APPENDIX B

EULER PARAMETERS

B.1 Euler Parameters Introduction

This appendix presents the Euler parameters formulation used to avoid the singularity problem associated with general rotations between coordinate frames. Body attached coordinate frames are convenient for defining the changes in orientation experienced by moving bodies. This change in orientation is referred to as a rotation. For the simplest case, this rotation occurs about a single axis shared by both frames. These single axis rotations are often referred to as simple rotations. Consider the simple rotation shown in Fig. B.1 where the frame $\widehat{\mathbf{A}}$ rotates with respect to an intuitively known and non-moving frame $\widehat{\mathbf{N}}$. From this diagram, a series of claims can be made defining the axes of the $\widehat{\mathbf{A}}$ frame with respect to the $\widehat{\mathbf{N}}$ frame:

$$\begin{aligned}
\widehat{\mathbf{A}}_1 &= cos(\theta)\widehat{\mathbf{N}}_1 + sin(\theta)\widehat{\mathbf{N}}_2 \\
\widehat{\mathbf{A}}_2 &= -sin(\theta)\widehat{\mathbf{N}}_1 + cos(\theta)\widehat{\mathbf{N}}_2 \\
\widehat{\mathbf{A}}_3 &= \widehat{\mathbf{N}}_3
\end{aligned} \tag{B.1}$$

These claims can be written in vector-matrix form:

$$\begin{bmatrix} \widehat{\mathbf{A}}_1 \\ \widehat{\mathbf{A}}_2 \\ \widehat{\mathbf{A}}_3 \end{bmatrix} = \begin{bmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{N}}_1 \\ \widehat{\mathbf{N}}_2 \\ \widehat{\mathbf{N}}_3 \end{bmatrix} \tag{B.2}$$

$$\widehat{\mathbf{A}} = {}^{\mathrm{N}}_{\mathrm{A}}R \, \widehat{\mathbf{N}}$$

where ${}^{\mathrm{N}}_{\mathrm{A}}R$ defines the rotation of frame $\widehat{\mathbf{A}}$ with respect to frame $\widehat{\mathbf{N}}$. Recognize that this matrix transforms vectors defined in the $\widehat{\mathbf{N}}$ frame into the $\widehat{\mathbf{A}}$ frame. For transforming vectors defined in the $\widehat{\mathbf{A}}$ frame into the $\widehat{\mathbf{N}}$ frame, the transpose of ${}^{\mathrm{N}}_{\mathrm{A}}R$ may be used:

$$\begin{aligned}
\widehat{\mathbf{N}} &= {}^{\mathrm{A}}_{\mathrm{N}}R \, \widehat{\mathbf{A}} \\
{}^{\mathrm{A}}_{\mathrm{N}}R &= {}^{\mathrm{N}}_{\mathrm{A}}R^T
\end{aligned} \tag{B.3}$$

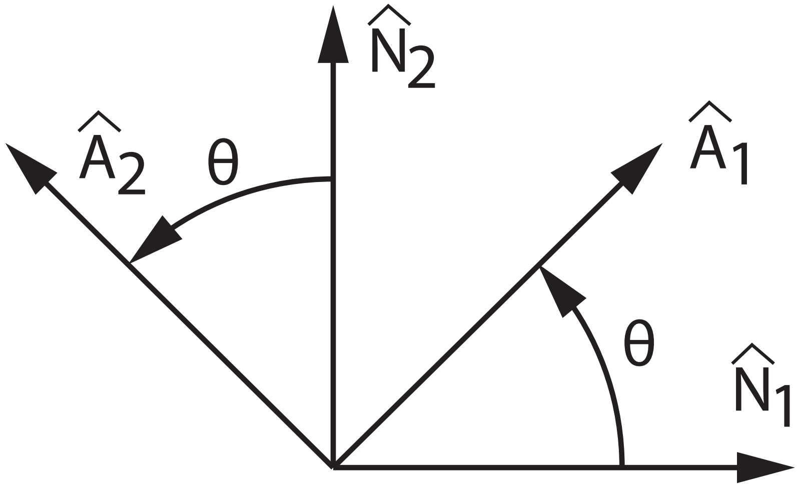

Figure B.1. Simple rotation about a shared $\widehat{\mathbf{3}}$ axis.

These simple rotations can also be staged in series across multiple frames:

$$\begin{aligned} \widehat{\mathbf{B}} &= {}^{\mathrm{A}}_{\mathrm{B}}R\, \widehat{\mathbf{A}} \\ \widehat{\mathbf{A}} &= {}^{\mathrm{N}}_{\mathrm{A}}R\, \widehat{\mathbf{N}} \\ \longrightarrow\ \widehat{\mathbf{B}} &= {}^{\mathrm{A}}_{\mathrm{B}}R\, {}^{\mathrm{N}}_{\mathrm{A}}R\, \widehat{\mathbf{N}} \ = \ {}^{\mathrm{N}}_{\mathrm{B}}R\, \widehat{\mathbf{N}} \end{aligned} \tag{B.4}$$

Defining the general rotation of one frame with respect to another frame is most intuitively done using Euler angles, a sequence of three simple rotations about any set of non-consecutive orthogonal vectors:

$$ {}^{\mathrm{N}}_{\mathrm{A}}R \ = \ R_1(\theta_1)\ R_2(\theta_2)\ R_3(\theta_3) \tag{B.5}$$

where each rotation angle θ_i is the Euler angle. This definition shows that for an arbitrary rotation, there are three degrees of freedom. In the dynamics of aircraft, these Euler angles are often referred to as roll, pitch, and yaw.

However, regardless of the sequence of vectors chosen - be they defined in inertial or body frame - the resulting rotation matrix is susceptible to singularities at certain orientations. These singularities occur when the middle angle is such that the first and third rotation axes are aligned. A singularity here refers to a mathematical event in which a matrix loses rank or a quantity becomes ambiguous. In the case of Euler angles, the individual angles become unrecoverable and ambiguous. This singularity can lead to failures in simulations and control mechanisms.

Many alternatives to Euler angles exist for defining general rotations, such as Principle Axis-Angles and Rodrigues Parameters. For further discussion on these two formulations, see [160]. Another alternative to Euler angles is Euler parameters, a set of four homogenous coordinates:

$$\mathbf{e} \;=\; [\; e_0 \; e_1 \; e_2 \; e_3 \;]^T \tag{B.6}$$

where $\mathbf{e}$ denotes the four-tuple of Euler parameters. These four Euler parameters can be related to the Principle Axis-Angles rotation definition. Considering the general rotation shown in Fig. B.2, the Euler parameters are defined by:

$$\begin{aligned} e_0 \;&=\; cos\Big(\frac{\phi}{2}\Big) \\ e_1 \;&=\; cos(\theta_1)\; sin\Big(\frac{\phi}{2}\Big) \\ e_2 \;&=\; cos(\theta_2)\; sin\Big(\frac{\phi}{2}\Big) \\ e_3 \;&=\; cos(\theta_3)\; sin\Big(\frac{\phi}{2}\Big) \end{aligned} \tag{B.7}$$

where

$$cos(\theta_1) \;=\; \widehat{\mathbf{n}} \cdot \widehat{\mathbf{N}}_1 \qquad cos(\theta_2) \;=\; \widehat{\mathbf{n}} \cdot \widehat{\mathbf{N}}_2 \qquad cos(\theta_3) \;=\; \widehat{\mathbf{n}} \cdot \widehat{\mathbf{N}}_3 \tag{B.8}$$

and $\widehat{\mathbf{n}}$ denotes the principle axis of rotation and ϕ is a rotation angle about that principle axis.

Recall that the definition of Euler angles shows that there are three degrees of freedom associated with an arbitrary rotation, so some constraint must exist amongst

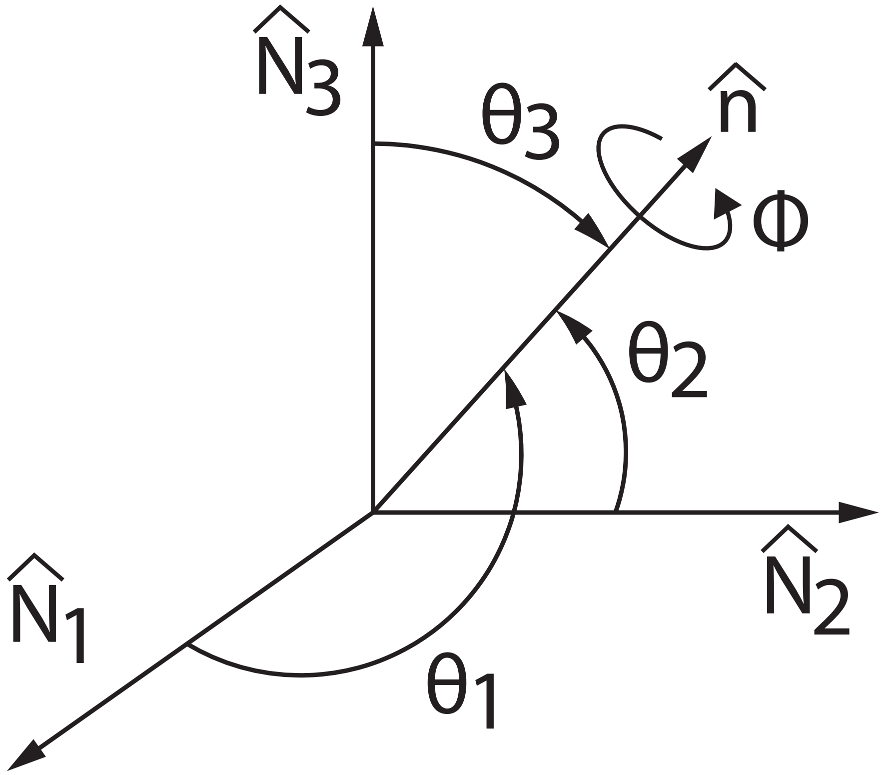

Figure B.2. Rotation about an arbitrary axis $\widehat{\mathbf{n}}$.

the Euler parameters to address the fourth coordinate. A normality constraint is often implemented of the form:

$$\begin{aligned} e_0^2 \; + \; e_1^2 \; + \; e_2^2 \; + \; e_3^2 \; &= \; 1 \\ \mathbf{e}^T\mathbf{e} \; &= \; 1 \end{aligned} \tag{B.9}$$

where the derivative of Eqn. (B.9) provides the useful speed constraint:

$$\begin{aligned} e_0\dot{e}_0 \; + \; e_1\dot{e}_1 \; + \; e_2\dot{e}_2 \; + \; e_3\dot{e}_3 \; &= \; 0 \\ \dot{\mathbf{e}}^T\mathbf{e} \; + \; \mathbf{e}^T\dot{\mathbf{e}} \; &= \; 0 \\ \rightarrow \; \dot{\mathbf{e}}^T\mathbf{e} \; &= \; 0 \end{aligned} \tag{B.10}$$

Given a set of Euler parameters $\mathbf{e}$, the rotation matrix associated with transforming vectors from the $\widehat{\mathbf{A}}$ frame to $\widehat{\mathbf{N}}$ frame is given by:

$$ {}^{\mathrm{A}}_{\mathrm{N}}R = 2 \begin{bmatrix} \frac{1}{2} - e_2^2 - e_3^2 & e_1 e_2 - e_3 e_0 & e_1 e_3 + e_2 e_0 \\ e_1 e_2 + e_3 e_0 & \frac{1}{2} - e_1^2 - e_3^2 & e_2 e_3 - e_1 e_0 \\ e_1 e_3 - e_2 e_0 & e_2 e_3 + e_1 e_0 & \frac{1}{2} - e_1^2 - e_2^2 \end{bmatrix} \tag{B.11} $$

The same rotation matrix may also be generated from:

$$ \begin{aligned} {}^{\mathrm{A}}_{\mathrm{N}}R &= I_3 \big[1 - 2\ \epsilon^T \epsilon \big] + 2\ \epsilon\ \epsilon^T + 2\ e_0 \epsilon \times \\ \epsilon &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}^T \end{aligned} \tag{B.12} $$

where I_3 denotes a 3×3 identity matrix.

B.2 Angular Velocity

The angular velocity vector $\boldsymbol{\omega}$ can be related to the time derivatives of the Euler parameters:

$$ \dot{\mathbf{e}} = \frac{1}{2} \begin{bmatrix} -e_1 & -e_2 & -e_3 & e_0 \\ e_0 & -e_3 & e_2 & e_1 \\ e_3 & e_0 & -e_1 & e_2 \\ -e_2 & e_1 & e_0 & e_3 \end{bmatrix} \bar{\boldsymbol{\omega}} $$

$$ \dot{\mathbf{e}} = \frac{1}{2} E \bar{\boldsymbol{\omega}} \tag{B.13} $$

where $\bar{\boldsymbol{\omega}}$ denotes $\boldsymbol{\omega}$ mapped to (4×1) space:

$$ \bar{\boldsymbol{\omega}} = \begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \tag{B.14} $$

Equation (B.13) may also be written as:

$$ \dot{\mathbf{e}} = \frac{1}{2} E \eta^T \boldsymbol{\omega} \tag{B.15} $$

where η^T is a 4×3 matrix used to map $\boldsymbol{\omega}$ to a 4×1 vector space:

$$\eta^T = \begin{bmatrix} I_3 \\ 0_{1\times 3} \end{bmatrix} \tag{B.16}$$

Note that the angular velocity $\boldsymbol{\omega}$ is defined strictly in the body frame. Equation (B.15) may be rearranged to solve for $\boldsymbol{\omega}$ in terms of $\dot{e}$:

$$\boldsymbol{\omega} = 2\eta E^T \dot{\mathbf{e}} \tag{B.17}$$

It can be seen from Eqns. (B.15) and (B.17) that η might be unncessary if E were defined as a 4×3 matrix. However, it is necessary to define E as a 4×4 matrix to ensure that the inverse of E is also the transpose:

$$E^{-1} = E^T \tag{B.18}$$

such that

$$E^T E = E E^T = I_4 \tag{B.19}$$

Taking the derivative of Eqn. (B.19) leads to the useful identity:

$$\dot{E}^T E + E^T \dot{E} = \dot{E} E^T + E \dot{E}^T = 0_{4\times 4} \tag{B.20}$$

B.3 Angular Acceleration

The angular acceleration can be found by taking the derivative of Eqn. (B.17)

$$\dot{\boldsymbol{\omega}} = 2\eta \left[E^T \ddot{\mathbf{e}} + \dot{E}^T \dot{\mathbf{e}} \right] \tag{B.21}$$

where $\dot{E}$ is defined by taking the derivative of E:

$$\dot{E} = \frac{d}{dt} E \tag{B.22}$$

Equation (B.21) may also be found by taking the derivative of Eqn. (B.15):

$$\ddot{\mathbf{e}} = \frac{1}{2}\left[E\eta^T \dot{\boldsymbol{\omega}} + \dot{E}\eta^T \boldsymbol{\omega} \right] \tag{B.23}$$

and by taking advantage Eqn. (B.19) and the useful identity:

$$\eta \, \eta^T = I_3 \tag{B.24}$$

the quantity $\dot{\boldsymbol{\omega}}$ may be isolated:

$$\begin{aligned}
\frac{1}{2} E \eta^T \dot{\boldsymbol{\omega}} \; &= \; \ddot{\mathbf{e}} \; - \; \frac{1}{2} \dot{E} \eta^T \boldsymbol{\omega} \\
\eta^T \dot{\boldsymbol{\omega}} \; &= \; 2 E^T \ddot{\mathbf{e}} \; - \; E^T \dot{E} \eta^T \boldsymbol{\omega} \\
\dot{\boldsymbol{\omega}} \; &= \; 2 \eta E^T \ddot{\mathbf{e}} \; - \; \eta E^T \dot{E} \eta^T \boldsymbol{\omega}
\end{aligned} \tag{B.25}$$

Inserting Eqn. (B.15) into Eqn. (B.25) yields:

$$\dot{\boldsymbol{\omega}} \; = \; 2 \eta E^T \ddot{\mathbf{e}} \; - \; 2 \eta E^T \dot{E} \eta^T \eta E^T \dot{\mathbf{e}} \tag{B.26}$$

Consider the second terms on the left hand sides of Eqn. (B.21) and (B.26). For those two equations to be equal, the following claim must be shown to be true:

$$2 \eta \dot{E}^T \; = \; -2 \eta E^T \dot{E} \eta^T \eta E^T \tag{B.27}$$

This equivalence can be shown by:

$$\begin{aligned}
\dot{E}^T \; &= \; -E^T \dot{E} \eta^T \eta E^T \\
E \dot{E}^T \; &= \; -\dot{E} \eta^T \eta E^T \\
E^T \dot{E}^T E \; &= \; -\dot{E} \eta^T \eta \\
E^T \dot{E}^T E \eta^T \; &= \; -\dot{E} \eta^T \\
E^T \dot{E}^T E \; &= \; -\dot{E} \\
\dot{E}^T E \; &= \; -E^T \dot{E} \\
\dot{E}^T E \; + \; E^T \dot{E} \; &= \; 0
\end{aligned} \tag{B.28}$$

which is the identity given in Eqn. (B.20). Equation (B.26) may therefore be expressed as

$$\dot{\boldsymbol{\omega}} \;=\; 2\eta E^T \ddot{\mathbf{e}} \;-\; 2\eta \dot{E}^T \dot{\mathbf{e}} \tag{B.29}$$

which is Eqn. (B.21).

B.4 Change in Angular Momentum

Consider Euler's second law:

$$\left(\sum \mathbf{M}\right) \;=\; \frac{d}{dt}\mathbf{H} \;=\; I\dot{\boldsymbol{\omega}} \;+\; \boldsymbol{\omega} \times I\boldsymbol{\omega} \tag{B.30}$$

Inserting Eqns. (B.17) and (B.21) into Eqn. (B.30) yields:

$$\dot{\mathbf{H}} \;=\; I(2\eta E^T \ddot{\mathbf{e}} - 2\eta \dot{E}^T \dot{\mathbf{e}}) \;+\; (2\eta E^T \dot{\mathbf{e}}) \times I(2\eta E^T \dot{\mathbf{e}}) \tag{B.31}$$

For convenience, an intermediate variable L and its time derivative $\dot{L}$ can be defined:

$$\begin{aligned} L \;&=\; 2\eta E^T \\ \dot{L} \;&=\; 2\eta \dot{E}^T \end{aligned} \tag{B.32}$$

Equation (B.31) may now be expressed as:

$$\dot{\mathbf{H}} \;=\; I(L\ddot{\mathbf{e}} \;+\; \dot{L}\dot{\mathbf{e}}) \;+\; (L\dot{\mathbf{e}}) \times I(L\dot{\mathbf{e}}) \tag{B.33}$$

For convenience, an intermediate variable K can be defined:

$$K \;=\; I\dot{L}\dot{\mathbf{e}} \;+\; (L\dot{\mathbf{e}}) \times I(L\dot{\mathbf{e}}) \tag{B.34}$$

Equation (B.33) may now be expressed as:

$$\dot{\mathbf{H}} \;=\; IL\ddot{\mathbf{e}} \;+\; K \tag{B.35}$$

It can be seen that L and K are both known quantities and are size 3×4 and 3×1, respectively.

B.5 Equations of Motion

Of the multiple methods for generating the equations of motion, consider the method formulated by Thomas Kane [150]. Kane's method is an iterative process exceptionally useful for generating the equations of motion when using Euler parameters. The scalar equation of motion for the i^{th} degree of freedom is given by:

$$F_i - F_i^* = 0 \tag{B.36}$$

$$F_i \;=\; \sum_{j}^{bodies} \Big[\; (\textstyle\sum \mathbf{F})_{\mathrm{j}} \cdot \dfrac{\partial \mathbf{V}_{\mathrm{j}}}{\partial \dot{q}_i} \;+\; (\textstyle\sum \mathbf{M})_{\mathrm{j}} \cdot \dfrac{\partial \boldsymbol{\omega}_j}{\partial \dot{q}_i} \;\Big] \tag{B.37}$$

$$F_i^* \;=\; \sum_{j}^{bodies} \Big[\; m_j \dot{\mathbf{V}}_{\mathrm{j}} \cdot \dfrac{\partial \mathbf{V}_{\mathrm{j}}}{\partial \dot{q}_i} \;+\; \dot{\mathbf{H}}_j \cdot \dfrac{\partial \boldsymbol{\omega}_j}{\partial \dot{q}_i} \;\Big] \tag{B.38}$$

where F_i and F_i^* are called the Generalized Active Forces and Generalized Inertia Forces, respectively. It can be seen that there are several quantities that must be defined:

1. Linear and angular velocity ($\mathbf{V}$ and $\boldsymbol{\omega}$)
2. Change in linear and angular momentum ($m\dot{\mathbf{V}}$ and $\dot{\mathbf{H}}$)
3. Sum of forces and moments ($(\sum \mathbf{F})$ and $(\sum \mathbf{M})$)

Consider an ungrounded body A with body attached frame $\widehat{\mathbf{A}}$. Body A has six degrees of freedom associated with translation and rotation. This body has a known mass m and inertia matrix I about its center of mass defined in body frame. Euler parameters are used to avoid singularity problems when defining the rotation matrix for the floating base. There are then seven generalized coordinates consisting of the three translations and four Euler parameters:

$$\mathbf{r} = [\; q_1 \; q_2 \; q_3 \;]^T \tag{B.39}$$

$$\mathbf{e} = [\; q_4 \; q_5 \; q_6 \; q_7 \;]^T \tag{B.40}$$

$$\mathbf{q} = [\; \mathbf{r}^T \; \mathbf{e}^T \;]^T \tag{B.41}$$

with the vectors $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ denoting the time derivatives. The position, velocity, and acceleration of the center of mass may be defined as:

$$\mathbf{P}_{\mathrm{NA}} = q_1\,\widehat{\mathbf{N}}_1 + q_2\,\widehat{\mathbf{N}}_2 + q_3\,\widehat{\mathbf{N}}_3 \tag{B.42}$$

$$\mathbf{V}_{\mathrm{A}} = \dot{q}_1\,\widehat{\mathbf{N}}_1 + \dot{q}_2\,\widehat{\mathbf{N}}_2 + \dot{q}_3\,\widehat{\mathbf{N}}_3 \tag{B.43}$$

$$\dot{\mathbf{V}}_{\mathrm{A}} = \ddot{q}_1\,\widehat{\mathbf{N}}_1 + \ddot{q}_2\,\widehat{\mathbf{N}}_2 + \ddot{q}_3\,\widehat{\mathbf{N}}_3 \tag{B.44}$$

The rotation matrix R, transformation matrix E, and its time derivative $\dot{E}$ are defined using $\mathbf{e}$ and $\dot{\mathbf{e}}$ according to Eqns. (B.11), (B.13), and (B.22). The angular velocity and angular accelerations are defined according to Eqns. (B.17) and (B.21) and are repeated here:

$$\boldsymbol{\omega} = 2\eta E^T\dot{\mathbf{e}}$$

$$\dot{\boldsymbol{\omega}} = 2\eta\left[\, E^T\ddot{\mathbf{e}} + \dot{E}^T\dot{\mathbf{e}}\,\right]$$

The change in angular momentum $\dot{\mathbf{H}}$ is given by Eqn. (B.35) and is repeated here:

$$\dot{\mathbf{H}} = IL\ddot{\mathbf{e}} + K$$

A known set of external forces and moments are defined as:

$$\mathbf{F} = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}^T \tag{B.45}$$

$$\mathbf{M} = \begin{bmatrix} M_1 & M_2 & M_3 \end{bmatrix}^T \tag{B.46}$$

All the quantities required for generating the equations of motion using Kane's method are now defined. For the three degrees of freedom associated with translation, the partial derivatives are:

$$\frac{\partial \boldsymbol{\omega}}{\partial \dot{\mathbf{q}}_{1-3}} = \mathbf{0} \tag{B.47}$$

$$\frac{\partial \mathbf{V}_{\mathrm{A}}}{\partial \dot{\mathbf{q}}_{1-3}} = I_3 \tag{B.48}$$

The Generalized Active Forces and Generalized Inertia Forces are then:

$$\mathbf{F}_{1-3} \;=\mathbf{F}\cdot I_3 \;=\; \mathbf{F} \tag{B.49}$$

$$\mathbf{F}^*_{1-3} \;=\; m\dot{\mathbf{V}}_{\mathrm{A}}\cdot I_3 \;=\; m\dot{\mathbf{V}}_{\mathrm{A}} \tag{B.50}$$

The equations of motion for the three translational degrees of freedom may be succinctly expressed as:

$$\mathbf{F}_{1-3} \;-\; \mathbf{F}^*_{1-3} \;=\; \mathbf{F} \;-\; m\dot{\mathbf{V}}_{\mathrm{A}} \;=\; \mathbf{0} \tag{B.51}$$

The equation of motion for the translation of the body in the $\widehat{\mathbf{N}}_1$ direction is then:

$$m\ddot{q}_1 \;=\; F_1 \tag{B.52}$$

For the three degrees of freedom associated with rotation, the partial derivatives are:

$$\frac{\partial\boldsymbol{\omega}}{\partial\dot{\mathbf{q}}_{4-7}} \;=\; \frac{\partial\boldsymbol{\omega}}{\partial\dot{\mathbf{e}}} \;=\; 2\eta E^T \tag{B.53}$$

$$\frac{\partial\mathbf{V}_{\mathrm{A}}}{\partial\dot{\mathbf{q}}_{4-7}} \;=\; \frac{\partial\mathbf{V}_{\mathrm{A}}}{\partial\dot{\mathbf{e}}} \;=\; \mathbf{0} \tag{B.54}$$

The Generalized Active Forces and Generalized Inertia Forces are then:

$$\mathbf{F}_{4-7} \;=\; \mathbf{M}\cdot(2\eta E^T) \;=\; 2(\eta E^T)^T\mathbf{M} \;=\; 2E\eta^T\mathbf{M} \tag{B.55}$$

$$\mathbf{F}^*_{4-7} \;=\; \dot{\mathbf{H}}\cdot(2\eta E^T) \;=\; 2(\eta E^T)^T(IL\ddot{\mathbf{e}}+K) \;=\; 2E\eta^T\big(IL\ddot{\mathbf{e}} \;+\; K\big) \tag{B.56}$$

The equations of motion for the four Euler parameters may be expressed as:

$$\mathbf{F}_{4-7} \;-\; \mathbf{F}^*_{4-7} \;=\; 2E\eta^T\mathbf{M} - 2E\eta^T(IL\ddot{\mathbf{e}}+K) \;=\; \mathbf{0} \tag{B.57}$$

Rearranging Eqn. B.57 yields:

$$\begin{aligned} 2E\eta^T IL\,\ddot{\mathbf{e}} \;+\; 2E\eta^T K \;&=\; 2E\eta^T\mathbf{M} \\ L^T IL\,\ddot{\mathbf{e}} \;+\; L^T K \;&=\; L^T\mathbf{M} \end{aligned} \tag{B.58}$$

which may be further rearranged to collect all known quantities:

$$L^T I L\, \ddot{\mathbf{e}} \;=\; L^T(\mathbf{M} - K) \tag{B.59}$$

Equation (B.58) can be represented by the general form:

$$M(\mathbf{e})\, \ddot{\mathbf{e}} \;+\; C(\mathbf{e}, \dot{\mathbf{e}}) \;=\; \mathbf{\Gamma}(\mathbf{e}) \tag{B.60}$$

These rotational equations of motion can be combined with the equations for translation shown in Eqn. (B.51). These combined equations of motion can then be represented by the general form:

$$M(\mathbf{q})\, \ddot{\mathbf{q}} \;+\; C(\mathbf{q}, \dot{\mathbf{q}}) \;=\; \mathbf{\Gamma}(\mathbf{q}) \tag{B.61}$$

where $M(\mathbf{q})$, $C(\mathbf{q}, \dot{\mathbf{q}})$, and $\mathbf{\Gamma}(\mathbf{q})$ denote the mass matrix, nonlinear velocity product terms, and the Generalized Active Forces, respectively, mapped to the 7-dimensional space corresponding to translation and the Euler parameters. Note that in the case of a spherical joint, the dimensions of Eqn. (B.60) are 4×1.

Consider the equation of motion for an ungrounded body A solved using the recursive formulation presented in Appendix A:

$$M_A^* \dot{\boldsymbol{\vartheta}}_A \;=\; -\mathbf{h}_A^* \tag{B.62}$$

The M_A^* term denotes a modified mass matrix and has dimensions 6×6. To solve for $\ddot{\mathbf{e}}_A$ in place of ${}^N\dot{\boldsymbol{\omega}}^A$, Eqn. (B.62) must be transformed to the Euler parameter vector space similar to Eqn. (B.59). This may be accomplished by:

$$Y_A^T M_A^* Y_A \dot{\boldsymbol{\vartheta}}_A^{\dagger} \;=\; -Y_A^T \mathbf{h}_A^* \tag{B.63}$$

$$Y_A \;=\; \begin{bmatrix} I_3 & 0_{3\times 4} \\ 0_{3\times 3} & L_A \end{bmatrix} \tag{B.64}$$

$$\dot{\boldsymbol{\vartheta}}_A^{\dagger} \;=\; \begin{bmatrix} \dot{\mathbf{V}}_{\mathrm{A}} \\ \\ \ddot{\mathbf{e}}_A \end{bmatrix} \tag{B.65}$$

Also, consider the equation of motion for the kinematic joint B shown by Eqn. (A.41), which is repeated here:

$$s_B^T M_B^* s_B \; \ddot{\mathbf{q}}_B \;=\; -s_B^T \big[M_B^* \phi_B \dot{\boldsymbol{\vartheta}}_A \;+\; \mathbf{h}_B^* \big] \tag{B.66}$$

If joint B is spherical, the dimensions of s_B are 6×3. The $s_B^T M_B^* s_B$ term denotes a modified mass matrix and has dimensions 3×3. Since the spatial acceleration of body A is calculated first it is known with respect to solving Eqn. (B.66). This equation then can be converted to a form similar to Eqn. (B.59) to solve for the Euler accelerations at that joint:

$$L_B^T s_B^T M_B^* s_B L_B \; \ddot{\mathbf{e}}_B \;=\; -L_B^T s_B^T \big[M_B^* \phi_B \dot{\boldsymbol{\vartheta}}_A \;+\; \mathbf{h}_B^* \big] \tag{B.67}$$

where the new mass matrix is given by $L_B^T s_B^T M_B^* s_B L_B$ and has the dimensions 4×4. Because the mass matrices in Eqns. (B.58), (B.63), and (B.67) have been mapped to the Euler parameter vector space, they are singular and non-invertible. However, there is the normality constraint associated with the Euler parameters, given by Eqn. (B.9), that can be applied to eliminate a dependent degree of freedom. Application of this constraint will reduce the size of the mass matrix by one, allowing that mass matrix to be inverted to solve for the independent accelerations. Those independent accelerations can then be used to solve for the dependent acceleration. These accelerations can be integrated to update states for the next integration step. Appendix C discusses constraints and presents the online constraint embedding procedure used to accommodate Euler parameters for the work shown in Chapter 5.