}_{-}(\mathbf{H})$ preserves $(2,1)$-limits and $(2,1)$-colimits, which are the properties defining $\mathbf{Cart}_2$.

color involves green and red arrows.

$$
\begin{tikzcd}
\mathbf{Cart}_2 \arrow[d, bend right=60, "\mathrm{Grp}_2"'] \arrow[d, "\mathrm{disc}_{1,2}" description] \arrow[d, bend left=60, "\mathrm{Grpd}_2"] \arrow[d, phantom, "\Rightarrow\quad\quad\Rightarrow"] \\
(2,1)\mathbf{Cat}_2
\end{tikzcd}
$$

We can define analogues of the diagram (??) replacing red arrows with green or blue arrows, i.e., replacing the 2-functor Grpd_2 with $\mathrm{disc}_{1,2}$ or Grp_2. The Yoneda embedding factors through the cyan and orange arrows, producing "green Yoneda 2-embedding" and "blue Yoneda 2-embedding", as follows.

$$
\begin{tikzcd}
\mathbf{Cat}_2 \arrow[r, "id"] \arrow[d, bend right, "2\,\mathrm{PShv}_{\mathrm{Grp}_2}"'] \arrow[d, bend left, "\mathrm{Grp}_2"] \arrow[d, phantom, "\overset{y}{\Leftarrow}"] & \mathbf{Cat}_2 \arrow[r, "id"] \arrow[d, bend right, "2\,\mathrm{PShv}_{\mathrm{disc}_{1,2}}"'] \arrow[d, bend left, "\mathrm{disc}_{1,2}"] \arrow[d, phantom, "\overset{y}{\Leftarrow}"] & \mathbf{Cat}_2 \arrow[d, bend right, "2\,\mathrm{PShv}_{\mathrm{Grpd}_2}"'] \arrow[d, bend left, "\mathrm{Grpd}_2"] \arrow[d, phantom, "\overset{y}{\Leftarrow}"] \\
(2,1)\mathbf{Cat}_2 \arrow[r, "id"] & (2,1)\mathbf{Cat}_2 \arrow[r, "id"] & (2,1)\mathbf{Cat}_2
\end{tikzcd}
$$

Now, take the delooping transformation $\mathbf{B} : \mathrm{Grp}_2 \Rightarrow \mathrm{Grpd}_2$ and transport it to a transformation $\mathbf{B}^{ext}_{PShv} : 2\,\mathrm{PShv}_{\mathrm{Grp}_2} \Rightarrow 2\,\mathrm{PShv}_{\mathrm{Grpd}_2}$ using the colored Yoneda 2-embeddings above, which is the external delooping for $(2,1)$-presheaf $(2,1)$-topoi. The case of Grothendieck $(2,1)$-topoi is analogously, but also transport with the localization transformation (2.3).

3.3 Equivalence

The relation between internal and external delooping is obtained noticing that we have a 2-equivalence, as below. Again, red color means that we are using the 2-functor Grpd_2. Furthermore, we also have analogue equivalence replacing red arrows with green arrows or blue arrows.

$$
\begin{tikzcd}
\mathbf{Cat}_2 \times (2,1)\mathbf{Cat}_2 \arrow[rr, "2\,\mathrm{PShv}_{(-)}(-)"] & & (2,1)\mathbf{Cat}_2 & (2,1)\mathbf{Cat}_2 \\
\mathbf{Cat}_2 \times \mathbf{Cart}_2 \arrow[u, "id \times \mathrm{Grpd}_2"] & & \mathbf{Cart}_2 \arrow[u, "\mathrm{Grpd}_2"'] \arrow[dll, Rightarrow, "\simeq"] & \\
\mathbf{Cat}_2 \times *_2 \arrow[u, "id \times \mathbf{Set}"] & \mathbf{Cat}_2 \arrow[l, "\simeq"] & (2,1)\mathbf{PShvTopos}_2 \arrow[l, hook'] \arrow[u, "U_{PShv}"'] & (2,1)\mathbf{PShvTopos}_2 \arrow[uu, bend left, "2\,\mathrm{PShv}_{\mathrm{Grpd}_2}"] \arrow[uu, bend right, "\mathrm{Grpd}_2 \circ U_{PShv}"'] \arrow[uu, phantom, "\overset{\simeq}{\Leftarrow}"]
\end{tikzcd}
\tag{3.3}
$$

Thus one can glue each colored diagram (3.2) with the corresponding colored diagram (3.3) of same color. Then, recalling the definition (3.1) of the internal delooping $\mathbf{B}_{PShv}$, we conclude that it is exactly the transportation of the external delooping $\mathbf{B}^{ext}_{PShv}$ along the 2-equivalences of diagrams (3.3). Objectwise this is represented by the commutativity of the following diagram. The case of Grothendieck $(2,1)$-topoi is obtained adding another vertical arrows corresponding to the localization transformation.

$$
\begin{tikzcd}
\mathrm{Grp}_2(\mathbf{C}) \arrow[r, hook, "y_{\mathbf{C}}"] \arrow[d, hook] \arrow[dd, bend right=60, "\mathbf{B}_{\mathbf{C}}"'] & 2\,\mathrm{PShv}_{\mathrm{Grp}_2(\mathbf{Grpd}_2)}(\mathbf{C}) \arrow[r, "\simeq"] \arrow[d, hook] \arrow[dd, bend left=40, "\mathbf{B}^{ext}_{PShv,\mathbf{C}}"] & \mathrm{Grp}_2(2\,\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})) \arrow[d, hook] \arrow[dd, bend left=60, "\mathbf{B}_{PShv,\mathbf{C}}"] \\
\mathrm{disc}_{1,2}(\mathbf{C}) \arrow[r, hook, "y_{\mathbf{C}}"'] \arrow[d, hook] & 2\,\mathrm{PShv}_{\mathbf{Gprd}_2}(\mathbf{C}) \arrow[r, "\simeq"'] \arrow[d, hook] & 2\,\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C}) \arrow[d, hook] \\
\mathrm{Grpd}_2(\mathbf{C}) \arrow[r, hook, "y_{\mathbf{C}}"'] & 2\,\mathrm{PShv}_{\mathrm{Grpd}_2(\mathbf{Grpd}_2)}(\mathbf{C}) \arrow[r, "\simeq"'] & \mathrm{Grpd}_2(2\,\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C}))
\end{tikzcd}
\tag{3.4}
$$

Theorem 3.3.1. *We have:*

- *for every cartesian (2,1)-category* $\mathbf{H}$ *a 2-functor* $\mathbf{B}_{\mathbf{H}} : \mathrm{Grp}_2(\mathbf{H}) \to \mathrm{Grpd}_2(\mathbf{H})$ *- the internal delooping;*
- *for every (2,1)-presheaf (2,1)-topos* $\mathbf{H} \simeq 2\,\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$ *a 2-functor*

  $$\mathbf{B}_{\mathbf{H},ext} : 2\,\mathrm{PShv}_{\mathrm{Grp}_2(\mathbf{Grpd}_2)}(\mathbf{C}) \to 2\,\mathrm{PShv}_{\mathrm{Grpd}_2(\mathbf{Grpd}_2)}(\mathbf{C}),$$

  the external delooping.

Furthermore, these 2-functors are 2-isomorphic in (2,1)-presheaf (2,1)-topoi. In addition, if $\mathbf{H}$ *is actually a Grothendieck (2,1)-topos, then the internal delooping factor through the localization.*

3.4 Delooping Objects

By the above, to every group object $G \in \mathrm{Grp}_2(\mathbf{H})$ in a cartesian $(2,1)$-category there corresponds an internal groupoid $\mathbf{B}_{\mathbf{H},int}G$. Thus, we also have a corresponding representing object $B_{\mathbf{H},int}G = t(\mathrm{coeq}(\mathbf{B}_{\mathbf{H},int}G)))$. This is the *internal delooping object* of G.

If $\mathbf{H}$ is a (2,1)-presheaf (2,1)-topos or a Grothencieck (2,1)-topos, then is other internal groupoid $\mathbf{B}_{\mathbf{H},ext}G$ and, therefore, other representing object: the *external delooping object* $\mathbf{B}_{\mathbf{H},int}G$. But it comes equipped with an equivalence $\mathbf{B}_{\mathbf{H},int}G \simeq B_{\mathbf{H},ext}G$ that therefore induces an isomorphism $B_{\mathbf{H},\mathbf{int}}G \simeq B_{\mathbf{H},ext}G$.

Since in a Grothendieck $(2,1)$-topos all groupoid object is effective, it follows that the delooping groupoids can be reconstructed from the corresponding delooping objects.

3.5 Higher Delooping

We saw that for every cartesian $(2,1)$-category $\mathbf{H}$ we have the internal delooping 2-functor $\mathbf{B}_{\mathbf{H}} : \mathrm{Grp}_2(\mathbf{H}) \to \mathrm{Grpd}_2(\mathbf{H})$ and the representing object 2-functor $t \circ \mathrm{coeq} : \mathrm{Grpd}_2(\mathbf{H}) \to \mathbf{H}$. Sometimes we would like to iterate the delooping object construction. Observe that $B^2_{\mathbf{H}}G = B_{\mathbf{H}}(B_{\mathbf{H}}G)$ is well-defined precisely if G belongs to the full sub-2-category $\mathrm{Del}^2_2(\mathbf{H}) \subset \mathrm{Grp}_2(\mathbf{H})$

of group object for which we have the following factorization:

$$\begin{tikzcd}
 & & & \mathbf{H} \\
 & & \mathrm{Grp}_2(\mathbf{H}) \arrow[ur, "B_{\mathbf{H}}"] \arrow[r, "\mathbf{B}_{\mathbf{H}}"'] \arrow[d, hook] & \mathrm{Grpd}_2(\mathbf{H}) \arrow[u, "t\circ\mathrm{coeq}"'] \\
 & & \mathbf{H} & \\
\mathrm{Del}_2^2(\mathbf{H}) \arrow[r, hook] \arrow[uurr, dashed, "B_{\mathbf{H}}"] \arrow[uuurrr, dotted, bend left, "B^2_{\mathbf{H}}"] & \mathrm{Grp}_2(\mathbf{H}) \arrow[ur, "B_{\mathbf{H}}"] \arrow[r, "\mathbf{B}_{\mathbf{H}}"'] & \mathrm{Grpd}_2(\mathbf{H}) \arrow[u, "t\circ\mathrm{coeq}"'] &
\end{tikzcd}$$

Recursively we define full sub-2-categories $\mathrm{Del}_2^k(\mathbf{H}) \subset \mathrm{Grp}_2(\mathbf{H})$, with $\mathrm{Del}_2^k(\mathbf{H}) \subset \mathrm{Del}_2^l(\mathbf{H})$ if $l \leq k$, consisting of those objects which are k-tuply deloopable, i.e., such that $B^k_{\mathbf{H}}G$ exists. Defining $\mathrm{Del}_2^1(\mathbf{H}) := \mathrm{Grp}_2(\mathbf{H})$ and $\mathrm{Del}_2^0(\mathbf{H}) := \mathbf{H}$ we have the following homotopy commutative diagram.

$$\begin{tikzcd}
\cdots \arrow[r, hook] & \mathrm{Del}_2^{k+1}(\mathbf{H}) \arrow[r, hook] \arrow[r, bend left, "B^1_{k+1}"] \arrow[rrrrrr, bend left=40, "B^{k+1}_{k+1}"] \arrow[rrrrr, bend left=20, "B^k_{k+1}"] \arrow[rr, bend right, "B^2_{k+1}"'] \arrow[rrrr, bend right=20, "B^{k-1}_{k+1}"'] \arrow[rrrrrr, bend right=40, "B^0_{k+1}"'] & \mathrm{Del}_2^k(\mathbf{H}) \arrow[r, hook] & \mathrm{Del}_2^{k-1}(\mathbf{H}) \arrow[r, hook] & \cdots \arrow[r, hook] & \mathrm{Del}_2^2(\mathbf{H}) \arrow[r, hook] \arrow[rr, bend left, "B^2_2"] \arrow[rr, bend right, "B^0_2"'] & \mathrm{Del}_2^1(\mathbf{H}) \arrow[r, shift left, "B^1_1"] \arrow[r, shift right, "B^0_1"'] & \mathrm{Del}_2^0(\mathbf{H})
\end{tikzcd}$$

This is clearly a diagram of a inverse $(2,1)$-limit in the $(2,1)$-category $(2,1)\mathbf{Cat}_2$. Its $(2,1)$-limit $\mathrm{Del}_2^\infty(\mathbf{H}) \subset \mathbf{H}$ is the sub-2-category of of the *infinitely deloopable objects*.

3.6 Eckmann-Hilton

Saying that the delooping BG exists we are automatically saying that G is a group object: an object with additional stuff satisfying additional properties. In turn, saying that B^2G exists we are saying that BG exists and that $B(BG)$ exists. So, both G and BG must be group objects. But BG is, in essence, the $(2,1)$-coequalizer of $G \to *$. Thus, a group structure in BG induces a second group structure in G which is compatible with the first one. Define a *k-tuply group object* as an object endowed that is a group object in $k > 0$ different, but compatible, ways. Let $\mathbf{Grp}_2^k(\mathbf{H}) \subset \mathbf{H}$ the sub-2-category of them. For $k = 0$, define $\mathbf{Grp}_2^0(\mathbf{H}) := \mathbf{H}$.

Lemma 3.6.1. *For every cartesian $(2,1)$-category $\mathbf{H}$ and every $k \geq 0$ we have $\mathbf{Grp}_2^k(\mathbf{H}) \simeq \mathrm{Del}_2^k(\mathbf{H})$.*

Since the sub-2-categories $\mathrm{Del}_2^k(\mathbf{H}) \subset \mathbf{H}$ are full, one should not expect that it consists of group objects with additional *structure*, since in this case the morphisms and 2-morphisms would be required to preserve them (meaning that the sub-2-category is not full). Thus, they should be defined by group objects fulfilling additional *properties*. However, the lemma above clearly characterize k-deloopable objects as group objects with additional $k-1$-group structure satisfying compatibility conditions. The fundamental fact is that all the k-group structure coincide and their compatibility conditions translates in terms of additional properties of the single group object structure. Formally, this is a $(2,1)$-categorical version of the classical Eckmann-Hilton argument [??,??], as we will briefly review.

Let $\mathbf{H}$ be a cartesian monoidal $(2,1)$-category. From Lemma (3.6.1) we have a sequence of inclusions, as below.

$$\cdots \hookrightarrow \mathrm{Grp}_2^k(\mathbf{H}) \hookrightarrow \cdots \hookrightarrow \mathrm{Grp}_2^3(\mathbf{H}) \hookrightarrow \mathrm{Grp}_2^2(\mathbf{H}) \hookrightarrow \mathrm{Grp}_2^1(\mathbf{H}) \hookrightarrow \mathrm{Grp}_2^0(\mathbf{H}) \tag{3.5}$$

Define a sequence of 2-categories $\mathrm{Mon}_2^k(\mathbf{H})$ as follows. For $k=0$ take $\mathrm{Mon}_2^0(\mathbf{H}) = \mathbf{H}$. For $k=1$, put $\mathrm{Mon}_2^1(\mathbf{H}) = \mathrm{Mon}_2(\mathbf{H})$. Finally, for $k>1$ define $\mathrm{Mon}_2^k(\mathbf{H}) = \mathbf{H}$ as the universal $(2,1)$-category that contains $\mathrm{Grp}_2^k(\mathbf{H})$ and for which there is the dotted arrow below.

$$\begin{tikzcd}
\mathrm{Mon}_2^k(\mathbf{H}) \arrow[r, dashed] & \mathrm{Mon}_2^{k-1}(\mathbf{H}) \\
\mathrm{Grp}_2^k(\mathbf{H}) \arrow[r, hook] \arrow[u, hook] & \mathrm{Grp}_2^{k-1}(\mathbf{H}) \arrow[u, hook]
\end{tikzcd}$$

Thus, the sequence of inclusions (3.5) extends to the following commutative diagram.

$$\begin{tikzcd}
\cdots \arrow[r, hook] & \mathrm{Mon}_2^k(\mathbf{H}) \arrow[r, hook] & \cdots \arrow[r, hook] & \mathrm{Mon}_2^3(\mathbf{H}) \arrow[r, hook] & \mathrm{Mon}_2^2(\mathbf{H}) \arrow[r, hook] & \mathrm{Mon}_2^1(\mathbf{H}) \arrow[r, hook] & \mathrm{Mon}_2^0(\mathbf{H}) \\
\cdots \arrow[r, hook] & \mathrm{Grp}_2^k(\mathbf{H}) \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, hook] & \mathrm{Grp}_2^3(\mathbf{H}) \arrow[r, hook] \arrow[u, hook] & \mathrm{Grp}_2^2(\mathbf{H}) \arrow[r, hook] \arrow[u, hook] & \mathrm{Grp}_2^1(\mathbf{H}) \arrow[r, hook] \arrow[u, hook] & \mathrm{Grp}_2^0(\mathbf{H}) \arrow[u, "\simeq"]
\end{tikzcd} \tag{3.6}$$

Theorem 3.6.2. *Let $\mathbb{H}$ be a cartesian monoidal $(2,1)$-category. For every $k \geq 0$ we have an equivalence $\mathrm{Mon}_2^k(\mathbf{H}) \simeq \mathbb{E}_k\,\mathrm{Alg}_2(\mathbf{H})$ between k-tuply monoid objects and $\mathbb{E}_k$-algebra objects which is natural, meaning that the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{E}_k\,\mathrm{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, "\simeq"'] & \mathbb{E}_{k-1}\,\mathrm{Alg}_2(\mathbf{H}) \arrow[d, "\simeq"] \\
\mathrm{Mon}_2^k(\mathbf{H}) \arrow[r] & \mathrm{Mon}_2^{k-1}(\mathbf{H})
\end{tikzcd}$$

Proof. This is ..... □

Therefore, diagram (3.6) extends to the following one.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{E}_k\,\mathrm{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, dotted, "\simeq"] & \cdots \arrow[r] & \mathbb{E}_3\,\mathrm{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, dotted, "\simeq"] & \mathbb{E}_2\,\mathrm{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, dotted, "\simeq"] & \mathbb{E}_1\,\mathrm{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, dotted, "\simeq"] & \mathbb{E}_0\,\mathrm{Alg}_2(\mathbf{H}) \arrow[d, dotted, "\simeq"] \\
\cdots \arrow[r, hook] & \mathrm{Mon}_2^k(\mathbf{H}) \arrow[r, hook] & \cdots \arrow[r, hook] & \mathrm{Mon}_2^3(\mathbf{H}) \arrow[r, hook] & \mathrm{Mon}_2^2(\mathbf{H}) \arrow[r, hook] & \mathrm{Mon}_2^1(\mathbf{H}) \arrow[r, hook] & \mathrm{Mon}_2^0(\mathbf{H}) \\
\cdots \arrow[r, hook] & \mathrm{Grp}_2^k(\mathbf{H}) \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, hook] & \mathrm{Grp}_2^3(\mathbf{H}) \arrow[r, hook] \arrow[u, hook] & \mathrm{Grp}_2^2(\mathbf{H}) \arrow[r, hook] \arrow[u, hook] & \mathrm{Grp}_2^1(\mathbf{H}) \arrow[r, hook] \arrow[u, hook] & \mathrm{Grp}_2^0(\mathbf{H}) \arrow[u, dotted, "\simeq"]
\end{tikzcd}$$

Moreover, from the universality of the groupal $\mathbb{E}_k$-algebras we find that the vertical equivalences factors as below.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{E}_k \operatorname{Alg}_2^{grp}(\mathbf{H}) \arrow[r] \arrow[d, hook] \arrow[ddd, dashed, bend left, "\simeq"] & \cdots \arrow[r] & \mathbb{E}_2 \operatorname{Alg}_2^{grp}(\mathbf{H}) \arrow[r] \arrow[d, hook] \arrow[ddd, dashed, bend left, "\simeq"] & \mathbb{E}_1 \operatorname{Alg}_2^{grp}(\mathbf{H}) \arrow[d, hook] \arrow[ddd, dashed, bend left, "\simeq"] & \\
\cdots \arrow[r] & \mathbb{E}_k \operatorname{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, dashed, "\simeq"] & \cdots \arrow[r] & \mathbb{E}_2 \operatorname{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, dashed, "\simeq"] & \mathbb{E}_1 \operatorname{Alg}_2(\mathbf{H}) \arrow[r] \arrow[d, dashed, "\simeq"] & \mathbb{E}_0 \operatorname{Alg}_2(\mathbf{H}) \arrow[d, dashed, "\simeq"] \\
\cdots \arrow[r] & \operatorname{Mon}_2^k(\mathbf{H}) \arrow[r] & \cdots \arrow[r] & \operatorname{Mon}_2^2(\mathbf{H}) \arrow[r] & \operatorname{Mon}_2^1(\mathbf{H}) \arrow[r] & \operatorname{Mon}_2^0(\mathbf{H}) \\
\cdots \arrow[r] & \operatorname{Grp}_2^k(\mathbf{H}) \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & \operatorname{Grp}_2^2(\mathbf{H}) \arrow[r] \arrow[u, hook] & \operatorname{Grp}_2^1(\mathbf{H}) \arrow[u, hook] &
\end{tikzcd}$$

In sum, we have the following result.

Theorem 3.6.3. *For every cartesian* $(2,1)$*-category* $\mathbf{H}$ *and every* $k \geq 0$ *there are natural equivalences of 2-categories*

$$\operatorname{Del}_2^k(\mathbf{H}) \simeq \operatorname{Grp}_2^k(\mathbf{H}) \simeq \mathbb{E}_k \operatorname{Alg}_2^{grp}(\mathbf{H}).$$

Joining this theorem with Theorem 2.5.3 we get a characterization of all higher deloopable objects.

Corollary 3.6.4. *Let* $X \in \mathbf{H}$ *be an object of a cartesian* $(2,1)$*-category. Then:*

1. *it is deloopable iff it is a group object;*
2. *it is doubly deloopable iff it is a braided group object;*
3. *it is triply deloopable iff it is a commutative group object;*
4. *it is* k*-tuply deloopable, with* $k > 3$*, iff if it is triply deloopable;*
5. *it is infinitely deloopable iff it is a commutative group object.*

3.7 Looping vs Delooping

Recall that a *based object* in a cartesian $(2,1)$-category $\mathbf{H}$ with terminal object $*$ is just a morphism $* \to X$, hence an object of the slice $(2,1)$-category $*/\mathbf{H}$. Equivalently, it is an object X endowed with a global element. There is the projection 2-functor $\pi : */\mathbf{H} \to \mathbf{H}$ that forgets the global elements. We also have a 2-functor $\Omega : */\mathbf{H} \to \mathbf{H}$ that takes the $(2,1)$-equalizer of $* \to X$, as below.

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & X
\end{tikzcd}$$

We say that ΩX is the *looping object* of X. The above diagram can be viewed as the diagram defining the Čech groupoid of the morphism $* \to X$ (note that $\Omega X = * \times_X *$). Since the object of objects of this groupoid object is the terminal object, it follows that it is actually the delooping of a group object, so ΩX is a group object. This means that the 2-functor Ω factors as below.

$$\begin{tikzcd}
& \mathrm{Grp}_2(\mathbf{H}) \arrow[d, hook] \\
*/\mathbf{H} \arrow[ur, dashed, "\Omega"] \arrow[r, "\Omega"'] & \mathbf{H}
\end{tikzcd}$$

On the other hand, for every group object $G \in \mathrm{Grp}_2(\mathbf{H})$ its internal delooping object $B_{\mathbf{H}}G$ is the $(2,1)$-coequalizer of the unique map $G \to *$. Thus, it comes equipped with a universal global element $* \to B_{\mathbf{H}}G$, as in the first diagram below. Thus means that the 2-functor $B_{\mathbf{H}}$ factors as in the second diagram.

$$\begin{tikzcd}
B_{\mathbf{H}}G & * \arrow[l] \\
* \arrow[u] & G \arrow[l] \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{H} & \mathrm{Grp}_2(\mathbf{H}) \arrow[l, "B_{\mathbf{H}}"'] \arrow[dl, dashed, "B_{\mathbf{H}}"] \\
*/\mathbf{H} \arrow[u, "\pi"] &
\end{tikzcd}$$

Joining diagrams above we then find the following diagram.

$$\begin{tikzcd}
\mathbf{H} & \mathrm{Grp}_2(\mathbf{H}) \arrow[l, "B_{\mathbf{H}}"'] \arrow[dl, dashed, bend left, "B_{\mathbf{H}}"'] \arrow[d, hook] \\
*/\mathbf{H} \arrow[u, "\pi"] \arrow[ur, dashed, bend right, "\Omega"'] \arrow[r, "\Omega"'] & \mathbf{H}
\end{tikzcd}$$

For every group object G we have $\Omega B_{\mathbf{H}}G \simeq G$ as represented in diagrams below.

$$\begin{tikzcd}
\Omega B_{\mathbf{H}}G \arrow[r] \arrow[d] & * \arrow[d] & \\
* \arrow[r] & B_{\mathbf{H}}G & * \arrow[l] \\
& * \arrow[u] & G \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[dr, dashed, "\simeq"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \Omega B_{\mathbf{H}}G \arrow[r] \arrow[d] & * \arrow[d] \\
& * \arrow[r] & B_{\mathbf{H}}G
\end{tikzcd}$$

Analogously one can show that for every $X \in \mathbf{H}$ we have an equivalence $X \simeq B_{\mathbf{H}}\Omega X$. This suggests that $B_{\mathbf{H}}$ and Ω are equivalences, being one the inverse of the other. This, however, is generally not the case:

1. for every group object G the delooping groupoid $\mathbf{B}_{\mathbf{H}}$ is effective, but the groupoid object $\boldsymbol{\Omega}X$ of ΩX need not be;

2. the equivalence $X \simeq B_{\mathbf{H}}\Omega X$ depends explicitly on the choice of a global element $* :\to X$.

The first condition is avoided if $\mathbf{H}$ is a $(2,1)$-topos, because for them every groupoid object is effective. The second one, in turn, is avoided restricting to the full sub-2-category $\mathrm{Conn}(*/\mathbf{H}) \subset */\mathbf{H}$ of connected based objects, establishing the following result.

Theorem 3.7.1. *For every Grothencieck $(2,1)$-topos $\mathbf{H}$ the internal delooping factor through connected based objects and define an equivalence of 2-categories*

$$B_{\mathbf{H}} : \mathrm{Grp}_2(\mathbf{H}) \simeq \mathrm{Conn}(*/\mathbf{H}).$$

Proof. This is the 2-truncation of Lemma 7.2.2.11, p. 727 of [HTT]. □

Remark 3.7.2. *The last result has an extension for higher delooping. In them, the group objects are replaced with k-tuply group objects (that by Theorem 3.6.3 are equivalent to groupal $\mathbb{E}_k$-algebra objects) and the connected based objects are replaced with k-connected based objects. See Theorem 5.2.6.10, around page 873 of [HA], or Theorem 1.3.6, pages 22-23 of [DAGIV], for the $(\infty, 1)$-version and then take the 2-truncation.*

Chapter 4

Cohesive 2-topoi

(...)

4.1 Cohesive Modalities

With the fundamental of $(2,1)$-topos theory reviewed, in this short section we recall the additional modalities that are added in a $(2,1)$-topos in order to improve its internal logic, providing the needed syntax to formalize differential geometry.

Let $f : \mathbf{H} \to \mathbf{H}'$ be a geometric morphism (f_*, f^*) of $(2,1)$-pretopoi. We say that f is *1-connected* if f^* is a fully-faithful 2-functor. If f^* has a further left-adjoint $f_! : \mathbf{H} \to \mathbf{H}'$ we say that f is a *locally* 1*-connected* geometric morphism (first diagram below). If, in addition, $f_!$ preserves finite products we say that f is *strongly 1-connected.* Every 1-connected geometric morphism is locally 1-connected, while a locally 1-connected geometric morphism $(f_*, f^*, f_!)$ is 1-connected iff $f_!$ preserves terminal objects [??]. Thus, we have the following inclusions:

$$\text{strongly 1-connected} \subset \text{1-connected} \subset \text{locally 1-connected}. \tag{4.1}$$

Dually, if f_* has a right-adjoint $f^! : \mathbf{H}' \to \mathbf{H}$, we say that f is a *locally local* geometric morphism (second diagram). If $f^!$ is a fully-faithful 2-functor, we say that f is local. By general facts on adjoint triples it follows that a locally local geometric morphism is local iff it is 1-connected [??]. Finally, we say that f is *locally cohesive* (fourth diagram) if it is locally 1-connected and locally local, and we say that f is *cohesive* if, in addition, it is strongly 1-connected (and, in particular, local).

$$\begin{tikzcd} \mathbf{H} \arrow[r, dashed, bend left=60, "f_!"] \arrow[r, bend right=20, "f_*"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=1, "\perp"] & \mathbf{H}' \arrow[l, "f^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{H} \arrow[r, bend right=20, "f_*"'] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathbf{H}' \arrow[l, bend right=20, "f^*"'] \arrow[l, dashed, bend left=60, "f^!"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{H} \arrow[r, dashed, bend left=60, "f_!"] \arrow[r, bend right=20, "f_*"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathbf{H}' \arrow[l, "f^*"'] \arrow[l, dashed, bend left=60, "f^!"] \end{tikzcd}$$

If $\mathbf{H}$ is a $(2,1)$-pretopos, the endo-2-functors $H : \mathbf{H} \to \mathbf{H}$ are know as *internal modalities.* If $X \in \mathbf{H}$ we say that $H(X) \in \mathbf{H}$ represents *the object X in the modality H*. Every pair of adjoint functors $(R, L) : \mathbf{C} \to \mathbf{D}$ between two 2-categories clearly defines an endofunctor $L \circ R : \mathbf{C} \to \mathbf{C}$ in $\mathbf{C}$. It then follows that every geometric morphism $f : \mathbf{H} \to \mathbf{H}'$ defines a modality $\flat_f : \mathbf{H} \to \mathbf{H}$, given by $\flat_f = f^* \circ f_*$ and known as *f-flat modality* [??].

By general facts on adjoint triples, a triple $L' \dashv L \dashv R$ of adjoint functors induce adjoint endofunctors $(L \circ L') \dashv (L \circ R)$. Thus, every locally 1-connected geometric morphism $(f_*, f^*, f_!) : \mathbf{H} \to \mathbf{H}'$ induces adjoint internal modalities $\int_f \dashv \flat_f$, where $\int_f = f^* \circ f_!$ is called *f-shape modality*. Similarly, a locally local geometric morphism $(f^!, f_*, f^*) : \mathbf{H} \to \mathbf{H}'$ produces adjoint modalities $\flat_f \dashv \sharp_f$, with $\sharp_f = f^! \circ f_*$ known as *f-sharp modality*. More generally, adjoint quadruples induce adjoint triples of endofunctors, so in a locally cohesive geometric morphism (in particular in a cohesive geometric morphism) $(f^!, f_*, f^*, f_!) : \mathbf{H} \to \mathbf{H}'$ we have an adjoint triple of modalities $\int_f \dashv \flat_f \dashv \sharp_f$, as below[1] [??].

$$\begin{tikzcd}
\mathbf{H} \arrow[r, shift left, "f_!"] \arrow[r, shift right, "f_*"'] \arrow[rr, bend left=50, "f^! f_!"] \arrow[rr, bend left=30, "f^* f_!"] \arrow[rr, bend right=30, "f^* f_*"'] \arrow[rr, bend right=50, "f^! f_*"'] & \mathbf{H}' \arrow[r, shift left, "f^*"] \arrow[r, shift right, "f^!"'] & \mathbf{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{H} \arrow[r, bend left=50, "\int_f"] \arrow[r, "\flat_f" description] \arrow[r, bend right=50, "\sharp_f"'] & \mathbf{H}
\end{tikzcd}$$

It is a well known fact that $\mathbf{Grpd}_2$ is a terminal object for the 2-category $(2,1)\mathbf{PTopos}_2$ of $(2,1)$-presheaf $(2,1)$-topoi. Thus, for every $\mathbf{H} \in (2,1)\mathbf{PTopos}_2$ we have an essentially unique geometric morphism $(\mathrm{El}, \mathrm{Cst}) : \mathbf{H} \to \mathbf{Grpd}_2$, known as *terminal geometric morphism* - see [below] for more details on its construction. Since the localization functor defining a Grothendieck (2,1)-topos $L : \mathbf{H}_0 \to \mathbf{H}$ is exact, it follows that $\mathbf{Grpd}_2$ is also a terminal object in $(2,1)\mathbf{ShvTopos}_2$, with terminal geometric morphism given by $(L \circ \mathrm{El}, L \circ \mathrm{Cst})$. Thus, in the context of cohesive Grothendieck $(2,1)$-topoi, it is natural to work with the modalities $\int \dashv \flat \dashv \sharp$ induced by the terminal geometric morphism.

4.2 Geometric Structures

For gauge theories the cohesive Grothencieck $(2,1)$-topos is that of smooth groupoids, and the modalities appear as follows:

- shape modality $\int$ is used define bundles;
- flat modality $\flat$ is used to define differential forms;
- sharp modality $\sharp$ is used to define differential moduli spaces;

Therefore:

- shape modality with flat modality $(\int + \flat)$ are used to define connection forms and curvature forms;
- shape modality with flat modality and sharp modality $(\int + \flat + \sharp)$ are used to define differential moduli spaces of connection forms.

This moduli space of connection forms is precisely the domain of the action functionals that define gauge theories. Therefore, the above formalization means that the kinematics of gauge theories can be internalize in any cohesive $(2,1)$-topos [??,??]. During the last years it has been argued that cohesion is the structure that allows variational calculus to be intrinsically defined, capturing therefore not only the kinematic, but also the dynamical aspects of gauge theories [??]. This project deals exclusively with the abstraction of the kinematic aspects of gauge theories, leaving dynamical aspects to a future project, as explained in the introduction.

In this section we will review the role of the modalities $\int \, dashv \flat \dashv \sharp$ in the synthetic description of the moduli space of connection forms. For a more complete exposition in the

[1] There is also a fourth modality $f^! \circ f_!$, but it is not needed in our context.

$(\infty,1)$-categorical setting, see Section 5.1 and Section 5.2 of [DCCT2], or Section 3.7, Section 3.8 and Section 3.9 of [DCCT1].

4.2.1 Bundles: $\int$

Let $2\mathbf{Top}_2$ be the $(2,1)$-category of (Hausdorff compactly generated) topological spaces, continuous maps and homotopy classes of homotopies - the trucation $\tau_1(2\mathbf{Top}_2)$ is precisely the homotopy category $\mathrm{Ho}(\mathbf{Top})$. In the classification of principal G-bundles, the bijection $\mathrm{Iso}_G(X) \simeq [X; BG]$ is obtained pulling back the universal G-bundle $u_G = EG \to BG$.

The interesting point is that this morphism can be realized in internal groupoids. Indeed, let $\mathrm{Grpd}_2(\mathbf{Top}_2)$ be the (2,1)-category of topological groupoids. It is a sub-2-category of $\mathrm{Grpd}_2(\mathbf{Set}_2) = \mathbf{Grpd}_2$. There is a 2-functor $|\cdot| : \mathrm{Grpd}_2(\mathbf{Set}_2) \to \mathbf{Top}_2$ assigning to every groupoid $\mathbf{G}$ its geometric realization $|\mathbf{G}|_{\mathbf{Top}_2}$. To every topological group $G \in \mathrm{Grp}_2(\mathbf{Top}_2)$ one can associate the discrete $(2,1)$-category $\mathrm{disc}_{2,1}(G)$ and the internal delooping groupoid $\mathbf{B}_{\mathbf{Top}_2}G$. There is a canonical morphism $\mu_G : \mathrm{disc}_{1,2}(G) \to \mathbf{B}_{\mathbf{Top}_2}G$ and its geometric realization is the universal G-bundle, i.e., $|\mu_G| = u_G$.

On the other hand, in the case of $(2,1)$-topos as $2\mathbf{Top}_2$, every groupoid is effective, being completely determined by its representing object. Thus, the geometric realization can be viewed not acting on the canonical functor $\mu_G : \mathrm{disc}_{1,2}(G) \to \mathbf{B}_{\mathbf{Top}_2}G$ between internal groupoids, but actually on the morphisms $* \to B_{\mathbf{Top}_2}G$ between their representing objects.

Thus, we can axiomatize the universal G-bundle in every cartesian (2,1)-category $\mathbf{H}$ in the form of a morphism $* \to B_{\mathbf{H}}G$. The G-bundles properly are then axiomatized by taking maps $f : X \to B_{\mathbf{H}}G$ pulling back with the abstract universal G-bundle, as below.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r, "f"'] & B_{\mathbf{H}}G \end{tikzcd}$$

Definition 4.2.1. *Let $\mathbf{H}$ be a cartesian $(2,1)$-category and let $G \in \mathrm{Grp}_2(\mathbf{H})$ be a group object. A morphism $X \to B_{\mathbf{H}}G$ is said to classify an abstract G-bundle over X. The G-bundle classified by f is the morphism $P \to X$ as in diagram above.*

To really makes sense of $P \to X$ as a bundle, one should build a rule assigning to each "abstract bundle" in $\mathbf{H}$ a "genuine/concrete bundle" in $\mathbf{Top}_2$. This means that we need a "geometric realization 2-functor" $|\cdot|_{\mathbf{H}} : \mathbf{H} \to \mathbf{Top}$ that maps the abstract universal G-bundle in the concrete ones. Thus, $|\cdot|_{\mathbf{H}}$ must preserve group objects, terminal objects and delooping objects, which are all the ingredients that define the universal bundles. It also need to factors through the classical geometric realization functor, in the sense that there is the functor $F : \mathbf{H} \to \mathbf{Grpd}_2$ as below.

$$\begin{tikzcd} \mathbf{H} \arrow[r, "F"'] \arrow[rr, bend left, "|\cdot|_{\mathbf{H}}"] & \mathbf{Grpd}_2 \arrow[r, "|\cdot|"'] & \mathbf{Top}_2 \end{tikzcd} \tag{4.2}$$

Groupoid objects are preserved by cartesian functors, which already preserves terminal objects. On the other hand, delooping objects are $(2,1)$-coequalizer, i.e., $(2,1)$-colimits. Thus, it is suffices that $|\cdot|_{\mathbf{H}}$ preserves finite products and $(2,1)$-colimits. It is a well known fact that the classical geometric realization $|\cdot| : \mathbf{Grpd}_2 \to \mathbf{Top}_2$ preserves finite products. Furthermore, it is a left adjoint to the fundamental groupoid 2-functor $\Pi_0 : \mathbf{Top}_2 \to \mathbf{Top}_2$.

Therefore, it also preserves $(2,1)$-colimits. Consequently, what we need is to find a functor $F : \mathbf{H} \to \mathbf{Grpd}_2$ preserving finite products and $(2,1)$-limits.

Recall that a locally 1-connected $(2,1)$-topos is one such that the terminal geometric morphism $\mathrm{LCst} \dashv \Gamma : \mathbf{H} \to \mathbf{Grpd}_2$ has an additional left adjoint $\Pi : \mathbf{H} \to \mathbf{Grpd}_2$. Being a left adjoint Π preserves $(2,1)$-colimits. By definition, a strongly 1-connected $(2,1)$-topos is one such that Π not only exists, but also preserves finite products. This motivate us to define, in every strongly 1-connected $(2,1)$-topos, the abstract geometric realization as the composition below.

$$\begin{tikzcd} \mathbf{H} \arrow[r, bend left=40, "\Pi"] \arrow[r, bend right=40, "\Gamma"'] \arrow[rr, bend left=50, "|\cdot|_{\mathbf{H}}"] & \mathbf{Grpd}_2 \arrow[l, "\mathrm{LCst}"'] \arrow[r, "|\cdot|"'] & \mathbf{Top}_2 \end{tikzcd}$$

In [??,??,??] it is argued that the 2-functor built in this way really plays the role of an internal geometric realization, mapping abstract bundles into topological bundles, preserving all the structure involved.

Definition 4.2.2. *Let $\mathbf{H}$ be a locally 1-connected (2,1)-topos and let $G \in \mathrm{Grp}_2(\mathbf{H})$ be a group object. The topological realization of an abstract bundle $f : X \to B_{\mathbf{H}}G$ is the topological bundle classified by $|f|_{\mathbf{H}} : |X|_{\mathbf{H}} \to B|G|_{\mathbf{H}}$.*

Remark 4.2.3. *Observe that, a priori, the abstract geometric realization of as abstract bundle $P \to X$ need not agree with the topological realization of P. The classical geometric realization is a left exact functor, so that it also preserves finite $(2,1)$-limits and, in particular, $(2,1)$-pullbacks. But the additional left adjoint $\Pi : \mathbf{H} \to \mathbf{Grpd}_2$ in a strongly 1-connected $(2,1)$-topos is not necessarily exact: it only preserves finite products, so $|\cdot|_{\mathbf{H}}$ need not to preserve $(2,1)$-pullbacks. A $(2,1)$-topos for which Π exists and preserve finite $(2,1)$-limits is called totally 1-connected.*

We could do similar discussion replacing the group object G with k-tuply deloopable object G.

Definition 4.2.4. *Let $\mathbf{H}$ be a cartesian $(2,1)$-category and let G be a k-tuply deloopable object, with $k \geq 0$. An abstract k-G-bundle over X is a morphism $P_k \to X$ classified by a morphism $f : X \to B^k_{\mathbf{H}}G$, as below. The topological realization of $P_k \to X$ is the topological bundle classified by $|f|_{\mathbf{H}} \to B|B^k_{\mathbf{H}}G|_{\mathbf{H}}$.*

4.2.2 Differential Forms: $\flat$

Let $\mathrm{LCst} \dashv \Gamma : \mathbf{H} \to \mathbf{Grpd}_2$ be a Grothencieck $(2,1)$-topos and let $\flat : \mathbf{H} \to \mathbf{H}$ be its flat modality. Recall that $\flat = \mathrm{LCst} \circ \Gamma$. The counit of the adjunction $\mathrm{LCst} \dashv \Gamma$ is a natural transformation $\epsilon : \flat \Rightarrow id$, so for every $A \in \mathbf{H}$ we have a canonical morphism $\epsilon_A : \flat A \to A$. Let $\flat_{dR}A \to \flat A$ denote the homotopy kernel of ϵ_A, i.e, the $(2,1)$-pullback below.

$$\begin{tikzcd} \flat_{dR}A \arrow[r] \arrow[d] & \flat A \arrow[d, "\epsilon_A"] \\ *_2 \arrow[r] & A \end{tikzcd}$$

We say that $\flat_{dR}A$ is the *de Rham refinement of* $\flat A$. Varying $A \in \mathbf{H}$ we get a 2-functor $\flat_{dR} : \mathbf{H} \to \mathbf{H}$, hence a new internal modality: the *flat-de Rham modality.* The reason of the name comes from the fact that $\flat_{dR}A$ serves as the "moduli stack of 0th A-valued flat differential forms" [??,??]. More generally, if A is k-tuply deloopable with $k \geq 0$ (a group object for $k = 1$, a braided group object for $k = 2$ and an abelian/symmetric group object for $k > 2$), then $\flat_{dR}B^kA$ is the "moduli stack of flat differential k-forms".

Of course, one expect some interaction between "A-valued *closed* forms" and "A-valued *flat* forms". With the later already formalized, we can define the former, as follows.

Definition 4.2.5. *For every $(2,1)$-pretopoi $\mathbf{H}$ and for every k-tuply deloopable object A, the moduli stack of closed A-valued k-forms as an object $\Omega^k_{cl}(-;A) \in \mathbf{H}$ endowed with a morphism $\sigma : \Omega^k_{cl}(-;A) \to \flat_{dR}B^kA$ such that:*

1. *the induced functors on hom-groupoids $\sigma_* : \mathbf{H}(X;\Omega^k_{cl}(-;A)) \to \mathbf{H}(X;\flat_{dR}B^kA)$ is an effective epimorphism in $\mathbf{Grpd}_2$[2];*

2. *it is universal relative to this property.*

One can characterize the moduli stack of A-valued closed k-forms as a sub-2-category $\boldsymbol{\Omega}^k_{cl}(-;A)$ of the slice-(2,1)-topos $\mathbf{H}/\flat_{dR}B^kA$ such that

1. its objects satisfy 2. above;

2. there is a unique equivalence between any two objects.

4.2.3 Connection Forms: $\int \dashv \flat$

As discussed above, we can talk about principal G-bundles internally to every locally 1-connected $(2,1)$-topos and, therefore, to every Grothendieck $(2,1)$-topos with the shape modality $\int$. Furthermore, the flat modality $\flat$ allows us to formalize flat k-forms. Here we will see that joining both we can formalize connection forms on principal bundles.

Recall that principal connections can be viewed both as vertical equivariant Lie(G)-valued differential 1-forms $D : TP \to \mathrm{Lie}(G)$ on a principal bundle $P \to X$ (the connection forms) or as families D_i of locally defined differential 1-forms on $D_i : U_i \to \mathrm{Lie}(G)$ (the gauge potential forms) that in the intersections transforms according to gauge transformations, i.e., $A_j = g^{-1}A_ig + g^{-1}dg$. The gauge transformations are nonlinear precisely due the term $\theta_G = g^{-1}dg$ which is an equivariant Lie(G)-valued 1-form in G: the Maurer-Cartan form.

Therefore, to internalize principal connections on $(2,1)$-topoi we first need to internalize the concepts of "algebra-valued differential form" and of "Maurer-Cartan form". Since we already have a formalization of algebra-valued forms, let us focus on the Maurer-Cartan forms. In the plain smooth case, it assigns to each element $g \in G$ a Lie(G)-valued 1-form, hence a function $\theta : G \to \Omega^1(G;\mathfrak{g})$. Thus, by the interpretation above the abstract Maurer-Cartan form should be a morphism $\theta : G \to \flat_{dR}BG$. That morphism is obtained by universality in any locally 1-connected $(2,1)$-topoi, as below. There we used the definition of the delooping object as a $(2,1)$-coequalizer and the fact that in an adjunction modality $\int \dashv \flat$ arising from a triple adjunction $\Pi \dashv \mathrm{LCst} \circ \Gamma$ the right modality $\flat$ preserves limits and, therefore, terminal

[2]Actually one only need to require this effectiveness hypothesis for the objects $X \in \mathbf{H}$ that are equivalent to internal manifolds.

objects.

$$
\begin{tikzcd}
\flat_{dR}BG \arrow[r] \arrow[d] & \flat BG \arrow[d] & \\
* \arrow[r, "x"] & BG & * \arrow[l] \\
& * \arrow[u] & G \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[rr] \arrow[dr, dashed, "\theta"] \arrow[ddr, bend right] & & \flat * \simeq * \arrow[d, "\flat x"] \\
& \flat_{dR}BG \arrow[r] \arrow[d] & \flat BG \arrow[d] \\
& * \arrow[r, "x"] & BG
\end{tikzcd}
$$

More generally, if G is $k > 0$-tuply deloopable, we get an universal morphism $\theta_k : B^{k-1}G \to \flat_{dR}B^kG$.

We are interested in connection forms that need not be flat. This means that they comes equipped with closed Lie(G)-valued 2-forms representing the curvature. On the other hand, recall that the Maurer-Cartan form also appear in the expression of the curvature of a principal connection, suggesting that the universal morphism $\theta_2 : BG \to \flat_{dR}B^2G$ of a braided group object models precise the universal curvature characteristics. Thus, the formalization of connection form lives in the "intersection" of Lie(G)-valued 2-forms with those Maurer-Cartan 2-forms. Since we are in a $(2,1)$-categorical setting, that intersection would be a homotopy intersection, hence a $(2,1)$-pullback.

Definition 4.2.6. *Let $\mathbf{H}$ be a locally 1-connected $(2,1)$-topoi. For a given braided group object $G \in \mathbf{H}$, the moduli stack of connection G-forms is the $(2,1)$-pullback below.*

$$
\begin{tikzcd}
B_{conn}G \arrow[r] \arrow[d] & \Omega^2_{cl}(-;). \arrow[d, "\sigma_2"] \\
BG \arrow[r, "\theta_2"'] & \flat_{dR}B^2G
\end{tikzcd}
$$

Previously we modeled abstract G-bundles over an object $X \in \mathbf{H}$ as being classified by morphisms $f : X \to BG$. We now define an abstract principal G-connection on the bundle classified by $f : X \to BG$ as a lifting to $B_{con}G$.

$$
\begin{tikzcd}
& B_{con}G \arrow[r] \arrow[d] & \Omega^2_{cl}(-;) \arrow[d, "\sigma_2"] \\
X \arrow[r, "f"'] \arrow[ur, dashed, "\hat{f}"] & BG \arrow[r, "\theta_2"'] & \flat_{dR}B^2G
\end{tikzcd}
$$

More generally, we can define connections on k-bundles, with $k \geq 0$, replacing the curvature characteristics with the kth Maurer-Cartan forms θ_k.

Definition 4.2.7. *Let $\mathbf{H}$ be a cartesian $(2,1)$-category and let $G \in \mathrm{Del}_2^{k+1}(\mathbf{H})$ be a $k+1$-tuply deloopable object, with $k \geq 0$. The moduli stack of G-connection k-forms is the first $(2,1)$-pullback below. If $P_k \to X$ is a $G-k$-bundle classified by $f : X \to B^k_{\mathbf{H}}G$, a connection on P_k is a lifting of f as in the second diagram below.*

$$
\begin{tikzcd}
B^k_{con}G \arrow[r] \arrow[d] & \Omega^{k+1}_{cl}(-;G) \arrow[d] \\
B^kG \arrow[r] & \flat_{dR}B^{k+1}G
\end{tikzcd}
\qquad
\begin{tikzcd}
& B^k_{con}G \arrow[d] \\
X \arrow[r, "f"'] \arrow[ur, "\hat{f}"] & B^kG
\end{tikzcd}
$$

The topological/concrete realization of those abstract connection k-forms on abstract bundles is then obtained applying the abstract geometric realization $|\cdot|_{\mathbf{H}} : \mathbf{H} \to \mathbf{Grpd}_2$. Observe, however, that the result is not a connection in the classical sense: the geometric realization returns topological data, while connections are smooth/differential structure.

4.2.4 Moduli of Connection Forms: $\int \dashv \flat \dashv \sharp$

Previously we reviewed how the adjoint modality $\int \dashv \flat$ can be used to axiomatize the moduli stack of connection forms. By a moduli stack we mean an object $BG_{con} \in \mathbf{H}$ such that for every other object $X \in \mathbf{H}$ the hom-groupoid $\mathbf{H}(X; BG_{con})$ is the groupoid of G-connection forms on X.

In the study of (higher) gauge theories we are interested in functionals which are defined in the "space of $G - k$-connections on X". Thus, to internalize (higher) gauge theories in a given language L, the later must contain these "spaces of k-connections" as objects of L. The obvious attempt to define the "space of k-connections" over X is as the hom-groupoid $\mathbf{H}(X; B^k_{con}G)$. But, in this case, we do not have an *object of* $\mathbf{H}$, but an *object of* $\mathbf{Grpd}_2$. Therefore, we need to replace the hom-groupoid $\mathbf{H}(X; B^k_{con}G)$ with some object $G\mathbf{Conn}^k(X) \in \mathbf{H}$.

Since a $(2,1)$-topos is a cartesian $(2,1)$-category, it comes equipped with its cartesian monoidal structure. It is closed, hence we have the hom-objects $[X; Y]_{\mathbf{H}} \in \mathbf{H}$. This is the natural choice to replace the hom-groupoids. However, as argued in [??] it is not the correct choice to describe the "space of k-connections". The problem appears even for $G = U(1)$ and $k = 1$ in the $(2,1)$-topos of smooth groupoids, as detailed discussed in Section 6.4.16 of [DCCT2]. The basic idea is the following.

Being an object of a Grothendieck $(2,1)$-topos, $U(1)\mathbf{Conn}(X)_{\mathbf{H}}$ should be a stack, hence a 2-functor $U(1)\mathbf{Conn}(X) : \mathbf{H}^{op} \to \mathbf{Grpd}_2$ satisfying descend conditions. On the other hand, it should describe the "space of $U(1)$-connections over X". There is a natural notion of morphism between connections, given by gauge transformations, so that they form a groupoid. Furthermore, connections satisfy descend conditions. Thus, we would like that $U(1)\mathbf{Conn}(X)_{\mathbf{H}}$ reproduce this natural structure. The hom-object $[X; B_{con}U(1)]_{\mathbf{H}}$ fail precisely in this point.

Luckily, as discussed in Section 6.4.16 of [DCCT2] for the $G = U(1)$ and $k = 1$ case, using the additional sharp modality $\sharp$ in the $(2,1)$-topos of smooth groupoids one can modify $[X; B_{con}U(1)]_{\mathbf{H}}$ to produce the correct moduli object. Furthermore, since the construction depends only on the locally local structure, in Section 5.2.13.4 of [DCCT2], the same discussion is used to define $G\mathbf{Conn}^k(X)$ for every G, k and X, in every locally local $(2,1)$-topos. In the following we review the construction.

(.......)

4.3 Geometric Cohomology

(....)

4.3.1 Intrinsic Cohomology

There is a zoo of notions of cohomology in many areas of mathematics. Many of them seem to be very different in nature. In his seminal paper [??] from 1973, Ken Brown showed, however, that the main examples of cohomology (as abelian sheaf cohomology, singular cohomology, generalized cohomology, group cohomology, chain/cochain cohomology and nonabelian abelian) are just flavors/realizations of a same concept: 0-truncation (i.e, connected components set) of hom-functors on a category of simplicial objects.

On the other hand, Lurie's noticed that the construction makes sense in every Grothendieck $(\infty, 1)$-topos $\mathbf{H}$, suggesting that every $\mathbf{H}$ acquires a canonical notion of "abstract/intrinsic cohomology" - see Section 7.7.2 (in particular Definition 7.2.2.14, page 723) of [HTT]. The general definition, makes sense, however, in every $(\infty, 1)$-category, as pointed out, for example, in [DCCT2].

In our $(2,1)$-categorical context this goes as follows.

Definition 4.3.1. *Let $\mathbf{H}$ be a $(2,1)$-category and let $A \in \mathbf{H}$ be an object. For every $X \in \mathbf{H}$ define the 0th-cohomology of X with coefficients in A as*

$$H^0(X; A) := \tau_{\leq 0}\mathbf{H}(X; A).$$

Suppose, now that $\mathbf{H}$ is a cartesian $(2,1)$-category and that A is k-tuply deloopable. Define the kth-cohology of X with coefficients in A as

$$H^k(X; A) := \tau_{\leq 0}\mathbf{H}(X; B^k_{\mathbf{H}}A) = H^0(X; B^k_{\mathbf{H}}A).$$

An immediate corollary of this definition is the following.

Lemma 4.3.2. *Let $\mathbf{H}$ be a cartesian $(2,1)$-category and let $A \in \mathbf{H}$ be an object. Then:*

1. *A is a group object iff the kth cohomology $H^k(X; A)$, with $k = 0, 1$ is defined for every $X \in \mathbf{H}$;*
2. *A is a braided group object iff the kth cohomology $H^k(X; A)$, with $k = 0, 1, 2$ is defined for every $X \in \mathbf{H}$;*
3. *A is a commutative group object iff the kth cohomology $H^k(X; A)$, with $k = 0, 1, 2, 3$ is defined for every $X \in \mathbf{H}$;*
4. *A is a commutative group object iff the kth cohomology $H^k(X; A)$ is defined for every $X \in \mathbf{H}$ and every $k \geq 0$.*

Proof. Direct from the characterization of k-tuply deloopable objects in Theorem ??. $\square$

For a fixed $X \in \mathbf{H}$, intrinsic cohomology of X defines 2-functor $H(X; -) : \mathbf{H}_1 \to \mathbf{Set}_1$ in the 2-category of coefficients as in the composition below.

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\mathbf{H}(X;-)"'] \arrow[rr, bend left, "H(X;-)"] & \mathbf{Grpd}_2 \arrow[r, "\tau_{\leq 0}"'] & \mathbf{Set}_2 \end{tikzcd}$$

This functor is the 0-truncation of a representable 2-functor, hence it is representable. On the other hand, representable functors preserve limits, so that they are cartesian $(2,1)$-functors and, therefore, preserve group objects, braided group objects and commutative group objects. Thus, if $A \in \mathbf{H}$ is a group object, a braided group object or a commutative group object, then $H(X; A)$ is so, for every $X \in \mathbf{H}$, meaning that there are the dotted arrows in diagram below.

$$\begin{tikzcd}
\mathrm{ComGrp}_2(\mathbf{H}) \arrow[d, hook] \arrow[rrddd, squiggly, "H(X;-)"] & & \\
\mathrm{BrGrp}_2(\mathbf{H}) \arrow[d, hook] \arrow[rrdd, dotted, "H(X;-)"] & & \\
\mathrm{Grp}_2(\mathbf{H}) \arrow[d, hook] \arrow[rrd, dashed, "H(X;-)"] & & \\
\mathbf{H} \arrow[r, "\mathbf{H}(X;-)"'] \arrow[rr, bend left, "H(X;-)"] & \mathbf{Grpd}_2 \arrow[r, "\tau_{\leq 0}"'] & \mathbf{Set}_2
\end{tikzcd}$$

But since $\mathbf{Set}_2$ is a discrete 2-category, its braided group objects are commutative. Thus:

Lemma 4.3.3. *Let $\mathbf{H}$ be a cartesian $(2,1)$-category and let $A \in \mathbf{H}$ be an object. Then*

1. *if $H^1(X;A)$ is defined for every $X \in \mathbf{H}$, then it is actually a group;*

2. *if $H^k(X;A)$ is defined for every $X \in \mathbf{H}$ and $k > 1$, then it is an abelian group.*

The result above motivates us to call $H^1(X;A)$ of *nonabelian cohomology* when $H^k(X;A)$ is not defined for $k > 1$.

4.3.2 Modal Cohomologies

We saw that differential geometry can be synthetically described in any $(2,1)$-topos with additional modalities. The presence of each modality induces a new flavor of cohomology.

Definition 4.3.4. *Let $\mathbf{H}$ be a cartesian $(2,1)$-category and let $F : \mathbf{H} \to \mathbf{H}$ be a modality in $\mathbf{H}$.*

1. *If $A \in \mathbf{H}$ is an arbitrary object, define the 0th F-cohomology of $X \in \mathbf{H}$ with coefficients in A as $H^0_F(X;A) := H^0(X;F(A))$.*

2. *If $F(A)$ is k-tuply deloopable, define the kth F-cohomology of $X \in \mathbf{H}$ with coefficients in A as $H^k_F(X;A) := H^0(X;B^kF(A))$.*

Note that If the modality $F : \mathbf{H} \to \mathbf{H}$ preserves $(2,1)$-coequalizers, then it commutes with taking delooping objects. This is the case, in particular, if F is the left-adjoint of an adjoint modality. As a consequence,

Lemma 4.3.5. *Let $F : \mathbf{H} \to \mathbf{H}$ be a modality in a cartesian $(2,1)$-category. If it preserves $(2,1)$-coequalizers and $A \in \mathbf{H}$ is k-tuply deloopable, then $F(A)$ is k-tuply deloopable and we have $B^kF(A) \simeq F(B^kA)$. Consequently, $H^k_F(X;A)$ is defined and $H^k_F(X;A) \simeq H^0_F(X;B^kA)$.*

In every Grothencieck $(2,1)$-topos $\mathbf{H}$ we have the flat modality $\flat : \mathbf{H} \to \mathbf{H}$ and its refinement to the flat-de Rham modality $\flat_{dR}$. Therefore, for every $A, X \in \mathbf{H}$ we have the *flat cohomology* $H^0_\flat(X;A) = H^0(X;\flat A)$ and the *flat-de Rham cohomology* $H^0_{dR}(X;A) = H^0(X;\flat_{dR}A)$.

In a locally 1-connected Grothendieck $(2,1)$-topos we have a left adjoint to the flat modality given by the shape modality $\int \dashv \flat$. Thus, it is defined the *shape cohomology* $H^0_\int(X;A) := H^0(X;\int A)$. In a locally local Grothendieck $(2,1)$-topos the flat modality has a right adjoint given by the sharp modality $\flat \dashv \sharp$. Therefore, we can define the *sharp modality* $H^0_\sharp(X;A) := H^0(X;\sharp A)$. In a locally cohesive $(2,1)$-topos we have both flat, shape and sharp cohomologies.

Proposition 4.3.6. *Let $\mathbf{H}$ be cartesian $(2,1)$-category and let $A \in \mathbf{H}$ be a k-tuply deloopable object.*

1. *If $\mathbf{H}$ is locally 1-connected, then $\int A$ is k-tuply deloopable and $B^k \int A \simeq \int B^k A$. In particular, $H^k_{\int}(X;A) \simeq H^0_{\int}(X;B^k A)$.*

2. *If $\mathbf{H}$ is locally local, then $\flat A$ and $\flat_{dR} A$ are k-tuply deloopable, and $B^k \flat A \simeq \flat B^k A$ and $B^k \flat_{dR} A \simeq \flat_{dR} B^k A$. In particular, $H^k_{\flat}(X;A) \simeq H^0_{\flat}(X;B^k A)$ and $H^k_{dR}(X;A) \simeq H^0_{dR}(X;B^k A)$.*

Proof. Directly from the last lemma and from the fact that, under the hypotheses, $\flat$, $\flat_{dR}$ and $\int$ are left adjoints. □

4.3.3 Differential Cohomology

Previously we saw that the moduli of G-connections on k-bundles $B^k_{con}G$, with $k \geq 0$, can be internalized in any locally 1-connected $(2,1)$-topos $\mathbf{H}$ when G is $k+1$-tuply deloopable. It is natural to ask if $G \mapsto B^k_{con}G$ extends to a modality in $\mathbf{H}$. More precisely, it would be interesting to find a 2-functor $(-)_{con} : \mathbf{H} \to \mathbf{H}$ such that $(B^k G)_{con} \simeq B^k_{con}G$. In this case, we could consider its induced modal cohomology. The problem is that $B^k_{con}G$ exists only if G is $k+1$-deloopable, hence the functor $(-)_{con}$ would not be defined for all objects, but only for group objects. Thus, the first point is to define $B^0_{con}A$ for an arbitrary object $A \in \mathbf{H}$.

By definition $B^0_{con}A$ is the $(2,1)$-pullback below. Thus, we see that the hypothesis of A to be a group is used only to consider the Maurer-Cartan form $\theta : A \to \flat_{dR} BA$. Thus, what we need is to extend the Maurer-Cartan form to arbitrary objects.

(diagram)

We sketch the construction of two possible extensions of the Maurer-Cartan form. The first one makes sense in every Grothendieck $(2,1)$-topos, while the second one makes sense only in locally 1-connected $(2,1)$-topos.

1. Recall, in addition, that looping and delooping form an equivalence between group objects and pointed connected objects. Applying the functor Ω we then see that Maurer-Cartan form is equivalent to a morphism $\Omega\theta : \Omega A \to \Omega \flat_{dR} BA$. But both $\flat_{dR} X$ and ΩX are defined in terms of $(2,1)$-pullbacks, which commute. Thus, $\Omega \flat_{dR} BA \simeq \flat_{dR} \Omega BA \simeq A$, where in the last identification we used again that looping and delooping are one the inverse of the other. Therefore, to the Maurer-Cartan form corresponds uniquely a morphism $\Omega\theta : \Omega A \to \flat_{dR} A$, which makes sense for every object $A \in \mathbf{H}$[3].

2. Recall the adjunction $\int \dashv \flat$. It extends to the de Rham refinements as $\int_{dR} \dashv \flat_{dR}$. Thus, the Maurer-Cartan form corresponds uniquely to a morphism $\vartheta : \int_{dR} A \to BA$. Applying again the looping functor, we find that the Maurer-Cartan form is equivalent to a morphism $\Omega\vartheta : \Omega \int_{dR} A \to A$, which makes sense for every $A \in \mathbf{H}$.

The following natural speculation should be developed during this project.

Conjecture 4.3.7. *The first (resp. second) construction above of an extension for the Maurer-Cartan form define a modality $(-)_{con,\flat} : \mathbf{H} \to \mathbf{H}$ (resp. $(-)_{con,\int} : \mathbf{H} \to \mathbf{H}$) in every locally 1-connected Grothendieck $(2,1)$-topos. Furthermore, they are adjoint modalities, with adjunction $(-)_{con,\int} \dashv (-)_{con,\flat}$ induced by the adjunction $\int \dashv \flat$.*

[3]This strategy was implicitly used in [??] in the case of the $(\infty,1)$-category of topological spaces, presented by the Homotopy Hypothesis. See Definition 4.38 of

Remark 4.3.8. *A third approach to extend the Maurer-Cartan form is to consider a "group replacement" (or "group completion") 2-functor* $\mathrm{gr}(-) : \mathbf{H} \to \mathrm{Grp}_2(\mathbf{H})$ *such that* A *and* $\mathrm{gr}(A)$ *are "universally closer", in some sense, and which is the identity when restricted to group objects. One could then consider the Maurer-Cartan form* $\theta : A \to \flat_{dR} B\,\mathrm{gr}(A)$ *for* $\mathrm{gr}(A)$.

Definition 4.3.9. *Let* $\mathbf{H}$ *be a locally 1-connected Grothendieck* $(2,1)$*-topos and let* G *be* $k+1$*-tuply group object. The*

4.3.4 Classified Structures

Recall that in the context of Grothendieck 1-topoi we have the notion of *classifying topos* for a theory T. This is a Grothendieck 1-topos $\mathbf{C}[T]$ with a model $M \in \mathrm{Mod}_{\mathbf{C}[T]}(T)$ of T in $\mathbf{C}[T]$ such that for every other Grothendieck 1-topos $\mathbf{C}'$ there is an equivalence of 1-categories

$$\mathrm{Mod}_{\mathbf{C}'}(T) \simeq 1\mathbf{ShvTopos}_2(\mathbf{C}'; \mathbf{C}[T])$$

obtained by composition with the given model M. Taking its 0-truncation, from the broad definition of intrinsic cohomology introduced above, we see that the right-hand side is precisely the intrinsic cohomology of

Notice, now, that the same definition could be reproduced internal to any 2-category $\mathbf{H}$ instead of in the 2-category $1\mathbf{ShvTopos}_2$ of Grothendieck 1-topos.

See also Section 9.3 of [dissertacao] and Section 5.1.10.1 of [DCCT2].