Chapter 8
Rigid Body Kinematics

From this chapter, we begin to introduce attitude kinematics and dynamics, attitude determination and control.

In this chapter we discuss the description of attitude kinematics using reference frames, rotation matrices, Euler parameters, Euler angles, and quaternion.

Recall the fundamental dynamics equations law of linear momentum and angular momentum

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \quad \mathbf{g} = \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}, \tag{8.1}$$

where $\mathbf{f}$ is force, $\mathbf{g}$ is torque, $\mathbf{p}$ is linear momentum and $\mathbf{h}$ is angular momentum. For both equations, we must relate momentum to kinematics.

Before we introduce attitude kinematics, we recall Translational Kinematics. Newton's Second Law can be written in first-order state-vector form as

$$\dot{\mathbf{r}} = \mathbf{p}/m, \quad \dot{\mathbf{p}} = \mathbf{f}. \tag{8.2}$$

Here, $\mathbf{p}$ is the linear momentum, defined by the kinematics equation; that is, $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$.

Thus, the kinematics differential equation allows us to integrate the velocity to compute the position. The kinetics differential equation allows us to integrate the applied force to compute the linear momentum.

In general, $\mathbf{f} = \mathbf{f}(\mathbf{r}, \mathbf{p})$. Consider $\mathbf{f} = m\mathbf{a}$ expressed in an inertial frame: $m\ddot{\mathbf{r}} = \mathbf{f} \Leftrightarrow [m\ddot{r}_1, m\ddot{r}_2, m\ddot{r}_3] = [f_1, f_2, f_3]$. Equivalently, $\dot{\mathbf{p}} = \mathbf{f} \Leftrightarrow [\dot{p}_1, \dot{p}_2, \dot{p}_3] = [f_1, f_2, f_2]$, and $\dot{\mathbf{r}} = \mathbf{p}/m \Leftrightarrow [\dot{r}_1, \dot{r}_2, \dot{r}_3] = [p_1/m, p_2/m, p_3/m]$. We need to develop rotational equations equivalent to the translational kinematics equations.

8.1 Attitude Notions

8.1.1 Attitude

Spacecraft Attitude is the orientation of the spacecraft with respect to other objects. Spacecraft Attitude Determination is the process of measuring and calculating the spacecraft attitude, i.e, “attitude knowledge.” Spacecraft Attitude Control is the generation of moments and torques required to place the spacecraft in the appropriate attitude.

S/C pointing is a mission's requirements for a variety of purposes. In scientific sensors, it is used for deep space observation of considerable time (Hubble), comet identification, scan Earth (ground imaging: Nadir swath Mapping, True scan), scan Earth limb (Aurora Borealis), space physics (Electron Density), Sun physics, and launch imaging.

It is used in communications such as geosynchronous communications satellites which must point antennae for communications gain, ground station communications for Uplink/Downlink, and inter-satellite communications.

It is also use for power, i.e, solar arrays must point towards the Sun.

And in a thermal system those thermally sensitive components must not be in direct sunlight.

Sensitive instruments which light sensitive components must not point at the Sun (star cameras, IR sensors, etc.).

It is also used for orbit transfer, in which orbit transfer burns must be made in the correct direction, spacecraft Rendezvous Docking port and collar alignment, end of life spacecraft disposal, final spacecraft disposal (de-orbit for LEO, higher orbit for GEO).

Attitude is defined by three parameters. Typically, the relative orientation of two orthogonal, co-origination vector triads. It requires two (single axis) or three (three axis) parameters (degrees-of-freedom) to define.

Single Axis Attitude means that one axis is pointing along a single vector; the motion around that axis is undefined. The examples are: Spin-stabilized spacecraft and Single axis sensors (Ladar pencil beam).

Table 8.1 The summary of linear and angular motion

$\mathbf{f} = \mathrm{d}\mathbf{p}/\mathrm{d}t$	$\mathbf{g} = \mathrm{d}\mathbf{h}/\mathrm{d}t$
Linear momentum	Angular momentum
$= \text{mass} \times \text{velocity}$	$= \text{inertia} \times \text{angular velocity}$
$\mathrm{d}/\mathrm{d}t$ (linear momentum)	$\mathrm{d}/\mathrm{d}t$ (angular momentum)
$= \text{applied forces}$	$= \text{applied torques}$
$\mathrm{d}/\mathrm{d}t$ (position)	$\mathrm{d}/\mathrm{d}t$ (attitude)
$= \text{linear momentum/mass}$	$= \text{“angular momentum/inertia”}$

Three Axis Attitude means that one axis may be pointing along a single vector, but the other two are also fixed. A complete definition of this kind of orientation requires three parameters. This approach is the most common and will be typically assumed for this book. Typically, we define the attitude as the relationship between a frame rigidly attached to the spacecraft's body (Body Frame) and either the Earth-Centered Inertial (ECI) frame or the Orbital frame, occasionally the Earth-Centered Earth-Fixed (ECEF) frame.

8.1.2 Frames

Here we introduce some coordinate, some of which may be listed already in Chapter 1, but they are defined in the field of attitude, maybe in the same name orby a different name.

For ECI (Earth-Centered Inertial) also called Geocentric Reference Frame (GCRF, its $\hat{X}$ axis points towards a fixed point in space, First Point of Aries, that is, a vector through the Sun from Earth center on March 21. Its $\hat{Z}$ axis is aligned along Earth's spin axis (North Pole), wobbles slightly, must define epoch. Then its $\hat{Y}$ axis makes a right-handed orthogonal frame. It is inertially fixed. It is the $\hat{I}\hat{J}\hat{K}$ which was defined in Chapter 1.

For ECEF (Earth-Centered Earth-Fixed) also known as Internally Terrestrial Reference (ITRF, its $\hat{X}$ axis is through Greenwich meridian at the equator. Its $\hat{Z}$ axis is aligned along the Earth's spin axis (North Pole), and its $\hat{Y}$ axis makes a right-handed orthogonal frame. It is noninertial frame which rotates with the Earth.

In an Orbital (roll, pitch, yaw), noninertial coordinate system, its $\hat{Z}$ axis points inward to the Earth's center (focus), $\hat{Y}$ axis is perpendicular to orbit plane, opposite direction of angular momentum, and its $\hat{X}$ axis makes a right-handed orthogonal frame which points roughly in the direction of motion for circular orbits. The orbital frame is similar to the frame $\hat{R}\hat{S}\hat{W}$ which was introduced in Chapter 1 with a transformation matrix (see Equation (8.116).

Body frame (rigidly attached to main S/C body) is often aligned along the principal axes of inertia.

It is normally denoted that reference frames are triads of mutually orthogonal unit vectors.

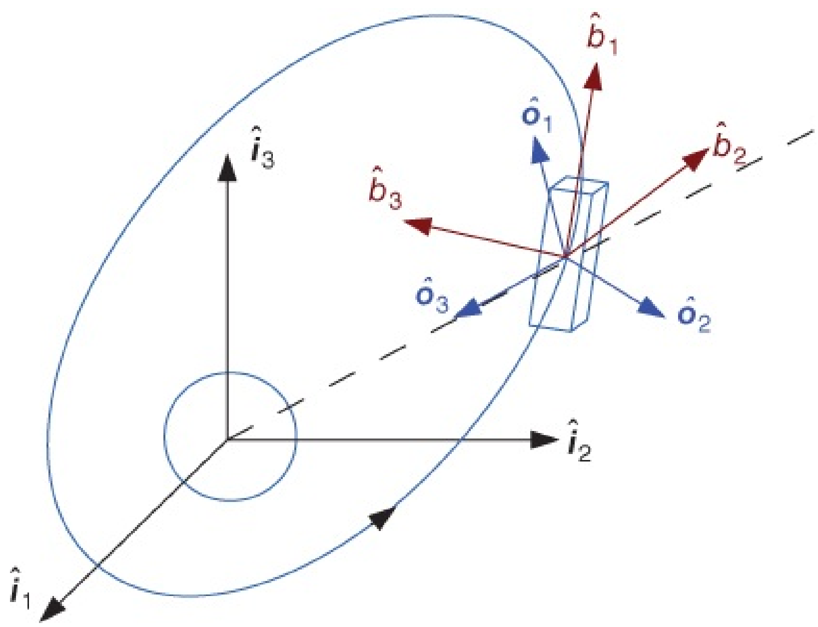

Figure 8.1 The inertial $\mathcal{F}_i : \hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3$, orbital $\mathcal{F}_o : \hat{o}_1, \hat{o}_2, \hat{o}_3$ and body $\mathcal{F}_b : \hat{b}_1, \hat{b}_2, \hat{b}_3$ frames

The definition of the frames are: the Inertial Frame, $\mathcal{F}_i$; the Orbital Frame, $\mathcal{F}_o$; the Body Frame: $\mathcal{F}_b$. See Figure 8.1. These axes are mutually perpendicular in each set, which satisfies the following relation as $\hat{\imath}_i$ for example.

$$\hat{\imath}_i \cdot \hat{\imath}_j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j, \end{cases} \tag{8.3}$$

which means $\hat{\imath}_i \cdot \hat{\imath}_j = \delta_{ij}$. For this right-handedness of reference frames:

$$\begin{array}{lll} \hat{i}_1 \times \hat{i}_1 = 0 & \hat{i}_1 \times \hat{i}_2 = \hat{i}_3 & \hat{i}_1 \times \hat{i}_3 = -\hat{i}_2 \\ \hat{i}_2 \times \hat{i}_1 = -\hat{i}_3 & \hat{i}_2 \times \hat{i}_2 = 0 & \hat{i}_2 \times \hat{i}_3 = \hat{i}_1 \\ \hat{i}_3 \times \hat{i}_1 = \hat{i}_2 & \hat{i}_3 \times \hat{i}_2 = -\hat{i}_1 & \hat{i}_1 \times \hat{i}_3 = 0. \end{array} \quad \textbf{8.4}$$

This means

$$\hat{i}_i \times \hat{i}_j = \varepsilon_{ijk} \hat{i}_k, \quad \textbf{8.5}$$

where

$$\hat{i}_i \cdot \hat{i}_j = \begin{cases} 1 & \text{for } i, j, k \text{ an even permutation of 1,2,3} \\ -1 & \text{for } i, j, k \text{ an odd permutation of 1,2,3} \\ 0 & \text{otherwise (i.e.,if any repetitions occur).} \end{cases} \quad \textbf{8.6}$$

The same thing happens in orbital and body frames.

Or even more succinctly as

$$\left\{\hat{i}\right\} \times \left\{\hat{i}\right\}^T = \left\{ \begin{array}{ccc} 0 & i_3 & -i_2 \\ -i_3 & 0 & i_1 \\ i_2 & -i_1 & 0 \end{array} \right\} = -\left\{\hat{i}\right\}^{\times}. \quad \textbf{8.7}$$

The []$\times$ notation defines a skew-symmetric 3×3 matrix whose 3 unique elements are the components of the 3×1 matrix []

The same notation applies if the components of the 3×1 matrix [] are scalars instead of vectors:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Rightarrow \mathbf{a}^{\times} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \quad \textbf{8.8}$$

The “skew-symmetry” property is satisfied since

$$(\mathbf{a}^{\times})^T = -\mathbf{a}^{\times}. \quad \textbf{8.9}$$

8.1.3 Vector

Reference frame is dextral triad of orthonormal unit vectors. A vector is an abstract mathematical object with two properties: direction and length.

Vectors used in this book include, for example, position, velocity, acceleration, force, momentum, torque, angular velocity. Vectors can be expressed in any reference frame.

Keep in mind that the term “state vector” refers to a different type of object—specifically, a state vector is generally a column matrix collecting all the system states.

Vector can be expressed as a linear combination of the unit vectors, but it must be clear about which reference frame is used. Orthogonal means the base vectors are perpendicular (orthogonal) to each other, and have unit length (normalized).

A vector can be uniquely represented as having a magnitude and direction. A vector is independent of any frame.

Frequently we collect the components of the vector into a matrix. $\mathbf{v} = [v_1, v_2, v_3]^T$. Note the absence of an overarrow here by using a bold character.

When we can easily identify the associated reference frame, we use the simple notation above; however, when multiple reference frames are involved, we use a subscript to make the connection clear.

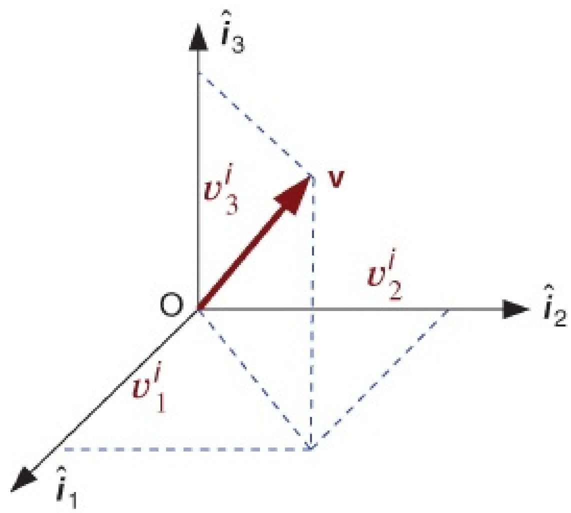

Figure 8.2 A vector in inertial frame $\mathbf{v}_i$

Examples: $\mathcal{F}_i$ denotes a vector in $\mathbf{v}_o$, $\mathcal{F}_o$ denotes a vector in $\mathbf{v}_b$, $\mathcal{F}_b$ denotes a vector in $\mathcal{F}_i$.

See Figure 8.2. A vector can be represented in any frame as:

$$\mathbf{v} = v_1^i \hat{i}_1 + v_2^i \hat{i}_2 + v_3^i \hat{i}_3. \qquad \textbf{8.10}$$

In this frame, the vector can be equivalently represented as a 3×1 matrix of scalars.

$$\mathbf{v}^i \equiv [v_1^i \; v_2^i \; v_3^i]^T. \qquad \textbf{8.11}$$

The scalars in the matrix are found as

$$v_1^i = \mathbf{v} \cdot \hat{i}_1, \quad v_2^i = \mathbf{v} \cdot \hat{i}_2, \quad v_3^i = \mathbf{v} \cdot \hat{i}_3.$$

Matrix multiplication arises frequently in dynamics and control, and an interesting application involves the 3×1 matrix of a vector's components and the 3×1 “matrix” of a frame's base vector.

The same vector can be represented in multiple frames as:

$$\mathbf{v} = v_1^i \hat{i}_1 + v_2^i \hat{i}_2 + v_3^i \hat{i}_3 \Rightarrow [v_1^i \; v_2^i \; v_3^i]^T \equiv \mathbf{v}^i \qquad \textbf{8.12}$$

$$\mathbf{v} = v_1^o \hat{o}_1 + v_2^o \hat{o}_2 + v_3^o \hat{o}_3 \Rightarrow [v_1^o \; v_2^o \; v_3^o]^T \equiv \mathbf{v}^o \qquad \textbf{8.13}$$

$$\mathbf{v} = v_1^b\hat{b}_1 + v_2^b\hat{b}_2 + v_3^b\hat{b}_3 \Rightarrow \left[v_1^b \; v_2^b \; v_3^b\right]^T \equiv \mathbf{v}^b. \quad (8.14)$$

For the same vector, the 3×1 matrixes differ between frames, i.e. normally

$$\left[v_1^i \; v_2^i \; v_3^i\right]^T \neq \left[v_1^o \; v_2^o \; v_3^o\right]^T \neq \left[v_1^b \; v_2^b \; v_3^b\right]^T. \quad (8.15)$$

The 2-norm (length) of each matrix is the same

$$\| \mathbf{v} \| = \| \mathbf{v}^i \| = \| \mathbf{v}^o \| = \| \mathbf{v}^b \| . \quad (8.16)$$

8.2 Attitude Parameters

8.2.1 Direct Cosine Matrix

Suppose we know components in body frame and we want to know components in inertial frame:

$$\mathbf{v} = v_1^i\hat{i}_1 + v_2^i\hat{i}_2 + v_3^i\hat{i}_3 = v_1^b\hat{b}_1 + v_2^b\hat{b}_2 + v_3^b\hat{b}_3. \quad (8.17)$$

Now we can find v_1^i by taking the dot product between $\mathbf{v}$ and $\hat{i}_1$, or:

$$\mathbf{v}_1^i = \hat{i}_1 \cdot \mathbf{v} = v_1^b(\hat{i}_1 \cdot \hat{b}_1) + v_2^b(\hat{i}_1 \cdot \hat{b}_2) + v_3^b(\hat{i}_1 \cdot \hat{b}_3). \quad (8.18)$$

Similarly, v_2^i and v_3^i can be found:

$$\mathbf{v}_2^i = \hat{i}_2 \cdot \mathbf{v} = v_1^b(\hat{i}_2 \cdot \hat{b}_1) + v_2^b(\hat{i}_2 \cdot \hat{b}_2) + v_3^b(\hat{i}_2 \cdot \hat{b}_3) \quad (8.19)$$

$$\mathbf{v}_3^i = \hat{i}_3 \cdot \mathbf{v} = v_1^b(\hat{i}_3 \cdot \hat{b}_1) + v_2^b(\hat{i}_3 \cdot \hat{b}_2) + v_3^b(\hat{i}_3 \cdot \hat{b}_3). \quad (8.20)$$

Notice that the terms in the parentheses are not dependent on the vector components of $\mathbf{v}$, just on the orientation between frames $\mathcal{F}_i$ and $\mathcal{F}_b$.

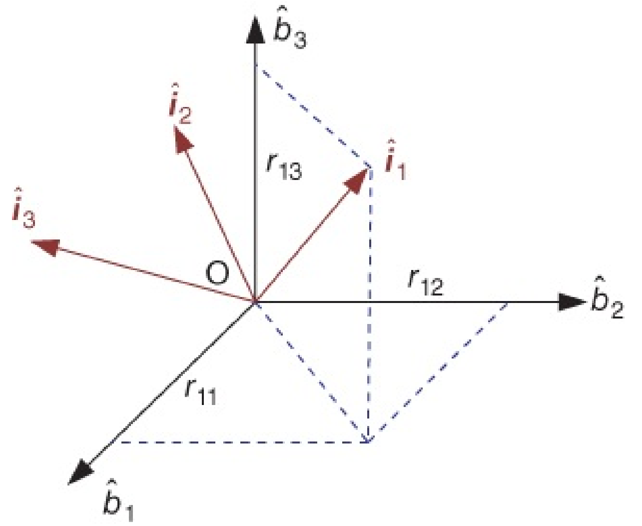

Figure 8.3 Direct cosine matrix between inertial $\mathbf{v}$ and body frame $\mathcal{F}_i$

This means that the matrix representation for $\mathbf{v}$ in $\mathcal{F}_i$ is linearly related to the matrix representation for $\mathbf{v}$ in $\mathcal{F}_b$:

$$\begin{bmatrix} v_1^i \\ v_2^i \\ v_3^i \end{bmatrix} = \begin{bmatrix} \hat{i}_1 \cdot \hat{b}_1 & \hat{i}_1 \cdot \hat{b}_2 & \hat{i}_1 \cdot \hat{b}_3 \\ \hat{i}_2 \cdot \hat{b}_1 & \hat{i}_2 \cdot \hat{b}_2 & \hat{i}_2 \cdot \hat{b}_3 \\ \hat{i}_3 \cdot \hat{b}_1 & \hat{i}_3 \cdot \hat{b}_2 & \hat{i}_3 \cdot \hat{b}_3 \end{bmatrix} \begin{bmatrix} v_1^b \\ v_2^b \\ v_3^b \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} v_1^b \\ v_2^b \\ v_3^b \end{bmatrix} \tag{8.21}$$

$$\Rightarrow [v]^i = \mathbf{R}^{ib} [v]^b . \tag{8.22}$$

See Figure 8.3. The equation $[v]^i = \mathbf{R}^{ib} [v]^b$ is a linear system of the form $\mathbf{Ax} = \mathbf{b}$. Thus, to determine the components in the inertial frame, we need to determine $\mathbf{R}^{ib}$ and solve the linear system. The components of $\mathbf{R}^{ib}$ are:

$$r_{jk}^{ib} = \hat{i}_j \cdot \hat{b}_k = \cos(\hat{i}_k, \hat{b}_j) \tag{8.23}$$

Thus the components of $\mathbf{R}$ are the direction cosines and the rotation matrix is also known as the Direction Cosine Matrix (DCM

Note also that

$$r_{jk}^{ib} = \hat{i}_j \cdot \hat{b}_k = r_{kj}^{bi} \quad \text{or} \quad \mathbf{R}^{ib} = (\mathbf{R}^{bi})^T . \tag{8.24}$$

Also

$$[v]^i = \mathbf{R}^{ib} [v]^b , \quad [v]^b = \mathbf{R}^{bi} [v]^i , \quad [v]^b = \mathbf{R}^{bi}\mathbf{R}^{ib} [v]^b , \tag{8.25}$$

$$\mathbf{R}^{bi}\mathbf{R}^{ib} = \mathbf{1}, \quad \mathbf{R}^{bi} = \mathbf{R}^{ib^{-1}} . \tag{8.26}$$

The inverse of a rotation matrix is simply its transpose:

$$\mathbf{R}^{ib} = \mathbf{R}^{bi^T} = \mathbf{R}^{bi^{-1}} . \tag{8.27}$$

A rotation matrix is an "orthonormal" matrix: its rows and columns are components of mutually orthogonal unit vectors.

Rotation about the $\hat{i}_3$ axis:

$$r_{jk}^{bi} = \hat{b}_j \cdot \hat{i}_k = \cos(\hat{b}_j, \hat{i}_k) \tag{8.28}$$

$$r_{11}^{bi} = \hat{b}_1 \cdot \hat{i}_1 = \cos(\hat{b}_1, \hat{i}_1) = \cos\theta \tag{8.29}$$

$$r_{12}^{bi} = \hat{b}_1 \cdot \hat{i}_2 = \cos(\hat{b}_1, \hat{i}_2) = \sin\theta \tag{8.30}$$

$$r_{21}^{bi} = \hat{b}_2 \cdot \hat{i}_1 = \cos(\hat{b}_2, \hat{i}_1) = -\sin\theta \tag{8.31}$$

$$r_{22}^{bi} = \hat{b}_2 \cdot \hat{i}_2 = \cos(\hat{b}_2, \hat{i}_2) = \cos\theta \tag{8.32}$$

$$r_{33}^{bi} = \hat{b}_3 \cdot \hat{i}_3 = \cos(\hat{b}_3, \hat{i}_3) = \cos(0) = 1 \tag{8.33}$$

$$r_{13}^{bi} = \hat{b}_1 \cdot \hat{i}_3 = \cos(\hat{b}_1, \hat{i}_3) = \cos(90°) = 0. \tag{8.34}$$

The rotation matrix $\mathbf{R}$ represents the attitude. It has 9 numbers, but they are not independent. There are 6 constraints on the 9 elements of a rotation matrix.

$$\mathbf{R}_i \cdot \mathbf{R}_j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}. \tag{8.35}$$

$\mathbf{R}_i, \mathbf{R}_j$ is a row or column of the rotation matrix. This rotation has 3 degrees of freedom. There are many different sets of parameters that can be used to represent or parameterize rotations, including Euler angles, Euler parameters (aka quaternions), Rodrigues parameters (aka Gibbs vectors), modified Rodrigues parameters.

8.2.2 Euler Angles

Leonhard Euler(1707–1783) reasoned that the rotation from one frame to another can be visualized as a sequence of three simple rotations about base vectors.

Each rotation is through an angle (Euler angle about a specified axis. Let's consider the rotation from $\mathcal{F}_i$ to $\mathcal{F}_b$ using three Euler angles, θ_1, θ_2, and θ_3.

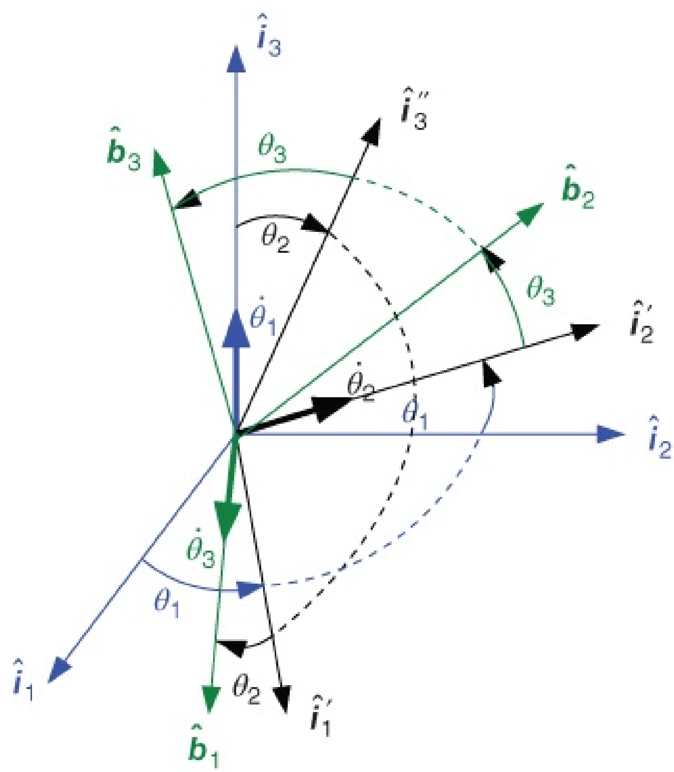

Figure 8.4 Euler angles and angle velocity for 3-2-1 sequence

The first rotation is about the axis $\hat{i}_3$, through angle θ_1. The resulting frame is denoted $\mathcal{F}_{i'}$ or $\{i'\}$. See Figure 8.4.

$$\mathbf{R}^{i'i} = \mathbf{R}_3(\theta_1) = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textbf{8.36}$$

$$\Rightarrow \mathbf{v}_{i'} = \mathbf{R}_3(\theta_1)\mathbf{v}_i. \qquad \textbf{8.37}$$

The second rotation is about the axis $\hat{i}'_2$, through angle θ_2. The resulting frame is denoted $\mathcal{F}_{i''}$ or $\{i''\}$.

The rotation matrix notation for the “simple” rotations is $\mathbf{R}_i(\theta_j)$ denotes a rotation about the i -axis. The subscript on $\mathbf{R}$ defines which simple rotation axis is used, and the subscript j on θ defines which of the three angles in the Euler sequence it is:

$$\mathbf{R}^{i''i'} = \mathbf{R}_2(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \qquad \textbf{8.38}$$

$$\Rightarrow \mathbf{v}_{i''} = \mathbf{R}_2(\theta_2)\mathbf{v}_{i'} = \mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_1)\mathbf{v}_i. \qquad \textbf{8.39}$$

The third rotation is a “1” rotation, through angle θ_3. The resulting frame is the desired body frame, denoted $\mathcal{F}_b$ or $\{\hat{b}\}$.

$$\mathbf{R}^{bi''} = \mathbf{R}_1(\theta_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{bmatrix} \qquad \textbf{8.40}$$

$$\Rightarrow \mathbf{v}_b = \mathbf{R}_1(\theta_3)\mathbf{v}_{i''} = \mathbf{R}_1(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_1)\mathbf{v}_i. \qquad \textbf{8.41}$$

We have performed a 3-2-1 rotation from $\mathcal{F}_i$ to $\mathcal{F}_{i''}$

$$\mathbf{R}^{bi} = \mathbf{R}_1(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_1) \qquad \textbf{8.42}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_3 & s\theta_3 \\ 0 & -s\theta_3 & c\theta_3 \end{bmatrix} \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 \\ 0 & 1 & 0 \\ s\theta_2 & 0 & c\theta_2 \end{bmatrix} \begin{bmatrix} c\theta_1 & s\theta_1 & 0 \\ -s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textbf{8.43}$$

$$= \begin{bmatrix} c\theta_1 c\theta_2 & s\theta_1 c\theta_2 & -s\theta_2 \\ -s\theta_1 c\theta_3 + c\theta_1 s\theta_2 s\theta_3 & c\theta_1 c\theta_3 + s\theta_1 s\theta_2 s\theta_3 & c\theta_2 s\theta_3 \\ s\theta_1 s\theta_3 + c\theta_1 s\theta_2 c\theta_3 & s\theta_1 s\theta_2 c\theta_3 - c\theta_1 s\theta_3 & c\theta_2 c\theta_3 \end{bmatrix}. \qquad \textbf{8.44}$$

We can select arbitrary values of the three angles and compute a rotation matrix.

For example, let $\theta_1 = \pi/6, \theta_2 = \pi/4$, and $\theta_3 = \pi/3$, then

$$\mathbf{R} = \begin{bmatrix} 0.6124 & 0.3536 & -0.7071 \\ 0.2803 & 0.7392 & 0.6124 \\ 0.7392 & -0.5732 & 0.3536 \end{bmatrix}. \qquad \textbf{8.45}$$

Conversely, given a rotation matrix, we can extract the Euler angles. Choose the easy one first:

$$R_{13} : -\sin\theta_2 = -0.7071 \Rightarrow \theta_2 = \sin^{-1} 0.7071 = 0.7854 \approx \pi/4 \tag{8.46}$$

$$R_{11} : \cos\theta_1 \cos\theta_2 = 0.6124 \Rightarrow \theta_1 = 0.5236 \approx \pi/6 \tag{8.47}$$

$$R_{23} : \cos\theta_2 \sin\theta_3 = 0.6124 \Rightarrow \theta_3 = 1.0472 \approx \pi/3. \tag{8.48}$$

Quadrant checks are generally necessary for the second and third calculations. For example, as $R_{12} > 0, \cos\theta_2 > 0$, then $\sin\theta_1 > 0$

We have just developed a (3-2-1) rotation, but there are other possibilities.

There are 3 choices for the first rotation, 2 choices for the second rotation, and 2 choices for the third rotation, so there are $3 \times 2 \times 2$ = 12 possible Euler angle sequences.

The Euler angle rotation sequences are

(1-2-3)(1-3-2)(2-3-1)(2-1-3)(3-1-2)(3-2-1)

(1-2-1)(1-3-1)(2-3-2)(2-1-2)(3-1-3)(3-2-3).

The sequence in the first row is sometimes called the asymmetric rotation sequence, and the sequence in the second row is called the symmetric sequence.

The roll-pitch-yaw sequence is an asymmetric sequence (3-2-1), whereas the $\Omega - i - \omega$ sequence (3-1-3) is a symmetric sequence.

In the 3-2-1 sequence (which we used earlier), θ_1 is the yaw angle ψ, θ_2 is the pitch angle θ, and θ_3 is the roll angle ϕ.

The 1-2-3 sequence leads to

$$\mathbf{R}^{bi} = \mathbf{R}_3(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{R}_1(\theta_1) \tag{8.49}$$

$$= \begin{bmatrix} c\theta_3 & s\theta_3 & 0 \\ -s\theta_3 & c\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 \\ 0 & 1 & 0 \\ s\theta_2 & 0 & c\theta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_1 & s\theta_1 \\ 0 & -s\theta_1 & c\theta_1 \end{bmatrix} \tag{8.50}$$

$$= \begin{bmatrix} c\theta_2 c\theta_3 & c\theta_1 s\theta_3 + s\theta_1 s\theta_2 c\theta_3 & s\theta_1 s\theta_3 - c\theta_1 s\theta_2 c\theta_3 \\ -c\theta_2 s\theta_3 & c\theta_1 c\theta_3 - s\theta_1 s\theta_2 s\theta_3 & c\theta_1 s\theta_2 s\theta_3 + s\theta_1 c\theta_3 \\ s\theta_2 & -s\theta_1 c\theta_2 & c\theta_1 c\theta_2 \end{bmatrix}, \tag{8.51}$$

where θ_1 is the roll angle ϕ, θ_2 is the pitch angle θ, and θ_3 is the yaw angle ψ.

Note that the two matrices are not the same, and the rotations do not commute. However, if we assume that the angles are small (appropriate for many vehicle dynamics problems), then the approximations of the two matrices are equal. For small angles, $\cos\theta \approx 1$ and $\sin\theta = \theta$.

For 3-2-1 sequence,

$$\mathbf{R}^{bi} \approx \begin{bmatrix} 1 & \theta_1 & -\theta_2 \\ -\theta_1 & 1 & \theta_3 \\ \theta_2 & -\theta_3 & 1 \end{bmatrix}. \quad \textbf{8.52}$$

In the 3-2-1 sequence (which we used earlier), θ_1 is the yaw angle, θ_2 is the pitch angle, and θ_3 is the roll angle.

For 1-2-3 sequence,

$$\mathbf{R}^{bi} \approx \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}. \quad \textbf{8.53}$$

In the 1-2-3 sequence (which we used earlier), θ_1 is the roll angle, θ_2 is the pitch angle, and θ_3 is the yaw angle. For all sequence,

$$\mathbf{R}^{bi} \approx 1 - \theta^{\times} \Rightarrow \mathbf{R}^{bi} = 1 - \begin{bmatrix} \text{roll} \\ \text{pitch} \\ \text{yaw} \end{bmatrix}^{\times}. \quad \textbf{8.54}$$

In summary, the subsequent transformations are important:

1. Given a sequence, say (3-1-2) for example, derive the Euler angle representation of $\mathbf{R}$. Be sure to get the order correct.

2. Given a sequence and some values for the angles, compute the numerical values of $\mathbf{R}$. Be sure to know the difference between degrees and radians.

3. Given the numerical values of R, extract numerical values of the Euler angles associated with a specified sequence. Be sure to make appropriate quadrant checks, and to check your answer.

8.3 Differential Equations of Kinematics

Given the velocity of a point and initial conditions for its position, we can compute its position as a function of time by integrating the differential equation $\dot{\mathbf{r}} = \mathbf{v}^b \equiv \mathbf{v}_b$.

We now need to develop the equivalent differential equations for the attitude when the angular velocity is known.

We use one frame-to-frame at a time, just as we did for developing rotation matrices. See Figure 8.4. (3-2-1) rotation from $\mathcal{F}_i$ to $\mathcal{F}_{i'}$ to $\mathcal{F}_{i''}$ to $\mathcal{F}_b$. 3-rotation from $\mathcal{F}_i$ to $\mathcal{F}_{i'}$ about $\hat{i}_3 \equiv \hat{i}'_3$ through θ_1. The angular velocity of $\mathcal{F}_{i'}$ with respect to $\mathcal{F}_i$ is

$$\omega^{i'i} = \dot{\theta}_1 \hat{i}_3 = \dot{\theta}_1 \hat{i}'_3. \quad \textbf{8.55}$$

We can express $\omega^{i'i}$ in any frame, but $\mathcal{F}_i$ and $\mathcal{F}_{i'}$ are especially simple:

$$\omega_i^{i'i} = [0\ 0\ \dot{\theta}_1]^T \quad \textbf{8.56}$$

$$\omega_{i'}^{i'i} = [0\ 0\ \dot{\theta}_1]^T. \quad \textbf{8.57}$$

Keep the notation in mind: $\omega_i^{i'i}$ is the angular velocity of $\mathcal{F}_{i'}$ with respect to $\mathcal{F}_i\mathcal{F}_i$, expressed in $\mathcal{F}_i$.

2-rotation from $\mathcal{F}_{i'}$ to $\mathcal{F}_{i''}$ about $\hat{i}'_2 \equiv \hat{i}''_2$ through θ_2

The angular velocity of $\mathcal{F}_{i''}$ with respect to $\mathcal{F}_{i'}$ is

$$\omega^{i''i'} = \dot{\theta}_2\hat{i}'_2 = \dot{\theta}_2\hat{i}''_2. \quad \textbf{8.58}$$

We can express $\omega^{i''i'}$ in any frame, but $\mathcal{F}_{i'}$ and$\mathcal{F}_i\mathcal{F}_{i''}$ are especially simple:

$$\omega_{i'}^{i''i'} = [0\ \dot{\theta}_2\ 0]^T \quad \textbf{8.59}$$

$$\omega_{i''}^{i''i'} = [0\ \dot{\theta}_2\ 0]^T. \quad \textbf{8.60}$$

Keep the notation in mind: $\omega_{i'}^{i''i'}$ is the angular velocity of $\mathcal{F}_{i''}$ with respect to $\mathcal{F}_{i'}$, expressed in $\mathcal{F}_{i'}$.

1-rotation from $\mathcal{F}_{i''}$ to $\mathcal{F}_b$ about $\hat{i}''_1 \equiv \hat{b}_1$ through θ_3

The angular velocity of $\mathcal{F}_{ib}$ with respect to $\mathcal{F}_{i''}$ is

$$\omega^{bi''} = \dot{\theta}_3\hat{i}''_1 = \dot{\theta}_3\hat{\mathbf{b}}_1. \quad \textbf{8.61}$$

We can express $\omega^{bi''}$ in any frame, but $\mathcal{F}_{i''}$ and $\mathcal{F}_b$ are especially simple:

$$\omega_{i''}^{bi''} = [\dot{\theta}_3\ 0\ 0]^T \quad \textbf{8.62}$$

$$\omega_b^{bi''} = [\dot{\theta}_3\ 0\ 0]^T. \quad \textbf{8.63}$$

Keep the notation in mind: $\omega_b^{bi''}$ is the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_{i''}$, expressed in $\mathcal{F}_b$.

Angular velocities are vectors and add like vectors:

$$\omega^{bi} = \omega^{bi''} + \omega^{i''i'} + \omega^{i'i}. \quad \textbf{8.64}$$

We have expressed these three vectors in different frames; to add them together, we need to express all of them in the same frame.

Typically, we want ω^{bi}, so we need to rotate the 3×1 matrices we have just developed into $\mathcal{F}_b$. We have:

$$\omega_i^{i'i} = \omega_{i'}^{i'i} = [0\ 0\ \dot{\theta}_1]^T \Rightarrow \text{need } \mathbf{R}^{bi} = \mathbf{R}_1(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_1) \quad \textbf{8.65}$$

$$\omega_{i'}^{i''i'} = \omega_{i''}^{i''i'} = [0\ \dot{\theta}_2\ 0]^T \Rightarrow \text{need } \mathbf{R}^{bi'} = \mathbf{R}_1(\theta_3)\mathbf{R}_2(\theta_2) \quad \textbf{8.66}$$

$$\omega_{i''}^{bi''} = \omega_b^{bi''} = [\dot{\theta}_3\ 0\ 0]^T \Rightarrow \text{need } \mathbf{R}^{bi''} = \mathbf{R}_1(\theta_3). \tag{8.67}$$

We previously developed all these rotation matrices, so we just need to apply them and add the results.

Carry out the matrix multiplications and additions. Then obtain

$$\omega^{bi} = \mathbf{R}^{bi''}\omega^{bi''} + \mathbf{R}^{bi'}\omega^{i''i'} + \mathbf{R}^{bi}\omega^{i'i} \tag{8.68}$$

$$= \begin{bmatrix} \dot{\theta}_3 - \sin\theta_2\dot{\theta}_1 \\ \cos\theta_3\dot{\theta}_2 + \cos\theta_2\sin\theta_3\dot{\theta}_1 \\ -\sin\theta_3\dot{\theta}_2 + \cos\theta_2\cos\theta_3\dot{\theta}_1 \end{bmatrix} \tag{8.69}$$

$$= \begin{bmatrix} -\sin\theta_2 & 0 & 1 \\ \cos\theta_2\sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_2\cos\theta_3 & -\sin\theta_3 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{8.70}$$

$$= \mathbf{S}(\theta)\dot{\theta} \tag{8.71}$$

or

$$\dot{\theta} = \mathbf{S}^{-1}(\theta)\omega = \begin{bmatrix} 0 & \sin\theta_3/\cos\theta_2 & \cos\theta_3/\cos\theta_2 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 1 & \sin\theta_3\sin\theta_2/\cos\theta_2 & \cos\theta_3\sin\theta_2/\cos\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix}. \tag{8.72}$$

For this Euler angle set (3-2-1), the Euler rates go to infinity when $\cos\theta_2 \to 0$. The reason is that near $\theta_2 = \pi/2$ the first and third rotations are indistinguishable.

For the “symmetric” Euler angle sequences (3-1-3, 2-1-2, 1-3-1, etc.) the singularity occurs when $\theta_2 = 0$ or π.

For the “asymmetric” Euler angle sequences (3-2-1, 2-3-1, 1-3-2, etc.) the singularity occurs when $\theta_2 = \pi/2$ or $3\pi/2$.

This kinematic singularity is a major disadvantage of using Euler angles for large-angle motion.

There are attitude representations that do not have a kinematic singularity, but 4 or more scalars are required.

If the Euler angles and rates are small, then $\sin\theta_i \approx \theta_i$ and $\cos\theta_i \approx 1$:

$$\omega_b^{bi} \approx \begin{bmatrix} -\theta_2 & 0 & 1 \\ \theta_3 & 1 & 0 \\ 1 & -\theta_3 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \approx \begin{bmatrix} \dot{\theta}_3 \\ \dot{\theta}_2 \\ \dot{\theta}_1 \end{bmatrix}. \tag{8.73}$$

Suppose you want to invert the $n \times n$ matrix $\mathbf{A}$, with elements A_{ij} The elements of the inverse are

$$\mathbf{A}_{ij}^{-1} = \frac{C_{ij}}{|\mathbf{A}|}, \tag{8.74}$$

where C_{ji} is the cofactor, computed by multiplying the determinant of the $(n-1) \times (n-1)$ minor matrix obtained by deleting the j_{th} row and i_{th} column from $\mathbf{A}$, by $(-1)^{i+j}$.

This formulation is absolutely unsuitable for calculating matrix inverses in numerical work, especially with larger matrices, since it is computationally expensive.

One normally uses LU decomposition instead. Elementary row reduction is essentially LU decomposition.

Note: In most cases, we do not need the inverse anyway; we need the solution to a linear system.

Let us invert the matrix $\mathbf{S}(\theta)$ in Equation (8.71):

$$\mathbf{S}(\theta) = \begin{bmatrix} -\sin\theta_2 & 0 & 1 \\ \cos\theta_2 \sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_2 \cos\theta_3 & -\sin\theta_3 & 0 \end{bmatrix}. \tag{8.75}$$

In row reduction, we augment the matrix with the identity matrix:

$$\begin{bmatrix} -\mathrm{s}\theta_2 & 0 & 1 & 1 & 0 & 0 \\ \mathrm{c}\theta_2\mathrm{s}\theta_3 & \mathrm{c}\theta_3 & 0 & 0 & 1 & 0 \\ \mathrm{c}\theta_2\mathrm{c}\theta_3 & -\mathrm{s}\theta_3 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8.76}$$

and apply simply row-reduction operations to transform the left 3×3 block to the identity, leaving the inverse in the right 3×3 block.

In this case, we must swap the first row with one of the other two rows, say the 3rd row, which amounts to a permutation by:

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{8.77}$$

Note that $\mathbf{P}^{-1} = \mathbf{P}$.

$$\begin{bmatrix} \mathrm{c}\theta_2\mathrm{c}\theta_3 & -\mathrm{s}\theta_3 & 0 & 0 & 0 & 1 \\ \mathrm{c}\theta_2\mathrm{s}\theta_3 & \mathrm{c}\theta_3 & 0 & 0 & 1 & 0 \\ -\mathrm{s}\theta_2 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}. \tag{8.78}$$

Multiply row 2 by $\sin\theta_3 / \cos\theta_3$ and add to row 1.

Multiply row 1 by $-\sin\theta_3 \cos\theta_3$ and add to row 2.

$$\begin{bmatrix} \mathrm{c}\theta_2\mathrm{c}\theta_3 + \mathrm{c}\theta_2\mathrm{s}\theta_3 \tan\theta_3 & 0 & 0 & 0 & \tan\theta_3 & 1 \\ 0 & \mathrm{c}\theta_3 & 0 & 0 & 1 - \sin^2\theta_3 & -\sin\theta_3\cos\theta_3 \\ -\mathrm{s}\theta_2 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}. \tag{8.79}$$

Divide row 1 by $c\theta_2 c\theta_3 + c\theta_2 s\theta_3 \tan\theta_3 = \cos\theta_2/\cos\theta_3$ and simplify.

Multiply resulting row 1 by $s\theta_2$, add to row 3, and simplify.

Divide row 2 by $c\theta_3$ and simplify

$$\begin{bmatrix} 1 & 0 & 0 & 0 & s\theta_3/c\theta_2 & c\theta_3/c\theta_2 \\ 0 & 1 & 0 & 0 & c\theta_3 & -s\theta_3 \\ 0 & 0 & 1 & 1 & s\theta_3 \tan\theta_2 & c\theta_3 \tan\theta_2 \end{bmatrix}. \tag{8.80}$$

8.3.1 Typical Problem Involving Angular Velocity and Attitude

Given initial conditions for the attitude (in any form), and a time history of angular velocity, compute $\mathbf{R}$ or any other attitude representation as a function of time.

This requires integration of one of the sets of differential equations involving angular velocity.

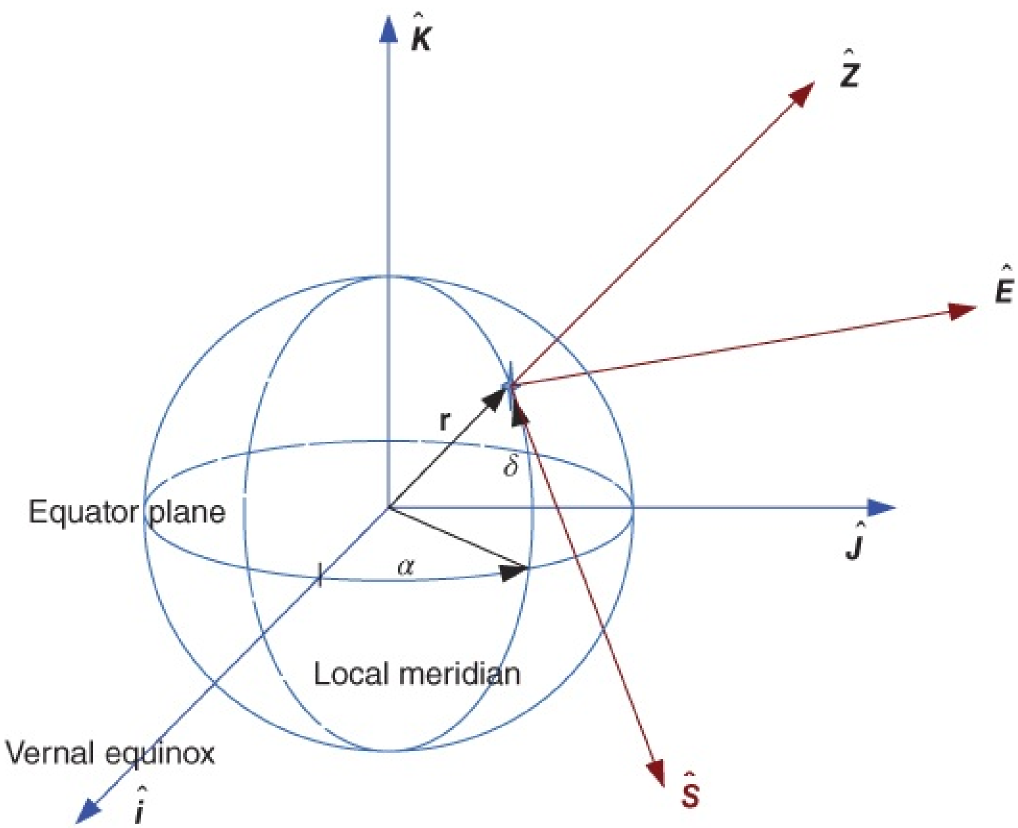

Figure 8.5 Rotation from $\mathcal{F}_i$

Use a rotation from $\mathcal{F}_b$ to α using two Euler angles, $\pi/2 - \delta$, and $\hat{I}\hat{J}\hat{K} \rightarrow \hat{S}\hat{E}\hat{Z}$. See Figure 8.5. The first (3) rotation is about the $\hat{i}_3$ axis, through angle α.

The second (2) rotation is about the $\hat{E}$ axis through $\pi/2 - \delta$. The three unit vectors $\hat{S}\hat{E}\hat{Z}$ (see Chapter 1) have derivatives:

$$\dot{\hat{S}} = \dot{\delta}\,\hat{Z} + \sin\delta\dot{\alpha}\,\hat{E} \tag{8.81}$$

$$\dot{\hat{E}} = -\cos\delta\dot{\alpha}\,\hat{Z} - \sin\delta\dot{\alpha}\,\hat{S} \quad \mathbf{8.82}$$

$$\dot{\hat{Z}} = \cos\delta\dot{\alpha}\,\hat{E} - \dot{\delta}\,\hat{S}. \quad \mathbf{8.83}$$

8.4 Euler's Theorem

The most general motion of a rigid body with a fixed point is a rotation about a fixed axis.

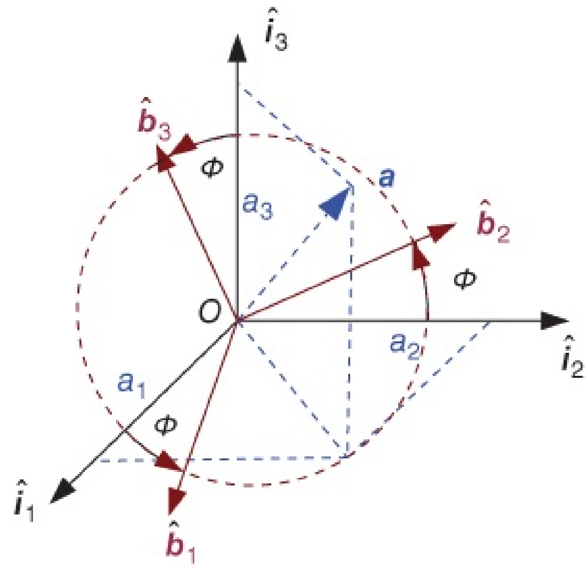

Figure 8.6 The geometry of Euler theorem

The axis, denoted **a** or $\hat{\mathbf{a}}$ is called the eigenaxis or Euler axis. The angle of rotation Φ is called the Euler angle or the principal Euler angle. See Figure 8.6.

$$\mathbf{a}^b = \mathbf{R}^{bi}\mathbf{a}^i = \mathbf{a}^i = [a_1, a_2, a_3]^T \quad \mathbf{8.84}$$

$$\mathbf{R}^{bi} = \cos\Phi\mathbf{1} + (1-\cos\Phi)\mathbf{a}\mathbf{a}^T - \sin\Phi\mathbf{a}^\times \quad \mathbf{8.85}$$

$$\mathbf{a}^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \quad \mathbf{8.86}$$

Let us see some observations regarding **a** and Φ.

Since $\mathbf{R}^{bi}\mathbf{a} = \mathbf{a}$, the Euler axis is the eigenvector of **R** associated with the eigenvalue 1. We can check this result:

$$\mathbf{R}\mathbf{a} = \left[\cos\Phi\mathbf{1} + (1-\cos\Phi)\mathbf{a}\mathbf{a}^T - \sin\Phi\mathbf{a}^\times\right]\mathbf{a} \quad \mathbf{8.87}$$

$$= \cos\Phi\mathbf{1}\mathbf{a} + (1-\cos\Phi)\mathbf{a}\mathbf{a}^T\mathbf{a} - \sin\Phi\mathbf{a}^\times\mathbf{a} \quad \mathbf{8.88}$$

$$= \cos\Phi\mathbf{a} + \mathbf{a}\mathbf{a}^T\mathbf{a} - \cos\Phi\mathbf{a}\mathbf{a}^T\mathbf{a} - \sin\Phi\mathbf{a}^\times\mathbf{a} \quad \mathbf{8.89}$$

$$= \cos\Phi\mathbf{a} + \mathbf{a} - \cos\Phi\mathbf{a} - \sin\Phi\mathbf{a}^{\times}\mathbf{a}(\mathbf{a}^T\mathbf{a} = 1) \quad 8.90$$

$$= \cos\Phi\mathbf{a} + \mathbf{a} - \cos\Phi\mathbf{a}(\mathbf{a}^{\times}\mathbf{a} = 0) \quad 8.91$$

$$= \mathbf{a}. \quad 8.92$$

Thus every rotation matrix has an eigenvalue that is equal to $+1$. This fact justifies the term eigenaxis for the Euler axis. This parameterization requires four parameters.

We can extract **a** and Φ from **R**. Just as we need to be able to compute Euler angles from a given rotation matrix, we need to be able to compute the Euler axis and Euler angle:

$$\Phi = \cos^{-1}\left[\frac{1}{2}(\operatorname{trace}\mathbf{R} - 1)\right] \quad 8.93$$

$$\mathbf{a}^{\times} = \frac{1}{2\sin\Phi}\left(\mathbf{R}^T - \mathbf{R}\right). \quad 8.94$$

Again for the example in Section 8.2.2, its $\Phi = 1.2105$, $\mathbf{a} = [0.6335, 0.7728, 0.0391]^T$.

One can show that the kinematics differential equations for $\mathbf{a}$ and Φ are:

$$\dot{\Phi} = \mathbf{a}^T\omega \quad 8.95$$

$$\dot{\mathbf{a}} = \frac{1}{2}\left[\mathbf{a}^{\times} - \coth(\boldsymbol{\Phi}/2)\mathbf{a}^{\times}\mathbf{a}^{\times}\right]. \quad 8.96$$

So this system of equations also has kinematic singularities, at $\Phi = 0$ and $\Phi = 2\pi$.

8.4.1 Another Four-Parameter Set

The Euler parameter set, also known as a quaternion, is a four-parameter set with some advantages over the Euler axis/angle set:

$$\mathbf{q} = \mathbf{a}\sin\frac{\Phi}{2} \quad 8.97$$

$$q_4 = \cos\frac{\Phi}{2}. \quad 8.98$$

The vector component, **q**, is a 3 × 1, whereas the scalar component, q_4, is, well, a scalar. The quaternion is denoted by $\bar{\mathbf{q}} = [\mathbf{q}^T, q_4]^T$, a 4×1 matrix.

To compute the rotation matrix using the quaternion:

$$\mathbf{R} = \left(q_4^2 - \mathbf{q}^T\mathbf{q}\right)\mathbf{1} + 2\mathbf{q}\mathbf{q}^T - 2q_4\mathbf{q}^{\times} \quad 8.99$$

$$= \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}. \quad \mathbf{8.100}$$

To compute the quaternion using the rotation:

$$q_4 = \pm\frac{1}{2}\sqrt{1 + \operatorname{trace}\mathbf{R}} \quad \mathbf{8.101}$$

$$\mathbf{q} = \frac{1}{4q_4}\begin{bmatrix} r_{23} - r_{32} \\ r_{31} - r_{13} \\ r_{12} - r_{21} \end{bmatrix}. \quad \mathbf{8.102}$$

Again for the example in Section 8.2.2, $\bar{\mathbf{q}} = [0.3604, 0.4397, 0.0223, 0.8224]^T$.

As with Euler angles, we are frequently interested in small attitude motions.

If Φ is small, then $\mathbf{q} = \mathbf{a}\sin(\Phi/2) \approx \mathbf{a}\Phi/2$, and $q_4 \approx 1$.

Hence for small Φ:

$$\mathbf{R} = (q_4^2 - \mathbf{q}^T\mathbf{q})\mathbf{1} + 2\mathbf{q}\mathbf{q}^T - 2q_4\mathbf{q}^\times \quad \mathbf{8.103}$$

$$\approx (1 - 0)\mathbf{1} + 2(0) - 2\mathbf{q}^\times \quad \mathbf{8.104}$$

$$\approx 1 - 2\mathbf{q}^\times. \quad \mathbf{8.105}$$

Compare this expression with the previously developed $\mathbf{R} \approx \mathbf{1} - \theta^\times$ for Euler angles. We see for 3-2-1 rotation, $q_1 = \theta_3/2$, $q_2 = \theta_2/2$, $q_3 = \theta_1/2$,

Small rotations are commutative:

$$\mathbf{R}^{cb}\mathbf{R}^{ba} \approx [\mathbf{1} - 2\mathbf{q}_2^\times][\mathbf{1} - 2\mathbf{q}_1^\times] \approx \mathbf{1} - 2\mathbf{q}_2^\times - 2\mathbf{q}_1^\times. \quad \mathbf{8.106}$$

Differential equations $\dot{\bar{\mathbf{q}}}$:

$$\dot{\bar{\mathbf{q}}} = \frac{1}{2}\begin{bmatrix} \mathbf{q}^\times + q_4\mathbf{1} \\ -\mathbf{q}^T \end{bmatrix}\omega = \mathbf{Q}(\bar{\mathbf{q}})\omega. \quad \mathbf{8.107}$$

Note that there is no kinematic singularity with these differential equations.

8.4.2 Summary and Extension of Kinematics Notation

Several equivalent methods of describing attitude or orientation are: rotation matrix, denoted as DCM, which is the vectors of one frame expressed in the other; it is also the dot products of vectors of one frame with those of the other.

Euler angles have 12 different sets, so getting the order right is required! Euler axis/angle are unit vector and angle. Euler parameters = quaternions : unit 4 $\times$ 1.

It is important to be able to compute one from the other for any given representation. There are other attitude representations.

We have seen direction cosines, Euler angles, Euler angle/axis, and quaternions.

Two other common representations are: Euler-Rodriguez parameters, denoted as $\mathbf{p}$:

$$\mathbf{p} = \mathbf{a} \tan \frac{\Phi}{2} \tag{8.108}$$

$$\mathbf{R} = \mathbf{1} + \frac{2}{1 + \mathbf{p}^T \mathbf{p}} (\mathbf{p}^\times \mathbf{p}^\times - \mathbf{p}^\times) \tag{8.109}$$

$$\bar{\mathbf{q}} = \frac{1}{\sqrt{1 + \mathbf{p}^T \mathbf{p}}} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} \tag{8.110}$$

$$\dot{\mathbf{p}} = \frac{1}{2} (\mathbf{p}\mathbf{p}^T + \mathbf{1} + \mathbf{p}^\times) \omega. \tag{8.111}$$

And Modified Rodriguez parameters, denoted as σ:

$$\sigma = \mathbf{a} \tan \frac{\Phi}{4} \tag{8.112}$$

$$\mathbf{R} = \frac{1}{1 + \sigma^T \sigma} \left[\left(1 - (\sigma^T \sigma)^2\right) \mathbf{1} + 2\sigma\sigma^T - 2\left(1 - (\sigma^T \sigma)^2\right) \sigma^\times \right] \tag{8.113}$$

$$\dot{\sigma} = \frac{1}{2} \left[\mathbf{1} - \sigma^\times + \sigma\sigma^T - \frac{1 + \sigma^T \sigma}{2} \mathbf{1} \right]. \tag{8.114}$$

Again for the example in Section 8.2.2, $\sigma = [0.1978, 0.2413, 0.0122]^T$.

8.4.2.1 An Illustrative Example

Let us develop the rotation matrix relating the Earth-centered inertial (ECI), that is, $\hat{I}\hat{J}\hat{K}$ frame $\mathcal{F}_i$ and the orbital frame $\mathcal{F}_o$. We consider the case of an elliptical orbit, with right ascension of the ascending node (or RAAN, Ω, inclination, i, and argument of latitude, $u = \omega + \theta$. Recall that argument of latitude is the angle frame the ascending node to the position of the satellite, and is especially useful for circular orbits, since argument of periapsis, ω, is not defined for circular orbits. See Figure 8.7.

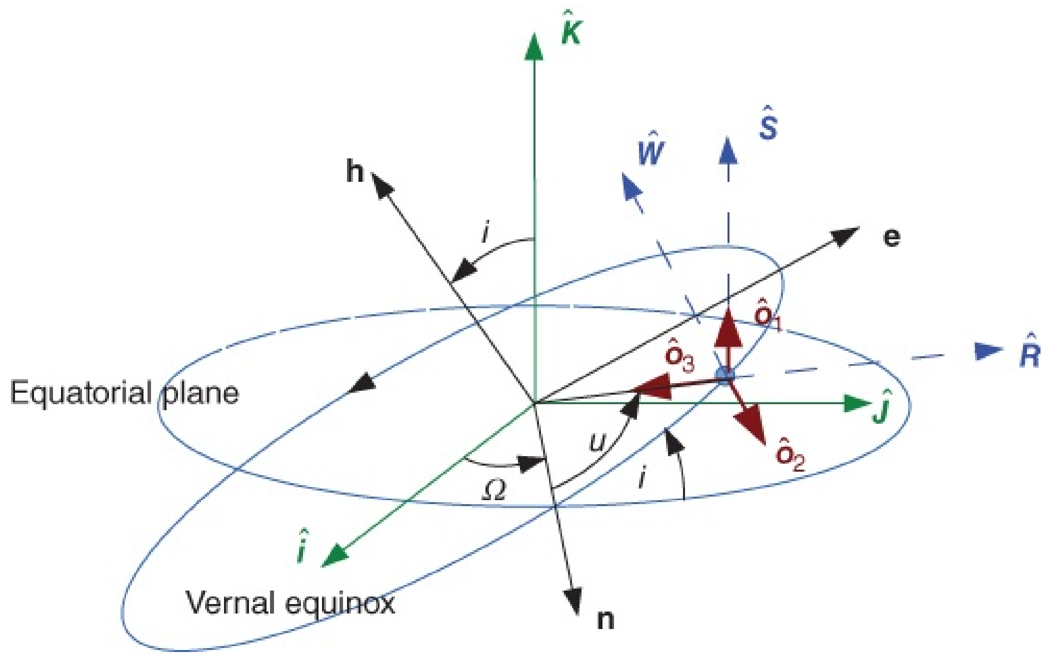

Figure 8.7 From inertial $\mathcal{F}_i$ to orbital frame $\mathcal{F}_o$ and $\hat{R}\hat{S}\hat{W}$ frame

We denote the ECI frame ($\mathcal{F}_i$) by $\{\hat{i}\}$, and the orbital frame ($\mathcal{F}_o$) by $\{\hat{o}\}$. Intermediate frames are designated using primes, as the Euler angle development above. We use a "3-1-3" sequence as follows: Begin with a "3" rotation about the inertial $\hat{i}_3$ axis through the RAAN, Ω. This rotation is followed by a "1" rotation about the $\hat{i}_1'$ axis through the inclination, i. The last rotation is another "3" rotation about the $\hat{i}_3''$ axis through the argument of latitude, u.

We denote the resulting reference frame by $\{\hat{o}'\}$, since it is not quite the desired orbital reference frame. Recall that the orbital reference frame for a circular orbit has its three vectors aligned as follows: $\{\hat{o}_1\}$ is in the direction of the orbital velocity vector (the $\mathbf{v}$ direction), $\{\hat{o}_2\}$ is in the direction opposite to the orbit normal (the $-\hat{h}(\hat{W})$ direction), and $\{\hat{o}_3\}$ is the nadir direction (or the $-\hat{r}(-\hat{R})$ direction). However, the frame resulting frame the (3-1-3) rotation developed above has its unit vectors aligned in the $\mathbf{r}, \mathbf{v}$ and $\mathbf{h}$ directions, respectively.

Now it is possible to go back and choose angles so that the (3-1-3) rotation gives the desired orbital frame; however, it is instructive to see how to use two more rotations to get frame the $\{\hat{o}'\}$ frame to the $\{\hat{o}\}$. Specifically, if we perform another "3" rotation about $\{\hat{o}_3'\}$ through 90° and a "1" rotation about $\{\hat{o}_1''\}$ through 270°, we arrive at the desired orbital reference frame. These final two rotations lead to an interesting rotation matrix, that is, from $\hat{R}\hat{S}\hat{W}$ to $\hat{O}_1\hat{O}_2\hat{O}_3(\mathcal{F}_o)$, see Figure 8.7:

$$\mathbf{R}^{oo'} = \mathbf{R}_1(270^\circ)\mathbf{R}_3(90^\circ) \quad \mathbf{8.115}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}. \quad \mathbf{8.116}$$

Careful study of this rotation matrix reveals that its effect is to move the second row to the first row. Negate the third row and move it to the second row, and negate the first row and move to the third row. So, the rotation matrix that takes vectors from the inertial frame to the orbital frame is

$$\mathbf{R}^{oi} = \mathbf{R}^{oo'}\mathbf{R}_3(u)\mathbf{R}_1(i)\mathbf{R}_3(\Omega), \tag{8.117}$$

which, when expanded, gives

$$\mathbf{R}^{oi} = \begin{bmatrix} -suc\Omega - cucis\Omega & -sus\Omega + cucic\Omega & cusi \\ -sis\Omega & sic\Omega & -ci \\ cuc\Omega + sucis\Omega & -cus\Omega - sucic\Omega & -susi \end{bmatrix}. \tag{8.118}$$

where $u = \omega + \theta$. This rotation matrix is not resulted from attitude motion; it is the rotation to Euler angles, not of the Euler angles. If the transformation is from Inertial Frame to Perifocal Frame, then the true anomaly $\theta = 0$.

Knowing the 9 numbers in the rotation matrix, we can compute the Euler angles.

$$i = \cos^{-1}(-R_{23}), \quad u = \tan^{-1}(-R_{33}/R_{13}), \quad \Omega = \tan^{-1}(-R_{21}/R_{22}) \tag{8.119}$$

Quadrant checks are imperative. It's better to use `|atan2(y,x)|` in $\mathrm{MATLAB}^{®}$.

8.5 Attitude Determination

8.5.1 Sensors

Attitude is measured by measuring the attitude of a celestial object that has a known orientation in the inertial coordinate system. One measured vector has only two degrees of freedom. Roll along the axis is undefined. Two measured vectors have four pieces of information, and only three are required for a unique solution. Hence just two independent measurements are needed.

Some items can be measured. The first is the direction from S/C to the sun, which must be out of umbra and the Sun must be in the S/C sun sensor field of view (FOV also. The second direction that can be measured is the direction to the center of the Earth from S/C. From HEO, such as GEO, the Earth can be assumed a centroid, but for LEO, two points should be considered, i.e. Earth limb. Lighting conditions make the limb difficult to see in visual bands. Earth limb ($\mathrm{CO_2}$) is very visible in infrared (IR.

Another item that can be measured is the direction to a star or set of stars. A single star (star tracker) limits attitude motion as its FOV is small. Multiple stars (star field imager) need a star map and image processing.

The Earth's magnetic field can also be measured. But it varies significantly in orbit and rotates with the Earth. Models of field accuracy vary.

Table 8.2 The comparison of sensors adapted from reference [1]

Sensor	Accuracy	Characteristics and applicability
Magneto-	1.0°	Attitude measured relative to Earth's local
meters	(5000 km alt)	magnetic field. Magnetic field uncertainties
	5.0°	and variability dominate accuracy. Usable below
	(200 km alt)	~ 6000 km, economical, orbital dependent
Earth-	0.05° (GEO)	Horizon uncertainties dominate accuracy.
sensor	0.1° (LEO)	Highly accurate units use scanning.
		orbit dependent, poor in yaw, relatively expensive
Sun sensor	0.01°	FOV : ±30°, simple, reliable, intermittent use, low cost
Star sensor	2″(arc-sec)	FOV: ±6°, heavy, complex, expensive, very accurate
Gyroscopes	0.001 deg/hr	Normal use involves periodically resetting reference.
		Rate only, power cost, can be heavy and expensive
Directional antennas	0.01° to 0.5°	Typically 1% of the antenna beamwidth

The Moon can also be measured too; it has wide variation in attitude and requires multiple sensors, and/or wide FOV.

Items you can measure can be the angular velocity vector, which does not tell actual attitude, but variations from an attitude can be found by integration methods when it is used in conjunction with other sensors, such as radio beacons.

Narrow band and intensity maps allow ground measurement of spacecraft attitude.

When using differential GPS signals, their accuracy is dependent on relative separation of receiving antennas. And when velocity of ~8 km/s, doppler effects appear, and also satellites in view change rapidly. There are also somelegal issues.

Let us introduce more about the two most frequently used sensors for S/C.

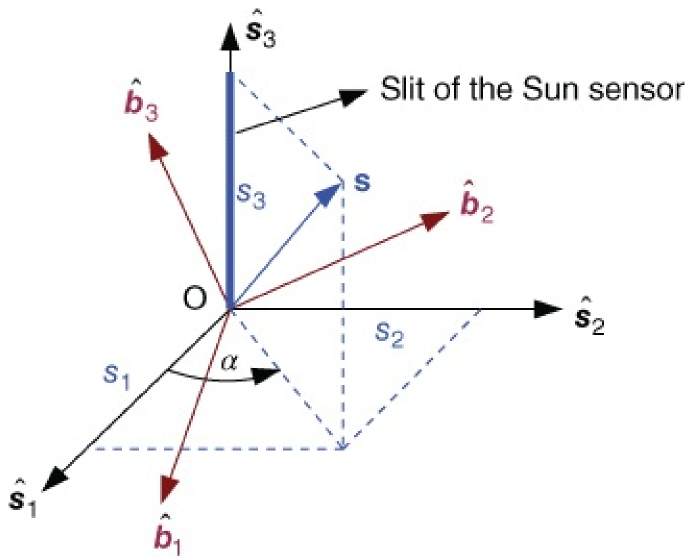

Figure 8.8 The frame of Sun sensor $\mathcal{F}_s$

8.5.1.1 Sun Sensor

See Figure 8.8. It is the frame of Sun sensor $\mathcal{F}_s$. What the Sun sensor measures is the angle α,

$$\alpha = \arctan\left(\frac{s_2}{s_1}\right), \tag{8.120}$$

where $\mathbf{s}_s = [s_1, s_2, s_3]^T$ are the components of the Sun direction in the $\mathcal{F}_s$. If $\mathbf{s}_i = [\cos\delta_s \cos\alpha_s, \cos\delta_s \sin\alpha_s, \sin\delta_s]^T$ is the Sun direction in $\mathcal{F}_i$, δ_s, α_s are the declination and right ascension of the Sun in celestial sphere, and $\mathbf{s}_s$ in $\mathcal{F}_s$, then

$$\mathbf{s}_s = \mathbf{R}^{sb}\mathbf{R}^{bo}\mathbf{R}^{oi}\mathbf{s}_i, \tag{8.121}$$

where $\mathbf{R}^{sb}$ is the fixed rotation matrix from body from $\mathcal{F}_b$ to the Sun sensor frame $\mathcal{F}_s$.

$$\mathbf{R}^{sb} = \begin{bmatrix} r_{ax} & r_{ay} & r_{az} \\ r_{bx} & r_{by} & r_{bz} \\ r_{cx} & r_{cy} & r_{cz} \end{bmatrix}. \tag{8.122}$$

Then component s_1, s_2 in Figure 8.8 can be obtained

$$s_1 = [r_{ax} r_{ay} r_{az}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\phi & \mathrm{s}\phi \\ 0 & -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} \mathrm{c}\theta & 0 & -\mathrm{s}\theta \\ 0 & 1 & 0 \\ \mathrm{s}\theta & 0 & \mathrm{c}\theta \end{bmatrix} \begin{bmatrix} \mathrm{c}\psi & \mathrm{s}\psi & 0 \\ -\mathrm{s}\psi & \mathrm{c}\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{ox} \\ s_{oy} \\ s_{oz} \end{bmatrix} \tag{8.123}$$

$$= [r_{ax} r_{ay} r_{az}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\phi & \mathrm{s}\phi \\ 0 & -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} \mathrm{c}\theta & 0 & -\mathrm{s}\theta \\ 0 & 1 & 0 \\ \mathrm{s}\theta & 0 & \mathrm{c}\theta \end{bmatrix} \begin{bmatrix} s_{ox} & s_{oy} & 0 \\ s_{oy} & -s_{ox} & 0 \\ 0 & 0 & s_{oz} \end{bmatrix} \begin{bmatrix} \mathrm{c}\psi \\ \mathrm{s}\psi \\ 1 \end{bmatrix} \tag{8.124}$$

$$s_2 = [r_{bx} r_{by} r_{bz}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{ox} \\ s_{oy} \\ s_{oz} \end{bmatrix} \tag{8.125}$$

$$= [r_{bx} r_{by} r_{bz}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} s_{ox} & s_{oy} & 0 \\ s_{oy} & -s_{ox} & 0 \\ 0 & 0 & s_{oz} \end{bmatrix} \begin{bmatrix} c\psi \\ s\psi \\ 1 \end{bmatrix}, \tag{8.126}$$

where $\mathbf{s}_o = [s_{ox}, s_{oy}, s_{oz}]^T = \mathbf{R}^{oi}\mathbf{s}_i$, by Equation (8.120),

$$k_1 \cos\psi + k_2 \sin\psi + k_3 = 0, \tag{8.127}$$

where

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} s_{ox} & s_{oy} & 0 \\ s_{oy} & -s_{ox} & 0 \\ 0 & 0 & s_{oz} \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \tag{8.128}$$

$$\cdot \left\{ \begin{bmatrix} s_{bx} \\ s_{by} \\ s_{bz} \end{bmatrix} - \tan\alpha \begin{bmatrix} s_{ax} \\ s_{ay} \\ s_{az} \end{bmatrix} \right\}, \tag{8.129}$$

then the yaw angle can be solved:

$$\psi = -\arctan\frac{k_3}{\sqrt{k_1^2 + k_2^2}} - \arctan\frac{k_1}{k_2} \tag{8.130}$$

Normally Equation (8.161) is a nonlinear equation. It is difficult to use directly. Its linearized equation of Equation (8.120) is

$$\alpha = \alpha_0 + \frac{\partial\alpha}{\partial\phi}\phi + \frac{\partial\alpha}{\partial\theta}\theta + \frac{\partial\alpha}{\partial\psi}(\psi - \psi_0). \tag{8.131}$$

By proper choice the $\mathbf{R}^{sb}$, such as unit matrix, as calculated bythe angle between the Sun's direction and the Earth nadir direction, i.e. $\cos\theta_{se} = r_{33}$ in $\mathbf{R}^{oi}$, λ_s is the angle from x_o to y_o in $\mathcal{F}_o$, the Sun's direction in $\mathcal{F}_o$ is

$$\mathbf{s}_o = \begin{bmatrix} \sin\theta_{se}\cos\lambda_s \\ \sin\theta_{se}\sin\lambda_s \\ \cos\theta_{se} \end{bmatrix}. \tag{8.132}$$

Then see Figure 8.8:

$$\alpha = \arctan\frac{-\sin\theta_{se}\sin(\psi_0 - \lambda_s) + \phi\cos\theta_{se} - (\psi - \psi_0)\sin\theta_{se}\cos(\psi_0 - \lambda_s)}{\sin\theta_{se}\cos(\psi_0 - \lambda_s) - \theta\cos\theta_{se} - (\psi - \psi_0)\sin\theta_{se}\sin(\psi_0 - \lambda_s)}. \tag{8.133}$$

At normal state, $\phi, \theta, \psi - \psi_0$ are all small, $\alpha_0 = \psi_0 - \lambda_s$. The linearized equation becomes

$$\alpha = \alpha_0 - \frac{\cos\alpha_0}{\tan\theta_{se}}\phi + \frac{\sin\alpha_0}{\tan\theta_{se}}\theta + \psi - \psi_0. \qquad \textbf{8.134}$$

The above equation is linear relation between α and ψ when the attitude angles are small.

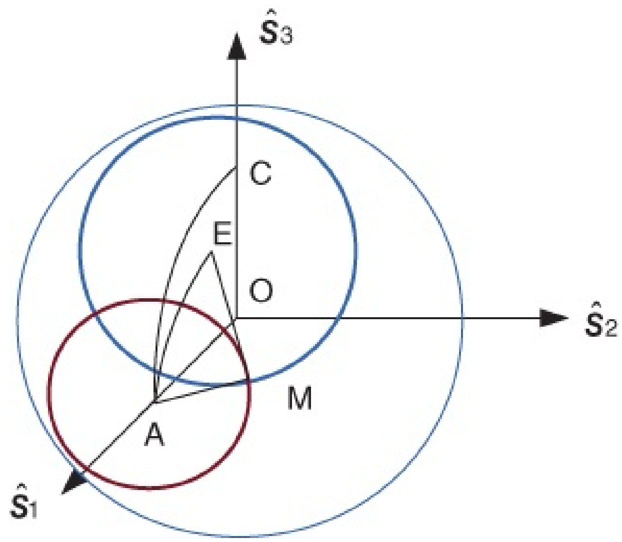

Figure 8.9 The frame of Earth sensor

8.5.1.2 Earth Sensor

A typical Earth sensor is an infrared scanning cone. See Figure 8.9. It is the frame of Earth sensor $\mathcal{F}_s$. In Figure 8.9, points A, M, E, C are all on the celestial sphere. The cone's symmetric axis is OA, which is on the axis $\hat{s}_1$, the Earth nadir is E, the intersection of the cone and the Earth is M, and $AM = \gamma$ is the cone angle of Earth sensor. $AE = \eta$ angle between the Earth nadir direction to the cone's axis. $ME = \rho = \arcsin(R_E/r_{sat})$, $\angle MAE = \mu/2$, $\angle CAE = \lambda$, where $\mu = \omega_{rot}(t_{out} - t_{in})$, $\lambda = \mu/2 - \omega_{rot}(t_{ref} - t_{in})$ Then the Earth nadir sensor frame is:

$$\mathbf{n}_s = \begin{bmatrix} \cos\eta \\ -\sin\eta\sin\lambda \\ \sin\eta\cos\lambda \end{bmatrix} \qquad \textbf{8.135}$$

$$\cos\rho = \cos\gamma\cos\eta + \sin\gamma\sin\eta\cos\frac{\mu}{2}. \qquad \textbf{8.136}$$

The nadir point is $[0, 0, 1]^T$

$$\mathbf{n}_s = \begin{bmatrix} \sin\theta \\ -\cos\theta\sin\phi \\ \cos\theta\cos\phi \end{bmatrix}. \qquad \textbf{8.137}$$

For small $\theta = \pi/2 - \eta$, $\phi = \lambda$.

8.5.2 Attitude Determination

There are many methods to do attitude determination [4]. One well-known method is the triad algorithm, which is a deterministic solution using two measured and known vectors. It assumes that one vector is known to be more precise than another. Some of the information in the problem is neglected.

Assume two known unit vectors in the ECI frame, Sun's direction $\mathbf{s}_i$ and Earth nadir direction $\mathbf{n}_i$. Using Sun sensors and Earth sensors, these vectors in the Body frame can be measured: $\mathbf{s}_b$ for the Sun and $\mathbf{n}_b$ for the Earth's nadir. See Figure 8.10.

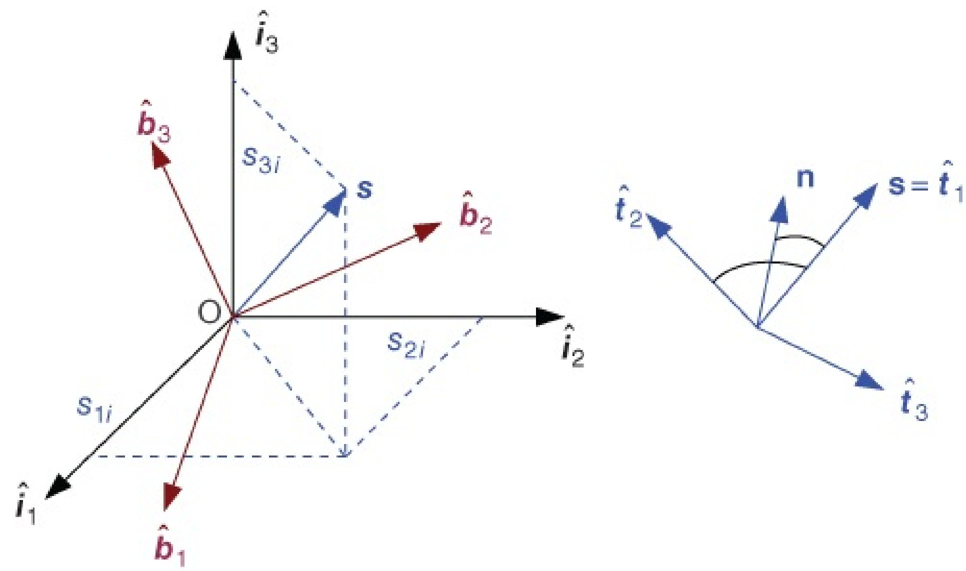

Figure 8.10 The geometry of triad algorithm

Define a new frame t, such that

$$\mathbf{t}_1 = \mathbf{s}, \quad \mathbf{t}_{1i} = \mathbf{s}_i, \quad \mathbf{t}_{1b} = \mathbf{s}_b, \tag{8.138}$$

$$\mathbf{t}_2 = \frac{\mathbf{s} \times \mathbf{n}}{|\mathbf{s} \times \mathbf{n}|}, \quad \mathbf{t}_{2i} = \frac{\mathbf{s}_i \times \mathbf{n}_i}{|\mathbf{s}_i \times \mathbf{n}_i|}, \quad \mathbf{t}_{2b} = \frac{\mathbf{s}_b \times \mathbf{n}_b}{|\mathbf{s}_b \times \mathbf{n}_b|}, \tag{8.139}$$

$$\mathbf{t}_3 = \mathbf{t}_1 \times \mathbf{t}_2, \quad \mathbf{t}_{3i} = \mathbf{t}_{1i} \times \mathbf{t}_{2i}, \quad \mathbf{t}_{3b} = \mathbf{t}_{1b} \times \mathbf{t}_{2b}. \tag{8.140}$$

Note that

$$[\mathbf{t}_{1i}\ \mathbf{t}_{2i}\ \mathbf{t}_{3i}] = \mathbf{R}^{it}\mathbf{I} = \mathbf{R}^{it} \tag{8.141}$$

$$[\mathbf{t}_{1b}\ \mathbf{t}_{2b}\ \mathbf{t}_{3b}] = \mathbf{R}^{bt}\mathbf{I} = \mathbf{R}^{bt} \tag{8.142}$$

or finally,

$$\mathbf{R}^{bi} = \mathbf{R}^{bt}\mathbf{R}^{ti} = \mathbf{R}^{bt}\left(\mathbf{R}^{it}\right)^T \tag{8.143}$$

$$\mathbf{R}^{bi} = [\mathbf{t}_{1b}\ \mathbf{t}_{2b}\ \mathbf{t}_{3b}]\,[\mathbf{t}_{1i}\ \mathbf{t}_{2i}\ \mathbf{t}_{3i}]^T. \tag{8.144}$$

Example

Suppose a spacecraft has two attitude sensors that provide the following measurements of the two vector $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{v}_{1b} = [0.8273,\ 0.5541,\ -0.0920]^T \qquad \textbf{8.145}$$

$$\mathbf{v}_{2b} = [-0.8285,\ 0.5522,\ -0.0955]^T. \qquad \textbf{8.146}$$

These vectors have known inertial frame components of

$$\mathbf{v}_{1i} = [-0.1517,\ -0.9669,\ 0.02050]^T \qquad \textbf{8.147}$$

$$\mathbf{v}_{2i} = [-0.8393,\ 0.4494,\ -0.3044]^T. \qquad \textbf{8.148}$$

Applying the Triad algorithm, we construct the components of the vectors $\mathbf{t}_j$, $j = 1, 2, 3$ in both the body and inertial frames:

$$\mathbf{t}_{1b} = [0.8273,\ 0.5541,\ -0.0920]^T \qquad \textbf{8.149}$$

$$\mathbf{t}_{2b} = [-0.0023,\ 0.1671,\ 0.9859]^T \qquad \textbf{8.150}$$

$$\mathbf{t}_{3b} = [0.5617,\ -0.8155,\ 0.1395]^T \qquad \textbf{8.151}$$

and

$$\mathbf{t}_{1i} = [-0.1517,\ -0.9669,\ 0.2050]^T \qquad \textbf{8.152}$$

$$\mathbf{t}_{2i} = [0.2177,\ -0.2350,\ -0.9473]^T \qquad \textbf{8.153}$$

$$\mathbf{t}_{3i} = [0.9641,\ -0.0991,\ 0.2462]^T. \qquad \textbf{8.154}$$

Using the above algorithm, we obtain the approximate rotation matrix:

$$\mathbf{R}^{bi} = \begin{bmatrix} 0.4156 & -0.8551 & 0.3100 \\ -0.8339 & -0.4943 & -0.2455 \\ 0.3631 & -0.1566 & -0.9185 \end{bmatrix}. \qquad \textbf{8.155}$$

Then we can have other representations for the attitude:

quaternion: $\mathbf{q} = [-0.8408, 0.5023, -0.2002, 0.0264]^T$,

Euler angle/axis: $\phi = 3.0887$, $\mathbf{a} = [-0.8411, 0.5025, -0.2002]^T$,

Rodriguez parameters: $\mathbf{p} = [-31.8031, 18.9991, -7.5711]^T$,

Modified Rodriguez parameters: $\sigma = [-0.8191, 0.4894, -0.1950]$,

Euler angle for 3-2-1 sequence: $\phi = -14.9614°$, $\theta = -18.0608°$, $\psi = 64.0834°$.

Applying this rotation matrix to $\mathbf{v}_{1i}$ gives $\mathbf{v}_{1b}$ exactly, because we used this condition in the formulation; however, applying it to $\mathbf{v}_{2i}$ does not gives $\mathbf{v}_{2b}$ exactly. If we know a priori that sensor 2 is more accurate than sensor 1, then we can use $\mathbf{v}_2$ as the exact measurement, hopefully leading to a more accurate estimate of $\mathbf{R}_{bi}$.

In the above triad algorithm, it is assumed the Sun and the Earth direction are measured directly. And also the satellite is inertial oriented. Practically, most modern satellites are Earth-oriented, the Sun and the Earth directions can not be measured directly.

8.5.2.1 Earth-oriented Satellite

For the sake of clarity, we can redefine the angle in Equation (8.44) of the 3-2-1 sequence which is shown in Figure 8.7. The axis definition is also given at the beginning of Section 8.1. We assumethat θ_1 is the yaw angle ψ for $\hat{z}$-axis, θ_2 is the pitch angle θ for $\hat{y}$-axis, θ_3 is roll ϕ for $\hat{x}$-axis, and rotation matrix $\mathbf{R}^{bo}$ is from orbital frame $\mathcal{F}_o$ to body frame $\mathcal{F}_b$, see Equation (8.44). The rotation from inertial form $\mathcal{F}_i$ to orbital frame $\mathcal{F}_o$ is $\mathbf{R}^{oi}$, see Equation (8.118). This time the rotation is from orbital frame $\mathcal{F}_o$ to body frame $\mathcal{F}_b$, still in 3-2-1 sequence.

$$\mathbf{R}^{bo} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{8.156}$$

$$= \begin{bmatrix} \mathrm{c}\psi\mathrm{c}\theta & \mathrm{s}\psi\mathrm{c}\theta & -\mathrm{s}\theta \\ -\mathrm{s}\psi\mathrm{c}\phi + \mathrm{c}\psi\mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\psi\mathrm{c}\phi + \mathrm{s}\psi\mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta\mathrm{s}\phi \\ \mathrm{s}\psi\mathrm{s}\phi + \mathrm{c}\psi\mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\psi\mathrm{s}\theta\mathrm{c}\phi - \mathrm{c}\psi\mathrm{s}\phi & \mathrm{c}\theta\mathrm{c}\phi \end{bmatrix}. \tag{8.157}$$

If a vector in $\mathcal{F}_i$ is $\mathbf{v}^i$, in $\mathcal{F}_b$ us $\mathbf{v}^b$, then

$$\mathbf{v}^b = \mathbf{R}^{bo}\mathbf{R}^{oi}\mathbf{v}^i. \tag{8.158}$$

If the orbit is a circular orbit, its angular orbital velocity is ω_o then $\omega^{bi} = \omega^{bo} + \mathbf{R}^{bo}\omega^{oi}$, $\dot{\theta} = \mathbf{S}^{-1}\omega^{bo}$. Then we have the relation between the Euler ϕ, θ, ψ and the angular velocity $\omega_x, \omega_y, \omega_z$.

$$\dot{\phi} = \omega_x + \sin\phi\tan\theta\,\omega_y + \cos\phi\tan\theta\,\omega_z + \sin\psi\sec\theta\,\omega_o \tag{8.159}$$

$$\dot{\theta} = \cos\phi\,\omega_y - \sin\phi\,\omega_z + \cos\psi\,\omega_o \tag{8.160}$$

$$\dot{\psi} = \sin\phi\sec\theta\,\omega_y + \cos\phi\sec\theta\,\omega_z + \tan\theta\sin\psi\,\omega_o. \tag{8.161}$$

8.5.2.2 Orbital Compass

When the attitudes ϕ, θ, ψ are small, Equations (8.159)–(8.161) become:

$$\dot{\phi} = \omega_x + \psi\,\omega_o \tag{8.162}$$

$$\dot{\theta} = \omega_y + \omega_o \quad \textbf{8.163}$$

$$\dot{\psi} = \omega_z - \phi\,\omega_o. \quad \textbf{8.164}$$

From Equations (8.162)–(8.164), θ is decoupled, and ϕ, ψ are coupled. Note that if $\omega = [\omega_x, \omega_y, \omega_z]^T = \mathbf{0}$, and the satellite is in circular orbit, ω_o is a constant, then the $\hat{y}$-axis of the body is inertially fixed space.

If the satellite initial time is at t_0, its roll and yaw angles are $\phi(t_0)$, $\psi(t_0)$

$$\phi(t) = \phi(t_0)\cos(\omega_o(t-t_0)) + \psi(t_0)\sin(\omega_o(t-t_0)) \quad \textbf{8.165}$$

$$\psi(t) = \psi(t_0)\cos(\omega_o(t-t_0)) - \phi(t_0)\sin(\omega_o(t-t_0)). \quad \textbf{8.166}$$

When $(t-t_0)\omega_o = \pi/2$, i.e., the orbit rotates 1/4 period, then $\phi(t) = \psi(t_0)$, $\psi(t) = -\phi(t_0)$. This means the roll angle and yaw angle are exchanged. From the view of measurement, with one sensor in the roll-yaw channel, the roll-yaw system is observable. In modern control theory, $C = [1\,0]$, $A = [0\,1; -1\,0]$, $\text{rank[C;CA]} = 2$.

In general cases, ω is not zero. We have three gyros in each axis of the body, it gives the angular velocities along each axis $\omega_x, \omega_y, \omega_z$ in $\mathcal{F}_b$. The gyro model is

$$g_x = \omega_x + N_{gx} \quad \textbf{8.167}$$

$$g_y = \omega_y + N_{gy} \quad \textbf{8.168}$$

$$g_z = \omega_z + N_{gz}, \quad \textbf{8.169}$$

where N_{gx}, N_{gy}, N_{gz} are measurement error, i.e. gyro noise and constant drift. By submitting the gyro model into Equations (8.162)–(8.164),

$$\dot{\phi} = \psi\,\omega_o + g_x - N_{gx} \quad \textbf{8.170}$$

$$\dot{\theta} = \omega_o + g_y - N_{gx} \quad \textbf{8.171}$$

$$\dot{\psi} = -\phi\,\omega_o + g_z - N_{gx}. \quad \textbf{8.172}$$

If we have two Earth sensors along the body $\hat{x}$ and $\hat{y}$ axis each, its measurement is denoted ϕ_H, ψ_H, then

$$\phi_H = \phi + N_{\phi H} \quad \textbf{8.173}$$

$$\theta_H = \theta + N_{\theta H}, \quad \textbf{8.174}$$

where $N_{\phi H}$, $N_{\theta H}$ are measurement noise of the Earth sensors.

From the view of modern control theory, Equations (8.170)–(8.172) are system state equations, where ϕ, θ, ψ are state variable, g_x, g_y, g_z are input variables, N_{gx}, N_{gy}, N_{gz} are system noises. Equations (8.173)–(8.174) are measurement equations, ϕ_H, θ_H are the outputs. The estimator can be designed:

$$\dot{\bar{\phi}} = \bar{\psi}\omega_o + g_x + K_\phi(\phi_H - \bar{\phi}) \tag{8.175}$$

$$\dot{\bar{\theta}} = \omega_o + g_y + K_\theta(\theta_H - \bar{\theta}) \tag{8.176}$$

$$\dot{\bar{\psi}} = -\bar{\phi}\omega_o + g_z + K_\psi(\phi_H - \bar{\phi}), \tag{8.177}$$

where $\bar{\phi}, \bar{\theta}, \bar{\psi}$ are estimates of the state, K_ϕ, K_θ, K_ψ are gains of the estimator. Normally they are small, about $0.005 - 0.05$(1/s). This set of equation is called “orbital compass,” as the yaw angle is not directly measured. Its state transfer function is

$$\bar{\phi} - \phi = \frac{s}{\Delta(s)}N_{gx} + \frac{\omega_o}{\Delta(s)}N_{gz} + \frac{K_\phi s + \omega_o K_\psi}{\Delta(s)}N_{\phi H} \tag{8.178}$$

$$\bar{\theta} - \theta = \frac{1}{s + K_\theta}N_{gy} + \frac{K_\theta}{s + K_\theta}N_{\theta H} \tag{8.179}$$

$$\bar{\psi} - \psi = -\frac{K_\psi + \omega_o}{\Delta(s)}N_{gx} + \frac{s + K_\phi}{\Delta(s)}N_{gz} + \frac{K_\psi s - \omega_o K_\phi}{\Delta(s)}N_{\phi H}, \tag{8.180}$$

where $\Delta(s)$ is the characteristic polynomial,

$$\Delta(s) = s^2 + K_\phi s + \omega_o(K_\psi + \omega_o), \tag{8.181}$$

by proper choice of K_ϕ, K_ψ, K_θ, $\Delta(s)$ and $s + K_\theta$ can be stable. If N_{gx}, N_{gy}, N_{gz} are constant drifts of gyros, the steady state errors are

$$(\bar{\phi} - \phi)_{t\to\infty} = \frac{1}{K_\psi + \omega_o}N_{gz} \tag{8.182}$$

$$(\bar{\theta} - \theta)_{t\to\infty} = \frac{1}{K_\theta}N_{gy} \tag{8.183}$$

$$(\bar{\psi} - \psi)_{t\to\infty} = -\frac{1}{\omega_o}N_{gx} + \frac{K_\phi}{\omega_o(K_\psi + \omega_o)}N_{gz}. \tag{8.184}$$

Large K_θ, K_ψ can reduce the steady state error, but is not effective in refraining the noise.

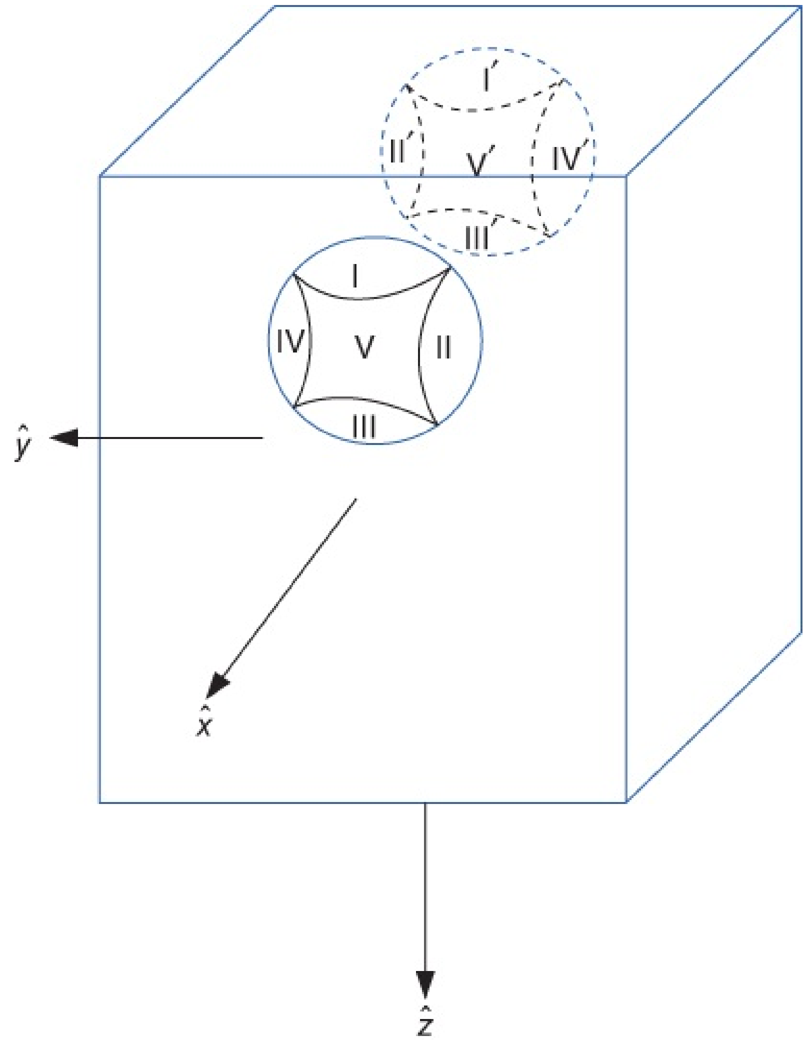

Figure 8.11 The “1/0” Sun sensor adapted from book [3]. The front sensor in the $\mathcal{F}_b$-direction includes $\hat{x}$, L_{11}, L_{12}, L_{13}, L_{14}; whereas the back sensor in the L_{15}-direction includes $-\hat{x}$, L_{21}, L_{22}, L_{23}, L_{24}. They are all in the body frame L_{25}

8.5.2.3 Attitude Capture

Another kind of Sun sensor is the “0/1” Sun sensor which is used for Sun capture. See Figure 8.11. It is in the body frame $\mathcal{F}_b$. If the Sun is in the region IV, let the satellite rotate around $\hat{x}$-axis; in the region II, rotate about -$\hat{x}$-axis; region III and V, about -$\hat{y}$-axis. At last the Sun is in region I near -$\hat{z}$-axis. If we use two “0/1” Sun sensors, the control logic is

$$S_x = L_{14} + L_{22}, \quad S_{-x} = L_{12} + L_{24}, \tag{8.185}$$

$$S_y = L_{23} + L_{25}, \quad S_{-y} = L_{15} + L_{13}. \tag{8.186}$$

The meaning, for example, is that if the Sun is in region IV of the front sensor and region II in the back sensor, i.e. $L_{14} = L_{22} = 1$ while others are zero, the satellite should rotate about the $\hat{x}$-axis, etc. If the Sun is in the direction of -$\hat{z}$-axis in $\mathcal{F}_b$, and $r_{ij} \in \mathbf{R}^{bo}$ in Equation (8.156), the Sun's vector in $\mathcal{F}_o$ is $\mathbf{s}_o = [s_{ox}, s_{oy}, s_{oz}]^T$, and $\mathbf{s}_b = [0, 0, -1]^T$ then $[r_{31}, r_{32}, r_{33}]^T = -[s_{ox}, s_{oy}, s_{oz}]^T$, also by the two Earth sensors.

$$r_{13} = \frac{-\cos\phi_H \sin\theta_H}{\sqrt{1-\sin^2\phi_H \sin^2\theta_H}} \quad \textbf{8.187}$$

$$r_{23} = \frac{\sin\phi_H \cos\theta_H}{\sqrt{1-\sin^2\phi_H \sin^2\theta_H}}. \quad \textbf{8.188}$$

As also for example $r_{12} = -r_{21}r_{22} + r_{23}r_{31}$, $r_{21} = -r_{12}r_{33} + r_{13}r_{32}$, then

$$r_{21} = \frac{r_{13}r_{32} - r_{32}r_{31}r_{33}}{r_{13}^2 + r_{23}^2} \quad \textbf{8.189}$$

$$r_{11} = -\frac{r_{23}r_{32} + r_{12}r_{31}r_{33}}{r_{13}^2 + r_{23}^2}. \quad \textbf{8.190}$$

More discussion about these models are given in reference [3].

8.6 Summary and Keywords

This chapter introduces satellite attitude notions and their representations by different methods and also describes a few attitude determination algorithms and their realizations.

Keywords: attitude kinematics, Euler angle, direct cosine matrix (DCM), quaternion, Euler theorem, satellite sensor, attitude determination, triad algorithm, orbital compass, attitude capture.

Problems

8.1 What are the 6 constraints on the 9 elements of a rotation matrix $\mathbf{R}$? List some other relations such as $r_{21} = -r_{12}r_{33} + r_{13}r_{32}$ and prove one of them.

8.2 Consider a unit vector $\hat{a}$, with components in a particular reference frame: $\hat{a} = [a_1, a_2, a_3]^T$. Show that $\hat{a}$ satisfies the following equations:

a. $(\hat{a}\hat{a}^T)^2 = \hat{a}\hat{a}^T$.

b. $\hat{a}^\times \hat{a} = 0$.

c. $\hat{a}\hat{a}^T\hat{a}^\times = 0$.

d. $\hat{a}^\times\hat{a}^\times = -\hat{a}^T\hat{a}\mathbf{1} + \hat{a}\hat{a}^T$.

8.3 Roll, pitch and yaw are Euler angles and are sometimes defined as a 3-2-1 sequence and sometimes defined as a 1-2-3 sequence What's the difference?

8.4 For Euler angle what happens when $\theta \to n\pi/2$, for odd n? What happens when the Euler angles and their rates are “small” ?

8.5 Using the relationship between the elements of the quaternion and the Euler angle and axis, verify that the expressions for **R**, in terms of $\bar{\mathbf{q}}$ and $\mathbf{a}, \Phi$ are equivalent.

8.6 Develop $\mathbf{S}(\theta)$ for a (2-3-1) rotation from $\mathcal{F}_i$ to $\mathcal{F}_b$, so that $\omega_b^{ib} = \mathbf{S}(\theta)\dot{\theta}$. Where is $\mathbf{S}(\theta)$ singular?

8.7 Consider the following rotation matrix $\mathbf{R}_{ab}$ that transforms vectors from $\mathcal{F}_b$ to $\mathcal{F}_a$:

$$\mathbf{R}^{ab} = \begin{bmatrix} 0.45457972 & 0.43387382 & -0.77788868 \\ -0.34766601 & 0.89049359 & 0.29351236 \\ 0.82005221 & 0.13702069 & 0.55564350 \end{bmatrix}. \quad \mathbf{8.191}$$

Use at least two different properties of rotation matrices to convince yourself that $\mathbf{R}^{ab}$ is indeed a rotation matrix. Remark on any discrepancies you notice. If it is 1-2-3 sequence, what are its Euler angles?

8.8 Determine the Euler axis $\mathbf{a}$ and Euler principal angle Φ directly from $\mathbf{R}^{ab}$. Verify your results using the formula for $\mathbf{R}(\mathbf{a}, \Phi)$. Verify that $\mathbf{Ra} = \mathbf{a}$.

8.9 Determine the components of the quaternion $\bar{\mathbf{q}}$, directly from $\mathbf{R}^{ab}$. Verify your results using the formula for $\mathbf{R}(\bar{\mathbf{q}})$. Verify your results using the relationship between $\bar{\mathbf{q}}$ and $(\mathbf{a}, \Phi)$.

8.10 Derive the formula for a (3-1-3) rotation. Determine the Euler angles for a (2-3-1) rotation, directly from $\mathbf{R}^{ab}$. Verify your results using the formula you derived.

8.11 What is the expression of small angle DCM, $\mathbf{R}$ for the $(\mathbf{a}, \Phi)$ representation? If $\phi = 0$, what do you do about the Euler axis/angle method?

8.12 Write a short paragraph discussing the relative merits of the three different representations of **R**.

8.13 This problem requires numerical integration of the kinematics equations of motion. You should have a look at the MATLAB® reference if needed. Suppose that $\mathcal{F}_b$ and $\mathcal{F}_a$ are initially aligned, so that $\mathbf{R}^{ba}(0) = \mathbf{1}$. At $t = 0$, $\mathcal{F}_b$ begins to rotate with angular velocity $\omega_b = e^{-4t}[\sin t, \sin 2t, \sin 3t]^T$ with respect to $\mathcal{F}_b$.

> **1.** Use a (1-2-3) Euler angle sequence and make a plot of the three Euler angles $vs.$ time for $t = 0$ to 10 s. Use a sufficiently small step size so that the resulting plot is “smooth.”
>
> **2.** Use the quaternion representation and make a plot of the four components of the quaternion vs. time for t = 0 to 10 s. Use a sufficiently small step size so that the resulting plot is “smooth.”
>
> **3.** Use the results of the two integrations to determine the rotation matrix at t = 10 s. Do this using the expression for $\mathbf{R}(\theta)$ and for $\mathbf{R}(\bar{\mathbf{q}})$. Compare the results.

8.14 Prove Equations (8.81)–(8.83).

References

[1] Wertz, J.R. (1985) *Spacecraft Attitude Determination and Control*. Boston: D. Reidel.

[2] Schaub, H. and Junkins, J. (2003) *Analytical Mechanics of Space Systems*, AIAA Education Series. Reston, VA: AIAA.

[3] Tu, S. (2001) *Satellite Attitude Dynamics and Control* (in Chinese). Beijing: China Astronautic Publishing House.

[4] Liu, C. and Hu, W. (2014) Relative pose estimation for cylinder shaped spacecraft using a single image, *IEEE Transactions on Aero and Electronic Systems*, **50**(4): 3036–56.