Lie Algebroidal Categories

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Plain

- 1. Categorification
- 2. Lie Algebroidal Categories
- 3. Connections with Lie Algebroids
- 4. Speculations

Lie Algebroids



Categorification

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Lie Algebroids

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Consists in two steps:

- 1. to describe the set-theoretic concept to be categorified terms of categorical structures in **Set**;
 - 2. to internalize the defining categorical structures in **Cat**.

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Lie Algebroids

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Categorification

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Lie Algebroids

Vertical Categorification

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Categorification

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 - 2. a function $*: X \times X \rightarrow X$;
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Lie Algebroidal Categories

Lie Algebroids

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Examples

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Categorification

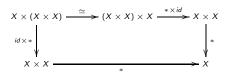
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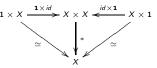
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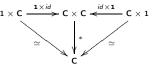
- ➤ a "monoid" is given by:
 - 1. a set *X*;
 - 2. a function $*: X \times X \to X$;
 - 3. a function $e: 1 \rightarrow X$, where 1 is a singleton;
 - 4. commutative diagrams





- ▶ a categorified monoid ("monoidal category") is given by:
 - 1. a category **C**;
 - 2. a functor $*: \mathbf{C} \times \mathbf{C} \to \mathbf{C}$;
 - 3. a functor $e: 1 \to \mathbf{C}$, where 1 is the categorical singleton;
 - 4. analogous commutative diagrams

$$\begin{array}{c|c}
C \times (C \times C) & \xrightarrow{\simeq} & (C \times C) \times C & \xrightarrow{* \times id} & C \times C \\
\downarrow id \times * & & & \downarrow * \\
C \times C & & & & & \downarrow C
\end{array}$$



set-theoretic concept	vertical categorification
monoid	monoidal category

Table: Examples of Vertical Categorification

set-theoretic concept	vertical categorification
monoid	monoidal category
commutative monoid	braided monoidal category
	symmetric monoidal category
group	monoidal category s.t $Ob(C) \subset Pic(C)$
free and fin. gen. monoid	rigid monoidal category
:	<u>:</u>

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- ► Folklore: Let "P" be some algebraic concept which can be
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Lie algebra	??

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Lie Algebroidal Categories

Lie Algebroids

Speculations

Categorification

- ► A "Lie algebra²" is given by:

²We are working over \mathbb{Z} .

$$\begin{bmatrix} x, y \end{bmatrix} + \begin{bmatrix} y, x \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$[x, [y, z]] + [z, [x, y]]$$

- ► A "Lie algebra²" is given by:
 - 1. an abelian group g;

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Remark: 3. is not categorical!



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set-theoretic concept	vertical categorification
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Lie algebra	Lie algebroidal category of Type I
	Lie algebroidal category of Type II
	Lie algebroidal category of Type III

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Table: Vertical Categorification of Lie Algebra

<u>Remark:</u> It seems (at least to me) that the corresponding categories LieCatI, LieCatII and LieCatIII are not equivalent.

- Consists in taking a set-theoretic concept "P" and finding for a class of categories Cat_P such that categories C ∈ Cat_P with a single object are equivalent to "P".
- ► In simple terms, horizontally categorifying "P" is to search for some way of considering a "many objects version of P"

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Examples

- ▶ If "P" is the concept of group, Cat_P is the category of all categories in which every morphism is an isomorphism.
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Table: Examples of Horizontal categorification

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group	group <u>oid</u>
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Table: Examples of Horizontal categorification

Categorification

► A *Lie algebroid* over a manifold *M* is given by:

- 1 a vector bundle $F \rightarrow M$
- 2. a Lie algebra structure on the space of gloal sections $\Gamma(E)$
- 3. a vector bundle morphism $\rho: E \to TM$ such that
- 3.1 it is a derivation relative to the action of the ring $C^{\infty}(M)$; 3.2 the induced map $\Gamma(\rho):\Gamma(E)\to\Gamma(TM)$ is a morphism of Lie
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set-theoretic concept	vertical categorification
group	groupoid
Lie algebra	Lie algebroid

Table: Horizontal Categorification of Lie Algebra

Speculations

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Categorification

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- ► This motivates us to ask:
- Question: Can we embed LieAlgd_M in some category of categories C ⊂ Cat such that when regarded as object of this new category "Lie algebroid" becomes a horizontal categorification of "Lie algebra"?

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From Lie Algebroids to Lie Algebras

- \blacktriangleright We propose a solution (at least after restriction to \mathbb{Z}):
- ▶ Theorem: For every manifold M the category $LieAlgd_{\mathbf{M}}^{\mathbb{Z}}$ of \mathbb{Z} -Lie algebroids over M can be fully embedded in LieCatIII. Furthermore, inside LieCatIII the one-object limit of $LieAlgd_{\mathbf{M}}^{\mathbb{Z}}$ coincides with the one-point limit.

► The idea is to use the following steps:

- 1 to consider a definition/result about Lie algebras:
- 2. to redefine/reprove the result using only categorical structure
- 3. to apply vertical categorification in order to get an analogous definition/result internal to LieCatIII:
- 4. to use the previous theorem in order to get a definition/result about Lie algebroids.

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- **Example:** what is the Lie algebroid version of the classification of complex semisimple Lie algebras?
- ▶ <u>OBS</u>: a "semisimple Lie algebroid" should be a Courant algebroid fulfilling additional conditions...
- **Example:** what is representation theory for Lie algebroids??
- and so on.

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- by the very notion of vertical categorification, the category of representations of a Lie algebra should belong to LieCatI, LieCatII or LieCatIII.
- can we build some version of Tannaka duality for "nonassociative algebraic objects"?
- ▶ in affirmative case, any Lie algebroidal category should be equivalent (under Tannaka duality) to the category of representations of some Lie algebra.
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