On A Mallows-type Model For (Ranked) Choices

Yifan Feng[12], and Yuxuan Tang[2]

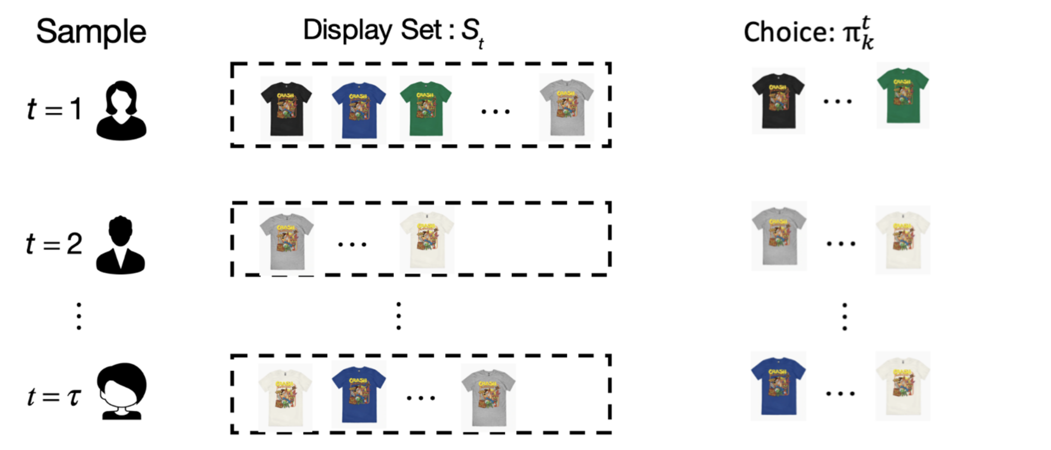
[1]Department of Analytics and Operations, and [2]Institute of Operations Research and Analytics, National University of Singapore, Singapore

Summary of Results

- We identify a new **distance-based (Mallows-type) ranking model**.
- It aggregates into a **simple closed-form** probability of any top-k subranking π_k among an arbitrary subset S, i.e., $\mathbb{P}(\pi_k|S)$.
- We develop **effective parameter learning** with theoretically proven **consistency**.

A Motivating Example

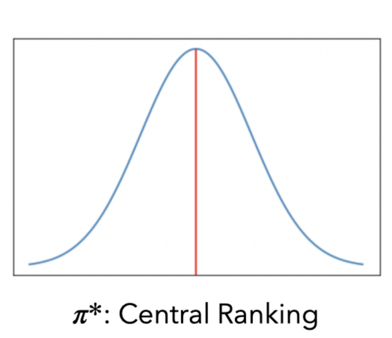
To learn the preference over n product prototypes, a company displays different subsets to customers and ask for top-k feedback.



How to aggregate the population's preferences from their ranked choices?

What are Mallows-type Models?

A collection of **Probability distribution of rankings**.



$$\lambda(\pi) = \frac{q^{d(\pi^*,\pi)}}{\sum_{\pi'} q^{d(\pi^*,\pi')}}$$

$d(\pi^*, \pi)$: Distance of π^* to π.

q: Dispersion parameter.

Figure 1: The probability of a ranking $\lambda(\pi)$ exponentially decays as its distance to the central ranking increases.

Different distance functions lead to different distributions.

Challenge

Under any existing Mallows-type model, it's difficult to obtain ranked choice probabilities even when $k = 1$.

A New Distance Function: Reverse Major Index (RMJ)

Assume the central ranking is the identity ranking, i.e, $\pi^* = (1, 2, \ldots, n)$.

$$\mathbf{d_R}(\pi) = \sum_{i=1}^{n-1} \mathbb{I}\{\pi(\mathbf{i}) > \pi(\mathbf{i+1})\} \cdot (\mathbf{n - i})$$

$k = 1$

Simple Closed-form Choice Probability

Let a display set $S = \{x_1, x_2, \ldots, x_M\}$ be such that $x_1 < x_2 < \ldots < x_M$. Under the RMJ-based ranking model

$$\mathbb{P}(x_i|S) = \frac{q^{i-1}}{1 + q + \ldots + q^{M-1}}$$

Parameter Learning: Maximum Likelihood Estimation

Given historical data $H_T = (S_1, x_1, \ldots, S_T, x_T)$. The MLE formulation is

$$\sum_{t=1}^{T} \log\left(\frac{1-q}{1-q^{|S_t|}}\right) + \log q \underbrace{\sum_{(i,j):i\neq j} \mathbb{I}\{j >_\pi i\} \cdot w_{ij}}_{\dagger} \tag{1}$$

- $w_{ij} := \sum_{t=1}^{T} \mathbb{I}\{\{i,j\} \subseteq S_t \text{ and } x_t = i\}$.
- Intuitively, a positive w_{ij} is an indication that item i should be preferred to item j.

$\dagger$ in (1) has an integer programming **reformulation**, and it's a **well-studied** weighted feedback arc set problem on tournaments.
Given π and set $\alpha = -\log q$, MLE is a **concave** function of α.
Under some mild conditions, the estimator $(\hat{\pi}, \hat{q})$ is consistent.

$k \geq 1$

Ranked Choice Probability and Its Learning

Given a display set S and a π_k such that $R(\pi_k) \subseteq S$, we have

$$\mathbb{P}(\pi_k|S) = q^{d_S(\pi_k)+L_S(\pi_k)} \cdot \frac{\psi(|S|-k,q)}{\psi(|S|,q)},$$

where $d_S(\pi_k) := \sum_{i=1}^{k-1} \mathbb{I}\{\pi_k(i) > \pi_k(i+1)\} \cdot (|S| - i)$, $L_S(\pi_k) := |\{x \in R^c(\pi_k) \cap S : x < \pi_k(k)\}|$, $\psi(n, q) := \prod_{i=1}^{n} \left(1 + q + \cdots + q^{(i-1)}\right)$.
Given historical data $H_T = (S_1, \pi_k^1, \ldots, S_T, \pi_k^T)$, where $\pi_k^t = (x_1^t, \ldots, x_k^t)$.
The **MLE** for the central ranking can be obtained from the same integer programming with a **generalized definition of** w_{ij} below

$$w_{ij} = \sum_{t=1}^{T} \left[\sum_{h=1}^{k-1} (|S_t| - h) \cdot \mathbb{I}\{x_h^t = i, x_{h+1}^t = j\} + \mathbb{I}\{x_k^t = i\} \cdot \mathbb{I}\{j \in S_t \backslash \{x_1^t, \ldots, x_{k-1}^t\}\} \right]$$

Sampling

Probability Distribution of Top-k

$$\lambda(\pi_k) = q^{d(\pi_k)+L(\pi_k)} \cdot \frac{\psi(n-k,q)}{\psi(n,q)},$$

where $d(\pi_k) := \sum_{i=1}^{k-1} \mathbb{I}\{\pi_k(i) > \pi_k(i+1)\} \cdot (n - i)$, $L(\pi_k) := |\{x \in R^c(\pi_k) : x < \pi_k(k)\}|$.

Sampling of Next Position

Given π_k such that $\pi_k(k) = z$, the conditional probability for the $(k+1)$-positioned item is

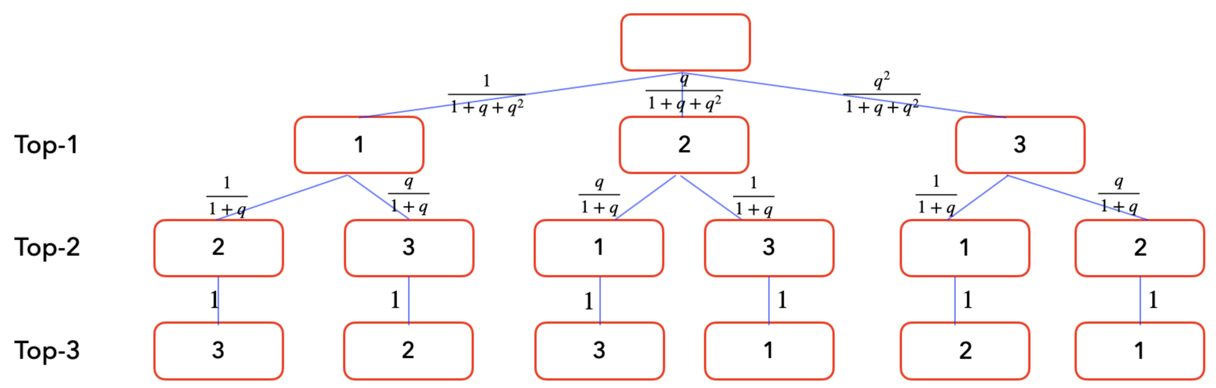
$$\mathbb{P}(\pi_{k+1} = \pi_k \oplus y \mid \pi_k) = \frac{q^{h(y|z)-1}}{1 + q + \cdots + q^{n-k-1}},$$

where

$$h(y|z) = \begin{cases} \sum_{x\in R^c(\pi_k)} \mathbb{I}\{z < x \leq y\} & \text{if } y > z, \\ n - k - \sum_{x\in R^c(\pi_k)} \mathbb{I}\{y < x < z\} & \text{if } y < z. \end{cases}$$

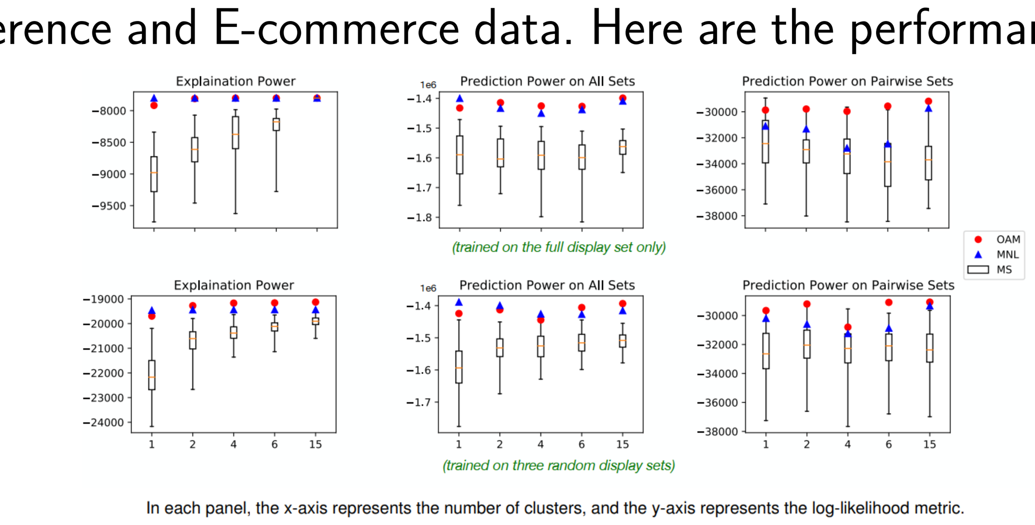
$O(nk)$

An example of 3 items, $\{1, 2, 3\}$.



Experiments

Real data. We test top-1 prediction power on Sushi preference and E-commerce data. Here are the performances:



We also conduct robustness check on top-k choice and test our estimation method when n is large.

Full Paper is Available at:

https://arxiv.org/abs/2207.01783